Rapid mixing on random regular graph with improved degree dependence (Note)

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Abstract

We analyze single-site Markov chains for sampling independent sets from the hard-core model on sparse random graphs. For a uniformly random Δ-regular graph G on n vertices and fugacity $\lambda \leq \frac{1-\delta}{2\sqrt{\Delta-1}}$, we show that the Gibbs measure $\mu_{G,\lambda}$ is (with high probability over G) *completely spectrally dominated* under arbitrary configuration pinning, which enables the restricted modified log-Sobolev (rMLSI) framework in a neighborhood of μ. With a deterministic warm start through systematic-scan, we show that the distributions generated by Glauber dynamiccs is uniformly C-completely bounded with $C = \exp\big(O(\sqrt{\Delta})\big)$, thus prove that Glauber dynamics mixes in total variation within $\exp\big(O(\sqrt{\Delta})\big) \cdot \Omega_\delta\Big(n \log \frac{n\, D_{\mathrm{KL}}(\nu_{-1} \| \mu)}{\varepsilon}\Big)$ steps from an arbitrary initial distribution ν_{-1}. This yields a sharper dependence on the maximum degree in the beyond-uniqueness random-regular regime.

1 Introduction

The hard-core model and Glauber dynamics. Given a graph $G = (V, E)$, the associated collection of all independent sets of G, and a fugacity parameter $\lambda > 0$, the hard-core model assigns each independent set $\sigma \in \mathcal{I}(G)$ probability

$$\mu_{G,\lambda}(\sigma) \propto \lambda^{|\sigma|}$$

. This distribution is a canonical Gibbs measure for lattice gases and also a central object in theoretical computer science, where sampling/counting independent sets serves as a benchmark for MCMC methodology and for phase-transition phenomena. One of the most canonical samplers is the single-site *Glauber dynamics*. At each step, the Glauber dynamics selects a vertex $v \in V$ uniformly, and updates the current independent set S_t as follows: Let $S' = S_t/\{v\}$. If $S' \cup \{v\} \notin \mathcal{I}(G)$, set $S_{t+1} = S'$, otherwise set $S_{t+1} = S' \cup \{v\}$ with probability $\frac{\lambda}{1+\lambda}$ and $S_{t+1} = S'$ with probability $\frac{1}{1+\lambda}$. Understanding when Glauber mixes in polynomial (or near-linear) time, and how its mixing depends on structural graph parameters such as the maximum degree Δ, has driven a long line of work.

Tree thresholds: uniqueness, correlation decay, and algorithms. A guiding principle is that, for bounded-degree graphs, the infinite Δ-regular tree often predicts thresholds for decay of correlations and algorithmic tractability. For the hard-core model, the tree uniqueness threshold

$$\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$$

already appears in [9]. Below $\lambda_c(\Delta)$, correlations on the tree decay and the Gibbs measure is unique; above it, multiple Gibbs measures arise.

For the hardcore model on a random regular graph, when $\lambda < \lambda_c(\Delta)$, recent works [5, 4, 3] have achieved an optimal mixing time $O(n \log n)$. When $\lambda = \lambda_c(\Delta)$, the latest work [2] gives an upper bound $O(n^{2+4e+O(1/\Delta)})$. When $\lambda > \lambda_c(\Delta)$, [2] gives a suboptimal bound $O(n^2 \log \Delta)$ in the case $\lambda < \frac{c}{\sqrt{\Delta}}$, and otherwise the mixing time grows exponentially with respect to dimension n [10]. Notably, [5, 3] also imply an optimal mixing time $O(n \log n) \cdot O(\exp \exp \Delta)$, but with a double exponential dependence on the maximum degree Δ.

This note refines the mixing analysis for the hard-core model on random Δ-regular graphs by making the dependence on the maximum degree significantly sharper in the warm-start-and-contraction pipeline. We show that both K-balanced Glauber dynamics and vanilla Glauber dynamics have the optimal mixing time with exponential dependence on the maximum degree Δ when the fugacity $\lambda \le \frac{c}{\sqrt{\Delta}}$.

2 Preliminary

The analysis is based on a restricted modified Sobolev inequality [1], where an entropy contraction is analyzed in a set of completely bounded distributions.

Definition 1 (partial configuration) *$\sigma_\Lambda \in \{0,1\}^{|\Lambda|}$ is a partial configuration on $\Lambda \subseteq [n]$. The conditional distribution $\mu_{\sigma_\Lambda}(\cdot) := \mathbb{P}_{\sigma\sim\mu}(\sigma \in \cdot | \sigma = \sigma_\Lambda$ restricted on $\Lambda)$*

Definition 2 ((Completely) bounded distributions) *Let μ be a probability distribution over $2^{[n]}$ and let $C \ge 1$. Define the class of C-bounded distributions with respect to μ by*

$$\mathcal{V}(C,\mu) := \{\nu \in AC_\mu | \frac{\nu(i)(1-\mu(i))}{\mu(i)(1-\nu(i))} \le C, \forall i \in [n]\}$$

We follow the result of [2] that when the graph is a good spectral expander, then the hardcore model with fugacity at most $\frac{1-\delta}{\sqrt{\Delta-1}}$ is spectrally independent (furthermore completely spectrally dominated).

