MATH 305

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Lecture 1

Why complex numbers?

Consider an equation of the form: $ax + b = 0$, where $a, b \in \mathbb{Z}$. The solution x does not always have an integer solution. This lead us to extend the set of numbers to include the rational numbers.

Consider the equation $x^2 - 2 = 0$. We know the solution is $\pm\sqrt{2}$, which is not a rational number. This lead us to extend the set of numbers to the real numbers $\mathbb{R}$.

Consider further $x^2 + 1 = 0$. The solution no longer belongs to the set of real numbers. To permit a solution, we expand the set of numbers to $\mathbb{C}$, the set of complex numbers.

The set of complex numbers, $\mathbb{C}$, is such that

- $\mathbb{C}$ extends $\mathbb{R}$, that is, $\mathbb{R} \subset \mathbb{C}$.
- Every polynomial with real coefficients have at least one root
- We can add and multiply complex numbers with the "usual rules" (distributivity, commutivity)

Complex numbers are also useful

- for handling periodic functions
- to compute integrals (some integrals cannot be computed using usual integral calculus methods)
- to compute sums (such as the harmonic series)
- to analyse dynamical systems, such as to determine stability of equilibrium
- in quantum mechanics

A complex number, $z \in \mathbb{C}$, is an expression of the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$, and i is the imaginary unit, $i = \sqrt{-1}$.

Complex arithmetic

Given $z = x + iy$, $w = u + iv$, addition:

$$z + w = (x + u) + i(y + v)$$

Multiplication:

$$z \cdot w = (xu - yv) + i(xv + yu)$$

Consider $i^2 = (0 + i1)^2$. Following the multiplication rule we just defined, we see $i^2 = -1$.

So $\pm i$ are two solutions of the equation

$$x^2 + 1 = 0$$

Here, we defined the operation to find that $i = \sqrt{-1}$. We could have also first defined i, then worked out the multiplication rule.

Lecture 2

Given a non-zero complex number $z = x + iy$, we have a multiplicative inverse:

$$z^{-1} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2}$$

such that

$$zz^{-1} = 1$$

Some more notation. Given $z = x + iy$, we define: $x = \text{Re}(z)$ and $y = \text{Im}(z)$. So $z = \text{Re}(z) + i\text{Im}(z)$.

$x + iy$ is naturally represented by a point in $(x, y) \in \mathbb{R}^2$. We describe the y-axis to be the "imaginary axis".

Using the representation of complex numbers as vectors in a complex plane, complex addition correspond to vector addition in $\mathbb{R}^2$.

But there is no suitable analogous vector multiplication operation to complex multiplication. This is why we work with complex numbers instead of representing everything with vectors.

We will see that multiplying complex numbers has to do with rotations and scaling in the plane.

The complex modulus:

$$|z| = \sqrt{z^2 + y^2}$$

The complex conjugate:

$$\bar{z} = x - iy$$

- Taking the complex conjugate is equivalent to making a reflection over the "real axis" (x-axis).

Useful properties:

- $|z|^2 = z\bar{z} = \bar{z}z$
  We can check that this is true:
  $$(x + iy)(x - iy) = (x^2 + y^2) + i(x(-y) + yx)$$
- $z^{-1} = \bar{z}/|z|^2$
  $$z\frac{\bar{z}}{|z|^2} = \frac{z\bar{z}}{|z|^2} = 1$$
- z is real iff $z = \bar{z}$
- z is purely imaginary iff $z = -\bar{z}$
- $\text{Re}(z) = (z + \bar{z})/2$
- $\text{Im}(z) = (z - \bar{z})/2i$

$$z - \bar{z} = 2iy$$

Triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Inverse triangle inequality:

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

Complex plane

Non-zero complex numbers can be expressed using complex coordinates

$$\begin{cases} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{cases}$$

where $r = |z|$, $\tan(\theta) = y/x$, and $\theta = \arg(z)$, $\theta \in [0, 2\pi)$. We do not consider $z = 0$. Since in this case $\arg(z)$ can take on any value.

We restrict $\arg(z) \in [0, 2\pi)$. This is one of many possible choices. This is the same as saying θ is always measured counterclockwise from the $+x$-axis.

For $x \in \mathbb{R}$, we define the complex exponential:

$$e^{ix} = \cos(x) + i\sin(x)$$

This is Euler's formula/identity. Here, we have made the choice to simply take this as an definition, arising from geometry.Properties:

- $z = |z|\exp(i\arg(z))$
  $$\begin{aligned} z &= r\cos(\theta) + ir\sin(\theta) \\ &= |z|(\cos(\theta) + i\sin(\theta)) \\ &= |z|e^{i\arg(z)} \end{aligned}$$
- $\exp(2\pi ik) = 1$ for all $k \in \mathbb{Z}$
- Since $\cos(\theta)$ and $\sin(\theta)$ are 2π periodic, we have
  $$e^{i(x+2\pi k)} = e^{ix}$$
  for all $k \in \mathbb{Z}$
- $\cos(x) = \left(e^{ix} + e^{-ix}\right)/2$
- $\sin(x) = \left(e^{ix} - e^{-ix}\right)/2i$

Given $z = |z|\exp(i\theta)$, $w = |w|\exp(i\phi)$,

$$\begin{aligned} zw &= |z||w|\left((\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi))\right) \\ &= |z||w|\exp(i(\theta + \phi)) \end{aligned}$$

We cannot assume that the exponential rules holds. Since the exponential rules were defined with real numbers. Formally, we use the trigonometric identity to show that this is the case.

We see that

- $|zw| = |z||w|$
- $\arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}$ such that $\arg(zw) \in [0, 2\pi)$

Lecture 3

Another standard choice for the argument is $(-\pi, \pi]$. When we choose this range,

$$\text{Arg}(z) \in (-\pi, \pi]$$

we use a capital A for the argument.

For example,

$$\text{Arg}(i) = \frac{\pi}{2} = \arg(i)$$

but

$$\text{Arg}(-i) = -\frac{\pi}{2}$$

while

$$\arg(-i) = \frac{3\pi}{2}$$

The choice of a certain argument can make our lives easier.

We can rederive trigonometric identities from the single Euler formula.

> The N-th root of unity is a complex number w such that $w^N = 1, \forall N \in \mathbb{N} = \{0, 1, \ldots\}$. Let $w_N = \exp(2\pi i/N)$, then
>
> $$1, w_N, w_N^2, \ldots, w_N^{N-1} \implies$$
> $$1^N, w_N^N, \left(w_N^2\right)^N, \ldots, \left(w_N^{N-1}\right)^N = 1$$
>
> are all the N-th roots of unity. (Since $w_N^N = 1$ we stop at $N-1$)

For $z \in \mathbb{C}$, $z = x + iy$,

$$e^z = e^{x+iy} = e^x\left(\cos(y) + i\sin(y)\right)$$

so the complex expontential is periodic in the imaginary direction.

- $|e^z| = e^{\text{Re}(z)}$

Sets and complex functions

Ω denotes a subset of $\mathbb{C}$.

