MATH 257

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Lecture 1

The difference between PDEs and ODEs is that PDEs have more than one independent variable.

Differential equations are equations that related the rate of change of a dependent variable to independent variables. We are interested in the function of dependent variable, not the function of the rate of change.

1st Order ODE

- Separable: $y' = P(x)Q(y)$
- Linear: $y' + P(x)y = Q(x)$

We will define the notation of a *differential operator* which map functions to functions involving derivatives. Let L be a differential operator,

$$L = \frac{d}{dx} + P(x) \tag{1}$$

such that $Ly = dy/dx + P(x)$. This is a notation that will come in handy later on.

We can solve a separable 1st order ODE by separating the differentials and integrating both sides.

$$\frac{dy}{dt} = P(x)Q(y)$$
$$\int_{y_0}^{y} \frac{1}{Q(y)}\,dy = \int_{x_0}^{x} P(x)\,dy$$

For example, to solve

$$\frac{dy}{dx} = y\cos(x)$$

We recognize that this is a separable equation,

$$\int \frac{1}{y}\,dy = \int \cos(x)\,dx + C$$
$$\ln|y| = \sin(x) + C$$
$$y = Ce^{\sin(x)} \qquad \text{General Solution}$$

To solve a first order linear ODE requires finding the integrating factor. We want a factor $\mu(x)$ such that multiplying both sides of the ODE by $\mu(x)$ resembles differentiating $\mu(x)y$.

$$y'\mu(x) + \mu'(x)y = \mu(x)Q(x)$$

This leads to the conclusion that $\mu'(x)$ must equal $\mu(x)P(x)$, which is a separable equation we can solve to obtain

$$\mu(x) = e^{\int P(x)\,dx} \tag{2}$$

We don't care about the the constants of integration since the factor in its current form satisfies out purpose. Then,

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$$
$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)\,dx$$

2nd Order Linear ODE

- Constant coefficient: $ay'' + by' + cy = 0$, a, b, c are constants
- Cauchy-Euler (Equidimentional): $x^2y'' + \alpha xy' + \beta y = 0$

We will define a second differential operator L where

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + C \tag{3}$$

Such that $Ly = ay'' + by' + cy$.

Solving the constant coefficient equation requires guessing a solution $y = e^{rx}$, where r is an unknown. Substituting in the higher derivatives of e^{rx}, we have

$$\begin{aligned} a(r^2e^{rx}) + b(re^{rx}) + c(e^{rx}) &= 0 \\ (ar^2 + br + c)e^{rx} &= 0 \\ ar^2 + br + c &= 0 \end{aligned}$$

The roots of the quadratic (characteristic) equation determines the two linear independent solutions of the ODE.

Let Δ be the discriminant,

$$\Delta = b^2 - 4ac$$

There are three cases

1. $\Delta > 0$: Two distinct real roots. Corresponding to

   $$r_1 = \frac{-b + \sqrt{\Delta}}{2a} \qquad r_1 = \frac{-b - \sqrt{\Delta}}{2}$$

   The general solution is

   $$y(x) = C_1e^{r_1x} + C_2e^{r_2x}$$

2. $\Delta < 0$: Two complex conjugate roots. Corresponding to

   $$r_{1,2} = \frac{-b}{2a} \pm \frac{i\sqrt{|\Delta|}}{2a} = \lambda + i\mu$$

   The general solution is

   $$y(x) = C_1e^{r_1x} + C_2e^{r_2x}$$

   We can use Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, to rewrite the general solution.

   $$\begin{aligned} y &= e^{\lambda}\left(C_1e^{i\mu x} + C_2e^{i\mu x}\right) \\ &= e^{\lambda x}\left[(C_1 + C_2)\cos(\mu x) + i(C_1 - C_2)\sin(\mu x)\right] \\ &= e^{\lambda x}(A\cos(\mu x) + B\sin(\mu x)) \end{aligned}$$

3. $\Delta = 0$: Only a single root. We use a trick to find the second solution.

   $$r_1 = \frac{-b}{2a}$$

   We guess the second solution to be $y_2 = xe^{r_1x}$. The general solution is

   $$y(x) = C_1e^{r_1x} + C_2xe^{r_1x}$$

   To derive the second solution, we can think of y as dependent on both x and r. Recall the differential operator.

   $$\begin{aligned} Ly(x, r) &= (ar^2 + b^2 + c)e^{rx} \\ &= a\left(r^2 - \frac{b}{a}r + \frac{c}{a}\right)e^{rx} \\ &\text{Completing the square,} \\ &= a\left[\left(r + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]e^{rx} \\ &= a\left(r + \frac{b}{2a}\right)^2 e^{rx} \\ &= a(r - r_1)e^{rx} \end{aligned}$$

   We have brought our equation to this simple form. Now the trick is derive this equation with respect to r.

   $$\frac{\partial}{\partial r}Ly = 2a(r - r_1)e^{rx} + ax(r - r_1)^2e^{rx}$$

   We see that when we plug $r = r_1$, the equation above evaluates to 0. Now we interchange the order of the differential operators

   $$\begin{aligned} \frac{\partial}{\partial r}Ly(r = r_1) &= 0 \\ L\left(\frac{\partial}{\partial r}y(r = r_1)\right) &= 0 \end{aligned}$$

   But this is the exactly the condition for a solution to the ODE. So the second solution is

   $$y_2 = \frac{\partial y}{\partial r}(r = r_1) = xe^{xr_1}$$

Solve the IVP:

$$y'' + 2y' + 5y = 0 \qquad y(0) = 0 \quad y'(0) = 2$$

Solving the characteristic equation gives $r_{1,2} = -1 \pm 2i$. The general solution is thus

$$y(x) = e^{-x}(A\cos(2x) + B\sin(2x))$$

We use our first initial condition to find that $A = 0$, which simplifies the equation. Deriving the simplified equation and plugging in the second initial condition gives $B = 1$, so our final answer is

$$y(x) = e^{-x}\sin(2x)$$

The Cauchy-Euler equation is a special 2nd order linear ODE with variable coefficient. The solution of the Cauchy-Euler equations also involves 3 cases. We guess that the solution has the form $y = x^r$. Substituting in the higher derivatives, we have

$$\begin{aligned} x^2(r^2 - r)x^{r-2} + \alpha xrx^{r-1} + \beta x^r &= 0 \\ (r^2 - r)x^r + \alpha rx^r + \beta x^r &= 0 \\ r^2 - (\alpha - 1)r + \beta &= 0 \end{aligned}$$

Let Δ be the discriminant.

$$\Delta = (\alpha - 1)^2 - 4\beta$$

The three cases are:

1. $\Delta > 0$: Two distinct real roots. The general solution is
$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$
2. $\Delta < 0$: There are 2 complex conjugate roots.
$$r_{1,2} = \lambda + i\mu$$
The general solution can be simplified to
$$y(x) = x^{\lambda}\left[A\cos(\mu\ln(x)) + B\sin(\mu\ln(x))\right]$$
3. $\Delta = 0$: There is a single root, but we can use our trick to finding a repeated solution.
$$y(x) = C_1 x^{r_1} + C_2 x^{r_1}\ln(x)$$

We know that the solutions to these ODEs should be linearly independent. Linear independence is checked by using the *Wronskian* $\neq 0$.

$$\mathbf{W}(y_1, y_2) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \tag{4}$$

Forcing: We will also consider a 2nd order linear constant coefficient ODE with forcing (such an equation is known as *inhomogeneous*), which has the form

$$y'' + by' + cy = F(x)$$

With the addition of the forcing term, we expect a solution general solution that consists of two linear independent terms (which forms the complimentary solution), and particular solution.

To find the general solution of these ODEs, we first solve the ODE without the forcing term, then take a guess to the particular solution (which as coefficients that are undetermined), plug the guess into the ODE to solve for the coefficients.

Let $F(x)$ be the forcing term, there are several cases depending on what F is

- Polynomial of order n: guess the solution to be polynomial of the same degree.
$$y_p = Ax^n + Bx^{n-1} + \cdots + D$$
If 0 is a solution to the characteristic equation, then multiply by x. If 0 is a repeated root, then multiply by x^2
- Exponential of form $Ce^{\tau x}$: guess a particular solution
$$y_p = Ae^{\tau x}$$
If τ is a root to the characteristic equation, multiply by x
- Trigonometric, for example $\cos(\omega x)$: choose a guess
$$y_p = A\cos(\omega x) + B\sin(\omega x)$$
If $\cos(\omega x)$ is a solution to homogeneous equation, then multiply by x
- Multiply the terms of the guess that is equal to a complementary solution by x until they are no longer equal.

Lecture 2

The Cauchy-Euler equation is a special 2nd order linear ODE with variable coefficient. We need other methods to solve linear ODEs. The we will use power series.

Power Series Infinitely differential and continuous functions allow for power series representations. For example, we can represent a function as a power series about $x = 0$, such that $f(x) = a_0 + a_1 x + a_2 x^2 \ldots$. Compactly,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

The coefficients are

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Which is formula for MacLaurin series. If we set the power series about another center, we get the Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

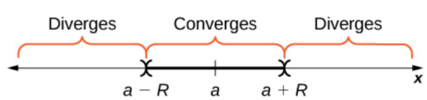
where

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

This isn't very useful when we do not know the function f explicitly - we need to determine the coefficients another way.

The *radius of convergence* of a power series is a number $R \geq 0$ such that the power series converges for all x such that $|x - x_0| < R$ and diverge for all x for $|x - x_0| > R$. The power series may or may not converge when $|x - x_0| = R$ (which requires manual checking).

The *interval of convergence* is all values of x for which the power series converges.



Finding R We use the *ratio test* find the radius of convergence. For the case that the power series is at $x = 0$, ratio test is such that

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n}\right| \tag{5}$$

This can be simplified to

$$\rho = |x - x_0| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

- If ρ is < 1, then the series converges
- If ρ is greater than 1, than the series diverges
- If ρ is equal to one, then the ratio test is inconclusive

We can define the constant τ

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

and $R = 1/\tau$.
This is true since if $\rho < 1$, then

$$|x - x_0| < \frac{1}{\tau}$$

And if $\rho > 1$, then

$$|x - x_0| > \frac{1}{\tau}$$

So $R = 1/\tau$ by definition. QED.

Some important MacLaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{6}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{7}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{8}$$

These three series converges for all values of x. These functions are called *analytic functions*.

Analytic Function: We say that $f(x)$ analytic at a point $x = x_0$ (notice that this is local definition) if $f(x)$ is infinitely differentiable at $x = x_0$, and $f(x)$ admits a power series representation which converges a radius of convergence.
$f(x)$ is *globally* analytic if it is analytic at $x = x_0$ for all x_0.
For a function $r(x) = P(x)/Q(x)$, r is analytic about $x = x_0$ if $P(x)$ and $Q(x)$ are analytic, and if the limit

$$\lim_{x\to x_0}\frac{P(x)}{Q(x)}$$

exists.

A single variable real function $f(x)$ is said to be analytic at $x = x_0$ if it has a Taylor expansion about x_0 which converges to $f(x)$ in some neighbourhood of x_0

$$\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x_0 - x)^n$$

Non C^∞ functions are naturally non-analytic. The set of analytic functions are a subset of C^∞ functions. In this sense, while all analytic functions, (We will denote as C^ω) are C^∞ functions. Not all C^∞ functions are C^ω functions

The hyperbolic functions have the following definitions

$$\sinh(a) = \frac{e^a - e^{-a}}{2} = \frac{e^{2a}-1}{2e^a} = \frac{1-e^{-2a}}{2e^{-a}} \tag{9}$$
$$\cosh(a) = \frac{e^a + e^{-a}}{2} = \frac{e^{2a}+1}{2e^a} = \frac{1+e^{-2a}}{2e^{-a}} \tag{10}$$
$$\tanh(a) = \frac{\sinh a}{\cosh a} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2a}-1}{e^{2a}+1} \tag{11}$$

The hyperbolic cotangent ($a \neq 0$), hyperbolic secant, and the hyperbolic cosecant ($a \neq 0$) are defined similar to the cotangent, secant, and cosecant functions.

Power series for hyperbolic cosine and sine functions are

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \tag{12}$$
$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{13}$$

$\cosh x$ is zero when $x = i\frac{\pi}{2} + \pi n$, $n \in \mathbf{Z}$. $\sinh x$ is zero when $x = i\pi n$, $n \in \mathbf{Z}$.

We can show that $y = e^x$ is a solution to $y' = y$ using power series. We know power series for e^x. We can differentiate term by term

$$y = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

The the first constant term vanishes, we start the indexing now at 1.

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(n+1)x^n}{(n+1)!s} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = y \end{aligned}$$

Series Solutions The idea is to solve ODEs by representing the unknown function y by a power series then use the ODE it self to gain information about the coefficients. We will get recursion on the coefficients (e.g. a_{n+1} will be expressed in terms of a_n). If the initial values are given (such as $y(0) = 0$..) these can often tell use about what a_0 and a_1.

We will use power series to solve $y' = 2y$. This ODE is can also be solved using separation variables (We will find that the solution is $y = C_0 e^{2x}$.)

Start by writing y as a power series.

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Notice we started indexing at $n = 1$ in y' since the series at $n = 0$ is simply 0.

Substitute into the ODE:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 2\sum_{n=0}^{\infty} a_n x^n$$
$$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = 2\sum_{n=0}^{\infty} a_n x^n$$

Rewriting,

$$\sum_{n=0}^{\infty} \left((n+1)a_{n+1} - 2a_n\right) x^n = 0$$

We reason that x^n terms cannot cancel each other out, which means that the coefficients must be zero

$$(n+1)a_{n+1} - 2a_n = 0$$

For each $n \geq 0$. We have the *recursion formula*,

$$a_{n+1} = \frac{2a_n}{n+1}$$

a_0 is free, we have

$$a_1 = \frac{2a_0}{0+1} = 2a_0$$
$$a_2 = \frac{2a_1}{1+1} = \frac{4a_0}{2}$$
$$a_3 = \frac{2a_2}{2+1} = \frac{8a_0}{3 \cdot 2 \cdot 1}$$

If we keep going we begin to see a pattern for $n \geq 1$.

$$a_n = \frac{2^n a_0}{n(n-1)\dots 1} = \frac{2^n a_0}{n!}$$

We see that this is also valid for $n = 0$.

Thus,

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \frac{2^n a_0}{n!} x^n \\ &= a_0 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \end{aligned}$$

The last term is equivalent to

$$y = a_0 e^{2x}$$

for any constant a_0. QED.

Even though we know we know that MacLaurin series for e^x is globally analytic, we can use to ratio test ot verify our results.

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}a_0}{(n+1)!} x^{n+1}}{\frac{2^n a_0}{n!} x^n} \right| = \lim_{n\to\infty} \left| \frac{2x}{n+1} \right|$$

We can see that for any fixed x, ρ must be less than 1, the convergence is true for each every x and $R = \infty$.

Solution to the Airy Equation Use power series to get the first terms of the Airy equation, $y'' - xy = 0$. The solutions are called the airy functions.

We begin by writing the power series for y, y', y'' about $x = x_0$.

$$y = \sum_{n=0}^{\infty} a_n (x)^n$$
$$y' = \sum_{n=1}^{\infty} a_n n (x)^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)(x)^{n-2}$$

Substituting into the equation,

$$\sum_{n=2}^{\infty} a_n n(n-1)(x)^{n-2} - x\sum_{n=0}^{\infty} a_n (x)^n = 0$$
$$\sum_{n=2}^{\infty} a_n n(n-1)(x)^{n-2} - \sum_{n=0}^{\infty} a_n (x)^{n+1} = 0$$
$$\sum_{n=-1}^{\infty} a_{n+3}(n+3)(n+2)(x)^{n+1} - \sum_{n=0}^{\infty} a_n (x)^{n+1} = 0$$

To equate the indices, we peel off a terms from the first sum

$$a_2(2)(1)x^0 + \sum_{n=0}^{\infty} (a_{n+3}(n+3)(n+2) - a_n)(x)^{n+1} = 0$$
$$2a_2 + \sum_{n=0}^{\infty} (a_{n+3}(n+3)(n+2) - a_n)(x)^{n+1} = 0$$

We have the equations that

$$a_2 = 0$$
$$a_{n+3}(n+3)(n+2) - a_n = 0$$

The recursive equation is

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)}$$

for $n \geq 0$.

Then,

$$\begin{aligned} a_0, a_1 &= \text{Free Parameters} \\ a_2 &= 0 \\ a_3 &= \frac{a_0}{(3)(2)} \\ a_4 &= \frac{a_1}{(4)(3)} \\ a_5 &= \frac{a_2}{(5)(4)} = 0 \\ a_6 &= \frac{a_3}{(6)(5)} = \frac{1}{(6)(5)} \frac{a_0}{(3)(2)} \end{aligned}$$

We see that coefficients $a_2, a_5, a_8, \ldots$ will be 0.

The solution will then take the form

$$y = a_0\left(1 + \frac{x^3}{(3)(2)} + \frac{x^6}{(6)(5)(3)(2)} + \ldots\right) + a_1\left(x + \frac{x^4}{(4)(3)} + \frac{x^7}{(7)(6)(4)(3)} + \ldots\right)$$

To check to convergence, we can use the ratio test

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+3}x^{n+3}}{a_n x^n}\right| < 1 \\ &= |x^3|\lim_{n\to\infty}\frac{a_n}{(n+3)(n+2)} = 0 < 1\end{aligned}$$

So the radius of convergence of y is ∞.

Lecture 3

This lecture we will go through singular points of a linear ODE, which will make the power series solutions more complicated.

Solve

$$(x-1)y'' + y' = 0$$

A tricky method ot solve this is to multiply both sides by $(x-1)$ to put the equation into the form of a shifted Cauchy-Euler equation.

We can guess a solution $y = (x-1)^r$.

$$\begin{aligned}(x-1)^2 r(r-1)(x-1)^{r-2} + (x-1)r(x-1)^{r-1} &= 0s \\ (r^2)(x-1)^r &= 0\end{aligned}$$

The characteristic equation has a single root $r = 0$. So we have the general solutions

$$\begin{aligned}y(x) &= C_1(x-1)^0 + (x-1)^0\ln(x-1) \\ &= C_1 + C_2\ln|x-1|\end{aligned}$$

Using a power series approach

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_{nx^n} \\ y' &= \sum_{n=1}^{\infty} na_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}\end{aligned}$$

Substituting and matching the exponents,

$$\begin{aligned}(x-1)\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{\infty} na_n x^{n-1} &= 0 \\ \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-1} & \\ + \sum_{n=1}^{\infty}(a_{n+1}(n+1)(n) + na_n)x^{n-1} &= 0 \\ (a_0(2)(1) + 1a_1)x^0 + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-1} & \\ + \sum_{n=2}^{\infty}(a_{n+1}(n+1)(n) + na_n)x^{n-1} &= 0\end{aligned}$$

Note that instead of peeling off a term in the second sum, we can decrement the counter in the first sum since the sum is zero when $n = 1$.

