# PHYS 350

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## Spherical coordinates

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \arctan(y/x) \end{cases}$$

$\theta$ is the polar angle, measured from $+z$ axis. $\phi$ is the azimuthal angle, measured from the $+x$ axis. Cartesian basis $\underline{e}_i$ are dependent on $\underline{s}_j(r,\theta,\phi)$

$$\begin{cases} \underline{e}_1 = \sin(\theta)\cos(\phi)\underline{s}_1 + \cos(\theta)\cos(\phi)\underline{s}_2 - \sin(\phi)\underline{s}_3 \\ \underline{e}_2 = \sin(\theta)\sin(\phi)\underline{s}_1 + \cos(\theta)\sin(\phi)\underline{s}_2 + \cos(\phi)\underline{s}_3 \\ \underline{e}_3 = \cos(\theta)\underline{s}_1 - \sin(\theta)\underline{s}_2 \end{cases}$$

Spherical basis in terms of Cartesian basis

$$\begin{cases} \underline{s}_1 = \sin(\theta)\cos(\phi)\underline{e}_1 + \sin(\theta)\sin(\phi)\underline{e}_2 + \cos(\theta)\underline{e}_3 \\ \underline{s}_2 = \cos(\theta)\cos(\phi)\underline{e}_1 + \cos(\theta)\sin(\phi)\underline{e}_2 - \sin(\theta)\underline{e}_3 \\ \underline{s}_3 = -\sin(\phi)\underline{e}_1 + \cos(\phi)\underline{e}_2 \end{cases}$$

$\underline{s}_i$ are dependent on $r, \theta, \phi$ themselves.

$$dV = dx\,dy\,dz \mapsto r^2\sin(\theta)\,dr\,d\theta\,d\phi$$

### Gradient

$$\underline{\nabla} u = \frac{\partial u}{\partial r}\underline{s}_1 + \frac{1}{r}\frac{\partial u}{\partial \theta}\underline{s}_2 + \frac{1}{r\sin(\theta)}\frac{\partial u}{\partial \phi}\underline{s}_3$$

### Divergence

$$\underline{\nabla}\cdot\underline{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)u_\theta\right) + \frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial \theta}$$

### Curl

$$\underline{\nabla}\times\underline{u} = \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial \theta}\left(\sin(\theta)u_\phi\right) - \frac{\partial u_\theta}{\partial \phi}\right]\underline{s}_1 + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r u_\phi\right)\right]\underline{s}_2 + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r u_\theta\right) - \frac{\partial u_r}{\partial \theta}\right]\underline{s}_3$$

### Laplacian

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 u}{\partial \phi^2}$$

### Speed and acceleration

$$\begin{cases} \dot{\underline{s}}_1 &= \dot{\theta}\underline{s}_2 + \sin(\theta)\dot{\phi}\underline{s}_3 \\ \dot{\underline{s}}_2 &= -\dot{\theta}\underline{s}_1 + \cos(\theta)\dot{\phi}\underline{s}_3 \\ \dot{\underline{s}}_3 &= -\dot{\phi}\left(\sin(\theta)\underline{s}_1 + \cos(\theta)\underline{s}_2\right) \end{cases}$$

Position:

$$\underline{r}(t) = r(t)\underline{s}_1$$

Velocity:

$$\underline{v}(t) = \dot{r}\underline{s}_1 + r\dot{\theta}\underline{s}_2 + r\sin(\theta)\dot{\phi}\underline{s}_3$$

## Cylindrical coordinates

$$\begin{cases} x = r\cos(\phi) \\ y = r\sin(\phi) \\ z = z \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \arctan(y/x) \\ z = z \end{cases}$$

$r$ is the distance to the closest point on the $z$ axis (different from spherical coordinates). $\phi$ is the azimuthal angle, measured from the $+x$ axis.

$$\begin{cases} \underline{e}_1 = \cos(\phi)\underline{s}_1 - \sin(\phi)\underline{s}_2 \\ \underline{e}_2 = \sin(\phi)\underline{s}_1 + \cos(\phi)\underline{s}_2 \\ \underline{e}_3 = \underline{s}_3 \end{cases} \qquad \begin{cases} \underline{s}_1 = \cos(\phi)\underline{e}_1 + \sin(\phi)\underline{e}_2 \\ \underline{s}_2 = -\sin(\phi)\underline{e}_1 + \cos(\phi)\underline{e}_2 \\ \underline{s}_3 = \underline{e}_3 \end{cases}$$

$$dx\,dy\,dz \mapsto r\,dr\,d\phi\,dz$$

### Gradient

$$\underline{\nabla} u = \frac{\partial u}{\partial r}\underline{s}_1 + \frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial z}\underline{s}_3$$

