MATH 305

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Lecture 1

Why complex numbers?

Consider an equation of the form: $ax + b = 0$, where $a, b \in \mathbb{Z}$. The solution x does not always have an integer solution. This lead us to extend the set of numbers to include the rational numbers.

Consider the equation $x^2 - 2 = 0$. We know the solution is $\pm\sqrt{2}$, which is not a rational number. This lead us to extend the set of numbers to the real numbers $\mathbb{R}$.

Consider further $x^2 + 1 = 0$. The solution no longer belongs to the set of real numbers. To permit a solution, we expand the set of numbers to $\mathbb{C}$, the set of complex numbers.

The set of complex numbers, $\mathbb{C}$, is such that

- $\mathbb{C}$ extends $\mathbb{R}$, that is, $\mathbb{R} \subset \mathbb{C}$.
- Every polynomial with real coefficients have at least one root
- We can add and multiply complex numbers with the "usual rules" (distributivity, commutivity)

Complex numbers are also useful

- for handling periodic functions
- to compute integrals (some integrals cannot be computed using usual integral calculus methods)
- to compute sums (such as the harmonic series)
- to analyse dynamical systems, such as to determine stability of equilibrium
- in quantum mechanics

A complex number, $z \in \mathbb{C}$, is an expression of the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$, and i is the imaginary unit, $i = \sqrt{-1}$.

Complex arithmetic

Given $z = x + iy$, $w = u + iv$, addition:

$$z + w = (x + u) + i(y + v)$$

Multiplication:

$$z \cdot w = (xu - yv) + i(xv + yu)$$

Consider $i^2 = (0 + i1)^2$. Following the multiplication rule we just defined, we see $i^2 = -1$.

So $\pm i$ are two solutions of the equation

$$x^2 + 1 = 0$$

Here, we defined the operation to find that $i = \sqrt{-1}$. We could have also first defined i, then worked out the multiplication rule.

Lecture 2

Given a non-zero complex number $z = x + iy$, we have a multiplicative inverse:

$$z^{-1} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2}$$

such that

$$zz^{-1} = 1$$

Some more notation. Given $z = x + iy$, we define: $x = \text{Re}(z)$ and $y = \text{Im}(z)$. So $z = \text{Re}(z) + i\text{Im}(z)$.

$x + iy$ is naturally represented by a point in $(x, y) \in \mathbb{R}^2$. We describe the y-axis to be the "imaginary axis".

Using the representation of complex numbers as vectors in a complex plane, complex addition correspond to vector addition in $\mathbb{R}^2$.

But there is no suitable analogous vector multiplication operation to complex multiplication. This is why we work with complex numbers instead of representing everything with vectors.

We will see that multiplying complex numbers has to do with rotations and scaling in the plane.

The complex modulus:

$$|z| = \sqrt{z^2 + y^2}$$

The complex conjugate:

$$\overline{z} = x - iy$$

- Taking the complex conjugate is equivalent to making a reflection over the "real axis" (x-axis).

Useful properties:

- $|z|^2 = z\overline{z} = \overline{z}z$
  We can check that this is true:
  $$(x + iy)(x - iy) = (x^2 + y^2) + i(x(-y) + yx)$$
- $z^{-1} = \overline{z}/|z|^2$
  $$z\frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = 1$$
- z is real iff $z = \overline{z}$
- z is purely imaginary iff $z = -\overline{z}$
- $\text{Re}(z) = (z + \overline{z})/2$
- $\text{Im}(z) = (z - \overline{z})/2i$

$$z - \overline{z} = 2iy$$

Triangle inequality:

$$|z_1 + z_2| \le |z_1| + |z_2|$$

Inverse triangle inequality:

$$|z_1 - z_2| \ge ||z_1| - |z_2||$$

Complex exponential inequality:

$$|e^z| \le e^{|z|}$$

Let $z = x + iy$. Then

$$|e^z| = e^x$$

while

$$e^{|z|} = e^{\sqrt{x^2+y^2}}$$

Complex plane

Non-zero complex numbers can be expressed using complex coordinates

$$\begin{cases} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{cases}$$

where $r = |z|$, $\tan(\theta) = y/x$, and $\theta = \arg(z)$, $\theta \in [0, 2\pi)$.

- Do not consider $z = 0$. Since $\arg(0)$ can take on any value
- By choice, we restrict $\arg(z) \in [0, 2\pi)$. This is the same as saying θ is always measured counterclockwise from the $+x$-axis.

For $x \in \mathbb{R}$, we define the complex exponential:

$$e^{ix} = \cos(x) + i\sin(x)$$

This is Euler's formula/identity. Here, we have made the choice to simply take this as an definition, arising from geometry. Properties:

- $z = |z|\exp(i\arg(z))$
  $$\begin{aligned} z &= r\cos(\theta) + ir\sin(\theta) \\ &= |z|(\cos(\theta) + i\sin(\theta)) \\ &= |z|e^{i\arg(z)} \end{aligned}$$
- $\exp(2\pi ik) = 1$ for all $k \in \mathbb{Z}$
- Since $\cos(\theta)$ and $\sin(\theta)$ are 2π periodic, we have
  $$e^{i(x+2\pi k)} = e^{ix}$$
  for all $k \in \mathbb{Z}$
- $\cos(x) = \left(e^{ix} + e^{-ix}\right)/2$
- $\sin(x) = \left(e^{ix} - e^{-ix}\right)/2i$

Given $z = |z|\exp(i\theta)$, $w = |w|\exp(i\phi)$,

$$\begin{aligned} zw &= |z||w|\left((\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi))\right) \\ &= |z||w|\exp\left(i(\theta + \phi)\right) \end{aligned}$$

We cannot assume that the exponential rules holds. Since the exponential rules were defined with real numbers. Formally, we use the trigonometric identity to show that this is the case.

We see that

- $|zw| = |z||w|$
- $\arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}$ such that $\arg(zw) \in [0, 2\pi)$

Lecture 3

Another standard choice for the argument is $(-\pi, \pi]$. When we choose this range,

$$\mathrm{Arg}(z) \in (-\pi, \pi]$$

we use a capital A for the argument.

For example,

$$\mathrm{Arg}(i) = \frac{\pi}{2} = \arg(i)$$

but

$$\mathrm{Arg}(-i) = -\frac{\pi}{2}$$

while

$$\arg(-i) = \frac{3\pi}{2}$$

The choice of a certain argument can make our lives easier.

We can rederive trigonometric identities from the single Euler formula.

> **The N-th root of unity:** A complex number w such that $w^N = 1$, $\forall N \in \mathbb{N} = \{0, 1, \dots\}$. Let $w_N = \exp(2\pi i/N)$, then
>
> $$1, w_N, w_N^2, \dots, w_N^{N-1} \implies$$
> $$1^N, w_N^N, \left(w_N^2\right)^N, \dots, \left(w_N^{N-1}\right)^N = 1$$
>
> are all the N-th roots of unity. (Since $w_N^N = 1$ we stop at $N - 1$)

For $z \in \mathbb{C}$, $z = x + iy$,

$$e^z = e^{x+iy} = e^x\left(\cos(y) + i\sin(y)\right)$$

so the complex exponential is periodic in the imaginary direction.

- $|e^z| = e^{\mathrm{Re}(z)}$

Sets and complex functions

Ω denotes a subset of $\mathbb{C}$.

A set is bounded if there exists $M > 0$, such that

$$|z| \leq M, \ \forall z \in \Omega$$

There exists a circle of radius a finite radius M in the complex plane such that Ω is contained within the circle.

A set, Ω, is open if it does not contain its boundary, $\partial\Omega$. For any point in Ω, there is a disk of finite radius such that all the points enclosed by the disk is is still in Ω.

Let r be the the radius of a ball in the complex plane. The open ball $B_r(z) = \{w \in \mathbb{C} : |w - z| < r\}$.

The closed ball: $\{w \in \mathbb{C} : |w - z| \leq r\}$ is not open. The closed ball is the union of Ω and $\partial\Omega$.

When we take the "closure" of a open set, we get a closed set.

A pointed disc:

$$\dot{B}_r(z) = B_r \backslash \{z\}$$

where the backslash denotes the difference of two sets.

An open set Ω is *connected* if there is a continuous path in Ω between any two points of Ω.

In two-D, we can imagine a connected set to be a continuous blob with a finite number of holes. A set is not connected when Ω looks like two or more distinct blobs.

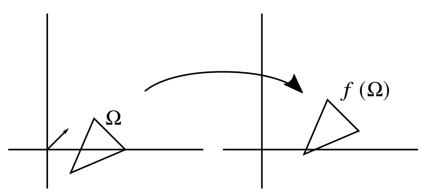
A complex function, $f(z)$, where $z \in \Omega$, takes the form

$$f(z) = \underbrace{u(z)}_{\mathrm{Re}(f(z))} + i \underbrace{v(z)}_{\mathrm{Im}(f(z))}$$

Lecture 4

A function $f : \Omega \to \mathbb{C}, \Omega \subset \mathbb{C}$ can be seen as a mapping that transform subsets of the complex plane. $f(\Omega)$ is also called the image of Ω under the mapping.

Map $z + w$**:** For $w \in \mathbb{C}$, translates Ω in the complex plane by w.



Map $e^{i\phi}z$:For $\phi \in \mathbb{R}$, $\phi > 0$, rotates Ω by an angle ϕ about the origin in the complex plane $\circlearrowleft$

Map λz**:** For $\lambda \in \mathbb{R}$, expands or compresses Ω by a factor of λ

Map $1/z$**:** Maps $\Omega = \dot{B}_1(0)$ to

$$f(\dot{B}_1(0)) = \{\zeta \in \mathbb{R} : |\zeta| > 1\}$$

Maps the open unit disk centered at 1

$$\widetilde{\Omega} = \{z \in \mathbb{C} : |z - 1| < 1\}$$

to

$$f(\widetilde{\Omega}) = \{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) > 0.5\}$$

If we pick $\Omega = \dot{B}_1(0)$ (pointed unit disk), $f(\dot{B}_1(0))$ is set of ζ such that $\zeta = z^{-1}$ for $z \neq 0$, $|z| < 1$. $|z| < 1$ implies that $|\zeta| > 1$. This means that the image of the pointed disk under f is

$$f(\dot{B}_1(0)) = \{\zeta \in \mathbb{R} : |\zeta| > 1\}$$

Consider

$$\widetilde{\Omega} = \{z \in \mathbb{C} : |z - 1| < 1\}$$

This is an open unit disk centered at $z = 1$. Under $f(z) = z^{-1} = \zeta$.

$$|z - 1| < 1 \iff |\zeta^{-1} - 1| < 1$$

we write $\zeta = u + iv$, and $\zeta^{-1} = \frac{u - iv}{u^2 + v^2}$, and

$$\begin{aligned} |\zeta^{-1} - 1|^2 &= \left|\left(\frac{u}{u^2 + v^2} - 1\right) + i\frac{v}{u^2 + v^2}\right|^2 \\ &= \left(\frac{u}{u^2 + v^2} - 1\right)^2 + \left(\frac{v}{u^2 + v^2}\right)^2 \\ &= \left(\frac{u}{u^2 + v^2}\right)^2 - \frac{2u}{u^2 + v^2} + 1 + \left(\frac{v}{u^2 + v^2}\right)^2 \\ &= \frac{u^2 + v^2 - 2u(u^2 + v^2) + (u^2 + v^2)^2}{(u^2 + v^2)^2} \end{aligned}$$

Applying the inequality $|\zeta^{-1} - 1| < 1$, results in

$$\frac{1}{2} < u$$

We arrive at

$$f(\widetilde{\Omega}) = \{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) > 0.5\}$$

which is the half of the complex plane shifted by 0.5. f mapped the shifted unit disk to the half-plane.

Joukowsky map:

$$f(z) = z + z^{-1}$$

Let $z = x + iy$ and $\zeta = u + iv$, and

$$\zeta = f(z) = (x + iy) + \frac{x - iy}{x^2 + y^2}$$

So

$$u = x + \frac{x}{x^2 + y^2} \qquad v = y - \frac{y}{x^2 + y^2}$$

The circle

$$\Omega = \{e^{i\theta} : \theta \in [0, 2\pi)\}$$

is mappped to $f(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$. So this becomes a line going from -2 to 2 on the complex plane. The general circle

$$\{z_0 + re^{i\theta}, \theta \in [0, 2\pi)\}$$

where z_0 defines the origin of the circle, r defines the radius, this turns out to be mapped to a shape of an air foil

Example: The complex function

$$f(z) = \frac{1}{-iz + 0.5}$$

where $\Omega = |z \in \mathbb{C} : \text{Im}(z) > 0|$ (The upper part of the complex plane). We can decompose $f(z)$ into three simpler functions ("Elementary maps").

$$f(z) = f_3(f_2(f_1(z)))$$

where

$$f_1(z) = -iz = e^{-i\pi/2}z$$

and

$$f_2(\zeta) = \zeta + 0.5$$

and

$$f_3(w) = w^{-1}$$

Under $f(z)$, Ω is

1. first rotated by clockwise by $\pi/2$ radians,
2. Translated to the right by $1/2$
3. The shifted half plane is mapped back to shifted open unit disk (see previous last examples)

Example: The map

$$f(z) = \frac{z-1}{z+1}$$

can be simplified to

$$f(z) = 1 - \frac{2}{z+1}$$

Lecture 5

Limits and continuity

We write

$$\lim_{z\to z_0} f(z) = L \qquad L \in \mathbb{C}$$

This means $f(z)$ and L can be made arbitrarily close, provided that z is close enough to z_0. Equivalently, $|f(z) - L|$ is very small, if $|z - z_0|$. (We are taking the modulus of the difference.)

There are different ways to for z to converge to z_0 in the complex plane. This means that $f(z)$ converges to L independent of how z approaches z_0.

Consider

$$\lim_{z\to i} \arg(z) = \frac{\pi}{2}$$

Consider

$$\lim_{z\to 1} \arg(z)$$

If z approches 1 from the upper half of the complex plane, then $\arg(z)$ approaches zero.

But if z approaches 1 from the lower half of the complex plane, thne $\arg(z)$ approches 2π. This means that this limit depends on the path z takes to approach 1. Therefore, this limit does not exist.

Consider

$$\lim_{z\to 1} \text{Arg}(z) = 0$$

By refining the discontinuity to the negative real axis, our limit exists.

A function is continuous at z if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

> We need to compute the limit for every path z can take to approach z_0.

We can say that $f(z) = \arg(z)$ is continuous on $\mathbb{C}\backslash[0,\infty)$ (the difference between the set of complex numbers and the positive real axis).

$f(z) = e^z$ is continuous on $\mathbb{C}$

$f(z) = |z|^2$ is continuous on $\mathbb{C}$

$f(z) = (z - w)^{-1}$ is contiuous on $\mathbb{C}$ except at $z = w$.

$$f(z) = \begin{cases} z/|z| & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is discontinuous at $z = 0$. Let's define

$$z_n = \frac{1}{n}e^{i\theta}$$

So the limit can be written as

$$\lim_{n\to\infty} f(z_n)$$

We see that

$$f(z_n) = \frac{e^{i\theta}/n}{1/n} = e^{i\theta}$$

The limit becomes

$$\lim_{n\to\infty} f(z_n) = e^{i\theta}$$

which is different for any given θ. So the limit does not exist.

Differentiability

A complex function f is differentiable at $z_0 \in \Omega$, if the following limit exists:

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

1. $f(z) = z^n$ for all $n \in \mathbb{N}$ is differentiable with

$$f'(z) = nz^{n-1}$$

2. $f(z) = \overline{z}$ is not differentiable anywhere. We can check that this is true. Let $z_0 = x + iy$, and $z = (x + h) + i(y + v)$, and define

$$R = \frac{f(z) - f(z_0)}{z - z_0} = \frac{((x+h) - i(y+v)) - (x - iy)}{h + iv} = \frac{h - iv}{h + iv}$$

   (a) When $v = 0$, $h \neq 0$, $h \to 0$, the limit approches 1

   (b) When $v \neq 0$, $v \to 0$, $h = 0$, the limit

$$\lim_{(0,v)\to(0,0)} R = -1$$

   Thus the limit does not exist!

Lecture 6

As a vector field, $f(z) = \overline{z}$,

$$f(x, y) = (x, -y)$$

if perfectly differentiable.

- If a function f is differentiable at every $x \in \Omega$, f is **holomorphic** in Ω.
- We denote the set of holomorphic functions in Ω is denoted $H(\Omega)$.
- When $\Omega = \mathbb{C}$, we say that f is an **entire** function.

We assume that $f \in H(\Omega)$, f is composed as by a real part, u and an imaginary part v. We want to know what f being holomorphic means for u and v.

Let's define $z_0 \in \Omega$, $z_0 = x + iy$, so the following limit exists.

