Draft

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1 Hardcore model in random graph

Definition 1 (hardcore model)

Definition 2 (partial configuration) *$\sigma_\Lambda \in \{0,1\}^{|\Lambda|}$ is a partial configuration on $\Lambda \subseteq [n]$. The conditional distribution $\mu_{\sigma_\Lambda}(\cdot) := \mathbb{P}_{\sigma\sim\mu}(\sigma \in \cdot | \sigma = \sigma_\Lambda$ restricted on $\Lambda)$*

Definition 3 ((Completely) bounded distributions) *Let μ be a probability distribution over $2^{[n]}$ and let $C \geq 1$. Define the class of C-bounded distributions with respect to μ by*

$$\mathcal{V}(C,\mu) := \{\nu \in AC_\mu | \frac{\nu(i)(1-\mu(i))}{\mu(i)(1-\nu(i))} \leq C, \forall i \in [n]\}$$

Theorem 1 (hardcore model on random Δ-regular graph) *Let $G = (V,E)$ be a graph, and let μ be the Gibbs distribution for the hardcore model on G with fugacity λ with $\lambda \leq \frac{1-\delta}{2\sqrt{\Delta-1}}$. Then with probability $1-o(1)$ the following holds: for any $\Lambda \subseteq [n]$ and the partial configuration $\sigma_\Lambda \in \{0,1\}^{|\Lambda|}$*

$$M_{\sigma_\Lambda} \preceq I + (1-\delta)\operatorname{diag}(r_{\sigma_\Lambda})^{-1}, \tag{1}$$

where

$$M_{\sigma_\Lambda}(i,j) := \begin{cases} \dfrac{\mu^{\sigma_\Lambda}(\{i,j\})\,\mu^{\sigma_\Lambda}(\emptyset)}{\mu^{\sigma_\Lambda}(\{i\})\,\mu^{\sigma_\Lambda}(\{j\})} - 1, & i \neq j, \\ 0, & i = j. \end{cases}$$

$$r_{\sigma_\Lambda}(i) := \frac{\mu^{\sigma_\Lambda}(\{i\})}{\mu^{\sigma_\Lambda}(\emptyset)} = \frac{\mu(\sigma_\Lambda \cup \{i\})}{\mu(\sigma = \sigma_\Lambda)}, \quad \forall\, i \notin \sigma_\Lambda.$$

Furthermore, for any $\Lambda \subseteq [n]$ and $\boldsymbol{\lambda} \in [0,1]^{n-|\Lambda|}$

$$Cov(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}) \preceq \operatorname{diag}(1-(1-\delta)\boldsymbol{\lambda})^{-1}\operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})) \tag{2}$$

Consequently μ is completely $(\frac{2}{\delta}, \frac{\delta}{2})$-spectrally dominated, equivalently, for any $\boldsymbol{\lambda} \in [0, 1+\frac{\delta}{2}]^{n-|\Lambda|}$ and $\Lambda \subseteq [n]$

$$Cov(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}) \preceq \frac{2}{\delta}\operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})). \tag{3}$$

proof The proof of inequality (1) and (2) is similar to [2]. We remark that in [2] they use μ_S instead of μ_{σ_Λ}, but the whole analysis remains valid when transferred to configuration pinning μ_{σ_Λ}, as detailed in section 11.1 in [3].

Since δ is arbitrary, the range of $\boldsymbol{\lambda}$ can be expanded to $[0, 1 + \frac{\delta}{2}]^n$, and thus

$$
\begin{aligned}
Cov(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}) &\preceq \operatorname{diag}(1 - (1-\delta)(1 + \frac{\delta}{2}))^{-1} \operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda})) \\
&\preceq \frac{2}{\delta} \operatorname{diag}(\boldsymbol{m}(\boldsymbol{\lambda} * \mu^{\sigma_\Lambda}))
\end{aligned}
$$

Note that the correlation matrix in [1] is defined as

$$
\Psi_\mu^{\text{cor}}(i,j) = \begin{cases} 1 - \mu(i), & \text{if } j = i, \\ \mu(j|i) - \mu(j), & \text{otherwise.} \end{cases}
$$

and $\Psi_\mu^{\text{cor}}(i,j) = \operatorname{diag}(\boldsymbol{m}(\mu))^{-1} Cov(\mu)$. Since

$$
\lambda_{max}(\Psi_\mu^{\text{cor}}) = \lambda_{max}(\operatorname{diag}(\boldsymbol{m}(\mu))^{-1/2} Cov(\mu) \operatorname{diag}(\boldsymbol{m}(\mu))^{-1/2})
$$

by definition we have μ is completely $(\frac{2}{\delta}, \frac{\delta}{2})$-spectrally dominated.

