Note on Hill's Equations and Coadjoint Orbits of the Virasoro-Bott group

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Abstract

A large class of Floquet systems are described by the Hill's equation, including the parametric oscillator as one of the most famous example. Depending on the driving protocol, the dynamics can be in the non-heating (stable) phase or the heating (unstable) phase. Each Hill's equation can be identified with a vector in the dual space of the Virasoro algebra, which allows us to determine the nature of the dynamics given the driving protocol without solving the equation.

Contents

1 Introduction

A general type of classical Floquet systems are described by the following Hill's equations

$$\left(\partial_\theta^2 + u(\theta)\right)\psi(\theta) = 0\,, \quad u(\theta + 2\pi) = u(\theta)\,, \tag{1}$$

where θ represents the time and $u(\theta)$ controls the periodic driving force with a period 2π. Regardless of the initial conditions, it has two independent solutions $(\psi_1, \psi_2)^T$. Although the equation is periodic, the solution does not have to be. When θ changes by 2π, it is possible that $(\psi_1, \psi_2)^T$ picks up a linear transformation, which is called the monodromy

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(\theta + 2\pi) = M \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(\theta)\,. \tag{2}$$

Physically, the monodromy matrix directly determines the nature of the dynamics. For example, if M has one eigenvalue that is larger than 1, ψ_1 and ψ_2 will grows exponentially with θ, which corresponds to a heating/unstable dynamics. If both eigenvalues of M have unit magnitude, ψ_1 and ψ_2 will oscillate, which is the feature of a non-heating/stable dynamics.

One of the most famous example is the Mathieu oscillator, for which the driving force is

$$u(\theta) = \left(\frac{\nu}{2}\right)^2 + \beta_0 + 2\beta_2 \cos 2\theta\,. \tag{3}$$

The general solutions are given by the even and odd Mathieu functions, which explicitly shows that both scenario can happen by tuning parameters. However, $u(\theta)$ is usually complicated and having analytical solutions can be a formidable task. Is it still possible to infer whether the system is heating or not, a.k.a. solving the type of the monodromy matrix, for a given $u(\theta)$?

The answer is positive, however the clue comes from a seemingly unrelated problem, which is to classify the coadjoint orbit of the Virasoro-Bott group. Let us call the group of orientation-preserving diffeomorphisms by $\mathrm{Diff}(S^1)$. The Virasoro-Bott group is its central extension, denoted by $\widehat{\mathrm{Diff}}(S^1)$. We will show that each Hill's equation can be identified with a coadjoint vector of $\widehat{\mathrm{Diff}}(S^1)$ and the conjugacy class of the monodromy matrix is a diffeomorphism invariant which labels the coadjoint orbit. Consequently, as long as we can determine the coadjoint orbit, to which the Hill's equation corresponds, we can read out the type of monodromy and

$u(\theta)=\nu^2/4$	Coadjoint orbit	Monodromy	dynamics
$\nu \in \mathbb{R}$	$\widehat{\mathrm{Diff}}(S^1)/S^1$	Elliptic	non-heating
$\nu \in \mathbb{Z}$	$\widehat{\mathrm{Diff}}(S^1)/\,\mathrm{SL}^{(n)}(2,\mathbb{R})$	$\pm\mathbb{I}$	non-heating
$i\nu \in \mathbb{R}$	$\widehat{\mathrm{Diff}}(S^1)/S^1$	Hyperbolic	heating with rate $i\nu$

Table 1: Coadjoint orbit and the corresponding monodromy of the Hill's equation for a constant $u(\theta)$. This is the non-driving limit. We can find that the elliptic and hyperbolic monodromy corresponds to the same type of coadjoint orbit. The degeneracy will be lifted as we turn on the driving.

$u(\theta)$	$f(\theta)\partial_\theta$	Coadjoint orbit	Monodromy	dynamics
	no zeros	$\widehat{\mathrm{Diff}}(S^1)/S^1$	Elliptic	non-heating
	simple zeros	$\widehat{\mathrm{Diff}}(S^1)/T_{(n,\Delta)}$	Hyperbolic	heating with rate Δ
	double zeros	$\widehat{\mathrm{Diff}}(S^1)/T_{(n,\pm)}$	Parabolic	transition
		$\widehat{\mathrm{Diff}}(S^1)/\,\mathrm{SL}^{(n)}(2,\mathbb{R})$	$\pm\mathbb{I}$	non-heating

Table 2: Coadjoint orbit and the corresponding monodromy of the Hill's equation for a generic $u(\theta)$. $f(\theta)\partial_\theta$ is the generator of the stabilizer subgroup of the coadjoint vector $u(\theta)$. Accordingly, u can be determined by f as explained in Sec. 4.2

the nature of the dynamics immediately. Before going into details, let us first list the full correspondence below in Tab. 1 and Tab. 2. More detailed explanations are in the following sections. A large portion of our discussion follows or is inspired by [1–3].

For clarity, let us list the notation for all the objects used in this note. Some might be nonstandard and are chosen just for the convenience of this paper.

- Let G, H represent groups, with their group elements denoted by $g \in G$, $h \in H$.
- The notation for a Lie algebra is written in Gothic font, such as $\mathfrak{g}$. The elements are called adjoint vectors and denoted by $x, y, z \in \mathfrak{g}$.
- Use $\mathfrak{g}^*$ for the dual space of a Lie algebra $\mathfrak{g}$. The elements are called coadjoint vectors and denoted by $u, v, w \in \mathfrak{g}^*$.
- Adjoint representation of the group G is $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$; adjoint representation of the Lie algebra $\mathfrak{g}$ is $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$.
- Coadjoint representation of the group is $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$; Coadjoint representation of the Lie algebra is $\mathrm{ad}^*_x : \mathfrak{g}^* \to \mathfrak{g}^*$.
- The orientation-preserving diffeomorphisms of the circle S^1 form a group called $\mathrm{Diff}(S^1)$. Our definition excludes maps that cannot be smoothly connected to the identity map. Most of the time, we use $\varphi : S^1 \to S^1$ for a diffeomorphism.
- Let $\mathrm{Vect}(S^1)$ be the set of vector fields on S^1, which is also naturally a Lie algebra. It generates the $\mathrm{Diff}(S^1)$ defined above. We use $f(\theta)\partial_\theta$, $g(\theta)\partial_\theta$ or $h(\theta)\partial_\theta$ for a vector field.

- The dual space of $\mathrm{Vect}(S^1)$ is called $\mathrm{Vect}^*(S^1)$ and its elements are denoted by $u(\theta)(d\theta)^2$, $v(\theta)(d\theta)^2$ or $w(\theta)\partial_\theta$.
- The central extension of $\mathrm{Vect}(S^1)$ by $\mathbb{C}$ with the central charge c is the Virasoro algebra $\mathfrak{vir}$. Its dual space is denoted by $\mathfrak{vir}^*$. The corresponding central extension of $\mathrm{Diff}(S^1)$ is the Virasoro-Bott group $\widehat{\mathrm{Diff}}(S^1)$.

2 Preliminaries

In this section, we will review the Hill's equation, diffeomorphism of the circle $\mathrm{Diff}(S^1)$, its Lie algebra $\mathrm{Vect}(S^1)$. At the end of this section, we will give a brief discussion on the adjoint and coadjoint orbit just for $\mathrm{Diff}(S^1)$, which is a warm-up for the later discussion on the cases with central extension. The definitions of all mathematics terminologies can be found in Appendix. A.

2.1 Hill's equations

In this section, we review some basic facts of the following Hill's equation

$$\left(\partial_\theta^2 + u(\theta)\right)\psi(\theta) = 0\,, \quad u(\theta+2\pi) = u(\theta)\,. \tag{4}$$

$\mathcal{H}_\theta = \partial_\theta^2 + u(\theta)$ is called the Hill's operator. We focus on its monodromy matrix and covariance under diffeomorphism. The later indicates its relation to the story of coadjoint orbits, and the former will be shown to classify the orbits.

Monodromy As a second order differential equation, Eq. (4) has two independent solutions when there is no restriction on the boundary/initial condition, which are assembled into a column vector $(\psi_1,\psi_2)^T$. Although the equation itself is periodic in 2π, its solutions do not have to respect this periodicity. If $\psi_{j=1,2}(\theta+2\pi)$ is a linear superposition of $\psi_1(\theta)$ and $\psi_2(\theta)$, they still satisfy the Hill equation with $\mathcal{H}_\theta$. Therefore, the most generic case is

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(\theta+2\pi) = M\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(\theta)\,, \tag{5}$$

where M is called the *monodromy matrix*. As mentioned in Sec. 1, if we consider Eq. (4) as describing a Floquet system, the spectrum of M is directly related to the nature of the dynamics. The naive expectation is that $M \in \mathrm{GL}(2,\mathbb{R})$ only has to be an invertible matrix. The following fact constrains that M must be an $\mathrm{SL}(2,\mathbb{R})$ matrix.

Lemma 2.1. *Consider the ODE $(\partial_x^2 + u(x))\psi(x) = 0$ defined on $\mathbb{R}$. The Wronskian of its two independent solutions $(\psi_1,\psi_2)^T$ is a constant*

$$W[\psi_1,\psi_2] = \psi_1'\psi_2 - \psi_1\psi_2' = \mathrm{const}\ . \tag{6}$$

Proof One can check that the derivative of $W[\psi_1,\psi_2]$ is zero, which proves the statement. □

Theorem 2.1. *Consider the ODE* $(\partial_x^2 + u(x))\psi(x) = 0$ *defined on* $\mathbb{R}$ *and its two independent solutions* $(\psi_1, \psi_2)^T$. *The potential* $u(x)$ *is related two the ratio of two solutions through a Schwarzian as*

$$u(\theta) = \frac{1}{2}\{\eta, x\} = \frac{1}{2}(S\eta)(x)\,, \quad \eta = \psi_1/\psi_2\,. \tag{7}$$

Proof We first notice that the Wronskian is a constant. Let us introduce $\eta = \psi_1/\psi_2$ and we have

$$\eta' = \frac{\psi_1'\psi_2 - \psi_1\psi_2}{\psi_2^2} = -\frac{C}{\psi_2^2}\,. \tag{8}$$

$$\eta'' = 2C\frac{\psi_2'}{\psi_2^3}\,, \quad \eta''' = 2C\frac{\psi_2''\psi_2 - 3\psi_2'^2}{\psi_2^4}\,. \tag{9}$$

Recalling the definition of the Schwarzian derivative, we have

$$\{\eta, x\} = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2 = -2\frac{\psi_2''\psi_2 - 3\psi_2'^2}{\psi_2^2} - 6\frac{\psi_2'^2}{\psi_2^2} = -2\psi_2''/\psi_2 = 2u(x)\,. \tag{10}$$

This relation is locally correct and thus also holds when the system is defined on a circle.