By decrementing the first counter,

$$\sum_{n=1}^{\infty}(n(n-1)a_n - (n+1)na_{n+1} + na_n)x^{n-1} = 0$$

So for $n \geq 1$,

$$n(n-1)a_n - (n+1)na_{n+1} + na_n = 0$$

Thus

$$a_{n+1} = \frac{na_n}{n+1}$$

Using this recursion formula, we get

$$\begin{aligned}a_2 &= \frac{a_1}{2} \\ a_3 &= \frac{2a_2}{2+1} = \frac{2}{3}\frac{a_1}{2} \\ a_4 &= \frac{3a_2}{3+1} = \frac{3}{4}\frac{2}{3}\frac{a_1}{2}\end{aligned}$$

We see the pattern that $a_n = a_1/a_n$ for $n \geq 1$.

The solution is then

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + \sum_{n=1}^{\infty} a_n x^n \\ &= a_0 + \sum_{n=1}^{\infty}\frac{a_1}{n}x^n \\ &= a_0 - a_1\ln|1-x|\end{aligned}$$

In the last step, we recognize that the series is a power series for $-\ln|1-x|$.

> To see that the series is a power series for $-\ln|1-x|$, we can simply recognize this, or use a geometric series to show this. Recall The geometric series
>
> $$\sum_{n=0}^{\infty} Ax^n = \frac{A}{1-x} \qquad |r| < 1$$
>
> Integrating the geometric series
>
> $$\int \frac{1}{1-x}\,dx = \int 1 + x + x^2 + x^3 + \ldots\,dx$$
>
> gives
>
> $$-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty}\frac{x^n}{n}$$

The radius of convergence for this answer is 1. This also happens to be the distance from the center $x = x_0$ to the nearest singularity at $x = 1$.

Let's attempt to solve the equation using a power series centred at $x_0 = 1$. We expect something to go wrong.

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n(x-1)^n \\ y' &= \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}\end{aligned}$$

Substituting in

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} = 0$$

Peeling a term off

$$\begin{aligned}a_1(x-1)^0 + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} & \\ + \sum_{n=2}^{\infty} na_n(x-1)^{n-1} &= 0\end{aligned}$$

We have the equations for $n \geq 2$

$$a_1 = 0 \qquad n^2 a_n = 0 \to a_n = 0$$

In the end

$$y = a_0 + \sum_{n=2}^{\infty} 0$$

Thus, we could only get the constant, and the power series failed to give the two independent solutions. The problems is that $x = 1$ is a singular point of the ODE.

Given an ODE of polynomial coefficients.

$$P(x)y'' + Q(x)y' + R(x) = 0$$

and the coefficients do not share a common factor, then a point $x = x_0$ is a singular point if $P(x_0) = 0$ but $Q(x_0) \neq 0$ or $R(x_0) \neq 0$.

Singular Points: In general, for a 2nd order linear ODE where P, Q, R are arbitrary functions of x,

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

We say $x = x_0$ is a singular point of the ODE if either Q/P or R/P is not analytic at $x = x_0$.
Any point is an ordinary point if it is not a singular point. We can find a solution using a power series if ordinary points. When $x = x_0$ is a singular point, then the power series solution may fail.
There are 2 classes of singular points

- Regular: using Frobenius series
- Irregular: solutions are beyond the scope

The motivation to be able to solve at non-singular points is Cauchy Euler Equation. Consider

$$(x - x_0)^2 y'' + \alpha(x - x_0)y' + \beta y = 0$$

We can find the singular points

$$y'' + \frac{\alpha(x - x_0)}{(x - x_0)^2}y' + \frac{\beta}{(x - x_0)^2}y = 0$$

And both these following terms are not analytic

$$\frac{\alpha}{(x - x_0)} \qquad \frac{\beta}{(x - x_0)^2}$$

Multiply both sides of the $y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)} = 0$ by $(x - x_0)^2$:

$$(x - x_0)^2 y'' + (x - x_0)^2 \frac{Q(x)}{P(x)}y' + (x - x_0)^2 \frac{R(x)}{P(x)} = 0$$

this resembles a Cauchy-Euler equation, where the second and third constants are now functions.

We say that x_0 is a regular singular point if both

$$p(x) = \frac{Q(x)}{P(x)}(x - x_0)$$
$$q(x) = \frac{R(x)}{P(x)}(x - x_0)^2$$

are analytic at $x = x_0$.
The moral is that singularities are regular singular points if they are are no worse than what we saw in Cauchy-Euler.
While Q/P, R/P, are not analytic, but p and q are, due the factors.

Suppose $x = x_0$ is a regular singular point, then we can write

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n$$
$$q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

We can write

$$(x - x_0)^2 y'' + (x - x_0)(p_0 + \text{H.O.T.})y' + (q_0 + \text{H.O.T.})y = 0$$

When we ignore the H.O.T. (higher order terms) terms, we have a Cauchy-Euler equation.

Indicial Equation: The *indicial* equation of a second order linear inhomogeneous ODE $(P(x)y'' + Q(x)y' + R(x)y = 0)$ is given by

$$r^2 + (p_0 - 1)r + q_0 \tag{14}$$

Where

$$p_0 = \lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \tag{15}$$
$$q_0 = \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \tag{16}$$

The equation has the same form irrespective of x_0.
The solutions to the indicial equation are called *exponents of singularity*.

The *Frobenius* method tells us that the solution to the original ODE (second order variable coefficient) at the **singular point** is the product between the a guess from the Cauchy-Euler equation and a correction term

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
$$= \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$$

To use the Frobenius solution,

1. Plug in the Frobenius series of $y(x)$ into $P(x)y'' + Q(x)y' + R(x)y = 0$
2. Rearrange the sum to find values of r; the indicial equation will appear in the terms you strip away, but only the terms with a_0 correspond to the indicial equation. The appearance of other terms with coefficient other than a_0 means that those coefficients are 0.
3. Find the recursion equation for coefficients a_n
4. Find the radius of convergence using ratio test

Convergence of Frobenius solutions: If x_0 is a regular singular point of a second order homogeneous ODE, $x > x_0$, then there exists a Frobenius series solution which converges for all x such that

$$0 < x - x_0 < R, \tag{17}$$

where R is the radius of convergence, and is least the distance to the nearest singular point.
This theorem, unlike what is for power series, does not guarantee convergence at x_0.

Lecture 4

The definition for singular points we have given are more general. In most cases, singular points occur when the denominator is zero.

Consider the equation

$$2x^2 y'' - xy' + (1 - x)y = 0$$

We rewrite this equation to eliminate the factor in front of the second derivative.

$$y'' - \frac{1}{2x}y' + \frac{(1 - x)}{2x^2}y = 0$$

By inspection, we can see that $x = 0$ is a singular point. Let's see if they are regular or irregular?

$$p(x) = \frac{1}{2x}(x - 0) \qquad q(x) = \frac{1 - x}{2x^2}(x - 0)^2$$

Since both p and q are analytic at $x = 0$, 0 is a regular singular point by definition.

We can now write out the Frobenius series and its first and second derivatives.

$$y = \sum_{n=0}^{\infty} a_n (x)^{n+r}$$
$$y' = \sum_{n=0}^{\infty} (n + r)a_n (x)^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n (x)^{n+r-2}$$

Substituting into the ODE,

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x)^{n+r-2}$$
$$-x\sum_{n=0}^{\infty}(n+r)a_n(x)^{n+r-1}$$
$$+(1-x)\sum_{n=0}^{\infty}a_n(x)^{n+r}=0$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x)^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n(x)^{n+r}$$
$$+\sum_{n=0}^{\infty}a_n(x)^{n+r}-\sum_{n=0}^{\infty}a_n(x)^{n+r+1}=0$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x)^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n(x)^{n+r}$$
$$+\sum_{n=0}^{\infty}a_n(x)^{n+r}-\sum_{n=1}^{\infty}a_{n-1}(x)^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(2(n+r)(n+r-1)-(n+r)+1)\,a_n(x)^{n+r}\right]$$
$$-\sum_{n=1}^{\infty}a_{n-1}(x)^{n+r}=0$$

$$(2(r)(r-1)-r+1)\,a_0(x)^r$$
$$+\sum_{n=1}^{\infty}2\left[((n+r)(n+r-1)-(n+r)+1)\,a_n(x)^{n+r}\right]$$
$$-\sum_{n=1}^{\infty}a_{n-1}(x)^{n+r}=0$$
$$(2(r)(r-1)-r+1)\,a_0(x)^r$$
$$+\sum_{n=1}^{\infty}[(2(n+r)(n+r-1)-(n+r)+1)\,a_n(x)^{n+r}$$
$$-a_{n-1}(x)^{n+r}]=0$$
$$(2r(r-1)-r+1)a_0x^r$$
$$+\sum_{n=1}^{\infty}[((2(n+r)(n+r-1)-(n+r)+1)\,a_n$$
$$-a_{n-1})(x)^{n+r}]=0$$

At this stage, we have power series that equals to zero. The only possibility is that all the powers are zero. Terms of different powers can cannot cancel each other out.

Thus we have

$$(2r(r-1)-r+1)a_0=0$$
$$(2(n+r)(n+r-1)-(n+r)+1)\,a_n-a_{n-1}=0$$

Find the roots of

$$2r^2-3r+1=0$$

The roots are $r_1=1/2$ and $r_2=1$.

$$a_n=\frac{a_{n-1}}{2(n+r)(n+r-1)-(n+r)+1}$$
$$=\frac{a_n}{(r+n)(2(r+n)-3)+1}$$

For $n\geq 1$.

> There are two cases to check. Either that the indicial equation is zero, or that a_0. We must consider both cases.

Now for the two roots, analyze the recursion equation. When $r=1$,

$$a_n=\frac{a_{n-1}}{(n+1)(2(n+1)-3)+1}$$
$$=\frac{a_{n-1}}{2n^2-n+2n-1+1}$$
$$=\frac{a_{n-1}}{(2n+1)n}$$

Our solution is

$$y_1(x)=x^1\sum_{n=0}^{\infty}a_nx^n$$
$$=x(a_0+a_1x+a_2x^2+\dots)$$
$$=x\left(a_0+\frac{a_0}{3}x+\dots\right)$$
$$=a_xx\left(1+\frac{x}{3}+\frac{x^2}{20}+\dots\right)$$

> Do not forget about the index n for which the recursive equation is defined. Do not forget that the solution by method of Frobenius is the usual power series solution, multiplied by a factor of $(x-x_0)^r$.

When $r=1/2$, we will find that

$$a_n=\frac{a_n-1}{n(2n-1)}$$

and

$$y_2=a_0x^{\frac{1}{2}}\left(1+x+\frac{x^2}{6}+\frac{x^3}{90}+\dots\right)$$

> **Lower bound of R:** Suppose that $x=x_0$ is an ordinary point. Then we have a usual power series solution.
> The radius of convergence for the solution is at least as large as the distance from the center $x=x_0$ to the nearest singularity of the ODE.
> When x_0 is a regular singular point, the solution by the *Frobenius* method has a radius of convergence that is at least as large as the distance to the nearest singular point.

Here is an example. The ODE

$$(x^2+1)y''+3xy'+5y=0$$

Say we have formulated a series solution to the ODE about $x=0$. The singularity of the ODE occurs when $x^2+1=0$, which on the complex domain occurs at $x=\pm i$. The *distance* to the nearest singularity is 1m so the radius of convergence is at least 1.

Here is another example. We want to find a lower bound on the radius of convergence for series representation of $y(x)$ centered at $x_0=2$ for the ODE

$$(4x^2+9)y''+x^3y'+y=0$$

Let us define functions p and q,

$$p(x)=\frac{x^3}{4x^2+9}\qquad q(x)=\frac{1}{4x^2+9}$$

At points $x=\pm i\sqrt{9/4}$, p and q are undefined. The lower bound of the radius of convergence is at least the distance between $x=2$ and $x=i\sqrt{9/4}$ or $x=2$ and $x=-i\sqrt{9/4}$.

$$\left\|\langle 2,0\rangle-\left\langle 0,\sqrt{9/4}\right\rangle\right\|=\sqrt{4+\frac{9}{4}}=\frac{5}{2}$$

Thus the radius of convergence is at least 5/2.

Lecture 5

We will introduce PDEs in this lecture. We begin by stating a notation for taking partial derivatives of a multivariate function. To take a partial with respect to an independent variable of x of a function $u(x,y,z,t)$, we will write

$$\frac{\partial u}{\partial x}=u_x$$

It is also useful to be reminded of the Laplacian operator, which in Euclidean space is

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\tag{18}$$

> **Laplacian in polar coordinates:** It is possible to derive the laplacian in polar coordinates via the chain rule. Let u be a function of (x,y). The polar coordinate transformation is
>
> $$x=r\cos(\theta)$$
> $$y=r\sin(\theta)$$
>
> Thus u can be written as $u(r\cos(\theta),r\sin(\theta))$. We will need to find u_r, u_{rr}, u_θ and $u_{\theta\theta}$. The formulae to find u_r is
>
> $$u_r=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$
>
> To find u_{rr}, the key is to recognize that even the x, y partials of u are still functions of x, y, so we must derive these partials with respect to x, y, then x, y with respect to r.
>
> $$u_{rr}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}\right)+\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right)$$
>
> The same goes of u_θ and $u_{\theta\theta}$.

In the PDEs we will discuss, we will reserve the symbol t in the independent variable for time. We will call variables that represent the spatial coordinates of the function as spatial variables.

The general form a first order linear PDE with 2 independent variables is

$$Au_x + Bu_y + Cu(x, y) = D \tag{19}$$

which is a *linear combination* of the possible partial derivatives of the function $u(x, y)$. The coefficients A, B, C, D do not have to be constants.

For a second order PDE with two independent variables, the general form is

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{20}$$

Again, the coefficients do not have to be constants.

We say that a second order PDE is homogeneous if G is zero. That is

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \tag{21}$$

We can categorize second order constant coefficient PDEs into three categories: parabolic; elliptic; hyperbolic.

Parabolic Equations:

$$u_t = u_{xx} \tag{22}$$

Examples are the heat equation, or the diffusion equation.

Elliptic Equations:

$$u_{xx} + u_{yy} = f(x, y) \tag{23}$$

Poisson's equation when $f \neq 0$, laplace's equation when $f = 0$.

Hyperbolic Equations:

$$u_{tt} - c^2 u_{xx} = 0 \tag{24}$$

As an introduction, we will study the *parabolic* PDE, the 1-D heat equation. The heat equation has the form

$$u_t = Ku_{xx}, \tag{25}$$

where K is some constant. We see that u depends on x and t. We call this "1-D" since it only depends on a single spatial dimension.

To derive this equation, we will need to know the Fourier law of heat transfer, as well as the conservation of energy. The scenario is a well insulated uniform rod that is heated in one end. There may or may not be generators of heat energy within the rod - these will matter little to our analysis.

We will define 3 quantities: thermal energy density; heat flux; internal heat generated.

Thermal energy density:we will call this quantity $e(x, t)$, which is a measure of the amount of heat energy at some position on the rod, with units J/m^3. For a small slice of the rod Δx, assuming that e is constant over this slice, the heat energy within the slice is approximately $e(x, t)A\Delta x$, where A is the cross-sectional area of the rod.

Heat flux:we will call this quantity $\phi(x, t)$, with units J/m^2, which represents the amount of heat energy that flows through a unit area per unit time. The positive direction of flux will be in the position x direction, or towards the right. The amount of heat energy which flows through a slice Δx is $\phi(x, y)A$.

Internal heat generated:we will call this quantity $Q(x, t)$, with units J/(s m^3), which is the amount of heat generated inside per unit volume, per unit time. The energy generated within a volume per unit time is then $Q(x, t)A\Delta x$.

Consider a slice of thickness Δx of the road. We reason that the amount of heat energy that enter at $x = a$ must equal the amount of heat energy that comes out the other end at $x = b$, where $\Delta x = b - a$.

$$\frac{d}{dt}\int_a^b e(x, t)\, dx$$

is the energy rate of change in energy density between $x = a$ and $x = b$. The conservation of energy says

$$\frac{d}{dt}\int_a^b e(x, t)\, dx = \phi(a, t) - \phi(b, t) + \int_a^b Q(x, t)\, dx \tag{26}$$

In that the time rate of change in thermal energy from a to b is equal to the difference in heat flux at a and b plus the heat generated from a to b.

For a function with continuous partials, we can bring the differential operator under the integral.

$$\frac{d}{dt}\int_a^b e(x, t)\, dx = \int_a^b \frac{d}{dt} e(x, t)\, dx \tag{27}$$

And by the fundamental theorem, we have

$$\phi(a, t) - \phi(b, t) = -\int_a^b \frac{d}{dx}\phi(x, t)\, dx \tag{28}$$

So we have

$$\int_a^b \frac{d}{dt} e(x, t)\, dx = -\int_a^b \frac{d}{dx}\phi(x, t)\, dx + \int_a^b Q(x, t)\, dx$$

We arranging, we have

$$\int_a^b \frac{d}{dt} e(x, t) - \frac{d}{dx}\phi(x, t) + Q(x, t)\, dx = 0$$

The integrand must be a constant or zero if the integral evaluates to zero. So

$$\frac{d}{dt} e(x, t) - \frac{d}{dx}\phi(x, t) + Q(x, t) = 0$$

With the current form, it still remains unclear as to how any of these quantities relate to temperature. Recall the formula $Q = mc\Delta T$, which says that the amount of heat energy Q is required to cause a temperature change of Δt in a material with mass m and specific heat c [J/kg·K] is the product of $mc\Delta T$. Thus

$$e(x, t) = \rho(x)(x)u(x, t)$$

, where $\rho(x)$ is the the density function, c is the specific heat function. In our case, these can be constants. So,

$$\frac{\partial e}{\partial t} = \rho(x)c(x)u_t$$

Next, the fourier law of heat transfer says that the heat flux is proportional the thermal conductivity of the material the negative temperature gradient.

$$\phi = -k_0 u_x$$

And differentiating both sides,

$$\frac{\partial \phi}{\partial t} = -k_0 u_{xx}$$

And finally,

$$c(x)\rho(x)u_t = k_0 u_{xx} + Q(x, t) \tag{29}$$

This equation reduces to the familiar seconder order PDE with constant coefficients when we take Q (internal heat generated) to be zero, and c and ρ to be constants.

$$u_t = \frac{k_0}{\rho c} u_{xx} \tag{30}$$

The equation in three dimensions is

$$c\rho u_t = k_0 \nabla^2 u + Q \tag{31}$$

At steady state, when $u_t = 0$, then

$$\frac{-Q}{k_0} = \nabla^2 u \tag{32}$$

which is a *Poisson equation*, an elliptic PDE.