Theorem 2 (Restricted MLSI) *Let $\eta \geq 1, \epsilon > 0$, and $C \geq 1$. Suppose that μ is a probability distribution on $2^{[n]}$ completely (η, ϵ)-spectrally independent. Then, for any $v \in \mathcal{V}(C, \mu), \theta \in (0,1), k \in \mathbb{N}$, and $P \in \{P_{k,\lceil nk\theta \rceil}^{proj}, P_\theta^{FD}\}$, we have*

$$
\mathcal{D}_{KL}(\nu P \| \mu P) \leq (1 - \kappa) \mathcal{D}_{KL}(\nu \| \mu),
$$

where $\kappa = (\theta/3)^{\eta'}$ with $\eta' = \max\{2\eta, \sqrt{log(C)/log(1+\epsilon}\}$.

Lemma 1 *Let $G = (V, E)$ be a graph on n vertices with maximum degree at most $\Delta \geq 3$. Let μ denote the distribution on $2^{|V|}$ corresponding to the hardcore model on G with fugacities $(\lambda_v)_{v \in V}$. Then, for any $\Lambda \subset V$, $v \in V$ and partial configuration $\sigma_\Lambda \in 2^{|\Lambda|}$ which leaves all neighbors of v unoccupied, we have*

$$
\frac{\lambda_v}{1 + \lambda_v} \Pi_{w \in N(v)} (\frac{1}{1 + \lambda_w}) \leq \mathbb{P}_\mu[v \in \sigma | \sigma_\Lambda] \leq \frac{\lambda_v}{1 + \lambda_v}
$$

Moreover, if $\lambda_v \leq \frac{c}{\sqrt{\Delta}}, \forall v \in V$ for some constant c, we have

$$
\Pi_{w \in N(v)} (\frac{1}{1 + \lambda_w}) \geq e^{-c\sqrt{\Delta}}
$$

Remark 1 *Let P^{SS} denote the Markov operator corresponding to a single pass of the systematic scan chain forμ, and let $v := v_0 P^{SS}$. Then v is C-completely bounded with respect to μ with $C = e^{O(\sqrt{\Delta})}$.*

proof of Remark 1 The proof is similar to proposition 52 in [1]: For any $v \in V$, define

$$\gamma_v := \max_{\sigma^-} \frac{\nu(\sigma^+)}{\nu(\sigma^-)},$$

where the maximum ranges over independent sets σ^- in V which do not include v and where σ^+ denotes the set $\sigma^- \cup \{v\}$. For any configuration x, we have

$$\frac{\mathbb{P}(x \to \sigma^+)}{\mathbb{P}(x \to \sigma^-)} = \frac{\mathbb{P}_\mu(v \in \sigma|\sigma^+(v))}{\mathbb{P}_\mu(v \in \sigma|\sigma^-(v))} \Pi_{i=1}^k \frac{\mathbb{P}_\mu(u_i \notin \sigma|\sigma^+(u_i))}{\mathbb{P}_\mu(u_i \notin \sigma|\sigma^-(u_i))}$$

where $u_1, \cdots, u_k$ denote the neighbors of site v which are updated after site v in systematic scan, and we let the notation $\sigma^-(u_i)$ denote the configuration right before the systematic scan updates site u_i, given that for all previous sites it updated according to configuration σ^-, and similarly for the notation $\sigma^+(u_i)$. Since $\frac{\mathbb{P}_\mu(v\in\sigma|\sigma^+(v))}{\mathbb{P}_\mu(v\in\sigma|\sigma^-(v))} \leq \lambda_v$ and $\Pi_{i=1}^k \frac{\mathbb{P}_\mu(u_i\notin\sigma|\sigma^+(u_i))}{\mathbb{P}_\mu(u_i\notin\sigma|\sigma^-(u_i))} \leq \Pi_{i=1}^k(1+\lambda_i) \leq e^{c\sqrt{\Delta}}$, we have $\frac{\mathbb{P}(x\to\sigma^+)}{\mathbb{P}(x\to\sigma^-)} \leq \lambda_v e^{c\sqrt{\Delta}}$. Now consider any external field $(\theta_v)_{v\in V} \in [0,1]^{|V|}$, we have