### Divergence

$$\underline{\nabla}\cdot\underline{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_1\right) + \frac{1}{r}\frac{\partial u_2}{\partial \phi} + \frac{\partial u_3}{\partial z}$$

### Curl

$$\underline{\nabla}\times\underline{u} = \left[\frac{1}{r}\frac{\partial u_3}{\partial \phi} - \frac{\partial u_2}{\partial z}\right]\underline{s}_1 + \left[\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial r}\right]\underline{s}_2 + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r u_2\right) - \frac{\partial u_1}{\partial \phi}\right]\underline{s}_3$$

### Laplacian

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

### Speed and acceleration

The direction of $\underline{s}_1$ and $\underline{s}_2$ are time dependent. Thus:

$$\begin{cases} \dot{\underline{s}}_1 &= -\sin(\phi)\dot{\phi}\underline{e}_1 + \cos(\phi)\dot{\phi}\underline{e}_2 = \dot{\phi}\underline{s}_2 \\ \dot{\underline{s}}_2 &= -\cos(\phi)\dot{\phi}\underline{e}_1 - \sin(\phi)\dot{\phi}\underline{e}_2 = -\dot{\phi}\underline{s}_1 \end{cases}$$

Position ($\underline{r}$ represents the position vector, not $r$, the distance from origin in $xy$-plane):

$$\underline{r}(t) = r(t)\underline{s}_1 + z(t)\underline{s}_3$$

Velocity:

$$\underline{v}(t) = \dot{r}\underline{s}_1 + r\dot{\phi}\underline{s}_2 + \dot{z}\underline{s}_3$$

## Derivative of single variable functions

For a 1-D function, $f: \mathbb{R}\to\mathbb{R}$, $\forall x\in\mathbb{R}$, recall that

$$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Recall the chain rule. Given $f(g(x))$,

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

We denote the first and second time derivatives of some quantity $x(t)$ as $\dot{x}$ and $\ddot{x}$.

> The number of dots on each term should equal to degree of time derivative we are taking.

## Derivative of multi-variable functions

Consider a function of $n$ independent variables, $f(x_1,\ldots,x_n)$,

$$\frac{\partial f}{\partial x_n} = \lim_{h\to 0}\frac{f(x_1,\ldots,x_n+h)-f(x_1,\ldots,x_n)}{h}$$

## Total derivatives

Consider a function $f(x,y,\ldots,t)$, where $x,y,\ldots$ can depend on $t$. Conventionally,

$$\frac{\partial f}{\partial t} = \lim_{h\to 0}\frac{f(x,y,\ldots,t+h)-f(x,y,\ldots,t)}{h}$$

> We define the total derivative of $f$ with respect to $t$, as
>
> $$\frac{df}{dt} = \lim_{h\to 0}\frac{f(x(t+h),\ldots,t+h)-f(x,\ldots,t)}{h}$$
>
> All arugments of $f$ that depend on $t$, including $t$, is being varied.
>
> $$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \cdots + \frac{\partial f}{\partial t}$$

- Energy is much more general and natural framework to to analyze physics
- Using the lanuage of Lagrangians and Hamiltonians, we can treat many aspects of physics under the same framework
- For more complex systems, it is much easier to use Lagrangian mechanics than Newtonian mechanics.

## Common ODEs

A second order, linear, constant coefficient ODE:

$$a\ddot{y} + b\dot{y} + cy = 0$$

Solved by guessing $y = e^{\lambda x}$, substituting:

$$\begin{aligned} 0 &= a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + c(e^{\lambda x}) \\ &= a\lambda^2 + b\lambda + c \end{aligned}$$

1. Two roots:
$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$
2. One repeated root:
$$y(x) = Ae^{\lambda x} + Bxe^{\lambda x}$$

A second order, linear, constant coefficient ODE with constant forcing:

$$a\ddot{y} + b\dot{y} + cy = F$$

General solution is the sum of a homogeneous and particular solution:

$$y(x) = y_h(x) + y_p(x)$$

$y_h$ is obtained by solving the equation with $F = 0$. Guess $y_p = D$.

