PHYS 350

Ruiheng Su 2021

Spherical coordinates

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \arctan(y/x) \end{cases}$$

θ is the polar angle, measured from $+z$ axis. ϕ is the azimuthal angle, measured from the $+x$ axis. Cartesian basis $\underline{e}_i$ are dependent on $\underline{s}_j(r,\theta,\phi)$

$$\begin{cases} \underline{e}_1 = \sin(\theta)\cos(\phi)\underline{s}_1 + \cos(\theta)\cos(\phi)\underline{s}_2 - \sin(\phi)\underline{s}_3 \\ \underline{e}_2 = \sin(\theta)\sin(\phi)\underline{s}_1 + \cos(\theta)\sin(\phi)\underline{s}_2 + \cos(\phi)\underline{s}_3 \\ \underline{e}_3 = \cos(\theta)\underline{s}_1 - \sin(\theta)\underline{s}_2 \end{cases}$$

Spherical basis in terms of Cartesian basis

$$\begin{cases} \underline{s}_1 = \sin(\theta)\cos(\phi)\underline{e}_1 + \sin(\theta)\sin(\phi)\underline{e}_2 + \cos(\theta)\underline{e}_3 \\ \underline{s}_2 = \cos(\theta)\cos(\phi)\underline{e}_1 + \cos(\theta)\sin(\phi)\underline{e}_2 - \sin(\theta)\underline{e}_3 \\ \underline{s}_3 = -\sin(\phi)\underline{e}_1 + \cos(\phi)\underline{e}_2 \end{cases}$$

$\underline{s}_i$ are dependent on r, θ, ϕ themselves.

$$dV = dx\,dy\,dz \mapsto r^2\sin(\theta)\,dr\,d\theta\,d\phi$$

Gradient

$$\underline{\nabla}u = \frac{\partial u}{\partial r}\underline{s}_1 + \frac{1}{r}\frac{\partial u}{\partial \theta}\underline{s}_2 + \frac{1}{r\sin(\theta)}\frac{\partial u}{\partial \phi}\underline{s}_3$$

Divergence

$$\underline{\nabla}\cdot\underline{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)u_\theta\right) + \frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial\theta}$$

Curl

$$\underline{\nabla}\times\underline{u} = \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial\theta}\left(\sin(\theta)u_\phi\right) - \frac{\partial u_\theta}{\partial\phi}\right]\underline{s}_1 + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r u_\phi\right)\right]\underline{s}_2 + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r u_\theta\right) - \frac{\partial u_r}{\partial\theta}\right]\underline{s}_3$$

Laplacian

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 u}{\partial\phi^2}$$

Speed and acceleration

$$\begin{cases} \dot{\underline{s}}_1 &= \dot\theta\underline{s}_2 + \sin(\theta)\dot\phi\underline{s}_3 \\ \dot{\underline{s}}_2 &= -\dot\theta\underline{s}_1 + \cos(\theta)\dot\phi\underline{s}_3 \\ \dot{\underline{s}}_3 &= -\dot\phi\left(\sin(\theta)\underline{s}_1 + \cos(\theta)\underline{s}_2\right) \end{cases}$$

Position:

$$\underline{r}(t) = r(t)\underline{s}_1$$

Velocity:

$$\underline{v}(t) = \dot r\underline{s}_1 + r\dot\theta\underline{s}_2 + r\sin(\theta)\dot\phi\underline{s}_3$$

Cylindrical coordinates

$$\begin{cases} x = r\cos(\phi) \\ y = r\sin(\phi) \\ z = z \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \arctan(y/x) \\ z = z \end{cases}$$

r is the distance to the closest point on the z axis (different from spherical coordinates). ϕ is the azimuthal angle, measured from the $+x$ axis.

$$\begin{cases} \underline{e}_1 = \cos(\phi)\underline{s}_1 - \sin(\phi)\underline{s}_2 \\ \underline{e}_2 = \sin(\phi)\underline{s}_1 + \cos(\phi)\underline{s}_2 \\ \underline{e}_3 = \underline{s}_3 \end{cases} \qquad \begin{cases} \underline{s}_1 = \cos(\phi)\underline{e}_1 + \sin(\phi)\underline{e}_2 \\ \underline{s}_2 = -\sin(\phi)\underline{e}_1 + \cos(\phi)\underline{e}_2 \\ \underline{s}_3 = \underline{e}_3 \end{cases}$$

$$dx\,dy\,dz \mapsto r\,dr\,d\phi\,dz$$

Gradient

$$\underline{\nabla}u = \frac{\partial u}{\partial r}\underline{s}_1 + \frac{1}{r}\frac{\partial u}{\partial\phi} + \frac{\partial u}{\partial z}\underline{s}_3$$

Divergence

$$\underline{\nabla}\cdot\underline{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_1\right) + \frac{1}{r}\frac{\partial u_2}{\partial\phi} + \frac{\partial u_3}{\partial z}$$

Curl

$$\underline{\nabla}\times\underline{u} = \left[\frac{1}{r}\frac{\partial u_3}{\partial\phi} - \frac{\partial u_2}{\partial z}\right]\underline{s}_1 + \left[\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial r}\right]\underline{s}_2 + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r u_2\right) - \frac{\partial u_1}{\partial\phi}\right]\underline{s}_3$$

Laplacian

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Speed and acceleration

The direction of $\underline{s}_1$ and $\underline{s}_2$ are time dependent. Thus:

$$\begin{cases} \dot{\underline{s}}_1 &= -\sin(\phi)\dot\phi\underline{e}_1 + \cos(\phi)\dot\phi\underline{e}_2 = \dot\phi\underline{s}_2 \\ \dot{\underline{s}}_2 &= -\cos(\phi)\dot\phi\underline{e}_1 - \sin(\phi)\dot\phi\underline{e}_2 = -\dot\phi\underline{s}_1 \end{cases}$$

Position ($\underline{r}$ represents the position vector, not r, the distance from origin in xy-plane):

$$\underline{r}(t) = r(t)\underline{s}_1 + z(t)\underline{s}_3$$

Velocity:

$$\underline{v}(t) = \dot r\underline{s}_1 + r\dot\phi\underline{s}_2 + \dot z\underline{s}_3$$

Derivative of single variable functions

For a 1-D function, $f:\mathbb{R}\to\mathbb{R}$, $\forall x\in\mathbb{R}$, recall that

$$\frac{df}{dx} = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

Recall the chain rule. Given $f(g(x))$,

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

We denote the first and second time derivatives of some quantity $x(t)$ as $\dot x$ and $\ddot x$.

> The number of dots on each term should equal to degree of time derivative we are taking.

Derivative of multi-variable functions

Consider a function of n independent variables, $f(x_1,\ldots,x_n)$,

$$\frac{\partial f}{\partial x_n} = \lim_{h\to0}\frac{f(x_1,\ldots,x_n+h)-f(x_1,\ldots,x_n)}{h}$$

Total derivatives

Consider a function $f(x,y,\ldots,t)$, where $x,y,\ldots$ can depend on t. Conventionally,

$$\frac{\partial f}{\partial t} = \lim_{h\to0}\frac{f(x,y,\ldots,t+h)-f(x,y,\ldots,t)}{h}$$

> We define the total derivative of f with respect to t, as
>
> $$\frac{df}{dt} = \lim_{h\to0}\frac{f(x(t+h),\ldots,t+h)-f(x,\ldots,t)}{h}$$
>
> All arugments of f that depend on t, including t, is being varied.
>
> $$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot x + \frac{\partial f}{\partial y}\dot y + \cdots + \frac{\partial f}{\partial t}$$

- Energy is much more general and natural framework to to analyze physics
- Using the lanuage of Lagrangians and Hamiltonians, we can treat many aspects of physics under the same framework
- For more complex systems, it is much easier to use Lagrangian mechanics than Newtonian mechanics.

Common ODEs

A second order, linear, constant coefficient ODE:

$$a\ddot{y} + b\dot{y} + cy = 0$$

Solved by guessing $y = e^{\lambda x}$, substituting:

$$\begin{aligned} 0 &= a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + c(e^{\lambda x}) \\ &= a\lambda^2 + b\lambda + c \end{aligned}$$

1. Two roots:
$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$
2. One repeated root:
$$y(t) = Ae^{\lambda t} + Bxe^{\lambda t}$$

A second order, linear, constant coefficient ODE with constant forcing:

$$a\ddot{y} + b\dot{y} + cy = F$$

General solution is the sum of a homogeneous and particular solution:

$$y(t) = y_h(t) + y_p(t)$$

y_h is obtained by solving the equation with $F = 0$. Guess $y_p = D$.

$$cD = F \iff D = F/c$$

- If $\lambda = 0$ is a solution to the characteristic equation, then guess $y_p = Dt$
- If 0 is a repeated root, then guess $y_p = Dt^2$

When forcing depends on time, if F is

- Polynomial of order n: guess the solution to be polynomial of the same degree.
$$y_p = At^n + Bt^{n-1} + \cdots + D$$
If 0 is a solution to the characteristic equation, then multiply by t. If 0 is a repeated root, then multiply by t^2
- Exponential of form $Ce^{\tau t}$: guess a particular solution
$$y_p = Ae^{\tau t}$$
If τ is a root to the characteristic equation, multiply by t
- Trigonometric, for example $\cos(\omega t)$: choose a guess
$$y_p = A\cos(\omega t) + B\sin(\omega t)$$
If $\cos(\omega t)$ is a solution to homogeneous equation, then multiply by t
- Multiply the terms of the guess that is equal to a complementary solution by t until they are no longer equal.

Newton's laws

Newton's first law: a free body

- at rest remains at rest
- in uniform motion remains in uniform motion

Inertial reference frame (IRF): reference frames where Newton's first law holds

- Space is homogeneous and isotropic
- Time is homogeneous

> *Galileo's relativity principle*: All frames moving uniformly in a straight line relative to an inertial frame is also an inertial frame

Between two inertial frames K and K', where K' moves at velocity $\underline{v}$ relative to K, the coordinates of a given point $\underline{r}$ in frame K and $\underline{r}'$ in K' is related by the *Galilean transformations*:

$$\begin{aligned} \underline{r} &= \underline{r}' + \underline{v}t \\ t &= t' \end{aligned}$$

The main technique to choosing an IRF is to tie it to the earth.

We will assume that the earth is always an IRF, even though it spins and orbits. The time scale of the problems we consider is too small so that any error introduced is small.

Newton's second law: in inertial frames, force is the rate of change in momentum. ($\underline{F} = m\underline{a} = md\underline{v}/dt$)

Newton's third law: every action has an equal and opposite reaction.

generalized coordinates

Degrees of freedom: independent quantities (distances, angles, ...) required to define the position of any system.

A system of N particles in 3-D requires $3N$ degrees of freedom (think that each particle is described by (x, y, z) coordinates).

A rigid body requires specifying 6 degrees of freedom.

