Market Efficiency with Many Investors*

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This version: April 10, 2025

Abstract

Modern financial markets contain many investors. In this context, we study the role of information in investor decision-making, and the informational efficiency and liquidity of the market. An equilibrium is characterized in closed-form for a continuous-time economy with many market participants and imperfect competition, in which investors receive private information with varying quality, and are heterogeneous in their misperception of the information quality. In equilibrium, investor heterogeneity in their misperception generates return predictability by investors' trading, and trading of different investors follows a simple factor structure with weak factors. To conduct empirical analysis that builds on these equilibrium implications, we develop a new big-data econometric method that utilizes the factor structure to accommodate the high-dimensionality of these implications. Applying the framework to price and institution holding data of the US stock market, we document that individual institution's trading with impotent predictive power can collectively generate significant return predictability that persists for about a quarter. We estimate dynamic price impact of around 0.25 at quarterly frequency, a moderate misperception of institutions on their information quality, and institutions' contributions to the informational efficiency of the market.

Keywords: Many Investors, Private Information, Return Prediction, Factor Model, Weak Factor, Big-data Econometrics, Market Efficiency, Price Impact

*I benefit tremendously from the guidances and advices from Zhiguo He, Dacheng Xiu, Ralph Koijen, Stefan Nagel, and Pietro Veronesi.

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1 Introduction

Tremendous success has been achieved in testing the semi-strong form of the market efficiency hypothesis Fama (1970) by estimating expectations as functions of public information, e.g., prices, firm characteristics, and macro variables. Yet, this common practice does not directly allow for testing the strong version of the market efficiency hypothesis and investigate the important function of financial markets of aggregating private information. Despite the enormous theoretical achievements regarding the role of private information in financial market since seminal Grossman and Stiglitz (1980) and Kyle (1985), empirically investigating the role of private information is not among the easiest tasks. Indeed, the model of Kyle (1985) powerfully reveal that, while her private information allows the informed trader to constantly profit, the price process is still a martingale under public information set, which means using data on public information alone can not possibly identify the presence of information asymmetry.

Koijen and Yogo (2019) demonstrates that richer economic implications can be extracted from the joint moments of prices and quantities under equilibrium frameworks. In the context of Kyle (1985) model, the trading of the informed (noise) trader would positively (negatively) correlate with future price movements, suggesting the potential of joint moments of prices and quantities in studying the role of private information.

Modern financial markets contain many investors. High-dimensionality of a financial market may have significant implications on both the behaviors of agents within the economy (Martin and Nagel (2022)) and on how econometricians shall analyze the data generated by the economy (Giglio, Kelly, and Xiu (2022)).

In this paper, we investigate, in the presence of many investors, the role of information in affecting investor decision-making, and the efficiency and liquidity of the market, as revealed by the joint moments of prices and investor trading. Building on the symmetric model of Kyle, Obizhaeva, and Wang (2018), we model a continuous-time economy which consists of many (but not a continuum of) investors with heterogeneous information and belief structure, trading a risk-free asset and a risky asset in a centralized market. The risky asset generates a cash flow with unobservable growth rate. Investors receive private signal flows of heterogeneous precision about the growth rate, potentially misperceive the precision of their own signals to different extents, and infer the future price movements utilizing both the private signals and the public information embedded in past price and cash flow. At each moment, investors make optimal consumption-portfolio decisions based on their inference, taking into account the the impact of their actions on prices. However, investors do not observe the actions of each other.

To tractably characterize the equilibrium under heterogeneity, we deviate from the standard equilibrium definition, denoted by "asymptotic equilibrium", by leveraging the presence of many investors. Specifically, in an asymptotic equilibrium, given the price function and others' strategies, an agent is willing to take a suboptimal strategy, as long as the difference in trading under the suboptimal strategy and an exactly optimal one, compared to the magnitude of trading itself, vanishes as the total number of investors grows. In other words, we only require investors to act optimally (under their subjective measures) to the leading order. These leading-order optimal strategies are much simpler than the exactly optimal strategies in the current context, which then allows for a tractable characterization of an equilibrium. Indeed, one crux of the tractability is that, with many investors, the total precision of public and private information is almost the same across investors with different private information quality and misperception, because the total precision mostly comes from the public part. In this regard, the presence of many investors allows investors in the economy to take simple trading strategies while still acting almost optimally.

In equilibrium, as in Kyle (1985), investors' trading rates, rather than positions, are proportional to their subjective expectations of future price changes, to avoid incurring large trading cost due to price impact. They also rebalance proportionally to their current position to optimize the risk exposure. An investor's expected price change, on the other hand, is proportional to the difference between her private signal in the recent past and the average of others' private signals, which is reflected through price, multiplied by the misperception-adjusted precision of her own signal. The price dependence of investors' trading in turn determines the equilibrium market liquidity and price impact that every investor takes into account.

The equilibrium has empirical implications in three aspects. In the cross section, because the willingness to trade, i.e., the private signal, is highly idiosyncratic, the trading across investors follows a weak factor model, with the factor, up to a constant scalar, being minus a weighted sum of the idiosyncratic shocks to private signals across investors and capturing the minus pricing error. Unlike standard factor models, the idiosyncratic shocks and the factor negatively correlate, necessary for the aggregate trading to be zero. The loading of an investor's trading on the factor depends on both her trading intensity and the information quality, where the two are not one-to-one mapped because of her misperception. The more intensively the investor trades, the more she contributes to the pricing error, whereas the more informative she is, the more she is able to eliminate the pricing error. In the same direction, the investor's trading positively (negatively) predict future price changes when she trades relatively conservatively (aggressively) compared to others given her information quality, which is in fact captured by whether her misperception is above or below the average level. Finally, the equilibrium model indicates that the time-series property of an investor's trading is governed by the size of the rebalancing effect and the persistence of predictable price changes.

The empirical implications leads to straightforward empirical strategies. The covariance matrix of trading across investors provide information about the magnitude of investor trading and the distribution of misperception across investors. The regression of future price changes on trading identifies the magnitude of the pricing error and how quickly the predictable price change decays. Using the autocorrelations of investor trading, econometricians can estimate the rebalancing magnitude and compare the trading-implied predictability persistence with the one from the predictive regression.

In the presence of many investors, however, the implications are of high dimension. Directly using the sample covariance matrix of trading and running predictive regression with trading of each institution would incur large estimation errors. Motivated by the factor structure of trading, and given that the factor is weak, our estimation procedures are centered around conducting dimension reduction with a modification of standard principal component analysis (PCA) approach. Specifically, we replace all the diagonal elements of the sample covariance matrix with zeros and conduct eigendecomposition afterwards. When private information is the only trading motive, the eigendecomposition would generate two (one) eigenvectors[1] if there is (is not) misperception heterogeneity, due to the factor-idiosyncratic shock correlation. There could be a few more eigenvectors when there are other motives such as public information driving the trading pattern. Regardless, because the eigenvectors span the pricing error factor, it would be sufficient to run predictive regression with the linear combinations of all the investors' trading using those small number of eigenvectors and achieve dimension reduction for this part as well. We further provide the procedure and conditions to identify misperception.

We then conduct empirical analysis using the stock-level data and 13F institution holding data of the US stock market. We find that institutions' trading collectively significantly predict excess returns, and the predictability persists for around a quarter. We estimate the dynamic price impact coefficient to be around 0.25 at quarterly frequency, meaning that selling 1% of the total outstanding shares of a stock over a quarter would push its price downward by .25% during the quarter. Across different types of institutions, investment advisors mostly have misperception below average, whereas banks and mutual funds mostly have above-average misperception. In addition, we find that, for all most institutions, especially investment advisors, a large part of their quarterly position change can be attributed to intra-quarter trading of relatively high frequency which does not target the more persistent predictable returns that we document. Finally, we find that, according to the equilibrium model, the pricing error, caused by the unobservable growth rate and reflected by the persistent predictable returns, has a standard deviation lower bounded by about 0.89%. Across different types of investors, investment advisors, household sectors, and mutual funds are most important for price informativeness. In the absence of trading by all the investment advisors and of trading by all the mutual funds, the standard deviation of the pricing error would increase by at least about 37% and 18%, respectively.

The theory part of our paper extends the symmetric model of Kyle, Obizhaeva, and Wang (2018) to allow for flexible heterogeneity in information quality and investor misperception in a dynamic economy, and contribute to the broad literature of asset pricing models with information asymmetry: Kyle (1989), Wang (1993), Wang (1994), He and Wang (1995), Vayanos (1999), Vives (2011), and Du and Zhu (2017). The empirically relevant many-investor setup indicates that investors take simple equilibrium strategies. It also indicates that, although in equilibrium there could be sizable pricing errors, each investor can at best profit out of a tiny portion of it. These results highlights the impact of high-dimensionality on agent behaviors and in this regard the paper is also related to Martin and

[1] Precisely, eigenvectors with nonzero eigenvalues.

Nagel (2022).

The econometric procedures in the paper build on and contribute to the evolving literature on the applications of statistical and machine learning in asset pricing, in particular on the topic of factor models and PCA approaches, e.g., Kelly and Pruitt (2013), Kozak, Nagel, and Santosh (2018), Kelly, Pruitt, and Su (2019), Kozak, Nagel, and Santosh (2020), Giglio and Xiu (2021), and Giglio, Xiu, and Zhang (2021). Complementary to this literature, the factors in our paper are weak yet pervasive and we propose a modification of the standard PCA accordingly. On the other hand, the way we conduct predictive regression shares the same spirit of Da, Nagel, and Xiu (2022), which demonstrates that weak predictors, if efficiently combined, can collectively generate significant predictive power. In our paper, however, the relative predictive power of each institution's trading is revealed by the trading correlations across institutions, which directly guides how we combine the predictors.

Moreover, our paper relates to the enormous literature on mutual funds, e.g., Berk and Green (2004), Fama and French (2010), Pástor and Stambaugh (2012), Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014, 2016), and Song (2020).[2] Utilizing PCA-based procedures, we document significant return predictability by efficiently combining trading of different institutions, despite that the aggregate institution trading barely predicts future returns. In addition, we demonstrate that how much predictive power econometricians can extract from investors' positions hinges on the dispersion of their misperception and that investors' portfolio returns and their contributions to market efficiency are not one-to-one mapped. For instance, in the fully symmetric case, even though investors are injecting their private information into the price, their trading has zero correlation with price movements and the alphas of their portfolios are all zero. We further provide estimates of the contributions to price informativeness by each type of institutions.

Finally, our paper is also connected to the literature that empirically estimates price elasticities of demand in financial market under various scenarios that are orders of magnitude smaller than what the standard models would imply,[3] including Harris and Gurel (1986), Shleifer (1986), Chang, Hong, and Liskovich (2015), Koijen and Yogo (2019), and Gabaix and Koijen (2021). Our structural model-based estimate of dynamic demand elasticity of around 4 at quarterly frequency is similar to the existing estimates in magnitude, suggesting that investors' low signal-to-noise ratio may provide a potential explanation to the small magnitude of estimated elasticities.

Our paper proceeds as follows. Sections 2 – 4 characterize the equilibrium model with many investors. Specifically, Section 2 sets up the economy, Section 3 characterizes the equilibrium under symmetry , and Section 4 generalizes the characterization to the scenario with heterogeneous information quality and misperception. Section 5 presents the empirical implications of the equilibrium model and develops econometric procedures to combine the implications with price and quantity data. Empirical analysis of the US stock market is conducted in Section 6. Section 7 concludes. The

[2]See also Wermers (2000), Berk and Van Binsbergen (2015), Pástor, Stambaugh, and Taylor (2015), Pástor, Stambaugh, and Taylor (2017), and Pástor, Stambaugh, and Taylor (2020).

[3]See, e.g., Petajisto (2009).

appendix provides technical details.

2 Setup of the Economy

2.1 Assets

There is a risky financial asset in the economy with zero total supply and its price at time t is denoted by $P_{i,t}$. In addition there exists a zero-return risk-free asset with risk-free rate r. The risky asset generates cash flow at rate $D_t + D_t^o$. It evolves according to

$$dD_t = -\xi^D D_t dt + G_t dt + \sigma^D dZ_t^D, \quad \text{and} \quad dD_t^o = -\xi^D D_t^o dt + G_t^o dt + \sigma^D dZ_t^{D,o} \tag{2.1}$$

Here ξ^D is the mean-reversion parameter, σ^D is the volatility parameter, and Z_t^D and $Z_t^{D,o}$ are mutually independent standard Brownian motions. The change rates of D_t and D_t^o are also affected by the dividend growth rates G_t and G_t^o, which both follow mean-reversion processes:

$$dG_t = -\xi^G G_t dt + \sigma^G dZ_t^G \quad \text{and} \quad dG_t^o = -\xi^o G_t^o dt + \sigma^o dZ_t^o, \tag{2.2}$$

where Z_t^G and Z_t^o are two other mutually independent standard Brownian motions, which do not depend on Z_t^D and $Z_t^{D,o}$ as well.

2.2 Preference

We index the many investors participating the economy by $j \in \mathcal{J}$. Investor j trades both risk-free and risky assets and we use $x_{j,t}^f$ to stand for the value of her time-t holding of risk-free asset, and denote by $x_{j,t}$ her time-t holding of the risky asset. Their time derivatives are denoted by by $\dot{x}_{j,t}^f$ and $\dot{x}_{j,t}$. The objective of investor j is

$$\max_{\{\dot{x}_{j,s}, c_{j,s}\}_{s \geq t}} \mathrm{E}_{j,t} \int_t^\infty e^{-\rho(s-t)} u(c_{j,s}) ds, \quad \text{with} \quad u(c_{j,s}) = -e^{-\gamma c_{j,s}}, \tag{2.3}$$

where the change of the risk-free asset value satisfies

$$\dot{x}_{j,t}^f = r x_{j,t}^f - c_{j,t} + x_{j,t}\big(D_t + D_t^o\big) - \dot{x}_{j,t} P_t \tag{2.4}$$

We use $\mathrm{E}_{j,t}(\cdot)$ to represent expectation under investor j' subjective measure, conditional on her time-t information set, which we will further discuss shortly. The trading rate $\dot{x}_{j,t}^f$ and $\dot{x}_{j,t}$ must also be measurable to the time-t information set. Market-clearing price P_t can depend on the investor's current and past trading, and the investor takes her price impact into account when making decisions.

2.3 Information and Belief

For the risky asset, we assume investors directly observe its price and cash flow components D_t and D_t^o. The cash-flow growth rate component G_t^o is also observable to each investor. However, investors misperceive how G_t^o drives cash-flow growth. Concretely, under investor j's subjective measure, the cash-flow rate follows

$$dD_t^o = -\xi^D D_t^o dt + \big(G_t + (1+\eta_j)G_t^o\big)dt + \sigma^D d\widehat{Z}_{j,t}^{D,o}, \tag{2.5}$$

where investor j considers $\widehat{Z}_{j,t}^{D,o} := Z_t^{D,o} - (\sigma^D)^{-1}\eta_j \int^t G_s^o ds$ a standard Brownian motion. On the other hand, investors do not observe the growth rate component G_t. Investor j instead receives a noisy signal flow $S_{j,t}$ about G_t, which satisfies

$$dS_{j,t} = G_t dt + \sigma_j^S dZ_{j,t}^S. \tag{2.6}$$

Here $Z_{j,t}^S$ is a standard Brownian motion independent of $(Z_t^D, Z_t^G$, and is also independent across assets. The volatility parameter σ_j^S controls the signal-to-noise ratio of $S_{j,t}$. The investor, however, perceive the drift of her signal as $\omega_j G_t$ in constructing her subjective expectation $\mathrm{E}_{j,t}(\cdot)$:

$$dS_{j,t} = \omega_j G_t dt + \sigma_j^S d\widehat{Z}_{j,t}^S.$$

where $\widehat{Z}_{j,t}^S := Z_{j,t}^S + (\sigma_j^S)^{-1}(1-\omega_j)\int^t G_s ds$ is considered by investor j as a standard Brownian motion. In other words, she has an incorrect understanding of the informativeness of her signals. This matches the econometric fact that it is hard to measure drift precisely without long time span, whereas volatility is perfectly measurable in continuous time. We assume that investors correctly perceive the dynamics of both asset fundamentals and others' private signals. Investors do not observe each other's private signals or trading actions. Therefore, investor j's subjective expectation $\mathrm{E}_{j,t}(\cdot)$ is only measurable to $\mathcal{F}_{j,t}$, the information set generated by $\{S_{j,s}, P_s, D_s, D_s^o, G_s^o\}_{s\le t}$, i.e., her private signals, the price, the cash flow, and the observable growth rate component, up to time t.

3 Equilibrium with Symmetric Information Structure

In this section, we conduct equilibrium analysis under the symmetric information structure.

3.1 Setup

For results in this section, we impose the following assumption:

Assumption 1. *The cash flow component D_t^o and the observable growth rate G_t^o are both zero. Moreover, signal noise parameter σ_j^S and the misperception parameter ω_j stay invariant across all investor $j \in \mathcal{J}$:*

$$\sigma_j^S = \sigma^S, \quad \text{and} \quad \omega_j = \omega, \quad \forall j \in \mathcal{J}. \tag{3.1}$$

Under Assumption 1, our setup matches that of Kyle, Obizhaeva, and Wang (2018), where an equilibrium with linear flow-strategies is elegantly characterized. We conduct an asymptotic exercise to understand the properties of the equilibrium of Kyle, Obizhaeva, and Wang (2018) as the total number of investors gets large. This matches our empirical goal as, for instance, in stock markets typical stocks are being traded by a large number of investors. Moreover, in this case we are able to obtain provide closed-form expressions of all the endogenous parameters, whereas in general numerical tools to required to solve key endogenous parameters. Specifically, we consider the following drifting sequence of exogenous parameters:

Assumption 2. *As J, i.e., the size of $\mathcal{J}$, increases,*

$$\left(\sigma^S\right)^2 = J\iota,$$

whereas ι and all the other exogenous parameters $(\xi^D, \xi^G, \sigma^D, \sigma^G, \omega, \rho, \gamma)$ stay unchanged.

The assumption is to prevent total precision of all investors' signals from exploding, which would lead to that asset price converges to the exact fundamental value and that price impact vanishes. It is notable that we keep γ constant. Arguably, absolute risk aversion changes with level of wealth. That the market consists of a lot of investors is connected with that each investor is relatively small and potentially has large absolute risk aversion. Fortunately, under the current setup, equilibria with different values of γ are isomorphic, and characterizing an equilibrium with any value of γ would be sufficient.[4]

Given the information structure, investors form their expected returns, which could depend on both their private signals and price history. On the other hand, however, price is endogenous and is affected by how investors trade based on the subjective expected returns. The equilibrium concept is formally defined as:

Definition 1: An equilibrium is a set of investor trading strategies $\{\dot{x}^f_{j,t}(\cdot), \dot{x}_{j,t}(\cdot)\}_{j\in\mathcal{J}}$ and price function $P_t\big(\{\dot{x}_{j,s}(\cdot), D_s, S_{j,s}, P_{s'}\}_{s\le t, s'<t, j\in\mathcal{J}}\big)$ such that

(i) For each investor $j \in \mathcal{J}$, given the price functions and the strategies of all the other investors, the trading strategy $\big(\dot{x}^f_{j,t}(\cdot), \dot{x}_{j,t}(\cdot)\big)$ solves the optimization problem specified by (2.3) and (2.4), subject to that $\dot{x}^f_{j,t}(\cdot)$ and $\dot{x}_{j,t}(\cdot)$ are measurable to the information set $\mathcal{F}_{j,t}$;

(ii) The risky asset market clears:

$$\sum_{j\in\mathcal{J}} x_{j,t} = 0.$$

Notably, in Definition 1 trading strategies can depend on contemporaneous prices. In other words, investors can trade in the form of submitting demand schedules, which allows the market to clear in the absence of market makers.

3.2 Equilibrium

We conjecture that in equilibrium all investors submit symmetric linear demand schedules as follows:

$$\dot{x}_{j,t} = \psi^D G^D_t + \psi^C D_t + \psi^S G_{j,t} - \psi^P P_t - \psi^H x_{j,t}, \tag{3.2}$$

[4] If we double the risk aversion, then all the price variables would not change and all the quantity variable would be halved. See Theorem 4 of Kyle, Obizhaeva, and Wang (2018).

where processes G_t^D and $G_{j,t}$ are constructed as follows:

$$\begin{aligned} G_t^D &= (\xi^P - \xi^G)\int_{-\infty}^{t} e^{-\xi^P(t-s)}(dD_s - \xi^G D_s ds), \quad (3.3) \\ G_{j,t} &= (\xi^P - \xi^G)\int_{-\infty}^{t} e^{-\xi^P(t-s)} dS_{j,s}, \quad (3.4) \end{aligned}$$

with the parameter ξ^P defined by

$$\xi^P = \sqrt{(\xi^G)^2 + (\sigma^G)^2\big((\sigma^D)^{-2} + \iota^{-1}\big)}. \quad (3.5)$$

In other words, G_t^D and $G_{j,t}$ are constructed using cash flow and private signals up to t, which both contain information about the growth rate G_t. The reason behind the specific way of construction will be explained shortly.

Market clearing forces both aggregate position and aggregate trading to be zero, which leads to

$$\dot{x}_{j,t} + \psi^H x_{j,t} = -\sum_{j\in\mathcal{J}:j'\neq j}(\dot{x}_{j',t} + \psi^H x_{j',t}). \quad (3.6)$$

Then, suppose all investors $j' \in \mathcal{J}$ with $j' \neq j$ submit demand schedule (3.2), the supply curve faced by investor j, which is a function of investor trading, would be

$$P_t(\dot{x}_{j,t}) = (\psi^P)^{-1}\Bigg(\psi^D G_t^D + \psi^C D_t + \psi^S \frac{1}{J-1}\sum_{j\in\mathcal{J}:j'\neq j} G_{j',t} + \frac{1}{J-1}\big(\dot{x}_{j,t} + \psi^H x_{j,t}\big)\Bigg). \quad (3.7)$$

Investor j's problem is to solve her optimization problem specified by (2.3) and (2.4) under (3.7). The equilibrium would be established if her optimal strategy also satisfy the conjecture (3.2).