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

where $z = (x + a) + i(y + b)$. Consider the horizontal limit, where $b = 0$, and we let $a \to 0$.

$$z - z_0 = a$$

and

$$f(z) - f(z_0) = (u(x + a, y) - u(x, y)) + i(v(x + a, y) - v(x, y))$$

thus,

$$\begin{aligned} f'(z_0) &= \lim_{a\to 0} \frac{1}{a}(u(x + a, y) - u(x, y)) \\ &+ \lim_{a\to 0} \frac{i}{a}(v(x + a, y) - v(x, y)) \\ &= \partial_x u(x, y) + i\partial_x v(x, y) \end{aligned}$$

Let's consider the vertical limit. Where $a = 0$, and $b \to 0$. Following the same steps, we have

$$\begin{aligned} f'(z_0) &= \lim_{b\to 0} \frac{1}{ib}(u(x, y + b) - u(x, y)) \\ &+ \lim_{b\to 0} \frac{1}{ib}(v(x, y + b) - v(x, y)) \\ &= -i\partial_y u(x, y) + \partial_y v(x, y) \end{aligned}$$

By our assumption, since f is holomorphic, the vertical and horizontal limits must be one in the same. We match the real and imaginary parts, and we find that

$$\partial_x u(x,y) = \partial_y v(x,y)$$

and

$$\partial_x v(x,y) = -\partial_y u(x,y)$$

Differentiability at a point: A complex function $f = u(x,y) + iv(x,y)$ defined over Ω is differentiable at a point $z_0 = x + iy$ if the first partial derivatives of u and v exist and are continuous at z_0, and satisfy the Cauchy-Riemann equations at z_0

$$\partial_x u(x,y) = \partial_y v(x,y)$$
$$\partial_y u(x,y) = -\partial_x v(x,y)$$

Differentiable in Ω (Holomorphic): When the first partial derivatives of u and v exist, are continuous, and satisfy the Cauchy-Riemann equations for all $z \in \Omega$, then f is holomophic in Ω.

There must be a disk, such that f is differentiable at every point in the disk. (Ω cannot be a path or a point!)

Differentiable in $\mathbb{C}$ (Entire): When the first partial derivatives of u and v exist and are continuous for all $\mathbb{R}^2$, then f is entire.

- If $f = u + iv$ is entire, then f^2 is also entire. When f is entire, then u and v are harmonic. Since f^2 is entire, $u^2 - v^2$ is harmonic, and $2uv$ is harmonic.

Differentiation

The first derivative of $f = u(x,y) + iv(x,y)$

$$f'(z_0) = \partial_x u(x,y) + i\partial_x v(x,y)$$
$$f'(z_0) = -i\partial_y u(x,y) + \partial_y v(x,y)$$

Example: Let's return to our example of $f(z) = \overline{z} = xi - y$. We can see that this is not differentiable from the Cauchy Euler equations.

Example: Consider

$$f(z) = |z|^2$$

If we let $z = x + iy$. Then, $f(z) = x^2 + y^2$. This means that we

$$u(x,y) = x^2 + y^2$$
$$v(x,y) = 0$$

We will find that the Cauchy-Euler equations can only be satisfied at $(0,0)$. This means $f(z)$ is only differentiable at $(0,0)$ or $z = 0 + 0i$.

As will find, this means that $f(z)$ is not holomorphic, or analytic everywhere.

Example: Differentiate: $f(z) = e^z = e^{x+iy}$

$$e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

$u(x,y) = e^x\cos(y)$, and $v(x,y) = e^x\sin(y)$. By the Cauchy Euler equations, f is differentiable everywhere. So f is entire. We can compute it's first derivative,

$$f'(z) = \partial_x u + i\partial_x v = e^z$$

Which is exactly what we would expect when if we took z as an ordinary real variable.

- Polynomials of z are entire.
- Ratio of polynomials in z, $f(z)/g(z)$, is holomorphic in $\{z \in \mathbb{C} : g(z) \neq 0\}$.

All rules of differentiation hold:

- $(f+g)'(z) = f'(z) + g'(z)$
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
- $f(g(z)) = f'(g(z))g'(z)$

Lecture 7

Recall that the real and imaginary parts of a complex number can be written as

$$x = \frac{1}{2}(z + \overline{z})$$

and

$$y = \frac{1}{2i}(z - \overline{z})$$

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \partial_x f \frac{\partial x}{\partial \overline{z}} + \partial_y f \frac{\partial y}{\partial \overline{z}} \\
&= \frac{1}{2}\left(\partial_x f - \frac{1}{i}\partial_y f\right) \\
&= \frac{1}{2}\left(\partial_z u + i\partial_x v - \frac{1}{i}\partial_y u - \partial_y v\right)
\end{aligned}$$

by the Cauchy-Riemann equations, this equation comes out to be zero. So when f is holomorphic, then it is a function of z only, and not of $\overline{z}$.

The real and imaginary part of a holomorphic function are related by the Cauchy-Riemann equation. The real part determines the imaginary part, and vice versa, up to a constant.

For example:

$$u(x,y) = x^3 - 3xy^2 + y$$

find $v(x,y)$ such that $f = u + iv$ is entire. Knowing u, we can solve the Cauchy-Riemann equation for v.

$$\partial_x v = -\partial_y u$$

The y partial of u is:

$$-\partial_y = 6xy - 1$$

So

$$\frac{\partial v(x,y)}{\partial x} = 6xy - 1$$

Integrating both sides:

$$v(x,y) = 3x^2 y - x + C(y)$$

We still have a missing function of y. We use the other Cauchy Riemann equation.

$$\partial_y v = 3x^2 + C'(y)$$

and

$$\partial_x u = 3x^2 - 3y^2$$

Then,

$$C'(y) = -3y^2$$

Integrating both sides w.r.t. y,

$$C(y) = -y^3 + B \qquad B \in \mathbb{R}$$

Finally,

$$\begin{aligned}
f(z) &= (x^3 - 3xy^2 + y) + i(3x^2 y - x - y^3 + B) \\
&= z^3 - i(z - B)
\end{aligned}$$

Can any differentiable function $u(x,y)$ be the real part of a holomorphic function?

$$\partial_x(\partial_x u) = \partial_x(\partial_y v) = \partial_y(\partial_x v) = -\partial_y(\partial_y u)$$

So we have

$$\partial^2_{xx} u + \partial^2_{yy} u = 0$$

If $f \in H(\Omega)$, $f = u + iv$, then,

$$\nabla^2 u = 0$$
$$\nabla^2 v = 0$$

u and v are *harmonic* functions, as they satisfy Laplace's equation. u and v are *harmonic conjugates* of each other.

The level curves of harmonic conjugates are perpendicular:

$$\underline{\nabla} u \cdot \underline{\nabla} v = \underline{0}$$

Gradients are orthogonal to level sets.

- Given u, we can see if it is possible to construct v such that $u + iv$ is $H(\Omega)$ by checking whether u is harmonic
- Two arbitrary harmonic functions do not make the real and imaginary parts of a holomorphic function. They would also need to be related by Cauchy-Riemann equation

We have seen harmonic functions in electrostatics.

Elementary functions derived from the exponential

Properties of the complex exponential:

$$e^z = e^{x+iy}$$

- The range of e^z is $\mathbb{C}\backslash\{0\}$
- Is periodic along the imaginary direction:
$$e^z + i2\pi n = e^z \qquad \forall n \in \mathbb{Z}$$
- e^z is entire
- The power series representation:
$$e^z = 1 + z + \frac{z^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
which is convergent for all $z \in \mathbb{C}$. In the sense of real analysis, e^z is analytic
- Trigonometric functions with a complex argument:
$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Lecture 8

Differntiation rules for the real argument sine and cosine still hold. We can also check that the trigonometric identities also hold.

$$\cos'(z) = -\sin(z)$$
$$\sin'(z) = \cos(z)$$

Similarly, the hyperbolic functions with a complex argument is still as defined:

$$\cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right)$$
$$\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right)$$

Hyperbolic trigonometric functions are rotated versions of the trigonometric functions in the complex plane.

$$\cosh(z) = \cos(iz)$$
$$\sinh(z) = -i\sin(iz)$$

Example: Consider the following trigonometric equation.

$$\cos(z) = 2$$

There are no real number for which the equality holds. Let's convert this problem into the complex exponentials.

$$2 = \frac{e^{iz} + e^{-iz}}{2}$$

Rearranging:

$$0 = e^{iz} + e^{-iz} - 4$$

Factor out e^{-iz}:

$$e^{-iz}\left(e^{2iz} - 4e^{iz} + 1\right) = 0$$

Since e^{-iz} cannot be zero, we are free to divide both sides by it. We come to the equation:

$$e^{2iz} - 4e^{iz} + 1 = 0$$

We see that e^{2iz} is square of e^{iz}. Let's complete the square:

$$\left(e^{iz} - 2\right)^2 - 3 = 0$$

Rearranging:

$$\left(e^{iz} - 2\right)^2 = 3$$

Divide by 3 on both sides:

$$\left(\frac{e^{iz} - 2}{\sqrt{3}}\right) = 1$$

This implies that

$$\frac{e^{iz} - 2}{\sqrt{3}} = \pm 1$$

We come to

$$e^{iz} = 2 \pm \sqrt{3}$$

In real variables, we can take the logarithm to invert the exponential. But can we do this when we have a complex exponential?

Since the exponential is periodic, this would mean that there is an infinite number of solutions!

Let's write

$$e^{iz} = e^{-y}e^{ix} = 2 \pm \sqrt{3}$$

Since $2 \pm \sqrt{3}$ is real and positive, we know that e^{ix} must simply be 1. This would mean that

$$x = 2\pi n \qquad \forall n \in \mathbb{Z}$$

This leaves us with:

$$e^{-y} = 2 \pm \sqrt{3}$$

Which has the solution:

$$-y = \ln\left(2 \pm \sqrt{3}\right)$$

Finally, the set of solutions

$$\left\{2\pi n - i\ln\left(2 \pm \sqrt{3}\right) : n \in \mathbb{Z}\right\}$$

In the reals, the function that takes any $x \in \mathbb{R}$ to e^x is one-to-one. We can always invert one-to-one maps. We define the natural log to be

$$\ln\left(e^x\right) = x$$

In the complex plane, the complex function that takes z to e^z is periodic. Since any complex number that differ in the imaginary part by an integer multiple of 2π will have the same complex exponential. We this a many-to-one function.

The function $f(z) = e^z$ is many-to-one. Since $e^{z+i2\pi n} = e^z$. We define the *principal strip* to be

$$\Omega_p = \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) < \pi\}$$

Restricting the domain of e^z to Ω_p makes $f : z \to e^z$ one-to-one from $\Omega_p \to \mathbb{C}\backslash\{0\}$.

This lets us define a inverse mapping:

$$\mathrm{Log} : \mathbb{C}\backslash\{0\} \to \Omega_p$$

And

$$\mathrm{Log}(z) = \ln(|z|) + i\mathrm{Arg}(z)$$

Let's check that

1. $\mathrm{Log}(z) \in \Omega_p$ for any $z \in \mathbb{C}\backslash\{0\}$
2. $e^{\mathrm{Log}(z)} = z$:
$$e^{\ln|z| + i\mathrm{Arg}(z)} = |z|e^{i\mathrm{Arg}(z)} = z$$
3. What is $\mathrm{Log}(e^z)$?
$$|e^z| = e^{\mathrm{Re}\{z\}}$$
and
$$\mathrm{Arg}(e^z) = \mathrm{Im}\{z\} + 2\pi n$$
where n is chosen so that the argument evaluates to be in $(-\pi, \pi]$ by definition.
$$\mathrm{Log}(e^z) = \ln\left(e^{\mathrm{Re}(z)}\right) + i\left(\mathrm{Im}\{z\} + 2\pi n\right) = z + 2\pi i n$$
When $z \in \Omega_p$, then n must be 0, so $\mathrm{Log}(e^z) = z$.

Example: Solve

$$\mathrm{Log}(z) = 2\pi i$$

There is no solution. Since the imaginary part of the Log must lie between $-\pi$ and π.

In general:

- $\mathrm{Log}(1/z) \neq -\mathrm{Log}(z)$: any real negative number would be a counter example
- $\mathrm{Log}(e^z) = z + 2\pi n$, $n \in \mathbb{Z}$

Lecture 9

Example: Let $-x$ be a negative real number. Equivalently, we can write

$$-x = xe^{i\pi}$$

The modulus of x is x, and $\text{Arg}(-x) = \pi$. The complex logarithm of $-x$ is then

$$\text{Log}(-x) = \ln(x) + i\pi \in \Omega_p$$

The complex logarithm permits negative real numbers.

- $\text{Log}(zw) = \text{Log}(z) + \text{Log}(w) + i2\pi n$, where $n = -1, 0, 1$ is chosen so that the imaginary part of $\text{Log}(zw)$ is within $(-\pi, \pi]$ This can be verified directly
- Log is discontinuous on $(-\infty, 0]$, where $\text{Arg}(z)$ is discontinuous. This discontinuity is called a *branch cut*.
- The end point of the branch cut, at $z = 0$ is called a *branch point*
- Away from the cut, Log is holomorphic, since e^z is holomorphic over Ω_p
- Another choice of the arugment would lead to other branch cuts
- If we chose $\arg(z) \in [0, 2\pi)$ instead of $\text{Arg}(z)$, we need to define Ω_p to be

  $$\{z \in \mathbb{C} : 0 < \text{Im}(z) < 2\pi\}$$

  so that

  $$\log(z) = \ln(|z|) + \arg(z)$$

- The complex Log provides two harmonic functions, its real and imaginary part.

Derivative of $\text{Log}(z)$:

$$\text{Log}'(z) = \frac{1}{z}$$

for all z not on a branch cut Antiderivative of $\text{Log}(z)$:

$$\frac{d}{dz}(z\text{Log}(z) - z) = \text{Log}(z)$$

Example: Find the steady state temperature distribution in the following situations.

1. An annulus of inner radius 1 and outer radius 2. With boundary conditions $T(1) = 20$ and $T(2) = -80$. Since the real part of $\text{Log}(z)$ is $\ln(|z|)$, it is constant along circles about the origin. Let's guess that the temperature distribution is given by

   $$\phi(x, y) = A\ln(|z|) + B$$

   We know that $\ln(|z|)$ is harmonic, so it solves laplace's equation. Substituting the boundary conditions give

   $$\phi(x, y) = \frac{100}{\ln(2)}\ln(r) - 80$$

2. An infinite wedge between $[\pi/4, -\pi/6]$ as measured from the positive real axis. The top edge has $T = 20$, and bottom edge has $T = -80$. We guess that

   $$\phi(x, y) = C\,\text{Arg}(z) + D$$

   Substituting the boundary conditions:

   $$C\frac{\pi}{4} + D = 20$$
   $$-C\frac{\pi}{6} + D = -80$$

   gives us the constants. We cannot have used $\arg(z)$ instead. As $\arg(z)$ has a branch cut in the domain.

Roots

For any $\alpha \in \mathbb{C}$, we define

$$z^\alpha = e^{\alpha \text{Log}(z)} \qquad \forall z \in \mathbb{C}\backslash\{0\}$$

In the reals, we could have written

$$\sqrt{z} = \left(e^{\ln(x)}\right)^{1/2} = e^{\ln(x)/2}$$

We can use the exponential and the logarithm to define roots.

Example :

$$\begin{aligned} 1^{1/2} &= e^{\text{Log}(1)/2} \\ &= e^{(\ln(1)+0)/2} \\ &= 1 \end{aligned}$$

$$\begin{aligned} (-1)^{1/2} &= e^{\text{Log(-1)}} \\ &= e^{i\pi/2} \\ &= i \end{aligned}$$

$$\begin{aligned} i^i &= e^{i\text{Log}(i)} \\ &= e^{i(0+i\pi/2)} \\ &= e^{-\pi/2} \end{aligned}$$

Square root

We define the square root using the priniple logarithm (Log).

$$\sqrt{z} = e^{\text{Log}(z)/2}$$

Let $z = re^{i\phi}$. Then

$$\sqrt{z} = \sqrt{r}e^{\text{Log}(e^{i\phi})/2}$$

The term in the exponential:

$$\begin{aligned} \text{Log}(e^{i\phi}) &= \ln|e^{i\phi}| + i\text{Arg}(e^{i\phi}) \\ &= i(-\pi < \phi \le \pi) \end{aligned}$$

Thus

$$\sqrt{z} = \sqrt{r}e^{i\phi/2}$$

The real part of the square root cannot be less than 0:

$$\text{Re}\left\{\sqrt{z}\right\} \ge 0 \qquad -\pi < \phi \le \pi$$

Example: The following equation has no solutions:

$$\sqrt{z} = -1 + i$$

since the real part of $\sqrt{z}$ must be positive or 0.

Lecture 10

What happens if we decide to define the range of the logarithm to be

$$\{\pi \le \text{Im}(z) < 4\pi\}$$

instead of the principle strip. Consider

$$\begin{aligned} 1^{1/2} &= e^{\frac{1}{2}(\ln(1) + i\arg_{2\pi,4\pi}(1))} \\ &= e^{i\pi} \\ &= -1 \end{aligned}$$

The branch cuts of composed functions are "inherited" from the branch cut of the Log.

Example: Consider

$$z^{1/3} = e^{\frac{1}{3}\text{Log}(z)}$$

This function is holomorphic on

$$\mathbb{C}\backslash\{\text{Re}(z) \in (-\infty, 0]\}$$

Example: Find the branch cut of the following:

$$\text{Log}(1 - z^2)$$

$\text{Log}(z)$ has a branch cut on $\text{Re}\{z\} \in (-\infty, 0]$. So we need to look for which values of z, such that $\text{Log}(1 - z^2)$ is on the brach cut.

$$\begin{cases} \text{Im}(1 - z^2) = 0 \\ \text{Re}(1 - z^2) \le 0 \end{cases}$$

Let's expand $1 - z^2$:

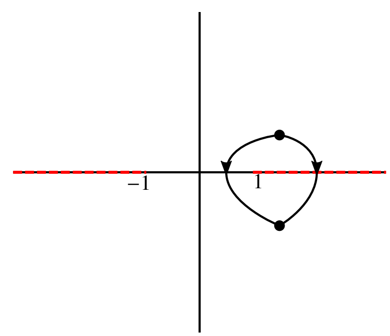
$$1 - z^2 = \underbrace{1 - x^2 + y^2}_{\text{Re}} - i\underbrace{2xy}_{\text{Im}}$$

So we need to solve

$$\begin{cases} xy = 0 \\ 1 - x^2 + y^2 \le 0 \end{cases}$$

If $x = 0$, this means that $1 + y^2 \le 0$. But y^2 is a positive number. So this case cannot be true.