The *generalized coordinates* of the system is the set of s degrees of freedom required to completely specify a the position of a system:

$$\begin{aligned} q &= \{q_1, q_2, \ldots, q_s\} \\ \dot{q} &= \{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s\} \qquad \textit{generalized velocities} \end{aligned}$$

Principal of least action

Mechanical systems are governed by the *Principal of least action*.

> Every mechanical system is characterized by a *Lagrangian*
>
> $$\mathcal{L}(q_1, \ldots, \dot{q}_1, \ldots, t) = \mathcal{L}(q, \dot{q}, t)$$
>
> such that for $t \in [t_1, t_2]$, system evolves such that its *action*:
>
> $$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt$$
>
> is at an extremum and $q = \{q_1, \ldots, q_s\}$ satisfies known boundary conditions
>
> $$\begin{cases} q(t_1) = q^{(1)} \\ q(t_2) = q^{(2)} \end{cases}$$

Functionals

A *functional*, $F[f, g, h, \ldots]$, maps functions to a scalar.

> $F[f, g, h, \ldots]$ is at an extremum (to the first order) if for any small variation to f, δf,
>
> $$\delta F = F[f + \delta f, g + \delta g, \ldots] - F[f, g, \ldots] = 0$$

Euler-Lagrange equations

Consider "small variations" to $q_1, \ldots, q_s$: $q(t) + \alpha\eta_1(t), \ldots, q_s(t) + \alpha\eta_s(t)$, where:

- $\alpha \in \mathbb{R}$ is a small parameter
- $\eta_i(t)$ are functions that evaluate to zero at the end points of $[t_1, t_2]$

> We are write the action as a function of α, $S(\alpha)$. The condition to extremize S is such that
>
> $$\left(\frac{dS}{d\alpha}\right)_{\alpha=0} = 0$$

It turns out that:

> $S[q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s, t]$ is at an extremum when the Lagrangian $\mathcal{L}$ satisfies
>
> $$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \qquad \forall i = 1, 2, \ldots, s$$

Lagrangian for a free particle

Since space is homogeneous and isotropic and time is homogeneous in an IRF, this implies that the Lagrangian **for a free particle**

- cannot explicitly contain $\underline{r}$ (the position vector of the particle)

- cannot depend on the direction of $\underline{\dot{r}} = \underline{v}$, so it depends on the speed only

This leads to

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underline{v}}\right) = \underline{0} \implies \frac{\partial \mathcal{L}}{\partial \underline{v}} = \text{constant} \implies \underline{v} = \text{constant}$$

Thus we showed that free motion in an IRF takes place with velocity constant in magnetude and direction.

Properties of the Lagrangian

Ignorable coordinates: Constant terms, and terms that do not depend on any of the generalized coordinates can be discarded.

Non-uniqueness: Two Lagrangians, $\mathcal{L}'$, $\mathcal{L}$, differing only by a total derivative in time of a function of the generalized coordinates and time describe the same mechanical system:

$$\mathcal{L}'(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

This result follows from Hamilton's principle, and our requirement that first variation of the action is 0.

$$\begin{aligned} S' &= \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt + f(q^{(2)}, t_2) - f(q^{(1)}, t_1) \\ &= S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1) \end{aligned}$$

And the terms, $f(q^{(2)}, t_2) - f(q^{(1)}, t_1)$, are constants, so they are zero on variation.

- Multiplying the Lagrangian by a constant gives the same equations of motion
- A system made of a sum of non-interacting subsystems numbered $1, 2, \ldots$ have the Lagrangian

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \ldots$$

- Terms that do not involve the generalized coordinates do not appear in the Euler-Lagrange equations; they can be thrown out

Solution steps

Given a mechanics problem,

1. Draw the system, determine the IRF
2. Determine the generalized coordinates, q
3. Determine the Lagrangian
4. Solve the Euler Lagrange equations
5. Reason about the answer

General principles

The definition of the Lagrangian can be reasoned from general principles

1. Superposition: $\mathcal{L}$ is a sum of terms to account for different bodies, interactions between bodies, interaction of the system with external fields
2. Galileo's principle: q holds under Galiean transformations
3. Correspondence principle: Agrees with Newton's laws and experiement
4. Symmetries: $\mathcal{L}$ reflects symmetries in the problem
5. "Beauty": the model with the least assumptions, simplest math

Collection of N self-interacting particles

$$\mathcal{L} = \sum_{i=1}^{N} \frac{m_i}{2} |v_i|^2 - U(\bar{r}_1, \bar{r}_1, \ldots, t)$$

U is the potential energy between the particles.

Potential energies do not depend on velocity since forces act instantaneously in non-relativistic physics

Common potential energies

Elastic potential energy

$$U(\ell) = \frac{k}{2}(\ell - \ell_0)^2$$

- ℓ_0 is the undistorted length of the spring
- k is the spring constant

Gravitational potential energy

$$U(r) = -\frac{G m_1 m_2}{r}$$

- G is the gravitational constant
- r is 2-norm distance between masses m_1 and m_2

Objects close to surface of the earth, z axis pointing upwards:

$$U(z) = mgh$$

The sign of $U(z)$ may flip depending on the orientation of the coordinate system.

Energy

Functions of q and $\dot{q}$ that remain constant during motion, and only depend on initial conditions are called *integrals of motion*.

The conservation of energy comes from homogeneity of time.

$$E = \sum_{i=1}^{s} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i\right) - \mathcal{L} = T + U$$

The homogeneity of time implies that $\mathcal{L}$ have no explicit time dependence. This means that

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

Consider now the total time derivative of $\mathcal{L}$:

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{s} \left(\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right)$$

Substituting the Euler-Lagrange equation and writing $\ddot{q}_i$ as $d\dot{q}_i/dt$:

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{s} \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\dot{q}_i\right)$$

by the product rule:

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{s} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i\right)$$

Moving $d\mathcal{L}/dt$ to the right hand side:

$$0 = \frac{d}{dt}\sum_{i=1}^{s}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}\right)$$

this implies that the series must be a constant:

$$E = \sum_{i=1}^{s}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}\right)$$

Momentum

A generalized coordinate, q_i, is said to be *cyclic* when

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

We define

- $p_i = \partial \mathcal{L}/\partial \dot{q}_i$: generalized momentum When q_i is an angle, we interpret P as angular momentum.
- $F_i = \partial \mathcal{L}/\partial q_i$: generalized force

The conservation of momentum comes from homogeneity of space.

In a closed mechanical system, momentum is conserved for cyclic quantities:

$$p = \sum_{C} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

The summation is over the set of cyclic quantities, C

This is a result that we can get from applying the Euler-Lagrange equations. When $\partial \mathcal{L}/\partial q_i = 0$, it follows that that $\partial \mathcal{L}/\partial \dot{q}_i$ must be a constant.

Homogeneity of space implies that the Lagrangian is not changed by a displacement of its coordinates.

$$\frac{d\mathcal{L}}{dq} = \sum_{i=1}^{s} \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Substituting in the Euler-Lagrange equation:

$$\frac{d\mathcal{L}}{dq} = \sum_{i=1}^{s} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 = \frac{d}{dt}\sum_{i=1}^{s}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

A quantity whose time derivative is zero must be a constant. Thus the following quantity, which we call the momentum, is conserved:

$$p = \sum_{i=1}^{s} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Since $\partial \mathcal{L}/\partial q_i = -\partial U/\partial q_i$, and conservative forces is the negative gradient of a potential, the requirement that $d\mathcal{L}/dq = 0$ is Newton's third law:

$$\sum_{i=1}^{s} F_i = 0$$

Euler-Lagrange equation in terms of generalized momentum and force:

$$\dot{p}_i = F_i \qquad \forall i = 1, 2, \ldots, s$$

The conservation of energy and moment are particular cases of Noether's theorem. Roughly, the statement is that: if $\mathcal{L}$ is invariant to a continuous transformations, then there is a conserved quantities associated with it.

Motion in 1-D

- The "dimension" of motion is equal to the number of dofs

Invoking the conservation of energy, we can often directly integrate

$$E = T(\dot{q}) + U(q) = \frac{1}{2}m\dot{x}^2 + U(x)$$

to find that

$$t = \sqrt{\frac{m}{2}}\int \frac{dx}{\sqrt{E(t_0) - U(x)}} + \text{const}$$

Motion is restricted in regions where

$$E = T + U \geq U$$

When $E = U$, we have turning points.

Finite motion in 1-D is oscillatory. We can find the period by integrating over turning points x_1, x_2.

$$T = 2\sqrt{\frac{m}{2}}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}$$

Knowing the graph of $U(q)$, E, $\dot{q}_0$, we can

- sketch $q(t)$ and $\dot{q}(t)$

Two body problem

1. Only two bodies are involved
2. The potential energy of the interaction between the two only depends on their relative distance from each other

We know the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m_1\dot{\underline{r}}_1^2 + \frac{1}{2}m_2\dot{\underline{r}}_2^2 - U(|\underline{r}_1 - \underline{r}_2|)$$

There is a clever way to avoid solving coupled ODES. This involves changing coordinates to

- relative displacement $\underline{r} = \underline{r}_1 - \underline{r}_2$
- center of mass position

$$\underline{R}_{\text{cm}} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{M}$$

where $M = m_1 + m_2$

Relation between coordinates: New dofs are related to $\underline{r}_1$ and $\underline{r}_2$ by:

$$\underline{r}_1 = \underline{R}_{\text{cm}} + \frac{m_2}{M}\underline{r}$$
$$\underline{r}_2 = \underline{R}_{\text{cm}} - \frac{m_1}{M}\underline{r}$$

Making Substitutions into our original Lagrangian, we will find that

$$\mathcal{L} = \frac{M\dot{R}_{\text{cm}}^2}{2} + \frac{\mu\dot{r}^2}{2} - U(r)$$

where μ is the **reduced mass**: m_1m_2/M.

Our 2 generalized coordinates are $q = \{\underline{r}, \underline{R}_{\text{cm}}\}$.

Two body Lagrangian: is the sum the center of mass and relative motion Lagrangians:

$$\mathcal{L} = \mathcal{L}_{\text{cm}}(\underline{R}_{\text{cm}}) + \mathcal{L}_{\text{rel}}(\underline{r})$$

Center of mass motion

Generically, $\mathcal{L}_{\text{cm}}$ resembles the kinetic energy of a particle of mass M. $\underline{R}_{\text{cm}}$ will be either stationary or move with constant velocity

$$\underline{R}_{\text{cm}}(t) = \underline{R}_{\text{cm}}(0) + \underline{V}_{\text{cm}}(0)t$$

If the potential can be put into a form

$$U = U(r) + U(\underline{R}_{\text{cm}})$$

then the center of mass Lagrangian gains an additional term. The center of mass may or may not move with constant velocity any more.

Relative motion

$\mathcal{L}_{\text{rel}}$, resembles a Lagrangian that describes motion of a single particle of mass μ in the field U that depends on the distance away from some reference point.

Motion in a central field: Relative motion takes place in the plane defined by $\underline{r}_0$ and $\underline{v}_0$ (Initial conditions on the relative displacement vector), since $U(r)$ resembles a central field

Since motion is 2-D, we can use polar coordinates to describe $\underline{r}$.