The next proposition presents the implications of large J on the equilibrium characterized by Kyle, Obizhaeva, and Wang (2018):

Proposition 1. *Suppose Assumptions 1 and 2 hold. Then there exists an equilibrium as in Definition 1 if J is sufficiently large, $\xi^G > r$, and $\omega > 2$.*[5] *The equilibrium has the following properties:*

(i) In equilibrium the price satisfies

$$P_t = \frac{D_t}{\xi^D + r} + \frac{\mathrm{E}(G_t|\{D_s, \bar{S}_s\}_{s\leq t})}{(\xi^D + r)(\xi^G + r)} + O_{\mathrm{P}}(J^{-1}), \quad \textit{with} \quad \bar{S}_s = \frac{1}{J}\sum_{j\in\mathcal{J}} S_{j,s}, \quad (3.8)$$

where $\mathrm{E}(\cdot)$ is the expectation under the objective measure. Moreover, it holds that

$$\mathrm{E}(G_t|\{D_s, \bar{S}_s\}_{s\leq t}) = \frac{(\sigma^D)^{-2} G_t^D + \iota^{-1}\bar{G}_t}{(\sigma^D)^{-2} + \iota^{-1}}, \quad \textit{with} \quad \bar{G}_t := \frac{1}{J}\sum_{j\in\mathcal{J}} G_{j,t}, \quad (3.9)$$

[5] This shows that the existence condition conjectured by Kyle, Obizhaeva, and Wang (2018), at least when the market is large, is correct.

(ii) The optimal trading strategy follows (3.2), with $\big((J-1)\psi^P\big)^{-1} = \zeta + O_{\mathrm{P}}(J^{-1})$ *and* $\psi^H = b + O_{\mathrm{P}}(J^{-1})$*, and the trading satisfies*

$$\dot{x}_{j,t} = a\frac{\xi^P + r}{(\xi^D + r)(\xi^G + r)}\frac{\omega(\sigma^S)^{-2}}{(\sigma^D)^{-2} + \iota^{-1}}(G_{j,t} - \bar{G}_t) - bx_{j,t} + O_{\mathrm{P}}(J^{-3/2}). \tag{3.10}$$

Here the endogenous parameters a*,* b*, and* ζ *are given by*

$$a = \frac{1}{\zeta\omega(\xi^P + r)}, \quad b = \frac{1}{2}(\omega - 2)(\xi^P + r), \quad \zeta = \frac{1}{2}\frac{r\gamma(\sigma^P)^2}{\big(b + \frac{1}{4}r\big)^2 - \frac{1}{16}r^2}, \tag{3.11}$$

where $\sigma^P = (\xi^D + r)^{-1}\sqrt{\big(\sigma^D\big)^2 + \big(\sigma^G\big)^2(\xi^G + r)^{-2}}$.

As revealed by property (i), when the market becomes large, the gap between the equilibrium price and what best reflects the present value of future cash flows converges to zero. In general, as demonstrated by Kyle, Obizhaeva, and Wang (2018), the equilibrium price is

$$P_t = \frac{D_t}{\xi^D + r} + \frac{\phi}{(\xi^D + r)(\xi^G + r)}\frac{1}{J}\sum_{j \in \mathcal{J}} \widetilde{\mathrm{E}}_{j,t}(G_t),$$

where ϕ is a endogenous parameter conjectured to be smaller than one, and $\widetilde{\mathrm{E}}_{j,t}(\cdot)$ is the investor j's expectation if she hypothetically observes everyone's private signal up to t. Therefore, the gap comes from two sources: the "price-dampening" parameter ϕ and that $\widetilde{\mathrm{E}}_{j,t}\big(G_t\big)$ differs from $\mathrm{E}(G_t|\{D_s, \bar{S}_s\}_{s\le t})$. The latter one naturally shrinks, because the difference between the two expectations only comes from investor j's misperception of her own signal, which plays a vanishing role in forming the expectations as J increases. The price dampening effect originates from that investor j understand (correctly) that other investors misperceive the precision of their signals in forming their growth rate estimates $\widetilde{\mathrm{E}}_{j,t}\big(G_t\big)$. If $\phi = 1$, the average expected return across investors would be negative (positive) when G_t is positive (negative). So would the aggregate demand. As a result, $\phi < 1$ is needed to clear the market. Hence, when the market is large, ϕ is pushed towards one as $\widetilde{\mathrm{E}}_{j,t}\big(G_t\big)$ converges to $\mathrm{E}(G_t|\{D_s, \bar{S}_s\}_{s\le t})$.

Equation (3.9) is a direct result of standard Kalman-Bucy filtering. $G_{j,t}$ is constructed using investor j's own signal, whereas G_t^D is intended to capture the information about the growth rate contained in the cash flow. The expectation of G_t is a weighted average of them, with the weights determined by their relative informativeness. The parameter ξ^P determines the relative weights on signals from recent and distant past. Indeed, when the information on G_t contained in the cash flow or the signal is of high quality, ξ^P increases and $G_{j,t}$ and G_t^D is mostly composed of very recent signals.

The equilibrium trading is quite interpretable as well. The gap between average growth rate estimate $\bar{G}_t$ to the true growth rate G_t mean-reverts to zero at rate ξ^P.[6] With G_t unobservable,

[6]Precisely speaking, when investor j makes her trading decisions, what matters should be the average growth rate

investor j measure the gap using $G_{j,t} - \bar{G}_t$ and shrink it by the factor $\omega(\sigma^S)^{-2}/\big((\sigma^D)^{-2} + \iota^{-1}\big)$, as she is aware of the noise of her own signal. The factor $\big(\xi^D + r\big)^{-1}\big(\xi^G + r\big)^{-1}$ reflects how price is connected to the average growth rate estimate. Finally, the investor chooses parameters a and b to balance capturing expected returns and alleviating trading costs, with the price impact $\big((J-1)\psi^P\big)^{-1} = \zeta + O_{\mathrm{P}}(J^{-1})$ being endogenously determined. The equilibrium trading resembles the one obtained by Gârleanu and Pedersen (2013, 2016) in partial equilibrium models with exogenous trading cost function, where investors directly derive utility from after-cost investment performance rather than from consumption.

4 Equilibrium with Heterogeneous Information Structure

This section is devoted to the equilibrium analysis under general heterogeneous information structure.

4.1 Setup

Motivated by the simplification of the equilibrium under large market demonstrated by Section 3.2, we restrict our analysis to the large market scenario, in order to generate a tractable characterization of the equilibrium in the presence of heterogeneity in belief and information structure. To regulate the asymptotic behaviors of various parameters, we impose the following assumption, which accommodates Assumption 2 as a special case.

Assumption 3. *As J, i.e., the size of $\mathcal{J}$, increases,*

$$\big(\sigma_j^S\big)^2 = J\iota_j,$$

whereas ι_j and exogenous parameters $(\xi^D, \xi^G, \sigma^D, \sigma^G, \omega_j, \rho, \gamma)$ stay unchanged. The belief parameters η_j satisfies

$$\eta_j = \bar{\eta} + \kappa_j, \quad \text{with} \quad \sum_{j \in \mathcal{J}} \kappa_j = 0.$$

where $\bar{\eta}$ can either stay constant or vary with J, and satisfies $\bar{\eta} = O(1)$. On the other hand, κ_j satisfies $\max_{j\in\mathcal{J}} |\kappa_j| = O(J^{-1/2})$.

In contrast to Assumption 2, here we explicitly allow for incorrect beliefs on G_t^o represented by η_j. The requirement that belief dispersion must not be larger than $\sim J^{-1/2}$ might sound restrictive, but it actually already allows the belief dispersion to generate almost arbitrarily strong correlations between investors' trading.

The main message from Proposition A1 is that, even though investors solve their optimization problems exactly, as the market becomes large, the equilibrium price and investor trading are dominated by leading-order terms of simple forms. Now we take one step further and only require the investors to approximately solve their optimization problems, that is, intuitively, as long as a strategy

estimate among all the *other* investors, but the difference is of second order when the market is large.

leads to trading that is sufficiently close to trading under an exactly optimal one, they are willing to take it. To formalize this idea, we update the equilibrium concept as follows:

DEFINITION 2: An asymptotic equilibrium is a set of trading strategies $\{\dot{x}^f_{j,t}(\cdot), \dot{x}_{j,t}(\cdot)\}_{j\in\mathcal{J}}$ and price functions $P_t\big(\{\dot{x}_{j,s}(\cdot), D_s, D^o_s, G^o_s, S_{j,s}, P_{s'}\}_{s\le t, s'<t, j\in\mathcal{J}}\big)$ such that

(i) For each investor $j \in \mathcal{J}$, given the price function and the strategies of all the other investors, $\big(\dot{x}^f_{j,t}(\cdot), \dot{x}_{j,t}(\cdot)\big)$ are measurable to the information set $\mathcal{F}_{j,t}$ and satisfy

$$\mathrm{E}_j\big(|\dot{x}_{j,t} - \dot{x}^*_{j,t}|^2\big) \lesssim J^{-2} \quad \text{and} \quad \mathrm{E}_j\big(|x_{j,t} - x^*_{j,t}|^2\big) \lesssim J^{-2},$$

for some strategy $\big(\dot{x}^{f*}_{j,t}(\cdot), \dot{x}^*_{j,t}(\cdot)\big)$ that solves the optimization problem specified by (2.3) – (2.4), subject to that $\dot{x}^{f*}_{j,t}(\cdot)$ and $\dot{x}^*_{j,t}(\cdot)$ are measurable to $\mathcal{F}_{j,t}$;

(ii) The risky asset market clears:

$$\sum_{j\in\mathcal{J}} x_{j,t} = 0.$$

We note that in Proposition 1 the magnitude of the leading terms for both trading rate $\dot{x}^*_{j,t}$ and position $x^*_{j,t}$ are $\approx J^{-1/2}$, which turns out to be also true in the asymptotic equilibrium studied in this section. The small magnitude comes from that investors' signals are highly noisy, under Assumptions 2 or 3. These leading terms, however, dominate the deviations from exactly optimal trading and position allowed by requirement (i) in Definition 2.

4.2 Equilibrium

As in Section 3.2, we conjecture that in equilibrium all investors submit linear demand schedules

$$\dot{x}_{j,t} = \psi^D_j G^D_t + \psi^C_j \big(D_t + D^o_t\big) + \psi^S_j G_{j,t} + \psi^o_j G^o_t - \psi^P_j P_t - \psi^H x_{j,t}. \tag{4.1}$$

All coefficients but ψ_H are j-dependent. The processes G^D_t and $G_{j,t}$ are also defined by (3.3) and (3.4), where ξ^P is now defined by

$$\xi^P = \sqrt{(\xi^G)^2 + (\sigma^G)^2\big((\sigma^D)^{-2} + \widetilde{\iota}^{-1}\big)}, \tag{4.2}$$

with

$$\widetilde{\iota} = \sum_{j\in\mathcal{J}} \pi_j^2 (\sigma^S_j)^2 \quad \text{and} \quad \pi_j = \omega_j (\sigma^S_j)^{-2} / \sum_{j'\in\mathcal{J}} \omega_{j'} (\sigma^S_{j'})^{-2}.$$

Under symmetric information structure (Assumption 2), weight π_j would reduce to equal weight, $\widetilde{\iota}$ would be equal to ι, and ξ^P would be the same as the one in (3.5). We will explain why the weight π_j appears later.

Suppose all investors $j' \in \mathcal{J}$ with $j' \neq j$ submit demand schedule (4.1). Similar to the symmetric case, the market clearing condition would lead to the following supply curve which investor j faces:

$$P_t(\dot{x}_{j,t}) = (\widetilde{\psi}_j^P)^{-1}\left(\widetilde{\psi}_j^D G_t^D + \widetilde{\psi}_j^C (D_t + D_t^o) + \widetilde{\psi}_j^o G_t^o + \frac{1}{J-1}\sum_{j\in\mathcal{J}:j'\neq j}\psi_{j'}^S G_{j',t} + \frac{1}{J-1}\left(\dot{x}_{j,t} + \psi^H x_{j,t}\right)\right), \tag{4.3}$$

where $\left(\widetilde{\psi}_j^P, \widetilde{\psi}_j^D, \widetilde{\psi}_j^C, \widetilde{\psi}_j^o\right) = (J-1)^{-1}\sum_{j\in\mathcal{J}:j'\neq j}\left(\psi_{j'}^P, \psi_{j'}^D, \psi_{j'}^C, \psi_{j'}^o\right)$. According to Definition 2, an asymptotic equilibrium exists if we find a set of parameters $\left\{\psi_j^P, \psi_j^D, \psi_j^C, \psi_j^o, \psi_j^P, \psi^H\right\}_{j\in\mathcal{J}}$ such that, for each investor $j \in \mathcal{J}$, (4.1) "almost" (as in requirement (i) of Definition 2) solves the optimization problem (2.3) and (2.4), under the supply curve (4.3). The following theorem presents the result.

Theorem 1. *Suppose Assumption 3 holds. Then there exists an asymptotic equilibrium as in Definition 2 if $\xi^G > r$ and $\widetilde{\omega} > 2$, where $\widetilde{\omega} = \sum_{j\in\mathcal{J}}\pi_j\omega_j$. The equilibrium has the following properties:*

(i) In equilibrium the price satisfies

$$P_t = \frac{D_t + D_t^o}{\xi^D + r} + \frac{\mathrm{E}(G_t|\{D_s, \bar{S}_s\}_{s\leq t})}{(\xi^D + r)(\xi^G + r)} + \frac{(1+\bar{\eta})G_t^o}{(\xi^D + r)(\xi^o + r)}, \quad \text{with} \quad \bar{S}_s = \sum_{j\in\mathcal{J}}\pi_j S_{j,s}, \tag{4.4}$$

and $\mathrm{E}(\cdot)$ is the expectation under the objective measure. Moreover, it holds that

$$\mathrm{E}(G_t|\{D_s, \bar{S}_s\}_{s\leq t}) = \frac{(\sigma^D)^{-2}G_t^D + \widetilde{\iota}^{-1}\bar{G}_t}{(\sigma^D)^{-2} + \widetilde{\iota}^{-1}}, \quad \text{with} \quad \bar{G}_t = \sum_{j\in\mathcal{J}}\pi_j G_{j,t}. \tag{4.5}$$

(ii) Investor j's trading strategy follows (4.1) with $\psi_j^P = \zeta^{-1}\pi_j$ and $\psi_j^H = b$, and the trading satisfies

$$\dot{x}_{j,t} = a\frac{\xi^P + r}{(\xi^D + r)(\xi^G + r)}\frac{\omega_j(\sigma_j^S)^{-2}}{(\sigma^D)^{-2} + \widetilde{\iota}^{-1}}(G_{j,t} - \bar{G}_t) + a^o\frac{\kappa_j}{\xi^D + r}G_t^o - bx_{j,t}. \tag{4.6}$$

Here the endogenous parameters a, a^o, b, and ζ are given by

$$a = \frac{1}{\zeta\widetilde{\omega}(\xi^P + r)}, \quad a^o = \frac{1}{2\zeta}\frac{1}{b + \xi^o + r}, \quad b = \frac{\widetilde{\omega} - 2}{2}(\xi^P + r), \quad \zeta = \frac{1}{2}\frac{r\gamma(\sigma^P)^2}{\left(b + \frac{1}{4}r\right)^2 - \frac{1}{16}r^2}, \tag{4.7}$$

where $\sigma^P = (\xi^D + r)^{-1}\sqrt{2(\sigma^D)^2 + (\sigma^G)^2\left(\xi^G + r\right)^{-2} + (1+\bar{\eta})^2(\sigma^o)^2\left(\xi^o + r\right)^{-2}}$ is the volatility of the equilibrium price.

The equilibrium properties resembles those in Proposition 1. The equilibrium price (4.4) takes the same form of its counterpart (3.8), where $\bar{S}_{j,t}$ is an average of $S_{j,t}$ weighted by π_j. Parameter $\widetilde{\iota}$ appearing in (4.5) and (4.2) is the squared volatility of $\bar{S}_{j,t}$, which reflects its noise level. The homogeneous misperception parameter ω that affects trading and price impact in (3.11) is replaced with $\widetilde{\omega}$ as in (4.7), which is the average of ω_j weighted by π_j as well. The weight π_j reflects that investors with higher signal precision and larger upward bias in perceiving it would play more important roles in determining equilibrium price and market liquidity. Its specific form originates from the $\omega_j(\sigma_j^S)^{-2}$ factor appearing in the trading (4.6). Comparing (4.6) with the equilibrium price (4.4) and (4.5), we clearly see that the trading can be generated using a linear demand schedule of form (4.1).

5 Econometric Analysis of the Equilibrium Model

This section studies the identification and estimation of the following parameters: price mean-reversion coefficient ξ^P, price impact parameter ζ, misperception parameters $\{\omega_j\}_{j\in\mathcal{J}}$, and information quality parameters $\{\sigma_j^S\}_{j\in\mathcal{J}}$. As we can not rule out other types of equilibria, we impose the following assumption:

Assumption 4. *Equilibrium price and investor trading are the ones characterized in Theorem 1.*

5.1 Empirical Content of the Equilibrium Model

The equilibrium has two major implications: how trading depends on private and public information (4.6), and the relation of prices to private information (4.4). They connect the parameters of interest to moments of holdings and prices. The equilibrium implication on trading can be written more compactly as

$$\dot{x}_{j,t} + bx_{j,t} = y_{j,t} := \phi^S \pi_j (G_{j,t} - \bar{G}_t) + \phi^o \kappa_j G_t^o,$$

where the definitions of constants ϕ^S and ϕ^o is clear from (4.6). Moreover, we introduce notation

$$\varepsilon_{j,t} = \phi^S(\xi^P - \xi^G) \int_{-\infty}^{t} e^{-\xi^P(t-s)} \sigma_j^S dZ_{j,s}^S, \quad f_t = -\sum_{j\in\mathcal{J}} \pi_j \varepsilon_{j,t}, \quad \text{and} \quad g_t = \phi^o G_t^o.$$

Then the definition of y_t can be further simplified into:

$$y_{j,t} = \pi_j f_t + \kappa_j g_t + \pi_j \varepsilon_{j,t}.$$

In other words, trading across investors follows a simple factor structure. Investor trades on f_t, which is the aggregation of the noise in each investor's private signal entering the price. Because observe f_t is not observable to investors, to load more on the factor f_t, an investor will have to load more on the noise of her own signal as well, which more intensively moves the price against herself. The factor g_t originates from the observable growth rate component G_t^o, on which there is a belief dispersion. The next proposition provides statistical moments regarding the above factor model.

Proposition 2. *Suppose Assumptions 3 and 4 hold. Then we have, with some constant ϕ that only depends on (ϕ^S, ξ^P, ξ^G) and some constant $\bar{\phi}$ that only depends on $(\phi^o, \xi^o, \sigma^o)$,*

$$\mathrm{E}(f_t^2) = \phi\widetilde{\iota}, \quad \mathrm{E}(\varepsilon_{j,t}^2) = \phi(\sigma_j^S)^2, \quad \mathrm{E}(f_t\varepsilon_{j,t}) = -\phi\pi_j(\sigma_j^S)^2, \quad \mathrm{E}(g_t^2) = \bar{\phi}.$$

The covariance matrix of rebalancing-adjusted trading rate across investors is

$$\mathrm{Cov}\big(y_t\big) = \phi\widetilde{\iota} \cdot \Big(\beta\beta^\intercal - \nu\nu^\intercal + \mathrm{diag}\big(\nu\big)\Big) + \bar{\phi} \cdot \kappa\kappa^\intercal. \tag{5.1}$$

Here $\nu = (\nu_1, \nu_2, ..., \nu_J)^\intercal$ *with* $\nu_j = \pi_j \widetilde{\omega}^{-1}\omega_j$ *and* $\beta = \pi - \nu$, *whereas* y_t, π, *and* κ *are all* J*-dimensional column vectors whose entries are clear from the context. The average misperception* $\widetilde{\omega}$ *is defined in Theorem 1.*

Because weight π_j is of order $\sim J^{-1}$, and the standard deviation of idiosyncratic shock $\pi_j\varepsilon_{j,t}$ is of order $\sim J^{-1/2}$, factor f is a weak one. As mentioned earlier, even though Assumption 3 impose a bound on the size of κ_j, it is not highly restrictive. Given the magnitude of the idiosyncratic shocks, Assumption 3 allows g_t to be a standard strong factor and the dominant driver of trading correlation patterns. On the other hand, the factors f_t and g_t in fact carry all the predictive power of trading on future price changes. To be concrete, we introduce $\Pi_t = P_t + \int^t (D_s + D_s^o - rP_s)ds$, the excess gain process of holding one unit of the risky asset.

Proposition 3. *Suppose Assumptions 3 and 4 hold and denote by* $\widetilde{\mathcal{F}}_t$ *the information set generated by* $\big\{x_{j,s}, f_s, g_s\big\}_{s\le t, j\in\mathcal{J}}$. *Then it holds that, for all* $\tau \geq 0$,

$$\mathrm{E}\big(d\Pi_{t+\tau}\big|\widetilde{\mathcal{F}}_t\big)/dt = e^{-\xi^P\tau}\big(a\widetilde{\omega}\big)^{-1} \cdot f_t - e^{-\xi^o\tau}(a^o)^{-1} \cdot \bar{\eta} g_t,$$

where a *is introduced in (4.7).*

Notably, even though g_t could dominate f_t in generating the cross-sectional comovement of trading, it possesses similar or smaller predictive power compared to f_t, depending on the size of average belief distortion $\bar{\eta}$. The reason is that investors do not observe f_t, which appears in the conditioning information set $\widetilde{\mathcal{F}}_t$, and they are unwilling to expeditiously trade on it because of the low quality of their private signals. Econometricians, on the other hand, can efficiently aggregate their private signals implied by their trading and obtain a much more precise estimate of f_t, as long as the misperception parameter ω_j differs across investors. Indeed, as a result of Proposition 3, whenever $\tau \geq 0$,

$$\mathrm{E}\big(y_t d\Pi_{t+\tau}\big)/dt = e^{-\xi^P\tau}\big(a\widetilde{\omega}\big)^{-1} \cdot \phi\widetilde{\iota}\beta - e^{-\xi^o\tau}(a^o)^{-1} \cdot \bar{\phi}\bar{\eta}\kappa. \tag{5.2}$$

If there is no heterogeneity in ω_j, then $\beta = 0$ and every investor's trading has zero correlation with future price movements, making it impossible for econometricians to extract any predictive power.

