PHYS 401

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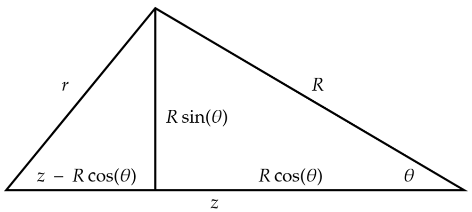
Fundamentals

The vector from a source point to a field point:

$$\underline{\mathscr{r}} := \underline{r} - \underline{r}'$$



Given z, R, θ, the length r follows from

$$\begin{aligned} r^2 &= (R\sin(\theta))^2 + (z - R\cos(\theta))^2 \\ &= z^2 + R^2 - 2Rz\cos(\theta) \end{aligned}$$

Common integrals

Omitting the constant of integration:

$$\int \sin^2(\theta)\, d\theta = \int \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$\int \cos^2(\theta)\, d\theta = \int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$\int \cos^4(\theta)\, d\theta = \frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32}$$

$$\int \left(\frac{3\cos^2(\theta) - 1}{2}\right) d\theta = \frac{1}{2}\int 2 - 3\sin^2(\theta)\, d\theta$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\int \frac{x}{(x^2+a^2)^{3/2}}\, dx = \frac{-1}{\sqrt{x^2+a^2}}$$

$$\int \frac{1}{(x^2+a^2)^{3/2}}\, dx = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}$$

$$\int \frac{x^3}{(x^2+a^2)^{3/2}}\, dx = \frac{x^2+2a^2}{\sqrt{x^2+a^2}}$$

$$\int_0^\pi \sin^2(\theta)\, d\theta = \int_0^\pi \cos^2(\theta)\, d\theta = \pi/2$$

$$\int_0^\pi \sin^4(\theta)\, d\theta = \int_0^\pi \cos^4(\theta)\, d\theta = 3\pi/8$$

Series expansion

Maclaurin series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

Integration

Line integrals

For $f(x, y, z)$ over a curve C described by

$$\underline{r}(t) = x(t)\underline{e}_1 + y(t)\underline{e}_2 + z(t)\underline{z}_3,$$

where

$$a \le t \le b$$

If $f = 1$, then following gives the arc-length.

$$\int_C f\, ds = \int_a^b f(x(t), y(t), z(t))\, \|\underline{r}'(t)\|\, dt$$

1-D integrals are effectively line integrals.

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

is an integral over the curve

$$\underline{r}(t) = x\underline{e}_1$$

for $a \le x \le b$. Let $\underline{T}$ denote the unit tangent of the curve $\underline{r}(t)$. Denote component wise differentiation w.r.t. t by sup prime.

$$\underline{T} = \underline{r}'/\|\underline{r}'\|$$

The line integral over C described by $\underline{r}(t)$ for the vector field $\underline{F}$:

$$\int_C \underline{F}\cdot d\underline{r} = \int_C \underline{F}\cdot\underline{T}\, ds = \int_a^b \underline{F}(\underline{r}(t))\cdot\underline{r}'\, dt$$

Surface integrals

If the surface is given as a graph, $z = g(x, y)$, we can use the graph to form a parameterization:

$$\underline{r}(x, y) = \langle x, y, g(x, y)\rangle$$

The integral of a scalar field over a surface, S, described by $\underline{r}(x, y)$:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\underline{r}(x, y))\, \|\underline{r}_{,1} \times \underline{r}_{,2}\|\, dA$$

Let $\underline{n}$ be the unit normal of the surface S described by $\underline{r}(x, y)$:

$$\underline{n} = \left(\underline{r}_{,1} \times \underline{r}_{,2}\right) / \|\underline{r}_{,1} \times \underline{r}_{,2}\|$$

The flux integral of $\underline{F}$ across S is

$$\iint_S \underline{F}\cdot d\underline{S} = \iint_S \underline{F}\cdot\underline{n}\, dS = \iint_D \underline{F}(\underline{r}(x, y))\cdot\left(\underline{r}_{,1}\times\underline{r}_{,2}\right) dA$$

Ignore the Jacobian in the right most integral.

Volume integrals

Volume under a surface given by $z = f(x, y)$ and above the region D

$$V = \iint_D f(x, y)\, dA$$

Trigonometric identities

$$\begin{aligned}
e^{j\theta} &= \cos(\theta) + j\sin(\theta) \\
\cos(-\alpha) &= \cos(\alpha) \\
\sin(-\alpha) &= -\sin(\alpha) \\
\cos(\omega t) &= \sin(\omega t + \pi/2) \\
\sin(\omega t) &= \cos(\omega t - \pi/2) \\
\cos(\omega t) &= -\cos(\omega t \pm \pi) \\
\sin(\omega t) &= -\sin(\omega t \pm \pi) \\
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
\cos(\alpha)\cos(\beta) &= \cos(\alpha+\beta)/2 + \cos(\alpha-\beta)/2 \\
\sin(\alpha)\cos(\beta) &= \sin(\alpha+\beta)/2 + \sin(\alpha-\beta)/2 \\
\cos^2(\theta) &= 1 - \sin^2(\theta) \\
\sec^2(\theta) &= \tan^2(\theta) + 1 \\
\tan^2(\theta) &= \sec^2(\theta) - 1
\end{aligned}$$

Combining $\cos(\omega t)$ and $\sin(\omega t)$:

$$A\sin(\omega t) + B\cos(\omega t) = \sqrt{A^2+B^2}\cos\left[\omega t - \tan(B/A)\right]$$

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Integral theorems

Divergence theorem

Let Ω be 3D solid with a piece-wise smooth $\partial\Omega$ (the boundary has a differentiable parameterization with nonzero normal), and $\underline{f}$ has continuous first partials. In direct notation

$$\iiint_\Omega \underline{\nabla}\cdot\underline{f}\, dV = \iint_{\partial\Omega} \underline{f}\cdot\underline{n}\, dS$$

In index notation

$$\iiint_\Omega f_{i,i}\, dV = \iint_{\partial\Omega} f_i n_i\, dS$$

Stokes' theorem

Let S be a piecewise smooth, oriented surface. And $\underline{F}$ has continuous first partials.

$$\int_{\partial S} \underline{F}\cdot d\underline{r} = \iint_\Omega \underline{\nabla}\times\underline{F}\cdot d\underline{S}$$

Green's theorem

Green's theorem is Stokes' theorem applied to a region in a $xy-$plane. Set normal vector to the z component of the orthonormal basis, and $F_3 = 0$.

$$\iint_\Omega \frac{\partial F_2}{\partial x} - \frac{\partial F}{\partial y}\, dA = \int_{\partial\Omega} \underline{F}\, d\underline{r}$$

Fundamental theorem of line integrals

If $\underline{F} = \underline{\nabla} f$, then $\underline{F}$ is a conservative vector field. For $\underline{F}$:

$$\int_a^b \underline{F}(\underline{r}(t))\cdot\underline{r}'\, dt = f(\underline{r}(b)) - f(\underline{r}(a))$$

$\underline{F}$ is path independent.

Integration by parts

Is the application of the divergence theorem and the product rule.

$$\int_\Omega w j_{i,i}\, dV = \int_\Omega (w j_i)_{,i} - w_{,i} j_i \, dV$$

By the divergence theorem:

$$= \int_{\partial\Omega} w j_i n_i \, dS - \int_\Omega w_{,i} j_i \, dV$$

Trigonometric substitution

$$\begin{cases} \sqrt{a^2 - x^2} & x = a\sin(\theta) \\ \sqrt{a^2 + x^2} & x = a\tan(\theta) \\ \sqrt{x^2 - a^2} & x = a\sec(\theta) \end{cases}$$

Index notation

Implied summation over repeated indices in a term:

$$\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ij}\delta_{ji} = \delta_{ii} = 3$$

Levi-Civita tensor:

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ is a cyclic permutation of 123} \\ -1 & ijk \text{ is a anticyclic permutation of 123} \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$$

Cyclic permutations of ijk do not change the sign of ϵ_{ijk}:

$$\epsilon_{ijk} = -\epsilon_{ikj} = -\epsilon_{kji} = -\epsilon_{jik}$$
$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

An identity:

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

If $n = k$, then

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix}$$

If $i = l$, $j = m$, $n = k$, then

$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

Orthonormal Cartesian basis vectors

$\underline{e}_i$

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

and

$$\underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\underline{e}_k$$

Vector $\underline{A}$ in Cartesian basis:

$$\underline{A} := A_i\underline{e}_i = A_1\underline{e}_1 + A_2\underline{e}_2 + A_3\underline{e}_3$$

Scalar product

$$\begin{aligned} \underline{A}\cdot\underline{B} &:= \sum_{i=1}^{3} A_i\underline{e}_i \cdot \sum_{j=1}^{3} B_i\underline{e}_j = \sum_{i,j} A_i B_j \underline{e}_i \underline{e}_j \\ &= \sum_{i,j} A_i B_j \delta_{ij} = \sum_i A_i B_i = A_i B_i \\ &= \|\underline{A}\|\|\underline{B}\|\cos(\theta) = \left(A_i A_i B_j B_j\right)^{1/2}\cos(\theta), \end{aligned}$$

where θ is a acute angle between $\underline{A}$ and $\underline{B}$; $\|A\| := (A_i A_i)^{1/2}$.