Relative motion Lagrangian:

$$\mathcal{L}_{\text{rel}} = \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\phi}^2}{2} - U(r)$$

where $\underline{\phi}$ is the polar angle, $\underline{r}$ be the radial vector. We know velocities can be decomposed into a tangential and radial component:

$$\dot{\underline{r}} = \dot{r}\underline{\hat{r}} + r\dot{\phi}\underline{\hat{\phi}}$$

Conservation of angular momentum:

$$\ell = \mu|\underline{r}_0 \times \underline{v}_0| = \mu r_0 v_0 \sin(\theta)$$
$$= \mu r^2\dot{\phi} = \text{const}$$

Rearranging:

$$\dot{\phi} = \frac{\ell}{\mu r^2}$$

Conservation of energy:

$$E_{\text{rel}} = \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\phi}^2}{2} + U(r) = \text{const}$$
$$= \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} + U(r)$$
$$= T_{\text{eff}} + U_{\text{eff}}$$

- If we know the initial position and velocities of the relative displacement vector, then we do not need to convert those values into polar coordinates.
- In Newtonian terms, since $\underline{F} = -\underline{\nabla}U$, $\underline{F} \parallel \underline{r}$, this means that the system has no net torque. Recall that

$$\frac{d\underline{\ell}}{dt} = \underline{\tau} = 0$$

so ℓ must be a conserved.

We can solve for the following:

1. $r(t)$:

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E_{\text{rel}} - U_{\text{eff}})}$$

2. Once we know $r(t)$, we know $\phi(t)$:

$$\frac{d\phi}{dt} = \frac{\ell}{\mu r^2}$$

3. We can get the "**trajectory equation**" by applying the inverse function theorem to dr/dt;

$$\frac{dt}{dr} = \pm\sqrt{\frac{\mu}{2(E_{\text{rel}} - U_{\text{eff}})}}$$

and combining:

$$\frac{d\phi}{dt}\frac{dt}{dr} = \frac{d\phi}{dr}$$

Planetary motion

Is a two body problem, with the gravitational potential:

$$U = -\frac{Gm_1m_2}{r} = -\frac{\alpha}{r}$$

The relative motion Lagrangian becomes:

$$\begin{aligned}\mathcal{L}_{\text{rel}} &= \frac{\mu\left(v_r^2 + v_\phi^2\right)}{2} + \frac{\alpha}{2} \\ &= \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\phi}^2}{2} + \frac{\alpha}{r}\end{aligned}$$

Invoking the conservation of energy and momentum:

$$E_{\text{rel}} = \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} - \frac{\alpha}{r}$$

When $dr/dt \geq 0$, The trajectory equation can be found by solving:

$$\frac{d\phi}{dr} = \frac{\ell}{\mu r^2}\sqrt{\frac{\mu}{2(E_{\text{rel}} - U_{\text{eff}})}}$$

Eccentricity:

$$e = \sqrt{1 + \frac{E_{\text{rel}}}{U_0}}$$

where $U_0 = \alpha/2p$, and e is unitless

Integrating:

$$\begin{aligned}\int d\phi &= \frac{\ell}{\sqrt{2\mu}}\int \frac{dr}{r^2\sqrt{E_{\text{rel}} - U_{\text{eff}}}} \\ &= \frac{\ell}{\sqrt{2\mu}}\int \frac{dr}{r^2\sqrt{E_{\text{rel}} - \ell^2/2\mu r^2 - \alpha/r}}\end{aligned}$$

Trajectory equation:

$$\frac{p}{r} = 1 + e\cos(\phi)$$

where $u = p/r$, and $du/dr = -p/r^2$.

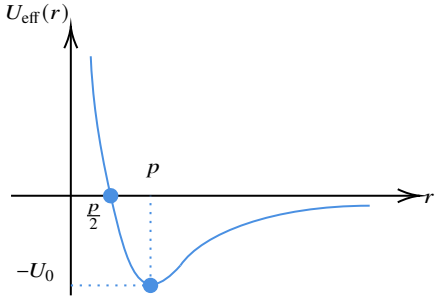
Trajectories

Let's consider when $\ell \neq 0$. If ℓ is zero, this implies that:

$$\underline{r}_0 \parallel \underline{v}_0$$

- Two bodies collide
- If there is enough kinetic energy, the bodies escape

Consider U_{eff}. For small r, U blows up like $1/r^2$. For large r, U is dominated by $1/r$.



- We can find p by looking for the minimum of U_{eff}.

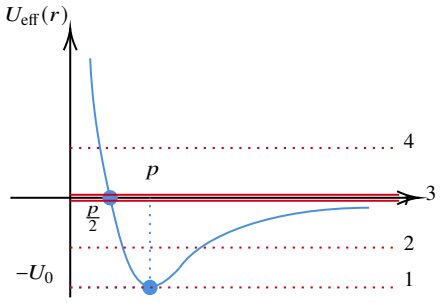
$$\frac{dU_{\text{eff}}}{dr} = 0 \implies p = \frac{\ell^2}{\mu\alpha}$$

- $-U_0$ is $-\alpha/2p$
- $U(p/2) = 0$

We know that motion is only possible when

$$E_{\text{rel}} = T_{\text{eff}} + U_{\text{eff}} \geq U_{\text{eff}}$$

Let's consider three cases for the total energy:



Case 1: When $E_{\text{rel}} = -U_0 = -\alpha/2p$

- $e = 0$
- r does not change, the distance between two objects is a constant ($\dot{r} = 0$; relative velocity is purely tangential)
- Trajectory is a circle: $p = r$, since $e = 0$
- $\dot{\phi} = \ell/\mu p^2 = \Omega = 2\pi/T$, which means that

$$\phi(t) = \phi(0) + \Omega t$$

Kepler's second law: particles in a central field sweeps out equal area in equal time. (dA/dt = const.)

$$\frac{dA}{dt} = \frac{\ell}{2\mu} = \text{const}$$

The derivation is based on the fact that $|\underline{r}_0 \times \underline{v}_0|$ can be though of as the area of a parallelogram spanned by $\underline{r}_0$ and $\underline{v}_0$, and the conservation of angular momentum.

For a circular orbit with period T,

$$v_0 = \frac{2\pi R}{T}$$

If the orbit is to be circular, the initial velocity must have no radial component, and of sufficient magnitude, such that

$$\begin{aligned}E(0) &= \frac{\ell^2}{2\mu r_0^2} - \frac{\alpha}{r_0} = \frac{\mu v_0^2}{2} - \frac{\alpha}{r_0} \\ &= \frac{-\alpha}{2r_0}\end{aligned}$$

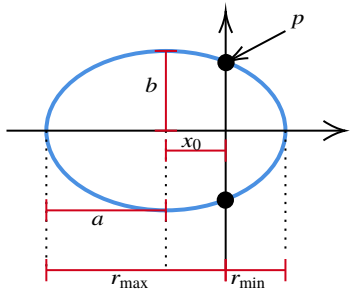
otherwise, the orbit will be elliptical. If the velocity is too high, the body may escape.

Case 2: when $-U_0 < E_{\text{rel}} < 0$:

- $0 < e < 1$
- Two turning points, so $r_{\min} \leq r \leq r_{\max}$
- Trajectory turns out to be an elipse

By letting $r\cos(\phi) = x$, we can find that

$$\frac{(x + x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$



where

$$a = \frac{p}{1 - e^2} = \frac{r_{\min} + r_{\max}}{2}$$

We can show that:

$$E_{\text{rel}} = \frac{-\alpha}{2a} = \frac{\alpha}{r_{\max} + r_{\min}}$$

- $b = \sqrt{pa}$
- $x_0 = ea$
- $r_{\min}$ is the *perihelion* of the orbit

$$\cos(\phi) = 1 \implies r_{\min} = \frac{p}{1+e}$$

- $r_{\max}$ is the *aphelion* of the orbit

$$\cos(\phi) = -1 \implies r_{\max} = \frac{p}{1-e}$$

Zero radial velocity: When $r = r_{\min}$ or $r = r_{\max}$, the radial velocity, $\dot{r}$, must be zero.

Given r_0, and desired $r_{\min}$, $r_{\max}$, we can solve for the magnitude of the initial velocity, v_0, and launch angle, θ, using the conservation of energy and angular momentum.

$$E(0) = \frac{\mu v_0^2}{2} - \frac{\alpha}{r_0} = \frac{\ell^2}{2\mu r_{\min}^2} - \frac{\alpha}{r_{\min}} = \frac{\ell^2}{2\mu r_{\max}^2} - \frac{\alpha}{r_{\max}}$$

We can find ℓ by equating

$$\frac{\ell^2}{2\mu r_{\min}^2} - \frac{\alpha}{r_{\min}} = \frac{\ell^2}{2\mu r_{\max}^2} - \frac{\alpha}{r_{\max}}$$

We can find v_0 by equating

$$\frac{\mu v_0^2}{2} - \frac{\alpha}{r_0} = \frac{-\alpha}{2a}$$

After v_0, ℓ are known. From $\ell = \mu r_0 v_0 \sin(\theta)$, we can find θ.

Kepler's first law: Planets move around their sun in an ellipse, with the sun at the focus.

We can show that this holds by consider $m_1 >> m_2$, where

- m_1 is the mass of the sun
- m_2 is the mass of a planet

In this case,

$$\underline{r}_1 = \underline{R}_{\text{cm}} + \frac{m_2}{M}\underline{r} \approx \underline{R}_{\text{cm}}$$
$$\underline{r}_2 = \underline{R}_{\text{cm}} + \frac{m_1}{M}\underline{r} \approx \underline{R}_{\text{cm}} + \underline{r}$$

Since is always possible to choose an IRF such that $\underline{R}_{\text{cm}}(t) = 0$ for all t. In this IRF, $\underline{r}_1$ is stationary, and $\underline{r}_2 = \underline{r}$, thus the planet orbits the sun in an ellipse.