$$cD = F \iff D = F/c$$

- If $\lambda = 0$ is a solution to the characteristic equation, then guess $y_p = Dx$
- If 0 is a repeated root, then guess $y_p = Dx^2$

## Newton's laws

*Newton's first law*: a free body

- at rest remains at rest
- in uniform motion remains in uniform motion

Inertial reference frame (IRF): reference frames where Newton's first law holds

- Space is homogeneous and isotropic
- Time is homogeneous

*Galileo's relativity principle*: All frames moving uniformly in a straight line relative to an inertial frame is also an inertial frame

Between two inertial frames $K$ and $K'$, where $K'$ moves at velocity $\underline{v}$ relative to $K$, the coordinates of a given point $\underline{r}$ in frame $K$ and $\underline{r}'$ in $K'$ is related by the *Galilean transformations*:

$$\begin{aligned} \underline{r} &= \underline{r}' + \underline{v}t \\ t &= t' \end{aligned}$$

The main technique to choosing an IRF is to tie it to the earth.

We will assume that the earth is always an IRF, even though it spins and orbits. The time scale of the problems we consider is too small so that any error introduced is small.

*Newton's second law*: in inertial frames, force is the rate of change in momentum. ($\underline{F} = m\underline{a} = md\underline{v}/dt$)

*Newton's third law*: every action has an equal and opposite reaction.

## Generalised coordinates

*Degrees of freedom*: independent quantities (distances, angles, ...) required to define the position of any system.

A system of $N$ particles in 3-D requires $3N$ degrees of freedom (think that each particle is described by $(x, y, z)$ coordinates).

A rigid body requires specifying 6 degrees of freedom.

The *generalised coordinates* of the system is the set of $s$ degrees of freedom required to completely specify a the position of a system:

$$\begin{aligned} q &= \{q_1, q_2, \dots, q_s\} \\ \dot{q} &= \{\dot{q}_1, \dot{q}_2, \dots, \dot{q}_s\} \qquad \textit{generalised velocities} \end{aligned}$$

## Principal of least action

Mechanical systems are governed by the *Principal of least action*.

Every mechanical system is characterized by a *Lagrangian*

$$\mathcal{L}(q_1, \dots, \dot{q}_1, \dots, t) = \mathcal{L}(q, \dot{q}, t)$$

such that for $t \in [t_1, t_2]$, system evolves such that its *action*:

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt$$

is at an extremum and $q = \{q_1, \dots, q_s\}$ satisfies known boundary conditions

$$\begin{cases} q(t_1) = q^{(1)} \\ q(t_2) = q^{(2)} \end{cases}$$

## Functionals

A *functional*, $F[f, g, h, \dots]$, maps functions to a scalar.

$F[f, g, h, \dots]$ is at an extremum (to the first order) if for any small variation to $f$, $\delta f$,

$$\delta F = F[f + \delta f, g + \delta g, \dots] - F[f, g, \dots] = 0$$

## Euler-Lagrange equations

Consider "small variations" to $q_1, \dots, q_s$: $q(t) + \alpha\eta_1(t), \dots, q_s(t) + \alpha\eta_s(t)$, where:

- $\alpha \in \mathbb{R}$ is a small parameter
- $\eta_i(t)$ are functions that evaluate to zero at the end points of $[t_1, t_2]$

We are write the action as a function of $\alpha$, $S(\alpha)$. The condition to extremize $S$ is such that

$$\left(\frac{dS}{d\alpha}\right)_{\alpha=0} = 0$$

It turns out that:

$S[q_1, \dots, q_s, \dot{q}_1, \dots, \dot{q}_s, t]$ is at an extremum when the Lagrangian $\mathcal{L}$ satisfies

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \qquad \forall i = 1, 2, \dots, s$$

### Lagrangian for a free particle

Since space is homogeneous and isotropic and time is homogeneous in an IRF, this implies that the Lagrangian **for a free particle**

- cannot explicitly contain $\underline{r}$ (the position vector of the particle)
- cannot depend on the direction of $\dot{\underline{r}} = \underline{v}$, so it depends on the speed only

This leads to

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underline{v}}\right) = \underline{0} \implies \frac{\partial \mathcal{L}}{\partial \underline{v}} = \text{constant} \implies \underline{v} = \text{constant}$$

Thus we showed that free motion in an IRF takes place with velocity constant in magnetude and direction.