A set is bounded if there exists $M > 0$, such that

$$|z| \le M, \forall z \in \Omega$$

There exists a circle of radius a finite radius M in the complex plane such that Ω is contained within the circle.

A set, Ω, is open if it does not contain its boundary, $\partial\Omega$. For any point in Ω, there is a disk of finite radius such that all the points enclosed by the disk is is still in Ω.

Let r be the the radius of a ball in the complex plane. The open ball $B_r(z) = \{w \in \mathbb{C} : |w - z| < r\}$.

The closed ball: $\{w \in \mathbb{C} : |w - z| \le r\}$ is not open. The closed ball is the union of Ω and $\partial\Omega$.

When we take the "closure" of a open set, we get a closed set.

A pointed disc:

$$\dot{B}_r(z) = B_r \backslash \{z\}$$

where the backslash denotes the difference of two sets.

An open set Ω is *connected* if there is a continuous path in Ω between any two points of Ω.

In two-D, we can imagine a connected set to be a continuous blob with a finite number of holes. A set is not connected when Ω looks like two or more distinct blobs.

A complex function, $f(z)$, where $z \in \Omega$, takes the form

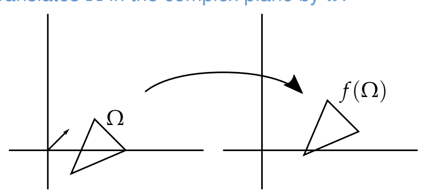
$$f(z) = \underbrace{u(z)}_{\text{Re}(f(z))} + i \underbrace{v(z)}_{\text{Im}(f(z))}$$

Lecture 4

A function $f : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ can be seen as a mapping that transform subsets of the complex plane. $f(\Omega)$ is also called the image of Ω under the mapping.

Some examples:

1. pick $w \in \mathbb{C}$ and define the complex function $f(z) = z + w$. This translates Ω in the complex plane by w.



2. Pick some $\phi \in \mathbb{R}$, and consider

$$f(z) = e^{i\phi}z$$

this rotates Ω by an ange ϕ about the origin in the complex plane

3. Pick a fixed $\lambda \in \mathbb{R}$, and define

$$f(z) = \lambda z$$

This "dilates" or compresses Ω by a factor of λ

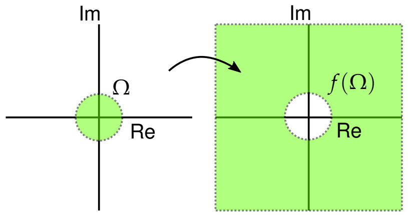
4. Consider $f(z) = z^{-1}$. If we pick $\Omega = \dot{B}_1(0)$ (pointed unit disk), what is $f(\dot{B}_1(0))$? We want to find the set of ζ such that $\zeta = z^{-1}$ for $z \ne 0$, $|z| < 1$. Since

$$|z| < 1$$

this implies that $|\zeta| > 1$. This means that the image of the pointed disk under f is

$$f(\dot{B}_1(0)) = \{\zeta \in \mathbb{R} : |\zeta| > 1\}$$

We had to use a pointed disk. Since 0^{-1} would be mapped to infinity. Since we defined the pointed disk to be an open set, the image will not contain the unit circle.



5. Consider

$$\widetilde{\Omega} = \{z \in \mathbb{C} : |z - 1| < 1\}$$

This is an open unit disk with its origin at $z = 1 + i0$. Under $f(z) = z^{-1} = \zeta$. We require all ζ to satisfy

$$|\zeta^{-1} - 1| < 1$$

we write $\zeta = u + iv$, and $\zeta = \frac{u - iv}{u^2 + v^2}$, and

$$|\zeta^{-1} - 1|^2 = \left|\left(\frac{u}{u^2 + v^2} - 1\right) + i\frac{v}{u^2 + v^2}\right|^2$$
$$= \left(\frac{u}{u^2 + v^2} - 1\right)^2 + \left(\frac{v}{u^2 + v^2}\right)^2$$
$$=$$

Applying the inequality $|\zeta^{-1} - 1| < 1$,

We arrive at

$$f(\widetilde{\Omega}) = \{\zeta \in \mathbb{C} : \text{Re}(\zeta) > 0.5\}$$

which is the half of the complex plane shifted by 0.5. f mapped the shifted unit disk to the half-plane.

6. *Joukowsky map*:

$$f(z) = z + z^{-1}$$

Let $z = x + iy$ and $\zeta = u + iv$, and

$$\zeta = f(z) = (x + iy) + \frac{x - iy}{x^2 + y^2}$$

So

$$u = x + \frac{x}{x^2 + y^2} \qquad v = y - \frac{y}{x^2 + y^2}$$

The circle

$$\Omega = \{e^{i\theta} : \theta \in [0, 2\pi)\}$$

is mappped to $f(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$. So this becomes a line going from -2 to 2 on the complex plane. The general circle

$$\{z_0 + re^{i\theta}, \theta \in [0, 2\pi)\}$$

where z_0 defines the origin of the circle, r defines the radius, this turns out to be mapped to a shape of an air foil

7. The complex function
$$f(z) = \frac{1}{-iz + 0.5}$$
where $\Omega = |z \in \mathbb{C} : \text{Im}(z) > 0|$ (The upper part of the complex plane). We can decompose $f(z)$ into three simpler functions ("Elementary maps").
$$f(z) = f_3(f_2(f_1(z)))$$
where
$$f_1(z) = -iz = e^{-i\pi/2}z$$
and
$$f_2(\zeta) = \zeta + 0.5$$
and
$$f_3(w) = w^{-1}$$
Under $f(z)$, Ω is
   (a) first rotated by clockwise by $\pi/2$ radians,
   (b) Translated to the right by $1/2$
   (c) The shifted half plane is mapped back to shifted open unit disk (see previous last examples)

Lecture 5

In the past lectures, we have looked at how complex functions are mappings on the complex plane.

Limits and continuity

We write
$$\lim_{z \to z_0} f(z) = L \qquad\qquad L \in \mathbb{C}$$
This means $f(z)$ and L can be made arbitrarily close, provided that z is close enough to z_0. Equivalently, $|f(z) - L|$ is very small, if $|z - z_0|$. (We are taking the modulus of the difference.)

There are different ways to for z to converge to z_0 in the complex plane. This means that $f(z)$ converges to L independent of how z approaches z_0.

Consider
$$\lim_{z \to i} \arg(z) = \frac{\pi}{2}$$
Consider
$$\lim_{z \to 1} \arg(z)$$
If z approches 1 from the upper half of the complex plane, then $\arg(z)$ approaches zero.

But if z approaches 1 from the lower half of the complex plane, thne $\arg(z)$ approches 2π. This means that this limit depends on the path z takes to approach 1. Therefore, this limit does not exist.

Consider
$$\lim_{z \to 1} \text{Arg}(z) = 0$$
By refining the discontinuity to the negative real axis, our limit exists.

A function is continuous at z if
$$\lim_{z \to z_0} f(z) = f(z_0)$$

> We need to compute the limit for every path z can take to approach z_0.

We can say that $f(z) = \arg(z)$ is continuous on $\mathbb{C} \backslash [0, \infty)$ (the difference between the set of complex numbers and the positive real axis).