Kepler's third law: $T^2 \propto a^3$ (the square of the orbit period is proportional to the cube of the length of the large semiaxis of the orbit.)

$$T = 2\pi\sqrt{\frac{\mu}{\alpha}}a^{3/2} = 2\pi\sqrt{\frac{a^3}{GM}}$$

We can show that this is true from:

- Finding $r(t)$ or $\phi(t)$
- Using Kepler's second law

Since

$$\frac{dA}{dt} = \frac{\ell}{2\mu} = \frac{A_{\text{total}}}{T}$$

and we know that $A_{\text{total}} = \pi ab$, rearranging,

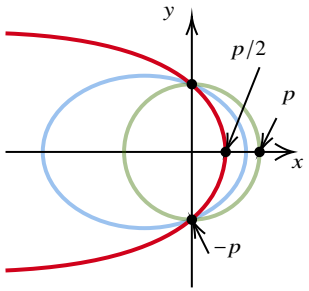
$$T = \frac{2\mu A_{\text{total}}}{\ell} = \frac{2\mu\pi ab}{\ell}$$

since $p = \ell^2/\mu\alpha$, we have $\sqrt{\mu\alpha p} = \ell$. Since $b = \sqrt{pa}$, we have $b/\sqrt{a} = \sqrt{p}$.

$$T = 2\mu\pi\frac{ab\sqrt{a}}{b\sqrt{\mu\alpha}} = 2\pi\sqrt{\frac{\mu}{\alpha}}a^{3/2}$$
$$= 2\pi\sqrt{\frac{m_1 m_2}{m_1+m_2}\frac{a^3}{Gm_1m_2}}$$
$$= 2\pi\sqrt{\frac{a^3}{GM}}$$

Case 3: when $E_{\text{rel}} = 0$:

- $e = 1$
- A single turning point at $r = p/2$
- Trajectory turns out to be a parabola
- Called an **escape trajectory**, the minimum energy for the bodies to escape each other



We will find that

$$x = \frac{p^2 - y^2}{2p}$$

Case 4: when $E_{\text{rel}} > 0$:

- $e > 1$
- One turning point, $r_{\min} \le r$
- Trajectory turns out to be a hyperbola

Small oscillations

$s = 1$ **with fixed external conditions**

General Lagrangian:

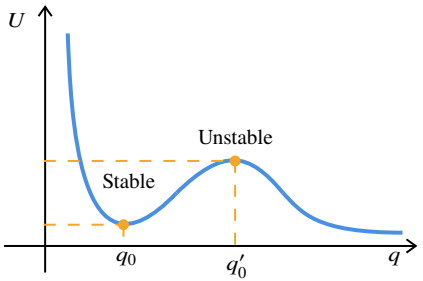
$$\mathcal{L}(q, \dot{q}) = \frac{\alpha(q)\dot{q}^2}{2} - U(q)$$

- energy is conserved, we can solve exactly

Rearranging:

$$\frac{dq}{dt} = \pm\sqrt{\frac{2(E - U(q))}{\alpha(q)}}$$

Our goal is to get an approximate solution.



Equilibrium points

A point q_0 is an **equilibrium point** when

$$\left[\frac{dU}{dq}\right]_{q=q_0} = 0$$

An equilibrium point is

1. Stable: when

$$\left[\frac{d^2U}{dq^2}\right]_{q=q_0} > 0$$

2. Unstable: when

$$\left[\frac{d^2U}{dq^2}\right]_{q=q_0} < 0$$

When the initial conditions are close to a stable equilibrium, we can get an approximation.

- Taylor series expansion to $\alpha(q)$ and $U(q)$

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n$$

- Truncate the series at the first term that will contribute to the Euler-Lagrange equations

We expand:

$$\alpha(q) \approx \alpha(q_0)$$

Since $T = \alpha(q_0)\dot{q}^2/2$ will contribute to the EL equations, we stop at this point.

$$U(q) \approx U(q_0) + U'(q_0)(q-q_0) + \frac{U''(q_0)}{2}(q-q_0)^2$$

$U'(q_0) = 0$ by definition, so the second term does not contribute. We stop at the second order terms. We are approximating the potential at q_0 by a parabola.

The Lagrangian becomes:

$$\mathcal{L}(q,\dot{q}) = \frac{\alpha(q_0)\dot{q}^2}{2} - \frac{U''(q_0)}{2}(q-q_0)^2$$

The EL-equation:

$$\ddot{q} = -\omega^2(q-q_0)$$

- $\omega = \sqrt{K/\alpha(q_0)}$
- $K = U''(q_0)$

Has solution

$$\begin{aligned} q(t) &= q_0 + A\cos(\omega t) + B\sin(\omega t) \\ &= q_0 + C\cos(\omega t + \phi) \end{aligned}$$

Forced oscillations

The potential energy/environment depends on time. General Lagrangian:

$$\mathcal{L}(q,\dot{q},t) = \frac{\alpha(q)\dot{q}^2}{2} - U(q) - U_{\text{ext}}(\dots,t)$$

If $q(t=0)$ is close to a stable equilibrium, q_0, we can taylor expand the terms in the Lagrangian. There are two cases for U_{ext}.

Case 1: If U_{ext} is of the form $\dot{q}g(t)$. Keep U_{ext} as is and try to simplify $U(q)$.

The EL equation:

$$\ddot{q} = -\omega^2(q-q_0) + f(t)$$

- $\omega^2 = K/\alpha(q_0)$
- $K = U''(q_0)$
- $f(t) = (dg/dt)/(\alpha(q_0))$

Case 2: If U_{ext} depends on q and not $\dot{q}$, expand

$$U_{\text{ext}} \approx -(q-q_0)F(t),$$

where $F(t)$ is

$$F(t) = -\left[\frac{\partial U_{\text{ext}}}{\partial q}\right]_{q=q_0}$$

Generically, the expansion for U_{ext} includes $U_{\text{ext}}(q_0,t)$. But since it does not depend on our generalized coordinate, we can discard it from our Lagrangian.

The Lagrangian becomes:

$$\mathcal{L}(q,\dot{q}) = \frac{\alpha(q_0)\dot{q}^2}{2} - \frac{U''(q_0)}{2}(q-q_0)^2 + (q-q_0)F(t)$$

The EL equation:

$$\ddot{q} = -\omega^2(q-q_0) + f(t)$$

- $\omega = \sqrt{K/\alpha(q_0)}$
- $K = U''(q_0)$
- $f(t) = F(t)/\alpha(q_0)$

General solution to the EL equation

Let $x = q - q_0$, and perform a change of variables:

$$\ddot{x} = -\omega^2 x + f(t)$$

A second order, linear, inhomogeneous ODE. The general solution is the sum of the solution to the homogeneous equation, and a particular solution:

$$x = x_h + x_p$$

We already know the homogeneous solution:

$$\begin{aligned} x_h &= A\cos(\omega t + \phi) \\ &= C\cos(\omega t) + D\sin(\omega t) \end{aligned}$$

Let $\zeta = \dot{x} + i\omega x$. Consider

$$\begin{aligned} \dot{\zeta} &= \ddot{x} + i\omega\dot{x} \\ &= -\omega^2 x + f(t) + i\omega\dot{x} \\ &= i\omega(\dot{x} + i\omega x) + f(t) \\ &= i\omega\zeta + f(t) \end{aligned}$$

Yet $\dot{\zeta} = i\omega\zeta + f(t)$, is a first order ODE, and we know

$$\zeta = A(t)e^{i\omega t}$$

Differentiating, we find

$$\begin{aligned} \dot{\zeta} &= i\omega A e^{i\omega t} + \dot{A}e^{i\omega t} \\ &= i\omega\zeta + f(t) \\ &= i\omega A e^{i\omega t} + f(t) \end{aligned}$$

Matching terms, we conclude that

$$\dot{A}e^{i\omega t} = f(t)$$

which is another ODE with solution:

$$A(t) = A(0) + \int_0^t f(\tau)e^{-i\omega\tau}\,d\tau$$

We know $\zeta(0) = A(0) = \dot{x}(0) + i\omega x(0)$, and

$$\begin{aligned} \zeta(t) &= e^{i\omega t}\left(\dot{x}(0) + i\omega x(0) + \int_0^t f(\tau)e^{-i\omega\tau}\,d\tau\right) \\ &= \dot{x} + i\omega x \end{aligned}$$

Finally,

$$q(t) = x(t) + q_0 = \frac{1}{\omega}\text{Im}(\zeta) + q_0$$

$s \geq 2$ with fixed external conditions

General Lagrangian:

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{s}\alpha_{ij}(q)\dot{q}_i\dot{q}_j - U(q),$$

where

- $q = \{q_1, q_2, \dots\}$;
- $\dot{q} = \{\dot{q}_1, \dot{q}_2, \dots\}$;
- α_{ij} is a symmetric matrix - $\alpha_{ij} = \alpha_{ji}$.

We want to find equilibrium points of U, and apply our small oscillation approximation.

A set of generalized coordinates, q, describes an equilibrium point,

$$q_0 = \left\{q_i^{(0)}, \dots, q_s^{(0)}\right\},$$

when

$$\left[\frac{\partial U}{\partial q}\right]_{q=q_0} = 0 \qquad \forall i = 1,\dots,s$$

Recall the *Hessian* of a scalar field:

$$\widetilde{K}_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad \forall i,j = 1,\dots,s,$$

- We only consider when $\underline{\widetilde{K}}$ is symmetric
- We can diagonalized $\underline{\widetilde{K}}$, we will call the diagonalized matrix $\underline{K}$:

$$\underline{K}(q) = \begin{bmatrix} \lambda_1(q) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_s(q) \end{bmatrix}$$

An equilibrium point q_0 is stable when all of $K_{ii}(q_0) \geq 0$, for all $i = 1, \ldots, s$.

Once we found a stable equilibrium point, q_0, we approximate

$$T(q) \approx \frac{1}{2}\sum_{i,j} \alpha_{ij}(q_0)\dot{q}_i\dot{q}_j$$

and

$$U(q) \approx \frac{1}{2}\sum_{i,j} K_{ij}\left(q_i - q_i^{(0)}\right)\left(q_j - q_j^{(0)}\right)$$

We know that the following terms drop out.

$$U(q_0) + \sum_{i=1} \underbrace{\left[\frac{\partial U}{\partial q_i}\right]_{q_i = q_i^{(0)}}}_{=0}\left(q_i - q_i^{(0)}\right)$$

This can be written in terms of matrix vector multiplication. Consider the column vector

$$\underline{q} - \underline{q}_0 = \begin{bmatrix} q_1 - q_1^{(0)} & \ldots & q_s - q_s^{(0)} \end{bmatrix}^\intercal$$

And

$$U(q) = \frac{1}{2}\left(\underline{q} - \underline{q}_0\right)^\intercal \underline{K}\left(\underline{q} - \underline{q}_0\right)$$

The Lagrangian (for q near q_0) becomes

$$\mathcal{L} \approx \frac{1}{2}\sum_{i,j} \alpha_{ij}(q_0)\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i,j} K_{ij}\left(q_i - q_i^{(0)}\right)\left(q_j - q_j^{(0)}\right)$$

The EL equations:

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{\partial\dot{q}_i}\right) = \frac{\partial\mathcal{L}}{\partial q_i} \qquad \forall i \in 1, \ldots, s$$

Consider the case of $s = 2$. The kinetic energy term consists of 4 terms:

$$\begin{aligned} 2T &= \alpha_{11}(q_0)\dot{q}_1\dot{q}_1 + \alpha_{12}(q_0)\dot{q}_1\dot{q}_2 \\ &\quad + \alpha_{21}(q_0)\dot{q}_2\dot{q}_1 + \alpha_{22}(q_0)\dot{q}_2\dot{q}_2 \end{aligned}$$

Since the generalized coordinates commute, and $\alpha_{ij} = \alpha_{ji}$,

$$T = \frac{\alpha_{11}(q_0)\dot{q}_1\dot{q}_1}{2} + \alpha_{12}(q_0)\dot{q}_1\dot{q}_2 + \frac{\alpha_{22}(q_0)\dot{q}_2\dot{q}_2}{2}$$

Finally,

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_1} = \alpha_{11}(q_0)\dot{q}_1 + \alpha_{12}(q_0)\dot{q}_2$$

We can repeat the same calculation for $\dot{q}_2$.