If $y = 0$, this means that $1 - x^2 \le 0$, or $x^2 \ge 1$.



We see a peculier property of complex functions. A particular math might involve jumping through a branch cut, whereas other paths do not.

Example: Find a branch of $(z^2-1)^{1/2}$, which is holomorphic on $\{|z|>1\}$. We can rewrite this function as

$$e^{\frac{1}{2}\mathrm{Log}(z^2-1)}$$

This is the standard way of how we take the squre root. But we can show that this function has branch cuts on both the real and imaginary axis.

Or we can write

$$\begin{aligned}\left(z^2-1\right)^{1/2} &= \left(z^2\left(1-\frac{1}{z^2}\right)\right)^2\\ &= z\left(1-\frac{1}{z^2}\right)^{1/2}\\ &= ze^{\frac{1}{2}\mathrm{Log}\left(1-\frac{1}{z^2}\right)}\end{aligned}$$

Let's check that

$$\left(ze^{\frac{1}{2}\mathrm{Log}\left(1-\frac{1}{z^2}\right)}\right)^2 = z^2-1$$

which is true.

This new form of $\left(z^2-1\right)^{1/2}$ has is on the branch cut of $\mathrm{Log}(z)$ when

$$\begin{cases}\mathrm{Im}\left(1-\frac{1}{z^2}\right)=0\\ \mathrm{Re}\left(1-\frac{1}{z^2}\right)\le 0\end{cases}$$

So $1/z^2$ must be real and greater than 1:

$$\frac{1}{z^2} = \frac{(x^2-y^2)-2ixy}{|z^2|^2}$$

So

$$\begin{cases}xy=0\\ \frac{x^2-y^2}{|z^2|^2}\ge 1\end{cases}$$

When $y=0$, then

$$x^2\ge x^4 \iff x^2\le 1$$

So the branch cut is on the real axis, on the interval $[-1,1]$.

A diffent choice of taking $(1-z^2)^{1/2}$ result in different branch cuts.

We can also define $\log(z)$ as the set of solutions of

$$\{w\in\mathbb{C}: e^z=w\}$$

this is the multivalued log. Defined by the infinite set of values:

$$\mathrm{Log}(z) = \ln(|z|) + i\left(\mathrm{Arg}(z)+2\pi n\right) \qquad n\in\mathbb{Z}$$

Lecture 11

Integration

Complex integration always takes place along curves.

A *smooth parameterized curve*, α, is a map:

$$\alpha: [a,b]\to\mathbb{C},$$

where $[a,b]$ is a real interval. Such that:

- α is differentialable
- α is continuous
- $\alpha'(t)\neq \forall t\in[a,b]$
- α is *oriented* from $\alpha(a)$ to $\alpha(b)$
- α is *simple*, when for $a<t\neq s<b$, we have $\alpha(t)\neq\alpha(s)$ (it does not intersect it self)
- α is closed, if it begins and end at the same point

Here we define a "curve", but it's also called "path", "contour", "arc".

We know that different parameterizations can trace out the same path. They will not matter in integration.

Example: A horizontal segment from -1 to 2. One parameterization is

- $\alpha(t)=t$, for $t\in[-1,2]$
- $\alpha(t)=3t-1$ for $t\in[0,1]$

we go three times as fast in the second parameterization than we do in the first parameterization

Example: A vertical segment from $1-i$ to $1-3i$. Since the real part stays fixed:

- $\alpha(t)=1-it$ for $t\in[1,3]$

Example: A circle of radius r centered at $z_0\in\mathbb{C}$ with *positive orientation* (counterclockwise, righthand rule).

Let's first consider at circle at the origin. Any point on the circle can be traced out by

$$\alpha(t)=re^{it}$$

for $t\in[0,2\pi]$. So to shift the origin of the circle, we can simply add z_0 to α:

$$\alpha(t)=z_0+re^{it}$$

Consider a continuous complex function, f, defined over Ω. We also define a path α in the domain. We define the integeral of f along α:

$$\int_\alpha f(z)\,dz = \int_a^b f(\alpha(t))\,\alpha'(t)\,dt$$

- The result only depends on the geometry of α, not the parameterization of α

Example: Consider α that goes from 1 to $2+i$ in a straight line. And the complex function $f(z)=\overline{z}$. (Recall that we know $\overline{z}$ is not holomorphic, but we it is continuous over α, so we can integrate.)

The first step is to parameterize α. Let's define for $t\in[0,1]$, $\alpha(t)=1+(1+i)t$. $\alpha'(t)=1+i$.

So the integral is

$$\begin{aligned}\int_\alpha f(z)\,dz &= \int_0^1 f(1+(1+i)t)(1+i)\,dt\\ &= \int_0^1 (1+(1-i)t)(1+i)\,dt\\ &= (1+i)\int_0^1 1+t-it\,dt\\ &= (1+i)\left(t+(1-i)\frac{t^2}{2}\right)_0^1\\ &= 2+i\end{aligned}$$

Example: Consider $z_0\in\mathbb{C}$, $n\in\mathbb{Z}$, and $\Omega=\mathbb{C}\backslash\{z_0\}$, $f(z)=(z-z_0)^n$ is holomorphic in Ω. α is a positively oriented circle of radius r, centered on z_0.

$$\alpha(t)=z_0+re^{it}$$

where $t\in[0,2\pi]$.

$$\alpha'(t)=rie^{it}$$

So the integral of $f(z)$ along α is

$$\begin{aligned}\oint_\alpha f(z)\,dz &= \int_0^{2\pi}\left(z_0+re^{it}-z_0\right)^n rie^{it}\,dt\\ &= \int_0^{2\pi}\left(re^{it}\right)^n rie^{it}\,dt\\ &= \int_0^{2\pi} r^ne^{nit}rie^{it}\,dt\\ &= ir^{n+1}\int_0^{2\pi} e^{(n+i)it}\,dt\end{aligned}$$

We need to consider two cases.

1. $n\neq -1$, this results in 0
2. $n=-1$

When $n=-1$ the integral reduces to

$$i\int_0^{2\pi} dt = 2\pi i$$

We see that

$$\oint_{|z-z_0|=r}\frac{1}{z-z_0}\,dz = 2\pi i$$

This result is independent of the radius.For $n\neq -1$, $(z-z_0)^n$ has an antiderivative along α,

$$\frac{1}{n+1}(z-z_0)^{n+1}$$

but not so for $n=-1$. Since the antiderivative of $(z-z_0)^{-1}$ is $\mathrm{Log}(z-z_0)$, and it has a branch cut along the negative real axis.

Lecture 12

When α is a piecewise-smooth curve we define:

$$\int_\alpha f(z)\,dx = \sum_{i=1}^{n} \int_{\alpha_i} f(z)\,dz$$

The length of $\alpha : [a, b] \to \mathbb{C}$ is given by

$$\ell(\alpha) = \int_a^b |\alpha'(t)|\,dt$$

Estimation lemma:

$$\left|\int_\alpha f(z)\,dz\right| \le \int_a^b |f(\alpha(t))||\alpha'(t)|\,dt$$
$$\le M(f)\ell(\alpha)$$

where

$$M(f) = \max\{|f(z)| : z \in \alpha\}$$

The estimation lemma: the modulus of the path integral is always bounded from the above by the maximum modulus of the function along the path, times the length of the path.

This the intermediate result comes from the "absolute value inequality" for integrals.

Example: Consider $\alpha = Re^{it}$ for $t \in [0, \pi]$. This is a half circle of radius R. Estimate the modulus of the following:

$$\int_\alpha \frac{z^{1/2}}{1+z^2}\,dz$$

Let's first compute the length of the path. We expect it to be πR.

$$\begin{aligned}\ell(\alpha) &= \int_0^\pi |iRe^{it}|\,dt \\ &= R\int_0^\pi dt \\ &= \pi R\end{aligned}$$

Now, let's find $M(f)$.

$$\left|\frac{z^{1/2}}{1+z^2}\right| = \frac{|\exp(\mathrm{Log}(z)/2)|}{|1+z^2|}$$

Along α, we know that $|z|$ is always equal to R. So the numerator is at most be $\sqrt{R}$. We then need to get an lower bound on $|1+z^2|$.

$$|1+z^2| \ge |z^2| - 1 = R^2 - 1$$

Thus, maximum modulus is

$$M(f) = \frac{\sqrt{R}}{R^2-1}$$

and the bound on our integral is

$$\left|\int_\alpha \frac{z^{1/2}}{1+z^2}\,dz\right| \le \pi R \frac{\sqrt{R}}{R^2-1}$$

We see that in the case for $R \to \infty$, the integral tends to zero!

A holomorphic function F is the antiderivative of f in a domain Ω is

$$f(z) = F'(z)$$

for all $z \in \Omega$.

Let α be a smooth curve, such that $\alpha(a) = z_i$, and $\alpha(b) = z_f$.

$$\frac{d}{dt}F(\alpha(t)) = F'(\alpha(t))\alpha'(t) = f(\alpha(t))\alpha'(t)$$

Let α be a curve in Ω; if f has an antiderivative F in a neighbourhood of α, then

$$\begin{aligned}\int_\alpha f(z)\,dz &= \int_a^b \frac{d}{dt}F(\alpha(t))\,dt \\ &= F(z_f) - F(z_i)\end{aligned}$$

- When α is a closed curve, the integral evaluates to zero
- The integral only depends on the end points of α
- For fixed z_f, z_i, we are free to change α and we will obtain the same results

This explains why

$$\oint_{|z-z_0|=r} (z-z_0)^n\,dz = 0$$

for $n \ne -1$. Since for $n \ne -1$, we have an antiderivative for $(z-z_0)^n$. But it does not explain the case where $n \ne -1$.

Lecture 13

Recall that if f has an antiderivative, F, and F is defined all along α, then

$$\int_\alpha f(z)\,dz = F(z_f) - F(z_i)$$

In particular, then α is closed, the line integral is 0.

Cauchy's theorem

Cauchy's theorem: Consider a disk, $B_r(z_0)$, and there is a function $f \in H(B_r(z_0))$. Then f has an unique antiderivative in $B_r(z_0)$ (up to a constant).

If $F' = f$ and $G' = f$, then

$$(F - G)' = 0$$

which implies that $F - G$ is a constant, thus F and G differ only by a constant.

Cauchy's theorem guarentees the existance of an antiderivative of a function in a disk. This means that for $f \in H(B_r(z_0))$,

$$\oint_\alpha f(z)\,dz = 0$$

for any α in $B_r(t_0)$.

More generally, for any Ω without holes, and $f \in H(\Omega)$, α is a simple closed curve in Ω, such that the interior of α is completely in Ω, then

$$\oint_\alpha f(z)\,dz = 0$$

and for any open, non-self-intersecting α,

$$\int_\alpha f(z)\,dz$$

is path independent.

....

But a zero closed loop integral does not imply the integrand is holomorphic everywhere in the interior of the loop. For example

$$\oint_{|z|=1} \frac{1}{z^2}\,dz = 0$$

But $1/z^2$ has a pole of order 2 at $z = 0$.

When α encloses a region of Ω containing a hole, then the region no longer holds.

Sketch of proof of Cauchy's theorem

We will proof Cauchy's theorem in the disk.

We first consider a triangle curve, τ, in $B_r(a)$, with perimeter L. We claim that

$$I = \oint_\tau f(z)\,dz = 0$$

By connecting the midpoint of each of the three segments of τ, we can partition τ into 4 new triangles. Let's denote the new triangles by a subscript, $\tau_1, \ldots, \tau_4$.

We can write I as

$$I = \sum_{k=1}^{4} \oint_{\tau_k} f(z)\,dz$$

This is true, since at shared boundaries, the direction of integration is always opposite, they cancel out.

Let $I^{(1)}$ be the integral along one of the four triangles with the largest modulus. Then, $|I|$ must be no greater than 4 times the modulus of $I^{(1)}$.

$$|I| \le 4|I^{(1)}|$$

For the triangle whose path integral is is $I^{(1)}$, we can connect the midpoints of its three sides. This time, we know that

$$I^{(1)} = \sum_{k=1}^{4} \oint_i f(z)\,dz$$

And let $I^{(2)}$ be the integral with the largest modulus, out of the four integrals that sum to $I^{(1)}$.

For $|I^{(1)}|$, we can again construct a bound:

$$|I^{(1)}| \le 4|I^{(2)}|$$

Relating this to our original bound on $|I|$, we see that

$$|I| \le 4|I^{(1)}| \le 4(4|I^{(2)}|)$$

If we repeat the same process, we can get another bound on $|I|$. If we repeat this n times, we have

$$|I| \le 4^n |I^{(n)}|$$

Clearly, as we continue to divide triangles, $|I^{(n)}|$ decreases with increasing n. Then, our goal is to show that $|I^{(n)}|$ decreases faster than 4^n increases, and $|I|$ is zero.

In the limit as $n \to \infty$, the triangles will converge to a single point. Let's denote this point by z_0.

Now we use our assumption that $f(z)$ is holomorphic at z_0. This means that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Before taking the limit, we can write:

$$f(z) = f(z_0) + (z - z_0) f'(z) + r(z, z_0)(z - z_0)$$

where

$$r(z, z_0) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z)$$

Then, in the limit as $z \to z_0$, $r(z, z_0)$ must converge to zero. Let's try to integrate $f(z)$ along τ_n, a very small triangle with the maximum modulus after n division procedures, as given by this expression.

The first term must be zero:

$$\oint_{\tau_n} f(z_0)\, dz = f(z_0) \oint_{\tau_n} dz = 0$$

this is true for any closed curve.

The second term:

$$f'(z_0) \int_{\tau_n} (z - z_0)\, dz = 0$$

We know that $(z - z_0)$ has an antiderivative. So the closed line integral must also be zero.

The last term:

$$\int_{\tau_n} r(z, z_0)(z - z_0)\, dz$$

is more difficult. But we can get an upper and lower bound on its modulus. Recall that

$$\left| \int_\alpha f(z)\, dz \right| \le M(f)\ell(\alpha)$$

where

$$M(f) = \max\{|f(z)| : z \in \alpha\}$$

The modulus of the path integral is always bounded from the above by the maximum modulus of the function along the path, times the length of the path.

In this case, we have

$$\left| \oint_{\tau_n} r(z, z_0)(z - z_0)\, dz \right| \le M(r(z, z_0)) M(z - z_0) \ell(\tau_n)$$

Every time we divide one triangle into four new triangles by the midpoint, its clear that its perimeter will be reduced by a factor of $1/2$. So after n divisions, τ_n has perimeter

$$\ell(\tau_n) = \frac{1}{2^n} L$$

We know that

$$M(z - z_0) < \frac{1}{2^n} L$$

So

$$\left| \oint_{\tau_n} r(z, z_0)(z - z_0)\, dz \right| \le M_n \left(\frac{1}{2^n} L \right)^2$$

Now, as $n \to \infty$, we have M_n must go to 0.

Recall that we bounded

$$\begin{aligned} |I| &\le 4^n |I^{(n)}| \\ &\le 4^n M_n \frac{L^2}{4^n} = L^n M_n \end{aligned}$$

Thus, $|I| = 0$ for any $f(z)$ along any triangle in $B_r(a)$.

Lecture 14

Assuming that a function $f(z)$ has an antiderivative, we showed that the path integral around any triangle must be zero.

We will now compute the antiderivative. Let's define two "vectors" in $B_r(z_0)$.

- z goes from z_0 to z
- h goes from z to $z + h$

z and h. The two span a parallelogram, and thus a triangle.

We claim that the antiderivative of $f(z)$, $F(z)$, in the disk, is

$$F(z) = \int_{[z_0, z]} f(z)\, dz$$

(integral along the segement from z_0 to z).

We showed that

$$\oint_\tau f(z)\, dz = 0$$

We can write this as a sum of terms:

$$\oint_\tau f(z)\, dz = F(z) + \int_{[z, z+h]} f(z)\, dz + \int_{[z+h, z_0]} f(z)\, dz$$

We see that

$$\begin{aligned} 0 = \int_{[z+h, z_0]} f(z)\, dz &= -\int_{[z_0, z+h]} f(z)\, dz \\ &= F(z + h) \end{aligned}$$

We can parameterize $[z, z+h]$ as $\alpha(t) = z + th$, for $t \in [0, 1]$. Thus,

$$F(z + h) - F(z) = \int_0^1 f(z + th) h\, dt$$

Hence,

$$\frac{F(z + h) - F(z)}{h} = \int_0^1 f(z + th)\, dt$$

In the limit as $h \to 0$, we have

$$\begin{aligned} \lim_{h \to 0} \frac{F(z + h) - F(z)}{h} &= \int_0^1 f(z)\, dt \\ &= f(z) \end{aligned}$$

Since $F'(z) = f(z)$, $F(z)$ is the antiderivative of $f(z)$.

Example: Compute:

$$\int_{\mathbb{R}} e^{-x^2} \cos(2kx)\, dx \qquad k \in \mathbb{R}$$

Consider $f(z) = e^{-z^2}$, which is entire, and the contour, that is a rectangle, whose two diagonal vertices are on $z = -R + i0$ and $R + ik$.