With ODEs, picard's theorem of existence and uniqueness guarantees a unique solution given an initial condition. These initial conditions where often results numbers. With PDEs, the required number of specifications that makes the PDE have unique solution depends on the order and the domain of the PDE. There is no general rule.

It can be shown that for the heat equation, $u_t = u_{xx}$, a unique solution can be obtained by specifying the initial and boundary conditions.

$$u(x, 0) = f(x) \qquad \text{(Initial Condition)}$$

The *Dirichlet* conditions specify the temperature bound condition

$$\begin{aligned} u(0, t) &= g_1(t) \\ u(L, t) &= g_2(t) \end{aligned} \qquad \text{(Boundary Conditions)}$$

The *Neumann* conditions specifies the boundary conditions in terms of the derivatives of the temperature function (heat flux). Written using the Fourier law:

$$\begin{aligned} -k_0 u_x(0, t) &= \phi_1(t) \\ -k_0 u_x(L, t) &= \phi_2(t) \end{aligned}$$

If the boundaries are perfectly insulated, we might have

$$\begin{aligned} u_x(0, t) &= 0 \\ u_x(L, t) &= 0 \end{aligned}$$

Lecture 6

The boundary conditions for the heat equation gives the behaviour of the temperature function on the bound of its domains. (In the 1-D case, the domain would be $0 \leq x \leq L$, and $0 \leq t < \infty$.)

A Dirichlet condition of $u(0,t) = 0$, $u(L,t) = 0$ would mean that the temperature of the end of the rods for all time is 0. This is equivalent to having a cold bath at both ends.

A Neumann condition of $u_x(0,t) = 0$, $u_x(L,t) = 0$, would mean that there is no heat flow into and out of ends of the rod.

There is a third kind of the boundary condition that involves specifying both u and a derivative of u. An example might be $u(0,t) = 0$, $u_x(L,t) = 0$, which would represent that one end of the rod is in a cold bath, and the other end of the rod has no heat flowing out.

We want to find the *equilibrium solutions* to the PDE that does not depend on time. That is, the time derivative for such a solution will zero.

Consider the following example:

$$\begin{aligned}
u_t &= \alpha^2 u_{xx} \\
u(0,t) &= T_0 \\
u(L,t) &= T_L
\end{aligned}$$

, where T_0 and T_L are constants. Since the equilibrium solution is time independent, we must have $u_t = 0 = \alpha^2 u_{xx}$.

What sort of function whose second spatial derivative is zero? A linear function. We reason that the form of the solution is then $u = ax + b$. Using the boundary conditions, $u(0,t) = T_0 = b$, and $u = aL + T_0 = T_L$, so $a = (T_L - T_0)/L$. Thus the solution is

$$u = \frac{T_L - T_0}{L} x + T_0$$

Consider a second example given with Neumann boundary conditions.

$$\begin{aligned}
u_t &= \alpha^2 u_{xx} \\
u_x(0,t) &= 0 \\
u_x(L,t) &= 0
\end{aligned}$$

Again, we reason that the solution is a linear function $u = ax + b$, $u_x = a$. Using the boundary conditions, we have $a = 0$. But cannot tell from our boundary conditions the value of b, so b is a free parameter.

So the solution is $u = b$, which are simply horizontal lines. The solution is not unique. We will require initial conditions to find the b. This is true if we consider the total heat energy of the rod per unit area (J/m^2) as a function of time.

$$H(t) = c\rho \int_0^L u(x,t)\, dx$$

If we take its time derivative:

$$\begin{aligned}
H'(t) &= c\rho \frac{d}{dt} \int_0^L u(x,t)\, dx \\
&= c\rho\alpha^2 \int_0^L u_{xx}\, dx \\
&= c\rho\alpha^2 u_x|_0^L \\
&= 0 \quad \text{by boundary conditions}
\end{aligned}$$

So this would mean the heat energy $H(t)$ is constant over time, and $\lim_{t\to\infty} H(x) = H(0)$.

$$c\rho \int_0^L u(0,t)\, dx = c\rho \int_0^L \lim_{t\to\infty} u(x,t)\, dx$$

But recognize that $\lim_{t\to\infty} u(x,t)$ is simply the equilibrium solution, b, and $u(x,0)$ is an initial condition. So,

$$c\rho \int_0^L u(x,0)\, dx = c\rho \int_0^L b\, dx = c\rho L b$$

We conclude that

$$b = \frac{1}{L} \int_0^L u(x,0)\, dx$$

so b is dependent on the initial condition.

In three dimensions, the domain of the function is a portion in space, which we will represent using Ω. The boundary of surface of this volume is represented by $\partial\Omega$. We give the boundary condition that the heat flux into out of the volume is zero by saying that the dot product between gradient and unit normal vector pointing outwards from the surface is zero.

$$\nabla u \cdot \mathbf{n} = 0$$

We will consider a case where symmetry helps us solve for the equilibrium solution.

$$\Omega = \left\{1 \leq x^2 + y^2 \leq 4\right.$$

which is an annulus with a inner radius of 1 and a outer radius of 2. The equation and boundary conditions are

$$\begin{aligned}
&\nabla^2 u(x,y,t) = 4 \\
&u(1,\theta,t) = 2 \\
&u(2,\theta,t) = 5 - \ln(2)
\end{aligned}$$

Notice that the boundary conditions are given in terms of polar coordinates. We will need to convert our equation, which is given in Cartesian coordinates to polar coordinates.

Laplacian in polar coordinates:(for 2 variables)

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) \tag{33}$$

Then, our equation becomes

$$4 = \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

But, since domain is radially symmetric, we reason that the equilibrium solution would depend on only r (and not on t or θ). Which would mean that the second partial with respect to θ is zero.

$$4 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

Then,

$$\begin{aligned}
4r &= \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \\
\int 4r\, dr + C &= \int \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) dr \\
2r^2 + C &= r\frac{\partial u}{\partial r} \\
\int 2r + C\, dr + B &= \int \frac{\partial u}{\partial r}\, dr \\
r^2 + C\ln(r) + B &= u
\end{aligned}$$

Using boundary conditions, we can solve for C and B.

Consider a heat equation with the domain

$$\Omega = \left\{a^2 \leq x^2 + y^2 \leq b^2\right.$$

If we have the boundary conditions

$$u_r(a,t) = p \quad u_r(b,t) = 1$$

for what values of p does an equilibrium exist? We will use radial symmetry to approach this problem, which allows us to get,

$$\begin{aligned}
0 &= \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \\
C &= r\frac{du}{dr}
\end{aligned}$$

Substituting in the initial conditions, we will find that $C = b$, and it follows that $p = a/b$. So there is an equilibrium solution when $p = a/b$.

We can show that if we are given the boundary conditions and the initial condition:

$$\begin{aligned}
u_t &= \alpha^2 u_{xx} \\
u(0,t) &= 0 \\
u(L,t) &= 0 \\
u(x,0) &= 0
\end{aligned}$$

then $u = 0$ (identically zero).

This is known as the *energy method*. We begin by defining a function

$$E(t) = \int_0^L \frac{1}{2}u^2\, dx$$

Then its time derivative is

$$\begin{aligned}
E'(t) &= \frac{d}{dt}\int_0^L \frac{1}{2}u^2\, dx \\
&= \int_0^L \frac{d}{dt}\frac{1}{2}u^2\, dx \\
&= \int_0^L u u_t\, dx \\
&= \int_0^L u\alpha^2 u_{xx}\, dx \\
&= \alpha^2 \int_0^L u u_{xx}\, dx
\end{aligned}$$

Using integration by parts, let $u = u$, $du/dx = u_x$, $dv = u_{xx}\,dx$, $v = u_x$. Then

$$\alpha^2 \int_0^L u u_{xx}\,dx = \alpha^2 \left(u u_x |_0^L - \int_0^L u_x u_x\,dx \right)$$
$$= -\alpha^2 \int_0^L u_x^2\,dx$$

since the $uu_x|_0^L$ term is zero due to boundary conditions.

We have $E(t)$ which must ≥ 0, and $E'(t) \leq 0$. By the boundary conditions, we have

$$E(0) = \int_0^L \frac{1}{2} u(x,0)^2\,dx = 0$$

So $E(t) = 0$ for all t. (It starts at zero, has a negative slope all the time, and cannot be negative. The only option is then $E = 0$.) Which would mean that $u(x,t) = 0$ for all t.

Uniqueness of the 1-D heat equation: We can use the result of the energy method to show that **there can only be a single solution to the heat equation if initial and boundary conditions are provided**. Let v_1 and v_2 be two solutions to the following

$$v_t = \alpha^2 v_{xx}$$
$$v(0,t) = g(t)$$
$$v(L,t) = h(t)$$
$$v(x,0) = f(x)$$

If we define $u = v_1 - v_2$, then

$$u_t = \alpha^2 u_{xx}$$
$$u(0,t) = 0$$
$$u(L,t) = 0$$
$$u(x,0) = 0$$

since both v_1 and v_2 satisfy the initial conditions. This is the condition that we proved where $u = 0$. So $v_1 = v_2$, and there is only a single unique solution.

Here is another example of the energy method. Consider the function w, which is the solution to the constant coefficient ODE

$$aw'' + bw' + cw = 0$$

where we know that $a > 0$, $c > 0$, $b \geq 0$. We want to show that if $w'(0) = 0$, and $w(0) = 0$, then $w(t) = 0$ for all t.

We define the energy function $E(t)$,

$$E(t) = \frac{1}{2} a (w')^2 + \frac{1}{2} c (w)^2$$

It is easy to see that E must be ≥ 0 for all t.

Find the time derivative of E,

$$E'(t) = aw'w'' + cww'$$
$$= -b(w')^2$$

we can see that E' must be ≤ 0.

By the initial conditions, we have $E(0) = 0$. Since $E \geq 0$, yet has a 0 or negative slope everywhere, it must be that $E = 0$. Since $a > 0$ and $c > 0$, it must be that $w = 0$. QED.

Lecture 7

We will discuss the finite difference method - a numerical method to solving PDEs.

The key idea is to replace derivatives with *finite difference quotients*. Recall the limit definition of derivatives

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If Δx is sufficiently small, then the derivative is well approximated by

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference: Assume that $\Delta x \geq 0$. Now, we can use a taylor series expansion to find the value of the function at $x + \Delta x$.

$$f(x+\Delta x) = \sum_{n=0}^{\infty} \frac{f^n(x)}{n!} (\Delta x)^n$$
$$= f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \ldots$$

We can rearrange this to solve for $f'(x)$.

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{\Delta x}{2!} f''(x) - \frac{(\Delta x)^2}{3!} f^3(x) + \ldots$$

The first term on the right is the finite difference quotient, and the rest of the terms are the *truncation error*. We can use the "big O" notation to write the above as

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)$$

The term $O(\Delta x)$ represents that the error is primarily controlled by the term Δx. We say that the order of accuracy is of 1st order.

If the error were $O((\Delta x)^k)$, then we would say that this is a k^{th} order accuracy. The higher the order, the better the accuracy. Say if you have a 10^{th} order accuracy, reducing the time step by one half reduces the error by 2^{10} times.

How can we make it a second order accuracy? We present the *central difference* method.

The forward difference is

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \frac{(\Delta x)^3}{3!} f^3(x) + \ldots$$

the backward difference is

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) - \frac{(\Delta x)^3}{3!} f^3(x) + \ldots$$

Let use subtract the second equation from the first:

$$f(x+\Delta x) - f(x-\Delta x) = 2(\Delta x) f'(x) + \frac{2(\Delta x)^3}{3!} f^3(x) + \ldots$$

We can now rearrange both sides and divide both sides by $2\Delta x$ to solve for $f'(x)$.

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{(\Delta x)^2}{3!} f^3(x) + \ldots$$
$$= \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + O((\Delta x)^2)$$

which is of second order accuracy.

What if you added the forward and backward difference?

$$f(x+\Delta x) + f(x-\Delta x) = 2f(x) + \frac{2(\Delta x)^2}{2!} f''(x) + \ldots$$
$$= 2f(x) + (\Delta x)^2 f''(x) + \frac{2(\Delta x)^4}{4!} f^4(x) + \ldots$$

we see that that the odd power terms are eliminated. We can rearrange and solve for $f''(x)$:

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} + O((\Delta x)^2)$$

this is the central difference for the second derivative.

Solving the 1-D heat equation via the finite difference method. We solve it over a "grid".

$$u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad t > 0$$

The boundary conditions are

$$u(0,t) = 0$$
$$u(1,t) = 0$$

and the initial conditions are

$$u(x,0) = f(x)$$

this tells us the temperature of the rod at all space at time equals to zero.

1. Discretize time and space. We will denote $u(x_n, t_k) = u_n^k$, where x_n denotes the space step n, and t_k is time step k. n ranges from 0 to $1/\Delta x$, where 1 represents the length of the rod.
2. Approximate u_n^k for all n and k using a discretized the heat equation. We discretize u_t using the forward difference, and the u_{xx} using the central difference.

   $$u_t(x,t) \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

   $$\alpha^2 u_{xx} \approx \alpha^2 \left[\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2} \right]$$

   we will rewrite this as

   $$u(x,t+\Delta t)$$
   $$= u(x,t) + \frac{\alpha^2 \Delta t}{(\Delta x)^2} \left[u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t) \right]$$

   recall that $t_{k+1} = t_k + \Delta t$. Similarly, $x_{n+1} = x_n + \Delta x$. So we can convert this to an index notation

   $$u_n^{k+1} = u_n^k + \frac{\alpha^2 \Delta t}{(\Delta x)^2} \left[u_{n+1}^k - 2u_n^k + u_{n-1}^k \right]$$

3. Solve for these $u_n^k = u(x_n, t_k)$ recursively.

Numerical Stability: for the heat equation, we must have $\Delta t \leq (\Delta x)^2/2\alpha^2$ for this scheme to be numerically stable. If this is true, then the coefficients we multiply is

$$\frac{\alpha^2 \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

which helps with convergence.

Consider laplace equation

$$u_{xx} + u_{yy} = 0 \quad 0 < x < 1, \quad 0 < y < 1$$

We would specify 4 boundary condition.

$$\begin{aligned} u(x,0) &= f_1(x) \\ u(x,1) &= f_2(x) \\ u(0,y) &= g_1(y) \\ u(1,y) &= g_2(y) \end{aligned}$$

Using the central difference method, the two second partials in the equation is written as

$$\frac{u(x+\Delta x, y) - 2u(x,y) + u(x-\Delta x, y)}{(\Delta x)^2} + \frac{u(x, y+\Delta y) - 2u(x,y) + u(x, y-\Delta y)}{(\Delta y)^2} = 0$$

If we assume that $\Delta x = \Delta y$, then

$$u_{n+1,m} - 2u_{n,m} + u_{n-1,m} + u_{n,m+1} - 2_{n,m} + u_{n,m-1}$$

Using the short hand $u(x_n, y_m) = u_{n,m}$. We did not use a superscript here since superscripts are reserved for time.

We can rearrange solve for $u_{m,n}$.

$$u_{n,m} = \frac{u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1}}{4}$$

We do not know what is in the center of the grid, but we do know the boundary. For the central values, we can just set it to 0. We can start our algorithm over the boundary and build it up.

Note on uniqueness: The 1-D heat equation: $u_t = u_{xx}$, is specified by an initial condition $u(x,0) = f(x)$, and two boundary conditions $u(0,t) = g(t)$, $u(L,t) = h(t)$. We have two boundary conditions because there are two boundary points. We have a single initial condition since we have the first partial in time. There are two boundary conditions since there are two boundary point.
For the 1-D wave equation, $u_{tt} = u_{xx}$, we an initial condition of $u(x,0)$ and $u_t(x,0)$. We still need two boundary condition.

Lecture 8

We will talk about a method to solve the wave and heat equation.

Guessing solutions to the heat equation. Consider the heat equation $u_t = \alpha^2 u_{xx}$ on the real line, so $x \in (-\infty, \infty)$.

How can we find some solutions? Let us guess a solution

$$u(x,t) = e^{kx+\sigma t}$$

where k and σ are some real numbers. We want to find for what coefficients the heat equations is satisfied.

We compute the derivatives

$$\begin{aligned} u_t &= (\sigma)e^{kx+\sigma t} \\ u_x &= (k)e^{kx+\sigma t} \\ u_{xx} &= (k^2)e^{kx+\sigma t} \end{aligned}$$

If we substitute our guess into the heat equation

$$\begin{aligned} u_t - \alpha^2 u_{xx} &= 0 \\ (\sigma - \alpha^2 k^2)e^{kx+\sigma t} &= 0 \end{aligned}$$

Since the exponential is never zero, we know that $\sigma - \alpha^2 k^2 = 0$. So we have a solution, which is a product of a function of x and a function t

$$u(x,t) = e^{kx+\sigma t} = e^{kx} e^{\alpha^2 k^2}$$

which works for any t.

But the problem is that the solution is not physical. As $t \to \infty$, we would find that the exponential with time would go off to infinity for each fixed x.

What if we change our guess to a complex exponential?

$$u(x,t) = e^{ikx+\sigma t}$$

where k, σ are still reals.

Take the derivatives of our guess,

$$\begin{aligned} u_t &= (\sigma)e^{ikx+\sigma t} \\ u_x &= (ik)e^{ikx+\sigma t} \\ u_{xx} &= ((ik)^2)e^{ikx+\sigma t} = -k^2 e^{ikx+\sigma t} \end{aligned}$$

Substitute term into the heat equation:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \\ \sigma e^{ikx+\sigma t} &= \alpha^2(-k^2)e^{ikx+\sigma t} \end{aligned}$$

so we have $\sigma = -\alpha^2 k^2$. And we can rewrite the heat equation as

$$u(x,t) = e^{ikx} e^{-\alpha^2 k^2 t}$$

Comparing our previous guess with the current one, we see that the current one will decay to zero as $t \to \infty$ - this makes more physical sense. In both cases, if we fix k, we have a solution to the heat equation. For any real k, we have a solution.

Let us call our current solution u_1. If we replace k with $-k$, the solution still solves the heat equation. We will call this u_2.

Using euler's identity, we can write the solution as

$$\begin{aligned} u_1(x,t) &= [\cos(kx) + i\sin(kx)]\, e^{-\alpha^2 k^2 t} \\ u_2(x,t) &= [\cos(kx) - i\sin(kx)]\, e^{-\alpha^2 k^2 t} \end{aligned}$$

By taking $(u_1 + u_2)/2$ and $(u_1 - u_2)/2i$, we get solutions of the form

$$\begin{aligned} u_3(x,t) &= \cos(kx)e^{-\alpha^2 k^2 t} \\ u_4(x,t) &= \sin(kx)e^{-\alpha^2 k^2 t} \end{aligned}$$

These solutions solve the heat equation even without boundary conditions and initial conditions.