We can verify that they become

Euler Lagrange equations for $s \geq 2$: Assuming time-independent external conditions:

$$\sum_{j=1}^{s} \alpha_{ij}(q_0)\ddot{q}_j = -\sum_{j=1}^{s} K_{ij}\left(q_i - q_i^{(0)}\right) \qquad \forall i = 1, \ldots, s$$

Which s second-order, inhomogeneous ODEs.

In matrix-vector form:

$$\underline{\alpha}(q_0)\ddot{\underline{q}} = -\underline{K}\left(\underline{q} - \underline{q}_0\right)$$

General solution

Consider displacements out of equilibrium:

$$x_j = q_j - q_j^{(0)}$$

Changing variables to x_j, we get homogeneous equations:

$$\sum_{j=1}^{s} \alpha_{ij}(q_0)\ddot{x}_j = -\sum_{j=1}^{s} K_{ij}x_j$$

If we consider solutions of the form

$$\begin{aligned} x_j(t) &= A_j\cos(\omega t + \phi) \\ \ddot{x}_j(t) &= -\omega^2 A_j\cos(\omega t + \phi) \end{aligned}$$

Then both sides will have the same time dependent terms, and if we carefully match the coefficients, the equality should hold.

$$-\omega^2\sum_{j=1}^{s} \alpha_{ij}(q_0)A_j\cos(\omega t + \phi) = -\sum_{j=1}^{s} K_{ij}A_j\cos(\omega t + \phi)$$

So we require

$$\omega^2\sum_{j=1}^{s} \alpha_{ij}(q_0)A_j = \sum_{j=1}^{s} K_{ij}A_j$$

Rearranging,

$$\sum_{j=1}^{s}\left(\omega^2\alpha_{ij}(q_0) - K_{ij}\right)A_j = 0$$

This is s linear equations, depending on our unknowns A_j and ω^2.

$$\begin{aligned} &\left(\omega^2\alpha_{11}(q_0) - K_{11}\right)A_1 + \cdots + \left(\omega^2\alpha_{1s}(q_0) - K_{1s}\right)A_s = 0 \\ &\qquad\vdots \\ &\left(\omega^2\alpha_{s1}(q_0) - K_{s1}\right)A_1 + \cdots + \left(\omega^2\alpha_{ss}(q_0) - K_{ss}\right)A_s = 0 \end{aligned}$$

In matrix-vector form:

$$\underline{M}\,\underline{A} = 0$$

We have a non-trivial $\underline{A}$ only for

$$\det\left(\underline{M}\right) = 0$$

This leads to s equations for s possible values of ω, denoted ω_α. We require $\omega_\alpha > 0$, $\forall\alpha = 1, \ldots, s$. ω_α are the **characteristic frequencies** (also called **eigenfrequencies, normal frequencies**)

If $\omega_\alpha^2 \leq 0$, either

- there were calculation errors
- q_0 is not a stable equilibrium

For s normal frequencies, we can determine A_j^α up an unknown coefficent, A_α.

Let $A_j^\alpha = A_\alpha a_j^\alpha$. The most general solution for $j = 1, \ldots, s$ is

$$x_j(t) = \sum_{\alpha=1}^{s} A_\alpha a_j^\alpha\cos(\omega_\alpha t + \phi_\alpha)$$

There are $2s$ remaining parameters (A_α, ϕ_α) that we can solve for using $2s$ initial conditions.

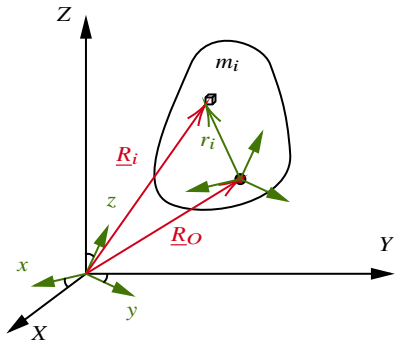
Rigid body motion

We require that the body does not deform

- distance between any two points on a body is constant in time

We need $s = 6$ to specify the motion of a rigid body.

- 3 to specify (X, Y, Z) a reference point O on the body measured in the IRF
- 3 to specify the "Euler angles" between the moving frame and IRF axes



A mass element on the rigid body, with respect to the reference point, is given by

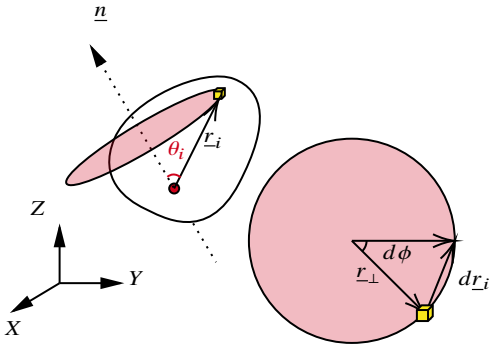
$$\underline{r}_i = \underline{R}_i - \underline{R}_O$$

The IRF coordinates are capitalized. Body reference frame coordinates in lowercase.

The velocity for any differential mass element, m_i, on the body is given by

$$\dot{\underline{R}}_i = \dot{\underline{R}}_O + \dot{\underline{r}}_i,$$

Since $\|\underline{r}_i\| = \text{const}$, $\dot{\underline{r}}_i$ is non-zero only when the direction of $\underline{r}_i$ changes (as measured in the IRF).



Over dt, $\underline{r}_i$ changes by $d\underline{r}_i$. Geometrically, we can see that

$$|d\underline{r}_i| = d\phi\,|\hat{\underline{n}}|\,|\underline{r}_i|\sin(\theta_i) = d\phi\,|\underline{r}_\perp|$$

and

$$d\underline{r}_i = d\phi\,(\hat{\underline{n}} \times \underline{r}_i)$$

When the body rotates about an axis with unit vector $\hat{\underline{n}}$, we can verify that

$$\frac{d\underline{r}_i}{dt} = \underline{\Omega} \times \underline{r}_i$$

where $\underline{\Omega} = \dot{\phi}\hat{\underline{n}}$ is the **angular velocity** of the body.

Velocity of any point on the body: Given a reference point O on the body

$$\dot{\underline{R}}_i = \dot{\underline{R}}_O + \underline{\Omega} \times \underline{r}_i$$

Angular velocity is independent of the reference point:

$$\underline{\Omega} = \frac{d\phi}{dt}\hat{\underline{n}}$$

is the same for any point on the rigid body, independent of O.

Consider two choices of reference points, A and B. By our formula,

$$\begin{aligned}\dot{\underline{R}}_i &= \dot{\underline{R}}_A + \underline{\Omega}_A \times (\underline{R}_i - \underline{R}_A)\\ &= \dot{\underline{R}}_B + \underline{\Omega}_B \times (\underline{R}_i - \underline{R}_B)\end{aligned}$$

Consider

$$\begin{aligned}\dot{\underline{R}}_i &= \dot{\underline{R}}_A + \underline{\Omega}_A \times (\underline{R}_i - \underline{R}_A + \underline{R}_B - \underline{R}_B)\\ &= \dot{\underline{R}}_A + \underline{\Omega}_A \times (\underline{R}_B - \underline{R}_A) + \underline{\Omega}_A \times (\underline{R}_i - \underline{R}_B)\\ &= \dot{\underline{R}}_B + \underline{\Omega}_A \times (\underline{R}_i - \underline{R}_B)\end{aligned}$$

Matching terms,

$$\dot{\underline{R}}_B + \underline{\Omega}_B \times (\underline{R}_i - \underline{R}_B) = \dot{\underline{R}}_B + \underline{\Omega}_A \times (\underline{R}_i - \underline{R}_B)$$

from which we can conclude that $\underline{\Omega}_B = \underline{\Omega}_A$, and the choice the reference point does not change the angular velocity.

Moment of inertia

To simplify our expression for kinetic energy requires the introduction of the **moment of inertia tensor**.

Moment of inertia tensor: Defined about a reference point, O,

$$I^O_{\alpha\beta} = \sum_i m_i \left(r_i^2 \delta_{\alpha\beta} - (r_\alpha)_i\,(r_\beta)_i\right)$$

where $\alpha, \beta = 1, 2, 3$, and $r^2 = x^2 + y^2 + z^2$, as measured in the body frame.

$$\underline{I}^O = \begin{bmatrix} \sum m_i(y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ \cdot & \sum m_i(x_i^2 + z_i^2) & -\sum m_i y_i z_i \\ \cdot & \cdot & \sum m_i(x_i^2 + y_i^2) \end{bmatrix}$$

is real symmetric.

- $\underline{I}$ is hermitian. So the eigenvectors of $\underline{I}$ form an eigenbasis
- If we choose our axes to align with the eigenvectors, then $\underline{I}$ becomes a diagonal matrix. When $\underline{I}$ is diagonal under our choice of axes, the axes are called **principle axes**.
- $\underline{I}$ is additive

Classification of rigid bodies: Given that $\underline{I}^O$ is diagonal, if

1. $I_{11} = I_{22} = I_{33}$: spherical top
2. $I_{11} = I_{22} \neq I_{33}$: symmetrical top
3. $I_{11} \neq I_{22} \neq I_{33}$: asymmetrical top

Parallel axis theorem: A body of mass M. Given the moment of inertia at the center of mass about an axis z, the moment of inertia at some point O about an axis parallel to some axis z is given by

$$I^O_{zz} = I^{\text{cm}}_{zz} + Md^2$$

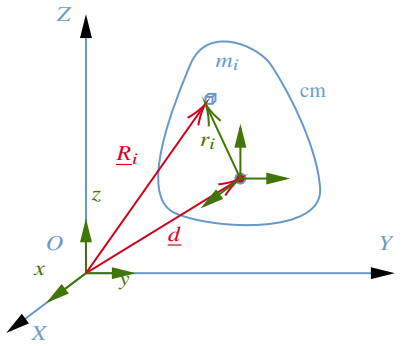
where d is the distance between O and cm projected into the plane spanned by vectors orthogonal to z.