By Cauchy's theorem, we know that

$$\int_\alpha f(z)\, dz = 0$$

for all $R > 0$. Expanding the integral:

$$\begin{aligned} 0 = &\int_{-R}^{R} f(x)\, dx + \int_0^k f(R + it) i\, dt \\ &- \int_{-R}^{R} f(x + ik)\, dx - \int_0^k f(-R + it) i\, dt \end{aligned}$$

In these types of problems, we hope that the original integrals we want shows up as one of the terms in the expansion, and that we know how to take care of the other integrals that show up.

Since our original integral was over $\mathbb{R}$, we take the limit as $R \to \infty$. The first integral is a Gaussian integral.

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

The second integral

$$\lim_{R \to \infty} \int_0^k e^{-(R+it)^2} i\, dt$$

The modulus of this integral happens to be bounded above by zero.

The fourth integral is also bounded from the above by zero.

The third integral

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-(x+ik)^2}\, dx &= \int_{-\infty}^{\infty} e^{-(x^2 - k^2)} (\cos(2kx) - i \sin(2kx))\, dx \\ &= e^{-k^2} \int_{-\infty}^{\infty} e^{-x^2} (\cos(2kx) - i \sin(2kx))\, dx \end{aligned}$$

In conclusion,

$$\int_{-\infty}^{\infty} e^{-x^2}\left(\cos(2kx) - i\sin(2kx)\right)\,dx = e^{-k^2}\sqrt{\pi}$$

we can equate the real and imaginary parts. We find that

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2kx)\,dx = e^{-k^2}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\sin(2kx)\,dx = 0$$

Our result also shows that fourier transform of a gaussian is a gaussian.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{x^2/2}e^{ikx} = e^{-k^2/2}$$

To compute our desired integral, we needed to compute the third integral, along the contour from $R + ik$ to $-R + ik$. But instead of computing this integral, we consider the three other edges. This is also called computing by "shifting the contours".

Lecture 15

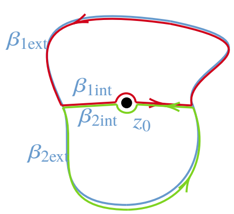
Recall that, for some $a \in \mathbb{C}$,

$$\oint_{|z-z_0|=r} \frac{a}{z - z_0}\,dz = 2\pi i a$$

This result was independent of the radius of the circle, r.

we see that Cauchy's theorem does not apply in this case. Since the function is undefined at $z = z_0$.

What about for another closed curve, β, about z_0?



Let's consider integration about positively oriented curves β_1 and β_2. By Cauchy's theorem, we know that

$$\oint_{\beta_1} \frac{a}{z - z_0}\,dz = \oint_{\beta_2} \frac{a}{z - z_0}\,dz = 0$$

We can decompose β_1 into the union of $\beta_{1\text{ext}}$, and $\beta_{1\text{int}}$. The same is true for β_2. So

$$\oint_{\beta_{1\text{ext}}} f(z)\,dz = -\oint_{\beta_{1\text{int}}} f(z)\,dz$$

Then, β is the union of $\beta_{1\text{ext}}$ and $\beta_{2\text{ext}}$.

$$\begin{aligned}\oint_{\beta} f(z)\,dz &= \oint_{\beta_{1\text{ext}}} f(z)\,dz + \oint_{\beta_{2\text{ext}}} f(z)\,dz \\ &= -\oint_{\beta_{1\text{int}}} f(z)\,dz - \oint_{\beta_{2\text{int}}} f(z)\,dz\end{aligned}$$

Since the straight parts of $\beta_{1\text{int}}$ and $\beta_{2\text{int}}$ are opposite in direction, their contributions will cancel out. All that is left is the the negatively oriented circle about z_0.

We find that

$$\oint_{\beta} f(z)\,dz = \oint_{|z-z_0|=\epsilon} f(z)\,dz$$

We can think of what we are doing here as "deforming the contour", β to a little circle about z_0. Indeed, we could have consider any simple contour in the interior of β. It did not need to be a small circle.

Principle of deformation of contours: If α and β are simple closed contours, and $\beta \in \text{int}(\alpha)$. Let f be holomorphic on both α and β, and at each point interior to α, but exterior to β, then

$$\oint_{\alpha} f(z)\,dz = \oint_{\beta} f(z)\,dz$$

Cauchy's integral formula

- α is a simple closed curve
- $f(z)$ is holomorphic on α and in its interior, denoted $\text{int}(\alpha)$
- the point w is in $\text{int}(\alpha)$

Consider

$$g(z) = \frac{f(z)}{z - w}$$

$g(z)$ is holomorphic in $\Omega\backslash\{w\}$. We state that

$$\oint_{\alpha} g(z)\,dz = \oint_{|z-w|=\epsilon} g(z)\,dz$$

for any $\epsilon > 0$ that is "small enough".

Consider the circle of radius ϵ about w: $w + \epsilon e^{it}$, for $t \in [0, 2\pi]$.

$$\begin{aligned}\oint_{\alpha} g(z)\,dz &= \int_0^{2\pi} \frac{f(w + \epsilon e^{it})}{(w + \epsilon e^{it}) - w} i\epsilon e^{it}\,dt \\ &= i\int_0^{2\pi} f(w + \epsilon e^{it})\,dt\end{aligned}$$

In the limit as $\epsilon \to 0$, the integrand becomes $f(w)$. And we conclude that

Cauchy's integral formula: Let f be a holomorphic function on $\alpha \cup \text{int}(\alpha)$, where α is any simple closed contour that winds around a fixed point $w \in \text{int}(\alpha)$ only once, then

$$f(w) = \frac{1}{2\pi i}\oint_{\alpha} \frac{f(z)}{z - w}\,dz$$

We can interpret this result as: "the values of f inside α are completely determined by the values of f along the curve α."

Lecture 16

Example: Compute

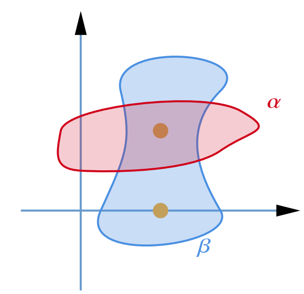
$$\oint_{|z-2|=3} \frac{e^z + \sin(z)}{z}\,dz$$

We see that the integrand is holomorphic everywhere in the interior of the circle, $|z - 2| = 3$, except at $z = 0$. We use Cauchy's integral formula. Let $f(z) = e^z + \sin(z)$, and $z = z - 0$. And we know that $0 \in \text{int}(|z - 2| = 3)$. So

$$\begin{aligned}\oint_{|z-2|=3} \frac{e^z + \sin(z)}{z}\,dz &= 2\pi i f(0) \\ &= 2\pi i\end{aligned}$$

Example: For arbitrary curves, α and β, compute

$$\oint_{\alpha \text{ or } \beta} \frac{z}{z^2 - (2+i)z + (1+i)}\,dz$$



This is not yet in a form we can apply Cauchy's integral theorem to. Consider the denominator:

$$z^2 - (2+i)z + (1+i) = (z - (1+i))(z - 1)$$

So the "poles" of the integrand is at $z = 1 + i$ and $z = 1$.

We will say that $z = 1 + i \in \text{int}(\alpha)$, but $z = 1$ is not in the interior of α.

Divide the numerator and denominator by $(z - 1)$:

$$\oint_{\alpha} \frac{z/(z-1)}{z - (1+i)}\,dz$$

We see that

$$f(z) = \frac{z}{z - 1}$$

is holomorphic in the interior of α. So

$$\oint_\alpha \frac{z/(z-1)}{z-(1+i)}\,dz = 2\pi i f(1+i) = 2\pi i \frac{1+i}{1+i-1} = 2\pi(1+i)$$

We will say that both the poles, $z=1, z=1+i$, of the integrand are in the interior of β.

The trick is that we will break up β into two parts, so one pole is in each of the two parts.

$$\oint_\beta \ldots\,dz = \oint_{\beta_1} \ldots\,dz + \oint_{\beta_2} \ldots\,dz$$

Let $z = 1+i$ be in the interior of β_1. We already know the result of this integral,

$$\oint_{\beta_1} \ldots\,dz = \oint_\alpha \ldots\,dz = 2\pi(1+i)$$

For β_2, we use the same trick as we did for α. Divide the numerator and denominator

$$\oint_{\beta_2} \frac{z/(z-(1+i))}{z-1}\,dz = 2\pi i \frac{1}{1-1+i} = 2\pi$$

So

$$\oint_\beta \ldots\,dz = 2\pi(2+i)$$

Consequences of Cauchy's integral formula

By CIF, for some simple closed α, and $f(z)$ is holomorphic in the neighbourhood of α, and in it's interior,

$$f(w) = \frac{1}{2\pi i}\oint_\alpha \frac{f(z)}{z-w}\,dz \qquad w \in \mathrm{int}(\alpha)$$

Differentiating w.r.t. w, we find that

$$f^n(w) = \frac{n!}{2\pi i}\oint_\alpha \frac{f(z)}{(z-w)^{n+1}}\,dz$$

Existence of higher derivatives: If $f \in H(\Omega)$, then all derivatives of f exist, and are analytic in Ω. Then, if $f = u+iv$, all partials of u and v exist, and are continuous in Ω.

Mean value property: The value of a holomorphic function f at the center of a circle is the mean value of f along that circle.

- There is no local maximum of $|f(z)|$ inside any Ω
- But there can be minimum of $|f(z)|$

Let α be a circle of radius of $r < R$. And $f(z_0)$ is holomorphic in the interior of α. α can be parameterized by $z_0 + re^{it}$ for $t \in [0, 2\pi)$.

If $f \in H(B_R(z_0))$ (holomorphic in an open unit disk of radius R, centered at z_0), then for any $r < R$:

$$f(z_0) = \frac{1}{2\pi i}\oint_\alpha \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{(z_0+re^{it})-z_0} ire^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt$$

Lecture 17

Maximum modulus principle: If Ω is a bounded domain and the non-constant $f \in H(\Omega)$ extends continuously to the boundary $\partial\Omega$, then $|f(z)|$ reaches its maximum on $\partial\Omega$, and not in Ω.

Liouville's theorem: If f is entire, and bounded, then f is constant.

- If f is entire, and $|f(z)| > 1$ everywhere, then $f(z)$ is constant. Because $g(z) = 1/f(z)$ is entire (since $f(z) \neq 0$ for all z) and less than 1 everywhere. By Liouville's theorem, $g(z)$ is constant since $g(z)$ is entire and bounded. Since $g(z)$ is constant, $f(z)$ is also constant.

If f is entire, and not constant, then it is unbounded. Then $|f(z)|$ must go to infinity somewhere.

Since f is holomorphic, its real and imaginary are harmonic, and harmonic functions also satisfy the maximum modulus principle. (Given without proof.)

Example: Find the maximum modulus of the function $f(z) = z^n e^{z^2}$ in $B_1(0)$, for $n = 0, 1, 2, \ldots$.

We see that $f(z)$ is a holomorphic function in the disk. It is in fact and entire function. The maximum modulus principle tells us that the maximum modulus of $f(z)$ will be found on the boundary of $B_1(0)$.

On the boundary of $B_1(0)$, we have $|z| = 1$.

Then, independent of n,

$$|f(z)| = \exp\left(\mathrm{Re}(z^2)\right)$$

The real part of z^2 is $\cos(2\phi)$, if $z^2 = e^{i2\phi}$.

The maximum modulus is then

$$\max\{e^{\cos(2\phi)} : \phi \in [0, 2\pi]\}$$

which occurs at $\phi = 0, \pi$, and $M(f) = e$ over the boundary of $B_1(0)$.

If $|f(z)|$ reaches a minimum at $z_0 \in \Omega$, then z_0 is a zero of f.

If $f(z) \neq 0$ for all z, then $1/f(z)$ is holomorphic, and hence $1/|f(z)|$ cannot reach its maximum of in Ω.

Minimum modulus principle: If the open domain Ω is bounded, and if the non-constant $f \in H(\Omega)$ extends continuously to the boundary $\partial\Omega$, then either $f(z_0) = 0$ for a $z_0 \in \Omega$, or

$$\min\{|f(z)| : z \in \Omega \cup \partial\Omega\} = \min\{|f(z)| : z \in \partial\Omega\}$$

and $|f(w)| > \min\{|f(z)| : z \in \partial\Omega\}$ for all $w \in \Omega$.

Lecture 18

Rouche's theorem (Weak form): Let α be a simple closed curve, and let f be holomorphic in a neighbourhood of $\alpha \cup \mathrm{int}(\alpha)$.

If $|f(z) - w| < |w|$ for some fixed $w \in \mathbb{C}$ and for all $z \in \alpha$, then $f \neq 0$ for all $z \in \alpha \cup \mathrm{int}(\alpha)$.

If $f(z) \in H(\mathrm{int}(\alpha))$, so too is $f(z) - w$. By the MMP, we know the maximum of $|f(z) - w|$ is attained for some $z \in \alpha$. By assumption, the maximum modulus must be less than $|w|$, implying that $|f(z) - w| < |w|$ for all $z \in \alpha \cup \mathrm{int}(\alpha)$.

So the distance between $f(z)$ and $z = w$ is less than $|w|$. So $f(z)$, $\forall z \in \alpha \cup \mathrm{int}(\alpha)$ is contained in an open disk of radius $|w|$ centered at $z = w$. Geometrically, we can see that $f(z) \neq 0$.

Algebraically, if $f(z) = 0$ for some $z \in \alpha \cup \mathrm{int}(\alpha)$, then $|0 - w = |w|$, which contradicts our assumption. So $f(z) \neq 0$ for any $z \in \alpha \cup \mathrm{int}(\alpha)$.

Example: The characteristic polynomial of a matrix $\underline{M}$ is given by

$$P_{\underline{M}}(T) = -t^5 + \frac{1}{12}z^4 - \frac{1}{4}t^2 + \frac{1}{6}t + 2$$

We claim that all the eigenvalues of M lie outside the unit disk. This means that $P_{\underline{M}}(t)$ has no zero in the unit disk. Let $f(z) = P_{\underline{M}}(z)/2$. Then,

$$|f(z) - 1| = \left|\frac{-z^5}{2} + \frac{z^4}{24} - \frac{z^2}{8} + \frac{z}{12}\right|$$

Applying the triangle inequality:

$$|f(z) - 1| \leq \frac{1}{2}|z|^5 + \frac{1}{24}|z|^4 + \frac{1}{8}|z|^3 + \frac{1}{12}|z|$$

On the unit circle, $|z| = 1$. Substituting $|z| = 1$, we have

$$\frac{1}{2} + \frac{1}{24} + \frac{1}{8} + \frac{1}{12} = \frac{3}{4} < 1$$

Since $3/4$ is less than 1, $f(z)$ satisfies Rouche's theorem, and there are no zeros of $f(z)$ in the unit disk.

Fundamental theorem of algebra: A polynomial (complex or real coefficients) of degree d has exactly d roots $\in \mathbb{C}$. They need not be distinct.

A polynomial of degree d has the form

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0$$

and $a_d \neq 0$, otherwise $P(z)$ is not a polynomial of degree d.

We will first prove that $P(z)$ has one root. To show this, we will suppose that $P(z) \neq 0$ for all $z \in \mathbb{C}$. If this is the case, then $1/P(z)$ is an entire function. Now

$$\frac{P(z)}{z^d} = a_d + \left(\frac{a_{d-1}}{z} + \dots + \frac{a_0}{z^d}\right)$$

We know that if

$$\lim |f(z)| = 0$$

then $\lim f(z) = 0$.

As $|z| \to \infty$, $P(z)$ approaches a_d, so

$$|P(z)| \geq \frac{1}{2}|a_d||z|^d$$

hence

$$\frac{1}{|P(z)|} \leq \frac{2}{|a_d|}\frac{1}{|z|^d}$$

For $|z| < R$, in a finite disk, the continuous function $1/|P(z)|$ is always bounded.

But if $1/P(z)$ is entire, and bounded, by Liouville's theorem, $1/P(z)$ is a constant.

But $1/P(z)$ cannot be a constant. So our assumption is invalid, and $|P(z)|$ must have at least 1 zero. We factor the single root $z = z_0$, and write

$$P(z) = (z - z_0)Q(z)$$

$Q(z)$ si a polynomial of degree $d-1$. If we repeat the same argument to $Q(z)$, we can conclude that $P(z)$ must have d roots.

Lecture 19

Analytic and meromorphic functions

Analytic and holomorphic functions are the same.

Recall the geometric series. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$$
$$= \frac{1}{1-r}$$

For any complex number $z \in B_1(0) \subset \mathbb{C}$,

$$\sum_{n=1}^{\infty} |z|^n < \infty$$

is a convergent series.

Recall that when the series is absolutely convergent, it follows that the series it self is convergent. So we can also write

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad \forall z \in B_1(0) \iff |z| < 1$$

The left hand side is a power series, but also a function that is perfectly well defined for all z in the unit circle.

Recall the set up for Cauchy's integral theorem.

Consider a domain $\Omega \subset \mathbb{C}$. Let $z_0 \in \Omega$, and Ω contains $B_r(z_0)$. Let w be a point in the interior of $B_r(z_0)$. Let γ be the circle on the boundary of $B_r(z_0)$.