Separation of variables: for a *linear and homogeneous* PDE with *linear and homogeneous* boundary conditions, it may be possible to solve the heat equation by guessing a solution $u(x,t) = X(x)T(t)$.

Consider the heat equation

$$\begin{aligned} u_t &= \alpha^2 u_{xx} && 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0 && \text{for all } t \geq 0 \\ u(x,0) &= f(x) && \text{for } 0 < x < L \end{aligned}$$

We need to also have $f(0) = 0$ and $f(L) = 0$ if the boundary conditions are for all $t \geq 0$, else $t > 0$.

We guess a solution that is the product of a function of time and function of space.

$$u(x,t) = X(x)T(t)$$

We can find the derivatives of our guess. We will denote a time derivative with dot, and a spatial derivative with a prime.

$$\begin{aligned} u_x &= X'T \\ u_{xx} &= X''T \\ u_t &= X\dot{T} \end{aligned}$$

Substitute into the heat equation

$$\begin{aligned} X(x)\dot{T}(t) &= \alpha^2 X''(x)T(t) \\ \frac{X''}{X(x)} &= \frac{\dot{T}(t)}{\alpha^2 T(t)} = \mu \end{aligned}$$

How can a function of x be equal to functions of t? The functions must be equal to some real constant, let's call it μ.

$$\begin{aligned} X''(x) &= \mu X(x) \\ \dot{T}(t) &= \mu\alpha^2 T(t) \end{aligned}$$

Look how we have replaced the origin PDE with 2 ODEs.

We can T ODE using a separation of variables

$$\int \frac{1}{T}\, dT = \mu\alpha^2 \int dt + C$$

we have $T(t) = C\exp\left(\mu\alpha^2 t\right)$.

For the X ODE, let us impose the boundary conditions.

$$\begin{aligned} 0 &= u(0,t) = X(0)T(t) \\ 0 &= u(L,t) = X(L)T(t) \end{aligned}$$

but $T(t)$ is never zero. So this forces $X(0) = X(L) = 0$.

We have an *eigenvalue problem*:

$$\begin{aligned} X''(x) - \mu X(x) &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

We want to find μ and a non zero $X(x)$ so which satisfies this condition. This is an eigenvalue problem since we can think of $X(x)$ as an eigenvector of the second derivative operator $\mathcal{L} = d^2/dx^2$.

$$\mathcal{L}X = \mu X$$

and μ is an eigenvalue. We require the eigenvectors (or we can call it an eigenfunction) to be non zero.

We will now solve this problem. We have 3 cases, when $\mu > 0$, when $\mu = 0$, when $\mu < 0$.

When $\mu > 0$. We write $\mu = \lambda^2$. We have

$$\mathcal{L}X - \lambda^2 X = 0$$

We guess $X = \exp(rx)$, and we have

$$r^2 - \lambda^2 = 0$$

So we have a solution

$$X = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

We can rewrite these in terms of the hyperbolic sine and cosine.

$$X(x) = A_1 \cosh(\lambda x) + A_2 \sinh(\lambda x)$$

We can figure out the constants A_1 and A_2 using the boundary conditions:

$$X(0) = A_1 \cosh(0) + A_2 \sinh(0) = 0$$

we obtain $A_1 = 0$. And

$$X(L) = 0 = A_2 \sinh(\lambda L) = 0$$

We know that $\lambda \neq 0$, and $L > 0$, this means that $\sinh(\lambda L) \neq 0$. So the only option is that $A_2 = 0$, so $X(x)$ is identically zero. There are no eigenfunctions in this case.

If $\mu = 0$, then the form of X must be linear since its second derivative results in 0.

$$\begin{aligned} X(x) &= Ax + B \\ X(0) = 0 &= A(0) + B \\ X(L) &= A(L) = 0 \end{aligned}$$

so both A and B are zero, thus there are no eigenfunctions in this case.

For $\mu < 0$. Let use write $\mu = -\lambda^2$ for $\lambda > 0$. The equation comes

$$X'' + \lambda^2 X = 0$$

Substitute $X = \exp(rx)$, and we find $r = \pm i\lambda$ to get

$$\begin{aligned} X(x) &= C_1 e^{i\lambda x} + C_2 e^{-i\lambda x} \\ &= A_1 \cos(\lambda x) + A_2 \sin(\lambda x) \end{aligned}$$

Using the first initial condition:

$$X(0) = A_1 \cos(0) + A_2 \sin(0) = 0$$

So

$$X(x) = A_2 \sin(\lambda x)$$

since we want a non zero solution, we may assume that $A_2 = 0$.

$$X(L) = 0 = A_2 \sin(\lambda L) = 0$$

So if $\sin(\lambda L) = 0$, then $\lambda L = \pi n$.

This is where our assumption of $\lambda > 0$ comes in. Since $\lambda > 0$ we know n is positive integer.

$$\lambda = \frac{\pi n}{L}$$

and thus

$$\mu = -\lambda^2 = -\left(\frac{\pi n}{L}\right)^2$$

So we have

$$X(x) = A_2 \sin\left(\frac{n\pi x}{L}\right)$$

Which is an eigenfunction. Here, A_2 is some arbitrary constant. It simply scales the eigenfunction. A scaled eigenfunction is still an eigenfunction. We will ignore A_2 since we will just take linear combinations later.

So we have a sequence of eigenfunctions:

$$\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

with the corresponding eigenvalues

$$\mu_n = -\left(\frac{\pi n}{L}\right)^2$$

And we have

$$u(x,t) = X(x)T(t) = \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t\right)$$

for any $n = 1, 2, 3, \ldots$.

Since the heat equation is linear, we can take any infinite linear combinations

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t\right)$$

which solves the heat equation and satisfies the boundary conditions by construction.

What about the initial condition? The only parameters we can vary is b_n. We substitute $u(x,0) = f(x)$ into the general form, and the time term vanishes:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi n}{L}\right)$$

The task then is given the function $f(x)$, we need to find the coefficients of $\{b_n\}_{n=1}^{\infty}$.

This leads to the subject of Fourier series. The goal is to represent functions as infinite sums involving trigonometric functions.

Given two vectors $\mathbf{v}, \mathbf{w}$, we say that the two vectors are orthogonal if their inner product is zero. In general, if the vectors $\mathbf{v}_n$, $n = 1, 2, 3, \ldots$ are mutually orthogonal if the inner product of v_i, v_j is zero for all $i \neq j$.
Consider the space of square integrable functions. We say that a function $f(x)$ is square integrable if over $[0, L]$ if

$$\int_0^L f(x)^2 \, dx < \infty$$

If we have functions f_1, f_2 on $[0, L]$, then we define the inner product (denoted $\langle ., . \rangle$) as

$$\langle f_1, f_2 \rangle = \int_0^L f_1(x) f_2(x) \, dx \tag{34}$$

It turns our that

$$\sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \cos\left(\frac{n\pi x}{L}\right)$$

for $n = 0, 1, 2, 3, \ldots$ form an orthogonal system for all square integrable function over $[-L, L]$.
That is

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 \tag{35}$$

for all m, n, including when $m = n$.
Over the *half range* $[0, L]$, we also have

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases} \tag{36}$$

We also have

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases} \tag{37}$$

We can remember that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}$$

by remember that the average value of $\sin^2(x)$ is $1/2$ over $[0, L]$.

$$\frac{1}{L} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}$$

Using the above facts, we can solve for b_n. **Assuming that $f(x)$ is**

square integrable, we integrate $f(x)$ against

$$\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \sum_{m=1}^{\infty}\int_0^L b_m \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

By orthogonality, the integral is zero for $m \neq n$. So only the $m = n$ term survives

$$\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^L b_n \sin^2\left(\frac{n\pi x}{L}\right)dx = b_n\frac{L}{2}$$

Rearranging gives the coefficients in the Fourier sine series

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{38}$$

We do an example:

$$\begin{aligned} u_t &= u_{xx} && 0 < x < L \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= x \end{aligned}$$

We will need to represent the function x by the Fourier sine series.

We find b_n by computing

$$b_n = 2\int_0^1 x\sin(n\pi x)\,dx = \frac{-2\cos(n\pi)}{n\pi} = \frac{-2(-1)^{n+1}}{n\pi}$$

via integration by parts.

So the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)e^{-(n\pi)^2 t}$$

as $t \to 0$, $u(x,t) \to 0$.

However, while this solution converges over $(0,1)$, it does not satisfy the initial conditions at the end points. (According to IC, $u(1,0) = 1$, but the series solution will be zero.)

Facts:

1. $\sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$ from an orthogonal basis for square integrable functions on the half range $[0, L]$
2. $\cos(n\pi x/L)$ for $n = 0, 1, 2, 3, \ldots$ from an orthogonal basis for square integrable functions on the half range $[0, L]$
3. $\sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$ from an orthogonal basis for square integrable **odd** functions on the full range $[-L, L]$
4. $\cos(n\pi x/L)$ for $n = 0, 1, 2, 3, \ldots$ from an orthogonal basis for square integrable **even** functions on the full range $[-L, L]$

On the half range, we the cosine Fourier series of the function represents the 2L-period *even* extension of the function. The sine Fourier series represents the 2L-period *odd* extensions of the function.

Post Lecture 8: Fourier Cosine Series

We used the Fourier sine series to solve the heat equation given Dirichlet boundary conditions. We will solve the same problem this time when given Neumann boundary conditions.

Consider the heat equation

$$\begin{aligned} u_t &= \alpha^2 u_{xx} && 0 \le x \le L \\ u_x(0,t) &= u_x(L,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

As before, we guess that the solution will be of the form

$$u(x,t) = X(x)T(t)$$

Finding its partials

$$\begin{aligned} u_t &= X(x)\dot{T}(t) \\ u_x &= X'(x)T(t) \\ u_{xx} &= X''(x)T(t) \end{aligned}$$

Substitute it into the heat equation

$$X(x)\dot{T}(t) = \alpha^2 X'' T(t)$$

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''}{X(x)} = \mu$$

we reason that the functions must equal to some real constant μ.

Here, we have 2 ODEs:

$$\begin{aligned} X'' - \mu X &= 0 \\ \dot{T}(t) &= \mu\alpha^2 T(t) \end{aligned}$$

We know that the solution to time equation to be we have $T(t) = C\exp\left(\mu\alpha^2 t\right)$.

We get an eigenvalue problem,

$$\begin{aligned} X'' - \mu X &= 0 \\ X'(0) &= 0 \\ X'(L) &= 0 \end{aligned}$$

For $\mu > 0$, we write $\mu = \lambda^2$. It can be shown that there are no eigenfunctions.

For $\mu = 0$, we have $X'' = 0$, and we know that X is some linear function.

$$\begin{aligned} X &= Ax + B \\ X' &= A \end{aligned}$$

From the boundary conditions we get that $A = 0$. So $X = B$. The representative eigenfunction is 1. (Since $\cos(0) = 1$ whereas $\sin(0) = 0$, we get a functions here as opposed to the sine series case).

For $\mu < 0$. Let use write $\mu = -\lambda^2$ for $\lambda > 0$. The equation comes

$$X'' + \lambda^2 X = 0$$

which is second order ODE with constant coefficients. And we found the general solution to this equation to be

$$X(x) = A_1\cos(\lambda x) + A_2\sin(\lambda x)$$

So we have found the forms of T and X. Now we can impose the Neumann boundary conditions to find to missing coefficients.

$$\begin{aligned} u &= X(x)T(t) \\ u_x &= T(t)X' \\ &= T(t)\left[-A_1\lambda\sin(\lambda x) + A_2\lambda\cos(\lambda x)\right] \end{aligned}$$

Plugging in the initial conditions

$$\begin{aligned} u_x(0,t) &= 0 = X'(0)T(t) \\ u_x(L,t) &= 0 = X'(L)T(t) \end{aligned}$$

which would mean that $X'(0) = 0$ and $X'(L) = 0$ since the T function is an exponential and is never zero. Then

$$X' = -A_1\lambda\sin(\lambda 0) + A_2\lambda\cos(\lambda 0) = 0$$

We get $A_2 = 0$.

$$X' = -A_1\lambda\sin(\lambda L) = 0$$

If we decide that $A_1 = 0$, then we would simply get the trivial solution. So if $A_1 \neq 0$, then the sine term must be zero. Sine is zero whenever its argument is a multiple of π. So we have

$$\lambda = \frac{n\pi}{L} \qquad \text{for } n = 1, 2, 3, \ldots$$

Notice that we have excluded the term $n = 0$, since $n = 0$ would return the trivial solution. Ignoring the constants A_1 - as we will take linear combinations of these - we have

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

for $n = 1, 2, 3, \ldots$. These are the eigenfunctions. The eigenvectors are $\mu = -(\pi n/L)^2$.

We will find that when $\mu = 0$ we would find that there are no solutions.

The complete set of eigenvectors and eigenvalues are

$$\mu = -\left(\frac{\pi n}{L}\right)^2 \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

$$X_0 = 1, X_n = \cos\left(\frac{n\pi x}{L}\right) \qquad \text{for } n = 1, 2, 3, \ldots$$

X_0 corresponds to the eigenfunction we get for the case when $\mu = 0$.

Since the heat equation is linear, we can take linear combinations of it's solutions. The most general solution is of the form

$$u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\exp\left(-\left(\frac{n\pi}{L}\right)^2\alpha^2 t\right)$$

We wrote the first term when $n = 0$ as $a_0/2$ by convention.

Substituting in the initial conditions,

$$u(x,0) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n\cos\left(\frac{n\pi x}{L}\right)$$

the exponential term is equal to one.

To find a_n for $n \neq 0$, we integrate $f(x)$ against $\cos(k\pi x)/L$

$$\begin{aligned}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)dx &= \int_0^L \frac{a_0}{2}\cos\left(\frac{k\pi x}{L}\right)dx\\ &+ \sum_{n=1}^{\infty}\int_0^L a_n\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)dx\end{aligned}$$

The first integral on the right is zero, and the second integral is non zero only when $n = k$, so

$$\begin{aligned}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)dx &= \int_0^L a_n\cos^2\left(\frac{n\pi x}{L}\right)dx\\ &= a_n\frac{L}{2}\end{aligned}$$

So the coefficients a_n for $n \neq 0$ are

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)dx$$

For a_0,

$$\int_0^L f(x)\,dx = \int_0^L \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\int_0^L a_n\cos\left(\frac{n\pi x}{L}\right)dx$$

The second integral on the right is zero by orthogonality, since $n \neq 0$. Since

$$a_n\int_0^L \cos\left(\frac{0\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0 \qquad n \neq 0$$

And we have

$$\int_0^L f(x)\,dx = \frac{La_0}{2}$$

and

$$a_0 = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{0\pi x}{L}\right)dx \tag{39}$$

So in general

$$a_k = \int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)dx \tag{40}$$

which works for $n = 0, 1, 2, 3, \ldots$

Lecture 9

Convergence of the Full Fourier Series: Let f and f' be *piecewise continuous* functions on the full interval $[-L, L]$, and let f have a period of 2L. Then f has a Fourier series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right) \tag{41}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \tag{42}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{43}$$

The series converges to $f(x)$ at all points for which f is continuous. Where f is discontinuous, the series converges to the *average values of the left and right hand limits* to the point of discontinuity, that is

$$\frac{1}{2}\left(\lim_{x\to x_{x+}} f(x) + \lim_{x\to x_0-} f(x)\right). \tag{44}$$

Any reasonable functions $f(x)$ over the half range $[0, L]$ can be represented as

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

for coefficients b_n.

When we approximated $f(x) = x$ using a Fourier sine series over $[0, 1]$, since the sine function here has a period of $2L$, and $L = 1$ in our case, sine is 2-periodic. The 2-periodic odd extension of x has jump discontinuities at $x = nL$, for $n = \mathbf{Z}$.

If we want no jump continuities, we need to require $f(0) = f(L) = 0$.

We present an example for the heat conduction on a ring. We can select a point on the ring as the origin with a length L to both sides of th origin, and the domain of the heat equation is $[-L, L]$.

The heat equation would be

$$\begin{aligned}u_t &= \alpha^2 u_{xx}\\ u(-L,t) &= u(L,t)\\ u_x(-L,t) &= u_x(L,t)\\ u(x,0) &= f(x)\end{aligned}$$

An example of periodic boundary conditions.

Using the seperation of variables, we get the same two ODE.

The eigenvalue problem is

$$\begin{aligned}X'' - \mu X &= 0\\ X(-L) &= X(L)\\ X'(-L) &= X'(L)\end{aligned}$$

For the case $\mu > 0$ we have no eigenfunctions.

For $\mu = 0$, we have $X = Ax + B$. Using that $X(-L) = X(L)$, we have $A(-L) + B = AL = B$, so $A = 0$. And we have equals to constant B is a eigenfunction.

For $\mu < 0$, we have

$$X(x) = A_1\cos(\lambda x) + A_2\sin(\lambda x)$$

Using $X(-L) = X(L)$, forces $A_2\sin(\lambda L) = 0$, but if we want A_2 to be non zero. So $\lambda L = n\pi$, for $n = 1, 2, 3, \ldots$.

So

$$X(x) = A_1\cos\left(\frac{\pi n x}{L}\right) + A_2\sin\left(\frac{\pi n x}{L}\right)$$

satisfies both the boundary and initial conditions. For the eigenvalue $\mu_n = -(\pi n/L)^2$, we get two linearly independent functions. (eigenvalue has multiplicity 2.)

So the solution is

$$\begin{aligned}u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\exp\left(\mu\alpha^2 t\right)\\ &+ \sum_{m=1}^{\infty} b_m\sin\left(\frac{m\pi x}{L}\right)\exp\left(\mu\alpha^2 t\right)\end{aligned}$$

Use the initial conditions and orthogonality to find a_m and b_n for ($n = 0, 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$).

Lecture 10

When $f(x)$ square integrable and piecewise continuous, defined on the half range $[0, L]$, its sine Fourier series representation is an odd extension of the function. The cosine Fourier series representation is an even extension of the function.

We will denote the full Fourier series representation for a square integrable function as

$$S_\infty(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

For a partial sum of the Fourier series

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

Fourier's Theorem: suppose $f(x)$ and its derivative is piecewise continuous, then $f(x) = S_\infty(x)$ for all points x where $f(x)$ is continuous.
When there is a jump discontinuity, such that

$$f(x_0^+) \neq f(x_0^-)$$

then $S_\infty(x_0) = (f(x_0^+) + f(x_0^-))/2$.