If we measure distance in uppercase coordinates about point O, and lowercase coordinates about cm, then

$$I^O_{zz} = \sum m_i \left(X_i^2 + Y_i^2\right)$$
$$I^{\text{cm}}_{zz} = \sum m_i \left(x_i^2 + y_i^2\right)$$

The position of cm as measured from O is related to $\underline{R}_i$ and $\underline{r}_i$ by

$$\underline{R}_i = \underline{r}_i + \underline{d}$$



It follows that

$$\begin{aligned}X_i &= x_i + d_x\\ Y_i &= y_i + d_y\\ Z_i &= x_i + d_z\end{aligned}$$

Substituting

$$\begin{aligned}I^O_{zz} &= \sum m_i \left((x_i + d_x)^2 + (y_i + d_y)^2\right)\\ &= \sum m_i \left(x_i^2 + y_i^2 + 2x_i d_x + 2y_i d_y + d_x^2 + d_y^2\right)\\ &= I^{\text{cm}}_{xx} + M\left(d_x^2 + d_y^2\right) + 2\sum m_i\left(x_i d_x + y_i d_y\right)\end{aligned}$$

And since

$$\begin{aligned}\sum m_i x_i d_x &= d_x \sum m_i x_i\\ &= M x_{\text{cm}}\\ &= 0\end{aligned}$$

In the center of mass frame, the x coordinate of the center of mass must be zero.

Homogeneous moments of inertia

A solid cylinder of radius R, high h:

$$I^{\text{cm}}_{zz} = \frac{MR^2}{2}$$

A hollow cylinder:

$$I^{\text{cm}}_{zz} = MR^2$$

A solid ball of radius R:

$$I^{\text{cm}}_{zz} = \frac{2}{5}MR^2$$

A hollow sphere:

$$I^{\text{cm}}_{zz} = \frac{2}{3}MR^2$$

A thin rod of length L, along z axis:

$$I_{xx} = I_{yy} = \frac{ML^2}{12} \qquad I_{zz} = 0$$

Kinetic energy

The kinetic energy of the rigid body can be found be summing the kinetic energies of all mass elements on the body

$$T = \frac{1}{2}\sum_i m_i \dot{R}_i^2$$

This can be expanded into

Kinetic energy of a rigid body: having chosen a reference point O; let $\underline{V}_O = \dot{\underline{R}}_O$:

$$T = \frac{MV_O^2}{2} + M\left(\underline{R}_{\text{cm}} - \underline{R}_O\right)\cdot\left(\underline{V}_O \times \underline{\Omega}\right) + \frac{1}{2}\underline{\Omega}^\intercal \underline{I}^O \underline{\Omega}$$

- Choosing $\underline{R}_O = \underline{R}_{\text{cm}}$ makes the middle term vanish
- Its convenient to use the body frame when measuring $\underline{\Omega}$
- if rigid body is undergoing planar movement, and $\exists$ a point s.t. $\underline{V}_O$ is zero at some time t, then choosing this point as the reference point makes the first and second terms vanish

Equations of motion for a rigid body

We want to show that Lagrangian mechanics for a rigid body recovers the the Newtonian equations of motion. Newton's second law states:

$$\underline{\dot{P}} = \underline{F}$$

where

$$\underline{F} = \sum_i \underline{f}_i \qquad \underline{P} = \sum_i m_i \underline{v}_i$$

The rotational equations of motion says that the rate of change in angular momentum equals the torque (about the center of mass)

$$\frac{d\underline{L}_{\text{cm}}}{dt} = \underline{\tau}_{\text{cm}}$$

where

$$\underline{\tau}_{\text{cm}} = \sum_i \underline{r}_i \times \underline{f}_i \qquad \underline{L}_{\text{cm}} = \sum_i m_i \underline{r}_i \times \underline{v}_i$$

Lagrangian of a rigid body: Let the center of mass be the reference point. The generalized coordinates $\left\{\underline{R}_{\text{cm}} = (X_{\text{cm}}, Y_{\text{cm}}, Z_{\text{cm}}), \underline{\Theta}\right\}$

$$\mathcal{L} = \frac{MV_{\text{cm}}^2}{2} + \frac{1}{2}\underline{\Omega}^\intercal \underline{I}^{\text{cm}} \underline{\Omega} - U\left(\underline{R}_{\text{cm}}, \underline{\Theta}\right),$$

- $\underline{\Theta}$ represents the three Euler angles, is such that

$$\dot{\underline{\Theta}} = \underline{\Omega}$$

Recovering Newton's second law: Let X_j for $j = 1, 2, 3$ represent the components of $\underline{R}_{\text{cm}}$. The EL-equations for $\underline{R}_{\text{cm}}$ that tells us

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}_j}\right) = \frac{\partial \mathcal{L}}{\partial X_j} \qquad \forall j = 1, 2, 3$$

Explicitly,

$$\frac{d}{dt}\left(M\dot{X}_j\right) = -\frac{\partial U}{\partial X_j}$$

By definition, $M\dot{X}_j$ is the momentum of the body along the j-axis. And

$$-\frac{\partial U}{\partial X_j} = F_j$$

is generalized force along the j-axis by construction. We recover Newton's second law.

The EL-equations involving $\underline{\Theta}$ is:

$$\frac{d}{dt}\underbrace{\left(\frac{\partial \mathcal{L}}{\partial \underline{\Omega}}\right)}_{\underline{L}_{\text{cm}}} = \underbrace{\frac{\partial \mathcal{L}}{\partial \underline{\Theta}}}_{\underline{\tau}_{\text{cm}}}$$

(Differentiation by Ω means to differentiate by each component of $\underline{\Omega}$.) In the body frame,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \Omega_x} &= \frac{\partial}{\partial \Omega_x}\left(\frac{1}{2}\Omega_x^2 I_{xx}^{\text{cm}} + \Omega_x\Omega_y I_{xy}^{\text{cm}} + \Omega_x\Omega_z I_{xz}^{\text{cm}} + \dots\right)\\ &= \Omega_x I_{xx}^{\text{cm}} + \Omega_y I_{xy}^{\text{cm}} + \Omega_z I_{xz}^{\text{cm}}\end{aligned}$$

Permuting the index cyclically:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \Omega_x} &= \Omega_x I_{xx}^{\text{cm}} + \Omega_y I_{xy}^{\text{cm}} + \Omega_z I_{xz}^{\text{cm}}\\ \frac{\partial \mathcal{L}}{\partial \Omega_y} &= \Omega_y I_{yy}^{\text{cm}} + \Omega_z I_{yz}^{\text{cm}} + \Omega_x I_{yx}^{\text{cm}}\\ \frac{\partial \mathcal{L}}{\partial \Omega_x} &= \Omega_z I_{zz}^{\text{cm}} + \Omega_x I_{zx}^{\text{cm}} + \Omega_y I_{zy}^{\text{cm}}\end{aligned}$$

This is a systems of equation which can express in matrix vector form. Also, by convention, we define the angular momentum of the system to be

$$\frac{\partial \mathcal{L}}{\partial \underline{\Omega}} = \underline{I}_{\text{cm}}\underline{\Omega} = \underline{L}_{\text{cm}}$$

We need to verify that this agrees with Newton's definition of angular momentum.

$$\underline{L}_{\text{cm}} = \sum_i m_i \underline{r}_i \times \underline{v}_i$$

Any point on the body of a rigid object, as measured in the IRF with the center of mass as the reference point, is given by

$$\dot{\underline{R}}_i = \dot{\underline{R}}_{\text{cm}} + \underline{\Omega} \times \underline{r}_i$$

Distributing

$$\begin{aligned}\sum_i m_i \underline{r}_i \times \underline{v}_i &= \sum_i m_i \underline{r}_i \times \dot{\underline{R}}_{\text{cm}}\\ &\quad + \sum_i m_i \underline{r}_i \times \left(\underline{\Omega} \times \underline{r}_i\right)\end{aligned}$$

Since

$$\sum_i m_i \underline{R}_i = M\underline{R}_{cm}$$

so

$$\begin{aligned}\sum_i m_i \underline{r}_i &= M\underline{R}_{\text{cm}} - M\underline{R}_{\text{cm}}\\ &= 0\end{aligned}$$

and the first term vanishes. We use the BAC-CAB rule to expand the second term:

$$\begin{aligned}\underline{L}_{\text{cm}} &= \sum_i m_i \underline{r}_i \times \left(\underline{\Omega} \times \underline{r}_i\right)\\ &= \sum_i m_i \left(\underline{\Omega} r_i^2 - \underline{r}_i\left(\underline{r}_i \cdot \underline{\Omega}\right)\right)\end{aligned}$$

Let's consider the first component:

$$\begin{aligned}L_x^{\text{cm}} &= \sum_i m_i\left(\Omega_x\left(x_i^2 + y_i^2 + z_i^2\right) - x_i\left(\Omega_x x_i + \Omega_y y_i + \Omega_z z_i\right)\right)\\ &= \Omega_x \sum_i m_i\left(x_i^2 + y_i^2\right) - \Omega_y \sum_i m_i x_i y_i - \Omega_z \sum_i m_i x_i z_i\\ &= \Omega_x I_{xx}^{\text{cm}} + \Omega_y I_{xy}^{\text{cm}} + \Omega_z I_{xz}^{\text{cm}}\end{aligned}$$

Doing the same expansion for the second and third componenet, we recover Newton's rotational equation of motion.

Angular momentum of a rigid body: About the center of mass,

$$\underline{L}_{\text{cm}} = \underline{I}_{\text{cm}}\underline{\Omega}$$

- Even under the choice of using principle axes, the diagonal entries of I_{cm} are often not equal
- Other than for spherical tops, $\underline{L}_{\text{cm}} \nparallel \underline{\Omega}$
- Where as $\underline{P} \parallel \underline{V}_{\text{cm}}$ for any body in general

Non-zero angular velocity: In the IRF, a body experiencing zero torque implies that $d\underline{L}_{\text{cm}}/dt = 0$, and so $\underline{L}_{\text{cm}}$ is constant. But Ω need not be constant.

To proof this, we consider Ω in the body frame, or in the inertial reference frame

Time derivatives in a rotating frame: Consider an IRF and a rotating frame, sharing the same origin O. The rate of change of $\underline{A}$ in the IRF is related to its rate of change in the rotating frame by

$$\left(\frac{d\underline{A}}{dt}\right)_{XYZ} = \left(\frac{d\underline{A}}{dt}\right)_{xyz} + \underline{\Omega} \times \underline{A}$$

where $\underline{\Omega}$ is the angular velocity of the rotating frame as seen in the IRF.

We can expand $\underline{A}$ in terms of three orthogonal basis vectors, $\underline{e}_j$, $j = 1, 2, 3$ in body frame:

$$\underline{A} = \sum_{j=1}^{3} a_j \underline{e}_j$$

The time derivative of $\underline{A}$ in the rotating frame is

$$\left(\frac{d\underline{A}}{dt}\right)_{xyz} = \sum_{j=1}^{3} \frac{da_j}{dt}\underline{e}_j$$

In the IRF, the vectors $\underline{e}_j$ are changing (rotating) in time. So

$$\left(\frac{d\underline{A}}{dt}\right)_{XYZ} = \sum_{j=1}^{3} \frac{da_j}{dt}\underline{e}_j + \sum_{j=1}^{3} a_j \left(\frac{d\underline{e}_j}{dt}\right)_{XYZ}$$

Where

$$\frac{d\underline{e}_j}{dt} = \underline{\Omega} \times \underline{e}_j$$

and we recover our result.

Euler's equations

Are the equations of motion of rotating object, given the torque acting on the body.