## Properties of the Lagrangian

Two Lagrangians, differing only a total time derivative of a function of coordinates and time, $df(q,t)/dt$, result in the same equations of motion.

- Multiplying the Lagrangian by a constant gives the same equations of motion
- A system made of a sum of non-interacting subsystems numbered $1, 2, \dots$ have the Lagrangian
$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \dots$$
- Terms that do not involve the generalised coordinates do not appear in the Euler-Lagrange equations; they can be thrown out

## Solution steps

Given a mechanics problem,

1. Draw the system
2. Determine the generalised coordinates, $q$

3. Determine the Lagrangian
4. Solve the Euler Lagrange equations
5. Reason about the answer

## General principles

The definition of the Lagrangian can be reasoned from general principles

1. Superposition: $\mathcal{L}$ is a sum of terms to account for different bodies, interactions between bodies, interaction of the system with external fields
2. Galileo's principle: $q$ holds under Galiean transformations
3. Correspondence principle: Agrees with Newton's laws and experiment
4. Symmetries: $\mathcal{L}$ reflects symmetries in the problem
5. "Beauty": the model with the least assumptions, simplest math

## Collection of $N$ self-interacting particles

$$\mathcal{L} = \sum_{i=1}^{N} \frac{m_i}{2} |v_i|^2 - U(\bar{r}_1, \bar{r}_1, \dots, t)$$

$U$ is the potential energy between the particles.

> Potential energies do not depend on velocity since forces act instantaneously in non-relativistic physics

## Common potential energies

Elastic potential energy

$$U(\ell) = \frac{k}{2}(\ell - \ell_0)^2$$

- $\ell_0$ is the undistorted length of the spring
- $k$ is the spring constant

Gravitational potential energy

$$U(r) = -\frac{Gm_1m_2}{r}$$

- $G$ is the gravitational constant
- $r$ is 2-norm distance between masses $m_1$ and $m_2$

Objects close to surface of the earth, $z$ axis pointing upwards:

$$U(z) = mgh$$

The sign of $U(z)$ may flip depending on the orientation of the coordinate system.

## Energy

Functions of $q$ and $\dot{q}$ that remain constant during motion, and only depend on initial conditions are called *integrals of motion*.

> The conservation of energy comes from homogeneity of time.
>
> $$E = \sum_{i=1}^{s} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i \right) - \mathcal{L} = T + U$$

The homogeneity of time implies that $\mathcal{L}$ have no explicit time dependence. This means that

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

Consider now the total time derivative of $\mathcal{L}$:

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{s} \left( \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right)$$

Substituting the Euler-Lagrange equation and writing $\ddot{q}_i$ as $d\dot{q}_i/dt$:

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{s} \left( \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d}{dt} \dot{q}_i \right)$$

by the product rule:

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{s} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i \right)$$

Moving $d\mathcal{L}/dt$ to the right hand side:

$$0 = \frac{d}{dt} \sum_{i=1}^{s} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \right)$$

this implies that the series must be a constant:

$$E = \sum_{i=1}^{s} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \right)$$

## Momentum

A generalised coordinate, $q_i$, is said to be *cyclic* when

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

We define

- $p_i = \partial \mathcal{L}/\partial \dot{q}_i$: generalised momentum When $q_i$ is an angle, we interpret $P$ as angular momentum.
- $F_i = \partial \mathcal{L}/\partial q_i$: generalised force

The conservation of momentum comes from homogeneity of space.

> In a closed mechanical system, momentum is conserved for cyclic quantities:
>
> $$p = \sum_{C} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$
>
> The summation is over the set of cyclic quantities, $C$

This is a result that we can get from applying the Euler-Lagrange equations. When $\partial \mathcal{L}/\partial q_i = 0$, it follows that $\partial \mathcal{L}/\partial \dot{q}_i$ must be a constant.