L^2 **Convergence:** For $f(x) \in L^2$, then

$$\lim_{N\to\infty}\int_{-L}^{L} |S_N(x) - f(x)|^2\,dx = 0$$

Bessel's Inequality For each N:

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \le \frac{1}{L}\int_{-L}^{L} |f(x)|^2 \, dx$$

Parseval's Identity

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{L}\int_{-L}^{L} |f(x)|^2 \, dx$$

When $f(x)$ is an odd functions on $[-L, L]$, then the Fourier series will only have sine terms. So

$$\frac{1}{L}\int_{-L}^{L} |f(x)|' \, dx = \frac{2}{L}\int_{0}^{L} |f(x)|^2 \, dx = \sum_{n=1}^{\infty} b_n^2$$

One application: Consider $f(x) = x$ on interval $[0, 2]$. The Fourier sine series representation is the 4-period odd extension of x.

$$f(x) = x = S_\infty(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

The 4L period extension goes from $[-2, 2]$. The product of odd functions is a even function. So we can applying the property of integrals of even functions over symmetric limits

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx \\ &= \int_{0}^{2} x \sin\left(\frac{n\pi x}{2}\right) dx \\ &= \frac{-4\cos(n\pi)}{n\pi} \\ &= \frac{4}{\pi}\frac{(-1)^{n+1}}{n} \end{aligned}$$

Parseval's identity leads us to

$$\begin{aligned} \sum_{n=1}^{\infty} b_n^2 &= \frac{1}{2}\int_{-2}^{2} |f(x)|^2 \, dx \\ &= \int_{0}^{2} x^2 \, dx \\ &= \frac{8}{3} \end{aligned}$$

The left hand side is

$$\sum_{n=1}^{\infty}\left(\frac{4}{\pi}\frac{(-1)^{n+1}}{n}\right)^2 = \frac{8}{3}$$

Some rearranging gives

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is an amazing formula (Basel Problem).

We have been able to use the method of separation of variables to solve linear and homogeneous PDEs with linear and homogeneous boundary conditions. However, when given a linear and homogeneous PDE with Dirichlet boundary conditions that are non zero, to trick is to first look for a steady state solution.

Consider the following PDE

$$\begin{aligned} u_t &= \alpha^2 u_{xx} && 0 < x < L, t > 0 \\ u(0,t) &= u_0 \\ u(L,t) &= u_L \\ u(x,0) &= g(x) && 0 < x < L \end{aligned}$$

Begin by searching for a steady state solution. The steady state solution $u_\infty(x)$ has no x dependence. So $0 = u_{xx}$. We have

$$\begin{aligned} u_\infty &= Ax + B \\ &= u_0 + \left(\frac{u_L - u_0}{L}\right) x \end{aligned}$$

The solution is the sum of the steady state solution and a solution that solves a homogeneous boundary condition.

$$u(x,t) = u_\infty(x) + v(x,t)$$

Differentiate the above expression, we find that u_∞ has a zero time derivative, and zero 2nd spatial derivative by construction.

So $v(x, 0)$ solves the PDE, but with initial conditions given by

$$\begin{aligned} v(x,0) &= u(x,0) - u_\infty(x) = g(x) - u_\infty \\ v(0,t) &= u(0,t) - u_\infty(0) = u_0 - u_0 = 0 \\ v(L,t) &= u(L,t) - u_E(L) = u_L - u_L = 0 \end{aligned}$$

This returns us to the realm where separation of variables is applicable.

The final note is that when presenting the final solution, don't forget to add the steady state solution and the homogeneous solution together.

Lecture 11

Consider the heat equation with inhomogeneous Neumann boundary conditions that are time independent

$$\begin{aligned} u_t &= \alpha^2 u_{xx} && 0 < x < L, t > 0 \\ u_x(0,t) &= A \\ u_x(L,t) &= B \\ u(x,0) &= g(x) \end{aligned}$$

The idea is to find a particular solution which satisfies with PDE and the boundary conditions. Then, we subtract the particular solution to reduce the problem to one with homogeneous Neumann boundary conditions.

We can make an attempt to find a steady-state solution u_∞ that satisfies $u_\infty''(x) = 0$. So the steady state must be linear function. Yet,

$$u_\infty' = (Cx + D)' = C$$

which does not satisfy the Neumann conditions. (The BC requires $C = B = A$).

To find particular solution, lets look into a quadratic polynomial. This is no longer a steady state solution. This new polynomial must depend on both x, t

$$\begin{aligned} w(x,t) &= ax^2 + bx + c + r(t) \\ &= ax^2 + bx + r(t) \end{aligned}$$

c can be absorbed into the $r(t)$ term.

Does this equation satisfy the heat equation?

$$\begin{aligned} w_t &= \alpha^2 w_{xx} \\ r'(t) &= 2\alpha^2 a \end{aligned}$$

Which is an ODE, and we can solve for $r(t)$:

$$r(t) = 2\alpha^2 at$$

we ignored the constant when solving for $r(t)$ as we only need a single solution.

So we have w to be

$$w(x,t) = ax^2 + bx + 2\alpha^2 at$$

which only depends on two free parameters a, b.

We solve for the parameters a, b using the boundary conditions

$$\begin{aligned} w_x(0,t) &= A = b \\ w_x(L,t) &= B = 2aL + b \end{aligned}$$

$$w(u,t) = \frac{B-A}{2L}x^2 + Ax + \alpha^2\frac{B-A}{L}t$$

The solution we want $u(x,t)$ is the sum of the particular and the homogeneous solution.

$$u(x,t) = w(x,t) + v(x,t)$$

$$\begin{aligned} w_t + v_t &= \alpha^2 w_{xx} + \alpha v_{xx} \\ v(t) &= \alpha^2 v_{xx} \end{aligned}$$

So the homogeneous solution also solves the PDE. From the boundary conditions

$$\begin{aligned} u_x(0,t) &= w_x(0,t) + v_x(0,t) \\ u_x(L,t) &= w_x(L,t) + v_x(L,t) \end{aligned}$$

Since both u and w must satisfy the BC, we have

$$\begin{aligned} v_x(0,t) &= 0 \\ v_x(L,t) &= 0 \end{aligned}$$

From the initial condition

$$\begin{aligned} u(x,0) &= w(x,0) + v(x,0) \\ g(x) &= w(x,0) + v(x,0) \\ v(x,0) &= g(x) - \frac{B-A}{2L}x^2 - Ax \end{aligned}$$

We get a new boundary value problem

$$\begin{aligned} v_t &= \alpha^2 v_{xx} \\ v_x(0,t) &= v_x(L,t) = 0 \\ v(x,0) &= g(x) - \frac{B-A}{2L}x^2 - Ax \end{aligned}$$

Using separation of variables, we will find that

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t\right)$$

which solves the PDE and the homogeneous boundary conditions. Now when we substitute in initial conditions

$$g(x) - \frac{B-A}{x^2} - Ax = v(x,0)$$

We can solve for a_n using orthogonality

$$a_n = \frac{2}{L}\int_0^L \left[g(x) - \frac{B-A}{x^2} - Ax\right] \cos\left(\frac{n\pi x}{L}\right) dx$$

Heat equation with source/sink: There ar two types

1. Time-independent source/sink

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + x \\ u(0,t) &= u_0 \\ u(L,t) &= u_L \\ u(x,0) &= g(x) \end{aligned}$$

   We can solve this type by long for a particular solution

2. Time-dependent source/sink

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + xt \\ u(0,t) &= 0 \\ u_x(L,t) &= 2 \\ u(x,0) &= h(x) \end{aligned}$$

   when the boundary conditions are also time dependent, its is also type two. We will need a new method called *eigenfunction expansion*

One example of type problem is

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + x \qquad 0 < x < L, t > 0 \\ u(0,t) &= u_0 \\ u(L,t) &= u_1 \\ u(x,0) &= g(x) \end{aligned}$$

We first look for a particular solution. A steady-state solution will work. We have

$$\alpha^2 u''_\infty + x = 0$$

Using the boundary conditions to find the constants, we will find that

$$u_\infty(x) = \frac{-x^2}{a\alpha^2} + \left(\frac{u_L - u_0}{L} + \frac{L^2}{6\alpha^2}\right) x + u_0$$

Again,

$$u(x,t) = u_\infty(x) + v(x,t)$$

which we can check that $v(x,t)$ solves the homogeneous heat equation.

So we have a new boundary value problem

$$\begin{aligned} v_t &= \alpha^2 v_{xx} \\ v(0,t) &= v(L,t) = 0 \\ v(x,0) &= g(x) - u_\infty(x) \end{aligned}$$

which we can solve using the method of separation of variables.

For a type 2 problem, we will solve a PDE with a time-dependent source with time independent Dirichlet condition.

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + S(x,t) \\ u(0,t) &= u_0 \\ u(L,t) &= u_L \\ u(x,0) &= g(x) \end{aligned}$$

We need to get rid of the inhomogeneous boundary conditions. The eigenfunction expansion method will only work with homogeneous boundary condition. We don't care about PDE at this point. We just want the boundary conditions to be satisfied.

$$q(x) = u_0 + \frac{u_L - u_0}{L} x$$

which does indeed satisfy BCs. Write

$$u(x,t) = q(x) + v(x,t)$$

substitute into the PDE

$$0 + v_t = \alpha^2(0 + v_{xx}) + s(x,t)$$

($q_t = q_{xx} = 0$) and v_t satisfies

$$v_t = \alpha^2 v_{xx} + S(x,t)$$

The boundary conditions give

$$\begin{aligned} u(0,t) &= q(0) + v(0,t) \\ u(L,t) &= q(L) + v(L,t) \\ u(x,0) &= q(x) + v(x,0) \end{aligned}$$

The new boundary value problem is

$$\begin{aligned} v_t &= \alpha^2 v_{xx} + S(x,t) \\ v(0,t) &= 0 = v(L,t) \\ v(x,0) &= g(x) - q(x) \end{aligned}$$

We have made the boundary conditions homogeneous, but the PDE is still non-homogeneous. The method of separation of variables will not work.

If there were no source term, then we would be able to find the eigenfunctions and values to be

$$\begin{aligned} X_n(x) &= \sin\left(\frac{n\pi x}{L}\right) \\ \mu_n &= -\left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\ldots \end{aligned}$$

Next write the source $S(x,t)$ in its eigenfunction expansion (which in this case involves sines):

$$\begin{aligned} S(x,t) &= \sum_{n=1}^{\infty} S_n(t) X_n(x) \\ & \sum_{n=1}^{\infty} s_n(t) \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

By orthogonality,

$$s_n(t) = \frac{2}{L}\int_0^L S(x,t)\sin\left(\frac{n\pi x}{L}\right) dx$$

Next also write the $v(x,t)$ in its eigenfunction expansion:

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} v_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} v_n(t)\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

Since we do not know the expression of $v(x,t)$ we cannot simply integrate against $v(x,t)$ to find the coefficients $v_n(t)$.

At this point, find v_t and v_{xx} and substitute the terms we found into the PDE. We get

$$\begin{aligned} v_t &= \sum_{n=1}^{\infty} \dot{v}_n(t)\sin\left(\frac{n\pi x}{L}\right) \\ v_{xx} &= -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 v_n(t)\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

We have

$$\sum_{n=1}^{\infty}\left[\dot{v}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2 v_n(t) - s_n(t)\right]\sin\left(\frac{n\pi x}{L}\right) = 0$$

By the linear independence of $\sin(n\pi x/L)$, we obtain

$$\dot{v}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2 v_n(t) - s_n(t) = 0$$

This is a first order inhomogeneous ODE. We can solve this using the integrating factor method.

$$\dot{v}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2 v_n(t) = s_n(t)$$

The integrating factor μ is

$$\mu(t) = \int \exp\left(\alpha^2\left(\frac{n\pi}{L}\right)^2\right) dt$$

$$\frac{d}{dt}\left(\mu(t)v_n(t)\right) = s_n(t)\mu(t)$$

Which can solve now for $v_n(t)$

$$\begin{aligned} v_n(t) &= \frac{1}{\mu(t)}\int_0^t s_n(z)\mu(z)\,dz + \frac{C_n}{\mu(t)} \\ &= \int_0^t s_n(z)\exp\left(-\alpha^2\left(\frac{n\pi}{L}\right)^2(t-z)\right) dz \\ &\quad + C_n\exp\left(-\alpha^2\left(\frac{n\pi}{L}\right)^2 t\right) \end{aligned}$$

So other than the coefficients C_n, we have found $v(x,t)$ which satisfies PDE and the boundary conditions.

$$v(x,t) = \sum_{n=1}^{\infty} [v_n(t)] \sin\left(\frac{n\pi x}{L}\right)$$

To find C_n, we use the initial condition. Notice the integral with s_n is from 0 to 0 when we plug in the initial condition.

$$\begin{aligned} v(x,0) &= g(x) - q(x) \\ &= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

which we can find by using the orthogonality relations.

Lecture 12

Consider the following heat equation (with source term) with time-dependent Dirichlet boundary conditions.

$$\begin{cases} u_t = \alpha^2 u_{xx} + S(x,t) \\ u(0,t) = \phi_0(t) \\ u(L,t) = \phi_L(t) \\ u(x,o) = f(x) \end{cases}$$

We have to do something to get rid of the inhomogeneous boundary conditions. Recall that if the boundary conditions were not time independent, we used a linear function to satisfy the boundary condition. We can do something similar here:

$$w(x,t) = \phi_o(t) + \left(\frac{\phi_L(t) - \phi_0(t)}{L}\right) x$$

Verify that w satisfies the boundary conditions:

$$\begin{aligned} w(0,t) &= \phi_o(t) \\ w(L,t) &= \phi_L(t). \end{aligned}$$

We will write our solution to be the sum of w and another function. Substitute our solution $u(x,t) = w(x,t) + v(x,t)$ into the heat equation

$$w_t + v_t = \underbrace{\alpha^2 w_{xx}}_{=0} + \alpha^2 v_{xx} + S(x,t).$$

Rearranging,

$$v_t = \alpha^2 v_{xx} + S(x,t) - w_t.$$

The source term gets shifted by w_t, which differs from the previous case of inhomogeneous constant Dirichlet boundary conditions.

Check the boundary conditions:

$$\underbrace{u(0,t)}_{=\phi_0(t)} = \underbrace{w(0,t)}_{=\phi_0(t)} + v(0,t) \Rightarrow v(0,t) = 0$$

$$\underbrace{u(L,t)}_{=\phi_L(t)} = \underbrace{w(L,t)}_{=\phi_L(t)} + v(L,t) \Rightarrow v(L,t) = 0$$

Check the initial conditions:

$$u(x,0) = f(x) = w(x,0) + v(x,0)$$

Rearranging,

$$v(x,0) = f(x) - w(x,0).$$

So we have an eigenvalue problem

$$\begin{cases} v_t = \alpha^2 v_{xx} + S(x,t) - w_t \\ v(0,t) = 0 \\ v(L,t) = 0 \\ v(x,0) = f(x) - w(x,0) \end{cases}$$

This this problem has the Dirichlet form, we can immediately conclude that the eigenfunctions for the problem without the source term $S(x,t) - w(t)$ are

$$\begin{aligned} &X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \\ &X(0) = X(L) = 0 \qquad n = 1, 2, 3, \ldots \end{aligned}$$

Similar to last lecture, we then write the eigenfunction expansion for the source term $S(x,t) - w_t$.

$$S(x,t) - w_t = \sum_{n=1}^{\infty} s_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

We can recover the coefficients via

$$s_n(t) = \frac{2}{L}\int_0^L (S(x,t) - w_t) \sin\left(\frac{n\pi x}{L}\right) dx$$

for $n = 1, 2, 3, \ldots$. Where

$$w_t = \dot{\phi}_0(t) + \left(\frac{\dot{\phi}_L(t) - \dot{\phi}_0(t)}{L}\right) x$$

We construct eigenfunction expansion for $v(x,t)$ itself

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} v_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

We compute its derivatives so we can substitute them into the heat equation:

$$\begin{aligned} v_t &= \sum_{n=1}^{\infty} \dot{v}_n(t) X_n(x) \\ v_{xx} &= -\sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi}{L}\right)^2 \end{aligned}$$

Substitute into the PDE,

$$v_t - \alpha^2 v_{xx} - S(x,t) + w_t = 0$$

We get

$$\begin{aligned} &\sum_{n=1}^{\infty} \dot{v}_n(t) X_n(x) + \alpha^2 \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi}{L}\right)^2 - \\ &\sum_{n=1}^{\infty} s_n(t) \sin\left(\frac{n\pi x}{L}\right) + \left[\dot{\phi}_0(t) + \left(\frac{\dot{\phi}_L(t) - \dot{\phi}_0(t)}{L}\right) x\right] = 0 \\ &\sum_{n=1}^{\infty} \left[\dot{v}_n(t) + \alpha^2 \left(\frac{n\pi}{L}^2\right) v_n(t) - s_n(t)\right] \sin\left(\frac{n\pi x}{L}\right) = 0 \end{aligned}$$

By the *linear independence* of sines, we can say

$$\dot{v}_n(t) + \alpha^2 \left(\frac{n\pi}{L}^2\right) v_n(t) - s_n(t) = 0,$$

which is an ODE, that we can solving via the method of integrating factor.

$$\begin{aligned} v_n(t) = &\int_0^t \exp\left(-\alpha^2 \left(\frac{\pi n}{L}\right)^2 (t-\tau)\right) s_n(\tau)\, d\tau \\ &+ C_n \exp\left(-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t\right) \end{aligned}$$

So the only unknown is the constants C_n which we get from solving the ODE. We can use the initial condition to solve for it

$$f(x) - w(x,0) = v(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

Using orthogonality relation

$$C_n = \frac{2}{L}\int_0^L (f(x) - w(x,0)) \sin\left(\frac{n\pi x}{L}\right) dx$$

We have found everything required to solve the PDE.

Wave Equation: We will move on to the 1-D wave equation. Consider a function u which measures the displacement of a particle away from its equilibrium position. The wave equation with homogeneous boundary conditions is

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

which is defined on $x \in [0, L]$, $t > 0$. This is the 1-D wave equation on a finite domain, which we can solve via the separation variable method.

There is an additional initial condition $u_t(x,0)$ to the wave equation

D'Alembert's Solution to the wave equation: Suppose we want to solve the wave equation on the real line.

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = u_0(x) \\ u_t(x,0) = v_0(x) \end{cases}$$

There is the domain for all $-\infty < x < \infty$, so we no longer have boundary conditions.

Let us guess and check to find some solutions to the equation. We see that any C^2 function of the form

$$u(x,t) = G(x \pm ct)$$

defined over the real line will solve the equation.

Since the PDE is linear, we take linear combinations of the solutions

$$u(x,t) = G(x+ct) + F(x-ct)$$

which can also be shown to satisfy the PDE.

We can show that these are the only solution.