Theorem 1 (hardcore model on random Δ-regular graph) *Let $G = (V,E)$ be a graph, and let μ be the Gibbs distribution for the hardcore model on G with* *fugacity* λ *with* $\lambda \leq \frac{1-\delta}{2\sqrt{\Delta-1}}$. *Then with probability* $1 - o(1)$ *the following holds: for any* $\Lambda \subseteq [n]$ *and the partial configuration* $\sigma_\Lambda \in \{0,1\}^{|\Lambda|}$

$$M_{\sigma_\Lambda} \preceq I + (1-\delta)\operatorname{diag}(r_{\sigma_\Lambda})^{-1}, \tag{1}$$

where

$$M_{\sigma_\Lambda}(i,j) := \begin{cases} \dfrac{\mu^{\sigma_\Lambda}(\{i,j\})\,\mu^{\sigma_\Lambda}(\emptyset)}{\mu^{\sigma_\Lambda}(\{i\})\,\mu^{\sigma_\Lambda}(\{j\})} - 1, & i \neq j, \\ 0, & i = j. \end{cases}$$

$$r_{\sigma_\Lambda}(i) := \frac{\mu^{\sigma_\Lambda}(\{i\})}{\mu^{\sigma_\Lambda}(\emptyset)} = \frac{\mu(\sigma_\Lambda \cup \{i\})}{\mu(\sigma = \sigma_\Lambda)}, \quad \forall\, i \notin \sigma_\Lambda.$$

Furthermore, for any $\Lambda \subseteq [n]$ *and* $\boldsymbol{\lambda} \in [0,1]^{n-|\Lambda|}$

$$Cov(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}) \preceq \operatorname{diag}(1-(1-\delta)\boldsymbol{\lambda})^{-1}\operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})) \tag{2}$$

Consequently μ *is completely* $(\frac{2}{\delta}, \frac{\delta}{2})$*-spectrally dominated, equivalently, for any* $\boldsymbol{\lambda} \in [0, 1+\frac{\delta}{2}]^{n-|\Lambda|}$ *and* $\Lambda \subseteq [n]$

$$Cov(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}) \preceq \frac{2}{\delta}\operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})). \tag{3}$$

proof The proof of inequality (1) and (2) is similar to [2]. We remark that in [2] they use μ_S instead of μ_{σ_Λ}, but the whole analysis remains valid when transferred to configuration pinning μ_{σ_Λ}, as detailed in section 11.1 in [6].

Since δ is arbitrary, the range of $\boldsymbol{\lambda}$ can be expanded to $[0, 1+\frac{\delta}{2}]^n$, and thus

$$\begin{aligned} Cov(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}) &\preceq \operatorname{diag}(1-(1-\delta)(1+\frac{\delta}{2}))^{-1}\operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})) \\ &\preceq \frac{2}{\delta}\operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})) \end{aligned}$$

Note that the correlation matrix in [1] is defined as

$$\Psi^{\mathrm{cor}}_\mu(i,j) = \begin{cases} 1-\mu(i), & \text{if } j = i, \\ \mu(j|i) - \mu(j), & \text{otherwise.} \end{cases}$$

and $\Psi^{\mathrm{cor}}_\mu(i,j) = \operatorname{diag}(\boldsymbol{m}(\mu))^{-1} Cov(\mu)$. Since

$$\lambda_{max}(\Psi^{\mathrm{cor}}_\mu) = \lambda_{max}(\operatorname{diag}(\boldsymbol{m}(\mu))^{-1/2} Cov(\mu)\operatorname{diag}(\boldsymbol{m}(\mu))^{-1/2})$$

by definition we have μ is completely $(\frac{2}{\delta}, \frac{\delta}{2})$-spectrally dominated.

The same result can be derived for hardcore model on Erdos-Renyi random graphs.

Theorem 2 (hardcore model on Erdos-Renyi random graphs $G(n,p)$) *Let $G = G(n,p)$ be a Erdos-Renyi random graph. Let $d = (n-1)p$ be the expected degree. Suppose $p > \frac{\log n}{n}$. Let μ be the Gibbs distribution for the hardcore model on G with fugacity $\lambda \leq \frac{1-\delta}{4\sqrt{d}-1}$. Then with probability $1-o(1)$ the following holds: For any $\Lambda \subseteq [n]$ and $\boldsymbol{\lambda} \in [0,1]^{n-|\Lambda|}$*

$$Cov(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}) \preceq \operatorname{diag}(1-(1-\delta)\boldsymbol{\lambda})^{-1} \operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})) \tag{4}$$

Consequently μ is completely $(\frac{2}{\delta}, \frac{\delta}{2})$-spectrally dominated, equivalently, for any $\boldsymbol{\lambda} \in [0, 1+\frac{\delta}{2}]^{n-|\Lambda|}$ and $\Lambda \subseteq [n]$

$$Cov(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}) \preceq \frac{2}{\delta} \operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})). \tag{5}$$

proof By [8, 7], with probability $1-o(1)$, we have $|\lambda_n(G)| \leq 4\sqrt{d}$ and $d_{max} \leq d + O(\sqrt{d \log n})$. The rest of the proof is similar to Theorem 1.