However, in reality econometricians do not observe y_t. They only observe investor positions at discrete time $\Delta x_t = x_t - x_{t-1}$. The following proposition connects their statistical moments.

Proposition 4. *Suppose Assumptions 3 and 4 hold. Suppose* $\xi^P = \xi^o$. *Then we have, for all* $\tau \geq 1$,

$$\Sigma := \mathrm{Cov}(\Delta x_t) = \lambda \mathrm{Cov}(y_t), \quad R_\tau := \mathrm{E}(\Delta x_t \Delta\Pi_{t+\tau}) = \bar{\lambda}\mathrm{E}(y_t\Delta\Pi_{t+\tau}), \tag{5.3}$$

where $\Delta\Pi_t = \Pi_t - \Pi_{t-1}$, $\bar{\lambda} = (\xi^P + b)^{-1}(1 - e^{-\xi^P})$, *and* λ *also only depends on* ξ^P *and* b*. Moreover, it holds that, for all* $\tau \geq 1$,

$$\rho_\tau := \mathrm{Corr}(x_{j,t}, x_{j,t+\tau}) = \frac{\xi^P e^{-b\tau} - b e^{-\xi^P \tau}}{\xi^P - b}. \tag{5.4}$$

Therefore, given ξ^P, by looking at the autocorrelation of position change $\Delta x_{j,t}$, the econometrician can identify b, which allows her to impute the moments involving y_t from those based on Δx_t.

Finally, in equilibrium a and $\widetilde{\omega}$ are connected to b, ξ^P, ζ, and r as in (4.7).

5.2 Identification and Estimation Procedure

Proposition 5. *Suppose Assumptions 3 and 4 hold and* ω_j *is not invariant across* j*. Also suppose* $\bar{\eta} = 0$ *and* $\xi^P = \xi^o$*. If the econometrician has access to risk-free rate* r*, the covariance matrix* Σ*, the expected return vector* R_τ*, and the trading autocorrelation* ρ_τ*, then using (5.1), (5.2), (5.3), (5.4) and (4.7), she can identify, for all* $j \in \mathcal{J}$*,*

$$\xi^P, \quad \pi_j, \quad \omega_j, \quad \zeta, \quad \text{and} \quad \phi\widetilde{\iota}.$$

Given π_j and ω_j, we can only obtain σ_j^S up to a constant common across j. In other words, only the relative magnitude of information quality is identified, because we do not know parameters such as ξ^D and ξ^G that connect growth rate to price. The quantity $\phi\widetilde{\iota}$ affects the amount of predictable return and is a combination of how much the average estimate of growth rate deviate from the true value and how much the growth rate affects price.

The identification is achieved through the following procedure:

Algorithm 1. *Inputs: risk-free rate* r*, covariance matrix* Σ*, portfolio price change vector* R_τ*, and trading autocorrelation* ρ_τ*.*

S1. Given Σ*, we can utilize (5.1) and (5.3) to obtain*

$$\widetilde{\nu} := (\lambda\phi\widetilde{\iota})\nu \quad \text{and} \quad \widetilde{\Sigma} := \lambda\phi\widetilde{\iota}\big(\beta\beta^\intercal - \nu\nu^\intercal\big) + \lambda\bar{\phi}\kappa\kappa^\intercal.$$

S2. Using $\widetilde{\nu}$ *and* $\widetilde{\Sigma}$*, we calculate* $\lambda\phi\widetilde{\iota}$ *using* $\lambda\phi\widetilde{\iota} = -\widetilde{\nu}^\intercal\widetilde{\Sigma}^{-1}\widetilde{\nu}$*.*[7] *We then obtain* ν *from* $\widetilde{\nu}$ *and* $\lambda\phi\widetilde{\iota}$*.*

S3. Further, using $\lambda\phi\widetilde{\iota}$*,* $\widetilde{\Sigma}$*, and* R_τ*, we obtain, with any* $\tau \geq 1$*,*

$$\beta = \frac{R_\tau}{\sqrt{\lambda\phi\widetilde{\iota}R_\tau^\intercal\widetilde{\Sigma}^{-1}R_\tau}}.$$

S4. From how R_τ *and* ρ_τ *change with* τ *specified by (5.2) and (5.4), we obtain* ξ^P *and* b*, and thereby* λ *and* $\bar{\lambda}$*.*

[7] $\widetilde{\Sigma}$ is singular and $\widetilde{\Sigma}^{-1}$ stands for its pseudo inverse.

S5. From β and ν, we obtain π and $\widetilde{\omega}^{-1}\omega$. From β, $\lambda\phi\widetilde{\iota}$, λ, $\bar{\lambda}$, and R_τ, we obtain $\phi\widetilde{\iota}$ directly and obtain $a\widetilde{\omega}$ using (5.2) and (5.3).

S6. Utilizing (4.7), we obtain $\widetilde{\omega}$ from b, ξ^P, and r. Then we obtain a from $a\widetilde{\omega}$. Using (4.7) again, we obtain ζ.

Outputs: ξ^P, π_j, ω_j, ζ, and $\phi\widetilde{\iota}$.

The above procedure does not rely on that the econometrician observe all the market participants.[8] In the case where we do observe every investor's trading, we do not need step 2 thanks to that $\sum_{j\in\mathcal{J}}\nu_j = 1$ by definition.

The natural implementation of Algorithm 1 is to construct empirical counterparts of the population moments. Given the large dimension of Σ and R_τ, using the sample covariance matrix and sample mean vector directly would incur large estimation erros. Motivated by the factor structure of trading manifested by (5.1), we propose an estimation method by modifying the standard principal component analysis (PCA) approach, which we call truncated PCA. Indeed, as discussed after Proposition 2, the eigenvalue generated by factor f_t is at the same order of magnitude as that from the idiosyncratic component $\varepsilon_{j,t}$. To bypass this issue, unlike the standard PCA that conducts eigendecomposition directly on covariance or correlation matrices, we replace all the diagonal elements of the sample version of Σ with zero and conduct eigendecomposition afterwards. The diagonal elements themselves can be used to estimate $\widetilde{\nu}$ directly, which is needed in in step 2 of Algorithm 1, because $\lambda\phi\widetilde{\iota}\nu$ is the dominating part of the diagonal elements of Σ.[9] The eigendecomposition would generate eigenvectors that span β, ν, and κ. As a result, as demonstrated by (5.2), the expected return vector is also spanned by those eigenvectors. Therefore, we only need to estimate the projection of R_τ on a small number of eigenvectors, which is therefore of low dimension. The next Algorithm presents the details.

Algorithm 2. *Inputs: position change Δx_t and gain change $\Delta\Pi_t$.*

S1. Construct sample covariance matrix $\widehat{\Sigma} = \widehat{\mathrm{Cov}}(\Delta x_t)$. Then estimate $\widetilde{\nu}$ using

$$\widehat{\widetilde{\nu}}_j = \widehat{\Sigma}_{j,j}.$$

S2. Replace diagonal elements of $\widehat{\Sigma}$ with zero. Conduct eigendecomposition and take d eigenvectors with large eigenvalues in absolute value. The eigenvectors and eigenvalues can be written as a

[8]The dimension of matrices and vectors involved in the procedure then apparently have dimensions being the number of observed investors, and we of course only identify π_j and ω_j for those observed investors.

[9]The estimates would be biased if κ is big enough. But in this case g_t becomes a strong factor and the danger would be easily detected.

$J \times d$ matrix $\widehat{\Lambda}$ and $d \times d$ diagonal matrix $\widehat{D}$,[10] *where Λ satisfies $\Lambda^\intercal \Lambda = I_d$. Estimate $\widetilde{\Sigma}$ as*

$$\widehat{\widetilde{\Sigma}} = \widehat{\Lambda}\widehat{D}\widehat{\Lambda}^\intercal.$$

S3. Estimate R_τ as $\widehat{R}_\tau = \widehat{\Lambda}\widehat{\mathrm{E}}\big(\widehat{\Lambda}^\intercal \Delta x_t \Delta\Pi_{t+\tau}\big)$.

Outputs: $\widehat{\widetilde{\nu}}$, $\widehat{\widetilde{\Sigma}}$, $\widehat{R}_\tau$, $\widehat{\Lambda}$, and $\widehat{D}$.

6 Empirical Study of the US Stock Market

6.1 Setup

Following Koijen and Yogo (2019), we use stock level data from the Center for Research in Security Prices (CRSP) and Compustat, and use institutional holdings data based on quarterly filings of Securities and Exchange Commission Form 13F. The stock level data include prices and four characteristics: book equity, dividends, profitability, and investment. The holdings data covers every institution with asset under management over \$100 million. The total number of these institutions is around 3,000 and together they own roughly 65% of the total US market. As in Koijen and Yogo (2019), the institutions are classified into six types: banks, insurance companies, investment advisors, mutual funds, pension funds, and others.

To connect to the equilibrium model, we construct the quantity of stock i held by investor j at time t as

$$x_{i,j,t} := \frac{\text{Shares of stock } i \text{ held by investor } j \text{ at time } t}{\text{Total shares of stock } i \text{ at time } t}.$$

The position change is accordingly $\Delta x_{i,j,t} = x_{i,j,t} - x_{i,j,t-1}$.

In a multi-asset scenario, there would be additional effects on trading and price, which can arise from return correlations between stocks, from heterogeneity in the market capitalizations of different stocks, or from the learning about one asset from other assets as in Admati (1985). Fully capturing these effects is beyond the scope of our model that includes only one risky asset. Moreover, as in standard asset pricing models with learning, our model adopts exponential utility and assumes that price has a normal distribution rather than a log-normal one. Therefore, it is the absolute price change rather than the return that is the central price variable in investors' decision making, which obviously vary greatly across big and small cap stocks. Moreover, investors have trading motives beyond information-driven ones, such as time-variations of risk premia, liquidity shocks, matching benchmarks, and wealth effects.

To operationalize the equilibrium model and aggregate the statistical power over the entire cross-section of assets, we need to take the following stances: (i) it is the residual price changes and the residual positions that can not be explained by the observable characteristics that the equilibrium model is describing; (ii) for each investor, the residual position changes are determined separately and

[10] We reuse the letter D which appears in the equilibrium model to represent cash flow rate. The context shall eliminate ambiguity.

independently for different assets; (iii) for each investor, the misperception parameter ω_j, the belief distortion η_j, and the mean reversion parameters (ξ^D, ξ^G, ξ^o) are invariant across assets, whereas the volatility parameters $(\sigma^D, \sigma^G, \sigma^o, \sigma^S)$ and absolute risk tolerance γ^{-1} change proportionally with the market capitalization of the asset. With these assumptions, the trading covariance matrix Σ, now based on the residual position changes, stays constant across assets, whereas and the covariance between trading and price change R_τ is is proportional to market capitalization. In other words, the predicted relative price change (excess return) the next quarter by an investor selling 1% of the total outstanding shares this quarter will be the same across big and small cap stocks. We thereafter scale R_τ by the market cap and construct it by replacing the gain change with the dividend-adjusted excess return over the period $(t+\tau-1, t+\tau)$, and we hence reuse $\Pi_{i,t}$ to denote cumulated dividend-adjusted excess return of stock i and $\Delta\Pi_{i,t} = \Pi_{i,t} - \Pi_{i,t-1}$ throughout the empirical analysis.

Table 1: Institutions Ranked by Total Quarterly Trading Volume 2010Q1 - 2017Q4

	Institution	$\sum_{i,t} \Delta x^2_{i,j,t}$
1	Fidelity Mgmt & Research	8.38
2	Wellington Mgmt Co LLP	4.40
3	Blackrock Inc.	3.37
4	T. Rowe Price Associates Inc.	3.09
5	Invesco	1.83
6	Bank of America Corp.	1.64
7	MSDW & Co.	1.61
8	Royce & Associates LP	1.52
9	J. P. Morgan Chase & Co.	1.39
10	Wells Fargo & Co.	1.25
11	Goldman Sachs & Co.	1.23
12	AXA Financial Inc.	1.18
13	Columbia Threadneedle Invts.	1.14
14	Bank of New York Mellon Crop.	1.10
15	Vanguard Group Inc.	1.06
	Total of the top 15:	34.21
	All Institutions:	113.45

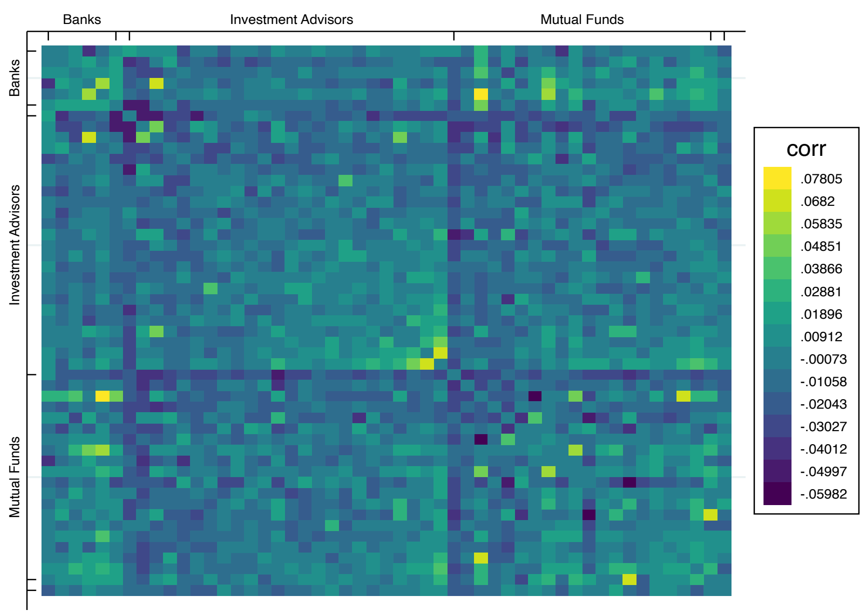
Note: This table reports the 15 institutions with the largest trading activity in terms of the sum of squared quarterly position changes across all the stocks over the period 2010Q1 - 2017Q4 and the sum of trading activity of all the institutions.

We then look at the distribution of trading activities across institutions, represented by the sum of squared quarterly position change across all the stocks over a period. As shown by Table 1, the trading activities are highly concentrated. 15 institutions contribute to 30 % of total institution trading activity. Across institution types, mutual funds and investment advisors are the two biggest contributors. We hence group smaller institutions in terms of trading of the same type and similar trading activities such that the group-level trading activities are similar. We have 51 institutions (groups) in total.

6.2 Cross-section of Trading

We construct the sample correlation matrix of investors' trading using their position changes. As demonstrated by Figure 1, the correlations between different investors' trading are quite small. One apparent pattern in the figure that the most active institutions, those in the top-left corners of investment advisor and the mutual fund blocks, have relatively large negative trading correlations with all the other investors, which is consistent with the $-\nu\nu^\intercal$ term appearing in (5.1). This originates from that more active investors push the price more, and therefore more strongly encourage other investors to trade in the opposite directions.

Figure 1: Correlation of Trading Across Investors 2010Q1 - 2017Q4



Note: The figure presents pairwise correlations in position changes $\Delta x_{i,j,t}$ between different institutions (groups) over the period 2010Q1 - 2017Q4. The ones on the diagonal are replaced with zeros. The type between banks and investment advisors is insurance companies. The one to the right of mutual funds are pension funds and others. Within each institution type, the top-left corner is the institution with largest trading activity, whereas the bottom-right corner represents the group of least active institutions.

The documented weak correlations shows that investor trading is dominated by its idiosyncratic component and validate the motivations behind the truncated PCA approach. Before we applied the Algorithm 2, we first regress position changes on a set of observable characteristics and investigate how much variation can be explained. Specifically, for each quarter and institution (group), we run cross-sectional regression of the position changes over the quarter onto book-to-market ratio, investment, and profitability at least 6 months prior to the beginning of the quarter, market beta and market equity the month before the quarter, and excess returns over the last month (lag 0) of the quarter and the other 8 months (lag 1 to lag 8) prior to it, with both the left-hand-side and right-hand-side variables standardized. The results are reported in Figure 2. The variables with highest explanatory power is returns lagging one to three months, and the total in-sample R^2 is on average only around 0.3%. The residual position change $\Delta\breve{x}_{i,j,t}$, in which all the aforementioned characteristics have been partialled out for each institution and quarter, will be the building block for our remaining empirical analysis.

Figure 2: Cross-sectional Regression of Trading on Characteristics by Institution and Quarter

Note: This figure reports the histograms of coefficients and total R^2 of regression of trading on characteristics for each institution (group) and quarter. All the variables are standardized.

6.3 Truncated PCA

Following steps 1 and 2 of Algorithm 2, we conduct truncated PCA on $\widehat{\text{Cov}}(\Delta\breve{x}_{i,j,t})$, the sample covariance matrix based on residual positions changes, to obtain the estimate of $\widetilde{\nu}$, and matrices Λ and D. The sample average is taken across all assets and time periods from 2010Q1 to 2017Q4.

As shown in Table 2, diagonal elements of $\widehat{\text{Cov}}(\Delta\breve{x}_{i,j,t})$ on average is larger than even the largest eigenvalue (in absolute value), which justifies the validity of our reduced-form weak factor model and again highlights the necessity of truncation. It also ensures that the diagonal elements of $\widehat{\text{Cov}}(\Delta\breve{x}_{i,j,t})$ are chiefly contributed by idiosyncratic shocks and provide a consistent estimate of $\widetilde{\nu}$, as discussed above Algorithm 2. That the first eigenvalue is negative reflects that trading is on average negatively correlated across investors for the market to clear, which echos the $-\phi\tilde{\iota}\nu\nu^{\intercal}$ term in (5.1). Indeed, as reveal by the spanning regression reported in the right half of the table, the first eigenvector strongly explains the estimate of $\widetilde{\nu}$. The 91.7% R^2 shows that, at least through the lens of vector ν, using only the first four eigenvectors is a reasonable choice in capturing the cross-sectional correlation of trading. Still $\widehat{\widetilde{\nu}}$ does load on other eigenvectors, as it is well known that PCA-based methods only identify loadings of a factor model up to a rotation because these loadings, in our case β, ν, and κ, are not necessarily orthogonal to each other.

Table 2: Eigendecomposition of Truncated Sample Covariance Matrix of Position Changes

$\widehat{\widehat{\nu}} \times 10{,}000$		Eigenvalues $\times$ 10,000				Regress $\widehat{\widehat{\nu}}$ on Eigenvectors				
mean	std	1st	2nd	3rd	4th	1st	2nd	3rd	4th	R^2
0.310	0.143	−0.181	0.123	0.072	0.067	0.935	0.186	0.120	0.005	0.917

Note: The table presents the mean and standard deviation of the diagonal elements of sample covariance matrix of position changes, the four eigenvalues of the truncated version of the sample covariance matrix with largest absolute values, and the spanning of the diagonal elements with the first four eigenvectors. Eigenvalues are diagonal components of D, and eigenvectors are columns of $\widehat{\Lambda}$. Both $\widehat{\widehat{\nu}}$ and eigenvalues are multiplied by 10,000. For the spanning regression, we regress $\widehat{\widehat{\nu}}$ on the first four eigenvectors, excluding the constant term. We normalize $\widehat{\widehat{\nu}}$ to make it a unit vector, and eigenvectors by design are also unit vectors.

6.4 Predictive Regression

Next, we study the predictive regression using position changes of institutions. Following step 3 of Algorithm 2, we combine institution trading using the four eigenvectors previously obtain. In other words, we construct four predictors that would summarize all the predictive power of position changes under our factor model (5.1) and prediction model (5.2). With each of the four quarterly trading-based predictors, we run Fama-MacBeth regression of monthly excess returns, over the 1-month T-bill rate onto the most recent trading-based predictor and lagged characteristics, over the period from April 2010 to December 2017. The lagged characteristics are log market equity, book-to-market equity, profitability, investment, market beta, and momentum, which are chosen to be the most recent ones that are public at month t to predict excess returns at month $t+1$. With all the four predictors together, we run Fama-MachBeth regression with the same controls again. Running predictive regression separately allows us to connect the regression coefficients directly to R_τ, β, and then structural parameters, as demonstrated in Algorithms 1 and 2. In contrast, the coefficients of the joint regression are also affected by the correlations between the trading-based predictors, which are generally nonzero and comes from both the factor and idiosyncratic parts of position changes.

The top panel of Table 3 shows that, even though the point estimates of predictive power are only moderate, they are statistically significant thanks to that the predictors are mostly trading on the idiosyncratic components of returns, leading to small standard error. Other than the first predictor, which represents roughly the total position changes of all the institutions in the sample, all the other predictors significantly predict future excess returns. Effectively, with the time unit being a quarter, the 4-dimensional vector consisting of the coefficients in the bottom-right panel of Table 3 is exactly $10,000 \times \widehat{\mathrm{E}}\big(\widehat{\Lambda}^\intercal \Delta\breve{x}_{i,t}\Delta\Pi_{i,t+1}\big)$,[11] indicating that vector $\widehat{R}_\tau$ mostly loads on the second and fourth eigenvectors.

[11] As in Section 5, the j-dimension is vectorized, i.e. $\Delta\breve{x}_{i,t}$ is a vector consisting of $\Delta\breve{x}_{i,j,t}$. 10,000 instead of 100 appears because excess returns are measured in percentage.