□

Corollary 2.2. *Consider an ODE* $(\partial_\theta^2 + u(\theta))\psi(\theta) = 0$ *defined on the circle* $\theta \in [0, 2\pi]$, $u(\theta) = u(\theta + 2\pi)$. *Denote two independent solutions as* $(\psi_1, \psi_2)^T$. *The monodromy* M *is given by an* $\mathrm{SL}(2,\mathbb{R})$ *matrix.*

Proof If we introduce $\eta = \psi_1/\psi_2$, we know that $u(\theta) = \frac{1}{2}\{\eta, \theta\}$. The periodicity of $u(\theta)$ implies that $\eta(\theta + 2\pi)$ and $\eta(\theta)$ are the same up to a $\mathrm{SL}(2,\mathbb{R})$ transformation, which proves the corollary.

□

Given a $\mathcal{H}_\theta$, the choice of independent solutions is not unique and one can do an arbitrary invertible linear transformation

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to S\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\,, S \in \mathrm{SL}(2,\mathbb{R})\,, \tag{11}$$

where we fix the normalization such that S belongs to $\mathrm{SL}(2,\mathbb{R})$. Accordingly, the monodromy matrix will change by a similar transformation

$$M \to SMS^{-1}\,. \tag{12}$$

As a result, only the *conjugacy class of the monodromy matrix* M is well-defined for given a $\mathcal{H}_\theta$ and it is uniquely labeled by the trace of $\mathrm{Tr}\,M$. For the $\mathrm{SL}(2,\mathbb{R})$ there are three classes

1. Elliptic $(\mathrm{Tr}\,M)^2 < 4$. The two eigenvalues are pure phases that are complex conjugation of each other. This corresponds to the non-heating/stable dynamics.

2. Hyperbolic $(\mathrm{Tr}\,M)^2 > 4$. The two eigenvalues are real. One is larger than 1 and the other is smaller than 1. This corresponds to the heating/unstable dynamics.

3. Parabolic $(\mathrm{Tr}\, M)^2 = 4$. The two eigenvalues are both 1. This corresponds to the critical regime between the first two cases.

Covariance under diffeomorphism The Hill's equation is covariant under a diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ with the following transformation laws[1]

$$\begin{aligned} &\left(\partial_\theta^2 + \widetilde{u}(\theta)\right) \widetilde{\psi}(\theta) = 0\,, \\ \widetilde{u}(\theta) = \left(\frac{d\varphi}{d\theta}\right)^2 u(\varphi(\theta)) + \frac{1}{2} S(\varphi)\,, &\quad \widetilde{\psi}(\theta) = \left(\frac{d\varphi}{d\theta}\right)^{-1/2} \psi(\varphi(\theta))\,. \end{aligned} \tag{14}$$

We require

$$\varphi'(\theta) > 0, \quad \varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi\,, \tag{15}$$

to make sure the transformed equation being well-defined and also for some reasons that will be explained in the next section. The motivation for these transformation laws comes from the study of the classical Liouville equation. In that context, $u(\theta)$ is the (holomorphic) stress-energy tensor and ψ is a primary field so that these transformation laws become natural. (See Appendix. D for details) Let us verify the statement by a brute force calculation

$$\begin{aligned} \partial_\theta^2 \widetilde{\psi}(\theta) =& \partial_\theta^2 \left[\left(\frac{d\varphi}{d\theta}\right)^{-1/2}\right] \psi(\varphi(\theta)) + 2\partial_\theta \left[\left(\frac{d\varphi}{d\theta}\right)^{-1/2}\right] \partial_\theta \psi(\varphi(\theta)) + \left(\frac{d\varphi}{d\theta}\right)^{-1/2} \partial_\theta^2 \psi(\varphi(\theta)) \\ =& -\frac{1}{2\varphi'^{1/2}} S(\varphi)\psi(\varphi) - \frac{\varphi''}{\varphi'} \partial_\varphi \psi(\varphi) + \frac{1}{\varphi'^{1/2}} \partial_\theta^2 \psi(\varphi(\theta)) \\ =& -\frac{1}{2\varphi'^{1/2}} S(\varphi)\psi(\varphi) + \varphi'^{3/2} \partial_\varphi^2 \psi(\varphi)\,, \end{aligned}$$

in the second and last step we change ∂_θ to ∂_φ by the chain rule. Using the condition that $(\partial_\varphi^2 + u(\varphi))\psi(\varphi) = 0$, we see that Eq. (14) is satisfied.

Under the diffeomorphism, ψ is only multiplied by a prefactor $\varphi'^{-1/2}$ that is non-vanishing and periodic, which implies that the monodromy must be invariant. Namely, *the conjugacy class of monodromy is a diffeomorphism invariant*, which hence can be used to label the orbit of Hill's operators $W_{\mathcal{H}_\theta}$. Here $W_{\mathcal{H}_\theta}$ denotes the collection of Hill's operators that are connected by diffeomorphism.

2.2 Definition of $\mathrm{Diff}(S^1)$, $\mathrm{Vect}(S^1)$ and $\mathrm{Vect}^*(S^1)$

Diffeomorphism group of S^1 Let us first consider what kind of maps are diffeomorphisms. Notice that all the smooth maps $\varphi : S^1 \to S^1$ are classified by the fundamental group $\pi_1(S^1) = \mathbb{Z}$. Not all of them are qualified for a diffeomorphism. For example, a trivial map $\varphi(\theta) = 0$ or a map with winding number equal to 2, like $\varphi(\theta) = 2\theta$, don't have an inverse and cannot be a

[1] To verify these laws, one can check an infinitesimal transformation $\varphi(\theta) = \theta + \epsilon f(\theta)$, for which we have

$$a\partial_\theta^2 \widetilde{\psi}(\theta) \approx a\psi'' + \epsilon(-\frac{3}{2} f' u\psi - f(u\psi)' - \frac{a}{2}\psi f''')\,,\ \widetilde{b}\widetilde{\psi} \approx u\psi + \epsilon(\frac{3}{2} f' u\psi + f(u\psi)' + \frac{a}{2}\epsilon f'''\psi) \tag{13}$$

where $a = t/12\pi$ and we have used $a\psi'' + u\psi = 0$. It is clear that $a\widetilde{\psi}'' + \widetilde{b}\widetilde{\psi} = 0$.

diffeomorphism. If we only consider those diffeomorphisms that can be smoothly connected to the identity map, then it requires them to be in the $\nu = 1$ homotopy sector and preserves the orientation (since the identity map does). They form a group and is denoted as $\mathrm{Diff}(S^1)$,

$$\mathrm{Diff}(S^1) = \left\{\varphi : S^1 \to S^1 \middle| \varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi \,, \varphi'(\theta) > 0\right\} \,. \tag{16}$$

The group multiplication is given by the composition of maps and the inverse is guaranteed by the definition of diffeomorphism.

Associated Lie algebra By considering an infinitesimal diffeomorphism, we get the corresponding Lie algebra $\mathrm{Vect}(S^1)$, which consists of all the smooth vector fields on S^1,

$$\mathrm{Vect}(S^1) = \{f(\theta)\partial_\theta \,|\, f(\theta + 2\pi) = f(\theta)\} \,. \tag{17}$$

However, not every diffeomorphism in $\mathrm{Diff}(S^1)$ can be generated by a single element in $\mathrm{Vect}(S^1)$. A more detailed discussion on the relation between them can be found in Appendix. B.1. The Lie bracket is inherited from the Lie bracket of vector fields

$$[f(\theta)\partial_\theta, g(\theta)\partial_\theta] = (fg' - f'g)\partial_\theta \,. \tag{18}$$

The adjoint representation is given by the way that a diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$ acts on a vector field $f(\theta)\partial_\theta$ (see Appendix. B.2 for details)

$$\mathrm{Ad}_{\varphi^{-1}} : f(\theta)\partial_\theta \mapsto f(\varphi(\theta))\partial_\varphi = f(\varphi(\theta))\left(\frac{d\varphi}{d\theta}\right)^{-1}\partial_\theta \,. \tag{19}$$

Notice that Eq. (19) uses φ^{-1} to make the formula look nicer. If we assume $\varphi = \theta + \epsilon f(\theta)$ to be an infinitesimal transformation and expand $\mathrm{Ad}_{\theta+\epsilon f}[g\partial_\theta]$ with respect to ϵ, the linear order term tells us the adjoint representation of $\mathrm{Vect}(S^1)$

$$\mathrm{ad}_{f\partial_\theta} : g(\theta)\partial_\theta \mapsto -[f(\theta)\partial_\theta,\, g(\theta)\partial_\theta] \,. \tag{20}$$

As expected, it is related to the Lie bracket but off by a minus sign. [This is different from the finite dimensional group, where the adjoint representation is the same as the Lie bracket.]

It is convenient to consider the complexified version of $\mathrm{Vect}(S^1)$, in which any element can be expanded into a Fourier series

$$f(\theta)\partial_\theta = \sum_{n\in\mathbb{Z}} f_n e^{in\theta}\partial_\theta \,. \tag{21}$$

Thus $L_n = ie^{in\theta}\partial_\theta$ form a complete basis for $\mathrm{Vect}_{\mathbb{C}}(S^1)$, which has the following commutator

$$[L_m, L_n] = (m - n)L_{m+n} \,, \tag{22}$$

This is called the *Witt algebra* and naturally arises when considering the conformal transformation of the complex plane.

The dual vector space Each vector space can have a dual vector space formed by all the linear functions of it. The dual space of $\mathrm{Vect}(S^1)$ consists of the quadratic differentials

$$\mathrm{Vect}^*(S^1) = \left\{u(\theta)(d\theta)^2 \,\middle|\, u(\theta+2\pi) = b(\theta)\right\}. \tag{23}$$

such that there is a natural pairing between them defined by

$$\langle *, * \rangle : \mathrm{Vect}^*(S^1) \times \mathrm{Vect}(S^1) \mapsto \mathbb{R}\,, \quad \langle u(\theta)(d\theta)^2\,,\, f(\theta)\partial_\theta \rangle = \int_0^{2\pi} d\theta\, u(\theta) f(\theta)\,. \tag{24}$$

The asymmetry of the definitions of $\mathrm{Vect}(S^1)$ and its dual $\mathrm{Vect}^*(S^1)$ implies that there is no (at least obvious) ismorphism between them, which will be more clear from the coadjoint representation. Following the definition of the coadjoint action we have

$$\begin{aligned}
\langle \mathrm{Ad}^*_{\varphi^{-1}}\, u(\theta)(d\theta)^2\,,\, u(\theta)\partial_\theta \rangle &:= \langle b(\theta)(d\theta)^2\,,\, \mathrm{Ad}_\varphi\, u(\theta)\partial_\theta \rangle \\
&= \int_0^{2\pi} d\theta\, b(\theta) u(\varphi^{-1}(\theta)) \left(\frac{d\varphi^{-1}}{d\theta}\right)^{-1} \\
&= \int_0^{2\pi} d\theta\, b(\theta) u(\varphi^{-1}(\theta)) \varphi'(\varphi^{-1}(\theta)) \\
&= \int_0^{2\pi} d\theta\, (\varphi'(\theta))^2 b(\varphi(\theta)) u(\theta)
\end{aligned} \tag{25}$$