Cross product

$$\begin{aligned} \underline{A}\times\underline{B} &= \epsilon_{ijk}A_i B_j \underline{e}_k \\ &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \end{aligned}$$

The k^{th} component of $\underline{A}\times\underline{B}$:

$$\left(\underline{A}\times\underline{B}\right)_k = \epsilon_{ijk}A_i B_j$$

Gradient

$$\underline{\nabla} u = \langle u_{,1}, u_{,2}, u_{,3} \rangle$$

The i^{th} component:

$$u_{,i}$$

Divergence

$$\underline{\nabla}\cdot\underline{u} = u_{i,i} = \frac{\partial u_i}{\partial x_i}$$

Curl

$$\underline{\nabla}\times\underline{u} = \epsilon_{ijk} v_{k,j}\underline{e}_i$$

The i^{th} component:

$$\epsilon_{ijk} v_{k,j}$$

Laplacian

In Cartesian coordinates:

$$\nabla^2 u = u_{,ii} = \frac{\partial^2 u}{\partial x_i \partial x_i}$$

Direct Notation

Dot product

$$\begin{aligned} &\underline{A}\cdot\underline{B} = \underline{B}\cdot\underline{A} \\ &\underline{A}\cdot\left(\underline{B}+\underline{C}\right) = \underline{A}\cdot\underline{B} + \underline{A}\cdot\underline{C} \\ &\underline{A}\cdot\underline{A} = A^2 \\ &\underline{A}\cdot\underline{B} = 0 \qquad \text{if } \underline{A}\perp\underline{B} \end{aligned}$$

Cross product

$$\begin{aligned} &\underline{A}\times\underline{B} = AB\sin(\theta)\underline{n} \qquad \|\underline{n}\| = 1; \underline{n}\perp\underline{A}/A; n\perp\underline{B}/B \\ &\|\underline{A}\times\underline{B}\| \qquad \text{area of parallelgram spanned by } \underline{A}, \underline{B} \\ &\underline{A}\times\left(\underline{B}+\underline{C}\right) = \underline{A}\times\underline{B} + \underline{A}\times\underline{C} \\ &\underline{A}\times\underline{B} = -\underline{B}\times\underline{A} \\ &\underline{A}\times\underline{A} = \underline{0} \end{aligned}$$

Triple Product

$$\|\underline{A}\cdot\left(\underline{B}\times\underline{C}\right)\| \qquad \text{volume of parallelepiped spanned by } \underline{A}, \underline{B}, \underline{C}$$

Polar coordinates

$$\begin{cases} x = r\cos(\phi) \\ y = r\sin(\phi) \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \arctan(y/x) \end{cases}$$

$$dA = dx\,dy \mapsto r\,dr\,d\phi$$

Spherical coordinates

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \arctan(y/x) \end{cases}$$

θ is the polar angle, measured from $+z$ axis. ϕ is the azimuthal angle, measured from the $+x$ axis.

$$\begin{cases} \underline{e}_1 = \sin(\theta)\cos(\phi)\underline{s}_1 + \cos(\theta)\cos(\phi)\underline{s}_2 - \sin(\phi)\underline{s}_3 \\ \underline{e}_2 = \sin(\theta)\sin(\phi)\underline{s}_1 + \cos(\theta)\sin(\phi)\underline{s}_2 + \cos(\phi)\underline{s}_3 \\ \underline{e}_3 = \cos(\theta)\underline{s}_1 - \sin(\theta)\underline{s}_2 \end{cases}$$

Cartesian basis $\underline{e}_i$ are dependent on $\underline{s}_j(r, \theta, \phi)$

$$\begin{cases} \underline{s}_1 = \sin(\theta)\cos(\phi)\underline{e}_1 + \sin(\theta)\sin(\phi)\underline{e}_2 + \cos(\theta)\underline{e}_3 \\ \underline{s}_2 = \cos(\theta)\cos(\phi)\underline{e}_1 + \cos(\theta)\sin(\phi)\underline{e}_2 - \sin(\theta)\underline{e}_3 \\ \underline{s}_3 = -\sin(\phi)\underline{e}_1 + \cos(\phi)\underline{e}_2 \end{cases}$$

$\underline{s}_i$ are dependent on r, θ, ϕ themselves.

$$dV = dx\,dy\,dz \mapsto r^2\sin(\theta)\,dr\,d\theta\,d\phi$$

Gradient

$$\underline{\nabla} u = \frac{\partial u}{\partial r}\underline{s}_1 + \frac{1}{r}\frac{\partial u}{\partial\theta}\underline{s}_2 + \frac{1}{r\sin(\theta)}\frac{\partial u}{\partial\phi}\underline{s}_3$$

Divergence

$$\underline{\nabla}\cdot\underline{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)u_\theta\right) + \frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial\theta}$$

Curl

$$\underline{\nabla} \times \underline{u} = \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial\theta}(\sin(\theta)u_\phi) - \frac{\partial u_\theta}{\partial\phi}\right]\underline{s}_1 + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(ru_\phi)\right]\underline{s}_2 + \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right]\underline{s}_3$$

Laplacian

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 u}{\partial\phi^2}$$

Cylindrical coordinates

$$\begin{cases} x = r\cos(\phi) \\ y = r\sin(\phi) \\ z = z \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \arctan(y/x) \\ z = z \end{cases}$$

r is the distance to the closest point on the z axis (different from spherical coordinates). ϕ is the azimuthal angle, measured from the $+x$ axis.

$$\begin{cases} \underline{e}_1 = \cos(\phi)\underline{s}_1 - \sin(\phi)\underline{s}_2 \\ \underline{e}_2 = \sin(\phi)\underline{s}_1 + \cos(\phi)\underline{s}_2 \\ \underline{e}_3 = \underline{s}_3 \end{cases} \qquad \begin{cases} \underline{s}_1 = \cos(\phi)\underline{e}_1 + \sin(\phi)\underline{e}_2 \\ \underline{s}_2 = -\sin(\phi)\underline{e}_1 + \cos(\phi)\underline{e}_2 \\ \underline{s}_3 = \underline{e}_3 \end{cases}$$

$$dx\,dy\,dz \mapsto r\,dr\,d\phi\,dz$$

Gradient

$$\underline{\nabla} u = \frac{\partial u}{\partial s}\underline{s}_1 + \frac{1}{r}\frac{\partial u}{\partial\phi} + \frac{\partial u}{\partial z}\underline{s}_3$$

Divergence

$$\underline{\nabla}\cdot\underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_1) + \frac{1}{r}\frac{\partial u_2}{\partial\phi} + \frac{\partial u_3}{\partial z}$$

Curl

$$\underline{\nabla}\times\underline{u} = \left[\frac{1}{r}\frac{\partial u_3}{\partial\phi} - \frac{\partial u_2}{\partial z}\right]\underline{s}_1 + \left[\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial r}\right]\underline{s}_2 + \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_2) - \frac{\partial u_1}{\partial\phi}\right]\underline{s}_3$$

Laplacian

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Vector identities

Assume A, B, and C, are vectors. f is a scalar function.

Triple products

$$A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)$$
$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

Product rules

$$\underline{\nabla}(A\cdot B) = A\times(\underline{\nabla}\times B) + B\times(\underline{\nabla}\times A) + (A\cdot\underline{\nabla})B + (B\cdot\underline{\nabla})A$$
$$\underline{\nabla}\cdot(fA) = f(\underline{\nabla}\cdot A) + A\cdot(\underline{\nabla}f)$$
$$\underline{\nabla}\cdot(A\times B) = B\cdot(\underline{\nabla}\times A) - A\cdot(\underline{\nabla}\times B)$$
$$\underline{\nabla}\times(fA) = f(\underline{\nabla}\times A) - A\times(\underline{\nabla}f)$$
$$\underline{\nabla}\times(A\times B) = (B\cdot\underline{\nabla})A - (A\cdot\underline{\nabla})B + A(\underline{\nabla}\cdot B) - B(\underline{\nabla}\cdot A)$$

We can proof $\underline{\nabla}\cdot(fA) = f(\underline{\nabla}\cdot A) + A\cdot(\underline{\nabla}f)$ using the index notation. In direct notation, $\underline{\nabla}\cdot(fA)$ says

$$\partial_x(fA_x) + \partial_y(fA_y) + \partial_z(fA_z)$$

In index notation (with Einstein summation), each of the terms give

$$f_{,i}A_i + fA_{i,i}$$

in direct notation, this says

$$(\nabla f)\cdot\underline{A} + f(\underline{\nabla}\cdot\underline{A})$$

Second derivatives

$$\underline{\nabla}\cdot(\underline{\nabla}\times A) = 0$$
$$\underline{\nabla}\times(\underline{\nabla}f) = 0$$
$$\underline{\nabla}\times(\underline{\nabla}\times A) = \underline{\nabla}(\underline{\nabla}\cdot A) - \nabla^2 A$$

Helmholtz representation

A vector field $\underline{F}$ can always be written as the sum of a curl-free and a divergence-free vector fields:

$$\underline{F} = -\underline{\nabla}V + \underline{\nabla}\times\underline{A}$$

Dirac delta function

$$\int_{a-\epsilon}^{a+\epsilon} f(x)\delta(x-a)\,dx = f(a),$$

for $\epsilon > 0$.

Equivalence property

Expressions $D_1(x), D_2(x)$ involving the delta function are equal if

$$\int_{-\infty}^{\infty} f(x)D_1(x)\,dx = \int_{-\infty}^{\infty} f(x)D_2(x)\,dx,$$

for all $f(x)$.

Scaling property

$$\delta(kx) = \delta(x)/|k|$$

So $\delta(-x) = \delta(x)$.

Three dimensions

$$\delta^3(\underline{r}) = \prod_{i=1}^{3}\delta(x_i)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\underline{r})\delta^3(\underline{r}-\underline{a})\,dV = f(\underline{a})$$

Point charge

$$\underline{\nabla}\cdot\frac{\underline{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\underline{\mathscr{r}})$$

Electrostatics

The force on a test charge Q from a resting point charge q is (Coulomb's law)

$$\underline{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^3}\underline{\mathscr{r}} \implies \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\underline{\hat{\mathscr{r}}}$$

Permittivity of free space:

$$\epsilon_0 = 8.85\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$$

Electric field

Is the force per unit charge:

$$\underline{F} = Q\underline{E},$$

where $\underline{E}$ is for a collection of n charges:

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\underline{\hat{\mathscr{r}}}_i$$

In the case of a continuum,

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_\Omega\frac{1}{\mathscr{r}^2}\underline{\hat{\mathscr{r}}}\,dq$$

Charge densities can have position dependence:

$$dq \mapsto \begin{cases} \lambda(\underline{r}')\,dl & \text{line charge} \\ \sigma(\underline{r}')\,dS & \text{surface charge} \\ \rho(\underline{r}')\,dV & \text{volume charge} \end{cases}$$

Divergence and curl of electrostatic fields

We can represent electric fields through field lines. The density of field lines ∝ field strength. The electric flux through a closed surface, $\partial\Omega$, is a measure of the field strength, and so the total charge enclosed.

$$\Phi_E = \int_{\partial\Omega} \underline{E} \cdot d\underline{S} = \frac{Q}{\epsilon_0}$$

If we apply the divergence theorem,

$$\int_\Omega \underline{\nabla} \cdot \underline{E}\, dV = \frac{1}{\epsilon_0} \int_\Omega \rho\, dV$$

We recover the differential form of Gauss's law by equating the integrands:

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{\epsilon_0}\rho$$

Curl of E

Any static electric fields are conservative. Recall the curl of conservative fields is zero. This implies that the components of $\underline{E}$ are such that

$$E_{i,j} = E_{j,i}$$

Given some $\underline{E}$ has continuous first partial derivatives, then

$$\underline{\nabla} \times \underline{E} = \underline{0}$$

is a check to see whether $\underline{E}$ is a possible E field.