$f(z) = e^z$ is continuous on $\mathbb{C}$

$f(z) = |z|^2$ is continuous on $\mathbb{C}$

$f(z) = (z - w)^{-1}$ is contiuous on $\mathbb{C}$ except at $z = w$.

$$f(z) = \begin{cases} z/|z| & z \neq 0 \\ 0 & z = 0 \end{cases}$$
is discontinuous at $z = 0$. Let's define
$$z_n = \frac{1}{n}e^{i\theta}$$
So the limit can be written as
$$\lim_{n \to \infty} f(z_n)$$
We see that
$$f(z_n) = \frac{e^{i\theta}/n}{1/n} = e^{i\theta}$$
The limit becomes
$$\lim_{n \to \infty} f(z_n) = e^{i\theta}$$
which is different for any given θ. So the limit does not exist.

Differentiability

A complex function f is differentiable at $z_0 \in \Omega$, if the following limit exists:
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

1. $f(z) = z^n$ for all $n \in \mathbb{N}$ is differentiable with
$$f'(z) = nz^{n-1}$$
2. $f(z) = \overline{z}$ is not differentiable anywhere. We can check that this is true. Let $z_0 = x + iy$, and $z = (x + h) + i(y + v)$, and define
$$R = \frac{f(z) - f(z_0)}{z - z_0} = \frac{((x + h) - i(y + v)) - (x - iy)}{h + iv} = \frac{h - iv}{h + iv}$$
   (a) When $v = 0, h \neq 0, h \to 0$, the limit approches 1
   (b) When $v \neq 0, v \to 0, h = 0$, the limit
$$\lim_{(0,v) \to (0,0)} R = -1$$
Thus the limit does not exist!

Lecture 6

As a vector field, $f(z) = \overline{z}$,
$$f(x, y) = (x, -y)$$
if perfectly differentiable.

- If a function f is differentiable at every $x \in \Omega$, f is *holomorphic* in Ω.
- We denote the set of holomorphic functions in Ω is denoted $H(\Omega)$.
- When $\Omega = \mathbb{C}$, we say that f is an *entire* function.

We assume that $f \in H(\Omega)$, f is composed as by a real part, u and an imaginary part v. We want to know what f being holomorphic means for u and v.

Let's define $z_0 \in \Omega$, $z_0 = x + iy$, so the following limit exists.
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
where $z = (x + a) + i(y + b)$. Consider the horizontal limit, where $b = 0$, and we let $a \to 0$.
$$z - z_0 = a$$
and
$$f(z) - f(z_0) = (u(x + a, y) - u(x, y)) + i(v(x + a, y) - v(x, y))$$
thus,
$$\begin{aligned} f'(z_0) &= \lim_{a \to 0} \left(\frac{1}{a} (u(x + a, y) - u(x, y)) + \frac{i}{a} (v(x + a, y) - v(x, y)) \right) \\ &= \partial_x u(x, y) + i\partial_x v(x, y) \end{aligned}$$
Let's consider the vertical limit. Where $a = 0$, and $b \to 0$. Following the same steps, we have
$$\begin{aligned} f'(z_0) &= \lim_{b \to 0} \left(\frac{1}{ib} (u(x, y + b) - u(x, y)) + \frac{1}{ib} (v(x, y + b) - v(x, y)) \right) \\ &= -i\partial_y u(x, y) + \partial_y v(x, y) \end{aligned}$$
By our assumption, since f is holomorphic, the vertical and horizontal limits must be one in the same. We match the real and imaginary parts, and we find that
$$\partial_x u(x, y) = \partial_y v(x, y)$$
and
$$\partial_x v(x, y) = -\partial_y u(x, y)$$

Differentiable at a point: A complex function $f = u(x,y) + iv(x,y)$ defined over Ω is differentiable at a point $z_0 = x + iy$ if the first partial derivatives of u and v exist and are continuous at z_0, and satisfy the Cauchy-Riemann equations at z_0

$$\partial_x u(x,y) = \partial_y v(x,y)$$
$$\partial_y u(x,y) = -\partial_x v(x,y)$$

Differentiable in Ω (Holomorphic): When the first partial derivatives of u and v exist, are continuous, and satisfy the Cauchy-Riemann equations for all $z \in \Omega$, then f is holomophic in Ω.

There must be a disk, such that f is differentiable at every point in the disk. (Ω cannot be a path or a point!)

Differentiable in $\mathbb{C}$ (Entire): When the first partial derivatives of u and v exist and are continuous for all $\mathbb{R}^2$, then f is entire.

Differentiation

The first derivative of $f = u(x,y) + iv(x,y)$

$$f'(z_0) = \partial_x u(x,y) + i\partial_x v(x,y)$$
$$f'(z_0) = -i\partial_y u(x,y) + \partial_y v(x,y)$$

1. Let's return to our example of $f(z) = \bar{z} = xi - y$. We can see that this is not differentiable from the Cauchy Euler equations.
2. Consider $f(z) = |z|^2$, where $u(x,y) = x^2 + y^2$, and $v(x,y) = 0$. Since v is zero everywhere, $\partial_x v$ and $\partial_y v$ is zero everywhere. But the partials of u is nonzero everywhere, except at $(0,0)$. This means $f(z)$ is only differentiable at $(0,0)$ or $z = 0 + 0i$.
3. $f(z) = e^z = e^{x+iy}$

$$e^x e^{iy} = e^x (\cos(y) + i\sin(y))$$

$u(x,y) = e^x \cos(y)$, and $v(x,y) = e^x \sin(y)$. By the Cauchy Euler equations, f is differentiable everywhere. So f is entire. We can compute it's first derivative,

$$f'(z) = \partial_x u + i\partial_x v = e^z$$

- Polynomials of z are entire.
- Ratio of polynomials in z, $f(z)/g(z)$, is holomorphic in $\{z \in \mathbb{C} : g(z) \neq 0\}$.

All rules of differentiation hold:

- $(f+g)'(z) = f'(z) + g'(z)$
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
- $f(g(z)) = f'(g(z))g'(z)$

Lecture 7

Recall that the real and imaginary parts of a complex number can be written as

$$x = \frac{1}{2}(z + \bar{z})$$

and

$$y = \frac{1}{2i}(z - \bar{z})$$

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \partial_x f \frac{\partial x}{\partial \bar{z}} + \partial_y f \frac{\partial y}{\partial \bar{z}} \\ &= \frac{1}{2}\left(\partial_x f - \frac{1}{i}\partial_y f\right) \\ &= \frac{1}{2}\left(\partial_z u + i\partial_x v - \frac{1}{i}\partial_y u - \partial_y v\right)\end{aligned}$$

by the Cauchy-Riemann equations, this equation comes out to be zero. So when f is holomorphic, then it is a function of z only, and not of $\bar{z}$.

The real and imaginary part of a holomorphic function are related by the Cauchy-Riemann equation. The real part determines the imaginary part, and vice versa, up to a constant.