In the third step, we use the identity $\frac{d}{d\theta}\varphi(\varphi^{-1}(\theta)) = \varphi'(\varphi^{-1}(\theta))\frac{d\varphi^{-1}}{d\theta} = 1$. In the last step, we substitute $\theta = \varphi(x)$ and rename x by θ. Therefore the coadjoint action takes the following natural form

$$\mathrm{Ad}^*_{\varphi^{-1}} : u(\theta)(d\theta)^2 \to u(\varphi(\theta))(d\varphi)^2 \to \left(\frac{d\varphi}{d\theta}\right)^2 u(\varphi(\theta))(d\theta)^2\,. \tag{26}$$

The difference between Eq. (19) and Eq. (26) directly implies that the adjoint and coadjoint representations are not isomorphic. This is in contrast to having a finite dimensional group/algebra, for which the coadjoint and adjoint representations are always isomorphic due to the existence of a Killing form. See Appendix. A for some concrete examples. An infinitesimal transformation $\varphi(\theta) = \theta + \epsilon f(\theta)$ yields the coadjoint representation of $\mathrm{Vect}(S^1)$

$$\mathrm{ad}^*_{f\partial_\theta} : u(\theta)(d\theta)^2 \mapsto -\left(2f'(\theta)u(\theta) + f(\theta)u'(\theta)\right)(d\theta)^2\,. \tag{27}$$

2.3 Warm up: adjoint and coadjoint orbits for $\mathrm{Diff}(S^1)$

As a warm up, we discuss the adjoint and coadjoint orbits for $\mathrm{Diff}(S^1)$ without the central extension. Let us start with the coadjoint orbit. Given a coadjoint vector $u(\theta)(d\theta)^2$, the orbit W_u will be isomorphic to the coset $\mathrm{Diff}(S^1)/H$, where $H \in \mathrm{Diff}(S^1)$ is the subgroup that leaves u invariant. An equivalent problem of solving W_u is thus to determine H. H is generated by vectors f satisfying

$$\mathrm{ad}^*_{f\partial_\theta}\left[u(d\theta)^2\right] = -(2f'u + fu') = 0 \quad \Rightarrow \quad f = u^{-1/2}\,. \tag{28}$$

Depending on whether $u(\theta)$ has zeros, there are two scenarios:

1. $u(\theta)$ has no zeros and thus $f = u^{-1/2}$ is not singular. Without loss of generality, let us assume $u(\theta) > 0$ and there is an associated $\mathrm{Diff}(S^1)$ invariant

$$u_0 = \frac{1}{2\pi}\int_0^{2\pi} d\theta \sqrt{u(\theta)}\,. \tag{29}$$

   Any such $u(\theta)$ can be set to a constant field $u(\theta) = u_0$ by a $\mathrm{Diff}(S^1)$ transformation. To do so, let us define σ by

$$\frac{d\sigma}{d\theta} = \sqrt{\frac{u(\theta)}{b_0}} > 0 \Rightarrow \sigma(\theta) = \int_0^\theta \sqrt{\frac{u(\theta)}{u_0}}\,, \tag{30}$$

   which shows that σ is monotonically increasing, $\sigma(\theta+2\pi) = \sigma(\theta)+2\pi$ and is a diffeomorphism that sets $u(\theta) = u_0$. Therefore the orbit of any positive definite $u(\theta)$ is the same as that of a certain constant u_0.

   For a constant u_0, only the rigid rotations of S^1 leave it invariant. We denote this subgroup by S^1 and identify W_{u_0} as $\mathrm{Diff}(S^1)/S_1$.

2. $u(\theta)$ has zeros and thus there is no non-zero vector field that leaves $u(\theta)$ invariant. The orbit W_u is then isomorphic to the whole group $\mathrm{Diff}(S^1)$.[2] In this case, there are also diffeomorphism invariants, which are

$$a_k = \int_{\theta_{k-1}}^{\theta_k} \sqrt{|u(\theta)|}d\theta\,, \quad u(\theta_k) = 0\,. \tag{31}$$

In summary, for positive/negative definite $b(\theta)$, $W_b \simeq \mathrm{Diff}(S^1)/S^1$ is labeled by a single diffeomorphism invariant b_0. For $b(\theta)$ that has zeros, $W_b \simeq \mathrm{Diff}(S^1)$ is labeled by two set of diffeomorphism invariants, the number of zeros and $\{a_k\}$.

The discussion on the adjoint orbits is similar. Let us first consider the subgroup H that leaves a vector field invariant. Given $u(\theta)$, such kind of subgroup is generated by $f(\theta)$ that satisfies

$$\mathrm{ad}_{f\partial_\theta}\,[g\partial_\theta] = -(g'f - gf') = 0 \Rightarrow f = g\,. \tag{32}$$

It follows that any vector field $g\partial_\theta$ is only invariant under the one-parameter subgroup generated by itself $H_{g\partial_\theta}$. Therefore the adjoint orbit is $W_g \simeq \mathrm{Diff}(S^1)/H_{g\partial_\theta}$.

3 Virasoro-Bott group, coadjoint representation and Hill's equation

In this section, we introduce the Virasoro-Bott group as the central extension of $\mathrm{Diff}(S^1)$. After showing its coadjoint representation, we will demonstrate that each coadjoint vector can be identified with a Hill's equation, which implies a connection between the classification of coadjoint orbits and the monodromy of Hill's equation.

[2] The coadjoint orbit of a group G being isomorphic to the whole group is unusual. It can never happen when there is an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$, because each element in $\mathfrak{g}^*$ is at least invariant under one element, namely itself.

3.1 Central extension of $\mathrm{Diff}(S^1)$, $\mathrm{Vect}(S^1)$ and $\mathrm{Vect}^*(S^1)$

In general, the central extension is not unique and is classified by the second cohomology. However, the central extension of $\mathrm{Diff}(S^1)$ and $\mathrm{Vect}(S^1)$ by $\mathbb{C}$ is completely fixed up to a coboundary. They are called Virasoro-Bott group $\widehat{\mathrm{Diff}}(S^1)$ and Virasoro algebra $\widehat{\mathrm{Vect}}(S^1) = \mathfrak{vir}$ respectively. In the following, we begin by considering the central extension of $\mathrm{Vect}(S^1)$ and its lift to the group level gives the central extension of $\mathrm{Diff}(S^1)$.

For the complexified $\mathrm{Vect}(S^1)$, it is convenient to use the basis $L_n = ie^{in\theta}\partial_\theta, n \in \mathbb{Z}$, which satisfies the commutator Eq. (22). By adding an appropriate coboundary term, the centrally extended commutator can be completely fixed to the following form

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \tag{33}$$

which is called the *Virasoro algebra* and c is a free parameter called the central charge. The math literature often adopts a different convention, where L_m is shifted by a coboundary term $L_m \to L_m + \frac{c}{24}\delta_{m,0}$ such that the commutator becomes

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}\,. \tag{34}$$

In this note, we will use the second version. More generally, we can write the central extension of $\mathrm{Vect}(S^1)$ by $\mathbb{C}$ with the central charge c as the following vector space

$$\mathfrak{vir} = \widehat{\mathrm{Vect}}(S^1) = \left\{ f(\theta)\partial_\theta + iac \,\middle|\, f(\theta+2\pi) = f(\theta), a \in \mathbb{R} \right\}\,, \tag{35}$$

with the commutator

$$[f\partial_\theta + ia_f c, g\partial_\theta + ia_g c] = (fg' - f'g)\partial_\theta + \frac{ic}{48\pi}\int_0^{2\pi} d\theta\,(fg''' - f'''g)\,. \tag{36}$$

The second term is called the *Gelfand–Fuchs cocycle*. For brevity, we refer to each element in $\mathfrak{vir}$ as (f,a). The above commutator also gives the adjoint representation of the algebra

$$\mathrm{ad}_{(f,a_f)} : (g, a_g) \mapsto \left(-(fg' - f'g),\, -\frac{1}{48\pi}\int_0^{2\pi} d\theta\,(fg''' - f'''g) \right)\,. \tag{37}$$

The dual space after the central extension can be naturally defined to consist of quadratic forms plus an additional term

$$\mathfrak{vir}^* = \left\{ u(\theta)(d\theta)^2 - it\widetilde{c} \,\middle|\, u(\theta+2\pi) = u(\theta), t \in \mathbb{R} \right\} \tag{38}$$

with $\widetilde{c}$ being dual to c. For brevity, we refer to each coadjoint vector as (u,t). If we choose $\widetilde{c}(c) = 1$, then the pairing between $\mathfrak{vir}$ and $\mathfrak{vir}^*$ takes the natural form

$$\langle (u,t),\, (f,a) \rangle = \int_0^{2\pi} d\theta\, u(\theta)f(\theta) + ta\,. \tag{39}$$

Accordingly, the coadjoint representation of $\mathfrak{vir}$, following its definition, is[3]

$$\mathrm{ad}^*_{(f,a_f)} : (u,t) \mapsto -\left(fu' + 2f'u + \frac{t}{24\pi} f''', 0 \right) . \tag{40}$$

Our next goal is to lift the above result to a central extension of the group $\mathrm{Diff}(S^1)$. It turns out that the central extension of $\mathrm{Diff}(S^1)$ corresponding to the Lie algebra $\mathfrak{vir}$ is topologically trivial and hence can be defined by a continuous group 2-cocycle, called the *Bott cocycle*

$$\begin{aligned} B : \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) &\to \mathbb{R} , \\ (\varphi, \psi) &\mapsto \frac{1}{48\pi} \int \log (\varphi \circ \psi)' \, d \log \psi' \end{aligned} \tag{41}$$

The prefactor $1/48\pi$ is chosen to be consistent with our normalization for the Lie algebra. Accordingly, the group multiplication is

$$(\varphi, a) \circ (\psi, b) = (\varphi \circ \psi, a + b + B(\varphi, \psi)) . \tag{42}$$

We can check that it satisfies the cocycle condition by a direct calculation. (See Appendix. C)

The adjoint and coadjoint representation of the group can be derived from the definition, which are listed below

$$\mathrm{Ad}_{\varphi^{-1}} : (g, a) \mapsto \left(\left(\frac{d\varphi}{d\theta} \right)^{-1} g(\varphi(\theta)),\, a + \frac{1}{24\pi} \int_0^{2\pi} d\theta S(\varphi^{-1}) g(\theta) \right) , \tag{43}$$

$$\mathrm{Ad}^*_{\varphi^{-1}} : (u, t) \mapsto \left(\left(\frac{d\varphi}{d\theta} \right)^{2} u(\varphi(\theta)) + \frac{t}{24\pi} S(\varphi) \, , \, t \right) . \tag{44}$$

The detailed derivation can be found in Appendix. C. One can see that, if we set $\varphi(\theta) = \theta + \epsilon f(\theta)$ to be an infinitesimal diffeomorphism, Eq. (43) and Eq. (44) goes back to Eq. (37) and Eq. (40) respectively, which provides a justification for the Bott cocycle.

