MECH 360

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Unit conversions

$$1 \text{ inch} = 0.0245 \text{ m}$$
$$1 \text{ ft} = 12 \text{ inch}$$
$$1 \text{ lb} = 4.448 \text{ N}$$
$$1 \text{ kip} = 1000 \text{ lb}$$
$$1 \text{ Psi} = 6894 \text{ N/m}^2$$

First moment of area

Use to determine the centroid, geometric center of an region Ω. The coordinate of the centroid:

$$\left(\frac{\int_\Omega x\,dA}{\int_\Omega dA}, \frac{\int_\Omega y\,dA}{\int_\Omega dA}\right)$$

Composite areas

Let $\tilde{x}$ and $\tilde{y}$ be the x and y distances to the centroid of each sub-area:

$$\left(\frac{\sum \tilde{x}A}{\sum A}, \frac{\sum \tilde{y}A}{\sum A}\right)$$

A area of no material can be considered to have negative area.

Second moment of area

I_x, I_y are first moments of area computed w.r.t. a horizontal and vertical axis, respectively.

$$I_x = \int_\Omega y^2\,dA \qquad I_y = \int_\Omega x^2\,dA$$

For square cross sections:

$$I_x = \frac{1}{12}bh^3 \qquad I_y = \frac{1}{12}hb^3$$

For circular cross sections:

$$I = \frac{\pi}{4}r^4$$

For a thin slice of area constant A (pull y^2 or x^2 out of integral):

$$I_x = y^2 A \qquad I_y = x^2 A$$

Perpendicular axis theorem

Moment of inertia w.r.t. the pole (z axis):

$$J = I_z = \int_\Omega r^2\,dA = I_x + I_y$$

Parallel axis theorem

Given the first moment of area $\overline{I}_{x'}$ about a centroidal axis x', the moment about a new axis $x = x' + d_y$ is:

$$I_x = \sum \overline{I}_{x'} + Ad_y^2$$

where A is the area of the region. Let $d = d_x^2 + d_y^2$,

$$J_0 = \overline{J}_C + Ad^2$$

Stress and strain

Hooke's law:

$$\sigma = E\epsilon$$

Where $\sigma = P/A$, and $\epsilon = \delta/L_0$.

When the cross-section of the beam changes along its length, we need to use

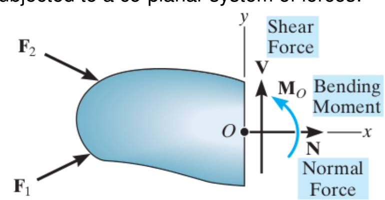
$$\epsilon = \frac{d\delta}{dx}$$

And

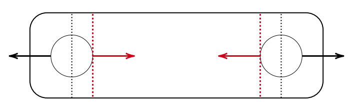
$$\delta = \int_0^L \frac{P}{EA(x)}\,dx$$

Coplanar loading

For a body subjected to a co-planar system of forces:



Use full cross section when link in compression, reduced cross section when link in tension.



Torsion

In linear elasticity, linear variation in shear strain, γ, leads to a linear variation in shear stress ($\tau = G\gamma$). For a uniform circular shaft, let ρ be distance from the shaft's axis:

$$\tau = \frac{T\rho}{J}$$

Symmetric beam bending

Neutral axis (NA) separates the compression and tension regions, and passes though the centroidal axis.
From the condition that the resultant force produced by the stress distribution over the cross section must be zero.
So for composite areas

$$\bar{y} = \frac{\sum \hat{y}A}{\sum A} \qquad \hat{y} = \text{distance to reference edge}$$

Flexure formula

Pure bending causes linear variation in strain, ϵ, and stress, σ. (Hooke's law: $\sigma = E\epsilon$)

$$\sigma = \frac{My}{I} \qquad y \text{ is the distance from the NA}$$

Composite beams

For a beam made of two materials with young's modulus E_1 and E_2, assume $E_1 > E_2$. We define

$$n = E_1/E_2$$

1. Enlarge the longitudinal dimension of the E_1 region of and treat the body as made of E_2
2. Compute the location of NA for the homogeneous beam
3. Apply the flexure formula
4. Multiply the stress calculated in the transformed region by n to obtain the actual stress.