For example:

$$u(x,y) = x^3 - 3xy^2 + y$$

find $v(x,y)$ such that $f = u + iv$ is entire. Knowing u, we can solve the Cauchy-Riemann equation for v.

$$\partial_x v = -\partial_y u$$

The y partial of u is:

$$-\partial_y = 6xy - 1$$

So

$$\frac{\partial v(x,y)}{\partial x} = 6xy - 1$$

Integrating both sides:

$$v(x,y) = 3x^2 y - x + C(y)$$

We still have a missing function of y. We use the other Cauchy Riemann equation.

$$\partial_y v = 3x^2 + C'(y)$$

and

$$\partial_x u = 3x^2 - 3y^2$$

Then,

$$C'(y) = -3y^2$$

Integrating both sides w.r.t. y,

$$C(y) = -y^3 + B \qquad B \in \mathbb{R}$$

Finally,

$$\begin{aligned}f(z) &= (x^3 - 3xy^2 + y) + i(3x^2 y - x - y^3 + B) \\ &= z^3 - i(z - B)\end{aligned}$$

Can any differentiable function $u(x,y)$ be the real part of a holomorphic function?

$$\partial_x(\partial_x u) = \partial_x(\partial_y v) = \partial_y(\partial_x v) = -\partial_y(\partial_y u)$$

So we have

$$\partial^2_{xx} u + \partial^2_{yy} u = 0$$

If $f \in H(\Omega)$, $f = u + iv$, then,

$$\nabla^2 u = 0$$
$$\nabla^2 v = 0$$

u and v are *harmonic* functions, as they satisfy Laplace's equation. u and v are *harmonic conjugates* of each other.

The level curves of harmonic conjugates are perpendicular:

$$\underline{\nabla} u \cdot \underline{\nabla} v = \underline{0}$$

Gradients are orthogonal to level sets.

- Given u, we can see if it is possible to construct v such that $u + iv$ is $H(\Omega)$ by checking whether u is harmonic
- Two arbitrary harmonic functions do not make the real and imaginary parts of a holomorphic function. They would also need to be related by Cauchy-Riemann equation

We have seen harmonic functions in electrostatics.

Elementary functions derived from the exponential

Properties of the complex exponential:

$$e^z = e^{x+iy}$$

- The range of e^z is $\mathbb{C}\backslash\{0\}$
- Is periodic along the imaginary direction:

$$e^z + i2\pi n = e^z \qquad \forall n \in \mathbb{Z}$$

- e^z is entire
- The power series representation:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \end{aligned}$$

  which is convergent for all $z \in \mathbb{C}$. In the sense of real analysis, e^z is analytic
- Trigonometric functions with a complex argument:

$$\begin{aligned} \cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\ \sin(z) &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \end{aligned}$$

Lecture 8

Differntiation rules for the real argument sine and cosine still hold. We can also check that the trigonometric identities also hold.

$$\begin{aligned} \cos'(z) &= -\sin(z) \\ \sin'(z) &= \cos(z) \end{aligned}$$

Similarly, the hyperbolic functions with a complex argument is still as defined:

$$\begin{aligned} \cosh(z) &= \frac{1}{2}\left(e^z + e^{-z}\right) \\ \sinh(z) &= \frac{1}{2}\left(e^z - e^{-z}\right) \end{aligned}$$

Hyperbolic trigonometric functions are rotated versions of the trigonometric functions in the complex plane.

$$\begin{aligned} \cosh(z) &= \cos(iz) \\ \sinh(z) &= -i\sin(iz) \end{aligned}$$

Example: Consider the following trigonometric equation.

$$\cos(z) = 2$$

There are no real number for which the equality holds. Let's convert this problem into the complex exponentials.

$$2 = \frac{e^{iz} + e^{-iz}}{2}$$

Rearranging:

$$0 = e^{iz} + e^{-iz} - 4$$

Factor out e^{-iz}:

$$e^{-iz}\left(e^{2iz} - 4e^{iz} + 1\right) = 0$$

Since e^{-iz} cannot be zero, we are free to divide both sides by it. We come to the equation:

$$e^{2iz} - 4e^{iz} + 1 = 0$$

We see that e^{2iz} is square of e^{iz}. Let's complete the square:

$$\left(e^{iz} - 2\right)^2 - 3 = 0$$

Rearranging:

$$\left(e^{iz} - 2\right)^2 = 3$$

Divide by 3 on both sides:

$$\left(\frac{e^{iz} - 2}{\sqrt{3}}\right) = 1$$

This implies that

$$\frac{e^{iz} - 2}{\sqrt{3}} = \pm 1$$

We come to

$$e^{iz} = 2 \pm \sqrt{3}$$

In real variables, we can take the logarithm to invert the exponential. But can we do this when we have a complex exponential?

Since the exponential is periodic, this would mean that there is an infinite number of solutions!

Let's write

$$e^{iz} = e^{-y}e^{ix} = 2 \pm \sqrt{3}$$

Since $2 \pm \sqrt{3}$ is real and positive, we know that e^{ix} must simply be 1. This would mean that

$$x = 2\pi n \qquad \forall n \in \mathbb{Z}$$

This leaves us with:

$$e^{-y} = 2 \pm \sqrt{3}$$

Which has the solution:

$$-y = \ln\left(2 \pm \sqrt{3}\right)$$

Finally, the set of solutions

$$\left\{2\pi n - i\ln\left(2 \pm \sqrt{3}\right) : n \in \mathbb{Z}\right\}$$

In the reals, the function that takes any $x \in \mathbb{R}$ to e^x is one-to-one. We can always invert one-to-one maps. We define the natural log to be

$$\ln\left(e^x\right) = x$$

In the complex plane, the complex function that takes z to e^z is periodic. Since any complex number that differ in the imaginary part by an integer multiple of 2π will have the same complex exponential. We this a many-to-one function.

The function $f(z) = e^z$ is many-to-one. Since $e^{z+i2\pi n} = e^z$. We define the *principal strip* to be

$$\Omega_p = \{z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi\}$$

Restricting the domain of e^z to Ω_p makes $f : z \to e^z$ one-to-one from $\Omega_p \to \mathbb{C}\backslash\{0\}$.

This lets us define a inverse mapping:

$$\text{Log} : \mathbb{C}\backslash\{0\} \to \Omega_p$$

And

$$\text{Log}(z) = \ln\left(|z|\right) + i\text{Arg}(z)$$

Let's check that

1. $\text{Log}(z) \in \Omega_p$ for any $z \in \mathbb{C}\backslash\{0\}$
2. $e^{\text{Log}(z)} = z$:

$$e^{\ln|z|z + i\text{Arg}(z)} = |z|e^{i\text{Arg}(z)} = z$$

3. What is $\text{Log}(e^z)$?

$$|e^z| = e^{\text{Re}\{z\}}$$

   and

$$\text{Arg}(e^z) = \text{Im}\{z\} + 2\pi n$$

   where n is chosen so that the argument evaluates to be in $(-\pi, \pi]$ by definition.

$$\begin{aligned} \text{Log}(e^z) &= \ln\left(e^{\text{Re}(z)}\right) + i\left(\text{Im}\{z\} + 2\pi n\right) \\ &= z + 2\pi in \end{aligned}$$

   When $z \in \Omega_p$, then n must be 0, so $\text{Log}(e^z) = z$.