Cauchy's integral formula tells us that, for a function $f(z) \in H(\Omega)$,

$$f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz \qquad \forall w \in \text{int}(\gamma)$$

A point, w, that is in the interior of γ must satisfy

$$|w - z_0| < r$$

And the radius, r, must be equal to the distance of any point z on γ from the center of the disk. So

$$|w - z_0| < r = |z - z_0| \qquad z \in \gamma$$

We want to somehow bring z_0 into our integral.

$$\frac{1}{z-w} = \frac{1}{(z-z_0)-(w-z_0)}$$
$$= \frac{1}{z-z_0}\left(\frac{1}{1-\frac{w-z_0}{z-z_0}}\right)$$

The modulus

$$\frac{|w-z_0|}{|z-z_0|} < 1$$

is true since the distance between a point on γ to z_0 must be greater than the distance between a point in int(γ) to z_0.

It follows that we can write the term in the bracket as a power series.

$$\frac{1}{z-w} = \frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n$$
$$= \sum_{n=0}^{\infty}\frac{(w-z_0)^n}{(z-z_0)^{n+1}}$$

Subtitute this term back into Cauchy's integral formula, and allowing ourselves to exchange the sum and integration, we get

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right)(w-z_0)^n$$

For some fixed z_0, we can say that

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = a_n$$

will be a number, so we might as well name it a_n

Then, we arrive at

$$f(w) = \sum_{n=1}^{\infty} a_n (w-z_0)^n$$

Recall that

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

And we actually have a taylor series.

$$f(w) = \sum_{n=1}^{\infty}\frac{f^{(n)}(z_0)}{n}(w-z_0)^n$$

By assuming that a function is holomorphic in a domain, we find that it also permits a convergent taylor series expansion, for any w, and a chosen point $z_0 \in \Omega$.

Analytic functions: If $f \in H(\Omega)$, then f is analytic.

Since f is equal to its Taylor series about any point $z_0 \in \Omega$, and the series is convergent in $B_r(z_0)$, at most for a radius r such that $B_r(z_0)$ is still contained within Ω. r is the **radius of convergence**.

Example: For $f(z) = e^z$, and $f^{(n)}(z) = e^z$, for all $n \in \mathbb{N}$. For $z_0 = 0$, we have

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \dots$$
$$= \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

This series has an infinite radius of convergence, since $f(z)$ is entire.

Some common MacLaurin series

$$e^z = 1 + z + \frac{z^2}{2!} + O(z^3) \qquad = \sum_{j=0}^{\infty}\frac{z^n}{j!}$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - O(z^6) \qquad = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}z^{2j}$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - O(z^7) \qquad = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}z^{2j+1}$$

Product of power series: Given two series representations for analytic functions f and g about $z = z_0$, their product is

$$f(z)g(z) = \sum_{n=0}^{\infty}\left(\sum_{i+j=n} a_i b_j\right)(z-z_0)^n$$

A consequence of Cauchy's integral formula was that if $f(z)$ is holomorphic, then its derivatives are also holomorphic.

Let's compute the taylor series for $f'(w)$ about z_0. We know that it takes the form

$$f'(w) = f'(z_0) + f''(z_0)(w-z_0) + \frac{f''(z_0)}{2}(w-z_0)^2 + \dots$$

What if we now differentiate each term in the Taylor series of $f(w)$ about z_0 with respect to w?

$$\frac{d}{dw}\left(f(z_0) + f'(z_0)(w - z_0) + \frac{f''(z_0)}{2}(w - z_0)^2 + \dots\right)$$

They are both equal.

- We are permitted to differentiate power series of holomorphic functions term by term

Example:

$$\begin{aligned}\frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right) &= \frac{d}{dz}\left(1 + \sum_{n=1}^{\infty}\frac{z^n}{n!}\right)\\ &= 0 + \sum_{n=1}^{\infty}\frac{nz^{n-1}}{n!}\\ &= \sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}\\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\end{aligned}$$

Which is exactly what we expect.

If function $g(z)$ can be represented by its power series every where in Ω,

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

then $g(z)$ is holomorphic in Ω with

$$g'(z) = \sum_{n=0}^{\infty} n b_n (z - z_0)^{n-1}$$

For this reason, an analytic function is also holomorphic.

Lecture 20

Example: The Fibonacci numbers are defined as the Taylor coefficients of the function

$$(1 - z - z^2)^{-1}$$

about $z_0 = 0$. We represent the set all Fibonacci numbers by $\{F_n : n \in \mathbb{N}\}$. So

$$(1 - z - z^2)^{-1} = \sum_{n=0}^{\infty} F_n z^n$$

The two roots of $(1 - z - z^2)^{-1}$ are the Golden ratios:

$$z_1, z_2 = -\frac{1 \pm \sqrt{5}}{2}$$

The radius of convergence is $-(1 + \sqrt{5})/2$. Recall that the radius of convergence is the radius of the largest disk about z_0.

We wish to compute F_n. We can do it directly, by repeated taking derivatives of $(1 - z - z^2)^{-1}$.

Consider

$$\begin{aligned}1 &= (1 - z - z^2)\sum_{n=0}^{\infty} F_n z^n\\ &= \sum_{n=0}^{\infty} F_n z^n - z\sum_{n=0}^{\infty} F_n z^n - z^2\sum_{n=0}^{\infty} F_n z^n\\ &= \sum_{n=0}^{\infty} F_n z^n - \sum_{n=0}^{\infty} F_n z^{n+1} - \sum_{n=0}^{\infty} F_n z^{n+2}\end{aligned}$$

Let's re-index the second and third series. If we increment the index under the summation, we need to decrement the index of the terms.

$$\sum_{n=0}^{\infty} F_n z^n - \sum_{n=1}^{\infty} F_{n-1} z^n - \sum_{n=2}^{\infty} F_{n-2} z^n$$

Rearranging:

$$\begin{aligned}1 &= F_0 + F_1 z - F_0 z + \sum_{n=2}^{\infty}(F_n - F_{n-1} - F_{n-2})z^n\\ &= F_0 + (F_1 - F_0)z + \sum_{n=2}^{\infty}(F_n - F_{n-1} - F_{n-2})z^n\end{aligned}$$

Let's identity to coefficents of the terms by associating them with the order the term of the form z^n.

The coefficient of F_0 is 1.

The coefficient of z is $0 = F_1 - F_0$.

The coefficient of z^n is $0 = F_n - F_{n-1} - F_{n-2}$, so we obtain a recursion relation:

$$F_n = F_{n-1} + F_{n-2}$$

This means that

$$\begin{aligned}F_0 &= 1\\ F_1 &= 1\\ F_2 &= F_1 + F_0 = 2\\ &\vdots\end{aligned}$$

More on the Log

Recall the deriviatives of $\mathrm{Log}(z)$:

$$\frac{d^j}{dz^j}\mathrm{Log}(z) = (-1)^{j-1}(j-1)!z^{-j}$$

Then, the taylor series at $z_0 = 1$ is

$$\mathrm{Log}(1) + (z - 1) - \frac{1}{2!}(z - 1)^2 + \frac{2!}{3!}(z - 1)^3 + \dots$$

(We chose $z_0 = 1$ since $z = 0$ is a branch point of the Log.) Alternatively,

$$\mathrm{Log}(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z - 1)^n$$

$\mathrm{Log}(z)$ is holomorphic everywhere, except at the branch cut $(-\infty, 0]$. So the radius of convergence of our taylor series is $r = 1$.

For example, if we considered the taylor series of the principle Log centered at $z_0 = -10 - i$, we would again find that the radius of convergence is 1.

If we want the taylor series of the first derivative of the Log, we can differentiate the series term by term, to find that:

$$\begin{aligned}&1 - (z - 1) + (z - 1)^2 - (z - 1)^3 + \dots\\ &= \sum_{n=0}^{\infty}(z - 1)^n\end{aligned}$$

By the geometric series formula, we get

$$\sum_{n=0}^{\infty}(z - 1)^n = \frac{1}{1 - z + 1} = \frac{1}{z}$$

as we expect for the derivative of the logarithm.

Zeros and poles

A number $z_0 \in \mathbb{C}$ such that for some function $f : \mathbb{C} \to \mathbb{C}$, $f(z_0) = 0$, is called a *zero* of f.

....

If f is analytic, then we can write the taylor series of f about z_0.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Since $f(z_0) = 0$, then $0 = a_0$. Possibly,

1. $a_n = 0$, and $f(z) = 0$ for all z; or
2. z_0 is a *zero of order* m when there is some coefficient a_m, for $m \geq 1$, that is non-zero. This means that while $a_0, a_1, \dots, a_{m-1} = 0$, and $a_m \neq 0$.

If we factor $(z - z_0)^m$ from the taylor series of $f(z)$, to get that

$$\begin{aligned}f(z) &= (z - z_0)^m (a_m + a_{m+1}(z - z_0)^{m+1} + \dots)\\ &= (z - z_0)^m g(z)\end{aligned}$$

Where $g(z)$ is also analytic, and $g(z_0) \neq 0$.

The equivalent requirement for the $m - 1$ first taylor coefficient to be 0 for z_0 to be a zero of order m, is that we require the first $m - 1$ derivatives of $f(z_0)$ to be zero

$$f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(m-1)}(z_0) = 0$$

Example: Consider

$$\begin{aligned}P(z) &= -(2 - 2i) + (5 - 3i)z - 3z^2 - (1 - i)z^3 + z^4\\ &= (z - 1)^2(z + 2)(z - (1 - i))\end{aligned}$$

So $z = 1$ is a zero of order 2. As the term $(z - 1)$ appears twice when we factored $P(z)$. We can also see this from expanding $P(z)$ about $z_0 = 1$.

Clearly, $g(z)$ is continous (a polynomial). The fact that $g(z_0) \neq 0$ implies that $g(z) \neq 0$ in a disk $B_r(z_0)$ around z_0. In $B_r(z_0)\backslash\{z_0\}$, $f(z)$ must then be non-zero.

Lecture 21

This leads us to say that

> A zero of finite order of an analytic function is always isolated in the complex plane. (They are never "grouped" together.)
>
> If we have a set of zeros that are not isolated, when the function must be zero!

> **The identity theorem:** Consider $f, g \in H(\Omega)$, and a simple curve α. If $f(z) = g(z)$ for all of $z \in \alpha$, then $f(z) = g(z)$ for all $z \in \Omega$.

As f, g are both analytic, if follows that their difference is also analytic. α is then a set of zeros for the function $f(z) - g(z)$. But these zeros are not isolated. So the only possiblity is that $f - g = 0$ for all $z \in \Omega$.

- In general, α can be replaced by a set of points that *accumulate somewhere*
- In a neighbourhood of a zero of order m, an analytic function behaves as

$$f(z) \approx (z - z_0)^m a_m$$

  Near z_0, $z - z_0$ is a small number. So the smallest power dominates.

Recall that we said all trigonometric identities hold in $\mathbb{C}$. This comes from the fact that $\sin(z)$ and $\cos(z)$ are entire functions. Since we know that the identities hold for all of $\mathbb{R}$, then they must hold for all $z \in \Omega$.

We know some functions are holomorphic in

$$\dot{B}_r(z_0) = B_r(z_0) \backslash \{z_0\}$$

What is the behaviour of these functions near z_0? We say that f has an **isolated singularity** at z_0 if $f \in H(\dot{B}_r(z_0))$ for some $z_0 \in \mathbb{C}$ and $r > 0$.

A function f has a pole of order m at z_0 if $1/f(z)$ is holomorphic in neighbourhood of z_0 and z_0 is a zero of order m of $1/f$

- A pole of order 1 is also known as a simple pole

Example: Consider

$$f(z) = \frac{\sin(z)}{(2z - 1)(z + (1 - 2i))^2}$$

The zeroes are at $n\pi$ for $n \in \mathbb{Z}$. Let's look at the case when $n = 0$.

The order of $z_0 = 0$ can be found by looking at how many derivative of $f(z)$ is non-zero, when evaluated at $z = 0$. It turns out that $f'(0) \neq 0$. So $z_0 = 0$ is a zero of order 1.

The complex sine can be written as

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

About $z = 0$,

$$e^{iz} = 1 + \frac{i}{1}z - \frac{1}{2!}z^2 + \dots$$

while

$$e^{-iz} = 1 - \frac{i}{1}z - \frac{1}{2!}z^2 + \dots$$

when we subtract them, the even powers cancel, and odd powers add. so

$$\frac{1}{i}\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(iz)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{6} + \dots$$

We can expand the denominator:

$$\begin{aligned}(2z - 1)(z + (1 + 2i))^2 &= (4i + 3) + z(-8 - 4i) + z^2(3 - 8i) + 2z^3 \\ &= (4i + 3)\left(1 + z\frac{-8 - 4i}{4i + 3} + \dots\right)\end{aligned}$$

Instead of computing the inverse directly,

$$\frac{1}{(4i + 3)} \frac{1}{1 + z\frac{-8-4i}{4i+3} + \dots}$$

we recognize that, ignoring $\dots$,

$$\frac{1}{1 + z\frac{-8-4i}{4i+3} + \dots}$$

appears like $1/(1 - z)$. This allows us to write it using the first two terms of this geometric series. Thus

$$\frac{1}{(2z - 1)(z + (1 + 2i))^2} = \frac{1}{4i + 3}\left(1 + z\frac{8 + 4i}{4i + 3} + \dots\right)$$

We get that

$$f(z) = \frac{1}{4i + 3}\left(1 + z\frac{8 + 4i}{4i + 3} + \dots\right)\left(z - \frac{z^3}{6} + \dots\right)$$

$1/2$ is a pole of order 1.

$2i - 1$ is a pole of order 2.

The radius of convergence of the series about $z = 0$ is the radius of the largest circle about $z = 0$ that do not contain any poles. This happens to be $1/2$.

Recall that $\sin(z)$ has zeros at $z = n\pi$. Its inverse,

$$\frac{1}{\sin(z)}$$

then has simple poles at $n\pi$.

Lecture 22

If an analytic function f has a pole of order m, we know that $1/f(z)$ has a zero of order m. Factoring out $(z - z_0)^m$, we have

$$\frac{1}{f(z)} = (z - z_0)^m g(z)$$

where $g(z_0) \neq 0$, and $g(z)$ is holomorphic near z_0. Taking the inverse, we find that

$$f(z) = \frac{1}{(z - z_0)^m g(z)}$$

Since zeroes of a function are always isolated, in a neighbourhood of z_0, $g(z) \neq 0$. It follows that $1/g(z)$ is analytic. We replace $1/g(z)$ by its taylor series about z_0.

$$f(z) = \frac{1}{(z - z_0)^m}\left(A_0 + A_1(z - z_0) + A_2(z - z_0)^2 + \dots\right)$$

Near z_0, a pole of $f(z)$, $f(z)$ is dominated by

$$\frac{A_0}{(z - z_0)^m}$$

which explains why all poles of order m have a similar appearance.

Expanding, we have

$$f(z) = \frac{A_0}{(z - z_0)^m} + \frac{A_1}{(z - z_0)^{m-1}} + \frac{A_2}{(z - z_0)^{m-2}} + \dots$$

> Even if $f(z)$ is not holomorphic at its pole of order m, at $z = z_0$, we can still find a power series representation of $f(z)$, involving negative powers of $z - z_0$ in the pointed disk centered at z_0.

Lecture 23

When our analytic function has a pole of finite order at $z = z_0$, we can still find a power series representation about z_0:

$$\begin{aligned}f(z) = &\frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots \\ &+ \frac{a_{-1}}{(z - z_0)} + a_0(z - z_0)^0 + \dots\end{aligned}$$

z_0 is a pole of f iff

$$\lim_{z \to z_0} |f(z)| = +\infty$$

We call the coefficient a_{-1} of our expansion the **residue**.

$$\text{Res}(f; z_0) = a_{-1}$$

A function that is holomorphic in a domain Ω with the exception of at isolated singularities $\{z_1, \dots, z_N\}$ is call **meromorphic** in Ω.

Consider a meromorphic function f in Ω. We have a simple closed curve, Ω, which only contains a single pole, z_0 of order m. The series representation of $f(z)$ about z_0 is

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{(z - z_0)^{-1}} + a_0(z - z_0)^0 + \dots$$

which can be decomposed in a analytic, and singular part:

$$f(z) = \sum_{j=m}^{1} \frac{a_{-j}}{(z-z_0)^j} + \sum_{j=0}^{\infty} a_j (z-z_0)^j$$

So

$$\oint_\alpha f(z)\,dz = \sum_{j=m}^{1} \oint_\alpha \frac{a_{-j}}{(z-z_0)^j}\,dz + \sum_{j=0}^{\infty} \oint_\alpha a_j (z-z_0)^j\,dz$$

When $j \neq 1$,

$$\frac{1}{(z-z_0)^j}$$

has an antiderivate along α. When $j = 1$,

$$\frac{1}{(z-z_0)}$$

does not have an antiderivate all along the curve, since the branch cut of the logarithm will eventually intersect α. When $j = 1$,

$$\oint_\alpha \frac{a_{-1}}{(z-z_0)}\,dz$$

The integral evaluates to $2\pi i a_{-1}$. Finally,

$$\oint_\alpha f(z)\,dz = 2\pi i \operatorname{Res}(f; z_0)$$

In the general case of having n singularities within α, we can "cut" the domain into n pieces such that each piece contains a portion of α, and onlt contains a single singularity.

> **Residue Theorem:** Let f be a meromorphic function in Ω, α is a simple closed curve in Ω. Then,
>
> $$\oint_\alpha f(z)\,dz = 2\pi i \sum_{z_j \in \operatorname{int}(\alpha)} \operatorname{Res}(f; z_j)$$

1. When there are no singularities:

$$\oint_\alpha f(z)\,dz = 0$$

this is Cauchy's theorem.