Homogeneity of space implies that the Lagrangian is not changed by a displacement of its coordinates.

$$\frac{d\mathcal{L}}{dq} = \sum_{i=1}^{s} \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Substituting in the Euler-Lagrange equation:

$$\frac{d\mathcal{L}}{dq} = \sum_{i=1}^{s} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = 0 = \frac{d}{dt} \sum_{i=1}^{s} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = 0$$

A quantity whose time derivative is zero must be a constant. Thus the following quantity, which we call the momentum, is conserved:

$$p = \sum_{i=1}^{s} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Since $\partial \mathcal{L}/\partial q_i = -\partial U/\partial q_i$, and conservative forces is the negative gradient of a potential, the requirement that $d\mathcal{L}/dq = 0$ is Newton's third law:

$$\sum_{i=1}^{s} F_i = 0$$

> Euler-Lagrange equation in terms of generalised momentum and force:
>
> $$\dot{p}_i = F_i \qquad \forall i = 1, 2, \dots, s$$

The conservation of energy and moment are particular cases of Noether's theorem. Roughly, the statement is that: if $\mathcal{L}$ is invariant to a continuous transformations, then there is a conserved quantities associated with it.

## Motion in 1-D

- The "dimension" of motion is equal to the number of dofs

Invoking the conservation of energy, we can often directly integrate

$$E = T(\dot{q}) + U(q) = \frac{1}{2} m\dot{x}^2 + U(x)$$

to find that

$$t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E(t_0) - U(x)}} + \text{Constant}$$

> Motion is restricted in regions where
>
> $$E = T + U > U(x)$$
>
> When $E = U$, we have turning points.

Finite motion in 1-D is oscillatory. We can find the period by integrating over turning points $x_1, x_2$.

$$T = 2\sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E-U(x)}}$$

Knowing the graph of $U(q)$, $E$, $\dot{q}_0$, we can

- sketch $q(t)$ and $\dot{q}(t)$

## The reduced mass

A problem is a *two body problem* iff:

1. Only two bodies are involved
2. The potential energy of the interaction between the two only depends on their relative distance from each other

We know the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m_1\dot{\underline{r}}_1^2 + \frac{1}{2}m_2\dot{\underline{r}}_2^2 - U(|\underline{r}_1 - \underline{r}_2|)$$

There is a clever way to reduce the number of dofs. This involves:

- Relative displacement vector $\underline{r}$
- Center of mass position ($\underline{R}_{\text{cm}}$)

Relative displacement points from $\underline{r}_2$ to $\underline{r}_1$:

$$\underline{r} = \underline{r}_1 - \underline{r}_2$$

Let $M$ be the total mass. The center of mass is given by:

$$\underline{R}_{\text{cm}} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{M}$$

The new dofs are related to $\underline{r}_1$ and $\underline{r}_2$ by

$$\underline{r}_1 = \underline{R}_{\text{cm}} + \frac{m_2}{M}\underline{r}$$
$$\underline{r}_2 = \underline{R}_{\text{cm}} - \frac{m_1}{M}\underline{r}$$

Making Substitutions into our original Lagrangian, we will find that

$$\mathcal{L} = \frac{M\dot{R}_{\text{cm}}^2}{2} + \frac{\mu\dot{r}^2}{2} - U(r)$$

where

- $\mu$ is the so called *reduced mass*: $m_1m_2/M$

Using

$$s = 2 \qquad q = \{\underline{r}, \underline{R}_{\text{cm}}\}$$

Interpret the Lagrangian as the sum of a the Lagrangian for the center of mass:

$$\mathcal{L}_{\text{cm}} = \frac{M\dot{R}_{\text{cm}}^2}{2}$$

and a Lagrangian for relative motion:

$$\mathcal{L}_{\text{rel}} = \frac{\mu\dot{r}^2}{2} - U(r)$$

Consider the center of mass Lagrangian. It resembles the kinetic energy of a particle of mass $M$. It will

- Be stationary
- Or move with constant velocity

In general, we know that

$$\underline{R}_{\text{cm}}(t) = \underline{R}_{\text{cm}}(0) + \underline{V}_{\text{cm}}(0)t$$

For relative motion Lagrangian, we need to realize that

Relative motion takes place in the plane defined by $\underline{r}_0$ and $\underline{v}_0$. (Initial conditions on the relative displacement vector)

- Lagrangian resembles motion of a particle of mass $\mu$ in the field $U$
- Since motion is 2-D, we can use polar coordinates to describe $\underline{r}$

In the plane of motion for $\underline{r}$, let

- $\underline{\phi}$ be the polar angle
- $\underline{r}$ be the radial vector

We know velocities can be decomposed into a tangential and radial component:

$$\dot{\underline{r}} = \dot{r}\underline{\hat{r}} + r\dot{\phi}\underline{\hat{\phi}}$$