3 Sharper degree independence via restricted MLSI

In this section, we first present a key theorem in [1] that when μ is completely spectrally independent, there is a restricted modified log Sobolev inequality restricted in C-bounded distributions. Then we show that the K-balanced Glauber dynamics and the vanilla Glauber dynamics yields a sequence of distribution $\{\nu_t\}_t$ that are uniformly bounded, thus the restricted MLSI theorem can be applied.

Theorem 3 (Restricted MLSI) *Let $\eta \geq 1, \epsilon > 0$, and $C \geq 1$. Suppose that μ is a probability distribution on $2^{[n]}$ completely (η, ϵ)-spectrally independent. Then, for any $v \in \mathcal{V}(C, \mu), \theta \in (0,1), k \in \mathbb{N}$, and $P \in \{P^{proj}_{k,\lceil nk\theta \rceil}, P^{FD}_{\theta}\}$, we have*

$$\mathcal{D}_{KL}(\nu P \| \mu P) \leq (1-\kappa) \mathcal{D}_{KL}(\nu \| \mu),$$

where $\kappa = (\theta/3)^{\eta'}$ with $\eta' = \max\{2\eta, \sqrt{log(C)/log(1+\epsilon}\}$.

The following lemma gives a bound of the marginal probability of μ.

Lemma 1 *Let $G = (V, E)$ be a graph on n vertices with maximum degree at most $\Delta \geq 3$. Let μ denote the distribution on $2^{|V|}$ corresponding to the hardcore model on G with fugacities $(\lambda_v)_{v \in V}$. Then, for any $\Lambda \subset V$, $v \in V$ and partial configuration $\sigma_\Lambda \in 2^{|\Lambda|}$ which leaves all neighbors of v unoccupied, we have*

$$\frac{\lambda_v}{1+\lambda_v} \Pi_{w \in N(v)} (\frac{1}{1+\lambda_w}) \leq \mathbb{P}_\mu[v \in \sigma | \sigma_\Lambda] \leq \frac{\lambda_v}{1+\lambda_v}$$

Moreover, if $\lambda_v \leq \frac{c}{\sqrt{\Delta}}, \forall v \in V$ for some constant c, we have

$$\Pi_{w \in N(v)} (\frac{1}{1+\lambda_w}) \geq e^{-c\sqrt{\Delta}}$$

Remark 1 *Let P^{SS} denote the Markov operator corresponding to a single pass of the systematic scan chain forμ, and let $v := v_0 P^{SS}$. Then v is C-completely bounded with respect to μ with $C = e^{O(\sqrt{\Delta})}$.*

proof of Remark 1 The proof is similar to proposition 52 in [1]: For any $v \in V$, define

$$\gamma_v := \max_{\sigma^-} \frac{\nu(\sigma^+)}{\nu(\sigma^-)},$$

where the maximum ranges over independent sets σ^- in V which do not include v and where σ^+ denotes the set $\sigma^- \cup \{v\}$. For any configuration x, we have

$$\frac{\mathbb{P}(x \to \sigma^+)}{\mathbb{P}(x \to \sigma^-)} = \frac{\mathbb{P}_\mu(v \in \sigma | \sigma^+(v))}{\mathbb{P}_\mu(v \in \sigma | \sigma^-(v))} \Pi_{i=1}^k \frac{\mathbb{P}_\mu(u_i \notin \sigma | \sigma^+(u_i))}{\mathbb{P}_\mu(u_i \notin \sigma | \sigma^-(u_i))}$$

where $u_1, \cdots, u_k$ denote the neighbors of site v which are updated after site v in systematic scan, and we let the notation $\sigma^-(u_i)$ denote the configuration right before the systematic scan updates site u_i, given that for all previous sites it updated according to configuration σ^-, and similarly for the notation $\sigma^+(u_i)$. Since $\frac{\mathbb{P}_\mu(v \in \sigma | \sigma^+(v))}{\mathbb{P}_\mu(v \in \sigma | \sigma^-(v))} \le \lambda_v$ and $\Pi_{i=1}^k \frac{\mathbb{P}_\mu(u_i \notin \sigma | \sigma^+(u_i))}{\mathbb{P}_\mu(u_i \notin \sigma | \sigma^-(u_i))} \le \Pi_{i=1}^k (1 + \lambda_i) \le e^{c\sqrt{\Delta}}$, we have $\frac{\mathbb{P}(x \to \sigma^+)}{\mathbb{P}(x \to \sigma^-)} \le \lambda_v e^{c\sqrt{\Delta}}$. Now consider any external field $(\theta_v)_{v \in V} \in [0,1]^{|V|}$, we have