Electric Potential

By definition, the potential (in J/C, or volts) is

$$V(\underline{r}) = -\int_O^r \underline{E} \cdot d\underline{l} \implies \underline{E} = -\underline{\nabla} V$$

Potentials differences are independent of the reference point O. By convention, for charge distributions that do not extend to infinity:

$$O = \infty$$

Potentials obey the superposition principle.

Poisson's equation and Laplace's equation

Since $\underline{E} = -\underline{\nabla} V$, we have

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} = -\underline{\nabla} \cdot \underline{\nabla} V = -\nabla^2 V$$

A Poisson's equation arises:

$$-\nabla^2 V = \frac{\rho}{\epsilon_0}$$

When $\rho = 0$, the equation reduces to a Laplace's equation.

The Potential of a localized charge distribution

Assuming a local (not infinite plane, wire) charge distribution:

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underline{r}')}{\mathscr{r}}\, dV$$

Boundary conditions

The electric potential is continuous across any boundary:

$$V_{\text{above}} = V_{\text{below}} \impliedby \lim_{\epsilon \to 0} -\int_{a-\epsilon}^{a+\epsilon} \underline{E} \cdot d\underline{l} = 0$$

Using the result from an infinite plane, the electric field is discontinuous across surfaces (in the normal direction)

$$\underline{E}_{\text{above}} - \underline{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\underline{n}$$

Since $-\underline{\nabla} V = E$,

$$\underline{\nabla} V_{\text{above}} - \underline{\nabla} V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\underline{n}$$

If we define

$$\frac{\partial V}{\partial n} = \underline{\nabla} \cdot \underline{n} \implies \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

$\underline{E}$ parallel to the surface is continuous.

Work and Energy in Electrostatics

The **minimum** amount of force to move a charge an electric field is

$$\underline{F} = -Q\underline{E}$$

In opposition of the electric force, equal to $Q\underline{E}$.

To actually set the charge in motion, you need to apply more than $-Q\underline{E}$, and this extract force you apply is converted to the kinetic energy of the charge.

Electrostatic force is conservative: since the work required to move a charge from $\underline{a}$ to $\underline{b}$ is path independent. To get from point $\underline{a}$ to $\underline{b}$ requires W amount of work:

$$W = \int_a^b \underline{F} \cdot d\underline{l} = -Q \int_a^b \underline{E} \cdot d\underline{l} = Q\left(V(\underline{b}) - V(\underline{a})\right)$$

The electric potential has the units of energy per unit charge. And the electric field has units of force per unit charge.

Interpretation of potential difference: the potential difference between two points is the **work per unit charge** required to move a particle between the two points

In an existing electric potential, when we want to bring charge from ∞ to $\underline{r}$, analogous to **assembling a system of a single charge**, we need to do work equal to

$$W = Q(V(\underline{r}) - V(\infty))$$

Interpretation of electric potential: choosing our reference point at ∞,

$$W = QV(\underline{r})$$

the electric potential tells us the **potential energy** per unit charge

The energy of a point charge distribution

Energy of a point charge configuration: The work required to assemble a system of n charges (the energy stored in such a configuration, **not including the energy of the charges themselves**) is:

$$W = \frac{1}{2}\sum_{i=1}^n q_i V(\underline{r}_i)$$

where $V(\underline{r}_i)$ is the potential due to all charges other than q_i.

Alternatively, this is also the energy you will get back, is you dismantled this system of n fixed charges.

Assuming no existing field, the energy required to bring in the first charge is 0.

The first charge has an electric potential given by

$$V(\mathscr{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{\mathscr{r}}$$

The energy required to being in the second charge q_2 to a distance $\mathscr{r}_{12}$ within the first charge is

$$q_2 V(\mathscr{r}_{12})$$

As we bring in the third charge, q_3, the potential it faces will be the sum of potentials due to q_1 and q_2.

The total work required is given by

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^n \sum_{j>i}^n \frac{q_i q_j}{\mathscr{r}_{ij}}$$

The energy of a continuous charge distribution

Energy of a charge distribution: Given a volume charge density ρ, the energy **stored in the** volume charge distribution (**including the work required to make the charges**) is

$$W = \frac{1}{2}\int \rho V\, d\tau$$

where V is the electric potential and $d\tau$ is the volume element. Applying the Gauss law and integration by parts, equivalently:

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2\, d\tau$$

The reason that this formula includes the work required make the charge is that V used the formula is the total electric potential. Unlike the discrete case, the contribution to total potential due to a single charge is very small, and effectively zero.

The total energy stored in a point charge as calculated using this formula is actually infinite.

$$E^2 = \frac{q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}$$

and

$$\int_0^\infty \frac{1}{r^4} r^2\, dr$$

diverges.

. . . .

We can write this in terms of the electric field instead of ρ and V.

We can write the charge density using in terms of $\underline{E}$ using the Gauss law:

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0}$$

This gives

$$W = \frac{\epsilon_0}{2} \int V \underline{\nabla} \cdot \underline{E}\, d\tau$$

Using a product rule, we can write

$$\underline{\nabla} \cdot (V\underline{E}) = \nabla V \cdot \underline{E} + V(\underline{\nabla} \cdot \underline{E})$$

Rearranging, we have

$$W = \frac{\epsilon_0}{2} \int \underline{\nabla} \cdot (V\underline{E}) - \nabla V \cdot \underline{E}\, d\tau$$

We can apply the divergence theorem to the first term in the integral

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \int (V\underline{E}) \cdot \underline{n}\, da - \frac{\epsilon_0}{2} \int \nabla V \cdot \underline{E}\, d\tau \\ &= \frac{\epsilon_0}{2} \left(-\int_\Omega \underline{E} \cdot \nabla V\, d\tau + \int_{\partial\Omega} (V\underline{E}) \cdot d\underline{a} \right) \end{aligned}$$

In electrostatics, $\nabla V = -\underline{E}$, so

$$W = \frac{\epsilon_0}{2} \left(\int_\Omega E^2\, d\tau + \int_{\partial\Omega} (V\underline{E}) \cdot d\underline{a} \right)$$

We realized that in original equation, we might as well have integrated the entire real space, instead of just Ω (where the blob of charge is located), since $\rho = 0$ for regions outside Ω, and their contributions to the integral would be 0.

As we increase Ω, volume integral would increase since $E^2 \geq 0$. But as we move further away from the charges, we should expect the electric field and electric potential to decay according to $1/r^2$ and $1/r$. It then makes sense that the surface integral over a region where V and E are increasingly small would $\to 0$.

Comments on electrostatic energy

Notice that work we calculate using

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2\, d\tau$$

is always positive. Yet the energy stored in a configuration of two charges of equal magnitude but opposite sign can be negative according to

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\underline{r}_i)$$

A point charge has electric potential:

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

In this case,

$$W = \frac{-1}{8\pi\epsilon_0} \frac{q^2}{r}$$

which is negative as given by the formula involving the point charge distribution.

Since electrostatic energy is proportional to the square of the electric field, by doubling the charge everywhere means that you quadruple the field everywhere (and thus also quadruple the energy everywhere)

$$\begin{aligned} W_{\text{tot}} &= \frac{\epsilon_0}{2} \int E^2\, d\tau = \frac{\epsilon_0}{2} \int (\mathbf{E}_1 + \mathbf{E}_2)^2\, d\tau \\ &= \frac{\epsilon_0}{2} \int \left(E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2\right) d\tau \\ &= W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2\, d\tau. \end{aligned}$$

Conductors

Conductors differ from insulators in that there are electrons that are free to roam.

1. All free electric charges in a conductor resides on its surface
2. $\underline{E}$ everywhere inside a conductor in static equilibrium is $\underline{0}$
3. $\underline{E}$ at the surface of a conductor is normal to the surface
4. $\underline{E}$ on the inner surface of a closed, grounded, conducting shell is zero
5. $\underline{E}$ everywhere inside a cavity of a closed, grounded, conducting shell is zero, even if that are charges external to the shell
6. $\underline{E}$ everywhere inside a cavity of a closed, insulated, conducting shell is zero, even if there are charges external to the shell
7. The charge induced by a charge distribution in the cavity of a closed conducting shell on the surface is uniform, independent of the charge distribution
8. A closed, insulated, conducting shell does not shield the exterior from fields created by charges in its cavity
9. A closed, grounded, conducting shell does shield the exterior from fields created by charges in its hollow region

Surface charge and the force on a conductor

For a solid conductor

$$\frac{V_{\text{above}}}{\partial n} - 0 = -\frac{\sigma}{\epsilon_0} \implies \sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Surface charges experience a force in the presence of electric fields. The force **per unit area** is given by

$$\underline{f} = \sigma \underline{E}_{\text{other}},$$

where $\underline{E}_{\text{other}}$ is the electric field due to other charges. Since

$$\begin{aligned} \underline{E}_{\text{above}} &= \underline{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\underline{n} \\ \underline{E}_{\text{below}} &= \underline{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0}\underline{n} \end{aligned}$$

Solving for $\underline{E}_{\text{other}}$ gives

$$\underline{E}_{\text{other}} = \frac{1}{2}\left(\underline{E}_{\text{above}} + \underline{E}_{\text{below}}\right) = \underline{E}_{\text{average}}$$

For a conductor, $\underline{E}_{\text{above}} = \underline{n}\sigma/\epsilon_0$, so

$$\underline{f} = \frac{\sigma^2}{2\epsilon_0}\underline{n}$$

The force per unit area on the surface does not depend on the sign of σ. The surface experiences an outward force regardless of the sign.