Perform a coordinate change:

$$\begin{aligned} r &= x + ct \\ s &= x - ct \\ x &= \frac{1}{2}(r+s) \\ t &= \frac{1}{2c}(r-s) \end{aligned}$$

For a function $h(x,t)$, we can find it's partial with respect to r by

$$\frac{\partial h}{\partial r} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial h}{\partial t}\frac{\partial t}{\partial r}$$

If we "remove" h from above, we have an *equality between differential operators*

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial t}\frac{\partial t}{\partial r}$$

Similarly,

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial t}\frac{\partial t}{\partial s}$$

We know what $\partial x/\partial s$, $\partial x/\partial r$, $\partial t/\partial s$, and $\partial t/\partial r$ is. Substituting

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \\ \frac{\partial}{\partial s} &= \frac{-1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \end{aligned}$$

Applying the operators back to back:

$$\frac{\partial}{\partial s}\frac{\partial}{\partial r}u = \frac{\partial^2 u}{\partial s \partial r} = \frac{-1}{4c}\left(\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}\right) = 0$$

Integrate the mixed partial twice, first with respect to r, then respect to s.

$$\begin{aligned} \frac{\partial u}{\partial s} &= \tilde{F}(s) \\ u &= \int_0^s \tilde{F}(\tau)\, d\tau + G(r) \\ &= F(s) + G(r) \\ &= F(s-ct) + G(s+ct) \end{aligned}$$

We recover our what we started with.

What exactly are the functions F and G? Recall that Our solution u must also satisfy the initial conditions.

$$\begin{aligned} u(x,0) &= u_0(x) = F(x) + G(x) \\ u_t(x,0) &= v_0(x) = -cF'(x) + cG'(x) \end{aligned}$$

Integrate the second expression with respect to x. (While the expression was originally obtained via a time derivative, since we have set $t=0$, we can integrate it with respect to x to recover G.

$$\int_0^x v_0(\tau)\, d\tau + A = -cF(x) + cG(x)$$

So we have

$$\begin{cases} F(x) + G(x) = u_0(x) \\ -cF(x) + cG(x) = \int_0^x v_0(\tau)\, d\tau + A \end{cases}$$

In matrix notation

$$\begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix}\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} u_0(x) \\ \int_0^x v_0(\tau)\, d\tau + A \end{bmatrix}$$

We can solve for F and G via linear row reduction. We obtain

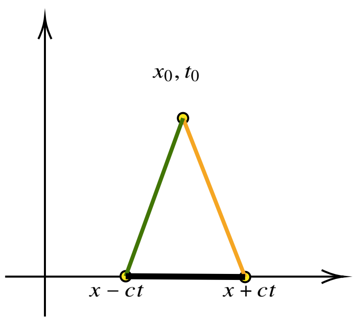
$$F(x,t) = \frac{1}{2c}\left[cu_0 - \left(\int_0^x v_0(\tau)\, d\tau + A\right)\right]$$

$$G(x,t) = \frac{1}{2c}\left[cu_0 + \left(\int_0^x v_0(\tau)\, d\tau + A\right)\right]$$

D'Alembert's solution is

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x-ct) \\ &= \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\tau)\, d\tau \end{aligned}$$

Characteristics for the wave equation



Three points on the x,t plane. The green line has slope $1/c$ and the orange line has slope $-1/c$. The green and orange lines are called *characteristic lines*.

Consider x,t plane. To find the value of the wave equation on the real line at some position x_0 and time t_0, we can use the D'Alembert's formula. Substituting,

$$\begin{aligned} u(x_0,t_0) = &\frac{1}{2}\left[u_0(x_0 - ct_0) + u_0(x_0 + ct_0)\right] \\ &+ \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} v_0(\tau)\, d\tau \end{aligned}$$

The first term is computing the average of the initial condition $u(x,0) = u_0(x)$ at the points $(x_0 + ct_0, 0)$, $(x_0 + ct_0, 0)$.

Since the average value of the initial velocity function v_0 over 2 points A and B is given by

$$\frac{1}{B-A}\int_A^B v_0(x)\, dx$$

We can see that the second term consists of multiplying the average value of the initial velocity $u_t(x,0) = v_0(x)$ over $(x_0 + ct_0, 0)$, $(x_0 + ct_0, 0)$ by t_0.

The wave equation has the property that the propagation of wave disturbance are finite.

Lecture 13

When initial velocity of the wave is zero ($u_{x,0} = 0$), the D'Alembert solution to the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = u_0(x) \\ u_t(x,0) = 0 \end{cases}$$

is

$$u(x_0,t_0) = \frac{1}{2}\left[u_0(x_0 - ct_0) + u_0(x_0 + ct_0)\right]$$

this can be thought of as the sum of a left moving and a right moving wave.

Wave operator:in 1-D is

$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) \tag{45}$$

The two factors which the wave operator decomposes into can be thought of as as as left and right moving pulses.

Wave Equation on a Finite String: Consider the following BVP with homogeneous Dirichlet boundary conditions.

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < L, t > 0 \\ u(0,t) = 0, u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Applying the separation of variables, we can write $u(x,t) = X(x)T(t)$. Substitute into the PDE,

$$X(x)\ddot{T}(t) = c^2 X''(x)T(t)$$

Dividing both sides by $c^2 X(x)T(t)$, we get

$$\frac{X''}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} = \mu$$

We have two ODEs:

$$X'' - \mu X = 0$$
$$\ddot{T}(t) = c^2 \mu T$$

Using the boundary conditions, we would find

$$u(0,t) = 0 = X(0)T(t) \quad \Rightarrow X(0) = 0$$
$$u(L,t) = 0 = X(L)T(t) \quad \Rightarrow X(L) = 0$$

We have the eigenvalue problem:

$$\begin{cases} X'' + \mu X = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases}$$

For $X(x)$, we know that the eigenvalues and eigenfunctions are

$$\mu_n = -\left(\frac{n\pi}{L}\right)^2 \qquad X_n = \sin\left(\frac{n\pi x}{L}\right) \qquad n \geq 1$$

For $T(x)$, we have the characteristic equation is

$$r^2 = c^2\mu \Rightarrow r = ic\sqrt{-\mu}$$

And we have the general solution

$$\begin{aligned} T_n(t) &= A\cos\left(c\sqrt{-\mu}t\right) + B\sin\left(c\sqrt{-\mu}t\right) \\ &= A\cos\left(\frac{cn\pi t}{L}\right) + B\sin\left(\frac{cn\pi t}{L}\right) \end{aligned}$$

So the solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) T_n(t)$$

we can combine the constants B_n with the constants in the solution for $T_n(t)$.

So the current solution which satisfies the PDE and the boundary condition

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{cn\pi t}{L}\right) + B_n\sin\left(\frac{cn\pi t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

when we apply the initial condition, we can find the coefficients

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

A_n is given by the Fourier sine series coefficients for f

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = b_n^f$$

If we differentiate $u(x,t)$ in time we can use the other initial condition

$$u_t = \left[\sum_{n=1}^{\infty} -A_n\sin\left(\frac{cn\pi t}{L}\right)\left(\frac{cn\pi}{L}\right) + B_n\cos\left(\frac{cn\pi t}{L}\right)\left(\frac{cn\pi}{L}\right)\right] \times \sin\left(\frac{n\pi x}{L}\right)$$

Substituting in

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n\frac{cn\pi}{L}\sin\left(\frac{n\pi x}{L}\right)$$

B_n is given by

$$\frac{cn\pi}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx = b_n^g$$

In the expression above, the right hand side is the Fourier cosine series coefficient. So B_n can be rearranged to be written as $B_n = b_n^g L/xn\pi$.

In summary, we can write the solution as

$$u(x,t) = \sum_{n=1}^{\infty}\left[b_n^f\cos\left(\frac{cn\pi t}{L}\right) + \frac{Lb_n^g}{\pi nc}\sin\left(\frac{cn\pi t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

How does this solution compare to the D'Alembert solution?

We can rewrite the solution using trigonometric identities.

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$
$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Let us denote $\gamma_n = n\pi/L$. So Our solution becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\frac{b_n^f}{2}\left[\sin(\gamma_n(x+ct)) + \sin(\gamma_n(x-ct))\right] \\ &+ \sum_{n=1}^{\infty}\frac{Lb_n^g}{2\pi nc}\left[\cos(\gamma_n(x-ct)) - \cos(\gamma_n(x+ct))\right] \\ &= u^f + u^g \end{aligned}$$

If our BVP where instead defined on the right line, then the D'Alembert solution to our equation would involve the 2L-periodic odd extensions of $f(x)$ and $g(x)$, so that the solution is

$$u(x,t) = \frac{1}{2}\left[f^O(x-ct) + f^O(x+c)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^O(\tau)\,d\tau$$

where the super script O denotes the 2L-periodic odd extension. The odd extension of a function can be represented by its Fourier sine series. Where

$$f^O(x) = \sum_{n=1}^{\infty} b_n^f \sin(\gamma_n x)$$
$$g^O(x) = \sum_{n=1}^{\infty} b_n^g \sin(\gamma_n x)$$

We can see that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f^O(x+xt) + f^O(x-ct)\right] \\ &+ \sum_{n=1}^{\infty}\frac{Lb_n^g}{2\pi nc}\left[\cos(\gamma_n(x-ct)) - \cos(\gamma_n(x+ct))\right] \end{aligned}$$

We claim that the second sum is

$$u^g = \frac{1}{2c}\int_{x-ct}^{x+ct} g^O(s)\,ds$$

We can show that this is the case by breaking up the integral and and expanding the integrand in its Fourier series

$$\begin{aligned} \int_{x-ct}^{x+ct} g^O(s)\,ds &= -\int_0^{x-ct} g^O(s)\,ds + \int_0^{x+ct} g^O(s)\,ds \\ &= -\int_0^{x-ct}\sum_{n=1}^{\infty} b_n^g\sin(\gamma_n s)\,ds \\ &+ \int_0^{x+ct}\sum_{n=1}^{\infty} b_n^g\sin(\gamma_n s)\,ds \\ &= \sum_{n=1}^{\infty}\left[\frac{b_n^g}{\gamma_n}\cos(\gamma_n s)\right]_0^{x-ct} \\ &- \sum_{n=1}^{\infty}\left[\frac{-b_n^g}{\gamma_n}\cos(\gamma_n s)\right]_0^{x+ct} \end{aligned}$$

which expands to equal

$$= \sum_{n=1}^{\infty}\frac{Lb_n^g}{\pi n}\left[\cos(\gamma_n(x-ct)) - \cos(\gamma_n(x+ct))\right]$$

Multiplying by the factor of $1/2c$ which we left out completes the equality.

We have showed that at least of the case of homogeneous Dirichlet boundary conditions, the solution given by separation of variables is equivalent to D'Alembert's solution when the 2L-periodic odd extension of the initial conditions are considered.

Consider the following example. The wave equation

$$\begin{cases} u_{tt} = 100u_{xx} & 0 < x < 4, t > 0 \\ u(0,t) = u(4,t) = 0 \\ u(x,t) = f(x) \\ u_x(x,0) = 0 \end{cases}$$

We only know that the initial amplitude function has values at

$$f(0) = 0$$
$$f(1) = 2$$
$$f(2) = 3$$
$$f(3) = 1$$

we want to find $u(1,1)$ and $u(3,9)$.

We know that the D'Alembert solution gives the solution to the same equation equation on the real line.

$$u(x,t) = \frac{1}{2}\left[f^O(x-ct) + f^O(x+c)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^O(\tau)\,d\tau$$

where f^O and g^O are 2-L periodic odd extensions of the original initial conditions.

2-L periodic odd extensions are defined such that

$$f^O(x+2Ln) = f^O(x) \qquad n \in \mathbf{Z}$$
$$f^O(-x) = -f^O(x)$$

We can exploit this property. We then know that

$$\begin{aligned} f^O(0) &= 0 & f^O(0) &= 0 \\ f^O(1) &= 2 & f^O(-1) &= -2 \\ f^O(2) &= 3 & f^O(-2) &= -3 \\ f^O(3) &= 1 & f^O(-3) &= -1 \end{aligned}$$

Substituting (1,1) into the D'Alembert solution ($c = 10$ and $g(x) = 0$ in our case)

$$\begin{aligned} u(1,1) &= \frac{1}{2}\left(f^O(-9) + f^O(11)\right) = \frac{1}{2}\left(f^O(-1) + f^O(3)\right) \\ &= \frac{1}{2}(-2+1) \end{aligned}$$

The solution for $u(3,9)$ can also be obtained in a similar way.

Laplace Equation: in 2-D is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is steady state solution to the 2-D heat equation. Since $u_t = 0 = u_{xx} + u_{yy}$. In 3-D is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Consider the laplace equation on a rectangular domain:

$$\begin{cases} 0 < x < a \\ 0 < y < b \\ \nabla^2 u = u_{xx} + u_{yy} = 0 \\ u(x,0) = f_1(x) \\ u(x,b) = f_2(x) \\ u(0,y) = g_1(y) \\ u(a,y) = g_2(y) \end{cases}$$

The trick is to decompose this problem into four sub problems, where each problem solves the laplace equation with one of the four inhomogeneous Dirichlet boundary conditions. We can do this since the equation is linear.

We will consider one of the four problems

$$\begin{cases} \nabla u^A = 0 \\ u(x,0) = f_1(x) \\ u(x,b) = 0 \\ u(0,y) = 0 \\ u(a,y) = 0 \end{cases}$$

Using separation of variables, $u(x,y) = X(x)Y(y)$, substituting into the equation

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

rearranging, since functions of different independent variables cannot be equal unless they are both equal to same constant μ

$$\frac{X''}{X} = \frac{-Y''}{Y} = \mu$$

Boundary conditions gives use the information that

$$\begin{aligned} u(x,0) &= X(x)Y(0) = f_1(x) \\ u(x,b) &= X(x)Y(b) = 0 \Rightarrow Y(b) = 0 \\ u(0,y) &= X(0)Y(y) = 0 \Rightarrow X(0) = 0 \\ u(a,y) &= X(a)Y(y) = 0 \Rightarrow X(a) = 0 \end{aligned}$$

The first boundary condition yields no useful information.

For the X problem, we know from solving the heat equation that the eigenvalues and eigenfunctions (for $n \ge 1$) are

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right) \qquad \mu_n = -\left(\frac{n\pi}{a}\right)$$

(We replaced "L" with a since the boundary conditions of X are from 0 to a.)

Since μ is negative, we know that a solution to the Y equation is

$$Y = C\cosh\left(\sqrt{-\mu}y\right) + D\sinh\left(\sqrt{-\mu}y\right)$$

Using the boundary condition $Y(b) = 0$,

$$Y(b) = C\cosh\left(\sqrt{-\mu}b\right) + D\sinh\left(\sqrt{-\mu}b\right) = 0$$

We can solve for C in terms of D

$$C = -D\frac{\sinh\left(\sqrt{-\mu}b\right)}{\cosh\left(\sqrt{-\mu}b\right)}$$

Substitute into the original equation

$$\begin{aligned} Y &= -D\frac{\sinh\left(\sqrt{-\mu}b\right)\cosh\left(\sqrt{-\mu}y\right)}{\cosh\left(\sqrt{-\mu}b\right)} + D\sinh\left(\sqrt{-\mu}y\right) \\ &= D\left[\frac{\sinh\left(\sqrt{-\mu}y\right)\cosh\left(\sqrt{-\mu}b\right) - \sinh\left(\sqrt{-\mu}b\right)\cosh\left(\sqrt{-\mu}y\right)}{\cosh\left(\sqrt{-\mu}b\right)}\right] \\ &= \frac{D}{\cosh\left(\sqrt{-\mu}b\right)}\sinh\left(\sqrt{-\mu}y - \sqrt{-\mu}b\right) \\ &= \bar{D}\sinh\left(\sqrt{-\mu}(y-b)\right) \end{aligned}$$

The last simplication uses hyperbolic trigonometric identity

$$\sinh(A \pm B) = \sinh(A)\cosh(B) \pm \sinh(B)\cosh(A)$$

To save ourselves some work, we could have built the condition $Y(b) = 0$ into the solution by guessing the solution

$$Y(y) = \bar{D}\sinh\left(\sqrt{-\mu}(y-b)\right)$$

right away. The justification for such a guess may be that the sinh function only has a single root at 0.

We know that the functions

$$u_n(x,y) = \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\sqrt{-\mu}(y-b)\right) \qquad n \ge 1$$

will satisfy our 3 of our boundary condition by construction. In order to match the last boundary condition, we will superimpose all solutions

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\sqrt{-\mu}(y-b)\right)$$

Substituting the last boundary condition

$$u(x,0) = f_1(x) = \sum_{n=1}^{\infty}\left[-B_n\sinh\left(\frac{n\pi b}{a}\right)\right]\sin\left(\frac{n\pi x}{a}\right)$$

We know the B_ns are give by

$$-B_n\sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f_1(x)\sin\left(\frac{n\pi x}{a}\right)dx = b_n^{f_1}$$

and

$$B_n = \frac{-b_n^f}{\sinh(n\pi b/a)}$$

In summary, the solution is

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty}\frac{-b_n^f\sinh(n\pi(y-b)/a)}{\sinh(n\pi b/a)}\sin\left(\frac{n\pi x}{a}\right) \\ &= \sum_{n=1}^{\infty}\frac{b_n^f\sinh(n\pi(b-y)/a)}{\sinh(n\pi b/a)}\sin\left(\frac{n\pi x}{a}\right) \end{aligned}$$

The solution to the sub-problem when $u(x,b) = f_2(x)$ is

$$u(x,y) = \sum_{n=1}^{\infty}\frac{b_n^{f_2}\sinh(n\pi y/a)}{\sinh(n\pi b/a)}\sin\left(\frac{n\pi x}{a}\right)$$

Where $b_n^{f_2}$ is

$$b_n^{f_2} = \frac{2}{a}\int_0^a f_2(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

Similarly, the solution to sub-problem when $u(0,y) = g_1(y)$

$$u(x,y) = \sum_{n=1}^{\infty}\frac{b_n^{g_1}\sinh(n\pi(a-x)/b)}{\sinh(n\pi a/b)}\sin\left(\frac{n\pi y}{b}\right)$$

Where $b_n^{g_1}$ is

$$b_n^{g_1} = \frac{2}{b}\int_0^b g_1(x)\sin\left(\frac{n\pi y}{b}\right)dy$$

Similarly, the solution to sub-problem when $u(a, y) = g_2(y)$

$$u(x, y) = \sum_{n=1}^{\infty} \frac{b_n^{g_2} \sinh(n\pi x/b)}{\sinh(n\pi a/b)} \sin\left(\frac{n\pi y}{b}\right)$$

Where $b_n^{g_2}$ is

$$b_n^{g_2} = \frac{2}{b}\int_0^b g_2(x) \sin\left(\frac{n\pi y}{b}\right) dy$$

Lecture 14

We will consider Laplace's equation in polar coordinates.

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Using the separation of variables

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\Theta'' = 0$$

Multiply both sides of the equation by $r^2/R\Theta$ to separate the R and Θ terms.

$$\frac{r^2R'' + rR'}{R} = \frac{-\Theta''}{\Theta} = \mu$$

We get 2 ODEs

$$\begin{cases} r^2R'' + rR' - \mu R = 0 \\ \Theta'' + \mu\Theta = 0 \end{cases}$$

The Θ equation is in the form we know how to solve.