As in seen in the IFR, the torque is

$$\underline{\tau}_{\text{cm}} = \left(\frac{d\underline{L}_{\text{cm}}}{dt}\right)_{XYZ} = \left(\frac{d\underline{L}_{\text{cm}}}{dt}\right)_{xyz} + \underline{\Omega} \times \underline{L}_{\text{cm}}$$

In matrix-vector notation:

$$\underline{\tau}_{\text{cm}} = \dot{\underline{L}}_{\text{cm}} + \underline{\Omega} \times \underline{L}_{\text{cm}}$$

These correspond to three equations in the rotating frame:

$$\begin{bmatrix} \tau_1^{\text{cm}} \\ \tau_2^{\text{cm}} \\ \tau_3^{\text{cm}} \end{bmatrix} = \begin{bmatrix} I_{11}\dot{\Omega}_1 + \left(I_{33}^{\text{cm}} - I_{22}^{\text{cm}}\right)\Omega_3\Omega_2 \\ I_{22}\dot{\Omega}_2 + \left(I_{11}^{\text{cm}} - I_{33}^{\text{cm}}\right)\Omega_1\Omega_3 \\ I_{33}\dot{\Omega}_3 + \left(I_{22}^{\text{cm}} - I_{11}^{\text{cm}}\right)\Omega_2\Omega_1 \end{bmatrix}$$

- Difficult to apply, since $\underline{\tau}$ in the rotating frame may be very complicated
- Often applied when $\underline{\tau} = \underline{0}$

Torque-free asymmetric top: free precession

Consider when $\underline{\tau} = \underline{0}$. For an asymetrical top, $I = I_{11}^{\text{cm}} = I_{22}^{\text{cm}} \neq I_{33}^{\text{cm}} = I_3$. Euler's equations tells us

$$\underline{0} = \begin{bmatrix} I\dot{\Omega}_1 + (I_3 - I)\,\Omega_3\Omega_2 \\ I\dot{\Omega}_2 + (I - I_3)\,\Omega_1\Omega_3 \\ I_3\dot{\Omega}_3 \end{bmatrix}$$

Since the time derivative of Ω_3 is 0, Ω_3 must be a constant. The two remaining equations are

$$\begin{aligned} 0 &= \frac{d\Omega_1}{dt} + \frac{\Omega_3}{I}(I_3 - I)\,\Omega_2 \\ &= \frac{d\Omega_2}{I} - \frac{\Omega_3}{I}(I_3 - I)\,\Omega_1 \end{aligned}$$

We define the **precession velocity** as

$$\Omega_{\text{pr}} = \frac{\Omega_3}{I}(I_3 - I)$$

So

$$\begin{aligned} \frac{d\Omega_1}{dt} &= -\Omega_{\text{pr}}\Omega_2 \\ \frac{d\Omega_2}{dt} &= +\Omega_{\text{pr}}\Omega_1 \end{aligned}$$

The symmetries of the equation suggests that Ω_1 and Ω_2 must be cosine and sines. We can verify that

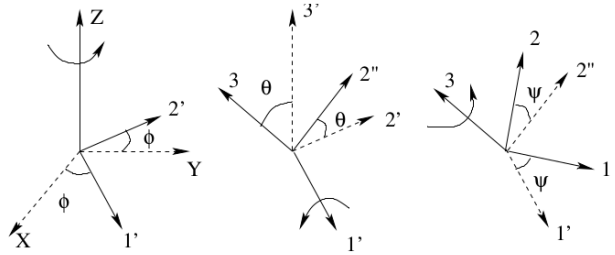
$$\begin{aligned} \Omega_1 &= A\cos(\Omega_{\text{pr}}t + \phi) \\ \Omega_2 &= B\sin(\Omega_{\text{pr}}t + \phi) \\ \Omega_3 &= \text{const} \end{aligned}$$

are solutions to Euler's equations.

In the body reference frame,

- the angular frequency vector "precesses" about the z-axis at a constant frequency Ω_{pr}
- $\underline{L}_{\text{cm}} = \underline{I}_{\text{cm}}\underline{\Omega}$ is also precessing in same plane as $\underline{\Omega}$, but they are not parallel (asymmetrical top)

Euler's angles



Are three angles, $\underline{\Theta} = (\phi, \theta, \psi)$ defined according to

1. Begin with the IRF and the body frame axes parallel to each other
2. Rotate the body frame about $\hat{\underline{z}}$ by ϕ
3. Rotate the body frame about $\underline{e}_{1'}$ by θ
4. Rotate the body frame about $\underline{e}_3$ by ψ

We can tell that

$$\underline{\Omega} = \dot{\phi}\hat{\underline{z}} + \dot{\theta}\underline{e}_{1'} + \dot{\psi}\underline{e}_3$$

We can find the projections of $\hat{\underline{z}}$, $\underline{e}_{1'}$, and $\underline{e}_3$ onto the IRF or body frame axes. In the IRF:

$$\underline{\Omega}_{XYZ} = \begin{bmatrix} \dot{\theta}\cos(\phi) + \dot{\psi}\sin(\theta)\sin(\phi) \\ \dot{\theta}\sin(\phi) - \dot{\psi}\sin(\theta)\cos(\phi) \\ \dot{\phi} + \dot{\psi}\cos(\theta) \end{bmatrix}$$

In the body frame:

$$\underline{\Omega}_{123} = \begin{bmatrix} \dot{\phi}\sin(\theta)\sin(\phi) + \dot{\theta}\cos(\psi) \\ \dot{\phi}\sin(\theta)\cos(\phi) - \dot{\theta}\sin(\psi) \\ \dot{\psi} + \dot{\phi}\cos(\theta) \end{bmatrix}$$

This allows us to express the rotational kinetic energy of the body (under the choice of principle axes)

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\underline{\Omega}^{\intercal}\underline{I}_{\text{cm}}\underline{\Omega} \\ &= \frac{1}{2}\left(I_{11}^{\text{cm}}\Omega_1^2 + I_{22}^{\text{cm}}\Omega_2 + I_{33}^{\text{cm}}\Omega_3^2\right) \end{aligned}$$

For a symmetric top $I_{11}^{\text{cm}} = I_{22}^{\text{cm}} = I \neq I_3$:

$$T_{\text{rot}} + \frac{I}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta)\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos(\theta)\right)^2$$

Noether's theorem

Rotational Invariance

Coordinate transformation can be **discrete** (such as mirror reflections, $x_i \to -x_i$) or **continuous** (such as rotations and translations).

Consider a particle of mass m in a central potential. The symmetry of the system allows us to rotate the IRF in any way we like without changing the form of the Lagrangian.

A rotation of a point by angle θ in the plane is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

For small angles, $\theta \approx \delta\theta$, the transformation becomes

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -\delta\theta \\ \delta\theta & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The Lagrangian is transformed under this rotation:

$$\mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to \mathcal{L}'(x', y', z, \dot{x}', \dot{y}', \dot{z}, t)$$

Explicitly, the transformed Lagrangian is

$$\mathcal{L}'(x - y\,\delta\theta, y + x\,\delta\theta, z, \dot{x} - \dot{y}\,\delta\theta, \dot{y} + \dot{x}\,\delta\theta, \dot{z}, t)$$

We can compute its Taylor series with respect to $\delta\theta$, about $\delta\theta = 0$. Let's first consider

$$\begin{aligned} \frac{\partial\mathcal{L}'}{\partial\delta\theta} &= \frac{\partial\mathcal{L}}{\partial(x - y\,\delta\theta)}(-y) + \frac{\partial\mathcal{L}}{\partial(y + x\,\delta\theta)}(x) \\ &+ \frac{\partial\mathcal{L}}{\partial(\dot{x} - \dot{y}\,\delta\theta)}(-\dot{y}) + \frac{\partial\mathcal{L}}{\partial(\dot{y} + \dot{x}\,\delta\theta)}(\dot{x}) \end{aligned}$$

Evaluating at $\delta\theta = 0$:

$$\left(\frac{\partial\mathcal{L}'}{\partial\delta\theta}\right)_{\delta\theta=0} = -y\frac{\partial\mathcal{L}}{\partial x} + x\frac{\partial\mathcal{L}}{\partial y} - \dot{y}\frac{\partial\mathcal{L}}{\partial\dot{x}} + \dot{x}\frac{\partial\mathcal{L}}{\partial\dot{y}}$$

So the Taylor expansion is

$$\mathcal{L}' = \mathcal{L} + \left[\left(x\frac{\partial\mathcal{L}}{\partial y} - y\frac{\partial\mathcal{L}}{\partial x}\right) + \left(\dot{x}\frac{\partial\mathcal{L}}{\partial\dot{y}} - \dot{y}\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)\right]\delta\theta + O(\delta\theta^2)$$

We expect that the Lagrangian for a particle in a central potential is invariant under planar rotation. This requires that

$$\mathcal{L} = \mathcal{L}'$$

This is only possible when the first order term in the expansion vanishes. Using the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$

we can substitute for $\partial\mathcal{L}/\partial y$ and $\partial\mathcal{L}/\partial x$:

$$\begin{aligned}\mathcal{L}' &= \mathcal{L} + \left[x\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - y\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \dot{y}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right]\delta\theta + O(\delta\theta^2)\\ &= \mathcal{L} + \left[\frac{d}{dt}\left(x\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{d}{dt}\left(y\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right]\delta\theta + O(\delta\theta^2)\\ &= \mathcal{L} + \frac{d}{dt}\left[x\frac{\partial \mathcal{L}}{\partial \dot{y}} - y\frac{\partial \mathcal{L}}{\partial \dot{x}}\right]\delta\theta + O(\delta\theta^2)\end{aligned}$$

By definition, $\partial\mathcal{L}/\partial\dot{q}_i$ is a generalized momentum. So the requirement for rotational Invariance about the z-axis leads to

$$\frac{d\ell_z}{dt} = \frac{d}{dt}\left(xp_y - yp_x\right) = 0$$

We can follow the same process for the X and Y axes, and find all the time derivative of $\underline{\ell}$ is 0. The invariance of a spherically symmetric Lagrangian under rotations the coordinate system about each of the Cartesian three axes for leads to the conservation of angular momentum.

Noether's theorem

Noether's theorem states that a "if a symmetry exists, then a corresponding constant of the motion exists".

We first need to find "a continuous family of transformations", that depend on continuous variables, such what when the variables are 0, recover the identity. If the Lagrangian is invariant under these transformations, then Noether's theorem can help us find the conserved quantities.

Consider a continuous symmetry transformation with n parameters, $\underline{\xi} = (\xi_1, \ldots, \xi_n)$, ξ_j, $j = 1, \ldots, n$. If $q_i(t)$ is a solution to the original Euler-Lagrange equations, then we denote

$$Q_i(\underline{\xi}, t)$$

to be the solution for transformed Lagrangian. Such that $Q_i(\underline{0}, t) = q_i(t)$.