2. If $f(z)$ has no singularity, but the integrand is

$$\frac{f(z)}{z-w}$$

which has a simple pole at w. We can expand the integral about $z = w$,

$$\begin{aligned}\frac{f(z)}{z-w} &= \frac{1}{z-w}\left(f(w) + f'(w)(z-w) + \frac{1}{2}f''(w)(z-w)^2 + \dots\right) \\ &= \frac{f(w)}{z-w} + f'(w) + \frac{1}{2}f''(w)(z-w) + \dots\end{aligned}$$

The residue in this case is $f(w)$. Thus

$$\oint_\alpha \frac{f(z)}{z-w}\,dz = 2\pi f(w)$$

which is Cauchy's integral formula

Computing the residue

Simple pole: When $f(z)$ has a simple pole at z_0, we get a series expansion of f

$$f(z) = \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

Multiplying both sides by $z - z_0$, we get

$$f(z)(z-z_0) = a_{-1} + a_1(z-z_0)^2 + \dots$$

Thus:

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z-z_0) f(z) = a_{-1}$$

> **Simple Pole:** Let z_0 be a simple pole of $f(z)/g(z)$, such that $f(z_0) \neq 0$ and $g(z_0) = 0$.
>
> $$\operatorname{Res}\left(\frac{f(z)}{g(z)}; z_0\right) = \frac{f(z)}{g'(z_0)}$$

Let $f, g \in H(B_r(z_0))$, and z_0 is a simple zero of g, $g(z_0) = 0$. $f(z_0) \neq 0$. The residue is

$$\lim_{z \to z_0} \frac{(z-z_0)}{g(z) - g(z_0)} f(z)$$

The limit

$$\lim_{z \to z_0} \frac{(z-z_0)}{g(z) - g(z_0)} = \frac{1}{g'(z_0)}$$

So we have

$$\operatorname{Res}\left(\frac{f(z)}{g(z)}; z_0\right) = \frac{f(z)}{g'(z_0)}$$

Pole of order $m \geq 2$: The series expansion about $z = z_0$

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \dots + a_{-1}(z-z_0)^{m-1} + \dots$$

Taking $m - 1$ derivaties on both sides:

$$\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right) = (m-1)! a_{-1} + \dots$$

Then

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

Example: Consider

$$f(z) = \frac{\sin(z)}{(2z-1)(z + (1-2i))^2}$$

$f(z)$ has a simple pole at $z = 1/2$. The residue is

$$\begin{aligned}\operatorname{Res}(f; 1/2) &= \lim_{z \to 1/2} (z - 1/2) \frac{\sin(z)}{2(z-1/2)(z+(1-2i))^2} \\ &= \frac{\sin(1/2)}{2(3/2 - 2i)^2}\end{aligned}$$

Lecture 24

The co-tangent function is useful for computing series.

$$\xi(z) = \pi \cot(\pi z) = \pi \frac{\cot(\pi t)}{\sin(\pi t)}$$

The sine function has simple zeros at $n \in \mathbb{Z}$, which means at $\xi(z)$ has simple poles at $n \in \mathbb{Z}$.

We can find the residues by differentiating the denominator:

$$\operatorname{Res}(\xi(z); n) = \pi \frac{\cos(\pi n)}{\pi \cos(\pi n)} = 1$$

Provided that f does not have a pole, or a zero at $z = n \in \mathbb{Z}$,

$$\operatorname{Res}(f(t)\xi(t); n) = f(n)$$

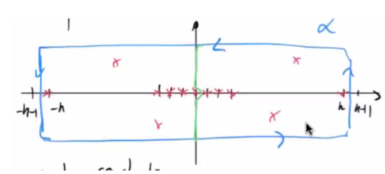
Example: In first year, we looked at whether certain series converged, but rarely computed their values. Using the residue theorem, we can compute certain series. Let f be a real valued function.

$$\sum_{n=0}^{\infty} f(n)$$

Recall the residue theorem,

$$\oint_\alpha f(z)\,dz = 2\pi i \sum_{z_j \in \operatorname{int}(\alpha)} \operatorname{Res}(f; z_j)$$

We pick a curve



can conclude

$$\frac{1}{2\pi i} \oint_\alpha \pi \cot(\pi z) f(z)\,dz = \sum_{j=m}^{n} f(j) + \sum (\text{other residues inside } \alpha)$$

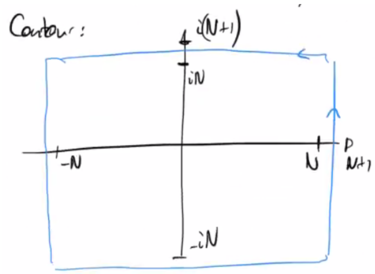
Example: Consider the case when

$$f(z) = \frac{1}{z^2 + a^2} \qquad a \notin i\mathbb{Z}$$

$f(z)$ has simple poles at $z = \pi i a$. The residue

$$\mathrm{Res}(f; \pi i a) = \lim_{z \to \pm i a} \frac{1}{(z \pm i a)} = \pm \frac{1}{2ia}$$

We choose the contour so that it does not intersect the integers.



We claim that

$$\lim_{N \to \infty} \oint_{\alpha_N} \pi \cot(\pi z) \frac{1}{z^2 + a^2}\, dz = 0$$

Since $|\cot(\pi z)|$ is bounded, away from the real integers. So its maximum modulus can be taken to be a constant.

$$\frac{1}{|z^2 + a^2|} \leq \frac{1}{|z|^2 - |a|^2} \approx \frac{1}{N^2}$$

Together

$$\left| \oint_{\alpha_N} \pi \cot(\pi z) \frac{1}{z^2 + a^2}\, dz \right| \leq \frac{\text{const}}{N^2} \ell(\alpha_N) \leq \frac{\text{const}}{N^2} 4(2N + 1)$$

We can see that in the limit as $N \to \infty$, the modulus of the integral is zero. So

$$0 = \sum_{j \in \mathbb{Z}} \frac{1}{j^2 + a^2} + \frac{\pi \cot(\pi i a)}{2ia} - \frac{\pi \cot(-\pi i a)}{2ia}$$

So

$$\begin{aligned}
\sum_{j \in \mathbb{Z}} \frac{1}{j^2 + a^2} &= -\frac{\pi \cot(\pi i a)}{ia} \\
&= -\frac{\pi}{a} \frac{e^{-\pi a} + e^{\pi a}}{e^{-\pi a} - e^{\pi a}} \\
&= \frac{\pi}{a} \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}}
\end{aligned}$$

In the series to the left, notice that the j and $-j$ terms add up

$$\sum_{j \in \mathbb{Z}} \frac{1}{j^2 + a^2} = \frac{1}{a^2} + 2 \sum_{j=1}^{\infty} \frac{1}{j^2 + a^2}$$

Rearranging,

$$\sum_{j=1}^{\infty} \frac{1}{j^2 + a^2} = \frac{1}{2a^2} \left(a\pi \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} - 1 \right)$$

In the limit as $a \to 0$, we have

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$$

Lecture 25

Example: Compute the following integral on the unit circle:

$$\oint_{|z|=1} \frac{1}{z^2 \sin(z)}\, dz$$

To find the residue, we first look at the zeros of the denominator. They are $z_0 = 0$, and $z \in n\pi$, for $n \in \mathbb{Z}$. We have $z_0 = 0$ as a zero of order 3.

Let's put the integrand into its series expansion.

$$\begin{aligned}
\frac{1}{z^2 \sin(z)} &= \frac{1}{z^2} \frac{1}{z - \frac{z^5}{6} + \dots} \\
&= \frac{1}{z^3} \frac{1}{1 - \frac{z^2}{6} + \dots} \\
&= \frac{1}{z^3} \left(1 + \frac{z^2}{6} \right) \\
&= \frac{1}{z^3} + \frac{1}{6z}
\end{aligned}$$

It's clear from the expansion that $z = 0$ is a pole of order 3 of the integrand. So we have

$$\mathrm{Res}\left(\frac{1}{z^2 \sin(z)}; 0 \right) = \frac{1}{6}$$

Instead of expanding the series, we could have also used the formula for the pole of order $m \geq 2$.

By the residue theorem, the integral evaluates to $2\pi i\,(1/6)$.

Example: Compute

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^4 + 4}$$

We pick $f(z) = 1/(z^4 + 4)$. The consider the contour α made of

$$\begin{aligned}
\alpha_1(t) &= t & t &\in [-R, R] \\
\alpha_2(t) &= Re^{it} & t &\in (0, \pi)
\end{aligned}$$

Our integral have simple poles at $z_j = \pm 1 \pm i$. and only $z_1 = 1 + i$, and $z_2 = -1 + i$ is in the interior of α.

The residues

$$\begin{aligned}
\mathrm{Res}\,(f; z_1) &= \lim_{z \to z_1} (z - z_1) f(z) \\
&= \frac{-1 - i}{16} \\
\mathrm{Res}\,(f; z_2) &= \frac{1 - i}{16}
\end{aligned}$$

The contribution of α_2

$$\begin{aligned}
\lim_{R \to \infty} \left| \int_{\alpha_2} f(z)\, dz \right| &\leq \lim_{R \to \infty} \pi R \max \left| \frac{1}{R^4 e^{4it} + 4} \right| \\
&\leq \lim_{R \to \infty} \frac{\pi R}{R^4 - 4} = 0
\end{aligned}$$

By the residue theorem, we also know

$$\oint_{\alpha} f(z)\, dz = 2\pi i \left(\frac{-1 - i}{16} + \frac{1 - i}{16} \right) = \frac{\pi}{4}$$

All together

$$\frac{\pi}{4} = \int_{-\infty}^{\infty} \frac{dz}{z^4 + 4}$$

If we have integrands that are a ratio of trigonometric polynomials:

$$\int_0^{2\pi} \frac{P(\sin(\phi), \cos(\phi))}{Q(\sin(\phi), \cos(\phi))}\, d\phi$$

where P and Q are polynomials. We can interpret it as a integral along the unit circle in $\mathbb{C}$.

We have the parameterization

$$\begin{aligned}
\alpha(\phi) &= e^{i\phi} \qquad \phi \in [0, 2\pi] \\
\frac{d\alpha(\phi)}{d\phi} &= i\alpha(\phi)
\end{aligned}$$

Under the parameterization, $z = \alpha(t)$, we can express

$$\begin{aligned}
\sin(\phi) &= \frac{1}{2i} \left(z - \frac{1}{z} \right) \\
\cos(\phi) &= \frac{1}{2} \left(z + \frac{1}{z} \right) \\
d\phi &\mapsto \frac{1}{iz}\, dz
\end{aligned}$$

Example: Compute

$$I = \int_0^{2\pi} \frac{1}{\frac{-9}{8} + \cos^2(\phi)}\, d\phi$$

We convert the real integral to a complex integral so we can use the residue theorem.

$$\begin{aligned}
I &= \oint_{\alpha} \frac{1}{\frac{-9}{8} + \left(\frac{1}{2} \left(z + \frac{1}{z} \right) \right)^2} \frac{1}{iz}\, dz \\
&= \frac{4}{i} \oint_{\alpha} \frac{1}{z \left(-\frac{9}{2} + z^3 + 2 + \frac{1}{z^2} \right)}\, dz \\
&= -4i \oint_{\alpha} \frac{z}{z^4 - \frac{5}{2} z^2 + 1}\, dz \\
&= -4i \oint_{\alpha} \frac{z}{(z^2 - 2)\left(z^2 - \frac{1}{2} \right)}\, dz
\end{aligned}$$

Since $\sqrt{2} > 1$, so the poles at $z = \pm\sqrt{2}$ are outside of α. But the poles are $\pm 1/\sqrt{2}$ are inside the contour.

Both the residues at $\pm 1/\sqrt{2}$ have the same value:

$$\mathrm{Res}\left(f; \pm \frac{1}{\sqrt{2}} \right) = \frac{1/\sqrt{2}}{\left(\frac{1}{2} - 2 \right) \frac{2}{\sqrt{2}}} = -\frac{1}{3}$$

In conclusion

$$I = 2\pi i\,(-4i) \left(\frac{-2}{3} \right) = -\frac{16\pi}{3}$$

Essential singularity

Consider the function $\sin(1/z)$. This function is singular at $z = 0$.

When $z \neq 0$, if we can substitute z for $1/z$ in the the Taylor series for $\sin(z)$ about $z_0 = 0$ to get the Taylor series for $\sin(1/z)$.

$$\sin\left(\frac{1}{z}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}\left(\frac{1}{z}\right)^{2j+1}$$
$$= \frac{1}{z} - \frac{1}{6}\frac{1}{z^3} + \frac{1}{120}\frac{1}{z^5} + \dots$$

We know that $z = 0$ is an isolated singularity. Yet $\sin(1/z)$ does not permit a series expansion where we go from the negative powers of z. to increasingly positive powers.

This can only mean that $z = 0$ is not a pole any finite order since there are z of arbitrarily large negative powers in the expansion. This is called an **essential singularity**.

- Looking at the series expansion, we can read off the residue
- But we dont have a formula for computing it directly, without going into a series expansion

Essential signularities are wild, in that

$$\lim_{z \to z_0} |f(z)| = \text{DNE}$$

For some point $w \in \mathbb{C}$, and $r > 0$, there is a serquence $z_n \in B_r(z_0)$ such that

$$f(z_n) \to w \qquad n \to \infty$$

So the function reaches all possible complex numbers near an essential singularity. (Maybe except at one point.)

Lecture 26

Laurent series

Is a series expansion of the form:

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

The singular part is

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

If z_0 is an essential singularity, then there is an infinite number non-zero a_n coefficients for $n \in (-\infty, -1]$.

The "regular part" is

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

The regular part is convergent for small enough $|z - z_0|$.

The singular part is convergent $|1/(z - z_0)|$ small enough. Equivalently, for $|z - z_0|$ that is large enough.

If a Laurent series is convergent, then it is convergent in an open annulus. We denote the region of convergence of the Laurent series about z_0 to be

$$C(z_0; a, b) = \{z \in \mathbb{C} : a < |z - z_0| < b\}$$

If a series is convergent in the pointed unit disk of radius r, in the new notaton,

$$\dot{B}_r(z_0) = C(z_0; 0, r)$$

Laurent's theorem: If $f \in H(C(z_0; a, b))$ for some $z_0 \in \mathbb{C}$, and $0 \leq a < b < \infty$, then f is equal to its Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where it is holomorphic. The Laurent coefficients can be computed using

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

If $f(z)$ where also holomorphic in every where in the "inner disk" (so $f \in H(B_b(z_0))$) we should expect that we recover the Taylor series. Consider what happens to

$$\frac{f(z)}{(z - z_0)^{n+1}}$$

(the integrand to coefficient a_n of the Laurent series) for $n \leq -1$.

- $n = -1$: the denominator is 1
- $n < -1$: the denominator becomes $(z - z_0)$ raised to some negative power

So for all $n < -1$, the integrand involves $f(z)$ times $(z - z_0)$ to some positive power. This means that the integrand is not singular for any $n \leq 1$, and the integrals evaluate to 0 by Cauchy's theorem.

- If $f \in H(B_b(z_0))$ instead, then

  $$a_n = 0 \qquad n \leq -1$$

  then the Laurent series is equivalent to the Taylor series
- If $f \in H(\dot{B}_b(z_0))$ (z_0 is an isolated singularity), then a_{-1} is $\text{Res}(f; z_0)$ by definition. We see that

  $$a_{-1} = \text{Res}(f; z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\, dz$$

  is exactly the residue theorem

Example: Consider

$$f(z) = \frac{1}{(z-1)(z+i)}$$

which has two simple poles at $-i$ and 1. We look at the region

$$C\left(1; 0; \sqrt{2}\right) = \left\{z \in \mathbb{C} : 0 < |z - 1| < \sqrt{2}\right\}$$

We want to find the Laurent series expansion of $f(z)$.

$$\frac{1}{z+i} = \frac{1}{(z-1) + (1+i)}$$
$$= \frac{1}{1+i}\frac{1}{1 + \frac{z-1}{1+i}}$$

Check that

$$\frac{|z-1|}{|1+i|} < 1$$

Since this holds, we can apply the geometric series expansion:

$$\frac{1}{z+i} = \frac{1}{1+i}\sum_{j=0}^{\infty}\left(\frac{z-1}{1+i}\right)^j (-1)^j$$

Thus

$$f(z) = \frac{1}{z-1}\frac{1}{1+i}\sum_{j=0}^{\infty}\left(\frac{z-1}{1+i}\right)^j (-1)^j$$
$$= \sum_{j=0}^{\infty} \frac{(-1)^j (z-1)^{j-1}}{(1+i)^{j+1}}$$
$$= \sum_{j=-1}^{\infty} \frac{(-1)^{j+1} (z-1)^j}{(1+i)^{j+2}}$$

This is not a true Laurent series, since our index begins at -1. From this, we can tell that $z_0 = 1$ is a simple pole, and read off $\text{Res}(f; 1) = 1/(1+i)$

Example: Consider the function

$$g(z) = \frac{1}{z(z-1)}$$

which has poles at $z = 0$ and $z = 1$ in the region $C(0; 1, \infty)$.

We cannot directly write the geometric series for $1/(z-1)$, since $|z| > 1$ in the region we are considering. But we can write the geometric series of $1/z$, which whose modulus is less than 1 in the region.

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1 - \frac{1}{z}}$$
$$= \frac{1}{z}\sum_{j=0}\left(\frac{1}{z}\right)^j$$

So

$$g(z) = \frac{1}{z^2}\sum_{j=0}\frac{1}{z^j}$$
$$= \sum_{j=0}^{\infty}\frac{1}{z^{j+2}}$$
$$= \sum_{n=-\infty}^{-2} z^n$$

Which is a Laurent series with only the singular part.

