IS5152 Data-driven Decision Making
SEMESTER 2, 2022-2023
Assignment 1
Suggested solution

1. (10 points) A company manufactures two products (1 and 2). Each unit of product 1 can be sold for \$15, and each unit of product 2 for \$25. Each product requires raw material and two types of labor (skilled and unskilled) as shown in the table below:

	Product 1	Product 2
Skilled labor	3 hours	4 hours
Unskilled labor	2 hours	3 hours
Raw material	1 unit	2 units

At present, the company has available 100 hours of skilled labor, 70 hours of unskilled labor, and 30 units of raw material. Because of marketing considerations at least 3 units of product 2 must be produced.

(a) (2 points) Let x_1 and x_2 be the number of units of product 1 and product 2 to be produced, respectively. Formulate a linear program to maximize total revenue.

$$\begin{aligned}
\max\ Z = 15x_1 + 25x_2 & & \\
\text{s.}t.\ 3x_1 + 4x_2 &\leq 100 \quad \text{Skilled labor constraint} \\
2x_1 + 3x_2 &\leq 70 \quad \text{Unskilled labor constraint} \\
x_1 + 2x_2 &\leq 30 \quad \text{Raw material constraint} \\
x_2 &\geq 3 \Leftrightarrow -x_2 \leq -3 \quad \text{Product 2 constraint} \\
x_1, x_2 &\geq 0
\end{aligned}$$

Note: $x_2 \geq 0$ is redundant, since x_2 must be at least 3. It is included here to fit standard LP formulation.

(b) (2 points) Show that the best production level is $x_1 = 24$ and $x_2 = 3$ by checking that all the necessary and sufficient conditions are satisfied.

KT conditions:

$$\begin{aligned}
-15 + 3u_1 + 2u_2 + u_3 - v_1 &= 0 \\
-25 + 4u_1 + 3u_2 + 2u_3 - u_4 - v_2 &= 0 \\
u_1(3x_1 + 4x_2 - 100) &= 0 \\
u_2(2x_1 + 3x_2 - 70) &= 0 \\
u_3(1x_1 + 2x_2 - 30) &= 0 \\
u_4(x_2 - 3) &= 0 \\
x_1 v_1 &= 0 \\
x_2 v_2 &= 0 \\
u_1, u_2, u_3, u_4, v_1, v_2 &\geq 0
\end{aligned}$$

Check feasibility given $x_1 = 24, x_2 = 3$:

$$\begin{aligned} 3x_1 + 4x_2 = 84 < 100 &\rightarrow u_1 = 0 \\ 2x_1 + 3x_2 = 57 < 70 &\rightarrow u_2 = 0 \\ 1x_1 + 2x_2 = 30 & \\ x_2 = 3 & \\ x_1 > 0 &\rightarrow v_1 = 0 \end{aligned}$$

We have the following equations:

$$\begin{aligned} -15 + u_3 &= 0 \\ -25 + 2u_3 - u_4 &= 0 \end{aligned}$$

Let $u_3 = 15, u_4 = 5, v_1 = 0$, then all the KT necessary and sufficient conditions are satisfied. Optimal objective function value = $Z = 15x_1 + 25x_2 = 15(24) + 25(3) = 435$.

(c) (2 points) How much would the company be willing to pay for an additional unit of each type of labor?

- For skilled labor: $u_1 = 0$.
- For unskilled labor: $u_2 = 0$.
- Since currently both labor constraints are not binding, having more labor will not improve the solution /increase the revenue.

(d) (4 points) State the dual of the linear program from part (a). What is the solution of this dual linear program?

$$\begin{aligned} \min\ W = 100u_1 + 70u_2 + 30u_3 - 3u_4 & \\ \text{s.}t.\ 3u_1 + 2u_2 + u_3 &\geq 15 \\ 4u_1 + 3u_2 + 2u_3 - u_4 &\geq 25 \\ u_1, u_2, u_3, u_4 &\geq 0 \end{aligned}$$

Solution: $u_1 = 0, u_2 = 0, u_3 = 15, u_4 = 5, W = (30 \times 15 - 3 \times 5) = 435 = Z$

2. (10 points) Find the solution of the quadratic programming problem:

$$\min\ 2x_1^2 - x_2$$

subject to

$$\begin{aligned} 2x_1 - x_2 &\leq 1 \\ x_1 + x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Show that all the necessary and sufficient optimality conditions are satisfied.

KT conditions:

$$\begin{aligned} 4x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 &= 0 \\ -1 - \lambda_1 + \lambda_2 - \lambda_4 &= 0 \\ \lambda_1(2x_1 - x_2 - 1) &= 0 \\ \lambda_2(x_1 + x_2 - 1) &= 0 \\ \lambda_3\, x_1 &= 0 \\ \lambda_4\, x_2 &= 0 \end{aligned}$$

Let $x_1 = 0$ and $x_2 = 1$ (why?), then $\lambda_1 = \lambda_4 = 0$. We have:

$$\begin{aligned} -1 - \lambda_1 + \lambda_2 - \lambda_4 &= 0 \\ \lambda_2 &= 1 \end{aligned}$$

and

$$\begin{aligned} 4x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 &= 0 \\ 0 + 0 + 1 - \lambda_3 &= 0 \\ \lambda_3 &= 1 \end{aligned}$$

The objective function:

$$\begin{aligned} f(x_1, x_2) &= 2x_1^2 - x_2 \\ \nabla f(x_1, x_2) &= \begin{pmatrix} 4x_1 \\ -1 \end{pmatrix} \\ H(x_1, x_2) &= \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T H(x_1, x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= 4x_1^2 \geq 0 \end{aligned}$$

The objective function is convex and the constraint are linear, hence $(0, 1)$ is a global solution.

3. (10 points) Consider the following linearly constrained optimization problem:

$$\text{maximize } f(x) = -\ln(x_1 + 1) - x_2^2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

where *ln* denotes natural logarithm.

(a) (5 points) Use the Kuhn-Tucker (KT) conditions to derive an optimal solution.

Consider the problem of minimizing $-f(x)$ instead.
KT conditions: Feasibility +

$$\begin{aligned} \frac{1}{x_1 + 1} + \lambda_1 - \lambda_2 &= 0 \\ 2x_2 + 2\lambda_1 - \lambda_3 &= 0 \\ \lambda_1(x_1 + 2x_2 - 3) &= 0 \\ \lambda_2 x_1 &= 0 \\ \lambda_3 x_2 &= 0 \end{aligned}$$

Let $x_1 = x_2 = 0$, then

$$\begin{aligned} \frac{1}{x_1+1} + \lambda_1 - \lambda_2 = 0 \quad &\Leftrightarrow \quad 1 + \lambda_1 - \lambda_2 = 0 \\ 2x_2 + 2\lambda_1 - \lambda_3 = 0 \quad &\Leftrightarrow \quad 2\lambda_1 - \lambda_3 = 0 \\ \lambda_1(x_1 + 2x_2 - 3) = 0 \quad &\Leftrightarrow \quad \lambda_1 = 0 \\ \lambda_2 x_1 = 0 \quad & \\ \lambda_3 x_2 = 0 \quad & \end{aligned}$$

Let $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda_3 = 0$, all KT conditions are satisfied.

(b) (5 points) Is the solution you obtain in part (a) a global solution or a local solution? Explain your answer.

Global solution. The minimum value of $\ln(x_1 + 1) = 0$, given the constraint $x_1 \geq 0$, and the minimum value of $x_2^2 = 0$ when $x_2 = 0$.