The invariance of Lagrangian under this symmetry transformation requires

$$\mathcal{L}\left(\underline{q}, \underline{\dot{q}}, t\right) = \mathcal{L}' = \mathcal{L}\left(\underline{Q}(\underline{\xi}, t), \underline{\dot{Q}}(\underline{\xi}, t), t\right)$$

If this is true, then $\mathcal{L}'$ must not depend on the parameters $\underline{\xi}$. So

$$\begin{aligned}\frac{d}{d\xi_j}\mathcal{L}\left(\underline{Q}(\underline{\xi}, t), \underline{\dot{Q}}(\underline{\xi}, t), t\right) &= \sum_{i=1}^{s}\frac{\partial \mathcal{L}}{\partial Q_i}\frac{\partial Q_i}{\partial \xi_j} + \sum_{i=1}^{s}\frac{\partial \mathcal{L}}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial \xi_j}\\ &= 0\end{aligned}$$

Invoking the Euler-Lagrange equations, we can substitute

$$\frac{\partial \mathcal{L}}{\partial Q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}_i}\right)$$

And

$$\frac{\partial \dot{Q}}{\partial \xi_j} = \frac{d}{dt}\frac{dQ_i}{d\xi_j}$$

So we have

$$\begin{aligned}\frac{d}{d\xi_j}\mathcal{L}\left(\underline{Q}(\underline{\xi}, t), \underline{\dot{Q}}(\underline{\xi}, t), t\right) &= \sum_{i=1}^{s}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}_i}\right)\frac{\partial Q_i}{\partial \xi_j} + \sum_{i=1}^{s}\frac{\partial \mathcal{L}}{\partial \dot{Q}}\frac{d}{dt}\left(\frac{\partial Q}{\partial \xi_j}\right)\\ &= \frac{d}{dt}\left(\sum_{i=1}^{s}\frac{\partial \mathcal{L}}{\partial \dot{Q}_i}\frac{\partial Q_i}{\partial \xi_j}\right)\\ &= 0\end{aligned}$$

Noether's theorem: If the Lagrangian is invariant under a continuous symmetry transformation, there are conserved quantities associated with that symmetry, one for each parameter of the transformation.

$$I_j(q, \dot{q}) = \left(\sum_{i=1}^{s} p_i\frac{dQ_i}{ds_j}\right)_{\xi_j = 1} = \text{const}$$

where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$.

Hamiltonian mechanics

The Hamiltonian formalism is even more flexible in the choice of coordinates than the Lagrangian formalism.

It is based on the Hamiltonian, $\mathcal{H}$, which so far equivalent to the total energy of the system.

The Hamiltonian formulation

- offers more flexibility in the choice of coordinates
- the Hamiltonian is a physically measurable quantity
- is widely used in quantum mechanics and other branches of modern physics
- opens possibility for more conservation laws
- results in a symmetric set of equations of q_i, p_i

The theory is reformulated by replacing $\dot{q}_i$ by **canonically conjugate momenta** (or generalized momenta). And the Lagrangian is replaced by Hamiltonian, $\mathcal{H}(q, p, t)$.

In the Lagrangian formalism, s generalized coordinates specifying a position in an s-dimensional **configuration space**. The $2s$ generalized coordinates and velocities define a point the **state space**.

The $2s$ generalized coordinates and momenta define a point in a $2s$-dimension **phase space**.

Legendre transformation

The Legendre transformation is a recipe for generating a function of a new variable, given the function for an initial variable.

Consider a "passive" variable, x, and an "active variable", y. The function $A(x, y)$ is explicitly known. We introduce a third variable, z, and define the function

$$B(x, y, z) = yz - A(x, y)$$

The total differential of B is

$$\begin{aligned}dB &= z\,dy + y\,dz - \frac{\partial A}{\partial x}\,dx - \frac{\partial A}{\partial y}\,dy\\ &= \left(z - \frac{\partial A}{\partial y}\right)dy + y\,dz - \frac{\partial A}{\partial x}\,dx\end{aligned}$$

If we now define z as a function of x and y,

$$z = z(x, y) = \frac{\partial A}{\partial y}$$

Then B is now a function of x, z alone, since small changes dz, dx alone cause a change dB.

$$dB = y\,dz - \frac{\partial A}{\partial x}\,dx$$

The total differential reveals that

$$\frac{\partial B}{\partial z} = y \qquad \frac{\partial B}{\partial x} = -\frac{\partial A}{\partial x}$$

To find B, we need to find invert the relation for $z(x, y)$, and solve for $y(x, z)$, and substitute into $B(x, y, z) \to B(x, y(x, z), z)$. Alternatively, if we know $B(x, z)$, then the z-partial derivative of B also gives y.

Given a function $A(x, y)$, by defining $z = \partial A/\partial y$, we can generate a new function $B(x, z)$.

The following example demonstrates the advantage of the Legendre transformation. Consider the function

$$A(x, y) = (1 + x^2)y^2$$

Following the procedure, we define a new variable $z(x, y)$

$$z = \frac{\partial A}{\partial y} = 2y(1 + x^2)$$

We isolate the active variable, y:

$$y = \frac{z}{2(1 + x^2)}$$

and substitute it back into $B(x, y, z)$:

$$\begin{aligned}B(x, z) &= \frac{z^2}{2(1 + x^2)} - (1 + x^2)\frac{z^2}{4(1 + x^2)^2}\\ &= \frac{z^2}{4(1 + x^2)}\end{aligned}$$

We can invert the transformation by using the partial derivative relations:

$$\frac{\partial B}{\partial z} = \frac{z}{2(1 + x^2)} = y$$

and

$$\frac{\partial B}{\partial x} = \frac{z^2}{4}(-1)(1+x^2)^{-2}(2x) = \frac{-z^2 x}{2(1+x^2)^2}$$
$$= \frac{-4y^2(1+x^2)^2 x}{2(1+x^2)^2} = -2y^2 x$$
$$= -\frac{\partial A}{\partial x}$$

Integrating with respect to x:

$$A = y^2 x^2 + C(y)$$

Since A is known, we can solve for $C(y)$ to recover A.

On the other hand, if we used an arbitrary substitution, $z = y$, then knowing

$$B(x,z) = yz - A(x,y)$$
$$= z^2 - (1+x^2)z^2$$
$$= x^2 z^2$$

There is no way to invert the transformation since information about $y(x,z)$ is lost after the substitution.

Why transform?

Starting the Lagrangian, $L(q,\dot{q})$ (omitting time dependence, without loss of generality), we want to generate a function $\mathcal{H}(q,p)$. The Legendre transformation tells us

$$\mathcal{H} = p\dot{q} - \mathcal{L}(q,\dot{q}) \qquad p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Instead of using $\dot{q}$, we use p, which is tangent to the Lagrangian because the principle of least action holds for independent variations of q and p, unlike variations of q and $\dot{q}$, which are not truly independent of each other in time. **(Proof to be considered.)**

The following is an alternative proof given in lecture.

A mechanical system can also be described by a set of generalized coordinates and generalized momenta. Consider the total differential of the Lagrangian $\mathcal{L}(q,\dot{q},t)$.

$$d\mathcal{L} = \sum_{i=1}^{s} \frac{\partial \mathcal{L}}{\partial q_i} dq_i + \sum_{i=1}^{s} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} dt$$

Recall that $\partial\mathcal{L}/\partial q_i$ is called a generalized momentum. So we can write

$$d\mathcal{L} = \sum_{i=1}^{s} \dot{p}_i\, dq_i + \sum_{i=1}^{s} p_i\, d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} dt$$

We introduce the following relation using the product rule, or a **Legendre transformation**

$$d(p_i\dot{q}_i) = p_i\, d\dot{q}_i + \dot{q}_i\, dp_i$$

Rearranging and substituting

$$d\mathcal{L} = \sum_{i=1}^{s} p_i\, dq_i + d\left(\sum_{i=1}^{s} p_i\,\dot{q}_i\right) - \sum_{i=1}^{s} \dot{q}_i\, dp_i + \frac{\partial \mathcal{L}}{\partial t} dt$$

We can group the total differentials

$$d\left(\sum_{i=1}^{s} p_i\dot{q}_i - \mathcal{L}\right) = -\sum_{i=1}^{s} p_i\, dq_i + \sum_{i=1}^{s} \dot{q}_i\, dp_i - \frac{\partial \mathcal{L}}{\partial t} dt$$

The Hamiltonian: given a choice of s generalized coordinates and s associated generalized momenta, the Hamiltonian is

$$\mathcal{H}(q,p,t) = \sum_{i=1}^{s} p_i\dot{q}_i - \mathcal{L}$$

We can see that the Hamiltonian is a function of q, p, and t. In the total differential of $\mathcal{H}$, we can see that a small change in $\mathcal{H}$ is equaled to sum of small changes in q_i, p_i and t.

$$d\mathcal{H} = -\sum_{i=1}^{s} p_i\, dq_i + \sum_{i=1}^{s} \dot{q}_i\, dp_i - \frac{\partial \mathcal{L}}{\partial t} dt$$
$$= \sum_{i=1}^{s} \frac{\partial \mathcal{H}}{\partial q_i} dq_i + \sum_{i=1}^{s} \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial t} dt$$

Equating the sums of different differentials give Hamiltonian's equations

Hamilton's equations: the Hamiltonian formalism gives $2s$ first order differential equations:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial p_i} \qquad \frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

Hamiltonian with no explicit time dependence is the conserved, total energy of the system.

The general steps to use the Hamiltonian formalism

1. Find the Lagrangian of the system
2. Compute the associated momentum canonically conjugate to each generalized coordinate: $\partial\mathcal{L}/\partial\dot{q}_i$
3. Solve for p_i in terms of q, $\dot{q}$
4. Eliminate all generalized velocities in the Hamiltonian

Momentum space

It's possible to construct a momentum space Lagrangian, $K(p,\dot{p},t)$, by performing a double Legendre transformation.

Theoretical mechanics

Canonical transformations are transformations from one set of canonical coordinates, q, and its conjugate momenta, p, to another conjugate set Q, P.

Under such a transformation, the form of the Hamiltonian changes, but the new set of Hamilton's equations may be easier to solve.

Recall that the Lagrangian is only defined up to an additive total time derivative of a function of the generalized coordinates and time. If two Lagrangian refer the same system if

$$\widetilde{\mathcal{L}}(Q,\dot{Q},t) = \mathcal{L}(q,\dot{q},t) - \frac{dF(q,Q,t)}{dt}$$

The function F can be used with any Lagrangian to generate a new, but equivalent description of a system.

Canonical transformation: For $F(\underline{q},\underline{Q},t)$ that is a function of s canonical coordinates, $\underline{q}$, s new coordinates, $\underline{Q}$, and possibly time, of the diagonal entries of the diagonalized Hessian matrix of $\underline{F}$ are non-zero, then

$$P = -\frac{\partial F}{\partial Q} \qquad p = \frac{\partial F}{\partial q}$$

and

$$\widetilde{H}(Q,P,t) = H(q(Q,P),p(Q,P),t) + \frac{\partial F(q(Q,P),Q,t)}{\partial t}$$