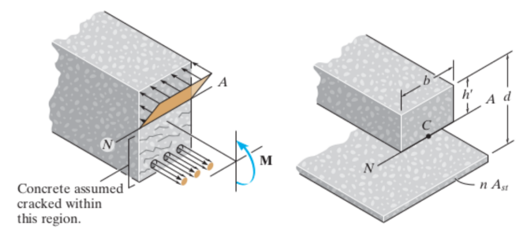
Steel reinforced concrete

Concrete is ≈ 12.5 times better in compression than in tension. In tension, is susceptible to cracking. So engineers reinforce concrete beams by placing steel rods longitudinal with the beam, far away from the Neutral axis.

1. Assume concrete does not act in tension
2. Transform steel area into concrete
3. Determine location of NA

$$\overline{y} = 0 = \frac{\sum \hat{y}A}{\sum A} \implies 0 = \sum \hat{y}A$$

from moment equilibrium (what is the formula when multiplied with $\sigma(y)$?)



Plastic bending

Given σ_Y, determine M_Y, M_P. For rectangular beams:

$$\text{Shape factor} = \frac{M_P}{M_Y}$$

The choice of the magnitude of shape factor depends the failure criterion.
Fully plastic bending, let A_c in area of beam in compression, and A_t be the area of the beam in tension. From force equilibrium,

$$\sigma_Y A_c - \sigma_Y A_T = 0$$

Elastic-plastic bending, where y_Y is the height of elastic core about NA

1. Find the location of resultant forces in the elastic and plastic regions
2. The applied moment must be in equilibrium with moment due to resultant forces about NA

Residual stress is zero at NA and two other points where the unloading stress cancels with the elastic plastic normal stress

Residual stress

Are stresses that remain in the material after the original cause of stress has been removed.

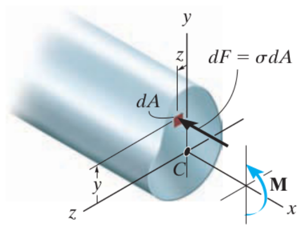
$$\sigma_R(y) = \sigma_{EP}(y) + \sigma_E(y),$$

where σ_R is the residual stress as a function of distance away from NA, σ_{EP} is the stress distribution in elastic plastic bending given some moment M, and σ_E is the elastic unloading of the member by $-M$

Asymmetric beam bending

Setting the origin of our coordinate system at the centroid of the cross section, we require the following for a general beam bending base

$$\begin{aligned} &\sum F && 0 = \int_\Omega \sigma \, dA \\ &\sum M_y && 0 = \int_\Omega z\sigma \, dA \\ &\sum M_z && M = \int_\Omega y\sigma \, dA \end{aligned}$$



Our assumption is that stress varies linearly from 0 at NA to σ_{max} at $y = c$. So $\sigma = -(y/c)\sigma_{\text{max}}$. Substituting this equation into the third condition gives the flexure formula. Substituting the flexure formula into the the second condition requires

$$I_{yz} = \int_\Omega yz \, dA = 0$$

We need to find an orientation of y and z axes such that the product of inertia is zero.

> The I_{yz} is positive upwards in the Mohr circle, as opposed downwards for τ_{xy}

1. Find I_y, I_z, I_{yz}. Compute $I_{avg} = (I_z + I_y)/2$
2. Connect a reference point (I_y, I_{yz}) to $(I_{avg}, 0)$
3. Find the angle θ to rotate the reference point counterclockwise so $I_{yz} = 0$
4. If I_y, I_{yz} was rotated to location of I_{min}, then assign I_{min} to I_y
5. Find the projection of M onto the principal axes
6. The location of maximum stress are furthest from the NA