Example: Solve

$$\text{Log}(z) = 2\pi i$$

There is no solution. Since the imaginary part of the Log must lie between $-\pi$ and π.

Lecture 9

Example: Let $-x$ be a negative real number. Equivalently, we can write

$$-x = xe^{i\pi}$$

The modulus of x is x, and $\text{Arg}(-x) = \pi$. The complex logarithm of $-x$ is then

$$\text{Log}(-x) = \ln(x) + i\pi \in \Omega_p$$

The complex logarithm permits negative real numbers.

- $\text{Log}(zw) = \text{Log}(z) + \text{Log}(w) + i2\pi n$, where $n = -1, 0, 1$ is chosen so that the imaginary part of $\text{Log}(zw)$ is within $(-\pi, \pi]$ This can be verified directly
- Log is discontinuous on $(-\infty, 0]$, where $\text{Arg}(z)$ is discontinuous. This discontinuity is called a *branch cut*.
- The end point of the branch cut, at $z = 0$ is called a *branch point*
- Away from the cut, Log is holomorphic, since e^z is holomorphic over Ω_p
- Derivative of $\text{Log}(z)$:

$$\text{Log}'(z) = \frac{1}{z}$$

  for all z not on a branch cut
- Another choice of the arugment would lead to other branch cuts
- If we chose $\arg(z) \in [0, 2\pi)$ instead of $\text{Arg}(z)$, we need to define Ω_p to be

$$\{z \in \mathbb{C} : 0 < \text{Im}(z) < 2\pi\}$$

  so that

$$\log(z) = \ln(|z|) + \arg(z)$$

- The complex Log provides two harmonic functions, its real and imaginary part.

Example: Find the steady state temperature distribution in the following situations.

1. An annulus of inner radius 1 and outer radius 2. With boundary conditions $T(1) = 20$ and $T(2) = -80$. Since the real part of $\text{Log}(z)$ is $\ln(|z|)$, it is constant along circles about the origin. Let's guess that the temperature distribution is given by

$$\phi(x, y) = A\ln(|z|) + B$$

   We know that $\ln(|z|)$ is harmonic, so it solves laplace's equation. Substituting the boundary conditions give

$$\phi(x, y) = \frac{100}{\ln(2)}\ln(r) - 80$$

2. An infinite wedge between $[\pi/4, -\pi/6]$ as measured from the positive real axis. The top edge has $T = 20$, and bottom edge has $T = -80$. We guess that

$$\phi(x, y) = C\text{Arg}(z) + D$$

   Substituting the boundary conditions:

$$C\frac{\pi}{4} + D = 20$$
$$-C\frac{\pi}{6} + D = -80$$

   gives us the constants. We cannot have used $\arg(z)$ instead. As $\arg(z)$ has a branch cut in the domain.

Roots

For any $\alpha \in \mathbb{C}$, we define

$$z^\alpha = e^{\alpha\text{Log}(z)} \qquad \forall z \in \mathbb{C}\backslash\{0\}$$

In the reals, we could have written

$$\sqrt{z} = \left(e^{\ln(x)}\right)^{1/2} = e^{\ln(x)/2}$$

We can use the exponential and the logarithm to define roots.

Example :

$$\begin{aligned}1^{1/2} &= e^{\text{Log}(1)/2}\\ &= e^{(\ln(1)+0)/2}\\ &= 1\end{aligned}$$

$$\begin{aligned}(-1)^{1/2} &= e^{\text{Log(-1)}}\\ &= e^{i\pi/2}\\ &= i\end{aligned}$$

$$\begin{aligned}i^i &= e^{i\text{Log}(i)}\\ &= e^{i(0+i\pi/2)}\\ &= e^{-\pi/2}\end{aligned}$$

Lecture 10

What happens if we decide to define the range of the logarithm to be

$$\{\pi \leq \text{Im}(z) < 4\pi\}$$

instead of the principle strip. Consider

$$\begin{aligned}1^{1/2} &= e^{\frac{1}{2}\left(\ln(1)+i\arg_{2\pi,4\pi}(1)\right)}\\ &= e^{i\pi}\\ &= -1\end{aligned}$$

The branch cuts of composed functions are "inherited" from the branch cut of the Log.

Example: Consider

$$z^{1/3} = e^{\frac{1}{3}\text{Log}(z)}$$

This function is holomorphic on

$$\mathbb{C}\backslash\{\text{Re}(z) \in (-\infty, 0]\}$$

Example: Find the branch cut of the following:

$$\text{Log}(1 - z^2)$$

$\text{Log}(z)$ has a branch cut on $\text{Re}\{z\} \in (-\infty, 0]$. So we need to look for which values of z, such that $\text{Log}(1 - z^2)$ is on the brach cut.

$$\begin{cases}\text{Im}(1 - z^2) = 0\\ \text{Re}(1 - z^2) \leq 0\end{cases}$$

Let's expand $1 - z^2$:

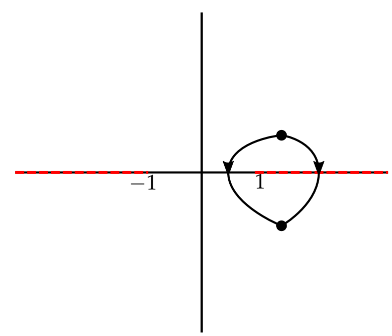
$$1 - z^2 = \underbrace{1 - x^2 + y^2}_{\text{Re}} - i\underbrace{2xy}_{\text{Im}}$$

So we need to solve

$$\begin{cases}xy = 0\\ 1 - x^2 + y^2 \leq 0\end{cases}$$

If $x = 0$, this means that $1 + y^2 \leq 0$. But y^2 is a positive number. So this case cannot be true.

If $y = 0$, this means that $1 - x^2 \leq 0$, or $x^2 \geq 1$.



We see a peculier property of complex functions. A particular math might involve jumping through a branch cut, whereas other paths do not.

Example: Find a branch of $(z^2-1)^{1/2}$, which is holomorphic on $\{|z|>1\}$. We can rewrite this function as

$$e^{\frac{1}{2}\text{Log}(z^2-1)}$$

This is the standard way of how we take the squre root. But we can show that this function has branchcuts on both the real and imaginary axis.

Of we can write

$$\begin{aligned}\left(z^2-1\right)^{1/2} &= \left(z^2\left(1-\frac{1}{z^2}\right)\right)^2\\ &= z\left(1-\frac{1}{z^2}\right)^{1/2}\\ &= ze^{\frac{1}{2}\text{Log}\left(1-\frac{1}{z^2}\right)}\end{aligned}$$

Let's check that

$$\left(ze^{\frac{1}{2}\text{Log}\left(1-\frac{1}{z^2}\right)}\right)^2 = z^2-1$$

which is true.