So

$$\mathcal{L}_{\text{rel}} = \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\phi}^2}{2} - U(r)$$

Conservation of angular momentum:

$$\ell = \mu|\underline{r}_0 \times \underline{v}_0| = \mu r^2\dot{\phi} = \text{const}$$

Rearraning:

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

Conservation of energy:

$$\begin{aligned} E_{\text{rel}} &= \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\phi}^2}{2} + U(r) = \text{const} \\ &= \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} - U(r) \\ &= T_{\text{eff}} + U_{\text{eff}} \end{aligned}$$

- If we know the initial position and velocities of the relative displacement vector, then we do not need to convert those values into polar coordinates.
- In Newtonian terms, since $\underline{F} = -\underline{\nabla}U$, $\underline{F} \parallel \underline{r}$, this means that the system has no net torque. Recall that

  $$\frac{d\ell}{dt} = \underline{\tau} = 0$$

  so $\ell$ must be a conserved.

**Kepler's second law:** particles in a central field sweeps out equal area in equal time. ($dA/dt = $ const.)

$|\underline{r}_0 \times \underline{v}_0|$ can be though of as the area of a parallelogram spanned by $\underline{r}_0$ and $\underline{v}_0$. Since of angular momentum is conserved,

$$\ell = p_\phi = \mu|\underline{r}_0 \times \underline{v}_0| = \text{const}$$

Then, we can show that

$$\frac{dA}{dt} = \frac{\ell}{2\mu} = \text{const}$$

We can solve for the following:

1. $r(t)$:

   $$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E_{\text{rel}} - U_{\text{eff}})}$$

2. Once we know $r(t)$, we know $\phi(t)$:

   $$\frac{d\phi}{dt} = \frac{\ell}{\mu r^2}$$

3. We can get the "*trajectory equation*" by applying the inverse function theorem to $dr/dt$;

   $$\frac{dt}{dr} = \pm\sqrt{\frac{\mu}{2(E_{\text{rel}} - U_{\text{eff}})}}$$

   and combining:

   $$\frac{d\phi}{dt}\frac{dt}{dr} = \frac{d\phi}{dr}$$

## Planetary motion

Is a two body problem, with the gravitational potential:

$$U = -\frac{Gm_1m_2}{r} = -\frac{\alpha}{r}$$

The relative motion Lagrangian becomes:

$$\mathcal{L}_{\text{rel}} = \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\phi}^2}{2} + \frac{\alpha}{r}$$

Invoking the conservation of energy and momentum:

$$E_{\text{rel}} = \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} - \frac{\alpha}{r}$$

When $dr/dt \geq 0$, The trajectory equation can be found by solving:

$$\frac{d\phi}{dr} = \frac{\ell}{\mu r^2}\sqrt{\frac{\mu}{2(E_{\text{rel}} - U_{\text{eff}})}}$$

Let's define a quantity called the *eccentricity*:

$$e = \sqrt{1 + \frac{E_{\text{rel}}}{U_0}}$$

where $U_0 = \alpha/2p$, and $e$ is unitless

Integrating:

$$\begin{aligned}\int d\phi &= \frac{\ell}{\sqrt{2\mu}}\int \frac{dr}{r^2\sqrt{E_{\text{rel}} - U_{\text{eff}}}} \\ &= \frac{\ell}{\sqrt{2\mu}}\int \frac{dr}{r^2\sqrt{E_{\text{rel}} - \ell^2/2\mu r^2 - \alpha/r}}\end{aligned}$$

Let $u = p/r$, and $du/dr = -p/r^2$. We will find that

$$\frac{p}{r} = 1 + e\cos(\phi)$$

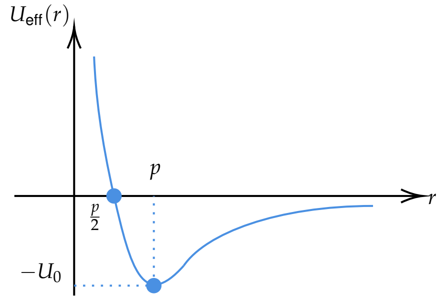
### Trajectories

Let's consider when $\ell \neq 0$. If $\ell$ is zero, this implies that:

$$\underline{r}_0 \parallel \underline{v}_0$$

- Two bodies collide
- If there is enough kinetic energy, the bodies escape

Consider $U_{\text{eff}}$. For small $r$, $U$ blows up like $1/r^2$. For large $r$, $U$ is dominated by $1/r$.