Table 3: Return Prediction with Eigenvector-weighted Institution Position Changes

	with Each Predictor				with All Predictors			
	1^{st}	2^{nd}	3^{rd}	4^{th}	1^{st}	2^{nd}	3^{rd}	4^{th}
mean	$-.015$	$-.072$	$-.062$	.092	$-.026$	$-.113$	$-.054$	.098
s.e.	.029	0.28	.030	.030	.038	.035	.030	.033
	Std of Predictors $\times$ 100				without Standardization $\times$ 100			
	1^{st}	2^{nd}	3^{rd}	4^{th}	1^{st}	2^{nd}	3^{rd}	4^{th}
mean	.62	.078	.58	.70	$-.012$	$-.056$	$-.035$	.066
s.e.	.07	.08	.03	.05	.018	.022	.017	.021

Note: The top panel of the table presents time-series means and standard errors of cross-sectional regression coefficients of monthly excess returns onto the most recent trading-based predictor and lagged characteristics, over the period from April 2010 to December 2017, with all the predictors and characteristics standardized each quarter. The top-left panel is the regression with each of the trading-based predictor and all the control variables. The top-right panel is the one with all the the trading-based predictors and controls. The bottom-left panel reports the size of the predictors, and the bottom-right panel redo the regression of the top-left panel without standardization. The monthly excess returns are measured in percentage.

We conclude this subsection by studying how quickly the return predictability decays, which in the equilibrium model is governed by the parameter ξ^P as in (5.2). In the context of predicting monthly return, we have $R^m_{(l+1)/3} = exp(-\xi^P l/3)R^m_{1/3}$, where $R^m_\tau = \mathrm{E}\big(\Delta\breve{x}_{i,t}\big(\Pi_{i,t+\tau} - \Pi_{i,t+\tau-1/3}\big)\big)$. Then we modify the previous Fama-MacBeth regression to predict, with quarter-t institution trading, the excess return over each of the six months after the end of quarter-t. The four predictors constructed above are now combined to boost statistical power, using the coefficients in the bottom-right panel of Table 3 as weights. The results are presented in the left panel of Table 4. The right panel of the table reports the GMM estimation of the parameter ξ^P using the cross-sectional regression coefficients as inputs, based on five moment conditions: $\mathrm{E}\big(R^{m,c}_{(l+1)/3} - exp(-\xi^P l/3)R^{m,c}_{1/3}\big) = 0$ for $l = 1, 2, 3, 4, 5$, where $R^{m,c}$ is an scalar that combines components of R^m in the aforementioned way.

6.5 Time Series of Trading

This subsection is devoted to the empirical analysis of the time-series properties of institution trading. As demonstrated by (5.4), the time-series structure provides identification of parameters b and ξ^P. We start by looking at the autocorrelation of long-run position change. Specifically, we choose the" long-run" as two years and investigate $\mathrm{Corr}\big(\sum_{s=t}^{t+7}\Delta\breve{x}_{i,j,s}, \sum_{\tau=t+8}^{t+15}\Delta\breve{x}_{i,j,\tau}\big)$ for each institution (group), reported in Figure 3.

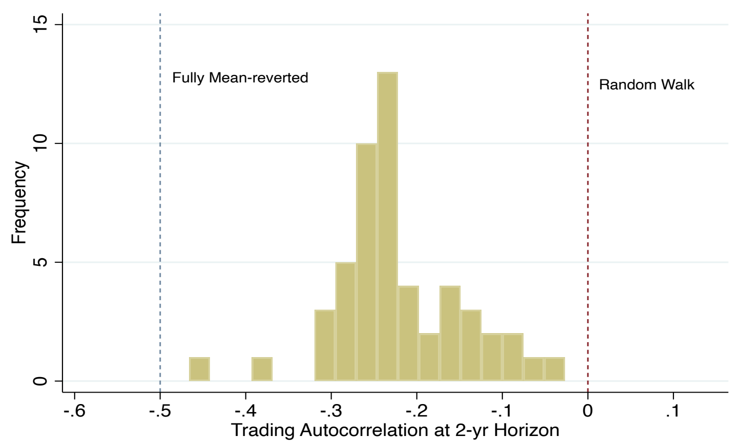
From (5.4), the slow mean reversion evident from the figure could come from either very small b or very small ξ^P. Because the parameter ξ^P, which is estimated through the decay rate of return predictability in Table 4, which is very large under 2-year time unit. Since it is b that is quite small, we expect that the short-run autocorrelation of quarterly position change shall be mostly determined

Table 4: Decay Rate of Predictable Returns with Institution Position Changes

	Predicted Return over Different Future Months						ξ^P
	1^{st} month	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	
mean	.154	.176	.066	.071	.044	$-.083$	.870
s.e.	.071	0.59	.060	.061	.060	.064	.737

Note: The left panel of the table reports the time-series means and standard errors of cross-sectional regression coefficients of future monthly excess returns over different future months onto the current trading-based predictor and the most recent lagged characteristics, over the period from April 2010 to December 2017, with the predictor and characteristics standardized each quarter. The right panel reports GMM estimation of structural parameter ξ^P using cross-sectional regression coefficients as input, based on five moment conditions.

Figure 3: Histogram of Sample Autocorrelation of Two-year Position Changes



Notes: The figure presents the histogram of sample autocorrelation, estimated using pooled estimation across all the stocks, of two-year position changes across institutions.

by ξ^P. We now estimate ξ^P using the autocorrelation of trading to provide comparison with the one from Table 4. It follows from (5.4) that when $l \geq 1$ and $b \times l$ is small we approximately have[12]

$$\gamma_{j,l} + \frac{1}{2}b = e^{-\xi^P(l-1)}\Big(\gamma_{j,1} + \frac{1}{2}b\Big), \quad \text{with} \quad \gamma_{j,l} = \text{Corr}(\Delta\breve{x}_{i,j,t}, \Delta\breve{x}_{i,j,t+l}).$$

In other words, the short-run correlation of $\Delta\breve{x}_{i,j,t}$, after removing the effect of long-run rebalancing by adding $b/2$, mimics that of a AR(1) process. To estimate ξ^P and b using trading autocorrelations, we first estimate sample autocorrelation $\widehat{\gamma}_{j,l}$ for $l = 1, 2, ..., 6$, and for each institution (group) j. Then we conduct GMM estimation of ξ^P and b imposing ten moment conditions: $\text{E}\big(\widehat{\gamma}_{j,l} + b/2 - e^{-\xi^P(l-1)}(\widehat{\gamma}_{j,1} + b/2)\big) = 0$ and $\text{E}\big(\big(\widehat{\gamma}_{j,l} + b/2 - e^{-\xi^P(l-1)}(\widehat{\gamma}_{j,1} + b/2)\big)\widehat{\gamma}_{j,1}\big) = 0$ for $l = 2, ..., 6$.[13]

The estimate of ξ^P matches the estimate based on decay of return predictability and the estimate

[12] The approximation is that the long-run rebalancing effect $-\frac{b}{2}e^{-bl}$ is approximately $\frac{b}{2}$ when bl is small.
[13] We are implicitly assuming the estimation error for $\widehat{\gamma}_{j,1}$ is negligible.

Table 5: Autocorrelation of Quarterly Position Changes

	Autocorrelations at Different Lags						Parameters		
	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6		ξ^P	b
mean	.056	-.004	-.026	-.030	-.030	$-.029$	estimate	1.29	.057
std.	.099	.039	.037	.023	.021	.020	s.e.	.074	.004

Note: The left panel of the table reports the mean and standard deviation of sample autocorrelations across institutions (groups). The right panel presents GMM estimates and standard errors of parameters ξ^P and b.

of b is consistent with Figure 3, reflecting that the relative magnitude of autocorrelations of different lags supports the implication of the equilibrium model. The left panel of the table, however, shows that the absolute magnitude of, say the autocorrelation of lag one, is small compared to that of an AR(1) model with mean reversion rate around one. Moreover, there is significant dispersion of the autocorrelation across institutions, evident from the magnitude of the standard deviation. This suggests that institutions, unsurprisingly, conduct "higher frequency" intra-quarter trading that is not targeted at the predictable returns documented by Tables 3 and 4, which are of "lower frequency".

6.6 Structural Implications

The presence of intra-quarter trading would affect the covariance matrix $\Sigma := \text{Cov}(\Delta \breve{x}_{i,t})$[14], breaks the first equation in (5.3), and contaminate the identification of parameters describing the trading that actually corresponds to the more persistent return predictability. To alleviate this issue, we instead estimate the parameters using the covariances between position changes over adjacent periods $\Sigma' := \text{Cov}(\Delta \breve{x}_{i,t}, \Delta \breve{x}_{i,t+1}) + \frac{b}{2}\text{Cov}(\Delta \breve{x}_{i,t})$,[15] which satisfies $\Sigma' = \lambda' \text{Cov}(y_t)$ for some constant λ' that only depends on ξ^P,[16] and on which the intra-quarter trading has less influence. Algorithm 1 applies as before, with λ replaced by λ'. R_τ has been estimated in Section 6.4, following step 3 of Algorithm 2. We now provide the procedure to estimate $\widetilde{\nu}' := (\lambda' \phi \widetilde{\iota})\nu$ and $\widetilde{\Sigma}' := \lambda' \phi \widetilde{\iota}\big(\beta\beta^\intercal - \nu\nu^\intercal\big) + \lambda' \bar{\phi}\kappa\kappa^\intercal$.

Algorithm 3. *Inputs: position change $\Delta \breve{x}_{i,t}$, autocorrelation $\widehat{\gamma}_{j,1}$, outputs of Algorithm 2 ($\widehat{\widetilde{\nu}}$, $\widehat{\Lambda}$, $\widehat{D}$), and rebalancing parameter estimate $\widehat{b}$.*

S1. Estimate $\widetilde{\nu}'$ using $\widehat{\widetilde{\nu}'}_j = \widehat{\widetilde{\nu}}_j\big(\widehat{\gamma}_{j,1} + \widehat{b}/2\big)$.

S2. Estimate $\widetilde{\Sigma}'$ as $\widehat{\widetilde{\Sigma}}' = \widehat{\Lambda}\widehat{D}'\widehat{\Lambda}^\intercal$, where $d \times d$ matrix $\widehat{D}'$ is

$$\widehat{D}' = \widehat{\text{Cov}}\big(\widehat{\Lambda}\Delta \breve{x}_{i,t}, \widehat{\Lambda}\Delta \breve{x}_{i,t+1}\big) - \widehat{\Lambda}\text{diag}\big(\widehat{\widetilde{\nu}'}\big)\widehat{\Lambda}^\intercal + \frac{1}{2}\widehat{b}\widehat{D}.$$

[14] See footnote 8.

[15] The second term is to remove the effect of rebalancing, which would depend on the magnitude of intra-quarter trading.

[16] $\lambda' = (\xi^P)^{-2}(1 - e^{-\xi^P})^2$.

Outputs: $\widehat{\widetilde{\nu}}'$ *and* $\widehat{\widetilde{\Sigma}}'$.

As in Algorithm 2, step 2 again conducts dimension reduction using $\widehat{\widetilde{\Sigma}}'$, which is valid under the assumption that the eigenvectors of $\widetilde{\Sigma}$ spans all the eigenvectors of $\widetilde{\Sigma}'$. As a partial test of this assumption, we regress $\widehat{\widetilde{\nu}}'$ on the four eigenvectors (the columns of $\widehat{\Lambda}$) without constant and obtain an R^2 of 85%. Following steps 2 and 3 of Algorithm 1, (where $\widetilde{\nu}$ and $\widetilde{\Sigma}$ are naturally replaced by $\widetilde{\nu}'$ and $\widetilde{\Sigma}'$, which in turn are estimated using Algorithm 3), we obtain estimates of β and ν across all the institutions (groups), presented in Figure 4.

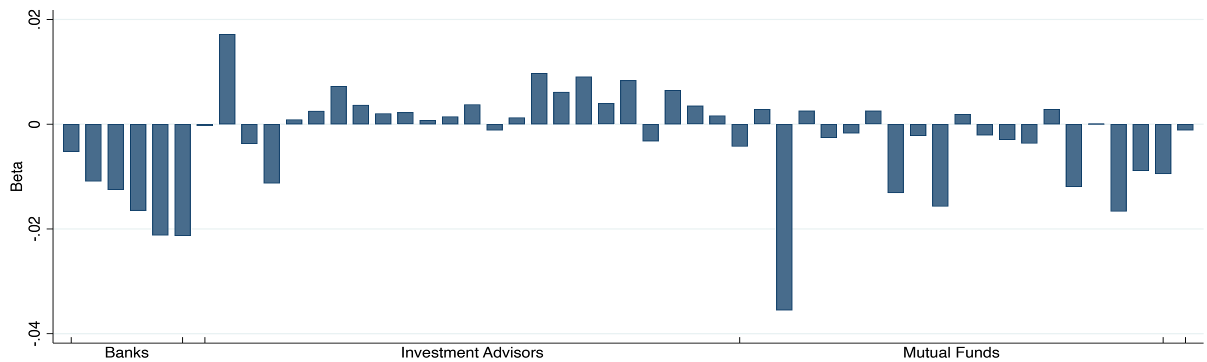
A few patterns emerge. All banks and most mutual funds trade against future price movement, which means they have more upward bias in perceiving their information quality within our model, whereas most investment advisors more correctly understand the precision of their private signals. If interpreted more broadly, it perhaps could be that a large part of bank trading is based on other motives such as hedging and liquidity considerations. On average, institutions have about zero investment performance in the current context. The average of β is -0.0026, meaning that market participants outside the data, which include households and smaller institutions, on average do not have a big advantage or disadvantage. Moreover, the dispersion of β is smaller than (about half of) the average magnitude of ν, suggesting that the deviation of misperception from average level is not too big.[17] On the other hand, the average misperception, however, must be greater than two for the equilibrium characterized here to exist. Further, because we construct institution groups such that they have similar trading activity, the distribution of ν suggests that mutual funds focus more on trading on price movements that are more persistent, whereas investment advisors devote a larger portion of their investment activities to intra-quarter trading, possibly due to that they on average have smaller sizes. Finally, the sum of ν_j across all the institutions in the data is 89.8%[18], indicating that institutions in the sample together may conduct trading on persistent price movement more than their coverage of the total market in terms of asset under management.

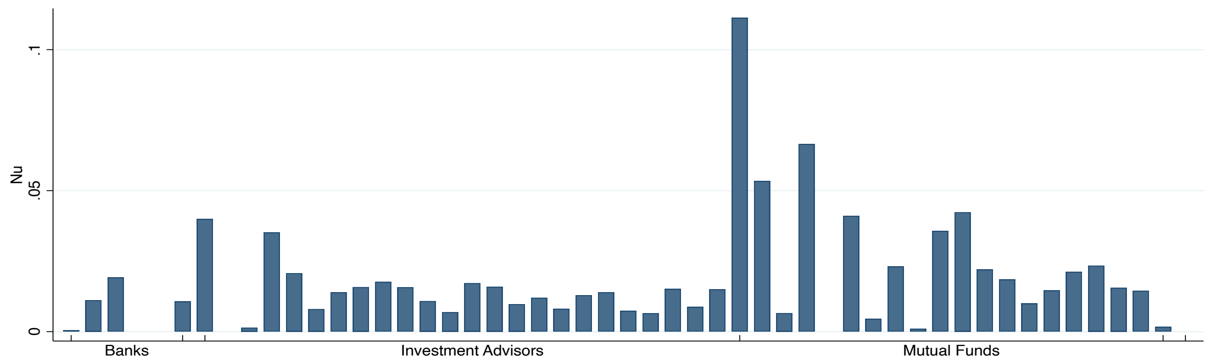
Table 6 collects the estimates of the other structural parameters estimated following steps 5 and 6 of Algorithm 1. The quantity $\phi\widetilde{\iota}$ reflects the magnitude of trading activity, and the number suggests that, ignoring rebalancing consideration, the total willingness to trade over a quarter for all the market participant, of which the motive is to capture the predictable return component that persists for about a quarter, is about 2.39% of the total outstanding shares for an average stock, which is then absorbed by appropriate price change to clear the market. The average misperception $\widetilde{\omega}$ is about 2.09, just exceeding the threshold 2 for our equilibrium to exist. The main driver between this number is that the rebalancing rate b is very small, compared to how quickly predictable return mean-reverts (the parameter ξ^P). Suppose price impact is less important compared to risk aversion, then, within our model, investors could rebalance more quickly, avoiding a large portion of risk exposure but capturing expected return almost equally well. In other words, small b reflects that price impact is of more importance in investors' tradeoff, indicating that $\widetilde{\omega}$ is not much higher than the threshold

[17] As in Proposition 2, $\beta_j/\nu_j = 1 - \widetilde{\omega}^{-1}\omega_j$.

[18] The sum of ν_j across all the market participants is one by definition.

Figure 4: Distribution of β and ν across Institutions





Note: The figure presents the estimated β and ν across institutions. The type between banks and investment advisors is insurance companies. The one to the right of mutual funds are pension funds and others. Within each institution type, the leftmost one is the institution with largest trading activity, whereas the rightmost one represents the group of least active institutions. For five institutions with negative estimates of ν, we replace the estimates with zeros.

2, at which the market liquidity completely goes away. The estimate of the price impact coefficient ζ suggests that selling a stock at the rate of 1% of the total shares per quarter would impose about 0.26% downward pressure on price during the selling. The estimate of a indicates that, facing a subjective predictable return of 1% that decays at rate ξ^P, an investor at the moment would buy the stock at rate 1.45% of the total shares per quarter.

In the predictive regression exercises, we as econometricians are essentially constructing a proxy for the predictive factor f_t that appears in Proposition 3 using investors position changes. The proxy would necessarily contain errors if a significant part of institutions' quarterly position changes originate for intra-quarter trading that is not directly connected to the persistent component of the predictable return. Even in the absence of intra-quarter trading, the precision of the proxy positively depends on the dispersion of investors' misperception. In the completely symmetric case, the positions or position changes does not correlate with f_t at all for all the investors. With the estimates of $\widetilde{\phi\iota}$, $\widetilde{\omega}$, and a, a back-of-the-envelop calculation using Proposition 3 indicates that perfect knowledge of f_t would be generate a predictable return of which the standard deviation of $4 \times (a\widetilde{\omega})^{-1}\sqrt{\widetilde{\phi\iota}} = 3.16\%$

Table 6: Estimates of Structural Parameters

$\sqrt{\phi\widetilde{\iota}} \times 100$	$\widetilde{\omega}$	ξ^P	ζ	a	b
2.39	2.09	1.29	.256	1.45	.057

Note: The table collects estimates of the six structural parameters. ξ^P and b are obtained from Table 5, whereas the other four are calculated based on steps 5 and 6 of Algorithm 1.

per quarter for an individual stock. If we further suppose f_t is purely uncorrelated across stocks and stock idiosyncratic volatility is about 30%, then with an investment universe of 1,600 stocks, an investor with perfect knowledge of f_t could generate s Sharpe ratio of around 4.1, suggesting that our private information interpretation is perhaps relevant.

From property (i) of Theorem 1 and under $\bar{\eta} = 0$, the pricing error arising from that G_t is unobservable satisfies

$$\mathrm{Var}\big(P_t - F_t\big) = 2\phi\widetilde{\iota}\zeta^2 \times \big(1 - \xi^G/\xi^P\big)^{-1}\big(1 + \widetilde{\iota}(\sigma^D)^{-2}\big) > 2\phi\widetilde{\iota}\zeta^2.$$

Here F_t is the present value of future cash flows when G_t is observable.[19] Then from Table 6, the estimate of $\sqrt{2\phi\widetilde{\iota}}\zeta$, the lower bound of the standard deviation of the pricing error, is around 0.86%. Because of the design of our empirical analysis articulated in Section 6.1, the number shall be interpreted as the error relative to the price itself. When ξ^G/ξ^P and $\widetilde{\iota}(\sigma^D)^{-2}$ are small, i.e., the growth rate does not mean-revert much over the course of a quarter and that the information contained in cash flow about the growth rate is little compared to that from the aggregation of private signals, $\sqrt{2\phi\widetilde{\iota}}\zeta$ provides a good approximation to the magnitude of the pricing error. In this case, the counterfactual standard deviation of the pricing error in the absence of investors j with $j \in \mathcal{J}_0$ would be the current standard deviation multiplied by factor $\sqrt{\sum_{j:j\notin\mathcal{J}_0}\nu_j}\big/\sum_{j:j\notin\mathcal{J}_0}\pi_j$.[20] We report in Table 7 the counterfactual pricing error magnitude relative to the current one in the absence of trading by each type of institutions and by the household sector.[21]

The type of institutions that is most important for improving the informational efficiency of the market is the type of investment advisors, followed by mutual funds. Households sector contribute to the price informativeness significantly as well. On the other hand, institutions that demand liquidity for other reasons such as risk hedging or balance sheet management such as banks, insurance companies, and pension funds, contribute negatively to the market efficiency.

[19]The expression of F_t is simply the one of P_t with $\mathrm{E}(G_t|\{D_s, \bar{S}_s\}_{s\le t})$ replaced by G_t.

[20]In general, the factor $\sqrt{\sum_{j:j\notin\mathcal{J}_0}\nu_j}\big/\sum_{j:j\notin\mathcal{J}_0}\pi_j$ is a lower (upper) bound of the ratio of the counterfactual pricing error magnitude to the current one when the factor is greater (less) than one.

[21]The household sector here includes all the households and non-13F institutions.

Table 7: Counterfactual Pricing Error Magnitude In the Absence of Certain Investors

Banks	Insurance Companies	Investment Advisors	Mutual Funds	Pension Funds	Other Institutions	Households
.95	.98	1.37	1.18	.99	1.00	1.24

Note: The table presents the counterfactual pricing error magnitude relative to the current one in the absence of trading by each type of institutions and by the household sector, using the estimates of β and ν presented in Figure 4.

7 Conclusion

Taking stock, our paper provides a new conceptual framework and appropriate econometric procedures to understanding the role of private information in the financial market utilizing price and quantity data together. The presence of many investors allows for a tractable equilibrium with heterogeneity in information and belief structure, and at the same time requires for properly designed econometric methods. Examining the equilibrium implications with the joint moments of price and institution holding data allows us to measure, among others, the magnitude of market inefficiency and the contributions of various investors to the price informativeness. More broadly, it would be interesting to investigate that to what extent the current empirical strategies, that are directly generated by our equilibrium model, can actually apply beyond under the current structural assumptions.