$$\frac{(\theta * \nu)(v \in \sigma | \sigma_\Lambda) \cdot (\theta * \mu)(v \notin \sigma | \sigma_\Lambda)}{(\theta * \nu)(v \notin \sigma | \sigma_\Lambda) \cdot (\theta * \mu)(v \in \sigma | \sigma_\Lambda)} \le \theta_v \lambda_v e^{c\sqrt{\Delta}} \cdot (\theta_v \lambda_v e^{-c\sqrt{\Delta}})^{-1} = e^{O(\sqrt{\Delta})}$$

Definition 3 (K-Balanced Glauber Dynamics) *For a distribution μ, at each time step, we keep track of a configuration σ_t and a tuple $(N_t(v))_{v \in V}$, described below. We initial $N_0(v) = 0$ for all $v \in V$. For each $t \ge 1$, sample a vertex I_t uniformly at random and update σ_{t-1} at I_t according to the distribution μ conditioned on $\sigma_{t-1,-I_t}$. Let $\sigma_{t,0}$ be the resulting configuration. We define $N_{t,0}(v) = N_{t-1}(v) + 1$ for each v adjacent to I_t and define $N_{t,0}(I_t) = 0$. For all other vertices u, we define $N_{t,0}(u) = N_{t-1}(u)$. Then for $j \ge 1$, as long as there is a vertex v with $N_{t,j-1}(v) > K\Delta$, we choose such a vertex with the smallest index (according to a fixed, but otherwise arbitrary ordering of the vertices) and resample $\sigma_{t,j-1}$ at v according to the distribution μ conditioned on $\sigma_{t,j-1,-v}$ to form $\sigma_{t,j}$. We then define $N_{t,j}$ by increasing $N_{t,j-1}$ at the neighbors of v by 1, setting $N_{t,j}(v) = 0$, and for all other vertices u, setting $N_{t,j}(u) = N_{t-1,j}(u)$. At j_t, when there are no vertices v with $N_{t,j_t}(v) > K\Delta$, we let $\sigma_t = \sigma_{t,j_{t-1}}$ and $N_t = N_{t,j_{t-1}}$.*

Theorem 4 (ATE by restrcited MLSI for field dynamics.) *Let μ be a probability distribution on $2^{[n]}$ which is completely (η, ϵ)-spectrally independent. Let $v \in \mathcal{V}^c(C, \mu)$ and $\theta \in (0,1)$, and let $f := dv/d\mu$. Suppose there exists $\kappa > 0$ such that for $\pi = \theta * \mu := (\theta, \cdots, \theta) * \mu$ and for all $R \subseteq [n]$ with $\pi(R) > 0$,*

$$\kappa \operatorname{Ent}_{\pi^{1_R}}(f) \le \mathcal{E}_{P_\theta}(f, \log f) \tag{6}$$

where P_θ is the transition matrix for the Glauber dynamics with respect to $\pi^{\mathbf{1}_\pi}$ and $\mathcal{E}_{P_\theta}$ is its corresponding Dirichlet form. Then we have

$$\mathrm{Ent}_\mu[f] \leq C' \sum_{v\in[n]} \mathbb{E}_\mu[\mathrm{Ent}_v[f]]$$

where $C' = \frac{C}{\kappa n} \times \frac{1}{\Omega(\theta)^{O(\eta')}}$, with $\eta' = \max\{2\eta, \sqrt{log(C)/log(1+\epsilon)}\}$

Proposition 1 (MLSI in easy regime) *Let μ denote the hardcore model on a graph of maximum degree at most $\Delta \geq 3$, with fugacity $\lambda_v \leq 1/2\Delta$ for all sites v, and let P denote the transition matrix of the Glauber dynamics. Then, the modified log-Sobolev constant $\rho_0(P)$ satisfies $\rho_0(P) \geq 1/4n$.*

Lemma 2 *For K-balanced Glauber dynamics P^{BG} and arbitrary distribution v that is absolutely continuous to μ, we have vP^{BG} is C-completely bounded with respect to μ for $C = e^{O(K\sqrt{\Delta})}$.*

proof. The proof follows a similar step as proof of Remark 1. Let ν_x denote the resulting measure on independent sets of G, starting from the initial distribution $\nu_0 = \mathbf{1}_x$, then it suffice to prove that

$$\frac{\nu_x(\sigma^+)}{\nu_x(\sigma^-)} \leq \tilde{C}\lambda_v$$

for $\tilde{C} = e^{O(K\sqrt{\Delta})}$. Similar to the proof of Remark 1, we have

$$\frac{\nu_x(\sigma^+)}{\nu_x(\sigma^-)} \leq \lambda_v \Pi_{i=1}^k (1+\lambda_{u_i}) \leq \lambda_v (1+\frac{c}{\sqrt{\Delta}})^{K\Delta} \leq e^{cK\sqrt{\Delta}}.$$