This new form of $(z^2-1)^{1/2}$ has is on the branch cut of Log(z) when

$$\begin{cases}\text{Im}\left(1-\frac{1}{z^2}\right)=0\\ \text{Re}\left(1-\frac{1}{z^2}\right)\le 0\end{cases}$$

So $1/z^2$ must be real and greater than 1:

$$\frac{1}{z^2}=\frac{(x^2-y^2)-2ixy}{|z^2|^2}$$

So

$$\begin{cases}xy=0\\ \frac{x^2-y^2}{|z^2|^2}\ge 1\end{cases}$$

When $y=0$, then

$$x^2\ge x^4 \iff x^2\le 1$$

So the branch cut is on the real axis, on the interval $[-1,1]$.

A diffent choice of taking $(1-z^2)^{1/2}$ result in different branch cuts.

We can also define $\log(z)$ as the set of solutions of

$$\{w\in\mathbb{C}: e^z=w\}$$

this is the multivalued log. Defined by the infinite set of values:

$$\text{Log}(z)=\ln(|z|)+i\left(\text{Arg}(z)+2\pi n\right)\qquad n\in\mathbb{Z}$$

Lecture 11

Integration

Complex integration always takes place along curves.

A *smooth parameterized curve*, α, is a map:

$$\alpha:[a,b]\to\mathbb{C},$$

where $[a,b]$ is a real interval. Such that:

- α is differentialable
- α is continuous
- $\alpha'(t)\neq\forall t\in[a,b]$
- α is *oriented* from $\alpha(a)$ to $\alpha(b)$
- α is *simple*, when for $a<t\neq s<b$, we have $\alpha(t)\neq\alpha(s)$ (it does not intersect it self)
- α is closed, if it begins and end at the same point

Here we define a "curve", but it's also called "path", "contour", "arc".

We know that different parameterizations can trace out the same path. They will not matter in integration.

Example: A horizontal segment from -1 to 2. One parameterization is

- $\alpha(t)=t$, for $t\in[-1,2]$
- $\alpha(t)=3t-1$ for $t\in[0,1]$

we go three times as fast in the second parameterization than we do in the first parameterization

Example: A vertical segment from $1-i$ to $1-3i$. Since the real part stays fixed:

- $\alpha(t)=1-it$ for $t\in[1,3]$

Example: A circle of radius r centered at $z_0\in\mathbb{C}$ with *positive orientation* (counterclockwise, righthand rule).

Let's first consider at circle at the origin. Any point on the circle can be traced out by

$$\alpha(t)=re^{it}$$

for $t\in[0,2\pi]$. So to shift the origin of the circle, we can simply add z_0 to α:

$$\alpha(t)=z_0+re^{it}$$

Consider a continuous complex function, f, defined over Ω. We also define a path α in the domain. We define the integral of f along α:

$$\int_\alpha f(z)\,dz=\int_a^b f(\alpha(t))\alpha'(t)\,dt$$

- The result only depends on the geometry of α, not the parameterization of α

Example: Consider α that goes from 1 to $2+i$ in a straight line. And the complex function $f(z)=\bar{z}$. (Recall that we know $\bar{z}$ is not holomorphic, but we it is continuous over α, so we can integrate.)

The first step is to parameterize α. Let's define for $t\in[0,1]$, $\alpha(t)=1+(1+i)t$. $\alpha'(t)=1+i$.

So the integral is

$$\begin{aligned}\int_\alpha f(z)\,dz &= \int_0^1 f(1+(1+i)t)(1+i)\,dt\\ &= \int_0^1 (1+(1-i)t)(1+i)\,dt\\ &= (1+i)\int_0^1 1+t-it\,dt\\ &= (1+i)\left(t+(1-i)\frac{t^2}{2}\right)_0^1\\ &= 2+i\end{aligned}$$

Example: Consider $z_0\in\mathbb{C}$, $n\in\mathbb{Z}$, and $\Omega=\mathbb{C}\backslash\{z_0\}$, $f(z)=(z-z_0)^n$ is holomorphic in Ω. α is a positively oriented circle of radius r, centered on z_0.

$$\alpha(t)=z_0+re^{it}$$

where $t\in[0,2\pi]$.

$$\alpha'(t)=rie^{it}$$

So the integral of $f(z)$ along α is

$$\begin{aligned}\oint_\alpha f(z)\,dz &= \int_0^{2\pi}\left(z_0+re^{it}-z_0\right)^n rie^{it}\,dt\\ &= \int_0^{2\pi}\left(re^{it}\right)^n rie^{it}\,dt\\ &= \int_0^{2\pi} r^ne^{nit}rie^{it}\,dt\\ &= ir^{n+1}\int_0^{2\pi}e^{(n+i)it}\,dt\end{aligned}$$

We need to consider two cases.

1. $n\neq -1$, this results in 0
2. $n=-1$

When $n=-1$ the integral reduces to

$$i\int_0^{2\pi}dt=2\pi i$$

We see that

$$\oint_{|z-z_0|=r}\frac{1}{z-z_0}\,dz=2\pi i$$

This result is independent of the radius.For $n\neq -1$, $(z-z_0)^n$ has an antiderivative along α,

$$\frac{1}{n+1}(z-z_0)^{n+1}$$

but not so for $n=-1$. Since the antiderivative of $(z-z_0)^{-1}$ is Log$(z-z_0)$, and it has a branch cut along the negative real axis.

Lecture 12

How do we integrate along piece wise smooth surves?For α that is made of a finite number of segments, such that the union of all the segments form a continuous curve, we define:

$$\int_\alpha f(z)\,dx = \sum_{i=1}^{n} \int_{\alpha_i} f(z)\,dz$$

The length of $\alpha : [a, b] \to \mathbb{C}$ is given by

$$\ell(\alpha) = \int_a^b |\alpha'(t)|\,dt$$

A useful bound:

$$\left| \int_\alpha f(z)\,dz \right| \leq M(f)\ell(\alpha)$$

where

$$M(f) = \max\{|f(z)| : z \in \alpha\}$$

The modulus of the path integral is always bounded from the above by the maximum modulus of the function along the path, times the length of the path.

Example: Consider $\alpha = Re^{it}$ for $t \in [0, \pi]$. This is a half circle of radius R. Estimate the modulus of the following:

$$\int_\alpha \frac{z^{1/2}}{1+z^2}\,dz$$

Let's first compute the length of the path. We expect it to be πR.

$$\begin{aligned}\ell(\alpha) &= \int_0^\pi |iRe^{it}|\,dt \\ &= R\int_0^\pi dt \\ &= \pi R\end{aligned}$$

Now, let's find $M(f)$.

$$\left|\frac{z^{1/2}}{1+z^2}\right| = \frac{|\exp(\mathrm{Log}(z)/2)|}{|1+z^2|}$$

Along α, we know that $|z|$ is always equal to R. So the numerator is at most be $\sqrt{R}$. We then need to get an lower bound on $|1+z^2|$.

$$|1+z^2| \geq |z^2| - 1 = R^2 - 1$$

Thus, maximum modulus is

$$M(f) = \frac{\sqrt{R}}{R^2-1}$$

and the bound on our integral is

$$\left|\int_\alpha \frac{z^{1/2}}{1+z^2}\,dz\right| \leq \pi R\frac{\sqrt{R}}{R^2-1}$$

We see that in the case for $R \to \infty$, the integral tends to zero!