3.2 Connection to Hill's equation

A comparison of Eq. (44) and Eq. (14) immediately shows that the coadjoint orbit of $(u, t) \in \mathfrak{vir}^*$ can be regarded as the orbit of a Hill's operators $\mathcal{H}_\theta = \frac{t}{12\pi} \partial_\theta^2 + u(\theta)$. Here an orbit is the collection of $\mathcal{H}_\theta$, which are connected by a diffeomorphism. As shown in Sec. 2.1, the monodromy of the Hill's equation is a diffeomorphism invariant and thus can be used to label the orbit of $\mathcal{H}_\theta$. With such an identification, we know that the monodromy can also be used label coadjoint orbits and vice versa. This will be the task for the next section.

[3]

$$\begin{aligned} \langle \mathrm{ad}^*_{(f,a_f)}[(u,t)],\, (g, a_g) \rangle &= - \langle (u,t),\, \mathrm{ad}_{(f,a_f)}[(g,a_g)] \rangle \\ &= - \left\langle (u,t),\, \left((fg' - f'g),\, \frac{1}{48\pi} \int_0^{2\pi} d\theta \, (fg''' - f'''g) \right) \right\rangle \\ &= - \int_0^{2\pi} d\theta \left(u(\theta)(fg' - f'g) + \frac{t}{48\pi} (fg''' - f'''g) \right) \\ &= \int_0^{2\pi} d\theta \left(u'f + 2uf' + \frac{t}{24\pi} f''' \right) g \end{aligned}$$

where we use integration by part in the last step. Notice that the central term disappears.

4 Classification of coadjoint orbit and the monodromy of Hill's equation

4.1 Perturbative analysis

The main idea is the same as the analysis done for $\mathrm{Diff}(S^1)$: given a coadjoint vector $(u, t) \in \mathfrak{vir}^*$, we solve for the subgroup $H_{(u,t)}$ that leaves it invariant, and the coset space $\widehat{\mathrm{Diff}(S^1)}/H$ is the orbit for (u, t). Here, we follow [1] and give a perturbative analysis.

Let us recall that the coadjoint representation of $\mathfrak{vir}$ Eq. (40), the vector field (f, a) that leaves (u, t) invariant should satisfy

$$fu' + 2f'u + \frac{t}{24\pi} f''' = 0\,. \tag{45}$$

In the following, we first solve the problem for special u and then turn on perturbations.

For simplicity, set the coadjoint vector to be constant $u = u_0 = \frac{\nu^2 t}{48\pi}$, for which the stabilizer equation is simplified to

$$(f')'' + \nu^2 f' = 0 \Rightarrow f' = C_0 e^{i\nu\theta} + C_1 e^{-i\nu\theta}\,. \tag{46}$$

There are three cases depending on the value of u_0:[4]

1. For generic $\nu \in \mathbb{R}$, the only periodic solution is f being a constant. Thus, generic constant u is left invariant by a rigid rotation generated by L_0. We denote this subgroup by S^1 and the orbit is $\widehat{\mathrm{Diff}}(S^1)/S^1$.

2. $\nu = n \in \mathbb{Z}$, f has two addition independent solutions $e^{\pm in\theta}$, corresponding to the generator $L_{\pm n}$ respectively. The subgroup generated by $L_0, L_{\pm n}$ is isomorphic to an n-fold cover of $\mathrm{SL}(2,\mathbb{R})$ which will be called $\mathrm{SL}^{(n)}(2,\mathbb{R})$. The coadjoint orbit is then $\widehat{\mathrm{Diff}(S^1)}/\,\mathrm{SL}^{(n)}(2,\mathbb{R})$.

3. ν is purely imaginary, the only periodic solution is f being a constant. Therefore the orbit is again $\widehat{\mathrm{Diff}}(S^1)/S^1$.

We know that the coadjoint orbit problem can be mapped to the Hill's equation. Before turning on a perturbation, let us make a parallel discussion on the Hill's equation. A constant

[4] For a more explicit analysis, we expand the vector field into the Fourier series

$$f(\theta)\partial_\theta = \sum_{n\in\mathbb{Z}} f_n L_n = i \sum f_n e^{in\theta} \partial_\theta\,, \tag{47}$$

which transforms a constant coadjoint vector (u_0, t) into

$$u_0 + \mathrm{ad}^*_{(f,*)}[u_0] = u_0 + \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} 4\pi n f_n \left(\frac{n^2 t}{48\pi} - u_0\right) e^{in\theta}\,. \tag{48}$$

It is clear that $f_{n=0}$ always keeps u_0 invariant. Unless $u_0 = \frac{k^2 t}{48\pi}$, none of the Fourier coefficients vanishes and $\{f_{n\neq 0}\}$ can be adopted as coordinates for the neighborhood of (u_0, t).

ν	Coadjoint orbit	Monodromy of Hill's equation
$\mathbb{R}$	$\widehat{\text{Diff}}(S^1)/S^1$	Elliptic
$\mathbb{Z}$	$\widehat{\text{Diff}}(S^1)/\,\text{SL}^{(n)}(2,\mathbb{R})$	$\pm\mathbb{I}$
purely imaginary	$\widehat{\text{Diff}}(S^1)/S^1$	Hyperbolic

Table 3: The coadjoint orbits and monodromy for special cases.

$b = \frac{\nu^2 t}{48\pi}$ gives the following Hill's equation and solutions

$$\left(\partial_\theta^2 + \frac{\nu^2}{4}\right)\psi(\theta) = 0 \Rightarrow \psi = C_0 e^{i\nu\theta/2} + C_1 e^{-i\nu\theta/2}\,. \tag{49}$$

Different ν then corresponds to different monodromy matrix and we have Tab. 3. The coadjoint orbits in the elliptic and hyperbolic monodromy cases are different but isomorphic.

To deal with the perturbation, we need to introduce another formalism. Let us rewrite the stabilizer equation into a matrix form

$$\frac{dF}{d\theta} = AF\,, \quad F = \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix}\,, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24\pi u'/t & -48\pi u/t & 0 \end{pmatrix}\,. \tag{50}$$

Given a coadjoint vector (u, t), the solution for f can be formally written as

$$F(\theta) = U(\theta)F(0)\,, \quad U(\theta) = \mathcal{P}\exp\left(\int_0^\theta d\theta' A(\theta')\right)\,, \tag{51}$$

where $\mathcal{P}$ is the path ordering operator. Vector fields are periodic in θ, which provides the following necessary condition for any valid solution

$$U(2\pi)F(0) = F(0)\,. \tag{52}$$

In our case, $u = \frac{n^2 t}{48\pi} + \delta u$ and we have

$$A = A_0 + \delta A\,, \quad A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -n^2 & 0 \end{pmatrix}\,, \delta A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24\pi\delta u'/t & -48\pi\delta u/t & 0 \end{pmatrix}\,. \tag{53}$$

To the leading order in δu we have

$$U(2\pi) = U_0(2\pi) + \delta U(2\pi) = \mathbb{I} + \int_0^{2\pi} d\theta' e^{(2\pi-\theta')A_0}\delta A(\theta') e^{\theta' A_0}\,. \tag{54}$$

Then the condition $U(2\pi)F(0) = F(0)$ requires $F(0)$ to be a null vector of $\delta U(2\pi)$. If we expand δu in a Fourier series

$$\delta u = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \beta_k e^{ik\theta}\,, \tag{55}$$

we find $\delta U(2\pi)$ only depends on the $k=0$ and $k=\pm n$ Fourier components

$$\delta U(2\pi)=\begin{pmatrix}-y/n & (z-x)/n^2 & 0\\ -x & 0 & (z-x)/n^2\\ ny & -z & y/n\end{pmatrix}\,,\qquad \begin{cases}x=&12\pi(\beta_n+\beta_{-n})/t\\ y=&-\,12\pi i(\beta_n-\beta_{-n})/t\,.\\ z=&24\pi\beta_0/t\end{cases} \tag{56}$$

Let us introduce a quadratic form $Q^2=x^2+y^2-z^2$ then the three eigenvalues of $\delta U(2\pi)$ are 0 and $\pm Q/n$. The eigenvector associated with the 0 eigenvalue is $(z-x\,,\,ny\,,\,n^2x)^T$. We can try the following ansatz for $f(\theta)$

$$f(\theta)=f_0+f_ne^{in\theta}+f_{-n}e^{-in\theta}\,. \tag{57}$$

At $\theta=0$, it should yield the null vector of $\delta U(2\pi)$ so that we have

$$\begin{aligned}f_0+f_n+f_{-n}&=C(z-x) & & & f_0&=Cz=C24\pi\beta_0/t\\ i(f_n-f_{-n})&=Cy & &\Rightarrow & f_n&=-C(x+iy)=C12\pi\beta_n/t\\ -(f_n+f_{-n})&=Cx & & & f_{-n}&=-C(x-iy)=-C12\pi\beta_{-n}/t\end{aligned} \tag{58}$$

where C is an unimportant normalization constant. Namely, the vector field that generates the stabilizer subgroup is completely fixed by the perturbation and is only one-dimensional. They can be classified into three cases:

1. $Q^2<0$, the perturbation as well as the stabilizer generator is *elliptic*. A typical element is $x=y=0, z\neq 0$, which by definition is just a constant shift of u_0. Accordingly, the only non-zero component of the stabilizer vector field is β_0, which generates rigid rotation. Thus we recover the statement that a constant coadjoint vector with generic u_0 is stabilized only by the rigid rotation. If we assume the stabilizer group does not change for the whole $Q^2<0$ regime, the coadjoint orbit for this class of perturbation is $\widehat{\text{Diff}}(S^1)/S^1$.

2. $Q^2>0$, the perturbation as well as the stabilizer generator is *hyperbolic*. For example $x\neq 0, y=z=0$, which yields a vector field $f(\theta)=\cos(n\theta)$. The corresponding one-parameter stabilizer subgroup is denoted by $T_{(n,\Delta)}$ and the corresponding coadjoint orbit is $\widehat{\text{Diff}}(S^1)/T_{(n,\Delta)}$. Here n represents the number of zeros of f and Δ is a diffeomorphism invariant that cannot be explained within the perturbation analysis but will be defined clearly later.

3. $Q^2=0$, the perturbation as well as the stabilizer generator is *parabolic*. For example $x=\pm z\neq 0, y=0$, which yields $f(\theta)=1\pm\cos(n\theta)$. The corresponding one-parameter stabilizer subgroup is called $T_{(n,\pm)}$ and the coadjoint orbit for this class of perturbation is $\widehat{\text{Diff}}(S^1)/T_{(n,\pm)}$, where n is the number of zeros of f.

