SWEN90004
Modelling Complex Software Systems

Lecture Cx.03
ODE Models I: Predator-Prey

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Recap

- complex systems
- building a mathematical model
  - assumptions
  - states
  - update rules
- population growth models
  - exponential
  - logistic
- behaviour of dynamic systems
  - fixed points
  - limit cycles
  - chaos

Objectives

- interpret mathematical equations describing a model
- understand how these equations can be solved numerically and/or analytically to give the model's dynamic behaviour
- describe the dynamic behaviour of the model, and how this depends on:
  - model parameters
  - initial condition

Outline

Motivation

Lynxes and hares

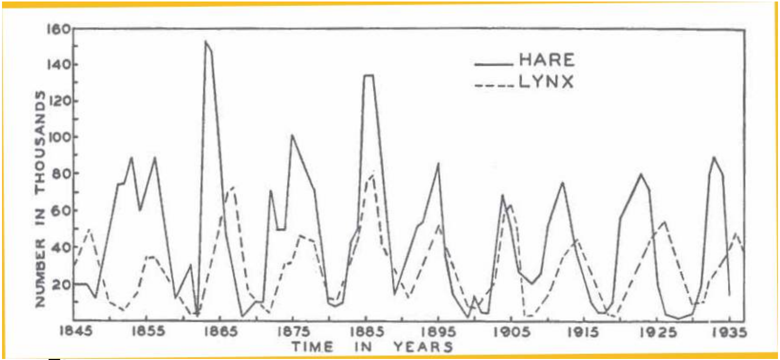
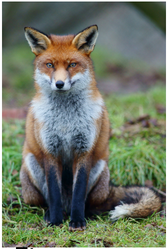


Figure90 years of hare and lynx populations in the Hudson Bay, Canada

History

Predator-prey relationships



FigureRed fox

FigureRabbit

The Lotka-Volterra model

Independently formulated in 1925–26 by Alfred Lotka, a US mathematician and chemist, and Vito Volterra, an Italian mathematician and physicist. Both were inspired by the logistic equation of Verhulst.

Assumptions

Predators: red foxes — Prey: rabbits

- the rabbit population is sustainable (they have ample food)
- the rabbit population grows at a rate proportional to its size if there are no red foxes
- the red foxes eat *only* rabbits
- the red fox population declines at a rate proportional to its size if there are no rabbits to eat
- the environment is static and has no effect on the rates of population growth and decay

Model formulation

Prey (rabbits):

$$\frac{dR}{dt} = \alpha R - \beta R F$$

Predators (red foxes):

$$\frac{dF}{dt} = \delta R F - \gamma F$$

Parameters:

Parameter	*Meaning*
α	growth rate of the rabbit population
β	rate at which foxes predate upon (eat) rabbits
δ	growth rate of the fox population
γ	decay rate of the fox population due to death and migration

Outline

An aside: solving ODEs using Euler method

Given a set of differential equations and an initial condition, how can we calculate the future states of the system?

In the system

$$y'(t) = f(t, y(t))$$

where f defines the relationshp between variables $\mathbf{y}$ and their derivatives, the Euler forward difference approach computes the sequence of approximations:

$$\begin{aligned} y_{n+1} &= y_n + \Delta t . f(t_n, y_n) \\ t_{n+1} &= t_n + \Delta t \end{aligned}$$

as a time-stepped *simulation* of the underlying behaviour

An aside: using the mid-point to improve accuracy

Rather that basing our estimate of the next point on the slope at the start of the interval, we can improve accuracy by using the slope at the mid-point:

$$
\begin{aligned}
y_{n+1} &= y_n + \Delta t . f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} f(t_n, y_n)\right) \\
t_{n+1} &= t_n + \Delta t
\end{aligned}
$$

This approach will diverge from the 'true' situation more slowly, but requires more calculations per iteration

An aside: solving ODEs using the Runge-Kutta method

The Euler method is often not accurate enough. Accuracy can be increased by making use of more points in the period between t_n and t_{n+1}

The Runge-Kutta family of approaches take this approach, with the most commonly used being RK4, which computes:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + (\frac{\Delta t}{6}).(k_1 + 2k_2 + 2k_3 + k_4)$$

where the k_n values are intermediate estimates of slope:

$$\begin{aligned}
k_1 &= f(t_n, \mathbf{y}_n) \\
k_2 &= f(t_n + \frac{\Delta t}{2}, \mathbf{y}_n + \frac{\Delta t}{2} k_1) \\
k_3 &= f(t_n + \frac{\Delta t}{2}, \mathbf{y}_n + \frac{\Delta t}{2} k_2) \\
k_4 &= f(t_n + \Delta t, \mathbf{y}_n + \Delta t . k_3)
\end{aligned}$$