Capacitors

The potential difference difference between two conductors is unambiguous since conductors are equipotential surfaces. We define capacitance, C, as

$$C = Q/\Delta V$$

V is the potential difference between the positive and negative conductor, Q is the charge of the positive conductor.

$$\Delta V = V_+ - V_- = -\int_{(-)}^{(+)} \underline{E} \cdot d\underline{l}$$

Energy in a capacitor

Charging a capacitor requires work since we need to move electrons from $+$ to $-$, overcoming the pull of $+q$ on the $+$ plate. The work required to transport dq given a q/C potential difference is

$$dW = \frac{q}{C}\, dq$$

The total work required to go from $q = 0$ to $q = Q$ is

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$

Potentials

Laplace's Equation

When we confine our attention to a local region where there is no charge, Poisson's equation reduces to Laplace's equation.

$$\nabla^2 V = 0$$

It has two interesting properties:

1. $V(x)$ is the average of its neighbours; for all a:
$$V(x) = \frac{V(x+a) + V(x-a)}{2},$$
2. Extreme values of V must only occur at the end points

Boundary conditions and the uniqueness theorem

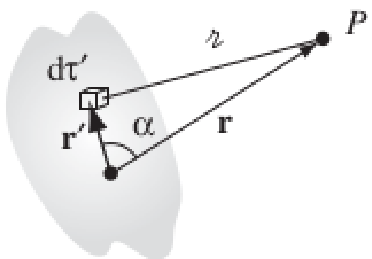
First uniqueness theorem:

> The solution of Laplace's equation ($\nabla^2 V = 0$) in Ω is uniquely determined if V is specified on $\partial\Omega$

Corollary:

> The electrostatic potential in Ω is uniquely determined if the charge density throughout Ω and the value of the potential on $\partial\Omega$.

Multipole expansion



The potential of an arbitrary finite charge distribution:

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\underline{r}')}{\mathscr{r}}\,dV$$

We can express $\mathscr{r}$ as the following from the cosine law.

$$\mathscr{r} = r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos(\alpha)} = r\sqrt{1+\epsilon}$$

Since $\epsilon << 1$ for points far from the charge distribution, we can expand $1/\mathscr{r}$ using the binomial expansion.

$$(x+y)^n = x^n\left(1+\frac{y}{x}\right)^n = x^n\left(1+n\frac{y}{x}+\frac{n(n-1)}{2!}\frac{y^2}{x^2}+\dots\right) = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x^{n-k}y^k$$

Expansion of $\mathscr{r} = r\sqrt{1+\epsilon}$: the first few terms of the expansion are

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1+\frac{-1}{2}\epsilon+\frac{3}{8}\epsilon^2-\frac{5}{16}\epsilon^3+\dots\right)$$

We will find that

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha),$$

where P_n is the n^{th} Legendre polynomial. So

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(\underline{r}')\,dV$$

$$4\pi\epsilon_0 V(\underline{r}) = \frac{1}{r}\int\rho(\underline{r}')\,dV + \frac{1}{r^2}\int r'\cos\alpha\rho(\underline{r}')\,dV + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\underline{r}')\,dV + \dots$$

Since the potential of monopoles fall like $1/r$, dipoles $1/r^2$, quadrupoles $1/r^3$, the expansion is called the multipole expansion.

Monopole and dipole terms

We can rewrite $r'\cos\alpha = \underline{\hat{r}}\cdot\underline{r}'$ so the dipole term can be written as

$$V = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\underline{\hat{r}}\cdot\int \underline{r}'\rho(\underline{r}')\,dV$$

We define the dipole moment as

$$\underline{p} = \int \underline{r}'\rho(\underline{r}')\,dV$$

so

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\underline{p}\cdot\underline{\hat{r}}}{r^2}$$

The dipole moment for a collection of point charges is

$$\underline{p} = \sum_{i=1}^{n} q_i\underline{r}'_i$$

A perfect dipole comes from taking the length of the separation vector between two opposite charges to 0, and letting $q \to \infty$.

Origin of coordinates in multipole expansions

In general, shifting the origin in a multipole expansion changes the terms in the expansion.

If we displace the original by $\underline{a}$, then our dipole moment changes by

$$\underline{p} = \int(\underline{r}' - \underline{a})\rho(\underline{r}')\,dV = \int\underline{r}'\rho(\underline{r}')\,dV - \underline{a}\int\rho(\underline{r}')\,dV = \underline{p} - Q\underline{a}$$

When the total charge of the configuration of zero, the dipole moment is independent of the choice of the origin.

Electric fields in matter

Polarization

All charges in insulators (also called dielectrics) are only allowed to move within the the molecules and atoms they are tied to. $\underline{E}$ distorts the charge distribution of a dielectric by stretching or rotating.

Induced dipoles: atoms

For $\underline{E}$ not too great (not enough to ionise the atom), mutual attraction of electrons and nucleus balances out with the separating force of $\underline{E}$, and the resulting induced dipole moment is $\propto \underline{E}$:

$$\underline{p} = \alpha\underline{E},$$

α is the atomic polarizability.

Induced dipoles: molecules

Molescules polarize more readily in some directions than in others. In general, let $\underline{p}$ be the polarization tensor, then,

$$E_i = p_{ij}E_j$$

for $i, j \in [1, 2, 3]$.

Alignment of polar molecules

Diople moments are induced in neurtal and molecules when placed in an electric field. Polar molecules with an dipole $\underline{p}$ in an electric field experiences no net force and a torque as long as $\underline{p} \nparallel \underline{E}$:

$$\underline{N} = \underline{p}\times\underline{E}$$

For a spatially varying field (field must change dramatically to be signicant in the space of a single molecule) $|+q\underline{E}|$ need not equal $|-q\underline{E}|$, so

$$\underline{F}_+ + \underline{F}_- = q(\underline{E}_+ - \underline{E}_+) = q(\Delta\underline{E})$$

Recall:

$$dT = (\underline{\nabla}T)\cdot d\underline{l}$$

then

$$\Delta E_i \approx (\underline{\nabla}E_i)\cdot\underline{d} \implies \Delta\underline{E} = (\underline{d}\cdot\underline{\nabla})\,\underline{E}$$

therefore,

$$\underline{F} = q\,(\underline{d}\cdot\underline{\nabla})\,\underline{E} = \left(\underline{p}\cdot\underline{\nabla}\right)\underline{E}$$

Polarization

1. Small dipole moments are induced in neutral atoms and molecules when placed in uniform $\underline{E}$, in the direction of $\underline{E}$ (for asymetric molecules $\underline{\hat{p}}$ may have component $\perp$ to $\underline{E}$, but these sum to zero due to randomness in orientation of the mocules)
2. A torque in induced in polar molecules so that $\underline{p} \parallel \underline{E}$

Polarization, $\underline{P}$, is dipole moment per unit volume.

Field of a polarized object

Given the polarization, $\underline{P}$, the dipole moment in each volume element dV is $\underline{P}\,dV$. The potential due to dipoles induced by an external field in a dielectric is:

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_{\Omega} \frac{\underline{P}(\underline{r}')\cdot\hat{\underline{\mathscr{r}}}}{\mathscr{r}^{\,2}}\,dV$$

Since the gradient w.r.t. the source coordinate is:

$$\underline{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^{\,2}}\hat{\underline{\mathscr{r}}}$$

Then:

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_{\Omega} \underline{P}\cdot\underline{\nabla}'\left(\frac{1}{\underline{r}'}\right)dV$$

Applying $\underline{\nabla}\cdot(fA) - f(\underline{\nabla}\cdot A) = A\cdot\underline{\nabla}f$, divergence theorem, and defining

$$\sigma_b = \underline{P}\cdot\underline{n}$$
$$\rho_b = -\underline{\nabla}\cdot\underline{P}$$

The potential of a polarized object is the sum of the potential due to a surface and volume bound charge densities:

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_{\partial\Omega}\frac{\sigma_b}{\mathscr{r}}\,dS + \frac{1}{4\pi\epsilon_0}\int_{\Omega}\frac{\rho_b}{\mathscr{r}}\,dV$$

Field inside a Dielectric

Microscopic field inside a dielectric in complicated. We are interested in the macroscopic field, where we average over the microscopic field over an appropriate volume (not too large to wash out important fluctuations.)

The macroscopic field for a point $\underline{r}$ is the sum of the field due to all charges outside the averaging volume, and due to all charges inside the averaging volume:

$$\underline{E} = \underline{E}_{\text{out}} + \underline{E}_{\text{in}}$$

The average field due to all the charges outside is the field they produce at the centre. Let Ω^a be the averaging volume so we can use the following formula.

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0}\int_{\Omega^a\cap\Omega}\frac{\underline{P}(\underline{r}')\cdot\hat{\underline{\mathscr{r}}}}{\mathscr{r}^{\,2}}\,dV$$

For the field inside, we can show that

$$\underline{E}_{in} = \frac{-1}{4\pi\epsilon_0}\frac{\underline{p}}{R^3}$$

This says that the average field over any sphere due to the charges inside is the same as the field at the center of a uniformly polarized sphere with the same total dipole moment.

The electric displacement

Electric fields induce bound charges on in Dielectrics. The total charge density is the sum of free and bound charges

$$\rho = \rho_b + \rho_f$$

Substitute into Gauss's law:

$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\epsilon_0} \implies \epsilon_0\underline{\nabla}\cdot\underline{E} = \rho_b + \rho_f$$
$$= -\underline{\nabla}\cdot\underline{P} + \rho_f$$

We define a new quantity called *electric displacement*, $\underline{D}$

$$\underline{D} = \epsilon_0\underline{E} + \underline{P},$$

such that

$$\underline{\nabla}\cdot\underline{D} = \rho_f \implies \int_{\partial\Omega}\underline{D}\cdot d\underline{S} = \int_{\Omega}\rho_f\,dV$$

$\underline{D}$ is not conservative

$$\underline{\nabla}\times\underline{D} = \epsilon_0\underline{\nabla}\times\underline{E} + \underline{\nabla}\times\underline{P} = \underline{\nabla}\times\underline{P},$$

unlike curl of electric field, curl of $\underline{D}$ is zero only when curl of $\underline{P}$ is zero.