References

Admati, A. R. (1985): "A noisy rational expectations equilibrium for multi-asset securities markets," *Econometrica: Journal of the Econometric Society*, pp. 629–657.

Berk, J. B., and R. C. Green (2004): "Mutual fund flows and performance in rational markets," *Journal of political economy*, 112(6), 1269–1295.

Berk, J. B., and J. H. Van Binsbergen (2015): "Measuring skill in the mutual fund industry," *Journal of financial economics*, 118(1), 1–20.

Chang, Y.-C., H. Hong, and I. Liskovich (2015): "Regression discontinuity and the price effects of stock market indexing," *The Review of Financial Studies*, 28(1), 212–246.

Chen, H.-L., N. Jegadeesh, and R. Wermers (2000): "The value of active mutual fund management: An examination of the stockholdings and trades of fund managers," *Journal of Financial and quantitative Analysis*, 35(3), 343–368.

Da, R., S. Nagel, and D. Xiu (2022): "The Statistical Limit of Arbitrage," Discussion paper, Technical Report, Chicago Booth.

Daniel, K., M. Grinblatt, S. Titman, and R. Wermers (1997): "Measuring mutual fund performance with characteristic-based benchmarks," *The Journal of finance*, 52(3), 1035–1058.

Du, S., and H. Zhu (2017): "What is the optimal trading frequency in financial markets?," *The Review of Economic Studies*, 84(4), 1606–1651.

Fama, E. F. (1970): "Efficient capital markets: A review of theory and empirical work," *The journal of Finance*, 25(2), 383–417.

Fama, E. F., and K. R. French (2010): "Luck versus skill in the cross-section of mutual fund returns," *The journal of finance*, 65(5), 1915–1947.

Gabaix, X., and R. S. Koijen (2021): "In search of the origins of financial fluctuations: The inelastic markets hypothesis," Discussion paper, National Bureau of Economic Research.

Gârleanu, N., and L. H. Pedersen (2013): "Dynamic trading with predictable returns and transaction costs," *The Journal of Finance*, 68(6), 2309–2340.

——— (2016): "Dynamic portfolio choice with frictions," *Journal of Economic Theory*, 165, 487–516.

Giglio, S., B. Kelly, and D. Xiu (2022): "Factor models, machine learning, and asset pricing," *Annual Review of Financial Economics*, 14, 337–368.

GIGLIO, S., AND D. XIU (2021): "Asset pricing with omitted factors," *Journal of Political Economy*, 129(7), 1947–1990.

GIGLIO, S., D. XIU, AND D. ZHANG (2021): "Test assets and weak factors," Discussion paper, National Bureau of Economic Research.

GROSSMAN, S. J., AND J. E. STIGLITZ (1980): "On the impossibility of informationally efficient markets," *The American economic review*, 70(3), 393–408.

HARRIS, L., AND E. GUREL (1986): "Price and volume effects associated with changes in the S&P 500 list: New evidence for the existence of price pressures," *the Journal of Finance*, 41(4), 815–829.

HE, H., AND J. WANG (1995): "Differential information and dynamic behavior of stock trading volume," *The Review of Financial Studies*, 8(4), 919–972.

KACPERCZYK, M., S. VAN NIEUWERBURGH, AND L. VELDKAMP (2014): "Time-varying fund manager skill," *The Journal of Finance*, 69(4), 1455–1484.

——— (2016): "A rational theory of mutual funds' attention allocation," *Econometrica*, 84(2), 571–626.

KELLY, B., AND S. PRUITT (2013): "Market expectations in the cross-section of present values," *The Journal of Finance*, 68(5), 1721–1756.

KELLY, B. T., S. PRUITT, AND Y. SU (2019): "Characteristics are covariances: A unified model of risk and return," *Journal of Financial Economics*, 134(3), 501–524.

KOIJEN, R. S., AND M. YOGO (2019): "A demand system approach to asset pricing," *Journal of Political Economy*, 127(4), 1475–1515.

KOZAK, S., S. NAGEL, AND S. SANTOSH (2018): "Interpreting factor models," *The Journal of Finance*, 73(3), 1183–1223.

——— (2020): "Shrinking the cross-section," *Journal of Financial Economics*, 135(2), 271–292.

KYLE, A. S. (1985): "Continuous auctions and insider trading," *Econometrica: Journal of the Econometric Society*, pp. 1315–1335.

——— (1989): "Informed speculation with imperfect competition," *The Review of Economic Studies*, 56(3), 317–355.

KYLE, A. S., A. A. OBIZHAEVA, AND Y. WANG (2018): "Smooth trading with overconfidence and market power," *The Review of Economic Studies*, 85(1), 611–662.

MARTIN, I. W., AND S. NAGEL (2022): "Market efficiency in the age of big data," *Journal of financial economics*, 145(1), 154–177.

Pástor, L., and R. F. Stambaugh (2012): "On the size of the active management industry," *Journal of Political Economy*, 120(4), 740–781.

Pástor, L., R. F. Stambaugh, and L. A. Taylor (2015): "Scale and skill in active management," *Journal of Financial Economics*, 116(1), 23–45.

——— (2017): "Do funds make more when they trade more?," *The Journal of Finance*, 72(4), 1483–1528.

——— (2020): "Fund tradeoffs," *Journal of Financial Economics*, 138(3), 614–634.

Petajisto, A. (2009): "Why do demand curves for stocks slope down?," *Journal of Financial and Quantitative Analysis*, 44(5), 1013–1044.

Shleifer, A. (1986): "Do demand curves for stocks slope down?," *The Journal of Finance*, 41(3), 579–590.

Song, Y. (2020): "The mismatch between mutual fund scale and skill," *The Journal of Finance*, 75(5), 2555–2589.

Van Nieuwerburgh, S., and L. Veldkamp (2010): "Information acquisition and under-diversification," *The Review of Economic Studies*, 77(2), 779–805.

Vayanos, D. (1999): "Strategic trading and welfare in a dynamic market," *The Review of Economic Studies*, 66(2), 219–254.

Vives, X. (2011): "Strategic supply function competition with private information," *Econometrica*, 79(6), 1919–1966.

Wang, J. (1993): "A model of intertemporal asset prices under asymmetric information," *The Review of Economic Studies*, 60(2), 249–282.

——— (1994): "A model of competitive stock trading volume," *Journal of political Economy*, 102(1), 127–168.

Wardlaw, M. (2020): "Measuring mutual fund flow pressure as shock to stock returns," *The Journal of Finance*, 75(6), 3221–3243.

Wermers, R. (2000): "Mutual fund performance: An empirical decomposition into stock-picking talent, style, transactions costs, and expenses," *The Journal of Finance*, 55(4), 1655–1695.

Appendix A Mathematical Proofs

A.1 Dynamic Trading with Price Impact

This section solves the investor j's optimization problem (2.3) and (2.4) under an exogenous supply curve, which provides foundation for the proofs of Proposition 1 and Theorem 1.

A.1.1 Setup

We consider the following form of supply curve faced by investor j:

$$P_t(\dot{x}_{j,t}) = \widetilde{P}_{j,t} + \zeta_j \dot{x}_{j,t} + \bar{\zeta}_j x_{j,t}, \tag{A.1}$$

Here $\widetilde{P}_{j,t}$ is the "intercept" of the supply curve that investor j faces and does not depend on her trading. Trading $\dot{x}_{j,t}$ not only affects contemporaneous price, with coefficient ζ_j, but also has persistent impact onto future prices, the intensity of which is measured by coefficient $\bar{\zeta}_j$. The intercept $\widetilde{P}_{j,t}$, on the other hand, is assumed to satisfy

$$d\widetilde{P}_{j,t} + D_t dt - r\widetilde{P}_{j,t} dt = \mu_{j,t} dt + \sigma_j^P d\widehat{Z}_{j,t}^P, \tag{A.2}$$

where, under investor j's subjective measure, $\mu_{j,t}$ is the drift and $\widehat{Z}_{j,t}^P$ is a standard Brownian motion.

Next, we specify the subjective dynamics of $\mu_{j,t}$ to have a two-component structure:

$$\mu_{j,t} = \mu_{j,t}^{(1)} + \mu_{j,t}^{(2)}, \tag{A.3}$$

and, for each $k \in \{1, 2\}$, $\mu_{j,t}^{(k)}$ follows a simple mean-reverting process:

$$d\mu_{j,t}^{(k)} = -\xi_j^{\mu,(k)} \mu_{j,t}^{(k)} dt + \sigma_j^{\mu,(k)} d\widehat{Z}_{j,t}^{\mu,(k)}, \tag{A.4}$$

where $\widehat{Z}_{j,t}^{\mu,(1)}$ and $\widehat{Z}_{j,t}^{\mu,(2)}$ are two standard Brownian motions, under investor j's subjective measure. The correlations between $\widehat{Z}_{j,t}^{\mu,(1)}$, $\widehat{Z}_{j,t}^{\mu,(2)}$, and $\widehat{Z}_{j,t}^P$ are such that

$$\mathrm{E}_{j,t}\big(d\mu_{j,t}^{(k)} d\mu_{j,t}^{(k')}\big) = \big(\Sigma_j^{\mu}\big)_{k,k'} dt, \quad \text{and} \quad \mathrm{E}_{j,t}\big(d\mu_{j,t}^{(k)} d\widetilde{P}_{j,t}\big) = \big(\Sigma_j^{(\mu,P)}\big)_k dt. \tag{A.5}$$

The problem is equally tractable when $\mu_{j,t}$ consists of any finite number of components mean-reverting at various rates. But allowing for two components is already sufficient for our equilibrium analysis.

A.1.2 Solution

This subsection solves the investor optimization problem (2.3) and (2.4) under the supply curve specified by (A.1), (A.3), (A.4), and (A.5). We present the solution in the following proposition:

Proposition A1. *Suppose the supply curve satisfies* (A.1)*,* (A.3)*,* (A.4)*, and* (A.5)*. If equations* (A.24)*,* (A.25)*, and* (A.26) *in Section A.1.3 has a solution satisfying* (A.28)*, then the optimal trading strategy that solves the problem* (2.3) *and* (2.4) *is*

$$\dot{x}_{j,t} = -b_j x_{j,t} + \sum_{k \in \{1,2\}} a_j^{(k)} \mu_{j,t}^{(k)}, \tag{A.6}$$

where the trading intensity parameters $(a_j^{(1)}, a_j^{(2)})$ and the rebalance parameter b_j are given by (A.29).

A.1.3 Proof of Proposition A1

Proof. Under the supply curve specified by (A.1), (A.3), (A.4), and (A.5), the value function of the optimization problem will only depend on $(x_{j,t}^f, x_{j,t}, \mu_{j,t}^{(1)}, \mu_{j,t}^{(2)}, \widetilde{P}_{j,t})$, and we denote it by $V(x_{j,t}^f, x_{j,t}, \mu_{j,t}^{(1)}, \mu_{j,t}^{(2)}, \widetilde{P}_{j,t})$. We denote $\Sigma_j^P = (\sigma_j^P)^2$. Using (2.3) and (2.4), we can write the HJB equation as:

$$
\begin{aligned}
\rho V \;=\; \max_{\dot{x}_t, c_t} \Bigg(& -\exp(-\gamma c_t) + \frac{\partial V}{\partial x_t^f}(r x_t^f + x_t D_t - c_t - P_t \dot{x}_t) + \frac{\partial V}{\partial x_t}\dot{x}_t \\
& - \sum_{k \in \{1,2\}} \xi^{\mu,(k)} \frac{\partial V}{\partial \mu_t^{(k)}} \mu_t^{(k)} + \frac{\partial V}{\partial \widetilde{P}_t}(\mu_t - D_t + r\widetilde{P}_t) \\
& + \frac{1}{2} \sum_{k,k' \in \{1,2\}} \frac{\partial^2 V}{\partial \mu_t^{(k)} \partial \mu_t^{(k')}} \Sigma_{k,k'}^{\mu} + \sum_{k \in \{1,2\}} \frac{\partial^2 V}{\partial \widetilde{P}_t \partial \mu_t^{(k)}} \Sigma_k^{(\mu,P)} + \frac{1}{2}\frac{\partial^2 V}{\partial \widetilde{P}_t^2}\Sigma^P \Bigg).
\end{aligned} \tag{A.7}
$$

Here and below in the proof, for simplicity of exposition, we suppress the argument of V and the subscript j. We conjecture the value function as

$$
V = -\exp\left(a_0 + a_1(x_t^f + \widetilde{P}_t x_t) + \sum_{k,k' \in \{1,2\}} (a_2)_{k,k'} \mu_t^{(k)} \mu_t^{(k')} + \sum_{k \in \{1,2\}} (a_3)_k \mu_t^{(k)} x_t + a_4 x_t^2 \right),
$$

where a_2 is a (2×2)-dimensional symmetric matrix, a_3 is a 2-dimensional vector, and a_0, a_1, and a_4 are scalars. Under the conjecture, we can calculate

$$
V^{-1}\frac{\partial V}{\partial x_t^f} = a_1, \quad V^{-1}\frac{\partial V}{\partial x_t} = a_1 \widetilde{P}_t + \sum_{k \in \{1,2\}} (a_3)_k \mu_t^{(k)} + 2a_4 x_t, \tag{A.8}
$$

$$
V^{-1}\frac{\partial V}{\partial \mu_t^{(k)}} = \sum_{k' \in \{1,2\}} (a_2)_{k,k'} \mu_t^{(k')}(1 + \delta_{k,k'}) + (a_3)_k x_t, \quad V^{-1}\frac{\partial V}{\partial \widetilde{P}_t} = a_1 x_t, \tag{A.9}
$$

$$
V^{-1}\frac{\partial^2 V}{\partial \mu_t^{(k)} \partial \mu_t^{(k')}} = (a_2)_{k,k'}(1 + \delta_{k,k'}) + V^{-2}\frac{\partial V}{\partial \mu_t^{(k)}}\frac{\partial V}{\partial \mu_t^{(k')}}, \tag{A.10}
$$

$$
V^{-1}\frac{\partial^2 V}{\partial \widetilde{P}_t \partial \mu_t^{(k)}} = a_1 x_t \frac{\partial V}{\partial \mu_t^{(k)}}, \quad V^{-1}\frac{\partial^2 V}{\partial \widetilde{P}_t^2} = a_1^2 x_t^2. \tag{A.11}
$$

The first-order condition w.r.t. c_t is

$$
\gamma \exp(-\gamma c_t) = \frac{\partial V}{\partial x_t^f} = a_1 V. \tag{A.12}
$$

Using (A.1), the first-order condition w.r.t. $\dot{x}_t$ is

$$\frac{\partial V}{\partial x_t^f}(-\widetilde{P}_t - \bar{\zeta} x_t - 2\zeta \dot{x}_t) = -\frac{\partial V}{\partial x_t}.$$

Substituting the derivatives given in (A.8) into the two first-order conditions, we obtain

$$a_1(-\widetilde{P}_t - \bar{\zeta} x_t - 2\zeta \dot{x}_t) + a_1 \widetilde{P}_t + \sum_{k\in\{1,2\}} (a_3)_k \mu_t^{(k)} + 2a_4 x_t = 0. \tag{A.13}$$

(A.13) further leads to

$$\begin{aligned} \dot{x}_t &= \frac{1}{2\zeta a_1}(-a_1\bar{\zeta} + 2a_4)x_t + \frac{1}{2\zeta a_1}\sum_{k\in\{1,2\}} (a_3)_k \mu_t^{(k)}, & \text{(A.14)} \\ a_1 \zeta \dot{x}_t V &= \frac{\partial V}{\partial x_t^f}(-P_t) + \frac{\partial V}{\partial x_t}. & \text{(A.15)} \end{aligned}$$

Substituting (A.12) and (A.15) into (A.7), we obtain

$$\begin{aligned} \rho V &= -\gamma^{-1} a_1 V + \frac{\partial V}{\partial x_t^f}(r x_t^f + x_t D_t + \gamma^{-1}\log(\gamma^{-1} a_1 V)) + a_1 \zeta \dot{x}_t^2 V \\ &\quad - \sum_{k\in\{1,2\}} \xi^{\mu,(k)} \frac{\partial V}{\partial \mu_t^{(k)}} \mu_t^{(k)} + \frac{\partial V}{\partial \widetilde{P}_t}(\mu_t - D_t + r\widetilde{P}_t) \\ &\quad + \frac{1}{2}\sum_{k,k'\in\{1,2\}} \frac{\partial^2 V}{\partial \mu_t^{(k)} \partial \mu_t^{(k')}} \Sigma^{\mu}_{k,k'} + \sum_{k\in\{1,2\}} \frac{\partial^2 V}{\partial \widetilde{P}_t \partial \mu_t^{(k)}} \Sigma_k^{(\mu,P)} + \frac{1}{2}\frac{\partial^2 V}{\partial \widetilde{P}_t^2}\Sigma^P. \end{aligned} \tag{A.16}$$

Substituting $\frac{\partial V}{\partial \widetilde{P}_t}$ and $\frac{\partial^2 V}{\partial \widetilde{P}_t^2}$ given by (A.9) and (A.11) into (A.16), we have

$$\rho V = V^{(1)} + V^{(2)} + a_1 \zeta \dot{x}_t^2 V, \tag{A.17}$$

where

$$\begin{aligned} V^{(1)} &= a_1 V \left(-\gamma^{-1} + r(x_t^f + x_t \widetilde{P}_t) + \gamma^{-1}\log(\gamma^{-1} a_1 V) + x_t \mu_t + \frac{1}{2} a_1 x_t^2 \Sigma^P\right), \\ V^{(2)} &= -\sum_{k\in\{1,2\}} \xi^{\mu,(k)} \frac{\partial V}{\partial \mu_t^{(k)}} \mu_t^{(k)} + \frac{1}{2}\sum_{k,k'\in\{1,2\}} \frac{\partial^2 V}{\partial \mu_t^{(k)} \partial \mu_t^{(k')}} \Sigma^{\mu}_{k,k'} + \sum_{k\in\{1,2\}} \frac{\partial^2 V}{\partial \widetilde{P}_t \partial \mu_t^{(k)}} \Sigma_k^{(\mu,P)}. \end{aligned}$$

Since $V^{(2)}$ does not contain x_t^f, setting the coefficient of x_t^f as zero leads to $a_1 = -\gamma r$, under which $V^{(1)}$ satisfies

$$V^{-1} V^{(1)} = r - r\log(r) - r a_0 - r\left(\frac{1}{2}\gamma a_1 \Sigma^P + a_4\right) x_t^2$$

$$-r\sum_{k,k'\in\{1,2\}}\mu_t^{(k)}\mu_t^{(k')}(a_2)_{k,k'}-r\sum_{k\in\{1,2\}}x_t\mu_t^{(k)}(\gamma+(a_3)_k). \tag{A.18}$$

On the other hand, using $\frac{\partial^2 V}{\partial\mu_t^{(k)}\partial\mu_t^{(k')}}$ and $\frac{\partial^2 V}{\partial\widetilde{P}_t\partial\mu_t^{(k)}}$ given by (A.10) and (A.11), we can decompose $V^{(2)}$ into

$$V^{(2)}=V^{(3)}+V^{(4)}+V^{(5)}, \tag{A.19}$$

where

$$\begin{aligned}
V^{(3)} &= -\sum_{k\in\{1,2\}}\xi^{\mu,(k)}\frac{\partial V}{\partial\mu_t^{(k)}}\mu_t^{(k)}+\frac{1}{2}V\sum_{k,k'\in\{1,2\}}(a_2)_{k,k'}(1+\delta_{k,k'})\Sigma^{\mu}_{k,k'},\\
V^{(4)} &= \frac{1}{2}V^{-1}\sum_{k,k'\in\{1,2\}}\frac{\partial V}{\partial\mu_t^{(k)}}\frac{\partial V}{\partial\mu_t^{(k')}}\Sigma^{\mu}_{k,k'},\\
V^{(5)} &= a_1x_t\sum_{k\in\{1,2\}}\frac{\partial V}{\partial\mu_t^{(k)}}\Sigma_k^{(\mu,P)}.
\end{aligned}$$

Using $\frac{\partial V}{\partial\mu_t^{(k)}}$ given by (A.9), we further write $V^{(3)}$, $V^{(4)}$, and $V^{(5)}$ as

$$\begin{aligned}
V^{-1}V^{(3)} &= -\sum_{k,k'\in\{1,2\}}\mu_t^{(k)}\mu_t^{(k')}\xi^{\mu,(k)}(a_2)_{k,k'}(1+\delta_{k,k'})\\
&\quad-\sum_{k\in\{1,2\}}x_t\mu_t^{(k)}\xi^{\mu,(k)}(a_3)_k+\frac{1}{2}\sum_{k,k'\in\{1,2\}}(a_2)_{k,k'}(1+\delta_{k,k'})\Sigma^{\mu}_{k,k'}, && \text{(A.20)}\\
V^{-1}V^{(4)} &= \frac{1}{2}\sum_{k,k'\in\{1,2\}}\mu_t^{(k)}\mu_t^{(k')}\sum_{l,l'\in\{1,2\}}\Sigma^{\mu}_{l,l'}(a_2)_{k,l}(a_2)_{k',l'}(1+\delta_{k,l})(1+\delta_{k',l'})\\
&\quad+\sum_{k\in\{1,2\}}x_t\mu_t^{(k)}\sum_{l,l'\in\{1,2\}}\Sigma^{\mu}_{l,l'}(a_2)_{k,l}(a_3)_{l'}(1+\delta_{k,l})+\frac{1}{2}x_t^2\sum_{l,l'\in\{1,2\}}\Sigma^{\mu}_{l,l'}(a_3)_l(a_3)_{l'}, && \text{(A.21)}\\
V^{-1}V^{(5)} &= a_1\sum_{k\in\{1,2\}}x_t\mu_t^{(k)}\sum_{l\in\{1,2\}}\Sigma_l^{(\mu,P)}(a_2)_{k,l}(1+\delta_{k,l})+a_1x_t^2\sum_{l\in\{1,2\}}\Sigma_l^{(\mu,P)}(a_3)_l. && \text{(A.22)}
\end{aligned}$$

