IS5152 Assignment 1 Wang Ruicong A0244297W

1. (a)

maximize $15x_1 + 25x_2$

subject to
$3x_1+4x_2 \le 100$
$2x_1+3x_2 \le 70$
$x_1+2x_2 \le 30$
$x_2 \ge 3$
$x_1 \ge 0$

which is equivalent to

$-$ minimize $-15x_1 - 25x_2$

subject to
$3x_1+4x_2 \le 100$ ①
$2x_1+3x_2 \le 70$ ②
$x_1+2x_2 \le 30$ ③
$-x_2 \le -3$ ④
$-x_1 \le 0$ ⑤

(b) First, we check whether the solution is feasible

$x_1=24,\ x_2=3$

① $3x_1+4x_2 = 84 < 100$
② $2x_1+3x_2 = 57 < 70$
③ $x_1+2x_2 = 30$
④ $-x_2 = -3$
⑤ $-x_1 = -24 < 0$

$\Rightarrow$ $x_1=24,\ x_2=3$ is a feasible solution

Next, we check the necessary and sufficient conditions

$-15+3\lambda_1+2\lambda_2+\lambda_3-\lambda_5=0$
$-25+4\lambda_1+3\lambda_2+2\lambda_3-\lambda_4=0$
$\lambda_1(100-3x_1-4x_2)=0$
$\lambda_2(70-2x_1-3x_2)=0$
$\lambda_3(30-x_1-2x_2)=0$
$\lambda_4(-3+x_2)=0$
$\lambda_5 x_1=0$
$\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5 \ge 0$

let $x_1=24,\ x_2=3$,
then $\lambda_1=0,\ \lambda_2=0,\ \lambda_5=0$

and $\begin{cases} -15+\lambda_3=0 \\ -25+2\lambda_3-\lambda_4=0 \end{cases} \Rightarrow \begin{cases} \lambda_3=15>0 \\ \lambda_4=5>0 \end{cases}$

Also, the objective function and constraints are all linear and convex.

therefore. $x_1=24,\ x_2=3$ satisfy all the necessary and sufficient conditions

Thus, (24,3) is the best production level

The profit is $15\times24+25\times3=435$

(c) ① For addition unit of skilled labor : $\lambda_1=0$
② For addition unit of unskilled labor : $\lambda_2=0$
③ For addition unit of raw material : $\lambda_3=15$

(d) dual:

minimize $100u_1+70u_2+30u_3-3u_4$

subject to
$3u_1+2u_2+u_3 \ge 15$
$4u_1+3u_2+2u_3-u_4 \ge 25$
$u_1,u_2,u_3,u_4 \ge 0$

The solution is $u_1=\lambda_1=0,\ u_2=\lambda_2=0$
$u_3=\lambda_3=15,\ u_4=\lambda_4=5$
$\therefore u=(0,0,15,5)$

The objective function is $100\times0+70\times0+30\times15-3\times5$
$=435$

which is the same with the maximum profit in (b)

2、KT conditions:

$4x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 = 0$ ①

$-1 - \lambda_1 + \lambda_2 - \lambda_4 = 0$ ②

$\lambda_1(1 - 2x_1 + x_2) = 0$ ③

$\lambda_2(1 - x_1 - x_2) = 0$ ④

$\lambda_3 x_1 = 0$ ⑤

$\lambda_4 x_2 = 0$ ⑥

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$

If $\lambda_2 = 0$, from ②, $\lambda_1 + \lambda_4 = -1$, which contradicts $\lambda_1, \lambda_4 \geq 0$

Thus $\lambda_2 > 0$, $x_1 + x_2 = 1$

If $\lambda_3 = 0$, from ①, $4x_1 + 2\lambda_1 + \lambda_2 = 0$, which contradicts $x_1 \geq 0$, $\lambda_1 \geq 0$, $\lambda_2 > 0$

Thus $\lambda_3 > 0$, $\boxed{x_1 = 0 \Rightarrow x_2 = 1}$, $\begin{cases} 2x_1 - x_2 = -1 \leq 1 \\ x_1 + x_2 = 1 \\ x_1 = 0 \\ x_2 = 1 > 0 \end{cases} \Rightarrow x = (0,1)$ is feasible solution

Therefore from ③④⑤⑥, $\lambda_1 = 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 = 0$

$\therefore \begin{cases} 4x_1 + \lambda_2 - \lambda_3 = 0 \\ -1 + \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_2 = 1 \\ \lambda_3 = 1 \end{cases}$ The solution is $x_1 = 0$, $x_2 = 1$

Therefore, $x_1 = 0$, $x_2 = 1$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\lambda_4 = 0$ satisfy the KT necessary condition.

Also, $2x_1^2$ is convex function and $(-x_2)$ is a convex function

Then $2x_1^2 - x_2$ is convex function

also all the constraints are linear and convex

Thus, the sufficient optimality conditions are also satisfied for $x = (0,1)$

The optimal solution is $(0,1)$.

3、(a) The problem is equivalent to

$-$ minimize $\ln(x_1+1) + x_2^2$

subject to $x_1 + 2x_2 \leq 3$

$x_1 \geq 0$

$x_2 \geq 0$

KT conditions:

$\frac{1}{x_1+1} + \lambda_1 - \lambda_2 = 0$

$2x_2 + 2\lambda_1 - \lambda_3 = 0$

$\lambda_1(3 - x_1 - 2x_2) = 0$

$\lambda_2 x_1 = 0$

$\lambda_3 x_2 = 0$

$\lambda_1, \lambda_2, \lambda_3 \geq 0$

Because $\frac{1}{x_1+1} > 0$, $\lambda_1 \geq 0$

then $\lambda_2 > 0$, $x_1 = 0$

If $x_2 > 0$, $\lambda_3 = 0$, $2x_2 + 2\lambda_1 - \lambda_3 > 0$ which contradicts. equation $\neq 0$

Thus $x_2 = 0$ $\lambda_3 > 0$ $\Rightarrow x = (x_1, x_2) = (0,0)$

$\therefore \begin{cases} \lambda_1 - \lambda_2 + 1 = 0 \\ 2\lambda_1 - \lambda_3 = 0 \\ \lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{cases}$

$x_1 + 2x_2 = 0 < 3$,

$x_1 = 0$, $x_2 = 0$ is also feasible

Therefore $x_1 = 0$, $x_2 = 0$ satisfies KT conditions, and thus $x = (0,0)$ is an optimal solution

(b) The solution is global solution.

$-\ln(x_1+1)$ reaches maximum 0 when $x_1 = 0$ given $x_1 \geq 0$

$-x_2^2$ reaches maximum 0 when $x_2 = 0$

Therefore, $-\ln(x_1+1) - x_2^2$ reaches maximum 0 when $x_1 = 0$, $x_2 = 0$

The solution in (a) is global solution