Lemma 3 (Lemma 55 in [1]) *let ν be the distribution of σ_{t-1} and let $f = d\nu/d\mu$ denotes the corresponding density. Define another Markov chain (ξ_t, N_t) where $\xi_0 \sim \mu$, and the coupled process $(\sigma_{t,i}, \xi_{t,i}, N_{t,i})$. Then, provided that the approximate tensorization of entropy estimate*

$$\mathcal{D}_{\mathrm{KL}}(\nu\|\mu) \leq \frac{1}{C}\sum_{v\in V} \mathbb{E}_\mu[\mathrm{Ent}_v(f)]$$

holds for some $C > 0$, we have

$$\mathcal{D}_{\mathrm{KL}}((\sigma_{t,0}, N_{t,0})\|(\xi_{t,0}, N_{t,0})) \leq \left(1 - \frac{C}{n}\right)\mathcal{D}_{\mathrm{KL}}((\sigma_{t-1}, N_{t-1})\|(\xi_{t-1}, N_{t-1})).$$

$$\mathcal{D}_{\mathrm{KL}}((\sigma_t, N_t)\|(\xi_t, N_t)) \leq \left(1 - \frac{C}{n}\right)\mathcal{D}_{\mathrm{KL}}((\sigma_{t-1}, N_{t-1})\|(\xi_{t-1}, N_{t-1})).$$

3.1 Main Results

By ensembling all the lemmas above, we are now ready to prove our main theorem for K-balanced Glauber dynamics.

Theorem 5 (Main theorem 1: fast mixing on random regular graph) *Let $G = (V, E)$ be a graph, and let μ be the Gibbs distribution for the hardcore model on G with fugacity λ with $\lambda \leq \frac{1-\delta}{2\sqrt{\Delta-1}}$. Let v_{-1} be an arbitrary initial distribution and $v_0 = v_{-1}P^{SS}$ be the distribution obtained by running a single round of the systematic scan. let (σ_t, N_t) be defined by the K-balanced Glauber dynamics, as above, starting from the initial distribution $v_0 \times (0)_{v\in V}$. Then with probability $1 - o(1)$, for any $\epsilon > 0$,*

$$d_{TV}(v_T, \mu) \leq \epsilon$$

for all $T = e^{O(K\sqrt{\Delta})} \cdot \Omega_\delta(n \log[n\mathcal{D}_{KL}(v_{-1}\|\mu)/\epsilon])$

proof of Main theorem 1. By Lemma 2, the distribution ν_t is C-completely bounded with respect to μ for all $t \geq 0$ with $C = e^{O(K\sqrt{\Delta})}$. Let $f_t = d\nu_t/d\mu$, by Theorem 4 (choosing external field $\theta = \frac{c}{\sqrt{\Delta}}$) combined with the fact that μ is $(\frac{2}{\eta}, \frac{\eta}{2})$-completely spectrally independent and the MLSI for Glauber dynamics in the easy regime is $\Omega(\frac{1}{n})$ (from Proposition 1), we have

$$\mathcal{D}_{\mathrm{KL}}(\nu_t\|\mu) \leq \mathcal{D}_{\mathrm{KL}}((\sigma_t, N_t)\|(\xi_t, N_t)) \leq \left(1 - \frac{C}{n}\right)^t \mathcal{D}_{\mathrm{KL}}(\nu_{-1}\|\mu),$$

where $C = e^{O(K\sqrt{\Delta})} \cdot (\sqrt{\Delta})^{\sqrt{K\sqrt{\Delta}}} = e^{O(K\sqrt{\Delta})}$.

Furthermore, by showing that the distributions $\{\nu_t\}_t$ generated by Glauber dynamics is uniformly C-bounded, we prove a similar result for the vanilla Glauber dynamics.

Corollary 1 *Let $G = (V, E)$ be a graph, and let μ be the Gibbs distribution for the hardcore model on G with fugacity λ with $\lambda \leq \frac{1-\delta}{2\sqrt{\Delta-1}}$. Let v_{-1} be an arbitrary initial distribution and $v_0 = v_{-1}P^{SS}$ be the distribution obtained by running a single round of the systematic scan. Let $\nu_{t+1} = \nu_t P^{GD}$ be the distribution sequence by running Glauber dynamics. Then with probability $1 - o(1)$, for any $\epsilon > 0$,*

$$d_{TV}(v_T, \mu) \leq \epsilon$$

for all $T = e^{O(\sqrt{\Delta})} \cdot \Omega_\delta(n \log[n\mathcal{D}_{KL}(v_{-1}\|\mu)/\epsilon])$