- We can find $p$ by looking for the minimum of $U_{\text{eff}}$.

$$\frac{dU_{\text{eff}}}{dr} = 0 \implies p = \frac{\ell^2}{\mu\alpha}$$

- $-U_0$ is $-\alpha/2p$
- $U(p/2) = 0$

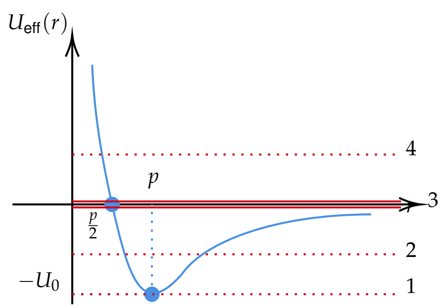
We can show that:

$$E_{\text{rel}} = -\frac{\alpha}{2a}$$

We know that motion is only possible when

$$E_{\text{rel}} = T_{\text{eff}} + U_{\text{eff}} \geq U_{\text{eff}}$$

Let's consider three cases for the total energy:




**Case 1:** When $E_{\text{rel}} = -U_0 = -\alpha/2p$

- $e = 0$
- $r$ does not change, the distance between two objects is a constant ($\dot{r} = 0$; relative velocity is purely tangential)
- Trajectory is a circle: $p = r$, since $e = 0$
- $\dot{\phi} = \ell/\mu p^2 = \Omega = 2\pi/T$, which means that

$$\phi(t) = \phi(0) + \Omega t$$

**Case 2:** when $-U_0 < E_{\text{rel}} < 0$:

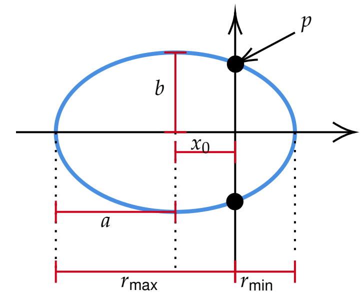
- $0 < e < 1$
- Two turning points, so $r_{\text{min}} \leq r \leq r_{\text{max}}$
- Trajectory turns out to be an ellipse

By letting $r\cos(\phi) = x$, we can find that

$$\frac{(x+x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

- $a = p/(1-e^2)$
- $b = \sqrt{pa}$
- $x_0 = ea$




$r_{\text{min}}$ is the *perihelion* of the orbit

$$\cos(\phi) = 1 \implies r_{\text{min}} = \frac{p}{1+e}$$

$r_{\text{max}}$ is the *aphelion* of the orbit

$$\cos(\phi) = -1 \implies r_{\text{max}} = \frac{p}{1-e}$$

**Kepler's first law**: Planets move around their sun in an ellipse, with the sun at the focus.

We can show that this holds by consider $m_1 >> m_2$, where

- $m_1$ is the mass of the sun
- $m_2$ is the mass of a planet

In this case,

$$\begin{aligned}\underline{r}_1 &= \underline{R}_{\text{cm}} + \frac{m_2}{M}\underline{r} \approx \underline{R}_{\text{cm}} \\ \underline{r}_2 &= \underline{R}_{\text{cm}} + \frac{m_1}{M}\underline{r} \approx \underline{R}_{\text{cm}} + \underline{r}\end{aligned}$$

Since is always possible to choose an IRF such that $\underline{R}_{\text{cm}}(t) = 0$ for all $t$. In this IRF, $\underline{r}_1$ is stationary, and $\underline{r}_2 = \underline{r}$, thus the planet orbits the sun in an ellipse.