Boundary conditions of the displacement field:

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$$
$$\underline{D}^{\parallel}_{\text{above}} - \underline{D}^{\parallel}_{\text{below}} = \underline{P}^{\parallel}_{\text{above}} - \underline{P}^{\parallel}_{\text{below}}$$

Problems with insufficient symmetry has nonzero curl of $\underline{D}$

Linear Dielectrics

For $\underline{E}$ not too strong, polarization is $\propto \underline{E}$ (total field) and χ_e, the electric susceptibility (that is material and temperature dependent):

$$\underline{P} = \epsilon_0\chi_e\underline{E}$$

assuming the medium is homogeneous (position independent) and isotropic (direction independent).

$$\underline{D} = \epsilon_0\underline{E} + \epsilon_0\chi_e\underline{E}$$
$$= \epsilon_0(1+\chi_e)\underline{E}$$

We can define the relative permittivity, or dielectric constant as

$$\epsilon_r = 1 + \chi_e$$

For ordinary materials, $\epsilon_r \geq 1$. We can define the permittivity as:

$$\epsilon = \epsilon_0(1+\chi_e) = \epsilon_0\epsilon_r$$

At dielectric and vaccuum interface, different dielectric constants causes

$$\oint \underline{P}\cdot d\underline{l} \neq 0$$

In the special case that the space is filled with homogeneous linear dielectric, the electric field everywhere is reduced by the a factor of ϵ (the polarization of dielectric medium partially shields charges):

$$\underline{E} = \frac{1}{\epsilon_0\epsilon_r}\underline{D} = \frac{1}{\epsilon_r}\underline{E}_{\text{vacuum}}$$

In general, some materials are more easily polarized in some directions than others. So any component of polarization in a linear dielectric is given by

$$P_i = \epsilon_0\chi_{e_{ij}}E_j$$

$\chi_{e_{ij}}$ are components to the susceptibility tensor.

Energy in dielectric systems

It takes more energy to charge dielectric-filled capacitors up to a desired potential since the the dielectric partially shields the electric field.

Starting with the unpolarized **linear** dielectric (this equation only holdes for linear dielectrics), the work required bring all the free charges to their locations (each free charge causes the dielectric to become increasingly polarized) requires doing

$$W = \frac{1}{2}\int_{\mathbb{R}^3}\underline{D}\cdot\underline{E}\,dV$$

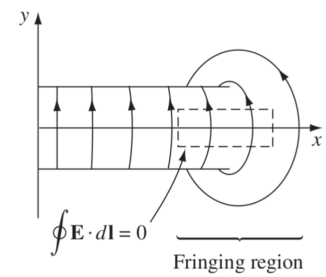
If we are trying to calculate the work required to bring in the free and bound charges, then

$$W = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3}E^2\,dV$$

as we established before.

Forces in dielectrics

Fringing fields exert a force on the dielectric.



If the electric field exerts a force $\underline{F}$ on the dielectric, the work done to pull the dielectric a distance dx opposite of the direction of $\underline{F}$ is

$$dW = -\underline{F}\,dx$$

So the electrical force on the slab is

$$F = -\frac{dW}{dx}$$

Let's assume the charge on the capacitor remains constant. So in terms of Q, we have

$$W = \frac{1}{2}\frac{Q^2}{C}$$

Then

$$F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

If V is assumed to be constant (say the capacitor was held at a constant potential difference by a battery), the battery also does work as the dielectric moves:

$$dW = -F\,dx + V\,dQ$$

Then,

$$F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx}$$

and we arrive at the same result.

Method of Images

Exploits the uniqueness theorem to solve Poisson's equation.

1. Identify the domain, Ω, and the boundary conditions over $\partial\Omega$
2. Introduce image charges of the opposite sign in $\Omega \cap \mathbb{R}^3$ such that the potential of the configuration satisfies the boundary conditions in Ω
3. The image charges need not have the same magnitude as the original charges
4. The energy of the charge and its image charge need not equal the energy of the original configuration.

Separation of variables

Cartesian coordinates

We consider solutions to the Laplace equation of the form

$$V(x, y, z) = X(x)Y(y)$$

Compute the second partials:

$$\partial_x^2 XYZ = YZ\partial_x^2 X$$
$$\partial_y^2 XYZ = XZ\partial_y^2 Y$$

Substitute into the Laplace equation:

$$YZX_{,xx} + XZY_{,yy} = 0$$

divide by XY on both sides. The resulting terms must be constant (or zero), since they are "independent of each other but sum to zero":

$$\frac{1}{X}X_{,xx} + \frac{1}{Y}Y_{,yy} = 0$$

Depending on the boundary condition, assign one of the terms to a positive or a negative constant. In general, we know from ODEs that

$$\frac{1}{f}f_{,uu} = \lambda^2 \implies f_{uu} - \lambda^2 f = 0$$

has a general solution

$$f(u) = Ae^{\lambda u} + Be^{-\lambda u}$$

And

$$\frac{1}{f}f_{,uu} = -\lambda^2 \implies f_{uu} + \lambda^2 f = 0$$

has a general solution

$$f(u) = C\sin(\lambda u) + B\cos(\lambda u)$$

- Assigning a positive constant means a exponential solution.
- A negative constant means a sinusoidal solution.
- V must vanish at infinity.

Orthogonality of sines and cosines:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

We also have

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \neq 0 \\ L & n = m = 0 \end{cases}$$

Completeness

A set of functions $f_n(y)$ is complete over an interval if any other function over the interval can be expanded in terms of a linear combination of them:

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$$

Orthogonality

A set of functions is orthogonal over the domain Ω if the integral of the product of any two different members of the set is zero:

$$\langle f_i, f_j \rangle := \int_\Omega f_i(y) f_j(y)\, dy = 0 \qquad \text{when } i \neq j$$

Spherical coordinates

In spherical coordinates, Laplace's equation becomes:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 V}{\partial \phi^2} = 0$$

If V is independent of ϕ, (V has azimuthal symmetry), then

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial V}{\partial \theta}\right) = 0$$

Let's consider solution that are products:

$$V(r, \theta) = R(r)\Theta(\theta)$$

Substitute into our equation, and dividing by $R\Theta$,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) = 0$$

Each term must be a constant. Let's write each term as

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)$$
$$\frac{1}{\Theta\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

This gives us the radial and angular equations. The radial equation is a Cauchy-Euler equation with the general solution:

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

The angular equation turns out to be Legendre differential equation. It has solutions given by

$$\Theta(\theta) = P_l(\theta)$$

The Legendre polynomials are given by the Rodrigues formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l \left(x^2 - 1\right)^l$$

The first couple of Legendre polynomials are given by

$$P_1(x) = x$$
$$P_2(x) = \left(3x^2 - 1\right)/2$$
$$P_3(x) = \left(5x^3 - 3x\right)/2$$
$$P_4(x) = \left(35x^4 - 30x^2 + 3\right)/8$$
$$P_5(x) = \left(63x^5 - 70x^3 + 15x\right)/8$$

It has the property that

$$P_l(1) = 1$$

Spherical Laplace's equation:

$$V(r, \theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

Legendre Polynomials

The polynomials form a complete set of function on the interval $-1 \leq x \leq 1$. They are also orthogonal to each other.

Orthogonality of Legendre Polynomials: For $i \neq j$, $i, j \leq 0$,

$$\int_{-1}^{1} P_i(x)P_j(x)\, dx = \int_0^{\pi} P_i(\cos(\theta))P_j(\cos(\theta))\sin(\theta)\, d\theta = \begin{cases} 0 & i \neq j \\ \frac{2}{2l+1} & i = j \end{cases}$$

Cylindrical coordinates

Let's consider the case where the potential does not depend on z. The cylindrical Laplacian of V becomes

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}$$

Consider a solution of the form

$$V = S(s)\Phi(\phi)$$

This results in two ODEs:

$$\frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) = n^2$$
$$\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -n^2$$

The first ODE is a Cauchy Euler equation. It has a the solution

$$S(s) = \begin{cases} A_0\ln(s) + B_0 & \text{n} = 0 \\ A_n s^n + B_n s^{-n} & \text{n} > 0 \end{cases}$$

The second ODE has the solution

$$\Phi(\phi) = C_n \cos(n\phi) + D_n \sin(n\phi)$$

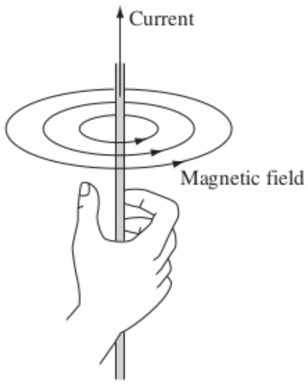
Cylindrical Laplace's equation:

$$V(s,\phi) = A_0 \ln(s) + B_0 + \sum_{n=1}^{\infty} \left(A_n s^n + B_n s^{-n}\right)\left(C_n \cos(\phi n) + D_n \sin(\phi n)\right)$$

Magnetostatics

The Lorentz force law

Given a static distribution of charge, we have learned to calculate the force exerted on a test charge (which may be moving).



While a stationary charge produces $\underline{E}$ in the surrounding space, a moving charge produces a magnetic field $\underline{B}$ in addition to $\underline{E}$.

Magnetic forces

Magnetic fields exert force on charges. The force on a charge Q moving with velocity $\underline{v}$ in a magnetic field $\underline{B}$ is

$$\underline{F}_{\text{mag}} = Q(\underline{v} \times \underline{B})$$

In the presence of both electric and magnetic fields, the net force on Q is

$$\underline{F} = Q\left[\underline{E} + (\underline{v} \times \underline{B})\right]$$

This is a fundamental "axiom" of electrodynamics.

Magnetic forces do no work

When Q moves by an amount $d\underline{l} = \underline{v}\, dt$, the work done is

$$dW_{\text{mag}} = \underline{F}_{\text{mag}} \cdot d\underline{l} = Q(\underline{v} \times \underline{B}) \cdot \underline{v}\, dt = 0$$

$\underline{B}$ only alters the direction in which particle moves but they cannot slow them down or speed them up.

Currents

Current is charge per unit time passing a given point. A line charge λ travelling down a wire at velocity $\underline{v}$ travels $\underline{v}dt$ in time dt. This constitutes a current vector

$$\underline{I} = \lambda \underline{v}$$

Current flow does not charge up wires. While electrons are flowing, there are just as many stationary plus charges as moving minus charges on any segment.