An aside: solving ODEs using Matlab

Matlab function for the Lotka-Volterra model:

```
function dx = lotka_volterra_function(t, x)
  dx = [0; 0];

  alpha     = 0.75;
  beta      = 0.2;
  delta     = 0.04;
  gamma     = 0.5;

  dx(1) = alpha * x(1) - beta * x(1) * x(2);
  dx(2) = delta * x(1) * x(2) - gamma * x(2);
return
```

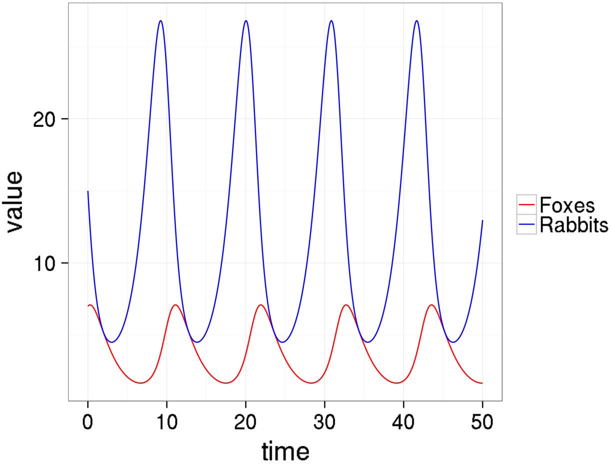
Matlab test script:

```
rabbits = 5;
foxes = 2;

[t, x] = ode45(@lotka_volterra_function, [0 200],
                [rabbits foxes], options);
```

Outline

Numerical solution of the Lotka-Volterra model



FigureRed fox (predator) and rabbit (prey) population levels over time

Numerical solution of the Lotka-Volterra model

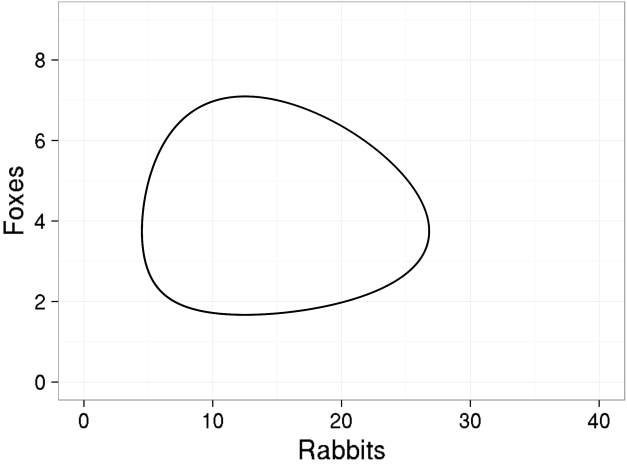


Figure The phase plane

Analysing the Lotka-Volterra model

What is the long term behaviour of the system?

equilibria: points at which the system variables (the population levels R and F) do not change; ie, where $\frac{dR}{dt} = 0$ and $\frac{dF}{dt} = 0$

$$R(\alpha - \beta F) = 0$$
$$F(\delta R - \gamma) = 0$$

Two equilibria:

- $R = 0, F = 0$
- $R = \frac{\gamma}{\delta}, F = \frac{\alpha}{\beta}$

Interpretation:

- equilibrium 1: both populations are extinct
- equilibrium 2: both populations sustained at non-zero levels indefinitely

Analysing the Lotka-Volterra model

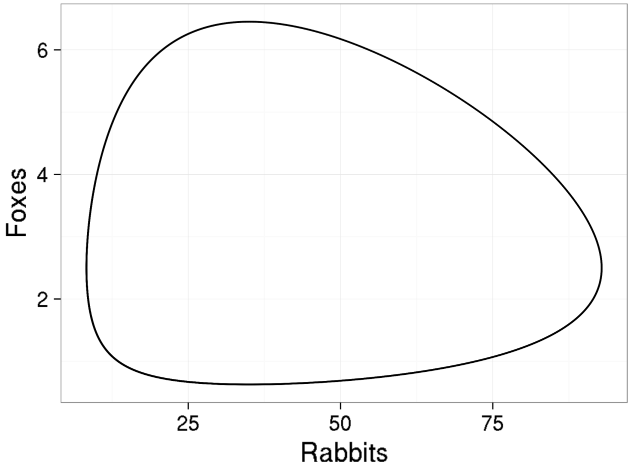
How does system behaviour depend on its initial condition?

Stability analysis reveals that:

- the first equilibrium (both populations going extinct) is *unstable*; it can only arise if the prey level is set to zero, at which point the predators also die out
- the second equilibrium is *stable*; specifically, it is *neutrally stable*, and surrounded by infinitely many periodic orbits

How realistic is this?

Another limitation



FigureAnother Lotka-Volterra phase plane

What is wrong with this picture?