**Kepler's third law:** $T^2 \propto a^3$ (the square of the orbit period is proportional to the cube of the length of the large semiaxis of the orbit.)

$$T = 2\pi\sqrt{\frac{\mu}{\alpha}}a^{3/2} = 2\pi\sqrt{\frac{a^3}{GM}}$$

We can show that this is true from:

- Finding $r(t)$ or $\phi(t)$
- Using Kepler's second law

Since

$$\frac{dA}{dt} = \frac{\ell}{2\mu} = \frac{A_{\text{total}}}{T}$$

and we know that $A_{\text{total}} = \pi ab$, rearranging,

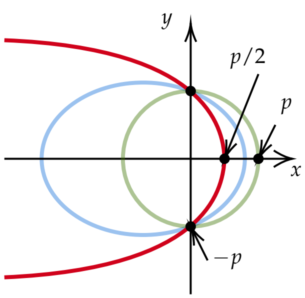
$$T = \frac{2\mu A_{\text{total}}}{\ell} = \frac{2\mu\pi ab}{\ell}$$

since $p = \ell^2/\mu\alpha$, we have $\sqrt{\mu\alpha p} = \ell$. Since $b = \sqrt{pa}$, we have $b/\sqrt{a} = \sqrt{p}$.

$$\begin{aligned} T = 2\mu\pi\frac{ab\sqrt{a}}{b\sqrt{\mu\alpha}} &= 2\pi\sqrt{\frac{\mu}{\alpha}}a^{3/2} \\ &= 2\pi\sqrt{\frac{m_1m_2}{m_1+m_2}\frac{a^3}{Gm_1m_2}} \\ &= 2\pi\sqrt{\frac{a^3}{GM}} \end{aligned}$$

**Case 3:** when $E_{\text{rel}} = 0$:

- $e = 1$
- A single turning point at $r = p/2$
- Trajectory turns out to be a parabola
- Called an *escape trajectory*, the minimum energy for the bodies to escape each other




We will find that

$$x = \frac{p^2 - y^2}{2p}$$

**Case 4:** when $E_{\text{rel}} > 0$:

- $e > 1$
- One turning point, $r_{\text{min}} \le r$
- Trajectory turns out to be a hyperbola

## Small oscillations

**$s = 1$ with fixed external conditions**

General Lagrangian:

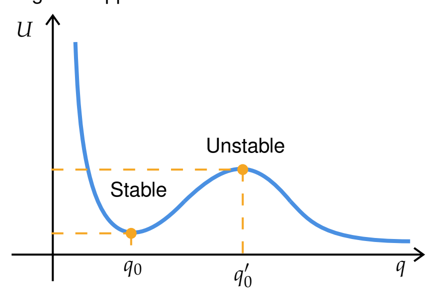
$$\mathcal{L}(q,\dot{q}) = \frac{\alpha(q)\dot{q}^2}{2} - U(q)$$

- energy is conserved, we can solve exactly

Rearranging:

$$\frac{dq}{dt} = \pm\sqrt{\frac{2(E-U(q))}{\alpha(q)}}$$

Our goal is to get an approximate solution.




Equilibrium points

A point $q_0$ is an **equilibrium point** when

$$\left[\frac{dU}{dq}\right]_{q=q_0} = 0$$

An equilibrium point is

1. Stable: when
$$\left[\frac{d^2U}{dq^2}\right]_{q=q_0} > 0$$
2. Unstable: when
$$\left[\frac{d^2U}{dq^2}\right]_{q=q_0} < 0$$

When the initial conditions are close to a stable equilibrium, we can get an approximation.

- Taylor series expansion to $\alpha(q)$ and $U(q)$
$$f(x) \approx \sum_{n=0}^{\infty}\frac{f^n(x_0)}{n!}(x-x_0)^n$$
- Truncate the series at the first term that will contribute to the Euler-Lagrange equations

We expand:

$$\alpha(q) \approx \alpha(q_0)$$

Since $T = \alpha(q_0)\dot{q}^2/2$ will contribute to the EL equations, we stop at this point.

$$U(q) \approx U(q_0) + U'(q_0)(q-q_0) + \frac{U''(q_0)}{2}(q-q_0)^2$$

$U'(q_0) = 0$ by definition, so the second term does not contribute. We stop at the second order terms.

- We are approximating the potential at $q_0$ by a parabola.

The Lagrangian becomes:

$$\mathcal{L}(q,\dot{q}) = \frac{\alpha(q_0)\dot{q}^2}{2} - \frac{U''(q_0)}{2}(q-q_0)^2$$

The EL-equation:

$$\ddot{q} = -\omega^2(q-q_0)$$

- $\omega = \sqrt{K/\alpha(q_0)}$
- $K = U''(q_0)$

Has solution

$$q(t) = q_0 + A\cos(\omega t) + B\sin(\omega t)$$