proof of corollary 1. By Lemma 4, the distribution ν_t is C-completely with respect to μ for all $t \geq 0$ with $C = e^{O(\sqrt{\Delta})}$. Let $f_t = d\nu_t/d\mu$, by Theorem 4 (choosing external field $\theta = \frac{c}{\sqrt{\Delta}}$) combined with the fact that μ is $(\frac{2}{\eta}, \frac{\eta}{2})$-completely spectrally independent and the MLSI for Glauber dynamics in the easy regime is $\Omega(\frac{1}{n})$ (from Proposition 1), we have

$$\mathrm{Ent}_\mu[f] \le C' \sum_v \mathbb{E}_\mu[\mathrm{Ent}_v[f]]$$

and thus

$$\begin{aligned}\mathcal{D}_{KL}[\nu_{t+1}\|\mu] &= \mathcal{D}_{KL}[\nu_t P^{GD}\|\mu P^{GD}] \\ &\le (1-\frac{1}{nC'})\mathcal{D}_{KL}[\nu_t\|\mu] \\ &\le (1-\frac{1}{nC'})^{t+1}\mathcal{D}_{KL}[\nu_0\|\mu]\end{aligned}$$

where $C' = e^{O(\sqrt{\Delta})} \cdot (\sqrt{\Delta})^{\Delta^{1/4}} = e^{O(\sqrt{\Delta})}$.

Lemma 4 *Let ν_t be arbitrary distribution and $\nu_{t+1} = \nu_t P$ be the distribution obtained by running a single round of the Glauber dynamics. Define*

$$\gamma_v^t := \max_{\sigma^-} \frac{\nu_t(\sigma^+)}{\nu_t(\sigma^-)}$$

where the maximum ranges over independent sets σ^- in V which do not include v and where σ^+ denotes the set $\sigma^- \cup \{v\}$. Similarly define

$$\gamma_v^{t+1} := \max_{\sigma^-} \frac{\nu^{t+1}(\sigma^+)}{\nu^{t+1}(\sigma^-)}$$

Then we have

$$\gamma_v^{t+1} \le \max\{O(\Delta), \gamma_v^t\}$$

Furthermore, let ν_{-1} be arbitrary initial distribution and $v_0 = v_{-1}P^{SS}$ be the distribution obtained by running a single round of the systematic scan. Let μ be the distribution of hardcore model on a random Δ-regular graph G with fugacity $\lambda \le \frac{1-\delta}{2\sqrt{\Delta}-1}$. Then $\gamma_v^t = e^{O(\sqrt{\Delta})}$ for $t \ge 0$, and consequently ν_t is C-completely bounded with $C = e^{O(\sqrt{\Delta})}$ for $t \ge 0$.

Theorem 6 (fast mixing on Erdos-Renyi graph) *Let $G = G(n,p)$ be a Erdos-Renyi random graph. Let $d = (n-1)p$ be the ex- pected degree. Suppose $p > \frac{\log n}{n}$. and let μ be the Gibbs distribution for the hardcore model on G with fugacity λ with $\lambda \le \frac{1-\delta}{4\sqrt{d}-1}$. Let v_{-1} be an arbitrary initial distribution and $v_0 = v_{-1}P^{SS}$ be the distribution obtained by running a single round of the systematic scan. let (σ_t, N_t) be defined by the K-balanced Glauber dynamics, as above, starting from the initial distribution $v_0 \times (0)_{v\in V}$. Then with probability $1-o(1)$, for any $\epsilon > 0$,*

$$d_{TV}(v_T, \mu) \le \epsilon$$

for all $T = e^{O(\sqrt{d+c\sqrt{d\log n}})} \cdot \Omega_\delta(n\log[n\mathcal{D}_{KL}(v_{-1}\|\mu)/\epsilon])$

proof The proof is the same as Theorem 5 by replacing the maximum degree Δ by $d + c\sqrt{d \log n}$

Proof of Lemma 4 Suppose that μ is the hardcore model on $G = (V, E)$. Let ν_t be a distribution supported on $2^{|V|}$, define

$$\gamma_v^t = \max_{\sigma^-} \frac{\nu_t(\sigma^+)}{\nu_t(\sigma^-)}$$

where the maximum ranges over independent sets σ^- in V which do not include v and where σ^+ denotes the set $\sigma \cup \{v\}$.

Define $\nu_{t+1} = \nu_t P$ where P is the transition kernel of Glauber dynamics, consider $\frac{\nu_{t+1}(\sigma^+)}{\nu_{t+1}(\sigma^-)}$ for any independent sets σ^-.