Magnetic force on a current carrying wire is: Magnetic fields exert force on moving particles.

$$\underline{F}_{\text{mag}} = I \int d\underline{l} \times \underline{B}$$

Surface current density describes current per unit width

$$\underline{K} = \frac{d\underline{I}}{dl_\perp} = \sigma \underline{v}$$

Volume current density describes current per unit area

$$\underline{J} = \frac{d\underline{I}}{da_\perp} = \rho \underline{v}$$

Continuity equation: Charge is locally conserved

$$\underline{\nabla} \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$$

We can show that,

$$\int \underline{J}(\underline{r}')\, d\tau' = \frac{d\underline{p}}{dt},$$

where $\underline{p}$ is the electric dipole moment. (P5.7 Griffiths)

The Biot-Savart law

Steady currents

Stationary charges produce constant electric fields. In magnetostatics, the parallel is that "steady currents produce constant magnetic field".

$$\partial\rho/\partial t = 0 \implies \partial\underline{J}/\partial t = \underline{0}$$

for all time and everywhere.

The magnitude of a steady current is the same everywhere along the wire (it does not change, and no charges pile up anywhere). So

$$\underline{\nabla} \cdot \underline{J} = 0$$

The magnetic field of a steady current

The magnetic field of a steady line current is given by the Biot-Savart law:

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{I} \times \hat{\underline{\mathscr{r}}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\underline{l}' \times \hat{\underline{\mathscr{r}}}}{\mathscr{r}^2}$$

Where $d\underline{l}'$ is a segment of the wire, and μ_0 is the permeability of free space:

$$\mu_0 = 4\pi \times 10^{-7}\ \text{N/A}^2$$

For surface currents

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{K}(\underline{r}') \times \hat{\underline{\mathscr{r}}}}{\mathscr{r}^2}\, da'$$

For volume currents:

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}') \times \hat{\underline{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

The strength of B is measured in Teslas.

- One coulomb of charge moving perpendicularly through a 1 T magnetic field at 1 m/s experiences 1 newton of force. 1 T = 1 N/(A·m)

Magnetic fields obey the principle of superposition.

The divergence and curl of the magnetic field

Straight-line currents

$\underline{B}$ has non-zero curl

Consider a line integral along a closed loop $\partial\Omega$ around the axis of a infinite wire:

$$\int_{\partial\Omega} \underline{B} \cdot d\underline{l} = \frac{\mu_0 I}{2\pi} \int_{\partial\Omega} \frac{1}{s} s\, d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

Consider a bundle of infinitely long wires

$$\int_{\partial\Omega} \underline{B} \cdot d\underline{l} = \mu_0 I_{\text{encl}}$$

If I_{encl} is represented by volume current densities, then

$$I_{\text{encl}} = \int_{\Omega} \underline{J} \cdot d\underline{a}$$

So we have

$$\int_{\partial\Omega} \underline{B} \cdot d\underline{l} = \mu_0 \int_{\Omega} \underline{J} \cdot d\underline{a}$$

Apply Stoke's theorem to the left hand side:

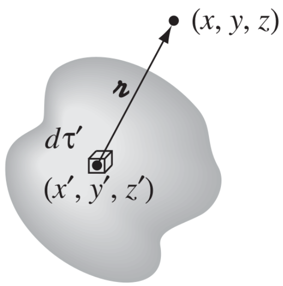
$$\int_{\partial\Omega} \underline{B} \cdot d\underline{l} = \int_{\Omega} \underline{\nabla} \times \underline{B} \cdot d\underline{a}$$

And we arrive at

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

(We have only showed this to be valid for infinite wires.)

The divergence and curl of the magnetic field



Biot-Savart law givens the magnetic field at a point (x, y, z) due to steady current:

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{r}') \times \hat{\underline{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

We are integrating over the the volume charge density. In Cartesian coordinates,

- $\underline{J}$ depends on (x', y', z')
- $\mathscr{r}$ is

$$\|(x - x', y - y', z - z')\|$$

Compute the divergence of both sides in the unprimed coordinates and using

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

gives

$$\underline{\nabla} \cdot \underline{B} = 0$$

The divergence of $\underline{B}$ is zero

In integral form:

$$\int_{\partial\Omega} \underline{B} \cdot d\underline{S} = 0$$

Similarly, taking the curl of both sides and applying the BACCAB rule gives

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

Ampere's law

The statement is exactly

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

or in integral form:

$$\int_{\partial\Omega} \underline{B} \cdot d\underline{l} = \mu_0 I_{\text{encl}}$$

($\partial\Omega$ is called the Amperian loop.) It can be helpful when solving for magnetic fields in high symmetry.

- Magnetic field of the toroid is circumferential at all points

Magnetic field of a long solenoid: An long solenoid of n turns per unit length and radius R has $\underline{B}$ field:

$$\underline{B} = \begin{cases} \mu_0 n I \hat{\underline{z}} & \text{Inside} \\ \underline{0} & \text{Outside} \end{cases}$$

Outside is zero comes from Ampere's law of a loop enclosing zero current.

Magnetic field of a toroid: The field of a toroid is circumferential,

$$\underline{B} = \begin{cases} \frac{\mu_0 N I}{2\pi s}\hat{\underline{\phi}} & \text{Inside} \\ \underline{0} & \text{Outside} \end{cases}$$

where N is the total number of turns.

Magnetic field of a counterclockwise circular current loop along the z axis

$$\underline{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2 + z^2)^{3/2}}\hat{\underline{z}}$$

Comparison of magnetostatics and electrostatics

In electrostatics,

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{\epsilon_0}\rho$$
$$\underline{\nabla} \times \underline{E} = \underline{0}$$
$$\underline{E}(r \to \infty) = 0$$

determines the electric field.

In magnetostatics

$$\underline{\nabla} \cdot \underline{B} = 0$$
$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \qquad \underline{B}(r \to \infty) = \underline{0}$$

determines the magnetic field.

1. Electric field diverge away from positive charge
2. Magnetic fields curls around a current
3. Magnetic fields have no beginning or end
4. There are no magnetic monopoles
5. Magnetic fields are due to charges in motion
6. It requires moving charges to field magnetic fields
7. $\underline{F}_{\underline{E}}$ is generally much greater than $\underline{F}_{\underline{B}}$, this is due to ϵ_0 and μ_0
8. $\underline{B}$ becomes apparent with large current, fast moving charges, and keeping the wire charge neutral

Magnetic vector potential

The vector potential

In electrostatics, the curl of $\underline{E} = 0$ permitted us to define $\underline{E} = -\underline{\nabla} V$.

Since $\underline{\nabla} \cdot \underline{B} = 0$, we can define the magnetic vector potential, $\underline{A}$:

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

By choice, we require

$$\underline{\nabla} \cdot \underline{A} = 0$$

Then, Ampere's law become:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) \implies \underline{\nabla}^2 \underline{A} = -\mu_0 \underline{J}$$

The laplace operator acts on a vector, so in Cartesian coordinates, this is equivalent to saying

$$\nabla^2 A_i \underline{e}_i = -\mu_0 J_i \underline{e}_i$$

Assuming $\underline{J}$ goes to zero at infinity, then

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{r}')}{\mathscr{r}}\, d\tau'$$

Ordinarily, the direction of $\underline{A}$ will match the direction of the current.

Boundary conditions

Magnetic fields are discontinuous at surface currents. Using the fact that the divergence of $\underline{B}$ is zero, and a pillbox of area A:

$$\int_{\partial\Omega} \underline{B} \cdot d\underline{S} = 0 = \underline{B}^{\perp}_{\text{above}} A - \underline{B}^{\perp}_{\text{below}} A$$

says that the perpendicular component of the magnetic field is continuous across surface currents:

$$\underline{B}^{\perp}_{\text{above}} = \underline{B}^{\perp}_{\text{below}}$$

Magnetic fields parallel to the surface and parallel to the current continuous, but is discontinuous when parallel to the surface and perpendicular to the current.

$$\underline{B}_{\text{above}} - \underline{B}_{\text{below}} = \mu_0 \left(\underline{K} \times \hat{\underline{n}}\right)$$

The magnetic vector potential is continuous across any boundary:

$$\underline{A}_{\text{above}} = \underline{A}_{\text{below}}$$

But

$$\partial_{,n}\underline{A}_{\text{above}} - \partial_{,n}\underline{A}_{\text{below}} = -\mu_0 \underline{K}$$

Multipole expansion of the vector potential

We can apply the power series expansion in $1/r$ to the vector potential of a current loop. Recall

$$\begin{aligned}\frac{1}{\mathscr{r}} &= \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos(\alpha)}}\\ &= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos(\alpha))\end{aligned}$$

We can then expand our integral equation for the vector potential.

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{\partial\Omega}(r')^n P_n(\cos(\alpha))\, d\underline{l}'$$

The monopole term is always zero since

$$\int_{\partial\Omega} d\underline{l}' = \underline{0}$$

The dipole term dominates, and is coordinate independent.

$$\underline{A}_{\text{dip}}(\underline{r}) = \frac{\mu_0 I}{4\pi r^2}\int_{\partial\Omega} r'\cos(\alpha)\, d\underline{l}' = \frac{\mu_0 I}{4\pi r^2}\int_{\partial\Omega}(\hat{\underline{r}}\cdot\underline{r}')\, d\underline{l}'$$

The integral can be written as

$$-\hat{\underline{r}}\times\int_{\Omega} d\underline{a}'$$

and if we define the magnetic moment, $\underline{m}$ to be

$$\underline{m} = I\int_{\Omega} d\underline{a}$$

Then we can write

$$\underline{A}_{\text{dip}}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{\underline{m}\times\hat{\underline{r}}}{r^2}$$

If $\underline{m}$ is in the $\hat{\underline{z}}$ direction, then

$$\underline{m}\times\hat{\underline{r}} = m\sin(\theta)\hat{\underline{\phi}}$$

So for a perfect magnetic dipole,

$$\underline{A}_{\text{dip}}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{m\sin(\theta)}{r^2}\hat{\underline{\phi}}$$

If we take the curl we get

$$\underline{B}_{\text{dip}}(\underline{r}) = \underline{\nabla}\times\underline{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos(\theta)\hat{\underline{r}}+\sin(\theta)\hat{\underline{\theta}}\right)$$

To produce a perfect magnetic dipole requires

- A infinitesimally small loop at the origin
- Infinite current, with $m = Ia$ held fixed

Magnetic fields in matter

Magnetization

Electrons orbiting around the nuclei form small current loops. We can treat them as magnetic dipoles. They usually cancel each other. But when a magnetic field is applied, a net alignment of magnetic dipoles occur, and the medium becomes magnetized.