Now let us apply the mapping and make another contact with the Hill's equation to examine whether the analysis above is consistent with what we have known. Let us consider the Mathieu's equation, which is a special Hill's equation

$$\left(\partial_\theta^2+\left(\frac{\nu}{2}\right)^2+\beta_0+2\beta_2\cos 2\theta\right)\psi(\theta)=0\,. \tag{59}$$

	β_0, β_2	coadjoint orbit	monodromy
special point	$\beta_0 = \beta_2 = 0$	$\widehat{\mathrm{Diff}}(S^1)/\,\mathrm{SL}^{(2)}(2,\mathbb{R})$	$\pm\mathbb{I}$
Perturbation	$\beta_0 \neq 0, \beta_2 = 0$	$\widehat{\mathrm{Diff}}(S^1)/S^1$	Elliptic
	$\beta_0 = 0, \beta_2 \neq 0$	$\widehat{\mathrm{Diff}}(S^1)/T_{(2,\Delta)}$	Hyperbolic
	$\beta_0 = \beta_2 \neq 0$	$\widehat{\mathrm{Diff}}(S^1)/\widetilde{T}_{(2,\pm)}$	Parabolic

Table 4: Coadjoint orbits and monodoromy of the Mathieu's equation. The data for the coadjoint orbits is from perturbative analysis and the data for the monodromy is from the exact solution of the Mathieu equation.

There is a perfect one-to-one correspondence between the coadjoint orbit and monodromy for $\nu = 2$ and small β's, which is summarized by Tab. 4. If we consider $\nu \geq 3$, the perturbative analysis will conclude that β_0 and β_2 do not affect the coadjoint orbits, while the exact solution says that heating phase does exist for $\nu = n \geq 3$. One may suspect whether the one-to-one correspondence between the coadjoint orbit and monodromy is a coincidence only for $\nu = 2$. In the following, we will take better advantage of the connection with the Hill's equation to give a full classification beyond perturbation, which shows that such a correspondence holds for arbitrary choices of $u(\theta)$.

4.2 Full classification of the coadjoint orbits

In this section, we give a full classification of the coadjoint orbit using the connection with the Hill's equation.

We know that for a Virasoro algebra, each coadjoint vector (u, t) can be identified with a Hill's equation $\left(\frac{t}{12\pi}\partial_\theta^2 + u(\theta)\right)\psi(\theta) = 0$ as well as their orbits. A coadjoint orbit is specified by the stabilizer subgroup while one important label for Hill's equations is the monodromy. From the perturbative analysis above, it implies that there is a correspondence between the two seemingly different characterization. It is natural to ask whether we can build a one-to-one correspondence between them on a full ground.

The answer is yes. That this is possible comes from the following fact. The stabilizer equation of a coadjoint vector (u, t) is

$$fu' + 2f'u + \frac{t}{24\pi}f''' = 0\,, \tag{60}$$

where u is a smooth vector field. Suppose that $(\psi_1, \psi_2)^T$ are two independent solutions of the corresponding Hill's equation, then the bilinear combination

$$\begin{pmatrix}\psi_1 & \psi_2\end{pmatrix} A \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}\,, \tag{61}$$

solves the stabilizer equation locally. Here A is a generic 2×2 matrix. Our definition contains huge redundancies, which will be fixed later. We can check this by a straightforward calculation.

Introduce $\xi = \psi_i\psi_j, i, j, = 1,2$

$$\begin{aligned}
\xi' =& \psi_i'\psi_j + \psi_i\psi_j' \\
\xi'' =& \psi_i''\psi_j + \psi_i\psi_j'' + 2\psi_i'\psi_j' = -\frac{24\pi}{t}b(\theta)\psi_i\psi_j + 2\psi_i'\psi_j' \\
\xi''' =& -\frac{24\pi}{t}b'\psi_i\psi_j - \frac{24\pi}{t}b(\psi_i\psi_j)' + 2\psi_i''\psi_j + 2\psi_i'\psi_j'' \\
=& -\frac{24\pi}{t}b'\psi_i\psi_j - \frac{48\pi}{t}b\psi_i\psi_j' - \frac{48\pi}{t}b\psi_i'\psi_j
\end{aligned}$$

We can see that ξ satisfies the stabilizer equation and this proves our statement.

$(\psi_1, \psi_2)^T$ generally have a nontrivial monodromy while f as a vector field must be periodic in 2π. This imposes a constraint on the choice of A, namely, it must satisfy

$$M^T A M = A \,. \tag{62}$$

Notice that A is not invariant under the linear transformation of $(\psi_1, \psi_2)^T$ while the coadjoint orbit is. This simplifies our discussion in the sense that we can pick any solutions we like and the final conclusion does not depend on our choice. Let us examine how this equation constrains A when M is in different conjugacy classes. We will fix the normalization of $(\psi_1, \psi_2)^T$ such that the Wronskian is

$$W[\psi_1, \psi_2] = \psi_1'\psi_2 - \psi_1\psi_2' = 1 \,. \tag{63}$$

1. Elliptic. We can always choose our solutions such that the monodromy takes the standard form

   $$M = \begin{pmatrix} \cos\nu\pi & -\sin\nu\pi \\ \sin\nu\pi & \cos\nu\pi \end{pmatrix} , \quad \nu \neq \mathbb{Z} \,. \tag{64}$$

   Therefore, A has to be an $SO(2)$ matrix

   $$A = \begin{pmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{pmatrix} \Rightarrow f = \cos^2\sigma(\psi_1^2 + \psi_2^2) \,. \tag{65}$$

   Although we have a continuous family of A, the corresponding vector fields with different coefficients $\cos^2\sigma$ should be identified, which leaves us with only one vector field that generates the one-parameter subgroup.

   Furthermore, notice that the fact the Wronskian $W[\psi_1, \psi_2] = 1$ guarantees that $\psi_1^2 + \psi_2^2$ is everywhere non-vanishing. Therefore, we conjecture that this subgroup is smoothly connected to the rigid rotation S^1. We can also define a diffeomorphism invariant that labels the stabilizer group

   $$\nu = \frac{1}{\pi}\int_0^{2\pi} d\theta \frac{1}{f} = \frac{1}{\pi}\int_0^{2\pi} d\theta \frac{1}{\psi_1^2 + \psi_2^2} \,. \tag{66}$$

Our notation suggests that it is related to the monodromy, which can be shown as follows

$$
\begin{aligned}
\nu =&\frac{1}{\pi}\int_0^{2\pi} d\theta\,\frac{\psi_1'\psi_2-\psi_1\psi_2'}{\psi_1^2+\psi_2^2}=\frac{1}{\pi}\int_0^{2\pi} d\theta\,\frac{\psi_1'/\psi_1-\psi_2'\psi_2}{\psi_1/\psi_2+\psi_2/\psi_1}\\
=&\frac{1}{\pi}\int\frac{dx}{e^x+e^{-x}}\qquad (x=\log\frac{\psi_1}{\psi_2})\\
=&\frac{1}{\pi}\arctan e^x\Big|_{x_0}^{x_{2\pi}}=\frac{i}{2\pi}\log\frac{1-ie^x}{1+ie^x}\Big|_{x_0}^{x_{2\pi}}\\
=&\frac{i}{2\pi}\log\frac{\psi_2-i\psi_1}{\psi_2+i\psi_1}\Big|_0^{2\pi}\\
=&\frac{i}{2\pi}\log e^{2i\nu\pi}=-\nu+n\,,n\in\mathbb{Z}\,.
\end{aligned}
\tag{67}
$$

2. Hyperbolic. We choose our solutions such that the monodromy is

$$
M=\begin{pmatrix} e^{\pi\lambda} & 0\\ 0 & e^{-\pi\lambda}\end{pmatrix}\,,\qquad \lambda\neq 0\,. \tag{68}
$$

Accordingly, A and the vector field have to be

$$
A=\begin{pmatrix} 0 & a_{12}\\ a_{21} & 0\end{pmatrix}\Rightarrow f=(a_{12}+a_{21})\psi_1\psi_2\,. \tag{69}
$$

Notice that the coefficient $a_{12}+a_{21}$ is redundant, and there is one vector field that generates a one-parameter subgroup.

In general, the two solutions $(\psi_1,\psi_2)^T$ by themselves can have zeros. The non-vanishing condition of the Wronksian requires that ψ_1 and ψ_2 cannot have double zeros and their simple zeros must be at different positions. Consequently, the vector field $f=\psi_1\psi_2$ can only have even number of simple zeros, which becomes the attractive and repulsive fixed points of the generated diffeomorphism. Furthermore, the derivatives of f at these zeros must have the same magnitude because

$$
f'=\psi_1'\psi_2+\psi_1\psi_2'=1+2\psi_1\psi_2'=-1+\psi_1'\psi_2\Rightarrow|f'|=1\text{ at }f=0\,. \tag{70}
$$

Since only simple zero appears, we can make an analogy with the elliptic case and define the following diffeomorphism invariant

$$
\Delta=\frac{1}{\pi}\text{P.V.}\int_0^{2\pi}d\theta\frac{1}{f}=\frac{1}{\pi}\text{P.V.}\int_0^{2\pi}d\theta\frac{1}{\psi_1\psi_2}\,. \tag{71}
$$

Here P.V. represents principle value. This quantity is also related to λ by massaging the formula in the following way

$$
\begin{aligned}
\Delta =&\frac{1}{\pi}\text{P.V.}\int_0^{2\pi}d\theta\,\frac{\psi_1'\psi_2-\psi_1\psi_2'}{\psi_1\psi_2}=\frac{1}{\pi}\text{P.V.}\int_0^{2\pi}d\theta\,\left(\frac{\psi_1'}{\psi_1}-\frac{\psi_2'}{\psi_2}\right)\\
=&\frac{1}{\pi}\int_0^{2\pi}d\log\frac{\psi_1}{\psi_2}=2\lambda\,.
\end{aligned}
\tag{72}
$$

3. Parabolic. We choose our solutions such that the monodromy is

$$M = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad p \neq 0 . \tag{73}$$

   Therefore, A has to be

$$A = \begin{pmatrix} 0 & -1 \\ 1 & y \end{pmatrix} \Rightarrow f = y\psi_2^2 . \tag{74}$$

   So there is one vector field that generates a one-parameter subgroup.

   In this case, the vector field f only contains an even number of double zeros, which, depending on the side it is approached and the sign in front of f, can be either repulsive or attractive.

4. $\mathbb{Z}_2$. In this scenario, any $\mathrm{SL}(2,\mathbb{R})$ matrix is as good. Therefore, there are three independent choices of vector fields. [This case is too special, may not be physically relevant.]

Given a vector field f, Eq. (60) can also be viewed as an equation for the coadjoint vector (u, t). Thus, we can reverse the process to suppose f is know and determine what b must look like. We introduce an auxiliary function

$$u = \frac{-t}{24\pi} f^{-2} h , \tag{75}$$

whence Eq. (60) becomes

$$h' = f f''' \Rightarrow h = f f'' - \frac{1}{2} f'^2 + \text{const} . \tag{76}$$

The constant can be fixed by the condition that u has to be a regular function so that we have the following three cases:

1. Elliptic: f has no zeros. In this case the constant can be arbitrary and we have

$$u = \frac{-t}{24\pi} \frac{f f'' - \frac{1}{2} f'^2 + \text{const}}{f^2} . \tag{77}$$

2. Hyperbolic: f only has simple zeros and $|f'| = 1$ at those zeros. In this case, we must have const $= 1/2$ to make sure h also vanishes. It follows that

$$u = \frac{-t}{24\pi} \frac{f f'' - \frac{1}{2}(f'^2 - 1)}{f^2} . \tag{78}$$

3. Parabolic: f only has double zeros. In this case, const $= 0$ and we have

$$u = \frac{-t}{24\pi} \frac{f f'' - \frac{1}{2} f'^2}{f^2} . \tag{79}$$

These results are summarized in Tab. 2, which is introduced at the very first beginning.