Finally, using (A.14), we can write

$$\begin{aligned}
a_1\zeta\dot{x}_t^2 &= \frac{1}{4\zeta a_1}\sum_{k,k'\in\{1,2\}}\mu_t^{(k)}\mu_t^{(k')}(a_3)_k(a_3)_{k'}\\
&\quad+\frac{-a_1\bar{\zeta}+2a_4}{2\zeta a_1}x_t\sum_{k\in\{1,2\}}\mu_t^{(k)}(a_3)_k+\frac{(-a_1\bar{\zeta}+2a_4)^2}{4\zeta a_1}x_t^2. && \text{(A.23)}
\end{aligned}$$

Substituting (A.18), (A.19), (A.20), (A.21), (A.22), and (A.23) into (A.17) and setting the coefficients of $\mu_t^{(k)}\mu_t^{(k')}$, $\mu_t^{(k)}x_t$, and x_t^2, and the constant term as zero would provide equations that (a_0,a_2,a_3,a_4)

must satisfy. Setting the coefficient of $\mu_t^{(k)}\mu_t^{(k')}$ as zero, we have, for $k, k' \in \{1,2\}$

$$-r(a_2)_{k,k'} + \xi^{\mu,(k)}(a_2)_{k,k'}(1+\delta_{k,k'}) - \frac{1}{4\zeta r\gamma}(a_3)_k(a_3)_{k'} = A^{(\mu,\mu)}_{k,k'}, \tag{A.24}$$

where

$$A^{(\mu,\mu)}_{k,k'} = -\frac{1}{2}\sum_{l,l'\in\{1,2\}} \Sigma^{\mu}_{l,l'}(a_2)_{k,l}(a_2)_{k',l'}(1+\delta_{k,l})(1+\delta_{k',l'}).$$

Setting the coefficient of $x_t\mu_t^{(k)}$ as zero, we have, for $k \in \{1,2\}$,

$$-r(\gamma + (a_3)_k) - \xi^{\mu,(k)}(a_3)_k - \frac{r\gamma\bar{\zeta} + 2a_4}{2\zeta r\gamma}(a_3)_k = A^{(x,\mu)}_k, \tag{A.25}$$

where

$$A^{(x,\mu)}_k = -\sum_{l,l'\in\{1,2\}} \Sigma^{\mu}_{l,l'}(a_2)_{k,l}(a_3)_{l'}(1+\delta_{k,l}) + r\gamma\sum_{l\in\{1,2\}} \Sigma^{(\mu,P)}_l(a_2)_{k,l}(1+\delta_{k,l}).$$

Setting the coefficient of x_t^2 as zero, we have

$$\frac{1}{2}r^2\gamma^2\Sigma^P - ra_4 - \frac{(r\gamma\bar{\zeta}+2a_4)^2}{4\zeta r\gamma} = A^{(x,x)}, \tag{A.26}$$

where

$$A^{(x,x)} = -\frac{1}{2}\sum_{l,l'\in\{1,2\}} \Sigma^{\mu}_{l,l'}(a_3)_l(a_3)_{l'} + r\gamma\sum_{l\in\{1,2\}} \Sigma^{(\mu,P)}_l(a_3)_l.$$

Setting the constant term as zero, we have

$$ra_0 = -\rho + r - r\log(r) + \frac{1}{2}\sum_{l,l'\in\{1,2\}} (a_2)_{l,l'}(1+\delta_{l,l'})\Sigma^{\mu}_{l,l'}. \tag{A.27}$$

Therefore, with $a_1 = -\gamma r$, we only need to find (a_2, a_3, a_4) that solve (A.24), (A.25), and (A.26). Then with a_2, (A.27) directly give a_0. For the strategy to be stationary, we need

$$r\gamma\bar{\zeta} + 2a_4 > 0 \tag{A.28}$$

according to (A.14), because $\zeta > 0$ and $a_1 = -\gamma r < 0$. Because $\gamma > 0$ and $\zeta > 0$, the second-order conditions hold. The stationarity of the optimal strategy and $r > 0$ indicates that Ponzi finance is ruled out. The transversality condition also holds because it follows from the HJB equation (A.16) and equations (A.24), (A.25), (A.26), (A.27), and $a_1 = -\gamma r$ that, for $t' > t$,

$$\mathrm{E}_{j,t}(e^{-\rho(t'-t)}V(x^f_{j,t'}, x_{j,t'}, \mu^{(1)}_{j,t'}, \mu^{(2)}_{j,t'}, \widetilde{P}_{j,t'})) = e^{-r(t'-t)}V(x^f_{j,t}, x_{j,t}, \mu^{(1)}_{j,t}, \mu^{(2)}_{j,t}, \widetilde{P}_{j,t}).$$

Given the solution (a_2, a_3, a_4) to equations (A.24), (A.25), and (A.26) with $a_1 = -\gamma r$, we directly obtain from (A.14) that

$$\dot{x}_{j,t} = -b_j x_{j,t} + \sum_{k\in\{1,2\}} a_j^{(k)} \mu_{j,t}^{(k)},$$

where

$$a_j^{(k)} = -\frac{(a_3)_k}{2\zeta r\gamma}, \quad b_j = \frac{1}{2\zeta r\gamma}(r\gamma\bar{\zeta} + 2a_4), \tag{A.29}$$

and the subscripts j of $a_j^{(k)}$ and b_j are added for clarity (we slightly abused the notation – $a_j^{(k)}$ and (a_2, a_3) are different). The proof ends. ∎

A.2 Proof of Proposition 1

Proof. The proof uses "conjecture and verify" approach. Step 1 conjectures investors' optimal strategy (demand schedule) and derives the supply curve investor j would face, should investor j' submits the conjectured demand schedule for all $j' \neq j$. Step 2 analyzes the dynamics of the (intercept) of the supply curve. In step 3, we study the dynamics of the supply curve under investor j's information set and subjective measure, derive her optimal strategy and show it is indeed the conjectured strategy. The existence condition, the endogenous price impact coefficient, and the equilibrium price are also derived in the this step. Steps 1 – 3 hence have already characterized the equilibrium. Step 4 proves the properties stated in the proposition based on steps 1 – 3.

Step 1. We conjecture

$$\dot{x}_{j,t} = \breve{a}\widetilde{\mu}_{j,t} - \breve{b}x_{j,t}, \tag{A.30}$$

where the process $\widetilde{\mu}_{j,t}$ is given by

$$\begin{aligned}
\widetilde{\mu}_{j,t} &= \frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D\breve{\xi}^G}\frac{J}{J-1}\left(\omega^{-1}\theta\breve{G}_{j,t} - \frac{\omega}{\omega+J-1}\left(\frac{\breve{\xi}^D\breve{\xi}^G}{\phi}\left(P_t - \frac{D_t}{\breve{\xi}^D}\right) - \theta^D\breve{G}_t^D\right)\right), &\text{(A.31)}\\
\breve{G}_{j,t} &= (\breve{\xi}^P - \xi^G)\int_{-\infty}^t e^{-\breve{\xi}^P(t-s)}dS_{j,s}, &\text{(A.32)}\\
\breve{G}_t^D &= (\breve{\xi}^P - \xi^G)\int_{-\infty}^t e^{-\breve{\xi}^P(t-s)}dS_s^D, \quad \text{with} \quad dS_t^D := dD_t + \xi^D D_t dt. &\text{(A.33)}
\end{aligned}$$

The parameters $\breve{a}$ and $\breve{b}$ involved in (A.30) and parameter ϕ appearing in (A.31) will be determined later on, and the parameters $(\breve{\xi}^D, \breve{\xi}^G, \theta, \theta^D, \breve{\xi}^P)$ used in (A.31) – (A.33) are defined as follows:

$$\breve{\xi}^D = \xi^D + r, \quad \breve{\xi}^G = \xi^G + r,$$

$$\theta = (\omega\widetilde{\sigma}^S/\sigma^S)^2, \quad \theta^D = (\widetilde{\sigma}^S/\sigma^D)^2, \tag{A.34}$$

$$(\widetilde{\sigma}^S)^2 = \frac{1}{(\sigma^D)^{-2} + (\omega^2 + J - 1)(\sigma^S)^{-2}}, \quad \breve{\xi}^P = \sqrt{(\sigma^G/\widetilde{\sigma}^S)^2 + (\xi^G)^2}. \tag{A.35}$$

Suppose, for all $j' \in \mathcal{J}$ with $j' \neq j$, investor j' submit the above demand schedule. Because the market clearing condition leads to

$$\dot{x}_{j,t} = -\sum_{j' \in \mathcal{J}: j' \neq j} \dot{x}_{j',t}, \quad x_{j,t} = -\sum_{j' \in \mathcal{J}: j' \neq j} x_{j',t},$$

we have

$$\breve{a} \sum_{j' \in \mathcal{J}: j' \neq j} \widetilde{\mu}_{j',t} = \sum_{j' \in \mathcal{J}: j' \neq j} (\dot{x}_{j',t} + \breve{b} x_{j',t}) = -\dot{x}_{j,t} - \breve{b} x_{j,t}. \tag{A.36}$$

Given (A.36), it follows from the definition of $\widetilde{\mu}_{j,t}$ that

$$\breve{a} \sum_{j' \in \mathcal{J}: j' \neq j} \frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D \breve{\xi}^G} \frac{J}{J-1} \left(\omega^{-1} \theta \breve{G}_{j',t} - \frac{\omega}{\omega + J - 1} \left(\frac{\breve{\xi}^D \breve{\xi}^G}{\phi} \left(P_t - \frac{D_t}{\breve{\xi}^D} \right) - \theta^D \breve{G}_t^D \right) \right) = -\dot{x}_{j,t} - \breve{b} x_{j,t}. \tag{A.37}$$

(A.37) directly gives P_t as a function of $(\{\breve{G}_{j',t}\}_{j' \in \mathcal{J}: j' \neq j}, \dot{x}_{j,t}, x_{j,t})$, i.e., the supply curve faced by investor j:

$$P_t = \frac{\omega + J - 1}{J \breve{a} \omega (\breve{\xi}^P + r)} (\dot{x}_{j,t} + \breve{b} x_{j,t}) + \frac{D_t}{\breve{\xi}^D} + \frac{\phi}{\breve{\xi}^D \breve{\xi}^G} \left(\theta^D \breve{G}_t^D + \frac{(\omega + J - 1)}{\omega^2} \frac{\theta}{J-1} \sum_{j' \in \mathcal{J}: j' \neq j} \breve{G}_{j',t} \right).$$

To facilitate exposition, we introduce

$$\widetilde{P}_{j,t} \coloneqq \frac{D_t}{\breve{\xi}^D} + \frac{\phi}{\breve{\xi}^D \breve{\xi}^G} \bar{G}_{j,t}, \quad \text{with} \quad \bar{G}_{j,t} \coloneqq \theta^D \breve{G}_t^D + \frac{(\omega + J - 1)}{\omega^2} \frac{\theta}{J-1} \sum_{j' \in \mathcal{J}: j' \neq j} \breve{G}_{j',t}. \tag{A.38}$$

Then the supply curve can be written as

$$P_t = \frac{\omega + J - 1}{J \breve{a} \omega (\breve{\xi}^P + r)} (\dot{x}_{j,t} + \breve{b} x_{j,t}) + \widetilde{P}_{j,t}, \tag{A.39}$$

in which the intercept $\widetilde{P}_{j,t}$ would be the focus of the next step.

Step 2. In this step, we focus on understanding the dynamics of $\widetilde{P}_{j,t}$. We start with the dynamics of $\bar{G}_{j,t}$, introduced in (A.38). We can write

$$\bar{G}_{j,t} = (\breve{\xi}^P - \xi^G) \int_{-\infty}^t e^{-\breve{\xi}^P (t-s)} \left(\theta^D dS_t^D + \frac{\omega + J - 1}{\omega^2} \theta d\bar{S}_{j,s} \right), \quad \text{with} \quad \bar{S}_{j,t} \coloneqq \frac{1}{J-1} \sum_{j' \in \mathcal{J}: j' \neq j} S_{j',t}. \tag{A.40}$$

Given the definitions of $(\theta^D, \theta, S_t^D)$ (see (A.33) and (A.34)) and the dynamics of $(D_t, S_{j,t})$, we have

$$\theta^D dS_t^D + \frac{\omega + J - 1}{\omega^2} \theta d\bar{S}_{j,t} = (\widetilde{\sigma}^S / \sigma^D)^2 dS_t^D + (\omega + J - 1)(\widetilde{\sigma}^S / \sigma^S)^2 d\bar{S}_{j,t}.$$

Hence, it holds that

$$\theta^D dS_t^D + \frac{\omega + J - 1}{\omega^2}\theta d\bar{S}_{j,s} = \bar{\phi} G_t dt + \bar{\sigma}^{G*} d\bar{Z}_{j,t}^G, \tag{A.41}$$

where $\bar{Z}_{j,t}^G$ is a standard Brownian motion and $(\bar{\phi}, \bar{\sigma}^{G*}, \bar{Z}_{j,t}^G)$ are defined by

$$\bar{\phi} = (\widetilde{\sigma}^S/\sigma^D)^2 + (\omega + J - 1)(\widetilde{\sigma}^S/\sigma^S)^2 \quad \text{and} \quad \bar{\sigma}^{G*} d\bar{Z}_{j,t}^G \coloneqq \frac{(\widetilde{\sigma}^S)^2}{\sigma^D} dZ_t^D + \frac{\omega + J - 1}{\sqrt{J-1}} \frac{(\widetilde{\sigma}^S)^2}{\sigma^S} d\bar{Z}_{j,t}^S, \tag{A.42}$$

and $\bar{Z}_{j,t}^S$ is the standard Brownian motion driving $\bar{S}_{j,t}$. Combining (A.40) and (A.41), we obtain that $\bar{G}_{j,t}$ evolves according to

$$d\bar{G}_{j,t} = -\breve{\xi}^P \bar{G}_{j,t} + (\breve{\xi}^P - \xi^G)\bar{\phi} G_t dt + (\breve{\xi}^P - \xi^G)\bar{\sigma}^{G*} d\bar{Z}_{j,t}^G.$$

Using that $dG_t = -\xi^G G_t + \sigma^G dZ_t^G$ by definition, we can further write

$$d(\bar{G}_{j,t} - \bar{\phi} G_t) = -\breve{\xi}^P (\bar{G}_{j,t} - \bar{\phi} G_t) dt + (\breve{\xi}^P - \xi^G)\bar{\sigma}^{G*} d\bar{Z}_{j,t}^G - \bar{\phi}\sigma^G dZ_t^G. \tag{A.43}$$

Next, based on the dynamics of $\bar{G}_{j,t}$, we derive the evolution of $\widetilde{P}_{j,t}$. We introduce short-hand notation

$$F_t = \frac{D_t}{\breve{\xi}^D} + \frac{\phi\bar{\phi}}{\breve{\xi}^D \breve{\xi}^G} G_t.$$

Since by definition it holds that

$$\widetilde{P}_{j,t} - F_t = \frac{\phi}{\breve{\xi}^D \breve{\xi}^G}(\bar{G}_{j,t} - \bar{\phi} G_t), \tag{A.44}$$

we use (A.43) to obtain

$$d(\widetilde{P}_{j,t} - F_t) = -\breve{\xi}^P (\widetilde{P}_{j,t} - F_t) dt + \phi \frac{(\breve{\xi}^P - \xi^G)\bar{\sigma}^{G*} d\bar{Z}_{j,t}^G - \bar{\phi}\sigma^G dZ_t^G}{\breve{\xi}^D \breve{\xi}^G}. \tag{A.45}$$

Moreover, from the dynamics of (D_t, G_t), it follows that F_t satifies

$$dF_t = rF_t dt - D_t dt + \frac{1 - \phi\bar{\phi}}{\breve{\xi}^D} G_t dt + \sigma^F dZ_t^F, \quad \text{with} \quad \sigma^F dZ_t^F \coloneqq \frac{\sigma^D dZ_t^D}{\breve{\xi}^D} + \frac{\phi\bar{\phi}\sigma^G dZ_t^G}{\breve{\xi}^D \breve{\xi}^G}. \tag{A.46}$$

Combining (A.45) and (A.46), we obtain

$$d\widetilde{P}_{j,t} = r\widetilde{P}_{j,t} dt - D_t dt + \frac{1 - \phi\bar{\phi}}{\breve{\xi}^D} G_t dt - (\breve{\xi}^P + r)(\widetilde{P}_{j,t} - F_t) dt + \sigma^{P*} dZ_{j,t}^{P*}, \tag{A.47}$$

where Z_t^{P*} is a standard Brownian motion and (σ^{P*}, Z_t^{P*}) are defined by

$$\sigma^{P*} dZ_{j,t}^{P*} := \frac{\sigma^D dZ_t^D}{\breve{\xi}^D} + \phi \frac{(\breve{\xi}^P - \xi^G)\bar{\sigma}^{G*} d\bar{Z}_{j,t}^G}{\breve{\xi}^D \breve{\xi}^G}. \tag{A.48}$$

Substituting (A.44) into (A.47), we obtain

$$d\widetilde{P}_{j,t} = -D_t dt + \frac{1-\phi\bar{\phi}}{\breve{\xi}^D} G_t dt - \frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D \breve{\xi}^G}(\bar{G}_{j,t} - \bar{\phi} G_t)dt + \sigma^{P*} dZ_t^{P*} j, t. \tag{A.49}$$

Step 3. In this step, we study the dynamics of $\mu_{j,t} := \mathrm{E}_{j,t}(d\widetilde{P}_{j,t})/dt + D_t - r\widetilde{P}_{j,t}$ and establish the equilibrium. Because the investor observes $\widetilde{P}_{j,t}$ and D_t, she effectively observes $\bar{G}_{j,t}$ according to (A.38). As a result, it follows from (A.49) that

$$\mu_{j,t} := \mathrm{E}_{j,t}(d\widetilde{P}_{j,t})/dt + D_t - r\widetilde{P}_{j,t} = \mu_{j,t}^{(1)} + \mu_{j,t}^{(2)}, \tag{A.50}$$

where

$$\mu_{j,t}^{(1)} := \frac{1-\phi\bar{\phi}}{\breve{\xi}^D} \mathrm{E}_{j,t}(G_t), \quad \text{and} \quad \mu_{j,t}^{(2)} = -\frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D \breve{\xi}^G}\left(\bar{G}_{j,t} - \bar{\phi}\mathrm{E}_{j,t}(G_t)\right). \tag{A.51}$$

We can further write

$$\begin{aligned} \mathrm{E}_{j,t}\left(d\mathrm{E}_{j,t}(G_t)\right) &= \mathrm{E}_{j,t}(dG_t) = -\xi^G \mathrm{E}_{j,t}(G_t)dt, \\ \mathrm{E}_{j,t}\left(d\bar{G}_{j,t} - \bar{\phi} d\mathrm{E}_{j,t}(G_t)\right) &= \mathrm{E}_{j,t}(d(\bar{G}_{j,t} - \bar{\phi}G_t)) = -\breve{\xi}^P \mathrm{E}_{j,t}(\bar{G}_{j,t} - \bar{\phi}G_t)dt, \end{aligned}$$

where the first equalities of both lines come from $\mathrm{E}_{j,t}\left(d\mathrm{E}_{j,t}(G_t)\right) = \mathrm{E}_{j,t}(dG_t)$ due to the law of iterated expectations. The second equality of the first line holds by the dynamics of G_t by assumption and the second equality of the second line is a result of (A.43). Substituting these results into (A.51), we obtain

$$\mathrm{E}_{j,t}(d\mu_{j,t}^{(1)}) = -\xi^G \mu_{j,t}^{(1)} dt, \quad \text{and} \quad \mathrm{E}_{j,t}(d\mu_{j,t}^{(2)}) = -\breve{\xi}^P \mu_{j,t}^{(2)} dt. \tag{A.52}$$

Note that the dynamics of $\widetilde{P}_{j,t}$ given by (A.49), (A.50), and (A.52) exactly matches the premises of Proposition A1 with $\xi^{\mu,(1)} = \xi^G$ and $\xi^{\mu,(2)} = \breve{\xi}^P$. Moreover, from the supply curve (A.39), it follows that

$$P_t = \widetilde{P}_{j,t} + \breve{\zeta}\dot{x}_{j,t} + \bar{\zeta} x_{j,t},$$

where

$$\breve{\zeta} = \frac{\omega + J - 1}{J\breve{a}\omega(\breve{\xi}^P + r)}, \quad \bar{\zeta} = \breve{b}\breve{\zeta}. \tag{A.53}$$

Then investor j's optimal trading strategy would be obtained if we can solve (A.24), (A.25), and (A.26), in which parameters would be functions of $(\zeta, \bar{\zeta}, \phi)$, which in turn are functions of $(\breve{a}, \breve{b}, \phi)$.