- **Paramagnets**: alignment is parallel to $\underline{B}$
- **Diamagnets**: alignment is opposite to $\underline{B}$
- **Ferromagnets**: retain their magnetization even after the external field is removed (Iron, Nickel, Cobalt)

Torques and forces on magnetic dipoles

Over a current loop, the torque is

$$\underline{N} = \underline{m}\times\underline{B}$$

This tends to align $\underline{m}$ with $\underline{B}$. When $\underline{B}$ is uniform, the net force on a current loop is zero (though there is a torque). When $\underline{B}$ is non-uniform, then

$$\underline{F} = \underline{\nabla}(\underline{m}\cdot\underline{B})$$

Unpaired electrons in atoms are subject to torque. (The torque on paired electrons cancel each other.) This is the mechanism for paramagnetism.

Effect of a magnetic field on atomic orbits

Consider the following model of the atomic nucleus.

- A single electron revolves around the positive nucleus at a constant speed v, and radius R

1. Placing a revolving electron with magnetic dipole moment $\underline{m}$ in a uniform magnetic field pointing in the opposite direction speeds up the electron.
2. If $\underline{B}$ were in the same direction as $\underline{m}$, then e would be slowed down
3. Change in orbital speed causes a in dipole moment opposite of the direction of $\underline{B}$

In the presence of a magnetic field, each atom picks up an addition dipole moment that are antiparallel to the field. This is the mechanism for diamagnetism that occurs in atoms with a even number of electrons (where paramagnetism is usually absent).

Magnetization

Magnetization $\underline{M}$ is a vector quantity that describes magnetic dipoles per unit volume.

- The effects of para- and diamagnetism are very weak.
- In a non-uniform field, paramagnets are attracted, and diagmagnets are repelled

The field of a magnetized object

Bound currents

Each volume element $d\tau'$ in a magnetized object carries a dipole moment $\underline{M}\, d\tau'$. The total vector potential is

$$\underline{A} = \frac{\mu_0}{4\pi}\int_{\Omega}\frac{\underline{M}(\underline{r}')\times\hat{\underline{r}}}{r^2}\, d\tau'$$

There is one identity exploit,

$$\underline{\nabla}'\frac{1}{\mathscr{r}} = \frac{\hat{\underline{\mathscr{r}}}}{\mathscr{r}^2}$$

We can show that by applying the product rule, divergence theorem,

$$\underline{\nabla}\times(f\underline{A}) = f(\underline{\nabla}\times\underline{A}) - \underline{A}\times(\underline{\nabla}f)$$

and defining

$$\begin{aligned}\underline{J}_b &= \underline{\nabla}\times\underline{M}\\ \underline{K}_b &= \underline{M}\times\hat{\underline{n}}\end{aligned}$$

The integral for the vector potential can be interpreted as being the sum of the vector potential due to a bound volume and surface current.

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_{\Omega}\frac{\underline{J}_b(\underline{r}')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\int_{\partial\Omega}\frac{\underline{K}_b(\underline{r}')}{\mathscr{r}}\, da'$$

The magnetic field of uniformly polarized sphere:

$$\underline{B} = \frac{2}{3}\mu_0\underline{M}$$

(Griffiths Ex. 6.1)

The auxiliary field

Ampere's law in magnetized materials

If we consider $\underline{J}$ to be the sum of contributions from free current, and from bound current, then

$$\frac{1}{\mu_0}(\underline{\nabla}\times\underline{B}) = \underline{J}_f + \underline{J}_b = \underline{J}_f + \underline{\nabla}\times\underline{M}$$

We define the auxiliary field, $\underline{H}$, by

$$\underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M}$$

Then,

$$\underline{\nabla}\times\underline{H} = \underline{J}_f$$

Integrating both sides over the surface and applying Stoke's theorem gives

$$\int_{\partial\Omega}\underline{H}\cdot d\underline{l} = I_{f_{encl}}$$

A deceptive parallel

1. When symmetry is absent, the curl of $\underline{H}$ is not enough to determine $\underline{H}$; we also require its divergence
2. In general,

$$\underline{\nabla}\cdot\underline{H} = -\underline{\nabla}\cdot\underline{M}$$

Boundary conditions

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -\left(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}\right)$$
$$\underline{H}^{\parallel}_{\text{above}} - \underline{H}^{\parallel}_{\text{below}} = \underline{K}_f \times \hat{\underline{n}}$$

When $\underline{J}_f = 0$ everywhere, $\underline{\nabla} \times \underline{H} = \underline{0}$. This lets us write

$$\underline{H} = -\underline{\nabla} W$$

Taking the divergence of both sides give

$$\underline{\nabla}^2 W = \underline{\nabla} \cdot \underline{M}$$

When $\underline{M}$ is uniform, its divergence is zero everywhere except maybe at the surface. So we have Laplace's equation for a scalar field W.

Since W is the scalar potential of $\underline{H}$,

$$W_{\text{in}} = \underline{W}_{\text{out}}$$

And

$$-\partial_{,n} W_{\text{above}} + \partial_{,n} W_{\text{below}} = -\left(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}\right)$$

Linear and nonlinear media

Magnetic susceptibility and permeability

Provided the magnetic field is to too strong, magnetization is proportional to $\underline{B}$ (and in turn, to $\underline{H}$). Magnetic susceptibility, χ_m, is a dimensionless constant of proportionality:

$$\underline{M} = \chi_m \underline{H}$$

- Paramagnets: $1 >> \chi_m > 0$
- Diamagnets: $\chi_m < 0$, $|\chi_m| << 1$
- Ferromagnets: $\chi_m > 0$, very large

Then with the assumption of a linear media,

$$\underline{B} = \mu_0(\underline{H} + \underline{M}) = \mu_0(1 + \chi_m)\underline{H}$$

If we define the relative permeability to be

$$\mu_r = 1 + \chi_m$$

and the permeability to be

$$\mu = \mu_0 \mu_r$$

We can say

$$\underline{B} = \mu \underline{H}$$

In general, the divergence of $\underline{H}$ or $\underline{M}$ still do not vanish (even $\underline{\nabla} \cdot \underline{B}$ is zero everywhere).

$$\underline{J}_b = \underline{\nabla} \times \underline{M} = \underline{\nabla} \times (\chi_m \underline{H}) = \chi_m \underline{J}_f$$

Ferromagnetism

Ferromagnets do not need external fields to sustain the magnetization.

- Magnetization is due to alignment of magnetic dipoles of associated with the spins of unpaired electrons
- Each dipole likes to point in the same direction as neighbouring dipoles (this is explained by QM)
- Nearly all dipoles are aligned in a ferromagnet

Ordinary pieces of iron consist of a many groups of dipoles aligned with each other, but in random directions, in *domains*. So there are no macroscopic magnetization.

- Placing the piece of iron in an external field pushes the domain boundaries until a single domain takes over
- Random thermal motions counteracts against the alignment, at 770° C, iron suddenly goes from being ferro- to paramagnetic

Electrodynamics

Electromotive force

Ohm's law

Current density is proportional to the force per unit charge.

$$\underline{J} = \sigma \underline{f},$$

where σ is the conductivity of the medium ($1/\sigma = \rho$, known as resistivity).

In general, we assume

- $\sigma = \infty$ for metals
- $\sigma = 0$ for insulators

In reality, there's a small finite conductivity for insulators too.

Recall

$$\underline{f} = \underline{E} + \underline{v} \times \underline{B}$$

Ohm's law: assuming that the velocity of the charges is sufficiently small, or $\underline{B}$ is sufficiently small, we can neglect $\underline{v} \times \underline{B}$ contribution to $\underline{f}$, which allows us to write

$$\underline{J} = \sigma \underline{E}$$

In electrostatics, we stated the electric field inside a perfect conductor is zero. If we assume that σ is effectively ∞ for metals, then $\underline{E} = \underline{J}/\sigma \to 0$ even for the the electrodynamic case.

A more familiar statement follows directly from Ohm's law.

$$V = IR$$

- R is the resistance: a function of geometry of the arrangement and the conductivity of the medium between the electrodes

For steady currents, and uniform conductivity,

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{\sigma} \underline{\nabla} \cdot \underline{J} = 0$$

(Recall the divergence of $\underline{J}$ is zero for uniform currents.)

- This was true for stationary charges
- Laplace's equation holds in homogeneous material carrying a steady current

Ohm's law suggests that in a constant electric field, the current density will also remain constant. Current density remains constant despite charges being constantly accelerated by an electric field since **charges collide as they move down the wire.** (Reference the Drude model.)

Electrons collide with each other as they travel down a wire. The work done by the electrical force is converted into heat in the resistor, and

$$P = VI = I^2 R$$

is the Joule heating law.

Electromotive force

When we drive a circuit using a battery, we assume that the current is the same all around the loop.

If current was not uniform, this implies that charge must be piling up some where. In this case, the charge that piles up would produce an electric field, with components that slow down any incoming charges, and components that pushes charges accumulated away.

This is the self correcting behaviour that occurs until the current flowing in and current flowing out are equal!

A force due to a source (i.e. a battery) and an electrostatic force is required to drive a circuit. In terms of force per unit charge:

$$\underline{f} = \underline{f}_s + \underline{E}$$

- The electrostatic force serves to smooth out the flow and communicate the influence of the source to distant parts of the circuit

Electromotive force (EMF): line integral of the force per unit charge around a circuit

$$\mathcal{E} = \oint \underline{f}_s + \underline{E} \cdot d\underline{l} = \oint \underline{f}_s \cdot d\underline{l}$$

EMF have units of **energy per unit charge**, the fact we call it a force is a pure convention.

Since electrostatic fields are conservative (i.e. curl-free), $\oint \underline{E} \cdot d\underline{l} = 0$.

Within an ideal EMF source (with no internal resistance), the net force on the charges is zero ($\underline{f} = 0 = \underline{f}_s + \underline{E}$), so $\underline{E} = -\underline{f}_s$.

The potential difference between two terminals a and b of a battery is then

$$V = -\int_a^b \underline{E} \cdot d\underline{l} = \int_a^b \underline{f}_s \cdot d\underline{l}$$
$$= \oint \underline{f}_s \cdot d\underline{l} = \mathcal{E}$$

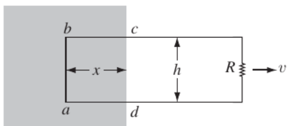
Since $\underline{f}_s$ is zero outside of the battery, we can extend $\int_a^b$ to $\oint$.