Recall that

$$\nu_{t+1}(\sigma^+) = \frac{1}{n}\sum_u (\nu_t(\sigma^+ \cup \{u\})K_u(\sigma^+ \cup \{u\} \to \sigma^+) + \nu_t(\sigma^+/\{u\})K_u(\sigma^+/\{u\} \to \sigma^+))$$

$$\nu_{t+1}(\sigma^-) = \frac{1}{n}\sum_u (\nu_t(\sigma^- \cup \{u\})K_u(\sigma^- \cup \{u\} \to \sigma^-) + \nu_t(\sigma^-/\{u\})K_u(\sigma^-/\{u\} \to \sigma^-))$$

Where $K_u(\tau \to \sigma)$ is the $(n-1, n)$ up walk from τ without knowing site u to σ. Thus σ is either $\tau \cup \{u\}$ for $u \notin \tau$ or $\tau/\{u\}$ for $u \in \tau$ or τ. When $u \in \tau$ (or $u \notin \tau$), we have $\nu(\tau \cup \{u\}) = \nu(\tau)$ (or $\nu(\tau/\{u\}) = \nu(\tau)$), because they represent the probability of same event: the vertices in τ and vertice u are occupied, and the other vertices are unoccupied (or the vertices in τ other than vertice u are occupied, and the other vertices are unoccupied).

Note that there are three types of transition from $\nu_t(\tau)$ to $\nu_{t+1}(\sigma^{\pm})$ when site u is chosen to be updated

- Case 1: $u = v$. Then

  $$\nu_t(\sigma^+ \cup \{u\})K_u(\sigma^+ \cup \{u\} \to \sigma^+) = \nu_t(\sigma^+)\frac{\lambda}{1+\lambda}$$

  and

  $$\nu_t(\sigma^+/\{u\})K_u(\sigma^+/\{u\} \to \sigma^+) = \nu_t(\sigma^-)\frac{\lambda}{1+\lambda}$$

  . Similarly

  $$\nu_t(\sigma^- \cup \{u\})K_u(\sigma^- \cup \{u\} \to \sigma^-) = \nu_t(\sigma^+)\frac{1}{1+\lambda}$$

  and

  $$\nu_t(\sigma^-/\{u\})K_u(\sigma^-/\{u\} \to \sigma^-) = \nu_t(\sigma^-)\frac{1}{1+\lambda}$$

- Case 2: $u \neq v$ and $u \notin N(v)$. We only consider the case when $\nu_t(\sigma^- \cup u) \neq 0$. Below we simply use the fact that $\nu_t(\{\sigma/u\}) = \nu_t(\sigma)$ if $u \notin \sigma$ and $\nu_t(\{\sigma \cup u\}) = \nu_t(\sigma)$ if $u \in \sigma$.

- **if u is in σ (i.e. $u \in \sigma^-$)**, then

$$\nu_t(\sigma^+ \cup \{u\})K_u(\sigma^+ \cup \{u\} \to \sigma^+) = \nu_t(\sigma^+)\frac{\lambda}{1+\lambda}$$

and

$$\nu_t(\sigma^+/\{u\})K_u(\sigma^+/\{u\} \to \sigma^+) = \nu_t(\{\sigma/u\}^+)\frac{\lambda}{1+\lambda}$$

. Similarly

$$\nu_t(\sigma^- \cup \{u\})K_u(\sigma^- \cup \{u\} \to \sigma^-) = \nu_t(\sigma^-)\frac{\lambda}{1+\lambda}$$

and

$$\nu_t(\sigma^-/\{u\})K_u(\sigma^-/\{u\} \to \sigma^-) = \nu_t(\{\sigma/u\}^-)\frac{\lambda}{1+\lambda}$$

.

- **if u is not in σ (i.e. $u \notin \sigma^-$)**, then

$$\nu_t(\sigma^+ \cup \{u\})K_u(\sigma^+ \cup \{u\} \to \sigma^+) = \nu_t(\{\sigma, u\}^+)\frac{1}{1+\lambda}$$

and

$$\nu_t(\sigma^+/\{u\})K_u(\sigma^+/\{u\} \to \sigma^+) = \nu_t(\sigma^+)\frac{1}{1+\lambda}$$

. Similarly

$$\nu_t(\sigma^- \cup \{u\})K_u(\sigma^- \cup \{u\} \to \sigma^-) = \nu_t(\{\sigma, u\}^-)\frac{1}{1+\lambda}$$

and

$$\nu_t(\sigma^-/\{u\})K_u(\sigma^-/\{u\} \to \sigma^-) = \nu_t(\sigma^-)\frac{1}{1+\lambda}.$$

- Case 3: $u \in N(v)$.
  - **If u is in σ^- (i.e. $u \in \sigma^-$)** , then $\nu_t(\sigma^+ \cup \{u\}) = K_u(\sigma^+/\{u\} \to \sigma^+) = 0$.
  - **If u is not in σ^- (i.e. $u \notin \sigma^-$)** , Then

$$\nu_t(\sigma^+ \cup \{u\})K_u(\sigma^+ \cup \{u\} \to \sigma^+) = 0$$

and

$$\nu_t(\sigma^+/\{u\})K_u(\sigma^+/\{u\} \to \sigma^+) = \nu_t(\sigma^+)$$

since $K_u(\sigma^+/\{u\} \to \sigma^+) = 1$ and u must not in σ. Similarly

$$\nu_t(\sigma^- \cup \{u\})K_u(\sigma^- \cup \{u\} \to \sigma^-) = \nu_t(\{\sigma, u\}^-)\frac{1}{1+\lambda}$$

and

$$\begin{aligned}\nu_t(\sigma^-/\{u\})K_u(\sigma^-/\{u\} \to \sigma^-) &= \nu_t(\sigma^-)(\frac{1}{1+\lambda}\mathbf{1}_{N(u)\cap\sigma=\emptyset} + \mathbf{1}_{N(u)\cap\sigma\neq\emptyset}) \\ &\geq \nu_t(\sigma^-)\frac{1}{1+\lambda}.\end{aligned}$$