For a rectangular region of area A with centroid located y_c and z_c from origin

$$I_{yz} = y_c z_c A$$

Transformation equations

Placing the origin at the centroid, the second moment of area and product of inertia as a function of angle θ measured ccw from the $+y$ axis:

$$\begin{aligned} I_{y'} &= \frac{I_y + I_z}{2} + \frac{I_y - I_z}{2}\cos(2\theta) + I_{yz}\sin(2\theta) \\ I_{z'} &= \frac{I_y + I_z}{2} - \frac{I_y - I_z}{2}\cos(2\theta) - I_{yz}\sin(2\theta) \\ I_{y'z'} &= \frac{I_y - I_z}{2}\sin(2\theta) + I_{yz}\cos(2\theta) \end{aligned}$$

This leads to a second moment of area mohr circle.

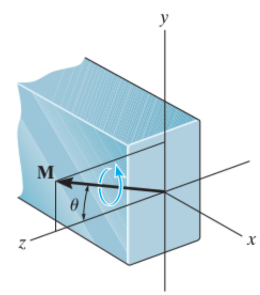
Rotation matrix

A point (y, z) is mapped to (y', z') by rotating the coordinate axes:

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},$$

θ is measured from the y axis.

Location of neutral axis



In the **transformed** axis, normal stress at any point due to $M_z = M\cos(\theta)$, $M_y = M\sin(\theta)$:

$$\sigma(z, y) = \frac{-M_z y}{I_z} + \frac{M_y z}{I_y}$$

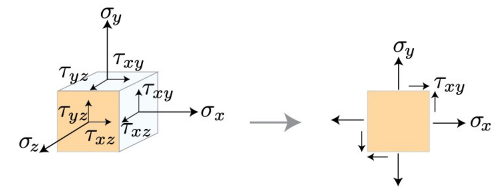
> The NA is at an angle ϕ from the z' axis
>
> $$\tan(\phi) = \frac{I_z}{I_y}\tan(\theta) = \frac{y}{z}$$
>
> θ the angle M makes with z', and is positive as measured in the figure above.

This comes from

$$\begin{aligned} 0 &= \frac{-M_z y}{I_z} + \frac{M_y z}{I_y} \\ &= \frac{-M\cos(\theta) y}{I_z} + \frac{M\sin(\theta) z}{I_y} \\ \implies y &= \frac{I_z}{I_y}\frac{\sin(\theta)}{\cos(\theta)} z \end{aligned}$$

2D Stress transformation

In 2-D case, general plane stress at a point is a combination of two component of normal stress, σ_x, σ_y, and one component of shear stress τ_{xy} shown in the following:



We can apply this to analyse surface elements in torsional loading.

Sign convention

Shear stress on some face is + if it rotates the element counter clockwise. Normal stress is + if tensile, − if compressive. Rotation by θ is positive counter clockwise.

Direct approach

Apply the equilibrium equations to x' and y' axis. Since area changes, we have to multiply each stress by the corresponding area of the face they act on before balancing them.

Transformation equations

$$\begin{aligned} \sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) \\ \sigma_{y'} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta) \\ \tau_{x'y'} &= -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) \end{aligned}$$

Principal stresses

If a material is prone ot fail by tensile cracking, it fail by cracking along the principal planes when the maximum principal stress exceeds *tensile strength*.

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

Principal stresses are $90°$ apart on plane diagram, and $180°$ apart on the Mohr circle:

$$\sigma_1, \sigma_2 = \sigma_{avg} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Planes of max and min normal stress:

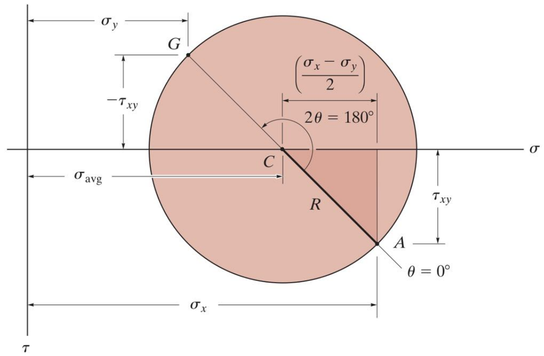
$$\tan(2\theta_p) = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

Maximum in plane shear stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum shear stress angle: θ_s and θ_p are $45°$ apart.