A holomorphic function F is the antiderivative of f in a domain Ω is

$$f(z) = F'(z)$$

for all $z \in \Omega$.

Let α be a smooth curve, such that $\alpha(a) = z_i$, and $\alpha(b) = z_f$.

$$\frac{d}{dt}F(\alpha(t)) = F'(\alpha(t))\alpha'(t) = f(\alpha(t))\alpha'(t)$$

Let α be a curve in Ω; if f has an antiderivative F in a neighbourhood of α, then

$$\begin{aligned}\int_\alpha f(z)\,dz &= \int_a^b \frac{d}{dt}F(\alpha(t))\,dt \\ &= F(z_f) - F(z_i)\end{aligned}$$

- When α is a closed curve, the integral evaluates to zero
- The integral only depends on the end points of α
- For fixed z_f, z_i, we are free to change α and we will obtain the same results

This explains why

$$\oint_{|z-z_0|=r} (z-z_0)^n\,dz = 0$$

for $n \neq -1$. Since for $n \neq -1$, we have an antiderivative for $(z-z_0)^n$. But it does not explain the case where $n \neq -1$.

Lecture 13

Recall that if f has an antiderivative, F, and F is defined all along α, then

$$\int_\alpha f(z)\,dz = F(z_f) - F(z_i)$$

In particular, then α is closed, the line integral is 0.

Cauchy's theorem

Consider a disk, $B_r(z_0)$, and there is a function $f \in H(B_r(z_0))$. Then f has an unique antiderivative in $B_r(z_0)$ (up to a constant). If $F' = f$ and $G' = f$, then

$$(F-G)' = 0$$

which implies that $F - G$ is a constant, thus F and G differ only by a constant.

Cauchy's theorem guarentees the existance of an antiderivative of a function in a disk. This means that for $f \in H(B_r(z_0))$,

$$\oint_\alpha f(z)\,dz = 0$$

for any α in $B_r(t_0)$.

More generally, for any Ω, and $f \in H(\Omega)$, α is a simple closed curve in Ω, such that the interior of α is completely in Ω, then

$$\oint_\alpha f(z)\,dz = 0$$

and for any open, non-self-intersecting α,

$$\int_\alpha f(z)\,dz$$

is path independent.

When α encloses a region of Ω containing a hole, then the region nolonger holds.

Sketch of proof of Cauchy's theorem

We will proof Cauchy's theorem in the disk.

We first consider a triangle curve, τ, in $B_r(a)$, with perimeter L. We claim that

$$I = \oint_\tau f(z)\,dz = 0$$

By connecting the midpoint of each of the three segments of τ, we can partition τ into 4 new triangles. Let's denote the new triangles by a subscript, $\tau_1, \ldots, \tau_4$.

We can write I as

$$I = \sum_{k=1}^{4} \oint_{\tau_k} f(z)\,dz$$

This is true, since at shared boundaries, the direction of integration is always opposite, they cancel out.

Let $I^{(1)}$ be the integral along one of the four triangles with the largest modulus. Then, $|I|$ must be no greater than 4 times the modulus of $I^{(1)}$.

$$|I| \leq 4|I^{(1)}|$$

For the triangle whose path integral is is $I^{(1)}$, we can connect the midpoints of its three sides. This time, we know that

$$I^{(1)} = \sum_{k=1}^{4} \oint_i f(z)\,dz$$

And let $I^{(2)}$ be the integral with the largest modulus, out of the four integrals that sum to $I^{(1)}$.

For $|I^{(1)}|$, we can again construct a bound:

$$|I^{(1)}| \leq 4|I^{(2)}|$$

Relating this to our original bound on $|I|$, we see that

$$|I| \leq 4|I^{(1)}| \leq 4(4|I^{(2)}|)$$

If we repeat the same process, we can get another bound on $|I|$. If we repeat this n times, we have

$$|I| \leq 4^n |I^{(n)}|$$

Clearly, as we continue to divide triangles, $|I^{(n)}|$ decreases with increasing n. Then, our goal is to show that $|I^{(n)}|$ decreases faster than 4^n increases, and $|I|$ is zero.

In the limit as $n \to \infty$, the triangles will converge to a single point. Let's denote this point by z_0.

Now we use our assumption that $f(z)$ is holomorphic at z_0. This means that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Before taking the limit, we can write:

$$f(z) = f(z_0) + (z - z_0)f'(z) + r(z, z_0)(z - z_0)$$

where

$$r(z, z_0) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z)$$

Then, in the limit as $z \to z_0$, $r(z, z_0)$ must converge to zero. Let's try to integrate $f(z)$ along τ_n, a very small triangle with the maximum modulus after n division procedures, as given by this expression.

The first term must be zero:

$$\oint_{\tau_n} f(z_0)\, dz = f(z_0) \oint_{\tau_n} dz = 0$$

this is true for any closed curve.

The second term:

$$f'(z_0) \int_{\tau_n} (z - z_0)\, dz = 0$$

We know that $(z - z_0)$ has an antiderivative. So the closed line integral must also be zero.

The last term:

$$\int_{\tau_n} r(z, z_0)(z - z_0)\, dz$$

is more difficult. But we can get an upper and lower bound on its modulus. Recall that

$$\left| \int_\alpha f(z)\, dz \right| \leq M(f)\ell(\alpha)$$

where

$$M(f) = \max\{|f(z)| : z \in \alpha\}$$

The modulus of the path integral is always bounded from the above by the maximum modulus of the function along the path, times the length of the path.

In this case, we have

$$\left| \oint_{\tau_n} r(z, z_0)(z - z_0)\, dz \right| \leq M(r(z, z_0))M(z - z_0)\ell(\tau_n)$$

Every time we divide one triangle into four new triangles by the midpoint, its clear that its perimeter will be reduced by a factor of $1/2$. So after n divisions, τ_n has perimeter

$$\ell(\tau_n) = \frac{1}{2^n} L$$

We know that

$$M(z - z_0) < \frac{1}{2^n} L$$

So

$$\left| \oint_{\tau_n} r(z, z_0)(z - z_0)\, dz \right| \leq M_n \left(\frac{1}{2^n} L \right)^2$$

Now, as $n \to \infty$, we have M_n must go to 0.

Recall that we bounded

$$\begin{aligned} |I| &\leq 4^n |I^{(n)}| \\ &\leq 4^n M_n \frac{L^2}{4^n} = L^n M_n \end{aligned}$$

Thus, $|I| = 0$ for any $f(z)$ along any triangle in $B_r(a)$.

Lecture 14

Assuming that a function $f(z)$ has an antiderivative, we showed that the path integral around any triangle must be zero.

We will now compute the antiderivative. Let's define two "vectors" in $B_r(z_0)$.

- z goes from z_0 to z
- h goes from z to $z + h$

z and h. The two span a parallelogram, and thus a triangle.

We claim that the antiderivative of $f(z)$, $F(z)$, in the disk, is

$$F(z) = \int_{[z_0, z]} f(z)\, dz$$

(integral along the segement from z_0 to z).