Then we have

$$\frac{\nu_{t+1}(\sigma^+)}{\nu_{t+1}(\sigma^-)} = \frac{\sum_{case1}\nu_{t+1}(\sigma^+|u) + \sum_{case2}\nu_{t+1}(\sigma^+|u) + \sum_{case3}\nu_{t+1}(\sigma^+|u)}{\sum_{case1}\nu_{t+1}(\sigma^-|u) + \sum_{case2}\nu_{t+1}(\sigma^-|u) + \sum_{case3}\nu_{t+1}(\sigma^-|u)}$$
$$\leq \max\{\frac{\sum_{case1}\nu_{t+1}(\sigma^+|u) + \sum_{case3}\nu_{t+1}(\sigma^+|u)}{\sum_{case1}\nu_{t+1}(\sigma^-|u) + \sum_{case3}\nu_{t+1}(\sigma^-|u)}, \frac{\sum_{case2}\nu_{t+1}(\sigma^+|u)}{\sum_{case2}\nu_{t+1}(\sigma^-|u)}\},$$

where $\nu_{t+1}(\sigma^\pm|u)$ is defined by $\nu_{t+1}(\sigma^\pm|u) := \frac{1}{n}(\nu_t(\sigma^\pm \cup \{u\})K_u(\sigma^\pm \cup \{u\} \to \sigma^\pm) + \nu_t(\sigma^\pm/\{u\})K_u(\sigma^\pm/\{u\} \to \sigma^\pm))$.

Note that in case 2, we have $\frac{\nu_{t+1}(\sigma^+|u)}{\nu_{t+1}(\sigma^-|u)} = \frac{\nu_t(\{\sigma,u\}^+)+\nu_t(\{\sigma/u\}^+)}{\nu_t(\{\sigma,u\}^-)+\nu_t(\{\sigma/u\}^-)} \leq \gamma_v^t$. On the other hand, we have

$$\begin{aligned}
&\frac{\sum_{case1}\nu_{t+1}(\sigma^+|u) + \sum_{case3}\nu_{t+1}(\sigma^+|u)}{\sum_{case1}\nu_{t+1}(\sigma^-|u) + \sum_{case3}\nu_{t+1}(\sigma^-|u)} \\
\leq&\frac{\nu_t(\sigma^-)\frac{\lambda}{1+\lambda} + \nu_t(\sigma^+)\frac{\lambda}{1+\lambda} + \sum_{u\in N(v)}\nu_t(\sigma^+)}{\nu_t(\sigma^-)\frac{1}{1+\lambda} + \nu_t(\sigma^+)\frac{1}{1+\lambda} + \sum_{u\in N(v)}\nu_t(\{\sigma,u\}^-)\frac{1}{1+\lambda} + \nu_t(\sigma^-)\frac{1}{1+\lambda}} \\
\leq&\frac{\nu_t(\sigma^-)\lambda + \nu_t(\sigma^+)\lambda + (1+\lambda)\sum_{u\in N(v)}\nu_t(\sigma^+)}{\nu_t(\sigma^-) + \nu_t(\sigma^+) + \sum_{u\in N(v)}\nu_t(\sigma^-)} \\
=&\frac{(1+\tilde{\gamma}_v)\lambda + (1+\lambda)\#(u \in N(v))\tilde{\gamma}_v}{1+\tilde{\gamma}_v + \#(u\in N(v))} \quad \text{set } \tilde{\gamma}_v := \frac{\nu_t(\sigma^+)}{\nu_t(\sigma^-)} \\
\leq&\frac{\lambda + (\lambda + \Delta(1+\lambda))\tilde{\gamma}_v}{1+\Delta+\tilde{\gamma}_v} \\
=&O(\Delta)
\end{aligned}$$

Thus in the case $\gamma_v^t = O(e^{\sqrt{\Delta}})$ we have

$$\gamma_v^{t+1} = \max_{\sigma^-}\frac{\nu_{t+1}(\sigma^+)}{\nu_{t+1}(\sigma^-)} \leq \max\{\underbrace{O(\Delta)}_{\text{by case 1 and case 3}}, \underbrace{\gamma_v^t}_{\text{by case 2}}\} = \gamma_v^t$$

Thus ν_t is always C'-completely bounded with respect to μ.

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