$$\frac{(\theta * \nu)(v \in \sigma|\sigma_\Lambda) \cdot (\theta * \mu)(v \notin \sigma|\sigma_\Lambda)}{(\theta * \nu)(v \notin \sigma|\sigma_\Lambda) \cdot (\theta * \mu)(v \in \sigma|\sigma_\Lambda)} \leq \theta_v \lambda_v e^{c\sqrt{\Delta}} \cdot (\theta_v \lambda_v e^{-c\sqrt{\Delta}})^{-1} = e^{O(\sqrt{\Delta})}$$

1.1 balanced Glauber dynamics

Definition 4 (K-Balanced Glauber Dynamics) *For a distribution μ, at each time step, we keep track of a configuration σ_t and a tuple $(N_t(v))_{v\in V}$, described below. We initial $N_0(v) = 0$ for all $v \in V$. For each $t \geq 1$, sample a vertex I_t uniformly at random and update σ_{t-1} at I_t according to the distribution μ conditioned on $\sigma_{t-1,-I_t}$. Let $\sigma_{t,0}$ be the resulting configuration. We define $N_{t,0}(v) = N_{t-1}(v) + 1$ for each v adjacent to I_t and define $N_{t,0}(I_t) = 0$. For all other vertices u, we define $N_{t,0}(u) = N_{t-1}(u)$. Then for $j \geq 1$, as long as there is a vertex v with $N_{t,j-1}(v) > K\Delta$, we choose such a vertex with the smallest index (according to a fixed, but otherwise arbitrary ordering of the vertices) and resample $\sigma_{t,j-1}$ at v according to the distribution μ conditioned on $\sigma_{t,j-1,-v}$ to form $\sigma_{t,j}$. We then define $N_{t,j}$ by increasing $N_{t,j-1}$ at the neighbors of v by 1, setting $N_{t,j}(v) = 0$, and for all other vertices u, setting $N_{t,j}(u) = N_{t-1,j}(u)$. At j_t, when there are no vertices v with $N_{t,j_t}(v) > K\Delta$, we let $\sigma_t = \sigma_{t,j_{t-1}}$ and $N_t = N_{t,j_{t-1}}$.*

Theorem 3 (ATE by restrcited MLSI for field dynamics.) *Let μ be a probability distribution on $2^{[n]}$ which is completely (η, ϵ)-spectrally independent. Let $v \in \mathcal{V}^c(C, \mu)$ and $\theta \in (0,1)$, and let $f := dv/d\mu$. Suppose there exists $\kappa > 0$ such that for $\pi = \theta * \mu := (\theta, \cdots, \theta) * \mu$ and for all $R \subseteq [n]$ with $\pi(R) > 0$,*

$$\kappa \operatorname{Ent}_{\pi^{\mathbf{1}_R}}(f) \leq \mathcal{E}_{P_\theta}(f, \log f) \tag{4}$$

where P_θ is the transition matrix for the Glauber dynamics with respect to $\pi^{\mathbf{1}_\pi}$ and $\mathcal{E}_{P_\theta}$ is its corresponding Dirichlet form. Then we have

$$\mathrm{Ent}_\mu[f] \le C' \sum_{v\in[n]} \mathbb{E}_\mu[\mathrm{Ent}_v[f]]$$

where $C' = \frac{C}{\kappa n} \times \frac{1}{\Omega(\theta)^{O(\eta')}}$, with $\eta' = \max\{2\eta, \sqrt{log(C)/log(1+\epsilon)}\}$

Proposition 1 (MLSI in easy regime) *Let μ denote the hardcore model on a graph of maximum degree at most $\Delta \ge 3$, with fugacity $\lambda_v \le 1/2\Delta$ for all sites v, and let P denote the transition matrix of the Glauber dynamics. Then, the modified log-Sobolev constant $\rho_0(P)$ satisfies $\rho_0(P) \ge 1/4n$.*