Further, the equilibrium can be established if we could find values of $(\breve{a}, \breve{b}, \phi)$ under which the optimal strategy coincides with (A.30), which also depends on $(\breve{a}, \breve{b}, \phi)$. In general, proving the existence of a solution to (A.24), (A.25), and (A.26) and finding a closed-form expression of it are both challenging. But with Assumption 2, we are able to show there exists a solution, and characterize its leading-order term in closed-form. Suppose $(A^{(\mu,\mu)}_{k,k'}, A^{(x,\mu)}_{k}, A^{(x,x)})$ are constants (the restriction would be removed later on) and $|A^{(x,x)}|$ is sufficiently small, then (A.24), (A.25), and (A.26) has the following solution:

$$\begin{aligned}
\frac{1}{2\breve{\zeta} r\gamma}(2a_4 + r\gamma\bar{\zeta}) &= \sqrt{\frac{1}{2\breve{\zeta}}(r\gamma\Sigma^P + r\bar{\zeta} - 2A^{(x,x)}r^{-1}\gamma^{-1}) + \frac{r^2}{4}} - \frac{r}{2}, && \text{(A.54)}\\
(a_3)_k &= -\frac{r\gamma + A^{(x,\mu)}_k}{\xi^{\mu,(k)} + \sqrt{\frac{1}{2\breve{\zeta}}(r\gamma\Sigma^P + r\bar{\zeta} - 2A^{(x,x)}r^{-1}\gamma^{-1}) + \frac{r^2}{4}} + \frac{r}{2}}, && \text{(A.55)}\\
(a_2)_{k,k'} &= \frac{1}{\xi^{\mu,(k)}(1+\delta_{k,k'}) - r}\left(\frac{1}{4\breve{\zeta} r\gamma}(a_3)_k(a_3)_{k'} + A^{(\mu,\mu)}_{k,k'}\right). && \text{(A.56)}
\end{aligned}$$

The last equation is well-defined because $\xi^{\mu,(k)} \geq \xi^G > r$, where $\xi^G > r$ holds by assumption. The optimal strategy, according to (A.14), is therefore

$$\dot{x}_{j,t} = \sum_{k\in\{1,2\}} \widetilde{a}^{(k)}\mu^{(k)}_{j,t} - \widetilde{b}x_{j,t}, \tag{A.57}$$

where

$$\begin{aligned}
\widetilde{a}^{(k)} &= \frac{1}{2\breve{\zeta}}\frac{1 + A^{(x,\mu)}_k (r\gamma)^{-1}}{\xi^{\mu,(k)} + \sqrt{\frac{1}{2\breve{\zeta}}(r\gamma\Sigma^P + r\bar{\zeta} - 2A^{(x,x)}r^{-1}\gamma^{-1}) + \frac{r^2}{4}} + \frac{r}{2}}, && \text{(A.58)}\\
\widetilde{b} &= \sqrt{\frac{1}{2\breve{\zeta}}(r\gamma\Sigma^P + r\bar{\zeta} - 2A^{(x,x)}r^{-1}\gamma^{-1}) + \frac{r^2}{4}} - \frac{r}{2}. && \text{(A.59)}
\end{aligned}$$

On the other hand, using (A.51), we obtain

$$\begin{aligned}
\widetilde{a}^{(1)}\mu^{(1)}_{j,t} + \widetilde{a}^{(2)}\mu^{(2)}_{j,t} &= \frac{1-\phi\bar{\phi}}{\breve{\xi}^D}\widetilde{a}^{(1)}\mathrm{E}_{j,t}(G_t) - \frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D\breve{\xi}^G}\widetilde{a}^{(2)}\left(\bar{G}_{j,t} - \bar{\phi}\mathrm{E}_{j,t}(G_t)\right)\\
&= \frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D\breve{\xi}^G}\left(-\widetilde{a}^{(2)}\bar{G}_{j,t} + \left(\frac{1-\phi\bar{\phi}}{\phi}\frac{\breve{\xi}^G}{\breve{\xi}^P + r}\widetilde{a}^{(1)} + \bar{\phi}\widetilde{a}^{(2)}\right)\mathrm{E}_{j,t}(G_t)\right). && \text{(A.60)}
\end{aligned}$$

Comparing (A.60) with (A.30) and (A.31), we seek to find $(\breve{a}, \breve{b}, \phi)$ such that (A.53), (A.58), and (A.59) hold, and

$$\widetilde{a}^{(2)} = \breve{a}, \quad \frac{1-\phi\bar{\phi}}{\phi}\frac{\breve{\xi}^G}{\breve{\xi}^P + r}\widetilde{a}^{(1)} + \bar{\phi}\widetilde{a}^{(2)} = \breve{a}, \quad \widetilde{b} = \breve{b}. \tag{A.61}$$

Now we incorporate that $(A^{(\mu,\mu)}_{k,k'}, A^{(x,\mu)}_{k}, A^{(x,x)})$ actually depend on (a_2, a_3, a_4). We consider the following algorithm: (1) set $(A^{(\mu,\mu)}_{k,k'}, A^{(x,\mu)}_{k}, A^{(x,x)})$ as zero, and solve $(\widetilde{a}^{(1)}, \widetilde{a}^{(2)}, \widetilde{b}, \phi, \breve{\zeta}, \bar{\zeta})$ from (A.53), (A.58), (A.59), and (A.61); (2) use the obtained $(\phi, \breve{\zeta}, \bar{\zeta})$ to calculate $(A^{(\mu,\mu)}_{k,k'}, A^{(x,\mu)}_{k}, A^{(x,x)})$, based on their definitions and equations (A.54) – (A.56); (3) update $(A^{(\mu,\mu)}_{k,k'}, A^{(x,\mu)}_{k}, A^{(x,x)})$ and resolve (A.53), (A.58), (A.59), and (A.61); (4) iterate the previous two steps. Because $|\Sigma^{\mu}_{k,k'}| \lesssim J^{-1}$ and $|\Sigma^{\mu,P}_{k}| \lesssim J^{-1}$, which we establish shortly using Assumption 2, we note that, as long as the solution satisfies that $(a_2, a_3, a_4, \phi, \breve{\zeta}, \bar{\zeta})$ are all bounded away from zero and infinity, in the lth iteration, the changes of $(a_2, a_3, a_4, \phi, \breve{\zeta}, \bar{\zeta})$ from the previous iteration are bounded by J^{-l}. Hence the sequences of $(a_2, a_3, a_4, \phi, \breve{\zeta}, \bar{\zeta})$ obtained in each iteration are Cauchy sequences. Hence the limits of the sequences exist and solve (A.53) – (A.61). Moreover, step 1 of the algorithm already gives us $(\widetilde{a}^{(1)}, \widetilde{a}^{(2)}, \widetilde{b}, \phi, \breve{\zeta}, \bar{\zeta})$ up to errors of order $O(J^{-1})$, which we calculate now.

Using $\Sigma^P = (\sigma^{P*})^2$, $\widetilde{b} = \breve{b}$, and $\bar{\zeta} = \breve{b}\breve{\zeta}$, and setting $A^{(x,x)}$ as zero, we obtain from (A.59) that

$$\breve{b} = \sqrt{\frac{1}{2\breve{\zeta}} r\gamma(\sigma^{P*})^2 + \frac{r\breve{b}}{2} + \frac{r^2}{4}} - \frac{r}{2}, \quad \Longrightarrow \quad \breve{b} = \sqrt{\frac{1}{2\breve{\zeta}} r\gamma(\sigma^{P*})^2 + \frac{1}{16}r^2} - \frac{1}{4}r. \tag{A.62}$$

Next, setting $A^{(x,x)} = A^{(x,\mu)}_{k} = 0$, we obtain from (A.53), (A.58), and $\widetilde{a}^{(2)} = \breve{a}$ that

$$\frac{J\omega}{\omega + J - 1}(\breve{\xi}^P + r) - 2\breve{\xi}^P = \sqrt{2\breve{\zeta}^{-1} r\gamma\Sigma^P + 2r\breve{b} + r^2} + r = 2\breve{b} + 2r. \tag{A.63}$$

Here the last equality comes from the first equation in (A.62). In light of the second equation in (A.62), we must have $\breve{b} > 0$ for the solution of $\breve{\zeta}$ to exist. We obtain the existence condition, $\breve{b}$, and $\breve{\zeta}$ as

$$\frac{J\omega}{\omega + J - 1} > 2, \quad \breve{b} = \frac{1}{2}\frac{J\omega}{\omega + J - 1}(\breve{\xi}^P + r) - \breve{\xi}^P - r, \quad \breve{\zeta} = \frac{1}{2}\frac{\gamma(\sigma^{P*})^2}{\left(\breve{b} + \frac{1}{4}r\right)^2 - \frac{1}{16}\rho^2}. \tag{A.64}$$

Note that the price volatility σ^{P*} can be expressed directly in terms of ϕ and exogenous parameters via (A.48) and (A.42). Then, from (A.58) it follows

$$\widetilde{a}^{(k)} = \frac{1}{2\breve{\zeta}} \frac{1}{\xi^{\mu,(k)} + \breve{b} + r}. \tag{A.65}$$

Substituting (A.65) into second equation in (A.61), we obtain

$$\phi^{-1} = \bar{\phi} + (1 - \bar{\phi}) \frac{\breve{\xi}^P + r}{\breve{\xi}^G} \cdot \frac{\xi^G + \breve{b} + r}{\breve{\xi}^P + \breve{b} + r}. \tag{A.66}$$

Substituting (A.66) into (A.60), we have

$$\widetilde{a}^{(1)}\mu^{(1)}_{j,t} + \widetilde{a}^{(2)}\mu^{(2)}_{j,t} = \frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D \breve{\xi}^G} \breve{a} \left(\mathrm{E}_{j,t}(G_t) - \bar{G}_{j,t} \right). \tag{A.67}$$

To finally verify $\widetilde{a}^{(1)}\mu_{j,t}^{(1)} + \widetilde{a}^{(2)}\mu_{j,t}^{(2)} = \breve{a}\widetilde{\mu}_{j,t}$ and establish the equilibrium, we derive the expression of $\mathrm{E}_{j,t}(G_t)$. The signals at the investor's disposal for learning about G_t are $\{S_{j,s}, \bar{S}_{j,s}, S_s^D\}_{s\leq t}$ (noting (A.38) and (A.40)), where

$$dS_{j,s} = \omega G_s dt + \sigma^S d\widehat{Z}_{j,s}^S, \quad d\bar{S}_{j,s} = G_s dt + \frac{\sigma^S}{\sqrt{J-1}} d\bar{Z}_{j,s}^S, \quad dS_s^D = G_s dt + \sigma^D dZ_s^D.$$

Because $\widehat{Z}_{j,s}^S$, $\bar{Z}_{j,s}^S$, and Z_s^D are mutually independent standard Brownian motions in investor j's subjective belief, the standard Kalman-Bucy filtering leads to

$$\mathrm{E}_{j,t}(G_t) = (\breve{\xi}^P - \xi^G) \int_{-\infty}^{t} e^{-\breve{\xi}^P (t-s)} d\widetilde{S}_{j,s}, \tag{A.68}$$

where

$$\widetilde{S}_{j,t} = (\widetilde{\sigma}^S)^2 (\omega(\sigma^S)^{-2} S_{j,t} + (J-1)(\sigma^S)^{-2} \bar{S}_{j,t} + (\sigma^D)^{-2} S_t^D). \tag{A.69}$$

Further, applying (A.32), (A.33), and (A.34), and using (A.40), we rewrite (A.68) as

$$\begin{aligned} \mathrm{E}_{j,t}(G_t) - \bar{G}_{j,t} &= \omega^{-1}\theta\breve{G}_{j,t} + \frac{J-1}{\omega+J-1}(\bar{G}_{j,t} - \theta^D \breve{G}_t^D) + \theta^D \breve{G}_t^D - \bar{G}_{j,t} \\ &= \omega^{-1}\theta\breve{G}_{j,t} - \frac{\omega}{\omega+J-1}(\bar{G}_{j,t} - \theta^D \breve{G}_t^D). \end{aligned} \tag{A.70}$$

Substituting (A.70) into (A.67), and in light of (A.38), we further have

$$\widetilde{a}^{(1)}\mu_{j,t}^{(1)} + \widetilde{a}^{(2)}\mu_{j,t}^{(2)} = \breve{a}\frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D \breve{\xi}^G} \left(\omega^{-1}\theta\breve{G}_{j,t} - \frac{\omega}{\omega+J-1} \left(\frac{\breve{\xi}^D \breve{\xi}^G}{\phi} \left(\widetilde{P}_{j,t} - \frac{D_t}{\breve{\xi}^D} \right) - \theta^D \breve{G}_t^D \right) \right). \tag{A.71}$$

On the other hand, it follows from (A.39) and (A.57) that

$$P_t = \frac{\omega+J-1}{J\breve{a}\omega(\breve{\xi}^P + r)} (\widetilde{a}^{(1)}\mu_{j,t}^{(1)} + \widetilde{a}^{(2)}\mu_{j,t}^{(2)}) + \widetilde{P}_{j,t}. \tag{A.72}$$

Combining (A.71) with (A.72), and noting $\breve{a} = \widetilde{a}^{(2)}$ by (A.61), we have

$$(\widetilde{a}^{(1)}\mu_{j,t}^{(1)} + \widetilde{a}^{(2)}\mu_{j,t}^{(2)}) \left(1 - \frac{1}{J}\right) = \breve{a}\frac{\phi(\breve{\xi}^P + r)}{\breve{\xi}^D \breve{\xi}^G} \left(\omega^{-1}\theta\breve{G}_{j,t} - \frac{\omega}{\omega+J-1} \left(\frac{\breve{\xi}^D \breve{\xi}^G}{\phi} \left(P_t - \frac{D_t}{\breve{\xi}^D} \right) - \theta^D \breve{G}_t^D \right) \right).$$

Comparing with (A.31), we finally obtain

$$\widetilde{a}^{(1)}\mu_{j,t}^{(1)} + \widetilde{a}^{(2)}\mu_{j,t}^{(2)} = \breve{a}\widetilde{\mu}_{j,t}. \tag{A.73}$$

In light of (A.66) and (A.73), we establsh that, when all investor j with $j' \neq j$ take the strategy

(A.30), it is optimal for investor j to take this strategy as well, where

$$\breve{a} = \frac{1}{2\breve{\zeta}} \frac{1}{\breve{\xi}^P + \breve{b} + r} + O(J^{-1}), \quad \breve{b} = \frac{1}{2} \frac{J\omega}{\omega + J - 1} (\breve{\xi}^P + r) - \breve{\xi}^P - r + O(J^{-1}), \tag{A.74}$$

$$\breve{\zeta} = \frac{1}{2} \frac{\gamma(\sigma^{P*})^2}{\left(\breve{b} + \frac{1}{4}r\right)^2 - \frac{1}{16}r^2} + O(J^{-1}), \quad \phi = \bar{\phi} + (1 - \bar{\phi}) \frac{\breve{\xi}^P + r}{\breve{\xi}^G} \cdot \frac{\xi^G + \breve{b} + r}{\breve{\xi}^P + \breve{b} + r} + O(J^{-1}). \tag{A.75}$$

The $O(J^{-1})$ terms originate from the argument above (A.62). On the other hand, it directly follows from (A.74), (A.75), and Assumption 2 that

$$\begin{aligned}
(\widetilde{\sigma}^S)^{-2} &= (\sigma^D)^{-2} + \iota^{-2} + O(J^{-1}), \\
\breve{\xi}^P &= \sqrt{(\xi^G)^2 + (\sigma^G)^2((\sigma^D)^{-2} + \iota^{-2})} + O(J^{-1}) = \xi^P + O(J^{-1}), && \text{(A.76)} \\
\breve{b} &= \frac{1}{2}(\xi^P + r)(\omega - 2) + O(J^{-1}) = b + O(J^{-1}), && \text{(A.77)} \\
\sigma^{P*} &= \sqrt{(\sigma^D)^2(\xi^D)^{-2} + (\sigma^G)^2(\xi^D\xi^G)^{-2}} + O(J^{-1}) = \sigma^P + O(J^{-1}), \\
\breve{\zeta} &= \frac{1}{2} \frac{\gamma(\sigma^P)^2}{\left(b + \frac{1}{4}r\right)^2 - \frac{1}{16}r^2} + O(J^{-1}) = \zeta + O(J^{-1}), \\
\breve{a} &= \frac{1}{2\breve{\zeta}} \frac{1}{\breve{\xi}^P + \breve{b} + r} + O(J^{-1}) = \frac{1}{\zeta\omega(\xi^P + r)} + O(J^{-1}) = a + O(J^{-1}), && \text{(A.78)} \\
\phi &= \left(1 + (\omega^2 - \omega)(\widetilde{\sigma}^S/\sigma^S)^2 \left(\frac{\breve{\xi}^P + r}{\breve{\xi}^G} \cdot \frac{\xi^G + \breve{b} + r}{\breve{\xi}^P + \breve{b} + r} - 1\right)\right)^{-1} + O(J^{-1}) = 1 + O(J^{-1}), && \text{(A.79)} \\
\theta &= \frac{\omega^2(\sigma^S)^{-2}}{(\sigma^D)^{-2} + \iota^{-2}} + O(J^{-2}), && \text{(A.80)} \\
\theta^D &= \frac{(\sigma^D)^{-2}}{(\sigma^D)^{-2} + \iota^{-2}} + O(J^{-1}), && \text{(A.81)}
\end{aligned}$$

Substituting (A.76), (A.79), (A.80), and (A.81) into (A.70) and (A.51), we obtain $|\Sigma^{\mu}_{k,k'}| \lesssim J^{-1}$ and $|\Sigma^{\mu,P}_{k}| \lesssim J^{-1}$. Hence the equilibrium stated in the proposition indeed exists as characterized above. The equilibrium price follows directly from (A.30) and (A.31) as

$$P_t = \frac{D_t}{\breve{\xi}^D} + \frac{\phi}{\breve{\xi}^D \breve{\xi}^G} \left(\theta^D \breve{G}^D_t + \frac{(\omega + J - 1)}{\omega^2} \frac{\theta}{J} \sum_{j \in \mathcal{J}} \breve{G}_{j,t}\right). \tag{A.82}$$

Step 4. We start with property (i). (3.9) is a direct result of Kalman-Bucy filtering. Substituting (A.76), (A.79), (A.80), and (A.81) into (A.82), we obtain

$$P_t = \frac{D_t}{\breve{\xi}^D} + \frac{1}{\breve{\xi}^D \breve{\xi}^G} \frac{1}{(\sigma^D)^{-2} + \iota^{-2}} \left((\sigma^D)^{-2} G^D_t + \iota^{-2} \frac{1}{J} \sum_{j \in \mathcal{J}} G_{j,t}\right) + O_{\mathrm{P}}(J^{-1}). \tag{A.83}$$

Combining (A.83) and (3.9), we obtain (3.8). Given (3.8), property (ii) directly follows if we combining (A.30) and (A.31) with (A.76), (A.77), (A.78), (A.79), and (A.80). The proof ends. ∎

A.3 Proof of Theorem 1

Proof. The proof again uses "conjecture and verify" approach with a structure parallel to the proof of Theorem 1.

Step 1. We conjecture

$$\dot{x}_{j,t} = a\widetilde{\mu}_{j,t} - bx_{j,t}, \tag{A.84}$$

where the process $\widetilde{\mu}_{j,t}$ is given by

$$\widetilde{\mu}_{j,t} = \frac{\xi^P + r}{\breve{\xi}^D \breve{\xi}^G} \omega_j^{-1} \theta_j \left(G_{j,t} - \bar{\lambda}^{-1} \left(\breve{\xi}^D \breve{\xi}^G \left(P_t - \frac{D_t}{\breve{\xi}^D} \right) - \theta^D G_t^D \right) \right), \tag{A.85}$$

Here $\breve{\xi}^D = \xi^D + r$, $\breve{\xi}^G = \xi^G + r$ and parameters (ξ^P, a, b) and processes $(G_t^D, G_{j,t})$ are introduced in the statement of the theorem, whereas parameters $(\theta_j, \theta^D, \bar{\lambda})$ are defined as follows:

$$(\widetilde{\sigma}^S)^2 = \frac{1}{(\sigma^D)^{-2} + \widetilde{\iota}^{-1}}, \quad \theta_j = (\omega_j \widetilde{\sigma}^S / \sigma_j^S)^2, \quad \theta^D = (\widetilde{\sigma}^S / \sigma^D)^2, \quad \bar{\lambda} = (\widetilde{\sigma}^S)^2 \widetilde{\iota}^{-1}. \tag{A.86}$$

Suppose, for all $j' \in \mathcal{J}$ with $j' \neq j$, investor j' submit the above demand schedule. Because the market clearing condition leads to

$$\dot{x}_{j,t} = -\sum_{j' \in \mathcal{J}: j' \neq j} \dot{x}_{j',t}, \quad x_{j,t} = -\sum_{j' \in \mathcal{J}: j' \neq j} x_{j',t},$$

we have

$$a \sum_{j' \in \mathcal{J}: j' \neq j} \widetilde{\mu}_{j',t} = \sum_{j' \in \mathcal{J}: j' \neq j} (\dot{x}_{j',t} + bx_{j',t}) = -\dot{x}_{j,t} - bx_{j,t}. \tag{A.87}$$

Given (A.87), it follows from the definition of $\widetilde{\mu}_{j,t}$ that

$$a \sum_{j' \in \mathcal{J}: j' \neq j} \frac{\xi^P + r}{\breve{\xi}^D \breve{\xi}^G} \omega_j^{-1} \theta_j \left(G_{j',t} - \bar{\lambda}^{-1} \left(\breve{\xi}^D \breve{\xi}^G \left(P_t - \frac{D_t}{\breve{\xi}^D} \right) - \theta^D G_t^D \right) \right) = -\dot{x}_{j,t} - bx_{j,t}. \tag{A.88}$$

We introduce $\lambda_j = \left(\widetilde{\omega} \sum_{j' \in \mathcal{J}: j' \neq j} \pi_{j'} \right)^{-1}$ and note

$$\sum_{j' \in \mathcal{J}: j' \neq j} \omega_j^{-1} \theta_j \bar{\lambda}^{-1} = \sum_{j' \in \mathcal{J}: j' \neq j} \omega_j (\sigma_j^S)^{-2} \widetilde{\iota} = \sum_{j' \in \mathcal{J}: j' \neq j} \pi_j \widetilde{\omega} = \lambda_j^{-1}. \tag{A.89}$$

Then (A.88) directly gives P_t as a function of $(\{G_{j',t}\}_{j' \in \mathcal{J}: j' \neq j}, \dot{x}_{j,t}, x_{j,t})$, i.e., the supply curve faced by investor j:

$$P_t = \frac{\lambda_j}{a(\xi^P + r)}(\dot{x}_{j,t} + bx_{j,t}) + \frac{D_t}{\breve{\xi}^D} + \frac{1}{\breve{\xi}^D \breve{\xi}^G}\Big(\theta^D G_t^D + \lambda_j \sum_{j' \in \mathcal{J}: j' \neq j} \omega_j \theta_j G_{j',t}\Big).$$