Mohr Circle



$$(\sigma_{x'} - \sigma_{avg})^2 + \tau_{x'y'}^2 = \tau_{max}^2$$

1. σ is positive to the right, τ is positive downwards
2. Plot $(\sigma_{avg}, 0)$; this is the origin of the Mohr circle
3. Plot (σ_x, τ_{xy}); draw a line between this point and the origin. This is represents $\theta = 0$.

3D Stress transformation

The stress matrix is symmetric:

$$\underline{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

The resultant stress on a plane with unit normal $\underline{n}$ is

$$\underline{s} = \underline{\sigma n}$$

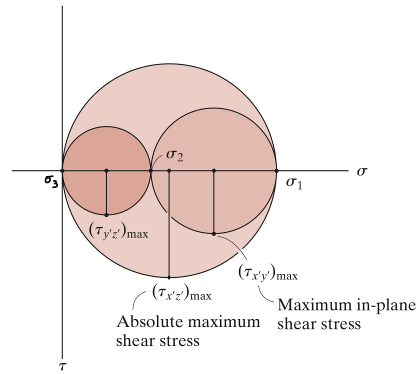
$\underline{n}$ can be specified by direction cosines.

$$\|n\| \cos(\alpha) = n_1$$
$$\|n\| \cos(\beta) = n_2$$
$$\|n\| \cos(\gamma) = n_3$$

$$\cos(\alpha) + \cos(\beta) + \cos(\gamma) = 1$$

> The principal stresses and direction of principal planes are given by the eigenvalues and eigenvectors of $\underline{\sigma}$. Principal planes are shear free.

Mohr Circle



Thin-walled pressure vessels

Cylindrical and spherical vessels are used in industry to serve as boilers or tanks. This analysis applies to vessels satisfying

$$r/t \geq 10,$$

where r is the **inner radius**, and t is the wall thickness.

Cylindrical vessel

For a gauge pressure (pressure above the atmospheric pressure), p, the hoop stress on an element is

$$\sigma_\theta = \frac{pr}{t}$$

The longitudinal stress is half of the hoop stress

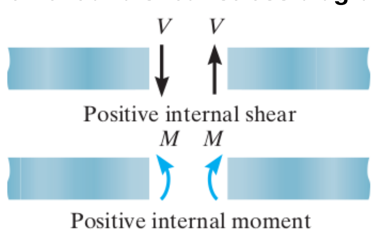
$$\sigma_a = \frac{pr}{2t}$$

So longitudinal joints need to be able to handle twice the stress than hoop joints.

Spherical vessel

$$\sigma_\theta = \sigma_a = \frac{pr}{2t}$$

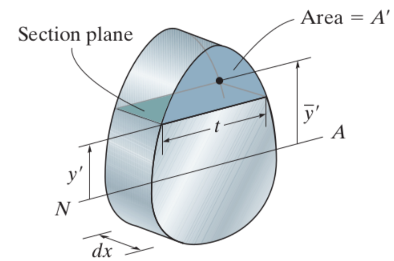
Bending moment and shear stress diagrams



1. Internal shear force is positive if it causes a clockwise rotation of the segment (the end with no shear force vector is considered to be a fixed end)
2. Bending moment is positive if it causes the top fibers to be compressed

Shearing stress in beam bending

While normal stresses dominate in the design of beams, shearing stress becomes dominant when the beam gets shorter and stubbier.



The shearing stress at a distance y' from the NA:

$$\tau = \frac{V(x)Q(y')}{It(y')}$$

- I is the second moment of area with respect to the NA
- $t(y')$ is the thickness of the cross section at y'
- $Q(y')$ is the first moment of area of region A' of the cross section whose centroid is located at $\bar{y}'$
- $V(x)$ is the transverse loading at a distance x away from a chosen end of the beam

Since shear stress is always coupled we also obtain the vertical shear stress at y'.