A Review on some facts of Lie group and Lie algebra

A.1 Adjoint and coadjoint representation of a Lie group

Representation A representation of a Lie group G on a vector space V is a linear map ρ of the group G that is smooth, namely

$$G \times V \to V\,, \quad (g, v) \mapsto \varphi(g)v \tag{80}$$

is smooth. (V, ρ) is called a real/complex representation if V is a real/complex vector space. If V is a Hilbert space, which has the inner product (not a bilinear product), (V, ρ) is called a unitary representation if the inner product is invariant under the action of G.

Adjoint representation Given a group element g, we can define a conjugation

$$c_g : h \to ghg^{-1}\,,\; h \in G\,, \tag{81}$$

which induces the following two adjoint representations, one for the Lie group and the other for the Lie algebra.

Let us assume h_t is a one-parameter subgroup labeled by $t \in [0, 1]$ with $h_{t=0} = \mathbb{I}$. By recalling the definition of a vector on a differential manifold, the derivative of h_t with respect to t at $t = 0$ also gives us a vector

$$x = \frac{dh_t}{dt}\Big|_{t=0}\,, \tag{82}$$

which can be identified with an element in the Lie algebra.[5] Such an operation can be naturally applied to the conjugated element $c_g(h_t)$, which leads to a map

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g, \quad \mathrm{Ad}_g(x) = \frac{d}{dt} g h_t g^{-1}\Big|_{t=0}. \tag{83}$$

This defines a representation of the group G on its Lie algebra $\mathfrak{g}$ and is called the *group adjoint representation*.[6] For a given $x \in \mathfrak{g}$, the collection of $\mathrm{Ad}_g(x)$ of all the group element in G is called the *adjoint orbit of G through x*. g is called a *stabilizer* of x if $\mathrm{Ad}_g(x) = x$.

Furthermore, we can use another one-parameter subgroup g_t for Ad and take the derivative over t at $t = 0$, which defines a map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})\,, \quad x \mapsto \mathrm{ad}_x\,, \quad \mathrm{ad}_x(y) = \frac{d}{dt}\,\mathrm{Ad}_{g_t}(y)\Big|_{t=0}. \tag{84}$$

and is called the *the adjoint representation of the Lie algebra* $\mathfrak{g}$.[7] This can be used to define the Lie algebra.

To define the coadjoint representation, we need the following concept:

[5] This can be chosen as a definition of the Lie algebra.

[6] Notice that Ad_g is an automorphism of $\mathfrak{g}$. An automorphism is an isomorphism from a mathematical object to itself. It is, in some sense, a symmetry of the object, and a way of mapping the object to itself while preserving all of its structure.

[7] Notice that $\mathrm{ad}_{\mathfrak{g}}$ is only an endomorphism. En endomorphism is a morphism from a mathematical object to itself. An endomorphism is not necessarily an automorphism unless it is an isomorphism.

Dual space Given any vector space V over a field F, we can define a **(algebraic) dual vector space** V^* as the set of all linear maps $\phi : V \to F$. Since V^* is also a vector space, it has to satisfy the following two properties:

$$\begin{aligned} (u+v)(x) =& u(x) + v(x) \\ (au)(x) =& au(x) \end{aligned} , \quad \forall u, v \in V^* , \, x \in V \, , \, a \in F \, . \tag{85}$$

Elements of the algebraic dual space V^* are sometimes called **covectors** or **one-forms**. The notation $u(x) = \langle u, x\rangle$ is often used, and is often called a *pairing*.

If $\dim V$ is finite, $\dim V^* = \dim V$ and one can define an isomorphism between V and V^*. This can be proved by noticing that specifying how u acts on each basis vector of V determines a basis of V^*, e.g. $u^i(e_j) = \delta^i_j$.

If $\dim V$ is infinite, the construction above still gives us a set of linearly independent elements in V^* but they do not necessarily form a complete basis. Namely, V^* can be larger than V. [RF: need better understanding.]

A Lie algebra $\mathfrak{g}$ is a vector space and thus admits the definition of the *dual of the Lie algebra*, denoted as $\mathfrak{g}^*$

Coadjoint representation The coadjoint representation of a group G is a linear map on the dual space of its Lie algebra

$$\begin{gathered} \mathrm{Ad}^* : \mathfrak{g}^* \to \mathrm{Aut}(\mathfrak{g}^*) \\ \langle \mathrm{Ad}^*_g u, x\rangle := \langle u, \mathrm{Ad}_{g^{-1}} x\rangle \, , g \in G, \, x \in \mathfrak{g}, \, u \in \mathfrak{g}^* \, . \end{gathered} \tag{86}$$

Notice that there is an inverse in the definition, which is necessary for the paring $\langle *, *\rangle$ to be invariant under the group action. For a given $u \in \mathfrak{g}^*$, the collection of $\mathrm{Ad}^*_g(u)$ for all the group element g is called the *coadjoint orbits of* G *through* u. g is a stabilizer of u if $\mathrm{Ad}^*_g(u) = u$.

Similar to the discussion for adjoint represention, we can also take derivative and define the *coadjoint representation of a Lie algebra*,

$$\begin{gathered} \mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*) \\ \langle \mathrm{ad}^*_v(b), u\rangle := -\langle b, \mathrm{ad}_v(u)\rangle \, , \quad u, v \in \mathfrak{g}, \, b \in \mathfrak{g}^* \, . \end{gathered} \tag{87}$$

In the following, we use some concrete examples to demonstrate the adjoint and coadjoint orbits. Here we focus on finite dimensional Lie groups and Lie algebra. The discussion on infinite dimensional Lie groups and Lie algebra can be found in Sec. 2.

For a finite dimensional (simple) Lie group and the corresponding Lie algebra, we can define an invariant quadratic form, for example $\mathrm{Tr}[uv]$, $u, v \in \mathfrak{g}$ in the fundamental representation. Therefore, there is a natural isomorphism between the dual Lie algebra $\mathfrak{g}^*$ and $\mathfrak{g}$. Accordingly, the adjoint and coadjoint representations and orbits are also isomorphic.

SU(2) If we pick the basis such that $\mathrm{Tr}(t_i t_j) = \delta_{ij}$, the invariant quadratic form is $x_1^2 + x_2^2 + x_3^2$. Therefore every adjoint orbit in $\mathfrak{su}(2)$ is a sphere $x_1^1 + x_2^2 + x_3^2 = R^2$ for some radius $R \geq 0$.

SL(2,R) If we pick the basis such that $\mathrm{Tr}(t_i t_j) = \delta_{ij}$, the invariant quadratic form is $x_1^2 - x_2^2 - x_3^2$. The adjoint orbits in $\mathfrak{sl}(2, \mathbb{R})$ are hyperboloid $x_1^2 - x_2^2 - x_3^2 = R^2$ and falls into three distinct classes:

- $R^2 > 0$. It consists of two disconnected paraboloids. For each paraboloid, the bottom/top point is invariant under a $U(1)$ rotation, which is a subgroup of $\mathrm{SL}(2,\mathbb{R})$. Thus each paraboloid can be identified with the coset space $\mathrm{SL}(2,\mathbb{R})/U(1)$. [How to understand: "the representation obtained by quantizing the hyperboloid are known as the discrete series"?]

- $R^2 < 0$. It consists of one connected hyperboloid. The associated representations are known as the continuous series.

- $R^2 = 0$.

For an infinite dimensional Lie group or Lie algebra, there is no natural isomorphism between the vector space and its dual. Thus the coadjoint representation is in general different from the adjoint representation.

A.2 Central extension of a Lie group and a Lie algebra

One can think of an **extension of a group** G as a new bigger group $\widetilde{G}$ fibered over the original group G with H being the fiber over $\mathbb{I} \in G$, as is described by the following exact sequence

$$\{\mathbb{I}\} \to H \to \widetilde{G} \to G \to \{\mathbb{I}\}\,. \tag{88}$$

The exactness at H implies that H is isomorphic to a normal subgroup of $\widetilde{G}$. The exactness at G implies that G is isomorphic to the quotient group $\widetilde{G}/H$. It is called a **central extension** if H belongs to the center of $\widetilde{G}$, i.e. $H \subset Z(\widetilde{G})$.

For simplicity, suppose G is a Lie group, H is abelian and the central extension $\widetilde{G}$ is a bigger Lie group that is topologically a product space $\widetilde{G} \simeq G \times H$. We can denote group elements of $\widetilde{G}$ by $\widetilde{g} = (g, h)$ and have the following multiplication rule

$$(g_1, h_1) \times (g_2, h_2) = (g_1 g_2, \gamma(g_1, g_2) h_1 h_2)\,, \tag{89}$$

where $\gamma : G \times G \to H$ is a smooth map. The following properties of γ are direct consequences of the definition:

- Without G, the multiplication within H is untwisted: $\gamma(1, 1) = 1$.

- H is in the center of $\widetilde{G}$:

$$(g, h) \times (1, h') = (1, h') \times (g, h) \Rightarrow \gamma(1, g) = \gamma(g, 1),\ \forall g \in G\,. \tag{90}$$

- Having a well-defined inverse, namely $(g, h)^{-1} \times (g, h) = (g, h) \times (g, h)^{-1} = (1, 1)$:

$$\gamma(g^{-1}, g) = \gamma(g, g^{-1})\,. \tag{91}$$

- Associativity

$$\begin{aligned}((g_1, h_1) \times (g_2, h_2)) \times (g_3, h_3) &= (g_1, h_1) \times ((g_2, h_2) \times (g_3, h_3)) \\ \Rightarrow \gamma(g_1 g_2, g_3)\gamma(g_1, g_2) &= \gamma(g_1, g_2 g_3)\gamma(g_2, g_3)\,.\end{aligned} \tag{92}$$

  This is called a *group 2-cocycle condition of G with values in H*. The set of γ satisfying this condition is denoted by $Z^2(G, H)$.

A 2-cocycle is called a *2-coboundary* if there exists a smooth map $\lambda : G \to H$ such that

$$\gamma(g_1, g_2) = \lambda(g_1)\lambda(g_2)\lambda(g_1 g_2)^{-1} . \tag{93}$$

The set of 2-coboundary is denoted by $B^2(G, H)$.

We have the following fact: there is a one-to-one correspondence between the set of central extensions of G by H that admit a smooth section and the *second cohomology group* $H^2(G, H) := Z^2(G, H)/B^2(G, H)$.