We have only considered the case when where only a single singularity is in the inner disk. When both singularities are in the inner disk, we cannot read off the residue of either of them from the series expansion. We see that coefficient a_{-1} is 0. We express it using the formula for calculate the Laurent series coefficients:

$$0 = \frac{1}{2\pi i}\oint_{|z|=r} f(z)\,dz$$

By direct application of the formulae for residues, we see that

$$\operatorname{Res}(g;0) + \operatorname{Res}(g;1) = 0$$

Lecture 27

Application to fluid dynamics

Consider **2-dimensional stationary, incompressible, and irrotational flow**.

- this means that the velocity field is time-indepedent

The velocity field, $\underline{v} = (v_1, v_2)$ is such that $v_1(x,y)$ and $v_2(x,y)$. The governing equations are

$$\begin{aligned}\underline{\nabla}\cdot\underline{v} &= 0\\ \underline{\nabla}\times\underline{v} &= 0\end{aligned}$$

In two dimensions, this is

$$\begin{aligned}\partial_x v_1 + \partial_y v_2 &= 0\\ \partial_y v_1 - \partial_x v_2 &= 0\end{aligned}$$

Complex analysis comes in when we introduce the **complex velocity**:

$$w(z) = v_1(x,y) - i v_2(x,y)$$

Assuming that $w(z)$ is holomorphic, the Cauchy-Riemann equations are

$$\begin{aligned}\partial_x v_1 &= \partial_y(-v_2)\\ \partial_y v_1 &= -\partial(-v_2)\end{aligned}$$

Rearranging, we can see that these are actually our governing equations.

We conclude that

> $w = v_1 - i v_2$ is holomorphic iff $\underline{v} = (v_1, v_2)$ satisfies the equation of stationary, incompressible, and irrotational flow
>
> $$\begin{aligned}\underline{\nabla}\cdot\underline{v} &= 0\\ \underline{\nabla}\times\underline{v} &= 0\end{aligned}$$

When w is holomorphic in a disk, w has an antiderivative (at least locally). By convention, we append a negative sign:

$$w(z) = -\phi'(z)$$

and $\phi(z)$ is self is holomorphic. We write ϕ in its real and imaginary parts:

$$\phi = \phi_1 - i\phi_2$$

and

$$\begin{aligned}w(z) &= v_1 - i v_2\\ &= -\partial_x\phi_1 + i\partial_x\phi_2\\ &= \partial_y\phi_2 + i\partial_y\phi_1\end{aligned}$$

We can write

$$\begin{aligned}v_1 &= -\partial_x\phi_1 = \partial_y\phi_2\\ v_2 &= -\partial_x\phi_2 = -\partial_y\phi_1\end{aligned}$$

We see

$$\underline{v} = (v_1, v_2) = -\underline{\nabla}\phi_1$$

The gradient of the imaginary part of ϕ (ϕ_2) can be thought of as the velocity field $\underline{v}^{\perp}$ that is perpendicular to $\underline{v}$

$$\underline{v}^{\perp} = (-v_2, v_1) = \underline{\nabla}\phi_2$$

- In vector analysis, we learned that "the gradient points in the direction of steepest descent", and that "gradient is perpendicular to the level curves".
- We can say that the level curves of ϕ_2 are perpendicular to ϕ_2, and is hence parallel to $\underline{v}$. The level curves of ϕ_2 is are the flow lines.

....

We try to solve for the flow outside of a fixed rigid body. We impose the constraint that $\underline{v}$ is tangential to the surface of the body ("The fluid cannot flow into the body", or ϕ_2 is constant on the the surface of the body).

We are considering a two dimensional case, so the "surface" we refer to is really the "outline" of the body. We denote the set of points that make up the outline of the body, b.

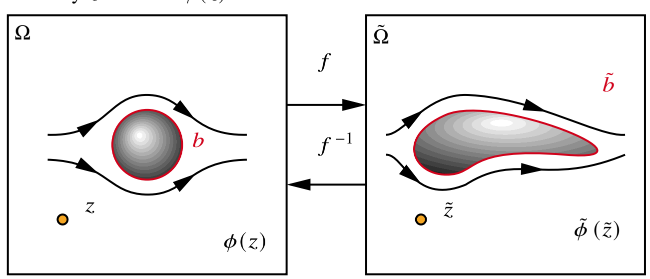
Let's use Ω to denote the domain outside of the rigid body. We can define a mapping $f : \mathbb{C} \to \mathbb{C}$, $f \in H(\Omega)$, and we require that $f'(z) \neq 0$ (we will see why).

Under f, the image of Ω is $f(\Omega) = \tilde{\Omega}$. Since f is holomorphic, f^{-1} is also holomorphic.

Say that ϕ is a solution in Ω. We define a new function $\tilde{\phi}$, such that

$$\tilde{\phi}(f(z)) = \tilde{\phi}(\tilde{z}) = \phi(z) = \phi(f^{-1}(\tilde{z}))$$

for every $z \in \Omega$ and $\tilde{\phi}(\tilde{z})$ is a solution in $\tilde{\Omega}$.



- Since both ϕ and f^{-1} are both holomorphic, and $\tilde{\phi}$ is a composition of two holomorphic functions, $\tilde{\phi} \in H(\tilde{\Omega})$
- Since $\phi_2(f^{-1}(\tilde{z}))$ is constant on the boundary of the body, so too is $\phi_2(\tilde{z})$ on $f(b) = \tilde{b}$, the image of b in $\tilde{\Omega}$

Lecture 28

If we know $\phi(z)$, then we can recover $\underline{v}$ by computing $-\underline{\nabla}\phi_1$.

Let the complex velocity field under the holomorphic mapping $\tilde{w}(\tilde{z})$.

$$\begin{aligned}\tilde{w}(\tilde{z}) &= -\tilde{\phi}'(\tilde{z})\\ &= -\phi'\left(f^{-1}(\tilde{z})\right)\end{aligned}$$

Since the superscript prime denotes differentiation w.r.t. $\tilde{z}$, we apply the chain rule:

$$\tilde{w}(\tilde{z}) = -\phi'\left(f^{-1}(\tilde{z})\right)\left(f^{-1}\right)'(\tilde{z})$$

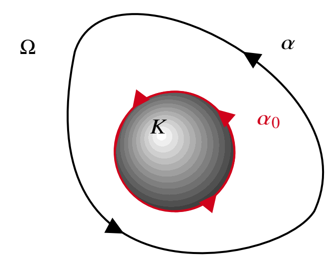
We can see that $-\phi'(f^{-1}(\tilde{z}))$ is exactly $w(z)$. And by the inverse function theorem, the derivative of f^{-1} is equal to $1/f'(z)$. So we get

$$\tilde{w}(\tilde{z}) = \frac{w(z)}{f'(z)}$$

The fact that we end up with $f'(z)$ in the denominator is where our assumption that $f'(z) \neq 0$ comes in handy.

> If we can find $w(z)$ around a body in a domain Ω, we can find the flow around a different body in $\tilde{\Omega}$, that is the image of Ω under a mapping f. We require $f \in H(\Omega)$ and $f'(z) \neq 0$ for all $z \in \Omega$.

....



An important parameter concerning fluid flow is **circulation**. We define circulation to be Z:

$$Z = \oint_{\alpha_0} w(z)\,dz$$

If w is a solution, then it must be holomorphic in Ω.

- By the principle of deformation of contours (a consequence of Cauchy's theorem), we can equivalently integrate along any simple closed curve α that encloses α_0, and $w(z) \in H(\mathrm{int}(\alpha))$
- Z is invariant under a holomorphic transformation. (We think of the holomorphic transformation as a "change of variables")

- If $Z \neq 0$, then w has some singularities inside of K

We can compute the force acting on K, $\underline{F} = (F_1, F_2)$ can be represented by the **complex force**

$$F(z) = F_1(x,y) + iF_2(x,y)$$
$$= \frac{i\rho}{2}\oint_\alpha w^2(z)\,dz$$

where ρ is the density of the fluid.

The force can be read off at ∞. If $\underline{v}(x,y)$ approaches some constant velocity $\underline{v}_\infty$ as $|(x,y)| \to \infty$, then

$$\underline{F} = -\rho Z \underline{v}_\infty^\perp$$

Here is the proof. Consider a circle of radius r, centered at $z=0$, such that the body K is completely enclosed in the circle's interior. The choice of $z=0$ is completely arbitrary. We could have picked any point z_0, but be might as well set it to $z=0$.

We may or may not know if w is completely holomorphic in the interior of the circle (since the circulation can be non zero), but we are certain that $w \in H(C(0; r, \infty))$. (Holomorphic in an annulus of inner radius r and outer radius ∞.)

Since $\underline{v}$ must approach a constant at large distances away from the body, we can reason that w (complex velocity) must also be bounded.

$w(z)$ on $C(0; r, \infty)$ has a Laurent series representation. We can think of the the regular part of the series as a bounded, entire function. This let's us conclude by Liouville's theorem that the regular part of the series must be a constant.

$$w(z) = w_0 + \sum_{n=1}^{\infty}\frac{w_{-n}}{z^n}$$

- no positive powers of z in the series, since w will no longer be bounded this way

Clearly, the series representation of $w(z)$ converges to w_0 as $|z| \to \infty$. By definition,

$$w_0 = v_{1\infty} - iv_{2\infty}$$

The integral formula for the Laurent coefficients tells us that

$$w_{-1} = \frac{1}{2\pi i}\oint_{|z|=r} w(z)\,dz = \frac{Z}{2\pi i}$$

In order to compute the force, we need to compute the integral of w^2 around a closed contour. By the residue theorem, this is equivalent to finding the residue of $w^2(z)$.

$$w^2(z) = \left(w_0 + \frac{w_{-1}}{z^1} + \frac{w_{-2}}{z^2} + \dots\right)\left(w_0 + \frac{w_{-1}}{z^1} + \frac{w_{-2}}{z^2} + \dots\right)$$
$$= w_0^2 + 2\frac{w_0 w_{-1}}{z^1} + \dots$$

So

$$\operatorname{Res}(w^2; 0) = \frac{Z}{\pi i}(v_{1\infty} - iv_{2\infty})$$

Altogether

$$F = \frac{i\rho}{2}\oint_\alpha (w_z)^2\,dz = \frac{i\rho}{2}2\pi i\frac{1}{i\pi}Z(v_{1\infty} - iv_{2\infty})$$
$$= \rho Z(iv_{1\infty} + v_{2\infty})$$
$$= \rho Z v_{2\infty} + i\rho Z v_{1\infty}$$
$$= F_1 - iF_2$$

So we have

$$\underline{F} = (\rho Z v_{2\infty}, -\rho Z v_{1\infty})$$
$$= -\rho Z \underline{v}_\infty^\perp$$

Instead of z and $\tilde{z}$, we use u and $z(u)$. The Joukowski map $u \mapsto z(u)$ where

$$z(u) = u + \frac{R^2}{u}$$

- singularity at $u = 0$
- the derivative

$$z'(u) = 1 - \left(\frac{R}{u}\right)^2$$

  is non zero as long as $|u| > R$

We can parameterize the circle $|u| = R$ by $u(\theta) = Re^{i\theta}$. Its image under the Joukowski map is

$$Re^{i\theta} + Re^{-i\theta} = 2R\cos(\theta)$$

which is purely real. The circle in the u-plane is mapped to the real interval $[-2R, 2R]$ in the z-plane.

We can solve for the inverse of $u \mapsto z(u)$. Multiplying both sides by u:

$$zu = u^2 + R^2$$

rearranging:

$$u^2 - zu + R^2 = 0$$

We can use the quadratic formula solve for the roots. We make the choice to pick the + sign in front of the square root.

$$u(z) = \frac{1}{2}\left(z + \sqrt{z^2 - 4R^2}\right)$$

Recall that we define the square root using the complex logarithm. Choosing the + sign places the branch cut exactly between $[-2R, 2R]$.

Luckily, the branch cut of $u(z)$ occurs exactly at the location where the velocity of the fluid isn't even defined.

As $|z| \to \infty$, we see that

$$u(z) \approx \frac{1}{2}\left(z + \sqrt{z^2}\right) = z$$

We can easily guess the flow along the plate.

$$v_1 = v_\infty, v_2 = 0$$

works. This is a constant flow in x direction only. So

$$w(z) = v_\infty = -\phi'(z)$$

and we reason that $\phi(z) = -v_\infty z$. Since w is a constant, the circulation, $Z = 0$. The force is proportional to Z. Since Z is 0, we can conclude that no force acts on the plane, as expected.

But now we can map the flow along the plate back into flow around the cylinder. So

$$\tilde{\phi}(u) = \phi(z(u))$$
$$= -v_\infty\left(u + \frac{R^2}{u}\right)$$

and the complex velocity field is

$$\tilde{w}(u) = v_\infty\left(1 + \left(\frac{R}{u}\right)^2\right)$$

By invariance, we conclude that $\tilde{Z} = 0 = \tilde{F}$.

We can also rotate the picture by an angle α. If we apply the holomorphic map which rotates a complex "vector" by α radians in the complex plane:

$$u \mapsto e^{i\alpha}u$$

$\tilde{\phi}(u)$ becomes

$$\tilde{\phi}(u) = -v_\infty\left(e^{i\alpha} + \frac{R^2}{e^{i\alpha}u}\right)$$

This time, we can map from the u plane (where the cylinder lives) to the z plane (where the thin plate lives), to find the rotated flow around the thin plate.

Lecture 29

A non-zero circulation implies that the interior of contour contains singularities.

Examples: Around the cylinder, the flow should be circularly symmetric. We can guess a complex velocity that has this property:

$$w(u) = \frac{Z}{2\pi i u}$$

The potential comes from finding the antiderivative of w:

$$\phi(u) = \frac{-Z}{2\pi i}\operatorname{Log}(u)$$
$$= \frac{Zi}{2\pi}(\ln(|u|) + i\operatorname{Arg}(u))$$

The negative imaginary part of the potential is

$$\phi_2 = -\operatorname{Im}(\phi(u))$$
$$= \frac{Z}{2\pi}\ln(|u|)$$

The level curves of ϕ_2 is constant along circles, implying that the flows are swirls around the cylinder.

Instead of thinking of fluid circulating around the cylinder, we can consider the cylinder to be rotating/translated. Suppose the flow is a superposition of linear and rotational potentials

$$\phi(u) = -v_\infty\left(u + \frac{R^2}{u}\right) - \frac{Z}{2\pi i}\mathrm{Log}(u)$$

This can be thought of as describing a rotating and translating cylinder. The complex velocity is

$$w(u) = -\phi'(u) = v_\infty\left(1 - \left(\frac{R}{u}\right)^2\right) + \frac{Z}{2\pi i u}$$

As $|u| \to \infty$, we see $w \to v_\infty$.

Since the velocity at infinity is non zero, and the circulation is non zero, a force arises.

$$\underline{F} = \rho Z \begin{bmatrix} 0 \\ -v_\infty \end{bmatrix}$$

A linear translation in the $+x$ direction and translation in the $+z$ direction leads to a force in the $-y$ direction. This is the **Magnus effect**.

The Joukowski map cylinders of radius R centered R centered at $z_0 \neq 0$ to "air foils". If we can compute the flow around a translated cylinder, we can can apply the Joukowski map to find the flow around the air foil.

Integration with branch cuts

Complex integration requires that the integrand is continous along α. This means that integration across a branch cut is generically prohibited.

Branch cuts can sometimes be used for our advantage. Consider

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$$

The denominator involves a square root. If we want to use complex analysis to solve this integral, we will eventually need to deal with a branch cut!

We set

$$f(z) = \frac{z^{-1/2}}{1+z}$$

from some choice of the branch of the Logarithm. It turns out that $\arg(z)$ is the correct choice.

$$z^{-1/2} = \exp\left(-\frac{1}{2}\left(\ln(z) + i\arg(z)\right)\right)$$

This seems counter intuitive, since the the branch cut exists between $[0, \infty)$.

If x is real and positive, then $\arg(x) = 0$ (since arg is defined to return some value in $[0, 2\pi)$). So this choice of the branch for the square root recovers the usual positive square root.

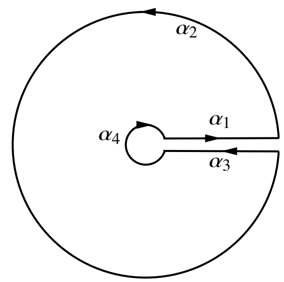
The function is defined on the branch cut, but discontinuous on the branch cut. Consider a point displaced a distance ϵ just above the cut.

$$\lim_{\epsilon \to 0+} (x + i\epsilon)^{-1/2} = \frac{1}{\sqrt{x}}$$

Consider now a point just below the branch cut.

$$\lim_{\epsilon \to 0-} (x - i\epsilon)^{-1/2} = \exp\left(-\frac{1}{2}\left(\ln(x) + i2\pi\right)\right) = \frac{1}{\sqrt{x}}e^{-i\pi} = \frac{-1}{\sqrt{x}}$$

The coutour we pick is



But α_1 and α_3 will not cancel, since they are on opposite sides of of branch cut.

$$\begin{aligned} \alpha_1 &= t + i\epsilon & t &\in [r, R] \\ \alpha_2 &= Re^{it} & t &\in [0, 2\pi] \\ -\alpha_3 &= t - i\epsilon & t &\in [r, R] \\ \alpha_4 &= re^{it} & t &\in [0, 2\pi] \end{aligned}$$

Our contour encloses a simple pole at $z_0 = -1$.

$$\begin{aligned} \mathrm{Res}\left(\frac{z^{-1/2}}{1+z}; -1\right) &= \exp\left(-\frac{1}{2}\left(\ln|-1| + i\pi\right)\right) \\ &= e^{-i\pi/2} \\ &= -i \end{aligned}$$

Integral along the entire contour α is

$$\oint_\alpha \frac{z^{-1/2}}{1+z}\,dz = 2\pi$$

Integral along α_2 (outer circle) is

$$\left|\int_{\alpha_1} \ldots\, dz\right| \leq 2\pi R\frac{R^{-1/2}}{R-1}$$

As $R \to \infty$, we see that the modulus of the integral goes to 0.