Lemma 2 *For K-balanced Glauber dynamics P^{BG} and arbitrary distribution v that is absolutely continuous to μ, we have vP^{BG} is C-completely bounded with respect to μ for $C = e^{O(K\sqrt{\Delta})}$.*

proof. The proof follows a similar step as proof of Remark 1. Let ν_x denote the resulting measure on independent sets of G, starting from the initial distribution $\nu_0 = \mathbf{1}_x$, then it suffice to prove that

$$\frac{\nu_x(\sigma^+)}{\nu_x(\sigma^-)} \le \tilde{C}\lambda_v$$

for $\tilde{C} = e^{O(K\sqrt{\Delta})}$. Similar to the proof of Remark 1, we have

$$\frac{\nu_x(\sigma^+)}{\nu_x(\sigma^-)} \le \lambda_v \Pi_{i=1}^k (1+\lambda_{u_i}) \le \lambda_v (1+\frac{c}{\sqrt{\Delta}})^{K\Delta} \le e^{cK\sqrt{\Delta}}.$$

Lemma 3 (Lemma 55 in [1]) *let ν be the distribution of σ_{t-1} and let $f = d\nu/d\mu$ denotes the corresponding density. Define another Markov chain (ξ_t, N_t) where $\xi_0 \sim \mu$, and the coupled process $(\sigma_{t,i}, \xi_{t,i}, N_{t,i})$. Then, provided that the approximate tensorization of entropy estimate*

$$\mathcal{D}_{\mathrm{KL}}(\nu\|\mu) \le \frac{1}{C} \sum_{v\in V} \mathbb{E}_\mu[\mathrm{Ent}_v(f)]$$

holds for some $C > 0$, we have

$$\mathcal{D}_{\mathrm{KL}}((\sigma_{t,0}, N_{t,0})\|(\xi_{t,0}, N_{t,0})) \le \left(1 - \frac{C}{n}\right) \mathcal{D}_{\mathrm{KL}}((\sigma_{t-1}, N_{t-1})\|(\xi_{t-1}, N_{t-1})).$$

$$\mathcal{D}_{\mathrm{KL}}((\sigma_t, N_t)\|(\xi_t, N_t)) \le \left(1 - \frac{C}{n}\right) \mathcal{D}_{\mathrm{KL}}((\sigma_{t-1}, N_{t-1})\|(\xi_{t-1}, N_{t-1})).$$

Theorem 4 (Main theorem 1) *Let $G = (V, E)$ be a graph, and let μ be the Gibbs distribution for the hardcore model on G with fugacity λ with $\lambda \leq \frac{1-\delta}{2\sqrt{\Delta}-1}$. Let v_{-1} be an arbitrary initial distribution and $v_0 = v_{-1}P^{SS}$ to be the distribution obtained by running a single round of the systematic scan. let (σ_t, N_t) be defined by the K-balanced Glauber dynamics, as above, starting from the initial distribution $v_0 \times (0)_{v \in V}$. Then with probability $1 - o(1)$, for any $\epsilon > 0$,*

$$d_{TV}(v_T, \mu) \leq \epsilon$$

for all $T = e^{O(K\sqrt{\Delta})} \cdot \Omega_\delta(n \log[n\mathcal{D}_{KL}(v_{-1}\|\mu)/\epsilon])$

proof of Main theorem 1. By Lemma 2, the distribution ν_t is C-completely bounded with respect to μ for all $t \geq 0$ with $C = e^{O(K\sqrt{\Delta})}$. Let $f_t = d\nu_t/d\mu$, by Theorem 3 combined with the fact that μ is $(\frac{2}{\eta}, \frac{\eta}{2})$-completely spectrally independent of and the MLSI for Glauber dynamics in the easy regime from Proposition 1 (by choosing external field $\theta = \frac{c}{\sqrt{\Delta}}$, we have

$$\mathcal{D}_{\mathrm{KL}}(\nu_t\|\mu) \leq \mathcal{D}_{\mathrm{KL}}((\sigma_t, N_t)\|(\xi_t, N_t)) \leq \left(1 - \frac{C}{n}\right)^t \mathcal{D}_{\mathrm{KL}}(\nu_{-1}\|\mu),$$

where $C = e^{O(K\sqrt{\Delta})} \cdot (\sqrt{\Delta})^{\sqrt{K\sqrt{\Delta}}} = e^{O(K\sqrt{\Delta})}$.

Corollary 1

References

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