> $$Q = \bar{y}'A' = \sum A_i'\bar{y}_i'$$
>
> When A' can be broken down into i pieces each with area A_i' and $\bar{y}_i'$.

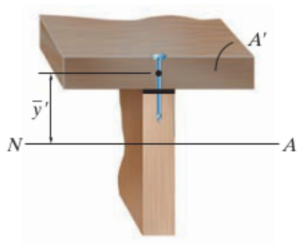
Limitations of the shear formula

Our formula assumes that the shear stress is uniformly distributed over the width t. In reality, the shear stress on the sides of the cross section depending on b/h (base divided by height). The formula is inaccurate for

- cross section with large b/h
- points of sudden cross sectional change (there would be a stress concentration there)
- points on an inclined boundary

Shear flow in built-up members

Components are often fastened to each other using nails, bolts, welding material, or glue. The design of these fasteners requires knowing the shear force that must be resisted by the member.



The shear force per unit length of the beam (shear flow):

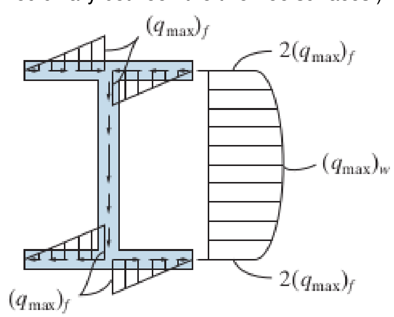
$$q = \frac{VQ}{I} = \frac{nF_{\text{nail}}}{\Delta x}$$

- Q is the first moment of area for region A'. It is the segment of that is connected to the beam at the "juncture" where the shear flow is to be calculated
- V is the resultant internal shear force
- I is the second moment of area for the entire cross section
- q changes with the y coordinate, since Q changes.
- n is the number of nails distributed along every Δx along the longitudinal axis of the beam
- F_{nail} is the longitudinal shear force on each nail

With sufficient symmetry, surfaces of zero shear stress can be treated as free surfaces.

Shear flow in thin-walled members

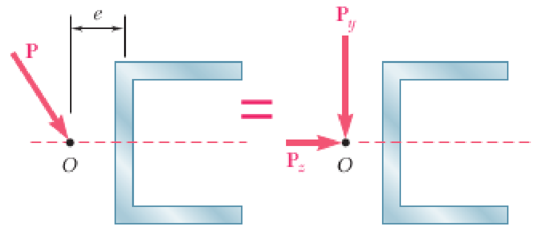
We neglect the vertical variation in shear stress. (Consider an I beam. The flanges have two horizontal free surfaces and the vertical shear stress would vary between the two free surfaces.)



- Thin-wall: the thickness of the member is small compared to its height and width
- q varies linearly along segments $\perp$ to the the direction of V
- q varies parabolically along segments inclined or parallel to V
- q is $\parallel$ to the walls of the member
- The integral of q along its path should balance the load V applied

Shear center

In the absence of a vertical plane of symmetry, we cannot directly apply the shear formula. An additional couple moment that twists the beam about the centroid will be induced.



When the applied force is perpendicular to the axis of symmetry, there exists a point for which the applied load produces an equal and oppose moment so that the beam does not twist.

Beam deflection

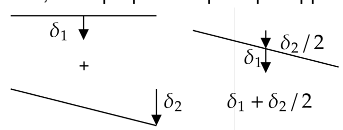
Euler-Bernoulli beam:

$$\frac{d^2y}{dx^2} - \frac{M(x)}{EI} = 0$$

y is the displacement field. The first derivative of y is the slope of the deflection

$$\theta = \frac{dy}{dx}$$

This is a linear ODE, so superposition principle applies.



