Note

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For the Poisson equation

$$-\Delta u = f,$$

where $f \in C_c^2(\mathbb{R}^n)$, consider Fourier transform of $f(x)$ and its inverse

$$\hat{f}(k) = \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)dx,$$
$$f(x) = \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} e^{ik\cdot x} \hat{f}(k)dk.$$

Since the eigenfunctions of Δ is $e^{ik\cdot x}$ with

$$\Delta e^{ik\cdot x} = -|k|^2 e^{ik\cdot x},$$

we then have

$$\begin{aligned} u &= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \frac{e^{ik\cdot x}}{|k|^2} \hat{f}(k)dk \\ &= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \frac{e^{ik\cdot x}}{|k|^2} \int_{\mathbb{R}^n} e^{-ik\cdot y} f(y)dy \\ &= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} f(y) \int_{\mathbb{R}^n} \frac{e^{ik\cdot(x-y)}}{|k|^2} dkdy. \end{aligned}$$

For $\int_{\mathbb{R}^n} \frac{e^{ik\cdot x}}{|k|^2} dk$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{e^{ik\cdot x}}{|k|^2} dk &= \int_0^\infty \int_{\partial B(0,1)} e^{ir\omega\cdot x} \frac{1}{r^2} r^{n-1} d\Omega dr \\ &= \int_0^\infty r^{n-3} \int_{\partial B(0,1)} e^{ir\omega\cdot x} d\Omega dr. \end{aligned}$$

By Fourier–Bessel integral, we have

$$\int_{\partial B(0,r)} e^{ir\omega\cdot x} d\Omega = (2\pi)^{\frac{n}{2}} (r|x|)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r|x|).$$

Then we have

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\begin{aligned}
\int_{\mathbb{R}^n} \frac{e^{ik\cdot x}}{|k|^2} dk &= \int_{\mathbb{R}^n} r^{n-3}(2\pi)^{\frac{n}{2}}(r|x|)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r|x|)dr \\
&= \int_{\mathbb{R}^n} (2\pi)^{\frac{n}{2}} r^{\frac{n}{2}-2}|x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r|x|)dr \\
&= 2^{n-2}\pi^{\frac{n}{2}}\Gamma(\frac{n-2}{2})\frac{1}{|x|^{n-2}}
\end{aligned}
$$

Finally we have

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\begin{aligned}
u &= \int_{\mathbb{R}^n} f(y)\frac{1}{4(\pi)^{\frac{n}{2}}}\Gamma(\frac{n-2}{2})\frac{1}{|x-y|^{n-2}}dy \\
&= \int_{\mathbb{R}^n} f(y)\frac{1}{(n-2)\omega(n)}\frac{1}{|x-y|^{n-2}}dy \\
&= \int_{\mathbb{R}^n} f(y)\phi(x-y)dy,
\end{aligned}
$$

which aligns with the representation formula.