To facilitate exposition, we introduce

$$\widetilde{P}_{j,t} := \frac{D_t}{\breve{\xi}^D} + \frac{\widetilde{G}_{j,t}}{\breve{\xi}^D \breve{\xi}^G}, \quad \text{with} \quad \widetilde{G}_{j,t} := \theta^D G_t^D + \lambda_j \sum_{j' \in \mathcal{J}: j' \neq j} \omega_j \theta_j G_{j',t}. \tag{A.90}$$

Then the supply curve can be written as

$$P_t = \frac{\lambda_j}{a(\xi^P + r)}(\dot{x}_{j,t} + bx_{j,t}) + \widetilde{P}_{j,t}. \tag{A.91}$$

Step 2. This step focuses on the dynamics of $\widetilde{P}_{j,t}$. We start with the dynamics of $\widetilde{G}_{j,t}$, introduced in (A.90). Given the definitions of θ^D, G_t^D, θ_j, and $G_{j,t}$, we have

$$\begin{aligned}
\frac{d\widetilde{G}_{j,t} + \xi^P \widetilde{G}_{j,t} dt}{\xi^P - \xi^G} &= \theta^D (dD_t - \alpha^G D_t dt) + \lambda_j \sum_{j' \in \mathcal{J}: j' \neq j} \omega_j \theta_j dS_{j,t} \\
&= (\widetilde{\sigma}^S / \sigma^D)^2 (G_t dt + \sigma^D dZ_t^D) + \lambda_j \sum_{j' \in \mathcal{J}: j' \neq j} \omega_{j'} (\widetilde{\sigma}^S / \sigma_{j'}^S)^2 dS_{j',t}.
\end{aligned} \tag{A.92}$$

We note that

$$\begin{aligned}
&(\widetilde{\sigma}^S / \sigma^D)^2 + \lambda_j \sum_{j' \in \mathcal{J}: j' \neq j} \omega_{j'} (\widetilde{\sigma}^S / \sigma_{j'}^S)^2 \\
= \;& (\widetilde{\sigma}^S)^2 (\sigma^D)^{-2} + (\widetilde{\sigma}^S)^2 \widetilde{\omega}^{-1} \sum_{j' \in \mathcal{J}} \omega_{j'} (\sigma_{j'}^S)^{-2} = (\widetilde{\sigma}^S)^2 (\sigma^D)^{-2} + (\widetilde{\sigma}^S)^2 \widetilde{\iota}^{-1} = 1.
\end{aligned} \tag{A.93}$$

Combining (A.92) and (A.93), we obtain

$$\frac{d\widetilde{G}_{j,t} + \xi^P \widetilde{G}_{j,t} dt}{\xi^P - \xi^G} = G_t dt + \widetilde{\sigma}_j^{G*} d\widetilde{Z}_{j,t}^G, \tag{A.94}$$

where $\widetilde{Z}_{j,t}^G$ is a standard Brownian motion and $\widetilde{\sigma}_j^{G*}$ and $\widetilde{Z}_{j,t}^G$ are defined by

$$\widetilde{\sigma}_j^{G*} d\widetilde{Z}_{j,t}^G := (\widetilde{\sigma}^S)^2 (\sigma^D)^{-1} dZ_t^D + \lambda_j \sum_{j' \in \mathcal{J}: j' \neq j} \omega_{j'} (\widetilde{\sigma}^S)^2 (\sigma_{j'}^S)^{-1} dZ_{j,t}^S. \tag{A.95}$$

Rewriting (A.94), we obtain that $\widetilde{G}_{j,t}$ evolves according to

$$d\widetilde{G}_{j,t} = -\xi^P \widetilde{G}_{j,t} + (\xi^P - \xi^G) G_t dt + (\xi^P - \xi^G) \widetilde{\sigma}_j^{G*} d\widetilde{Z}_{j,t}^G.$$

Using that $dG_t = -\xi^G G_t + \sigma^G dZ_t^G$ by definition, we can further write

$$d(\widetilde{G}_{j,t} - G_t) = -\xi^P(\widetilde{G}_{j,t} - G_t)dt + (\xi^P - \xi^G)\widetilde{\sigma}_j^{G*} d\widetilde{Z}_{j,t}^G - \sigma^G dZ_t^G. \tag{A.96}$$

Next, based on the dynamics of $\widetilde{G}_{j,t}$, we derive the evolution of $\widetilde{P}_{j,t}$. We introduce short-hand notation

$$F_t = \frac{D_t}{\breve{\xi}^D} + \frac{G_t}{\breve{\xi}^D \breve{\xi}^G}.$$

Since by definition it holds that

$$\widetilde{P}_{j,t} - F_t = \frac{1}{\breve{\xi}^D \breve{\xi}^G}(\widetilde{G}_{j,t} - G_t), \tag{A.97}$$

we use (A.96) to obtain

$$d(\widetilde{P}_{j,t} - F_t) = -\xi^P(\widetilde{P}_{j,t} - F_t)dt + \frac{(\xi^P - \xi^G)\widetilde{\sigma}_j^{G*} d\widetilde{Z}_{j,t}^G - \sigma^G dZ_t^G}{\breve{\xi}^D \breve{\xi}^G}. \tag{A.98}$$

Moreover, from the dynamics of (D_t, G_t), it follows that F_t satifies

$$dF_t = rF_t dt - D_t dt + \sigma^F dZ_t^F, \quad \text{with} \quad \sigma^F dZ_t^F \coloneqq \frac{\sigma^D dZ_t^D}{\breve{\xi}^D} + \frac{\sigma^G dZ_t^G}{\breve{\xi}^D \breve{\xi}^G}. \tag{A.99}$$

Combining (A.98) and (A.99), we obtain

$$d\widetilde{P}_{j,t} = r\widetilde{P}_{j,t} dt - D_t dt - (\xi^P + r)(\widetilde{P}_{j,t} - F_t)dt + \sigma_j^{P*} dZ_{j,t}^{P*}, \tag{A.100}$$

where $Z_{j,t}^{P*}$ is a standard Brownian motion and $(\sigma_j^{P*}, Z_{j,t}^{P*})$ are defined by

$$\sigma_j^{P*} dZ_{j,t}^{P*} \coloneqq \frac{\sigma^D dZ_t^D}{\breve{\xi}^D} + \frac{(\xi^P - \xi^G)\widetilde{\sigma}_j^{G*} d\widetilde{Z}_{j,t}^G}{\breve{\xi}^D \breve{\xi}^G}. \tag{A.101}$$

Substituting (A.97) into (A.100), we obtain

$$d\widetilde{P}_{j,t} = r\widetilde{P}_{j,t} dt - D_t dt - \frac{\xi^P + r}{\breve{\xi}^D \breve{\xi}^G}(\widetilde{G}_{j,t} - G_t)dt + \sigma_j^{P*} dZ_{j,t}^{P*}. \tag{A.102}$$

Step 3. In this step, we study the dynamics of $\mu_{j,t} \coloneqq \mathrm{E}_{j,t}(d\widetilde{P}_{j,t})/dt + D_t - r\widetilde{P}_{j,t}$ and establish the equilibrium. Because the investor observes $\widetilde{P}_{j,t}$ and D_t, she effectively observes $\widetilde{G}_{j,t}$ according to (A.90). As a result, it follows from (A.102) that

$$\mu_{j,t} \coloneqq \mathrm{E}_{j,t}(d\widetilde{P}_{j,t})/dt + D_t - r\widetilde{P}_{j,t} = -\frac{\xi^P + r}{\breve{\xi}^D \breve{\xi}^G}\left(\widetilde{G}_{j,t} - \mathrm{E}_{j,t}(G_t)\right), \tag{A.103}$$

We can further write

$$\mathrm{E}_{j,t}\left(d\widetilde{G}_{j,t} - d\mathrm{E}_{j,t}(G_t)\right) = \mathrm{E}_{j,t}(d(\widetilde{G}_{j,t} - G_t)) = -\xi^P \mathrm{E}_{j,t}(\widetilde{G}_{j,t} - G_t)dt,$$

where the first equality comes from $\mathrm{E}_{j,t}\left(d\mathrm{E}_{j,t}(G_t)\right) = \mathrm{E}_{j,t}(dG_t)$ due to the law of interated expectations. The second equality is a result of (A.96). Substituting this result into (A.103), we obtain

$$\mathrm{E}_{j,t}(d\mu_{j,t}) = -\xi^P \mu_{j,t} dt. \tag{A.104}$$

Note that the dynamics of $\widetilde{P}_{j,t}$, given by (A.102), (A.103), and (A.104), exactly matches the premises of Proposition A1 with $\xi^{\mu,(1)} = \xi^P$ and $\mu_{j,t}^{(2)} = 0$. Moreover, from the definition of (a, b) given by (4.7) and the definition of λ_j given after (A.88), it follows that

$$\frac{\lambda_j}{a(\xi^P + r)} = \lambda_j \widetilde{\omega} \zeta = \frac{\zeta}{\sum_{j' \in \mathcal{J} : j' \neq j} \pi_{j'}}.$$

Comparing with the supply curve (A.91), we obtain

$$P_t = \widetilde{P}_{j,t} + \zeta_j(\dot{x}_{j,t} + b x_{j,t}), \quad \text{with} \quad \zeta_j = \frac{\zeta}{\sum_{j' \in \mathcal{J} : j' \neq j} \pi_{j'}}.$$

Then, it directly follows from Proposition A1, (A.54) – (A.59), and the argument after (A.61) that investor j's exact optimal trading strategy $x_{j,t}^*$ is given by

$$\dot{x}_{j,t}^* = a_j^* \mu_{j,t} - b_j^* x_{j,t}^*, \tag{A.105}$$

where

$$a_j^* = \frac{1}{2\zeta_j} \frac{1}{\xi^P + b_j^* + r} + O(J^{-1}), \quad b_j^* = \sqrt{\frac{1}{2\zeta_j} r\gamma (\sigma_j^{P*})^2 + \frac{rb}{2} + \frac{r^2}{4}} - \frac{r}{2} + O(J^{-1}). \tag{A.106}$$

Moreover, it follows from Assumption 3 that

$$\max_{j \in \mathcal{J}} |\zeta_j - \zeta^*| = O(J^{-1}), \quad \max_{j \in \mathcal{J}} |\sigma_j^{P*} - \sigma^P| = O(J^{-1}). \tag{A.107}$$

Here the second result comes from the fact, which can be obtained by direct calculations, that σ^P defined after (4.7) is the volatility of the process

$$\frac{\sigma^D}{\breve{\xi}^D} Z_t^D + \frac{\xi^P - \xi^G}{\breve{\xi}^D \breve{\xi}^G} \frac{(\widetilde{\sigma}^S)^2}{\sigma^D} Z_t^D + \frac{\xi^P - \xi^G}{\breve{\xi}^D \breve{\xi}^G} (\widetilde{\sigma}^S)^2 \widetilde{\omega}^{-1} \sum_{j \in \mathcal{J}} \omega_j (\sigma_j^S)^{-1} Z_{j,t}^S. \tag{A.108}$$

Given (A.107), and substituting the definitions of b (see (4.7)) into (A.106), we obtain, under the existence condition $\widetilde{\lambda}_i > 2 + 2r/\xi_i^P$ as stated in the theorem,

$$\max_{j\in\mathcal{J}} |b_j^* - b| = O(J^{-1}). \tag{A.109}$$

According to Definition 2, to establish the equilibrium, we only need to show the trading strategy $x_{j,t}$ given by (A.84) satisfies

$$\max_{j\in\mathcal{J}} \mathrm{E}_j((\dot{x}_{j,t}^* - \dot{x}_{j,t})^2) = O(J^{-2}), \quad \text{and} \quad \max_{j\in\mathcal{J}} \mathrm{E}_j((x_{j,t}^* - x_{j,t})^2) = O(J^{-2}).$$

For this purpose, in light of (A.103) and (A.105), we first derive the expression of $\mathrm{E}_{j,t}(G_t)$. We introduce

$$\widetilde{S}_{j,t} = \lambda_j \sum_{j'\in\mathcal{J}:j'\neq j} \omega_j \theta_j dS_{j,t}. \tag{A.110}$$

Then signals that the investor can access for learning about G_t are $\{S_s^D, S_{j,s}, \widetilde{S}_{j,s},\}_{s\leq t}$ (noting (A.90) and (A.92)), where

$$dS_s^D = G_s dt + \sigma^D dZ_s^D, \quad dS_{j,s} = \omega_j G_s dt + \sigma_j^S d\widehat{Z}_{j,s}^S, \quad d\widetilde{S}_{j,s} = \bar{\lambda} G_s dt + \widetilde{\sigma}_j^S d\widetilde{Z}_{j,s}^S.$$

Here $\widetilde{Z}_{j,s}^S$ is a standard Brownian motion, and, by the definition of $\widetilde{S}_{j,s}$ in (A.110), the drift term is $\bar{\lambda} G_s$ (using (A.89))and

$$\widetilde{\sigma}_j^S = \lambda_j (\widetilde{\sigma}^S)^2 \sqrt{\sum_{j'\in\mathcal{J}:j'\neq j} \omega_{j'}^2 (\sigma_{j'}^S)^{-2}}. \tag{A.111}$$

Because $\widehat{Z}_{j,s}^S$, $\bar{Z}_{j,s}^S$, and Z_s^D are mutually independent standard Brownian motions in investor j's belief, the standard Kalman-Bucy filtering leads to

$$\mathrm{E}_{j,t}(G_t) = (\breve{\xi}_j^P - \xi^G) \int_{-\infty}^t e^{-\breve{\xi}_j^P (t-s)} d\breve{S}_{j,s}, \tag{A.112}$$

where

$$\begin{aligned}
\breve{S}_{j,t} &= (\breve{\sigma}_j^S)^2((\sigma^D)^{-2} S_t^D + \omega_j (\sigma_j^S)^{-2} S_{j,t} + \bar{\lambda}(\widetilde{\sigma}_j^S)^{-2} \widetilde{S}_{j,t}), & \text{(A.113)}\\
\breve{\sigma}_j^S &= ((\sigma^D)^{-2} + \omega_j^2 (\sigma_j^S)^{-2} + \bar{\lambda}^2 (\widetilde{\sigma}_j^S)^{-2})^{-1/2}, & \text{(A.114)}\\
\breve{\xi}_j^P &= \sqrt{(\sigma^G/\breve{\sigma}_j^S)^2 + (\xi^G)^2}. & \text{(A.115)}
\end{aligned}$$

Next, we conduct asymptotic analysis on $(\breve{S}_{j,t}, \breve{\sigma}_j^S, \breve{\xi}^P)$. Clearly by Assumption 3 we have

$$\bar{\lambda} \simeq 1, \quad \max_{j\in\mathcal{J}} \lambda_j \simeq \min_{j\in\mathcal{J}} \lambda_j \simeq 1, \quad \widetilde{\sigma}^S \simeq 1.$$

(Note $z \simeq 1$ stands for that z is positive and satisfies $z = O(1)$ and $z^{-1} = O(1)$.) It follows from

(A.111), the definition of λ_j, and Assumption 3 that

$$\max_{j\in\mathcal{J}}|(\widetilde{\sigma}^S)^4(\widetilde{\sigma}_j^S)^{-2}-\widetilde{\iota}| = \max_{j\in\mathcal{J}}\left|\frac{(\lambda_j)^{-2}}{\sum_{j'\in\mathcal{J}:j'\neq j}\omega_{j'}^2(\sigma_{j'}^S)^{-2}}-\widetilde{\iota}\right|$$
$$= \max_{j\in\mathcal{J}}\left|\frac{\widetilde{\iota}^2\left(\sum_{j'\in\mathcal{J}:j'\neq j}\omega_{j'}(\sigma_{j'}^S)^{-2}\right)^2}{\sum_{j'\in\mathcal{J}:j'\neq j}\omega_{j'}^2(\sigma_{j'}^S)^{-2}}-\widetilde{\iota}\right| = O(J^{-1}). \tag{A.116}$$

We can further write

$$\max_{j\in\mathcal{J}}|(\breve{\sigma}_j^S)^{-2}-(\widetilde{\sigma}^S)^2| = \max_{j\in\mathcal{J}}|\omega_j^2(\sigma_j^S)^{-2}+\widetilde{\iota}^{-2}(\widetilde{\sigma}^S)^4(\widetilde{\sigma}_j^S)^{-2}-\widetilde{\iota}^{-1}|$$
$$= \max_{j\in\mathcal{J}}|\omega_j^2(\sigma_j^S)^{-2}+\widetilde{\iota}^{-2}(\widetilde{\sigma}^S)^4(\widetilde{\sigma}_j^S)^{-2}-\widetilde{\iota}^{-1}| = O(J^{-1}). \tag{A.117}$$

where the first equality holds by (A.114) and the definition of $\widetilde{\sigma}^S$ in (A.86), the second equality comes from (A.116) and Assumption 3. It directly follows from (A.115) and (A.117) that

$$\max_{j\in\mathcal{J}}|\breve{\xi}_j^P-\xi^P| = O(J^{-1}). \tag{A.118}$$

Moreover, using (A.116) and (A.117), we obtain the following results regarding the coefficients of $(S_t^D, S_{j,t}, \widetilde{S}_{j,t})$ in (A.113):

$$\max_{j\in\mathcal{J}}|(\breve{\sigma}_j^S)^2(\sigma^D)^{-2}-\theta^D| = O(J^{-1}), \tag{A.119}$$
$$\max_{j\in\mathcal{J}}|(\breve{\sigma}_j^S)^2\omega_j(\sigma_j^S)^{-2}-\omega_j^{-1}\theta_j| = O(J^{-2}), \tag{A.120}$$
$$\max_{j\in\mathcal{J}}|(\breve{\sigma}_j^S)^2\bar{\lambda}(\widetilde{\sigma}_j^S)^{-2}-1| = O(J^{-1}). \tag{A.121}$$

Substituting (A.118) - (A.121) into (A.112) and (A.113), and using the relation between $\widetilde{G}_{j,t}$ and $(G_t^D, \widetilde{S}_{j,t})$ from (A.92) and (A.110), we obtain

$$\max_{j\in\mathcal{J}}\mathrm{E}_j\left(\left(\mathrm{E}_{j,t}(G_t)-\omega_j^{-1}\theta_jG_{j,t}-\widetilde{G}_{j,t}\right)^2\right) = O(J^{-2}). \tag{A.122}$$

On the other hand, given the supply curve (A.91), submitting demand schedule (A.84) means that the equilibrium price satisfies

$$P_t = \frac{D_t}{\breve{\xi}^D}+\frac{1}{\breve{\xi}^D\breve{\xi}^G}\left((\widetilde{\sigma}^S)^2(\sigma^D)^{-2}G_t^D+(\widetilde{\sigma}^S)^2\widetilde{\omega}^{-1}\sum_{j\in\mathcal{J}}\omega_j(\sigma_j^S)^{-2}G_{j,t}\right). \tag{A.123}$$

Substituting (A.123) into (A.85), we have

$$\widetilde{\mu}_{j,t} = \frac{\xi^P + r}{\breve{\xi}^D \breve{\xi}^G} \omega_j^{-1} \theta_j \left(G_{j,t} - \bar{\lambda}^{-1} \widetilde{\omega}^{-1} \omega_j^{-1} \theta_j G_{j,t} - \bar{\lambda}^{-1} \frac{1}{\widetilde{\omega} \lambda_j} (\widetilde{G}_{j,t} - \theta^D G_t^D) \right).$$

Then it follows from Assumption 3 that

$$\max_{j \in \mathcal{J}} \mathrm{E}_j \left(\left(\widetilde{\mu}_{j,t} - \frac{\xi^P + r}{\breve{\xi}^D \breve{\xi}^G} \omega_j^{-1} \theta_j G_{j,t} \right)^2 \right) = O(J^{-2}). \tag{A.124}$$

Combining (A.122), (A.124), and (A.103), we obtain

$$\max_{j \in \mathcal{J}} \mathrm{E}_j((\widetilde{\mu}_{j,t} - \mu_{j,t})^2) = O(J^{-2}). \tag{A.125}$$

On the other hand, from (A.84) and (A.105) it follows that

$$x_{j,t}^* = a_j^* \int_{-\infty}^t e^{-b_j^*(t-s)} \mu_{j,s} ds, \quad x_{j,t} = \frac{1}{2\zeta} \frac{1}{\xi^P + b + r} \int_{-\infty}^t e^{-b(t-s)} \widetilde{\mu}_{j,s} ds.$$

Then, using (A.107), (A.109), and (A.125), we obtain

$$\max_{j \in \mathcal{J}} \mathrm{E}_j((x_{j,t} - x_{j,t}^*)^2) = O(J^{-2}). \tag{A.126}$$

Combining (A.126) with (A.84) and (A.105), and again using (A.107), (A.109), and (A.125), we have

$$\max_{j \in \mathcal{J}} \mathrm{E}_j((\dot{x}_{j,t} - \dot{x}_{j,t}^*)^2) = O(J^{-2}). \tag{A.127}$$

With (A.126) and (A.127), the equilibrium is established.

Step 4. In this step we prove the properties stated in the theorem. (4.5) is a direct result of Kalman-Bucy filtering. Moreover, because $\widetilde{\omega}^{-1} \sum_{j \in \mathcal{J}} \omega_j (\sigma_j^S)^{-2} = \widetilde{\iota}^{-1}$ by definition, it follows from (A.123) that the equilibrium price satisfies (4.4). Substituting (4.4) and (4.5) into (A.85), we obtain

$$\widetilde{\mu}_{j,t} = \frac{\xi^P + r}{\breve{\xi}^D \breve{\xi}^G} \omega_j^{-1} \theta_j \left(G_{j,t} - \bar{\lambda}^{-1} \frac{\iota^{-1} \bar{G}_t}{(\sigma^D)^{-2} + \iota^{-1}} \right) = \frac{\xi^P + r}{\breve{\xi}^D \breve{\xi}^G} \omega_j^{-1} \theta_j (G_{j,t} - \bar{G}_t),$$

hence (4.6) is also verified. Finally, comparing (A.123) with (A.108), we have that σ^P is indeed the volatility of P_t in equilibrium. The proof ends. ∎