The nice thing about a Lie group is that it allows for differentiation. Given a central extension of a Lie group, differentiation of it yields a corresponding central extension of its Lie algebra. After being established, we will find that the central extension of a Lie algebra can be defined by itself. Just like a Lie algebra may not correspond to a Lie group, the central extension of a Lie algebra may not corresponds to a central extension a Lie group.

[How to derive the following statement from the central extension of a Lie group?]

A *central extension of a Lie algebra* $\mathfrak{g}$ by a vector space $\mathfrak{n}$ is a new bigger Lie algebra $\widetilde{\mathfrak{g}}$ whose vector space $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{n}$ is equipped with the following Lie braket

$$[(X, u), (Y, v)] = ([X, Y], \omega(X, Y)) , \tag{94}$$

where $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{n}$ is a smooth bilinear map. We require $\mathfrak{n}$ belongs to the center of $\widetilde{\mathfrak{g}}$. From the definition, we can derive the following properties:

- $\mathfrak{n}$ belongs to the center:

$$[(0, u), (X, v)] = 0 \Rightarrow \omega(0, X) = \omega(X, 0) = 0. \tag{95}$$

- Skew-symmetry of the Lie braket:

$$[(X, u), (Y, v)] = -[(Y, v), (X, u)] \Rightarrow \omega(X, Y) = -\omega(Y, X) . \tag{96}$$

- Jacobi identity (which is related to the associativity of group multiplication):

$$\begin{aligned} [(X, *), [(Y, *), (Z, *)]] + [(Y, *), [(Z, *), (X, *)]] + [(Z, *), [(X, *), (Y, *)]] = 0 \\ \Rightarrow \omega(X, [Y, Z]) + \omega(Y, [Z, X]) + \omega(Z, [X, Y]) = 0 \end{aligned} \tag{97}$$

  This is called the *2-cocycle identity*.

The set of ω that are bilinear, anti-symmetric and satisfy the 2-cocycle identity is called the *2-cocycle on the Lie algebra* $\mathfrak{g}$, denoted by $Z^2(\mathfrak{g}, \mathfrak{n})$.

A cocycle ω is called a *coboundary*, if there exists a smooth map $\alpha : \mathfrak{g} \to \mathfrak{n}$ such that $\omega(X, Y) = \alpha([X, Y])$, the collection of which is denoted as $B^2(\mathfrak{g}, \mathfrak{n})$. When we have a coboundary α in the extension, a redefinition of everything by $(X, u) \to (X, u - \alpha(X))$ shows that such kind of extension is trivial.

As a result, the central extension of a Lie algebra $\mathfrak{g}$ by $\mathfrak{n}$ is in one-to-one correspondence with the second cohomology $H^2(\mathfrak{g}, \mathfrak{n}) = Z^2(\mathfrak{g}, \mathfrak{n})/B^2(\mathfrak{g}, \mathfrak{n})$.

B Some details about $\mathrm{Diff}(S^1)$ and $\mathrm{Vect}(S^1)$

B.1 Relation between $\mathrm{Diff}(S^1)$ and $\mathrm{Vect}(S^1)$

Intuitively, taking a "derivative" or "integral" relates $\mathrm{Diff}(S^1)$ and $\mathrm{Vect}(S^1)$ with some subtleties addressed by the following two questions: [4]

- Given a vector field $u(\theta)\partial_\theta \in \mathrm{Vect}(S^1)$, can we find a diffeomorphism $f \in \mathrm{Diff}(S^1)$ that is generated by $u\partial_\theta$? The answer is yes by the following reasoning. Given a (smooth) manifold M and a vector field $X \in \mathcal{X}(M)$, any point $x_0 \in M$ can follow the vector field X and flow to another point $\sigma(t, x_0)$ that is determined by the ODE

  $$\frac{d}{dt}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0))\,. \tag{98}$$

  It follows from the uniqueness of ODE that

  $$\sigma(t_2, \sigma(t_1, x_0)) = \sigma(t_1 + t_2, x_0)\,, \quad \sigma(t, \sigma(-t, x_0)) = \sigma(0, x_0) = \mathbb{I}\,. \tag{99}$$

  If we consider t as a parameter, $\sigma_t : M \to M$ is an abelian group with the following rules

  $$\begin{aligned} &(i).\ \sigma_0 = \text{identity map} \\ &(ii).\ \sigma_t \circ \sigma_s = \sigma_{t+s} \\ &(iii).\ (\sigma_t)^{-1} = \sigma_{-t} \end{aligned} \tag{100}$$

  and therefore is by definition a diffeomorphism. In this way, any vector and generate a diffeomorphism.

- Given a diffeomorphism $f \in \mathrm{Diff}(S^1)$, can we write it as a flow of a vector field $u(\theta)\partial_\theta \in \mathrm{Vect}(S^1)$? The answer is no. By construction, $f \in \mathrm{Diff}(S^1)$ can be written as $f_t(\theta)$ with $f_{t=0}(\theta) = \theta$ and $f_{t=1}(\theta) = f(\theta)$, it does not necessarily mean that f_t form a one-parameter family group. Here is an counter example from Milnor's paper "*Remarks on infinite-dimensional Lie groups*":

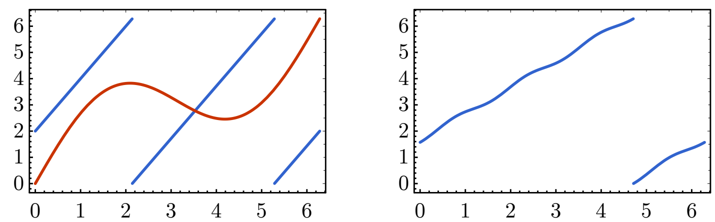
  $$\begin{aligned} &f : S^1 \to S^1 \\ &f(x) = x + \pi/n + \epsilon \sin^2(nx)\,, \quad x \in [0, 2\pi] \end{aligned} \tag{101}$$

  $f(x)$ for n large enough and ϵ small enough is a diffeomorphism but does not to any one-parameter subgroup of $\mathrm{Diff}(S^1)$ and thus cannot be a flow of certain vector field. See below for a proof.

The several examples mentioned above can be found in Fig. 1.

Proof of Milnor's example If we assume Milnor's example is in a one-parameter subgroup, then the group structure implies that $f(x)$ can be written as the composition $g \circ g$. The essential idea is to prove this composition is impossible.

Let us introduce $a_k = k\pi/n$, then we have $f(a_k) = a_{k+1}$ and $f(x) - a_{k+1} > x - a_k$ for $a_k < x < a_{k+1}$. Then we define $b_k = g(a_k)$. It follows from the definition of g that $f(b_k) = b_{k+1}$. We also have $b_{k+1} - a_{k+1} > b_k - a_k$. Then it leads to a contradiction because $b_0 - a_0 > b_{n-1} - a_{n-1} > \cdots > b_0 - a_0$.



(a) Smooth maps that don't belong to Diff(S^1)

(b) Milnor's example that a diffeomorphism don't correspond to a vector field

Figure 1: (a) The blue curves has winding number equal to 2, the red curve does not preserve the orientation. Therefore they are not diffeomorphism. (b) We choose $n = 2$, $\epsilon = 0.2$ to make this plot.

B.2 Adjoint representation of Diff(S^1)

In this section, we will follow the definition and present the detailed derivation of the adjoint representation of Diff(S^1). We work with the most general case first and specialize to S^1 then.

Let us assume a generic differential manifold M. Given a point $x_0 \in M$ and a vector field $V \in \mathcal{X}(M)$, we can let the point x_0 flow to another point $\sigma(t, x_0)$, with its motion governed by the following equation

$$\frac{d}{dt}\sigma^\mu(t, x_0) = V^\mu(\sigma(t, x_0)) \,. \tag{102}$$

If we treat t as a parameter, then $\sigma_t(x_0) = \sigma(t, x_0)$ can be regarded as a continuous family of diffeomorphism parametrized by t. We may need M to be compact for this diffeomorphism to be well-defined over the whole manifold. We can also reverse the above process. Given $\sigma_t : M \to M$ as a one-parameter family of diffeomorphism with $\sigma_{t=0}$ being the identity map, we can do a Taylor expanding near $t = 0$

$$\sigma^\mu_\epsilon(x) = \sigma^\mu(\epsilon, x) = x^\mu + \epsilon V^\mu(x) \,. \tag{103}$$

Namely, the derivative of σ_t at $t = 0$ defines a corresponding vector field at x

$$\frac{d}{dt}\sigma^\mu_t(x)\Big|_{t=0} = V^\mu(x) \,. \tag{104}$$

Since a vector field can be defined through its action on a scalar field, it seems that we need the diffeomorphism σ to have the following action on a scalar function f for things to be self-consistent

$$\sigma[f](x) := f(\sigma(x)) \,. \tag{105}$$

Then the abstract definition of the diffeomphism-induced vector field is

$$V[f](x) = \frac{d}{dt}\sigma_t[f](x)\Big|_{t=0} = \frac{d}{dt}f(\sigma(x))\Big|_{t=0} \,. \tag{106}$$

If we choose $f = x^\mu$ and notice that $V[x^\mu](x) = V^\mu(x)$, $x^\mu(\sigma(x)) = \sigma^\mu(x)$, this abstract definition goes back to Eq. (104).

This concept prepares us to discuss the adjoint representation of a diffeomorphism on a vector field. Let us assume the vector field V can be generated by a diffeomorphism σ_t as described in Eq. (104) or Eq. (106). Then under the adjoint action of a diffeomorphism φ, it transforms in the following way

$$\mathrm{Ad}_\varphi : V \to \widetilde{V} = \frac{d}{dt}\varphi \circ \sigma_t \circ \varphi^{-1}\Big|_{t=0}, \tag{107}$$

Then following Eq. (106) and choose $f = x^\mu$, on the right hand side we have

$$\begin{aligned}\varphi \circ \sigma_\epsilon \circ \varphi^{-1}[x^\mu](x) =& x^\mu\left(\varphi(\sigma_\epsilon(\varphi^{-1}(x)))\right)\\ =&\varphi^\mu\left(\varphi^{-1}(x) + \epsilon V(\varphi^{-1}(x))\right)\\ =&x^\mu + \epsilon V^\nu(\varphi^{-1}(x))\frac{\partial \varphi^\mu(y)}{\partial y^\nu}\Big|_{y=\varphi^{-1}(x)}\end{aligned} \tag{108}$$

Then the μ-th component of the transformed vector field is

$$\widetilde{V}^\mu(x) = V^\nu(\varphi^{-1}(x))\frac{\partial \varphi^\mu(y)}{\partial y^\nu}\Big|_{y=\varphi^{-1}(x)}. \tag{109}$$

If we use φ^{-1}, we can get a nicer form

$$\mathrm{Ad}_{\varphi^{-1}} : V \to \widetilde{V} = \frac{d}{dt}\varphi^{-1} \circ \sigma_t \circ \varphi\Big|_{t=0}, \quad \widetilde{V}^\mu(x) = V^\nu(\varphi(x))\left(\left(\frac{\partial \varphi}{\partial x}\right)^{-1}\right)^\mu_{\ \nu}, \tag{110}$$

where $\frac{\partial\varphi}{\partial x}$ represents the Jacobian matrix. Furthermore, let us calculate the adjoint representation of the corresponding algebra by considering an infinitesimal diffeomorphism $\varphi^\mu(x) \approx x^\mu + \epsilon W^\mu(x)$. We can expand the above expression to the linear order in ϵ and have

$$\widetilde{V}^\mu(x) \approx V^\nu(x^\rho + \epsilon W^\rho(x))\left(\delta^\mu_\nu - \epsilon\frac{\partial W^\mu}{\partial x^\nu}\right) = V^\mu(x) + \epsilon\left(W^\nu\frac{\partial V^\mu}{\partial x^\nu} - \frac{\partial W^\mu}{\partial x^\nu}V^\nu\right), \tag{111}$$

The definition of $\mathrm{ad}_{W\partial}$ is simply the linear term of $\mathrm{Ad}_{1+\epsilon W}$ (Notice that we used $\mathrm{Ad}_{\varphi^{-1}} = \mathrm{Ad}_{1-\epsilon W}$ in the above calculation)

$$\mathrm{ad}_{W^\mu\partial_\mu} : V^\mu \mapsto -[W,V]^\mu = W^\nu\frac{\partial V^\mu}{\partial x^\nu} - \frac{\partial W^\mu}{\partial x^\nu}V^\nu, \tag{112}$$

which is the same as the Lie bracket but with an additional minus sign. [This might be a special feature of the infinite dimensional manifold or Lie group.]