Indeterminant beams

1. Determine the redundant reactions; pick ones where we can easily get the deflection or slope at
2. Decompose the loading as a superposition of difference loadings
3. Determine the deflection and slope at the location of the redundant reactions
4. Solve for the reactions using the slope and deflection equations

Boundary conditions

Simple supports

$$y(a) = 0$$

Cantilever beam:

$$y(a) = 0 \qquad \theta(a) = 0$$

Macaulay Angle Bracket/Singularity functions

$$\langle x-a\rangle^n = \begin{cases} (x-a)^n & x \ge a \\ 0 & \text{otherwise} \end{cases}$$

Relation to Dirac delta

$$\int_0^x \delta(x-a) = \langle x-a\rangle^0$$

Integration

$$\int_0^x \langle x-a\rangle^n \, dx = \frac{1}{n+1}\langle x-a\rangle^{n+1}$$

Using this allows us to write out the distributed loads, shear forces, and bending moments along a beam as a function of x. Recall that

$$\frac{dV}{dx} = w \qquad \frac{dM}{dx} = V$$

Follow the internal moment convention when writing out the moment equations

Energy methods

External work and strain energy

Work of a force

Strain energy is area under the curve in a load-deformation graph:

$$U_e = \int_0^\Delta P\,dx$$

In the elastic region,

$$U = \frac{1}{2}Px$$

If P is already applied to the bar, and another load displaces the body by Δ', P does additional work equal to

$$U_e' = P\Delta'$$

Work of a concentrate couple

$\underline{M}$ does work when the body undergoes an angular displacement $d\theta$ along its line of action.

External work done by a couple moment

$$U_e = \int_0^\theta M \, d\theta$$

Linear elasticity:

$$U_e = \frac{M\theta}{2}$$

If a couple moment is already applied, and another load further rotates the body by θ', M does additional work equal to

$$U_e' = M\theta'$$

Elastic strain energy for different types of loading

Axial load

We can define the strain energy density, u,

$$u = \frac{U}{V} = \int_0^{x_1} \frac{P}{A}\frac{dx}{L} = \int_0^{\epsilon_1} \sigma_x \, d\epsilon_x$$

In the elastic region, we can apply Hooke's Law, $\sigma_x = E\epsilon_x$

$$u = \int_0^{\epsilon_1} E\epsilon_x \, d\epsilon_x = \frac{E\epsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

Strain energy is the integral of u over a volume.

$$U = \int u \, d\tau$$

If we are only considering elastic deformations,

$$U = \int \frac{\sigma_x^2}{2E} d\tau$$

Under an axial load, P, we can write $\sigma_x = P/A$. This gives

$$U = \int_0^L \frac{P^2}{2AE} \, dx$$

Bending moment/transverse load

We can apply the flexure formula, to $U = \int \frac{\sigma_x^2}{2E} d\tau$. This gives

Strain energy under bending moment

$$U = \int_0^L \frac{M^2(x)}{2EI} \, dx$$

- This formula neglects the effect of shearing stress. See Beer Example 11.05

Conservation of energy

An external elastic load does mechanical work on a body equal to the strain energy stored in the body. When the load is removed strain energy restores the body to its original shape.

$$U_e = U_{\text{internal}}$$

- When a single force/moment is applied, we can determine the deformation of the body in the direction of the load

Castigliano's second theorem

Assumptions:

- Constant temperature
- Linear elasticity

Strain energy under multiple loads

1. Consider each load as being applied sequentially
2. Compute the total work done by applying the loads

Mathematically, consider n loads, $P_1, \ldots P_n$. The deflection of the point of application of P_j along the *line of action of* P_j is

$$x_j = \sum_k a_{jk} P_k$$

The strain energy of the structure is

$$U = \frac{1}{2}\sum_i \sum_k \alpha_{ik} P_i P_k$$

We can show that

$$\frac{\partial U}{\partial P_j} = x_j$$

this is *Castigliano's theorem*.

$$x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_j} \, dx$$

$$\theta = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M'} \, dx$$

To determine the deflection of a joint in the direction of a force P at the joint:

$$y = \sum_i \frac{N_i}{AE}\left(L_i \frac{\partial N_i}{\partial P}\right)$$

1. Find the support reactions first
2. Consider each joint, and find all internal forces
3. Applying Castigliano's theorem