Integral along α_4 (inner circle) is

$$\left|\int_{\alpha_4} \ldots\, dz\right| \leq 2\pi r\frac{r^{-1/2}}{r-1}$$

which goes to zero as we take $r \to 0$.

Integral along α_1 is

$$\int_{\alpha_1} \ldots\, dz = \int_r^R \frac{(t+i\epsilon)^{-1/2}}{1+t+i\epsilon}\,dt$$

as $\epsilon \to 0$,

$$\int_{\alpha_1} \ldots\, dz = \int_r^R \frac{1}{\sqrt{t}\,(1+t)}\,dt$$

Integral along α_3 is

$$\int_{\alpha_3} \ldots\, dz = -\int_r^R \frac{(t-i\epsilon)^{-1/2}}{1+(t-i\epsilon)}\,dt$$

as $\epsilon \to 0$, we get an additional negative sign from $(t-i\epsilon)^{-1/2}$. So

$$\int_{\alpha_3} \ldots\, dz = \int_r^R \frac{1}{\sqrt{t}\,(1+t)}\,dt$$

Instead of cancelling, the integrals just above and just below the real line add up. Finally, we send $R \to \infty$.

$$2\pi = 2\int_0^\infty \frac{1}{\sqrt{t}\,(1+t)}\,dt$$

which is twice the original integral we wanted to compute.

Lecture 30

Argument principle

Let $P(z)$ be a polynomial. If z_0 is a zero of arbitrary multiplicity, then it is a simple pole of

$$\frac{P'(z)}{P(z)}$$

and

$$\mathrm{Res}\left(\frac{P'(z)}{P(z)}; z_0\right) = m$$

where m is the multiplicity of z_0. And so

$$\oint_\alpha \frac{P'(z)}{P(z)}\,dz = 2\pi i(N \leq m)$$

where N is the number of zeros inside α, a simple, closed, and positively oriented contour.

Lecture 31

If $f(z)$ has a zero of order m at z_0, then

$$f(z) = (z - z_0)^m g(z)$$

and $g(z) \in H(B_r(z_0))$, and $g(z) \neq 0$ for all $z \in B_r(z_0)$.

We see

$$f'(z) = m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)$$

And the ratio

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)}{(z - z_0)^m g(z)} = \frac{m}{(z - z_0)} + \frac{g'(z)}{g(z)}$$

Since $g'(z)/g(z)$ is holomorphic in the disk of radius r centered at z_0, a zero of arbitrary order, z_0, is always a simple pole of $f'(z)/f(z)$. And

$$\text{Res}\left(\frac{f'(z)}{f(z)}; z_0\right) = m$$

If f is a pole of order m at z_0, then

$$f(z) = \frac{1}{(z - z_0)^m h(z)}$$

and $h \in H(B_r(z_0))$, $h(z) \neq 0$ for all $z \in B_r(z_0)$.

We see

$$f'(z) = -m(z - z_0)^{-m-1}(h(z))^{-1} - (z - z_0)^{-m}(h(z))^{-2} h'(z) = \frac{-m}{(z - z_0)^{m+1} h(z)} - \frac{h'(z)}{(z - z_0)^m h^2(z)}$$

And the ratio

$$\frac{f'(z)}{f(z)} = -\frac{m}{(z - z_0)} - \frac{h'(z)}{h(z)}$$

When z_0 is a pole of order m of $f(z)$, it is simple pole of $f'(z)/f(z)$ and

$$\text{Res}\left(\frac{f'(z)}{f(z)}; z_0\right) = -m$$

> **Argument principle:** If f is meromorphic in Ω, and if α is a simple, positively oriented, closed curve in Ω, such that $\text{int}(\alpha) \subset \Omega$, and if $f(z)$ has **no** zero or pole along α, then
>
> $$\frac{1}{2\pi i} \oint_\alpha \frac{f'(z)}{f(z)} \, dz = (\text{\# of zeros of } f \text{ in } \alpha) - (\text{\# of poles of } f \text{ in } \alpha)$$

We want to understand what it means to integrate

$$\oint_\alpha \frac{f'(z)}{f(z)} \, dz$$

For any z such that $f(z)$ is not on the branch cut of the Log,

$$\frac{f'(z)}{f(z)} = \frac{d}{dz} \text{Log}(f(z)) = \frac{d}{dz} (\ln|f(z)| + i\text{Arg}(f(z)))$$

When we integrate along α, we

1. Start at some z_0 on α
2. Integrate along a segment of α, such that the argument does not change by more than 2π. Call this segment α_n
3. There exists a branch of the logarithm so that $\log^{\alpha_n}(f(z))$ is continuous over α_n
4. So along α_n,

$$\begin{aligned}\int_{\alpha_n} \frac{f'(z)}{f(z)} \, dz &= \int_{\alpha_n} \frac{d}{dz} (\log^{\alpha_n}(f(z))) \, dz \\ &= (\ln|f(z)| + i\arg^{\alpha_n}(f(z)))_{z_0}^{z_1} \\ &= (\ln|z_1| - \ln|z_0|) \\ &+ i(\arg^{\alpha_n}(f(z)) - \arg^{\alpha_n}(f(z)))\end{aligned}$$

5. Successively traverse α in segments

In the end, only the terms involving the various arguments is left.

> **Argument principle:** Around a simple, positively oriented, closed curve in Ω, the difference in the number of zeros and the number of poles of $f(z)$ in $\text{int}(\alpha)$ is the change of the argument of $f(z)$ around α.
>
> $$\frac{1}{2\pi i} \oint_\alpha \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi} \Delta_\alpha \arg(f(z))$$

Lecture 32

Example: Consider the functions $f(z) = z^3$ or $g(z) = 1/z^3$, and α representing the unit circle centered at $z_0 = 0$.

We split α into 6 equal pieces.

$$a_j(t) = e^{it}$$

for

$$t \in \left[j\frac{\pi}{3}, (j+1)\frac{\pi}{3}\right] \qquad j = 0, \ldots, 5$$

Let's choose the argument such that

$$\arg_{(0,2\pi]}(z) \in (0, 2\pi]$$

The change in the argument of z^3 over α_1 is

$$\begin{aligned}\Delta_{\alpha_1} \arg(z^3) &= \arg_{(0,2\pi]}\left(e^{3i\frac{2\pi}{3}}\right) - \arg_{(0,2\pi]}\left(e^{3i\frac{3\pi}{3}}\right) \\ &= \arg_{(0,2\pi]}\left(e^{2\pi i}\right) - \arg_{(0,2\pi]}\left(e^{i\pi}\right) \\ &= \pi\end{aligned}$$

We consider each α_j, and each time (maybe) choosing a different argument if needed, so we do not cross a branch cut. The end result is that

$$\Delta_\alpha \arg\left(z^3\right) = 6\pi$$

This result is consistent with the argument principle.

$$\frac{1}{2\pi i} \oint_\alpha \frac{3z^2}{z^3} \, dz = \frac{1}{2\pi} 6\pi = 3$$

For $g(z) = 1/z^3$,

$$\Delta_{\alpha_1} \arg(g(z)) = -\pi$$

the total change in argument is -6π. This is consistent with the argument principle.

We can understand

$$\frac{1}{2\pi} \Delta_\alpha(f(z))$$

as the number of times the closed curve, $f(\alpha)$ winds around the origin.

We can consider $f(z)$ as map acting on α. $f(z) = z^3$ maps the unit circle $\alpha = e^{i\theta}$, $\theta \in (0, 2\pi]$ to $\beta = e^{i\phi}$, $\phi \in (0, 6\pi]$. The curve β goes around the origin three times.

In general, if we can plot $f(\alpha)$, by looking at how many times $f(\alpha)$ goes aroud origin, we can read off the total change in argument of the $f(z)$ around α.

For polynomials (no poles), by plotting $P(\alpha)$, we can find the number zeros of P in the interior of α by reading off the number of times $P(\alpha)$ goes around the origin. (Condition on P having no zeros directly on α.)

> **Winding number:** Let f be holomorphic, and have no zeros on a closed, smooth, (need not be simple) curve, α. The number of times $f(\alpha)$ winds around z_0 is
>
> $$\omega(f(\alpha), z_0) = \frac{1}{2\pi i} \oint_\alpha \frac{f'(z)}{f(z) - z_0} \, dz$$

Example: Consider the following integral around $\alpha = \{|z| = 2\}$

$$\oint_{|z|=2} \frac{z^5}{z^6 + 2z} \, dz = \oint_{|z|=2} \frac{z^4}{z^5 + 2} \, dz$$

We can define the function $f(z) = z^5$. Then, $f'(z) = 5z^4$, and

$$\oint_{|z|=2} \frac{z^4}{z^5 + 2} \, dz = \frac{1}{5} \oint_\alpha \frac{f'(z)}{f(z) - (-2)} \, dz$$

By the argument principle, the integral is equal to the number of times $f(\alpha)$ winds around $z = -2$, divided by 5. The $f(\alpha)$ happens to wind around $z = -2$ five times.

$$\frac{1}{5} \frac{1}{2\pi i} \oint_\alpha \frac{f'(z)}{f(z) - (-2)} \, dz = \frac{5}{5}$$

Finally,

$$\oint_{|z|=2} \frac{z^4}{z^5 + 2} \, dz = 2\pi i$$

Lecture 33

Example: Consider

$$P(z) = 9z^4 + z^3 + \frac{8}{5}z^2 + 4z + 1$$

By plotting $P(C_r)$, where

$$C_r = \{re^{i\phi}, \phi \in [0, 2\pi)\}$$

for different choices of r. By counting the number of times $P(C_r)$ winds around the origin, we can find how many zeroes are in the interior of C_r.

The Nyquist Criterion

Consider a discrete-time linear dynamical system:

$$\underline{x}_{n+1} = \underline{A}\underline{x}_n \qquad n = 0, 1, 2, \ldots$$

Given $\underline{x}_0$, the recursion can be solved:

$$\underline{x}_n = \underline{A}^n \underline{x}_0$$

Let's suppose that $\underline{A}$ is diagonalizable, with eigenvalue and eigenvector pairs: $\lambda_1, \underline{v}_1, \ldots$. We can expand $\underline{x}_0$ in terms of its eigenbasis:

$$\underline{x}_0 = \sum \alpha_j \underline{v}_j$$

So,

$$\begin{aligned}\underline{x}_n &= A^n \sum \alpha_j \underline{v}_j \\ &= \sum \alpha_j \lambda^n \underline{v}_j\end{aligned}$$

Provided that $|\lambda_j| < 1$, that is, all the zeros of the characteristic polynomials lie within the unit disk, then

$$\lim_{n\to\infty} \|\underline{x}_n\| = 0$$

this is an example of when we might be interested in learning about the locations of zeros.

Consider a n^{th} order linear differential equation of the form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^1 + a_0 y = 0$$

Suppose we want to know steady-state behaviour. The tool of choice is Transform. The laplace transform of a function $y(t)$ is

$$Y(z) = \int_0^\infty y(t)e^{-zt}\, dt$$

$Y(z)$ is function of a complex variable z.

Integrating by parts:

$$\begin{aligned}\int_0^\infty y'(t)e^{-zt}\, dt &= \left(y(t)e^{-zt}\right)_0^\infty - \int_0^\infty y(t)(-z)e^{-zt}\, dt \\ &= -y(0) + zY(z)\end{aligned}$$

The left hand side is exactly the laplace transform of y', and we see that

$$\mathcal{L}(y^{(n)}) = z^n Y(z) - z^{n-1}y(0) - z^{n-2}y(0) - \cdots - y^{(n-1)}(0)$$

Applying the laplace transform to our ODE, we get:

$$\left(z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right) Y(z) + \widetilde{P}(z) = 0$$

where $\widetilde{P}(z)$ is a polynomial of degree $n-1$ involving initial conditions of the ODE.

Isolating $Y(z)$,

$$Y(z) = \frac{-\widetilde{P}(z)}{P(z)} = \frac{-\widetilde{P}(z)}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}$$

Lecture 34

Inverse Laplace transform:

$$y(t) = \sum_{j=1}^{n} \text{Res}\left(e^{tz}Y(z); z_j\right)$$

where z_j are the singularities of $Y(z)$.

If we have simple poles, the residues of $e^{tz}Y(z)$ has the form

$$\lim_{z\to z_j} (z - z_j)\, e^{tz}Y(z) = e^{tz_j}\text{const}$$

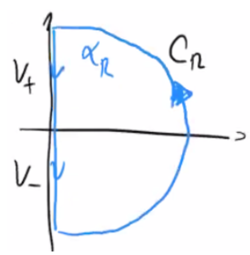
For large t,

$$\lim_{t\to\infty} |e^{tz_j}| = \lim_{t\to\infty} e^{t\text{Re}(z_j)} = \begin{cases} 0 & \text{Re}(z_j) < 0 \\ 1 & \text{Re}(z_j) = 0 \\ \infty & \text{otherwise} \end{cases}$$

So as along as all of z_j (zeroes of $P(z)$) lie in the left-half plane, $y(t) \to 0$ for $t \to \infty$ regardless of initial conditions.

To find whether zeroes of actually do lie in the left half-plane, we choose to compute the winding of $P(\alpha_R)$ about the origin, in the limit as $R \to \infty$.

$$\alpha_R = C_R \cup V_+ \cup V_-$$



This reveals the number of zeroes of $P(z)$ in the right half plane. Since $P(z)$ is a polynomial of order n, the number of zeroes in the left-half plane can be deduced.

$$\frac{1}{2\pi i}\int_{\alpha_R} \frac{P'(z)}{P(z)}\, dz = N$$

Where N is the number of zeros of $P(z)$ inside α_R. We assume in this case that no zeros of $P(z)$ lie on the imaginary axis. We cannot use the argument principle otherwise.

Consider what happens along C_R. For $R \to \infty$, the modulus of z goes to infinity. For large $|z|$, the highest order term of $P(z)$ dominates.

$$P(z) \approx z^n$$

The change in argument of $P(z)$ along C_R is

$$\begin{aligned}\lim_{R\to\infty} \Delta_{C_R}\arg(P(z)) &= \lim_{R\to\infty} \Delta_{C_R}\arg(z^n) \\ &= \lim_{R\to\infty}\left(\text{Arg}(Re^{in\pi/2}) - \text{Arg}(Re^{-in\pi/2})\right) \\ &= \frac{n\pi}{2} + \frac{n\pi}{2} \\ &= n\pi\end{aligned}$$

We chose Arg as opposed to arg so C_R does not cross a branch cut. Using the argument principle, we can conclude that

$$N = \frac{1}{2\pi}\left(n\pi + \Delta_{V_+}\arg(P(z)) + \Delta_{V_-}\arg(P(z))\right)$$

Example: Consider

$$P(z) = z^3 - 2z^2 + 4$$

This is a third order polynomial, so $n = 3$. We expect

$$2\pi N = 3\pi + \Delta_{V_+}\arg(P(z)) + \Delta_{V_-}\arg(P(z))$$

As we compute $\Delta_{V_+}\arg(P(z)) + \Delta_{V_-}\arg(P(z))$ we will go across $z = 0$, $P(0) = 4$. This tells us that

$$\arg(P(0)) = 0$$

We can parameterize V_- by $-iy$, $y \in (0, R]$, and V_+ by iy, $y \in (0, R]$. Since

$$P(-iy) = \overline{P(iy)}$$

for $y \in \mathbb{R}$, $y > 0$, the arguments of $P(z)$ along V_+ is related the argument of $P(z)$ along V_- by

$$\text{Arg}(P(-iy)) = -\text{Arg}(P(iy))$$

Complex conjugation is the same as a reflection over the real axis.

Along V_+, we approach 0 from infinity. Whereas alng V_-, we go to $-\infty$ from 0. So

$$\Delta_{V_+}\text{Arg}(P(z)) = 0 - \left(\lim_{t\to\infty}\arg(P(iy))\right)$$

where as

$$\Delta_{V_-}\text{Arg}(P(z)) = \left(\lim_{t\to\infty}\arg(P(-iy))\right) - 0$$

But since $\text{Arg}(P(-iy)) = -\text{Arg}(P(iy))$, we have

$$\Delta_{V_+}\text{Arg}(P(z)) = \Delta_{V_-}\text{Arg}(P(z))$$

Along V_+:

$$P(iy) = -iy^3 + 2y^2 + 4$$

The ratio between the imaginary and real part

$$\lim_{y\to\infty} \frac{-y^3}{2y^4 + 4} = -\infty$$

So we conclude that

$$\lim_{y\to\infty} \text{Arg}\left(P(iy)\right) = -\pi/2$$

Finally,

$$2\pi N = 3\pi + \pi/2 + \pi/2$$

And

$$N = 2$$

So there is $3 - 2 = 1$ zeros of $P(z)$ in the left half plane.