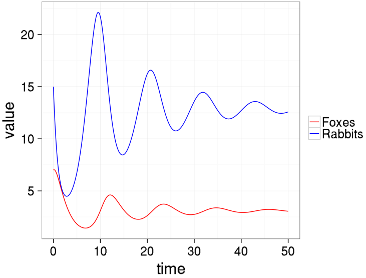
Outline

Lotka-Volterra model with competition

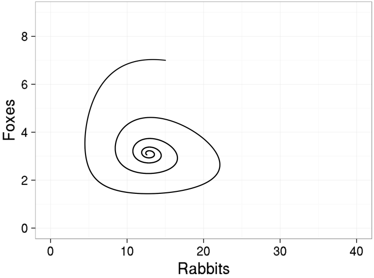
- There is no notion of *carrying capacity* for the prey species
- The logistic equation captured *intraspecific* competition
- The base Lotka-Volterra model captues *interspecific* competition
- We can combine these ideas by adding an additional term to the Lotka-Volterra model to account for competition among members of the same species:

$$\frac{dR}{dt} = \alpha R - \beta RF - aR^2$$
$$\frac{dF}{dt} = \delta RF - \gamma F - bF^2$$

Lotka-Volterra model with competition



FigureTime series



FigurePhase plane

Other model refinements have been proposed that increase the ability to capture observed phenomena

Lotka-Volterra model with multiple species

The two-species competitive model can also be written as:

$$\frac{dx_1}{dt} = r_1 x_1 \Big(1 - \Big(\frac{x_1 - \alpha_{12} x_2}{K_1}\Big)\Big)$$
$$\frac{dx_2}{dt} = r_2 x_2 \Big(1 - \Big(\frac{x_2 - \alpha_{21} x_1}{K_2}\Big)\Big)$$

where α_{ij} is the effect that species j has on species i.

This can then be generalised to an n-species model:

$$\frac{dx_i}{dt} = r_i x_i \Big(1 - \frac{\sum_{j=1}^{N} \alpha_{ij} x_j}{K_i}\Big)$$

where x_i represents the i-th species and α_{ij} is a matrix of interaction terms.

Lotka-Volterra model with multiple species

Another representation of the multi-species model is:

$$\frac{dx_i}{dt} = x_i\Big(r_i + \sum_{j=1}^{N} A_{ij}(1 - x_j)\Big)$$

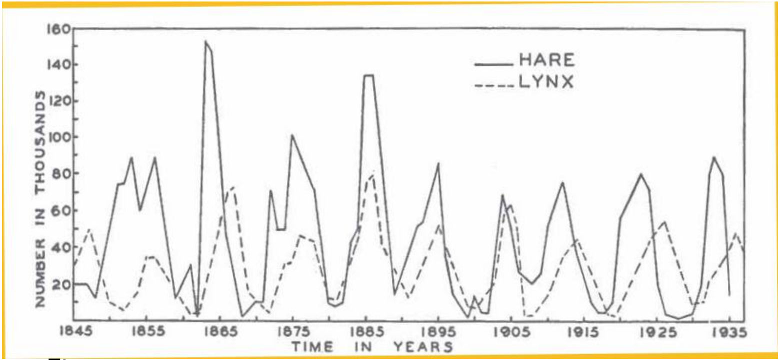
This version pulls the carrying capacities K_i and interaction terms α_{ij} inside the interaction matrix A

For three species, growth rates $r_1 = 1.1; r_2 = -0.5; r_3 = \alpha + 0.2$, and the interaction matrix:

$$A = \begin{pmatrix} 0.5 & 0.5 & 0.1 \\ -0.5 & -0.1 & 0.1 \\ \alpha & 0.1 & 0.1 \end{pmatrix}$$

can produce chaotic behaviour for values of $\alpha \geq 1.5$.

Lynxes and hares



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Outline

Recap on dynamic behaviours

- Dynamic systems can exhibit different types of long-term behaviour, including fixed points and limit cycles
- Fixed points and limit cycles can be:
  - stable and attracting (often referred to as *attractors*)
  - unstable and repelling
- A single dynamic system can exhibit *multiple* stable and unstable fixed points
- In a system with more than one stable fixed point, the long-term behaviour will depend on the initial conditions
- The set of initial conditions that approach an attractor are known as a basin of attraction

Broader applicability of Lotka-Volterra model

In the 1960s, Richard Goodwin, a mathematician and economist, proposed a model analogous to the Lotka-Volterra model for the relationship between wages and employment in an economy:

- *wages* $=$ *predators*
- *employment* $=$ *prey*

The system operates as follows:

- at high levels of employment, employees can bargain for higher wages, which reduces profits
- as profits shrink, fewer workers will be employed, which increases profits
- once profit levels are higher, more workers will be employed and employment will increase, leading to cyclic behaviour.

Complex systems often exhibit underlying patterns of behaviour across multiple domains