Dirichlet: For $\Theta'' + \mu\Theta = 0$, given boundary conditions

$$u(r, 0) = 0 \Rightarrow \Theta(0) = 0$$
$$u(r, \alpha) = 0 \Rightarrow \Theta(\alpha) = 0$$

on a domain

$$a < r < b$$
$$0 < \theta < \alpha$$

The eigenvalues and eigenfunctions are

$$\mu_n = \lambda^2 \Rightarrow \lambda_n = \frac{n\pi}{\alpha} \qquad n = 1, 2, 3, \dots$$
$$\Theta_n(\theta) = \sin(\lambda_n\theta)$$

Neumann: For $\Theta'' + \mu\Theta = 0$, given boundary conditions

$$\Theta'(0) = \Theta'(\alpha) = 0$$

on a domain

$$a < r < b$$
$$0 < \theta < \alpha$$

The eigenvalues and eigenfunctions are

$$\mu_n = \lambda^2 \Rightarrow \lambda_0 = 0, \lambda_n = \frac{n\pi}{\alpha}$$
$$\Theta_n(\theta) = \cos(\lambda_n\theta) \qquad n = 0, 1, 2, 3, \dots$$

Periodic: For $\Theta'' + \mu\Theta = 0$, given boundary conditions

$$\Theta(-\pi) = \Theta(\pi)$$
$$\Theta'(-\pi) = \Theta'(\pi)$$

These are non-trivial conditions. Since in the polar coordinate space, $-\pi \neq \pi$.
The eigenvalues and eigenfunctions are

$$\mu_n = \lambda^2 \Rightarrow \lambda_n = \frac{n\pi}{\pi} = n \qquad n = 0, 1, 2, 3, \dots$$
$$\Theta_n(\theta) \in [1, \cos(\lambda_n\theta), \sin(\lambda_n\theta)]$$

Mixed BVP Type I: For $\Theta'' + \mu\Theta = 0$, given boundary conditions

$$\Theta(0) = 0 = \Theta'(\alpha)$$

The eigenvalues and eigenfunctions are

$$\mu_n = \lambda^2 \Rightarrow \lambda_n = \frac{(2n-1)\pi}{2\alpha} \qquad n = 1, 2, 3, \dots$$
$$\Theta_n(\theta) = \sin(\lambda_n\theta)$$

Mixed BVP Type II: For $\Theta'' + \mu\Theta = 0$, given boundary conditions

$$\Theta'(0) = 0 = \Theta(\alpha)$$

The eigenvalues and eigenfunctions are

$$\mu_n = \lambda^2 \Rightarrow \lambda_n = \frac{(2n-1)\pi}{2\alpha} \qquad n = 1, 2, 3, \dots$$
$$\Theta_n(\theta) = \cos(\lambda_n\theta)$$

We still have the ODE

$$r^2R'' + rR' - \mu R = r^2R'' + rR' - \lambda^2 R = 0$$

Which is an Cauchy-Euler equation. For any of 5 eigenvalue problems, for $\lambda^2 = \mu > 0$, we have

1. $\mu \neq 0$: The general solution is

$$R(r) = Ar^{\lambda} + Br^{-\lambda}$$

2. $\mu = 0$: The general solution is

$$\begin{aligned} R(r) &= A(1) + Br^0 \ln|r| \\ &= A + B\ln(r) \end{aligned}$$

We removed the absolute value sign since the domain of r satisfies the requirement $r > 0$.

Recall that we wrote $u(r, \theta) = R\Theta$. We can write

$$\begin{aligned} u(r, \Theta) &= R\Theta \\ &= R_0\frac{b_n}{2} + \sum_{n=1}^{\infty} R_n b_n \cos(\lambda_n\theta) + \sum_{n=1}^{\infty} R_n c_n \sin(\lambda_n\theta) \end{aligned}$$

From this stage we can combine constants to get the most general solution.

The most general solution is

$$\begin{aligned} u(r, \theta) &= (A_0 + \alpha_0 \ln(r))\cos(0) \\ &+ \sum_{n=1}^{\infty}\left(A_n r^{\lambda_n} + \alpha_n r^{-\lambda_n}\right)\cos(\lambda_n\theta) \\ &+ \sum_{n=1}^{\infty}\left(B_n r^{\lambda_n} + \beta_n r^{-\lambda_n}\right)\sin(\lambda_n\theta) \end{aligned}$$

When the domain of r includes 0, we assume that $\lim_{r\to 0+}|u| < \infty$. We must have $\alpha_0, \alpha_n, \beta_n = 0$, for each $n \geq 1$ or else their terms would blow up.
When the domain in infinite, then we assume that $\lim_{r\to\infty}|u| < \infty$. We must have $A_n, B_n, \alpha_0 = 0$ for $n \geq 1$.

Consider the problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(r, 0) = 0 = u_\theta(r, \pi) \\ u \to 0 \text{ as } r \to 0 \\ u(a, \theta) = f(\theta) \end{cases}$$

We know that is a case of the Mixed BVP type I. So the eigenvalues can functions are

$$\mu_n = \lambda^2 \Rightarrow \lambda_n = \frac{(2n-1)\pi}{2\alpha} \qquad n = 1, 2, 3, \dots$$
$$\Theta_n(\theta) = \sin(\lambda_n\theta)$$

In our case, since θ ranges from 0 to π, we have

$$\lambda_n = \frac{2n-1}{2} \qquad \Theta_n = \sin\left(\lambda_n = \frac{(2n-1)\theta}{2}\right)$$

Recall the most general solution,

$$
\begin{aligned}
u(r,\theta) &= (A_0 + \alpha_0 \ln(r))\cos(0) \\
&+ \sum_{n=1}^{\infty} \left(A_n r^{\lambda_n} + \alpha_n r^{-\lambda_n}\right)\cos(\lambda_n\theta) \\
&+ \sum_{n=1}^{\infty} \left(B_n r^{\lambda_n} + \beta_n r^{-\lambda_n}\right)\sin(\lambda_n\theta)
\end{aligned}
$$

We remove the first and second term since there are no cosine eigenfunctions. We also remove β_n term, since we have the additional condition that $u \to 0$ as $r \to 0$. So the solution is

$$
u(r,\theta) = \sum_{n=1}^{\infty} \left(B_n r^{\lambda_n}\right)\sin(\lambda_n\theta)
$$

To solve for coefficients B_n, we use the initial condition $u(a,\theta) = f(\theta)$.

$$
B_n a^{\lambda_n} = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin(\lambda_n\theta)\, d\theta = b_n^f
$$

B_n is given by the Fourier sine series coefficients divided by a^{λ_n}.

$$
B_n = \frac{b_n^f}{a^{\lambda_n}}
$$

We conclude that the solutions is

$$
\sum_{n=1}^{\infty} \frac{b_n^f}{a^{\lambda_n}} r^{\lambda_n}\sin(\lambda_n\theta) = \sum_{n=1}^{\infty} b_n^f \left(\frac{r}{a}\right)^{(2n-1)/2}\sin\left(\frac{2n-1}{2}\theta\right)
$$

Suppose that we know $u(a,\theta) = \sin(\theta/2)$. Then,

$$
\begin{aligned}
u(a,\theta) = \sin(\theta/2) &= \sum_{n=1}^{\infty} b_n^f \sin\left(\frac{2n-1}{2}\theta\right) \\
&= 1\sin\left(\frac{\theta}{2}\right) + 0 + \dots
\end{aligned}
$$

So it must be that

$$
b_n^f = \delta_{n1}
$$

So we conclude the solution to be

$$
u(r,\theta) = \left(\frac{r}{a}\right)^{1/2}\sin\left(\frac{\theta}{2}\right)
$$

it is clear that this solution satisfies the condition that u goes to zero as r goes to zero.

Consider another example of Laplace's equation on the domain

$$
\begin{aligned}
0 < r < a \\
0 < \theta < \alpha
\end{aligned}
$$

with the boundary and initial conditions

$$
\begin{aligned}
u_\theta(r,0) &= 0 \\
u(r,\alpha) &= u_1 \\
u(a,\theta) &= f(\theta)
\end{aligned}
$$

where u_1 is a constant.

The first step we need to do is get rid of the inhomogeneous boundary conditions. We guess a function w

$$
\begin{aligned}
w &= A\theta + B \\
w' &= A
\end{aligned}
$$

By the first boundary condition, we have $A = 0$. From the second boundary condition, we have $B = u_1$. So w is a constant function.

We write

$$
u(r,\theta) = v(r,\theta) + w
$$

We have

$$
\begin{aligned}
u_{rr} &= v_{rr} \\
u_r &= v_r \\
u_{\theta\theta} &= v_{\theta\theta}
\end{aligned}
$$

And by the boundary conditions

$$
\begin{aligned}
v_\theta(r,0) &= 0 \\
v(r,\alpha) &= 0
\end{aligned}
$$

By the initial condition:

$$
v(a,\theta) = f(\theta) - u_1
$$

We already know how to solve this problem. This is a mixed BVP type II.

Consider the following example

$$
\begin{cases}
\nabla^2 u = 0 \\
u_\theta(r,0) = 0 \\
u_\theta(r,\alpha) = 0 \\
u(b,\theta) = 0 \\
u(a,\theta) = f(\theta)
\end{cases}
$$

This is a Neumann type problem, so we have

$$
\lambda_n = \frac{\pi n}{\alpha} \qquad \Theta_n = \cos\left(\frac{n\pi}{\alpha}\theta\right) \qquad n = 0, 1, 2, \dots
$$

Since the eigenfunctions only consists of cosines, we have the solution to be

$$
\begin{aligned}
u(r,\theta) &= (A_0 + \alpha_0 \ln(r))\cos(0) \\
&+ \sum_{n=1}^{\infty} \left(A_n r^{\lambda_n} + \alpha_n r^{-\lambda_n}\right)\cos(\lambda_n\theta)
\end{aligned}
$$

We can find the coefficients A_n and α_n, we use $u(b,\theta) = 0$,

$$
\begin{aligned}
u(b,\theta) &= (A_0 + \alpha_0 \ln(b)) \\
&+ \sum_{n=1}^{\infty} \left(A_n b^{\lambda_n} + \alpha_n b^{-\lambda_n}\right)\cos(\lambda_n\theta) = 0
\end{aligned}
$$

By linear independence of $1, \cos(\lambda_n\theta)$, we have

$$
\begin{aligned}
A_0 + \alpha_0 \ln(b) &= 0 \\
A_n b^{\lambda_n} + \alpha_n b^{-\lambda_n} &= 0
\end{aligned}
$$

From the second equality we find

$$
\alpha_n = \frac{-A_n b^{\lambda_n}}{b^{-\lambda_n}} = -A_n b^{2\lambda_n}
$$

So

$$
u(r,\theta) = \alpha_0 \ln\left(\frac{r}{b}\right) + \sum_{n=1}^{\infty} A_n \left(r^{\lambda_n} - b^{2\lambda_n} r^{-\lambda_n}\right)\cos(\lambda_n\theta)
$$

Use the initial condition $u(a,\theta) = f(\theta)$, we have

$$
\begin{aligned}
u(a,\theta) &= f(\theta) \\
&= \alpha_0 \ln\left(\frac{a}{b}\right) + \sum_{n=1}^{\infty} A_n \left(a^{\lambda_n} - b^{2\lambda_n} a^{-\lambda_n}\right)\cos(\lambda_n\theta)
\end{aligned}
$$

We can "match" this expression with the cosine Fourier series for $f(\theta)$.

$$
f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^f \cos(\lambda_n)
$$

We solve for the coefficients to find that

$$
a_n^2 = \frac{2}{\alpha}\int_0^{\alpha} f(\theta)\cos(\mu_n\theta)\, d\theta
$$

So we have

$$
\alpha_0 \ln\left(\frac{a}{b}\right) = \frac{a_0^f}{2} \Rightarrow \alpha_0 = \frac{a_0^f}{2\ln(a/b)}
$$

and

$$
A_n\left(a^{\lambda_n} - b^{2\lambda_n} a^{-\lambda_n}\right) = a_n^f
$$

Thus

$$
A_n = \frac{a_n^f}{a^{\lambda_n} - b^{2\lambda_n} a^{-\lambda_n}}
$$

for $n \geq 1$.

Lecture 15

Sturm-Liouville Theory: Sturm-Liouville boundary value problems are generalizations to real second order linear ODEs. The Sturm-Liouville form is

$$\mathcal{L}y = -\left(p(x)y'\right)' + q(x)y = \lambda r(x)y \tag{46}$$

alternatively,

$$-\left(p(x)y'\right)' + \left(q(x) - \lambda r(x)\right)y = 0 \tag{47}$$

where $0 \le x \le L$ and we assume $p(x), q(x), p', r(x)$ are continuous over x, and that $p(x) \ge 0, r(x) \ge 0$ over x.

Sturm-Liouville problems must also have a separated BC at two different points.

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(L) + \beta_2 y'(L) = 0 \end{cases}$$

Regular SL Problem: if $p(x), r(x)$ are both > 0 on $[0, L]$, and $L < \infty$.

Singular SL Problem: if $p(x), r(x)$ are both $= 0$ on $[0, L]$, or $L = \infty$.

The theory and properties we cover will only consider regular SL problems.

Problems with a periodic boundary condition is not a Sturm-Liouvile since the boundary values are not separated.

Consider the general second order linear ODE.

$$-P(x)y'' - Q(x)y' + R(x)y = \lambda y$$

If we multiply both sides by an integrating factor μ,

$$-\mu P(x)y'' - \mu Q(x)y' + \mu R(x)y = \mu\lambda y$$

and compare this to the expanded Sturm-Liouville form,

$$-p(x)y'' - p'(x)y' + q(x)y = \lambda r(x)y$$

and we would these two forms to be equivalent, then we require

$$\begin{cases} p(x) = \mu P(x) \\ p' = \mu Q(x) \end{cases}$$

Differentiate the first equation and equating it to the second

$$p'(x) = \mu Q = \mu' P(x) + \mu P'(x)$$

Rearranging

$$\mu' P(x) + \mu\left(P'(x) - Q(x)\right) = 0$$
$$\mu' + \mu\left(\frac{P'(x)}{P} - \frac{Q(x)}{P}\right) = 0$$

This is a first order ODE which we can solve by using the separation of variables.

We would get

$$\mu = K\exp\left(\int \frac{Q(x)}{P(x)} - \frac{P'(x)}{P(x)}\,dx\right)$$
$$= \frac{K}{P(x)}\exp\left(\int \frac{Q(x)}{P(x)}\,dx\right)$$

We can omit the constant K since we only need one integrating factor.

Convert into Sturm-Liouville form: Multiplying a second order ODE of the form

$$-P(x)y'' - Q(x)y' + R(x)y = \lambda y$$

by the factor

$$\mu = \frac{1}{P}\exp\left(\int \frac{Q(x)}{P(x)}\,dx\right)$$

converts the ODE into Sturm-Liouville form.

ODEs in Sturm-Liouville form

$$-\left(p(x)y'\right)' + q(x)y = \lambda r(x)y$$

1. Infinite sequence of ordered eigenvalues λ_j, and associating eigenfunctions ϕ_j. Such that
$$\lambda_1 > \lambda_2 > \lambda_3 > \ldots$$
2. $\lambda_n \to \infty$ as $n \to \infty$
3. λ_j are distinct and real
4. ϕ_j are mutually orthogonal with respect to the weight function $r(x)$.
$$\int_0^L r(x)\phi_i(x)\phi_j(x)\,dx = 0 \qquad i \ne j$$
5. The set of eigenfunctions is complete, in a sense that any C^∞ function $f(x)$ admits an eigenfunction expansion
$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$
6. For $f(x)$ and $f'(x)$ that is piecewise continuous, then eigenfunction expansion converges to $f(x)$ for every x where f is continuous, and converges to $f(x^+) + f(x^-)/2$ at point of discontinuity.

In addition, if it has boundary conditions

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(L) + \beta_2 y'(L) = 0 \end{cases}$$

then

7. Then each eigenvalue is greater than zero provided $\alpha_1/\alpha_1 < 0$ and $\beta_1/\beta_2 > 0$.

To find the coefficients c_k, we use the property that the eigenfunctions are orthogonal with respect to $r(x)$. Multiply both sides of the expression by the weight function and an eigenfunction

$$\int_0^L r(x)f(x)\phi_n(x)\,dx = \sum_{k=1}^{\infty} c_k \int_0^L r(x)\phi_k(x)\phi_n(x)\,dx$$

By orthogonality,s the integral on the right is non-zero only when $k = n$. Rearranging

$$c_n = \frac{\int_0^L r(x)f(x)\phi_n(x)\,dx}{\int_0^L r(x)\phi_n^2(x)\,dx}$$

Consider the following eigenvalue value problem with an *Robin* boundary condition.

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < 1 \\ X(0) = 0 \\ X'(1) + X(1) = 0 \end{cases}$$

To find the eigenvalues of this problem, we need test three cases.

When $\lambda < 0$, the general solution involves exponentials or hyperbolic functions.

$$X(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$$

Let us denote $\sqrt{-\lambda}$ by a constant μ. Using the first boundary condition forces A to be zero. The second boundary condition

$$X'(1) + X(1) = \mu B\cosh(\mu 1) + B\sinh(\mu 1) = 0$$

This also forces $B = 0$. So for the case when $\lambda < 0$, there are no eigenvalues or eigenfunctions.

When $\lambda > 0$, then the general solution must involve sines and cosines. Let us denote $\sqrt{\lambda}$ with a constant μ. Thus

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

The first boundary condition forces A to be zero.

$$X'(x) = \mu B\cos(\mu x)$$

The second boundary condition

$$X'(1) + X(1) = \mu B\cos(\mu 1) + B\sin(\mu 1) = 0$$

Rearranging,

$$\tan(\mu) = -\mu$$

which is an equation we won't be able to solve by hand. It turns our that

$$\mu_n \approx \frac{(2n-1)\pi}{2}$$

is a good approximation for eigenvalues. So we conclude that

$$X_n(x) = \sin(\mu_n x)$$
$$\mu_n \approx \frac{(2n-1)\pi}{2} \qquad n = 1, 2, 3, \ldots$$

are the eigenfunctions and eigenvalues.

Lecture 16

Consider the following *Robin* problem.

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < L \\ X'(0) = h_1 X(0) & h_1 \geq 0 \\ X'(L) = -h_2 X(L) & h_2 \geq 0 \end{cases}$$

In case both h_1, h_2 are zero, then it is the Neumann boundary condition. So Let us assume $h_1 \neq 0$ for our analysis.