We showed that

$$\oint_\tau f(z)\, dz = 0$$

We can write this as a sum of terms:

$$\oint_\tau f(z)\, dz = F(z) + \int_{[z, z+h]} f(z)\, dz + \int_{[z+h, z_0]} f(z)\, dz$$

We see that

$$\begin{aligned} 0 = \int_{[z+h, z_0]} f(z)\, dz &= -\int_{[z_0, z+h]} f(z)\, dz \\ &= F(z + h) \end{aligned}$$

We can parameterize $[z, z+h]$ as $\alpha(t) = z + th$, for $t \in [0, 1]$. Thus,

$$F(z + h) - F(z) = \int_0^1 f(z + th)h\, dt$$

Hence,

$$\frac{F(z + h) - F(z)}{h} = \int_0^1 f(z + th)\, dt$$

In the limit as $h \to 0$, we have

$$\begin{aligned} \lim_{h \to 0} \frac{F(z + h) - F(z)}{h} &= \int_0^1 f(z)\, dt \\ &= f(z) \end{aligned}$$

Since $F'(z) = f(z)$, $F(z)$ is the antiderivative of $f(z)$.

Example: Compute:

$$\int_{\mathbb{R}} e^{-x^2} \cos(2kx)\, dx \qquad k \in \mathbb{R}$$

Consider $f(z) = e^{-z^2}$, which is entire, and the contour, that is a rectangle, whose two diagonal vertices are on $z = -R + i0$ and $R + ik$.

By Cauchy's theorem, we know that

$$\int_\alpha f(z)\, dz = 0$$

for all $R > 0$. Expanding the integral:

$$\begin{aligned} 0 &= \int_{-R}^R f(x)\, dx + \int_0^k f(R + it)i\, dt \\ &- \int_{-R}^R f(x + ik)\, dx - \int_0^k f(-R + it)i\, dt \end{aligned}$$

In these types of problems, we hope that the original integrals we want shows up as one of the terms in the expansion, and that we know how to take care of the other integrals that show up.

Since our original integral was over $\mathbb{R}$, we take the limit as $R \to \infty$. The first integral is a Gaussian integral.

$$\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$$

The second integral

$$\lim_{R \to \infty} \int_0^k e^{-(R+it)^2} i\, dt$$

The modulus of this integral happens to be bounded above by zero.

The fourth integral is also bounded from the above by zero.

The third integral

$$\begin{aligned} \int_{-\infty}^\infty e^{-(x+ik)^2}\, dx &= \int_{-\infty}^\infty e^{-(x^2 - k^2)} \left(\cos(2kx) - i\sin(2kx)\right)\, dx \\ &= e^{-k^2} \int_{-\infty}^\infty e^{-x^2} \left(\cos(2kx) - i\sin(2kx)\right)\, dx \end{aligned}$$

In conclusion,

$$\int_{-\infty}^\infty e^{-x^2} \left(\cos(2kx) - i\sin(2kx)\right)\, dx = e^{-k^2}\sqrt{\pi}$$

we can equate the real and imaginary parts. We find that

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2kx)\,dx = e^{-k^2}\sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-x^2}\sin(2kx)\,dx = 0$$

Our result also shows that fourier transform of a gaussian is a gaussian.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{x^2/2}e^{ikx} = e^{-k^2/2}$$

To compute our desired integral, we needed to compute the third integral, along the contour from $R+ik$ to $-R+ik$. But instead of computing this integral, we consider the three other edges. This is also called computing by "shifting the contours".

Lecture 15

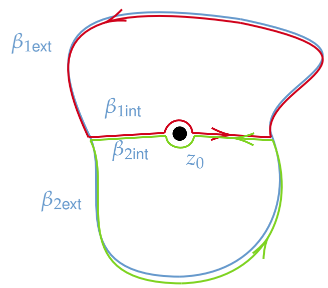
Recall that, for some $a \in \mathbb{C}$,

$$\oint_{|z-z_0|=r} \frac{a}{z-z_0}\,dz = 2\pi i a$$

This result was independent of the radius of the circle, r.

we see that Cauchy's theorem does not apply in this case. Since the function is undefined at $z = z_0$.

What about for another closed curve, β, about z_0?



Let's consider integration about positively oriented curves β_1 and β_2. By Cauchy's theorem, we know that

$$\oint_{\beta_1} \frac{a}{z-z_0}\,dz = \oint_{\beta_2} \frac{a}{z-z_0}\,dz = 0$$

We can decompose β_1 into the union of $\beta_{1\text{ext}}$, and $\beta_{1\text{int}}$. The same is true for β_2. So

$$\oint_{\beta_{1\text{ext}}} f(z)\,dz = -\oint_{\beta_{1\text{int}}} f(z)\,dz$$

Then, β is the union of $\beta_{1\text{ext}}$ and $\beta_{2\text{ext}}$.

$$\begin{aligned}\oint_{\beta} f(z)\,dz &= \oint_{\beta_{1\text{ext}}} f(z)\,dz + \oint_{\beta_{2\text{ext}}} f(z)\,dz \\ &= -\oint_{\beta_{1\text{int}}} f(z)\,dz - \oint_{\beta_{2\text{int}}} f(z)\,dz\end{aligned}$$

Since the straight parts of $\beta_{1\text{int}}$ and $\beta_{2\text{int}}$ are opposite in direction, their contributions will cancel out. All that is left is the the negatively oriented circle about z_0.

We find that

$$\oint_{\beta} f(z)\,dz = \oint_{|z-z_0|=\epsilon} f(z)\,dz$$

We already know the result.

We can think of what we are doing here as "deforming the contour", β to a little circle about z_0.

Integration along a simple closed curve about a singularity in Ω of a function $f(z)$ that is holomorphic everywhere in Ω except at the singularity, will yield the smae answer.

Cauchy's integral formula

- α is a simple closed curve
- $f(z)$ is holomorphic on α and in its interior, denoted int(α)
- the point w is in int(α)

Consider

$$g(z) = \frac{f(z)}{z-w}$$

$g(z)$ is holomorphic in $\Omega\backslash\{w\}$. We state that

$$\oint_{\alpha} g(z)\,dz = \oint_{|z-w|=\epsilon} g(z)\,dz$$

for any $\epsilon > 0$ that is "small enough".

Consider the circle of radius ϵ about w: $w + \epsilon e^{it}$, for $t \in [0, 2\pi]$.

$$\begin{aligned}\oint_{\alpha} g(z)\,dz &= \int_0^{2\pi} \frac{f(w+\epsilon e^{it})}{(w+\epsilon e^{it}) - w} i\epsilon e^{it}\,dt \\ &= i\int_0^{2\pi} f(w+\epsilon e^{it})\,dt\end{aligned}$$

In the limit as $\epsilon \to 0$, the integrand becomes $f(w)$. And we conclude that

Cauchy's integral formula:

$$f(w) = \frac{1}{2\pi i}\oint_{\alpha} \frac{f(t)}{z-w}\,dz \qquad w \in \text{int}(\alpha)$$

where α is a simple closed contour.

The values of f inside α are completely determined by the values of f along the curve α.

- α must be simple enough such that it winds around w only once