- The electrostatic field is analogous to gravity
- The force per unit charge due to the battery plays the role of a water pump, lifting the water up against gravity as it flows in a closed set of pipes

Motional EMF

Electrical generators exploit motional EMF, which arise when we move a wire through a magnetic field.

To derive the flux rule, consider a loop in a uniform magnetic field pointing into the page:



Consider the system at some moment in time. As the loop is being pulled to the right, charges in segment ab experience a force pointing upwards.

$$\underline{f}_{\text{mag}}^{ab} = v\hat{\underline{x}} \times (-B\hat{\underline{z}})$$
$$= vB\hat{\underline{y}}$$

In segments bc and ad, the charges also experience an upwards force, but since the charges in bc and ad cannot flow upwards, these forces have no contribution to the motional EMF.

$$\mathcal{E} = \oint \underline{f}_{\text{mag}} \cdot d\underline{l} = vbh$$

Magnetic forces never do work! Let's say that the charges are already moving in the loop. Velocity of charges in ab is given by

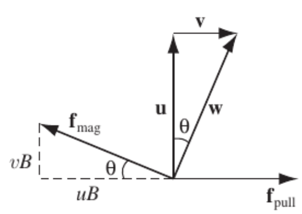
$$\underline{u} + \underline{v}$$

coming from moving the vertical direction and from the person pulling the loop.

The magnetic force has two components, given by

$$(\underline{u} + \underline{v}) \times \underline{B}$$

Clearly, the magnetic field would exert a force to the left in response to the vertical component of velocity.



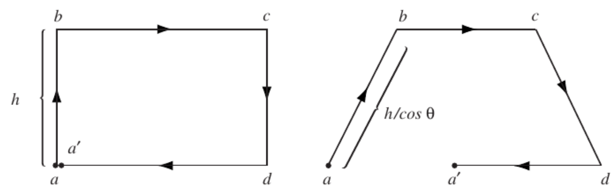
In this diagram, we see that

- To keep the loop moving to the right, the person must also exert a force uB per unit charge
- $\underline{w} = \underline{u} + \underline{v}$, which the velocity of a unit of charge in ab
- The total magnetic force has two components, vB and uB. But in total, $\underline{f}_{\text{mag}}$ is perpendicular to $\underline{w}$

Since $\underline{f}_{\text{mag}}$ is perpendicular to $\underline{w}$, it does not do any work! But it does contribute to the EMF.

On the other hand $\underline{f}_{\text{pull}} = uB\hat{\underline{x}}$ is not orthogonal to $\underline{w}$. It's the person pulling on the loop that is doing the work, and supplying the energy that heats the resistors. But $\underline{f}_{\text{pull}}$ is does not contribute to the EMF since it is perpendicular to $d\underline{l}$ around the circuit.

The work done per unit charge is exactly equal to the EMF, but the two are calculated using integrals around entirely different paths!



(a) Integration path for computing $\mathcal{E}$ (follow the wire at one instant of time).

(b) Integration path for calculating work done (follow the charge around the loop).

By definition, EMF should be calculated using a snap shot system in time. ($d\underline{l}$ is independent of whether the loop is moving in time not.)

When

- there is a single wire loop (current flows in a well define path)
- no switches, sliding contacts, of extended conductors allowing a variety of current paths
- though the loop is allowed to move, rotated, stretch

Motional EMF: we can apply the flux rule to calculating the motional EMF, given that the above assumptions hold

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

If we stick to the righthand rule, a negative EMF would mean that the current is flowing in the negative direction around the loop.

Electromagnetic induction

Faraday's law

Faraday performed three experiments,

1. Stationary $\underline{B}$, pulled a loop though the magnetic field
2. Stationary loop, pulled $\underline{B}$ away from the loop
3. Stationary wire and stationary $\underline{B}$. But $\underline{B}$ varied with time

in all cases, a current flowed through the loop. All three cases can be explained using the flux rule for motional EMF.

The second and third experiment is less obvious. If the charges are stationary, they cannot feel any magnetic force. Since the electric field is the only field that can exert forces on stationary charges, it must be the case that

A changing magnetic field induces an electric field

Empirically, Faraday postulated that

$$\mathcal{E} = -\frac{d\Phi}{dt} = \oint \underline{E} \cdot d\underline{l}$$

A change in magnetic flux is related to an induced electric field.

This is the integral form of Faraday's law. To get the differential form, we first rewrite the flux term,

$$\oint \underline{E} \cdot d\underline{l} = -\int_{\Omega} \frac{\partial \underline{B}}{\partial t} \cdot d\underline{a}$$

then apply Stoke's theorem to the line integral of the induced electric field

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

This induced electric field is not an electrostatic field, since it's closed loop integral is non-zero. In the case that $\partial \underline{B}/\partial t$ is 0, it reduces to

$$\underline{\nabla} \times \underline{E} = 0$$

which is analogous to saying that electrostatic $\underline{E}$ is conservative.

Faraday's experiment revealed another mechanism underlying the flux rule. In the first case, moving charges in an stationary magnetic field feels an EMF. In the second case, stationary charges feels a force due to an electric field induced by a changing magnetic field and feels an EMF.

Lenz's law is a rule of thumb. In that it states

Nature counteracts changes in flux

The induced electric field

If $\underline{E}$ is purely due to a changing $\underline{B}$ (and $\rho = 0$), then

$$\underline{\nabla} \cdot \underline{E} = 0$$

(much like $\underline{\nabla} \cdot \underline{B} = 0$) and

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

(much like $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$.)

This gives rise to an analog to the Biot-Savart law,

$$\underline{E} = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\underline{B} \times \hat{\underline{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau$$

If there is sufficient symmetry, we can use

$$\int_{\partial\Omega} \underline{E} \cdot d\underline{l} = -\frac{d\Phi}{dt}$$

as we do with Ampere's law ($\underline{E}$ replaced $\underline{B}$, and the time derivative replaced $\mu_0 I_{\text{encl}}$)

Quasistatic approximation

Faraday's law involves a changing $\underline{B}$ field. But to find the expression for $\underline{B}(t)$, we used magnetostatic equations. The error becomes significant when

- field rapidly fluctuates
- we are looking at points far away from the source

Inductance

A steady current though a loop of wire produces a magnetic field. The Biot-Savart law reveals that the magnetic field produced is proportional to the current I_1. The magnetic flux, Φ_2,through a second loop is then also proportional to the current I.

$$\Phi_2 \propto I_1 \implies \Phi_2 = M_{21} I_1$$

M_{21} is the mutual inductance between loop 2 and loop 1. We can derive the Neumann formula:

$$M_{12} = \frac{\mu_0}{4\pi} \oint\oint \frac{d\underline{l}_1 \cdot d\underline{l}_2}{\mathscr{r}}$$

- M purely geometrical
- $M_{12} = M_{21}$, so the flux through 2 when we run I around 1 is the same as flux around 1 when we run I around 2

If we vary I_1 *slowly* (quasistatic condition), then Φ_2 changes, and there a EMF will be induced in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M \frac{dI_1}{dt}$$

Self inductance: L, is the constant of proportionality between the magnetic flux through the loop, and the current around the loop

$$\Phi = LI$$

L has units of henries, $(\mathrm{V} \cdot \mathrm{s/A})$

A current through a wire loop induces a magnetic field. Changing the current through the loop changes the magnetic flux through the loop, and thus induces an EMF on the loop it self.

Back EMF: EMF induced in a wire loop due to a changing current:

$$\mathcal{E} = -L\frac{dI}{dt}$$

the EMF is induced in the direction to oppose any change in current

Energy in magnetic fields

By definition, the work done on unit charge against the back EMF around the circuit once is $-\mathcal{E}$ (not positive as it is the work done by us, not by the EMF). The work done per unit time is (we can get this from dimensional analysis)

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

If $I(0) = 0$, and $I(\infty) = I$, then

$$W = \frac{1}{2}LI^2$$

This can be shown to be equivalent to:

$$W_{\text{mag}} = \frac{1}{2\mu_0}\int_{\mathbb{R}^3} B^2\, d\tau$$

- It takes work to set up an magnetic field; though magnetic fields do not work
- A changing magnetic field induces an electric field, and the $\underline{E}$ field can do work

Maxwell's equations

Maxwell's addition of the displacement current removes a flaw from Ampere's law.

$$\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t}\right)$$

Maxwell's equations in Matter

Electric polarization leads to a boundary charge density. And magnetization results in a bound current.

In the dynamics, change in the polarization results in current flow from $-\sigma$ to $+\sigma$

$$\underline{J}_p = \frac{\partial \underline{P}}{\partial t}$$

The total charge density is a sum of the free and bound charges

$$\rho = \rho_f + \rho_b = \rho_f - \underline{\nabla} \cdot \underline{P}$$

And the current density becomes a sum of the free, bound, and polarization currents

$$\underline{J} = \underline{J}_f + \underline{J}_b + \underline{J}_p$$

Maxwell's equations:

- Gauss's law

$$\underline{\nabla} \cdot \underline{D} = \rho_f \qquad \int_S \underline{D} \cdot d\underline{a} = Q_{f_{\text{encl}}}$$

- No magnetic monopoles

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \int_S \underline{B} \cdot d\underline{a} = 0$$

- Faraday's law

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \oint \underline{E} \cdot d\underline{l} = -\frac{\partial}{\partial t}\int_S \underline{B} \cdot d\underline{a}$$

- Ampere's law

$$\underline{\nabla} \times \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}$$

$$\oint \underline{H} \cdot d\underline{l} = I_{f_{\text{encl}}} + \frac{\partial}{\partial t}\int_S \underline{D} \cdot d\underline{a}$$

Constitutive relations for linear media:

$$\underline{P} = \epsilon_0 \chi_e \underline{E} \qquad \underline{M} = \chi_m \underline{H}$$

$$\underline{D} = \epsilon \underline{E} \qquad \underline{H} = \frac{1}{\mu}\underline{B}$$

Boundary conditions

General boundary conditions:

$$D_1^\perp - D_2^\perp = \sigma_f \qquad B_1^\perp = B_2^\perp$$

$$\underline{H}_1^\parallel - \underline{H}_2^\parallel = \underline{K}_f \times \hat{\underline{n}} \qquad \underline{E}_1^\parallel = \underline{E}_2^\parallel$$