For this boundary condition, we can tell that our eigenvalues must be positive. Since $-h_1/h_2 < 0$ and $1/1 > 0$. The the general solution to equation is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Let us denote $\mu = \sqrt{\lambda}$. We have

$$\begin{aligned} X(x) &= A\cos(\mu x) + B\sin(\mu x) \\ X'(x) &= -A\mu\sin(\mu x) + \mu B\cos(\mu x) \end{aligned}$$

Plugging the boundary condition

$$\mu B = h_1 A \Rightarrow A = \frac{\mu B}{h_1}$$

For the second boundary condition

$$\begin{aligned} &-A\mu\sin(\mu L) + \mu B\cos(\mu L) = \\ &-h_2\left(A\cos(\mu L) + B\sin(\mu L)\right) \end{aligned}$$

Substitute in $A = \mu B/h_1$.

$$\begin{aligned} &-\frac{\mu B}{h_1}\mu\sin(\mu L) + \mu B\cos(\mu L) = \\ &-h_2\frac{\mu B}{h_1}\cos(\mu L) - h_2 B\sin(\mu L) \end{aligned}$$

we can factor out B from both sides, and eliminate. This is valid since if B were zero, there would no eigenfunctions.

$$\begin{aligned} &-\frac{\mu^2}{h_1}\sin(\mu L) + \mu\cos(\mu L) = \\ &-h_2\frac{\mu}{h_1}\cos(\mu L) - h_2\sin(\mu L) \end{aligned}$$

Further factoring

$$\begin{aligned} &\left(-\frac{\mu^2}{h_1} + h_2\right)\sin(\mu L) + \left(\mu + h_2\frac{\mu}{h_1}\right)\cos(\mu L) = 0 \\ &\left(\frac{-\mu^2 + h_1 h_2}{h_1}\right)\sin(\mu L) + \left(\frac{\mu(h_1 + h_2)}{h_1}\right)\cos(\mu L) = 0 \end{aligned}$$

Rearranging gives

$$\tan(\mu L) = \frac{\mu(h_1 + h_2)}{\mu^2 - h_1 h_2}$$

For the case that $h_2 > 0$, the eigenfunctions are

$$X_n(x) = \frac{\mu_n}{h_1}\cos(\mu_n x) + \sin(\mu_n x)$$

turns out that the eigenvalues are

$$\mu_n \approx \frac{n\pi}{L}$$

and this approximation gets better for large n.

For the case that $h_2 = 0$, then

$$\begin{aligned} \tan(\mu_n L) &= \frac{h_1}{\mu_n} \\ \mu\sin(\mu_n L) &= h_1\cos(\mu_n L) \end{aligned}$$

Substituting,

$$\begin{aligned} X_n(x) &= \frac{\mu_n}{h_1}\cos(\mu_n x) + \sin(\mu_n x) \\ &= \frac{\mu_n\cos(\mu_n x) + h_1\sin(\mu_n x)}{h_1} \end{aligned}$$

Multiply numerator and denominator by $\cos(\mu_n L)$ gives

$$X_n(x) = \frac{\mu_n\cos(\mu_n x)\cos(\mu_n L) + h_1\cos(\mu_n L)\sin(\mu_n x)}{h_1\cos(\mu_n L)}$$

recall that we found $\mu\sin(\mu_n L) = h_1\cos(\mu_n L)$,

$$\begin{aligned} X_n(x) &= \frac{\mu_n\cos(\mu_n x)\cos(\mu_n L) + \mu\sin(\mu_n L)\sin(\mu_n x)}{\mu\sin(\mu_n L)} \\ &= \frac{\cos(\mu_n(x - L))}{\sin(\mu_n L)} \end{aligned}$$

the last manipulation requires the identity

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

For the case that $h_1 \longrightarrow \infty$ (h_1 is large), and $h_2 \neq 0$. Then eigenfunctions and eigenvalues turn out be

$$\begin{aligned} X_n(x) &= \sin(\mu_n x) \qquad n = 1, 2, 3, \ldots \\ \mu_n &\approx \frac{(2n-1)\pi}{2L} \end{aligned}$$

Consider the heat equation on a robin boundary condition

$$\begin{cases} u_t = \alpha^2 u_{xx} & 0 < x < 1 \\ u(0, t) = 1 \\ u_x(1, t) + u(1, t) = 0 \\ u(x, 0) = f(x) \end{cases}$$

Notice that we have $u(0, t) = 1$, which is inhomogeneous. We can get rid of it by finding steady state solution $u_\infty(x)$. We must have

$$\begin{aligned} u_\infty(x) &= Ax + B \\ u'_\infty(x) &= A \end{aligned}$$

By the BCs, we have

$$\begin{aligned} &u_\infty(0) = 1 = B \\ &A + A + 1 = 0 \Rightarrow A = -1/2 \end{aligned}$$

Thus we have

$$u_\infty(x) = \frac{-x}{2} + 1$$

We write $u(x, t) = u_\infty(x) + v(x, t)$. By substituting into the original equation, we have

$$\begin{cases} v_t = \alpha^2 v_{xx} \\ v(0, t) = 0 \\ v_x(1, t) + v(1, t) = 0 \\ v(x, 0) = f(x) - u_\infty \end{cases}$$

This we know how to solve by using the separation of variables, $v(x, t) = XT$. We would find that

$$\frac{\dot{T}}{\alpha^2 T} = \frac{X''}{X} = -\lambda$$

We can easily solve the T equation using separation of variables.

$$T = \exp\left(-\lambda\alpha^2 t\right)$$

The X equation is an eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X'(1) + X(1) = 0 \end{cases}$$

We solved this equation in the previous lecture. Let $\sqrt{-\lambda} = \mu$ The eigenfunctions are

$$X = \sin(\mu_n x)$$

and the eigenvalues are the roots to $\tan(\mu_n) = -\mu_n$. Let us take linear combinations of the solution

$$v(x, t) = \sum_{n=1}^{\infty} c_n\sin(\mu_n x)\exp\left(-\mu^2\alpha^2 t\right)$$

We still have the initial condition we can use to solve for the constants c_n.

$$v(x, 0) = f(x) - \mu_\infty = \sum_{n=1}^{\infty} c_n\sin(\mu_n x)$$

By orthogonality, we have

$$c_n = \frac{\int_0^1 (f(x) - \mu_\infty)\sin(\mu_n x)\,dx}{\int_0^1 \sin(\mu_n x)^2\,dx}$$

Then the final answer is

$$u(x, t) = \frac{-x}{2} + 1 + \sum_{n=1}^{\infty} c_n\sin(\mu_n x)\exp\left(-\mu^2\alpha^2 t\right)$$

Sturm-Liouville Problems involving the Cauchy-Euler equation

Consider eigenvalue problem

$$\begin{cases} x^2 y'' + xy' + \lambda y = 0 & 1 < x < 2 \\ y(1) = 0 \\ y'(2) = 0 \end{cases}$$

The characteristic equation for such an equation is

$$r^2 + \lambda = 0$$

For the case that $\lambda > 0$, the general solution to Cauchy-Euler equation is

$$y(x) = A\cos\left(\sqrt{\lambda}\ln(x)\right) + B\sin\left(\sqrt{\lambda}\ln(x)\right)$$

Let us denote $\sqrt{\lambda}$ by μ.

The first boundary condition forces $A = 0$. The second boundary condition gives

$$\mu = \frac{(2n-1)\pi}{2\ln(2)} \qquad n = 1, 2, 3, \dots$$
$$y_n = \sin(\mu_n \ln(x))$$

It can be shown that there are no eigenvalues for the case $\lambda < 0$ and $\lambda = 0$.

We can take infinite linear combinations of the solutions.

$$y(x) = \sum_{n=1}^{\infty} c_n \sin(\mu_n \ln(x))$$

Realize that this is also a Sturm-Liouville ODE with the weight function $1/x$. By definition

$$\int_1^2 r(x)\sin(\mu_m \ln(x))\sin(\mu_n \ln(x))\,dx = \begin{cases} 0 & m \neq n \\ \frac{ln(2)}{2} & m = n \end{cases}$$

Lecture 17

Consider Laplace's equation on a semi-annular region.

$$\begin{cases} 1 < r < 2 \\ 0 < \theta < \pi \end{cases}$$

The equation and boundary conditions are

$$\begin{cases} \nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(r, 0) = 0 \\ u(r, \pi) = f(r) \\ u(1, \theta) = 0 \\ u_r(2, \theta) = 0 \end{cases}$$

Using separation of variables $u = R\Theta$, we have

$$\frac{r^2 R'' + rR}{R} = \frac{-\Theta''}{\theta} = -\lambda$$

and we get two ODEs

$$r^2 R'' + rR'' + R\lambda = 0$$
$$\Theta'' - \lambda\Theta = 0$$

It follows from the 3rd and 4th boundary condition that

$$u(1, \theta) = 0 = R(1)\Theta \Rightarrow R(1) = 0$$
$$u_r(2, \theta) = 0 = R'(2)\Theta \Rightarrow R'(2)$$

Notice that our first ODE is a Cauchy-Euler equation. We found in the last lecture that it has eigenfunctions and eigenvalues equal to

$$\sqrt{\lambda} = \mu = \frac{(2n-1)\pi}{2\ln(2)}$$
$$R(r) = \sin(\mu\ln(r)) \qquad n = 1, 2, 3, \dots$$

We immediately know that

$$\Theta'' - \mu^2\Theta = 0$$

And it's general solution is

$$\Theta(\theta) = A\cosh(\mu\theta) + B\sinh(\mu\theta)$$

From the first boundary condition

$$\Theta(0) = 0 = A$$
$$\Theta(\theta) = B\sinh(\mu\theta)$$

The representative eigenfunction is

$$\Theta_n = \sinh(\mu\theta)$$

One solution is

$$u(r, \theta) = \sin(\mu\ln(r))\sinh(\mu\theta)$$

Taking linear combinations

$$u(r, \theta) = \sum_{n=1}^{\infty} c_n \sin(\mu\ln(r))\sinh(\mu\theta)$$

By $u(r, \pi) = f(r)$, we have

$$u(r, \pi) = f(r) = \sum_{n=1}^{\infty} c_n \sin(\mu\ln(r))\sinh(\mu\pi)$$

We learned that Cauchy Euler ODEs of this form has weight function $1/r$. By property of Sturm-Liouville ODEs, we have

$$\int_1^2 \frac{f(r)\sin(\mu\ln(r))}{r}\,dr$$
$$= \sum_{n=1}^{\infty} c_n \sinh(\mu\pi)\int_1^2 \frac{\sin(\mu_n\ln(r))\sin(\mu_m\ln(r))}{r}\,dr$$

The integral on the right is nonzero only for $n = m$. So

$$\int_1^2 \frac{f(r)\sin(\mu\ln(r))}{r}\,dr$$
$$= c_n\sinh(\mu\pi)\int_1^2 \frac{\sin^2(\mu_n\ln(r))}{r}\,dr$$

By letting $u = \ln(r)$, and and applying orthogonality, we have

$$\int_1^2 \frac{\sin^2(\mu_n\ln(r))}{r}\,dr = \frac{\ln(2)}{2}$$

In summary we the expression for c_n

$$c_n = \frac{\int_1^2 \frac{1}{r}f(r)\sin(\mu\ln(r))\,dr}{\sinh(\mu\pi)\ln(2)/2}$$

Bessel's Function: are solutions to Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

ν is a constant. When ν is zero, the equation is known as Bessel's equation of order zero.

Consider Bessel's equation of order zero

$$x^2y'' + xy' + x^2y = 0$$

We can Frobenius series solution to show that this equation has two linearly independent solutions

$$y_1(x) = J_0(x) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

known as Bessel function of the first kind of order zero. And

$$y_2(x) = Y_0(x) = \frac{2}{\pi}\left[(\gamma + \ln(x/2))J_0(x) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}x^{2n}\right]$$

Bessel function of second kind of order zero. We define

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

and

$$\gamma = \lim_{n\to\infty}(H_n - \ln(n)) \approx 0.5772$$

γ is amazing, since it is the difference between the logarithm and the harmonic series, which both diverge on their own, but is finite when subtracted against each other.

Consider the eigenvalue problem

$$xy'' + y' + \lambda xy = 0$$

Lecture 18

Consider the following problem of Laplace's equation on the quarter of an annulus:

$$\begin{cases} 1 < r < 2 \\ 0 < \theta < \pi/2 \end{cases}$$

With the boundary conditions

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u_\theta(r,0) = 0 \\ u_\theta(r,\pi/2) = 0 \\ u(1,\theta) = 0 \\ u(2,\theta) = \cos(4\theta) \end{cases}$$

Let us use the separation of variables. $u(r,\theta) = R\Theta$. Using the boundary conditions, we can get some information about R and Θ.

$$\begin{aligned} u_\theta(r,0) &= R(r)\Theta'(0) = 0 && \Rightarrow \Theta'(0) = 0 \\ u_\theta(r,\pi/2) &= R(r)\Theta'(\pi/2) = 0 && \Rightarrow \Theta'(\pi/2) = 0 \\ u(1,\theta) &= R(1)\Theta(\theta) = 0 && \Rightarrow R(1) = 0 \end{aligned}$$

From this we can tell that this is going to involve a Neumann boundary condition on Θ.

The separation of variables tells us

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

For the Θ equation, the eigenvalue problem is

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta'(0) = 0, \Theta'(\pi/2) = 0 \end{cases}$$

For this problem, we get eigenfunctions when $\lambda > 0$. Let us denote $\sqrt{\lambda} = \mu$. The general solution is

$$\Theta(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$$

We know eigenfunctions and eigenvalues will be

$$\mu_n = \frac{n\pi}{\pi/2} = 2n$$

$$\Theta_n = \cos\left(\frac{n\pi\theta}{\pi/2}\right) = \cos(2n\theta)$$

For the R equation, we only know

$$\begin{cases} r^2R'' + rR' - \lambda R = 0 \\ R(1) = 0 \end{cases}$$

We have found in past lectures that

$$\begin{aligned} u(r,\theta) = (A_0 + \alpha_0\ln(r))\cos(0) \\ + \sum_{n=1}^{\infty}(A_n r^{\mu_n} + \alpha_n r^{-\mu_n})\cos(\mu_n\theta) \\ + \sum_{n=1}^{\infty}(B_n r^{\mu_n} + \beta_n r^{-\mu_n})\sin(\mu n\theta) \end{aligned}$$

We don't need to include the second sum involving sines. Since we only have cosine eigenfunctions. Using the condition $u(1,\theta) = 0$:

$$0 = u(1,\Theta) = (A_0 + \alpha_0\ln(1)) + \sum_{n=1}^{\infty}(A_n 1 + \alpha_n 1)\cos(\mu_n\theta)$$

Using the linear independence of cosines,

$$\begin{aligned} A_0 + \alpha_0\ln(1) = 0 &\Rightarrow A_0 = 0 \\ A_n + \alpha_n = 0 &\Rightarrow A_n = -\alpha_n \qquad n \geq 1 \end{aligned}$$

So we have

$$u(r,\theta) = \alpha_0\ln(r) + \sum_{n=1}^{\infty}(-\alpha_n r^{\mu_n} + \alpha_n r^{-\mu_n})\cos(\mu_n\Theta)$$

Using $u(2,\Theta) = \cos(4\Theta)$,

$$\begin{aligned} u(2,\theta) &= \cos(4\theta) \\ &= \alpha_0\ln(2) + \sum_{n=1}^{\infty}(-\alpha_n 2^{\mu_n} + \alpha_n 2^{-\mu_n})\cos(\mu_n\Theta) \end{aligned}$$

Matching the coefficients above with the cosine fourier series for $\cos(4\theta)$, we have

$$\alpha_0\ln(2) = 0 \Rightarrow \alpha_0 = 0$$

$$\alpha_n(2^{-\mu_n} - 2^{2n}) = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases}$$

So

$$\alpha_2 = \frac{1}{2^{-4} - 2^4}$$

In summary

$$u(r,\Theta) = \left(-\alpha_2 r^4 + \alpha r^{-4}\right)\cos(2n\Theta)$$

Consider the following Sturm-Liouville problem. Find the eigenvalues and eigenfunctions

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X'(1) = -X(1) \end{cases}$$

If $\lambda = 0$, then $X = Ax + B$, and $X' = A$. Using the condition $X'(0) = 0$ forces $A = 0$. Using the second condition $X'(1) = -X(1)$ forces $B = 0$. So we only have trivial solutions in this case.

If $\lambda < 0$, then the general solution is

$$\begin{aligned} X(x) &= Ae^{\mu x} + Be^{-\mu x} \\ X'(x) &= \mu Ae^{\mu x} - \mu Be^{-\mu x} \end{aligned}$$

where we denoted $\mu = \sqrt{-\lambda}$. Using $X'(0) = 0$, we have

$$\mu A - \mu B = 0 \Rightarrow A = B$$

So we can write

$$\begin{aligned} X(x) &= Ae^{\mu x} + Ae^{-\mu x} \\ X'(x) &= \mu Ae^{\mu x} - \mu Ae^{-\mu x} \end{aligned}$$

Using $X'(1) = -X(1)$, so

$$\begin{aligned} \mu Ae^{\mu} - \mu Ae^{-\mu} &= -Ae^{\mu} + Ae^{-\mu} \\ \mu e^{\mu} - \mu e^{-\mu} &= -e^{\mu} + e^{-\mu} \end{aligned}$$

For $\lambda > 0$, let us denote $\sqrt{\lambda} = \mu$. We have the general solution

$$\begin{aligned} X &= A\cos(\mu x) + B\sin(\mu x) \\ X' &= -\mu A\sin(\mu x) + \mu B\cos(\mu x) \end{aligned}$$

The first condition givens $B = 0$. The second consider, assuming $A \neq 0$ gives the equation

$$\tan(\mu) = \frac{1}{\mu}$$

This function has finitely many ordered eigenvalues. The eigenfunction is

$$X_n = \cos(\mu x)$$

Solve Laplace's equation on the semi-infinite strip

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < 1, 0 < y < \infty \\ u_x(0,y) = 0 \\ u_x(1,y) + u(1,y) = 0 \\ u(x,0) = 1 \\ \lim_{y\to\infty}|u| = 0 \end{cases}$$

Using the separation of variables gives rise an eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X'(1) = X(1) \end{cases}$$

We have already solved this problem in part 1.

For the equation in Y, we have

$$\begin{aligned} Y'' - \lambda Y &= 0 \\ Y'' - \mu_n^2 Y &= 0 \end{aligned}$$

The solution to this problem involves exponential since μ_n^2 is positive. So

$$Y_n(y) = A_n e^{\mu_n y} + B_n e^{\mu_n y}$$

From boundary conditions we have find that

$$Y_n = e^{-\mu y}$$

So the solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n\cos(\mu x)e^{\mu y}$$

Using the last boundary condition $1 = u(x,0)$ and the orthogonality property of Sturm-Liouville eigenfunctions to solve for the constants c_n. In our case, the weight function is simply 1. So,

$$c_n = \frac{\int_0^1 \cos(\mu_n x)\,dx}{\int_0^1 \cos^2(\mu_n x)\,dx}$$