On a one-dimensional manifold, the formula gets simplified. For example, on a circle we have

$$\begin{aligned}&\mathrm{Ad}_\varphi : V \to \widetilde{V} = \widetilde{V}(\theta)\partial_\theta, \quad \widetilde{V}(\theta) = \left(\frac{d\varphi^{-1}(\theta)}{d\theta}\right)^{-1} X(\varphi^{-1}(\theta)),\\ &\mathrm{Ad}_{\varphi^{-1}} : V \to \widetilde{V} = \widetilde{V}(\theta)\partial_\theta, \quad \widetilde{V}(\theta) = X(\varphi(\theta))\left(\frac{d\varphi}{d\theta}\right)^{-1}.\end{aligned} \tag{113}$$

This justifies the Eq. (19) used in the main text.

C Sanity check on the Bott cocycle

In this section, we check the Bott cocycle defined by Eq. (41) satisfies the cocycle identity and does correspond to the central extension of $\mathrm{Vect}(S^1)$. For brevity, we use 1/2 as the prefactor for the Bott cocycle in this section.

For the central extension of $\mathrm{Diff}(S^1)$, the group multiplication is written as

$$(\varphi, a) \circ (\psi, b) = (\varphi \circ \psi, a + b + B(\varphi, \psi)) . \tag{114}$$

The cocycle identity follows the associativity and is

$$B(\varphi, \psi) + B(\varphi \circ \psi, \eta) = B(\varphi, \psi \circ \eta) + B(\psi, \eta) . \tag{115}$$

Let us check this by a straightforward calculation. On the left hand side, we have

$$\begin{aligned} B(\varphi \circ \psi, \eta) =& \frac{1}{2} \int \log(\varphi \circ \psi \circ \eta)' \, d \log \eta' \\ =& \frac{1}{2} \int \log \left(\varphi'(\psi(\eta(x))) \psi'(\eta(x)) \eta'(x) \right) \, d \log \eta'(x) \\ =& \frac{1}{2} \int \log \left(\varphi'(\psi(\eta(x))) \right) \, d \log \eta'(x) + B(\psi, \eta) \end{aligned}$$

On the right hand side, we have

$$\begin{aligned} B(\varphi, \psi \circ \eta) =& \frac{1}{2} \int \log(\varphi \circ \psi \circ \eta)' \, d \log(\psi \circ \eta)' \\ =& \frac{1}{2} \int \log \left(\varphi'(\psi(\eta(x))) \psi'(\eta(x)) \eta'(x) \right) \, \left(d \log \psi'(\eta(x)) + d \log \eta'(x) \right) \\ =& \frac{1}{2} \int \log \varphi'(\psi(\eta(x))) d \log \psi'(\eta(x)) + \log \varphi'(\psi(\eta(x))) \, d \log \eta'(x) + \log(\psi \circ \eta)' d \log(\psi \circ \eta)' \\ =& B(\varphi, \psi) + \frac{1}{2} \log \varphi'(\psi(\eta(x))) \, d \log \eta'(x) \end{aligned}$$

It directly shows that the two sides of Eq. (115) are equal to each other.

Then, we derive the corresponding adjoint representation. Let us consider a one-parameter family of diffeomorphism σ_t, which generates the vector field g. The adjoint representation can be deduced by computing

$$\frac{d}{dt} (\varphi, *) \times (\sigma_t, *) \times (\varphi^{-1}, *) \Big|_{t=0} = \frac{d}{dt} \left(\varphi \circ \sigma_t \circ \varphi^{-1}, B(\varphi, \sigma_t) + B(\varphi \circ \sigma_t, \varphi^{-1}) \right) \Big|_{t=0} , \tag{116}$$

where $*$ are constant central elements and can be ignored. The expression has two parts. The first part is the same as the result before central extension. Only the second part involving the Bott cocycle comes from the central extension. Notice that $\sigma_{t=0}$ is the identity map, we have

$$\begin{aligned} \frac{d}{dt} B(\varphi, \sigma_t) \Big|_{t=0} =& \frac{1}{2} \frac{d}{dt} \int \log(\varphi \circ \sigma_t)' d \log(\sigma_t)' = \frac{1}{2} \int \log \varphi' dg' \\ =& - \frac{1}{2} \int \frac{\varphi''}{\varphi'} g' d\theta \end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}B(\varphi\circ\sigma_t,\varphi^{-1})\Big|_{t=0} =& \frac{1}{2}\frac{d}{dt}\int \log(\varphi\circ\sigma_t\circ\varphi^{-1})' d\log(\varphi^{-1})' \\
=& \frac{1}{2}\int \left(\frac{d}{dt}\varphi\circ\sigma_t\circ\varphi^{-1}\right)' d\log(\varphi^{-1})' \\
=& \frac{1}{2}\int \frac{d}{d\theta}\left[\varphi'(\varphi^{-1}(\theta))g(\varphi^{-1}(\theta))\right] d\log\left(\varphi^{-1}(\theta)\right)' \\
=& \frac{1}{2}\int \frac{1}{\varphi'(\alpha)}\frac{d}{d\alpha}\left[\varphi'(\alpha)g(\alpha)\right] d\log\frac{1}{\varphi'(\alpha)}\,, \quad (\alpha=\varphi(\theta)) \\
=& -\frac{1}{2}\int \left[\left(\frac{\varphi''}{\varphi'}g+\frac{\varphi''}{\varphi'}g'\right)\right] d\alpha
\end{aligned}
$$

Summing them up gives us

$$
\frac{d}{dt}\left(B(\varphi,\sigma_t)+B(\varphi\circ\sigma_t,\varphi^{-1})\right)\Big|_{t=0} = \int S(\varphi)g(\theta)d\theta\,. \tag{117}
$$

Therefore, the adjoint representation for $\widehat{\mathrm{Diff}}(S^1)$ is

$$
\mathrm{Ad}_\varphi : (g,a) \mapsto \left(\left(\frac{d\varphi^{-1}(\theta)}{d\theta}\right)^{-1} g(\varphi^{-1}(\theta)),\; a+\int_0^{2\pi} S(\varphi)g(\theta)d\theta\right)\,. \tag{118}
$$

If we assume $\varphi(\theta)=\theta+\epsilon f(\theta)$ is an infinitesimal transformation, we can derive the adjoint representation for $\widehat{\mathrm{Vect}}(S^1)$

$$
\mathrm{ad}_{f\partial_\theta}(g,a) \mapsto \left(-(fg'-f'g),\; -\int_0^{2\pi} fg'''\right) \tag{119}
$$

For the coadjoint representation, we follow its definition and have (we use φ^{-1} here)

$$
\begin{aligned}
\langle \mathrm{Ad}^*_{\varphi^{-1}}(u,t),\,(g,a)\rangle :=& \langle (u,t),\,\mathrm{Ad}_\varphi(g,a)\rangle \\
=& \int d\theta u(\theta)\left(\frac{d\varphi^{-1}(\theta)}{d\theta}\right)^{-1} g(\varphi^{-1}(\theta)) + t\int_0^{2\pi} S(\varphi)g(\theta)+at \\
=& \int d\theta\left(\frac{d\varphi(\theta)}{d\theta}\right)^2 u(\varphi(\theta))g(\theta)+t\int_0^{2\pi} S(\varphi)g(\theta)+at
\end{aligned} \tag{120}
$$

in the last step, we introduce $\alpha=\varphi^{-1}(\theta)$ to the first term and rename α as θ. So the coadjoint representation is

$$
\mathrm{Ad}^*_{\varphi^{-1}}(u,t) \mapsto \left(\left(\frac{d\varphi(\theta)}{d\theta}\right)^2 u(\varphi(\theta))+tS(\varphi),t\right)\,. \tag{121}
$$

The coadjoint representation can be obtained by choosing φ to be an infinitesimal diffeomorphism. By now, we have justified the statement in the main text.

D Hill's equations and classical Liouville CFT

In this section, we will discuss how the Hill's equation naturally arises in the discussion of the classical Liouville CFT. We will show that Hill's equations govern a special primary field and $u(\theta)$ can be interpreted as the stress energy tensor such that the appearance of the $1/2$ scaling and the Schwarzian derivative in Eq. (14) become natural. In this context, the monodromy has a different physical meaning, which is related to the type of background geometry. Throughout our discussion, we will work with Euclidean time and use the complex coordinate

$$\begin{aligned} ds^2_{flat} &= dx^2 + dy^2 = dz d\bar{z}\,, \\ \partial_x = \partial_z + \partial_{\bar{z}}\,&,\; \partial_y = i(\partial_z - \partial_{\bar{z}})\,, \\ \Delta &= \partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}}\,. \end{aligned} \tag{122}$$

Our discussion follows [5].

D.1 Review of the classical limit of Liouville theory

Let us assume the Lagrangian formulation of the Liouville CFT, which is defined by the following action

$$\begin{aligned} S[\sigma, \widehat{g}] &= \frac{1}{2\pi\hbar}\int d^2x\sqrt{\widehat{g}}\left(\frac{1}{2}\widehat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \widehat{R}\sigma(x) + \Lambda e^\sigma\right)\,, \\ \frac{1}{\hbar} &= \frac{26-c}{48}\,, \quad \Lambda = \hbar\Lambda_c\,. \end{aligned} \tag{123}$$

We drop out some surface terms that can be important but not quite relevant for our discussion. Here $\hbar$ is the effective Plank constant. The first two terms come from the interaction between the conformal matters and the curved metric $g_{\mu\nu} = e^\sigma\widehat{g}_{\mu\nu}$. $\widehat{g}$ is the background that encodes the data of topology and σ is the dilaton field. The last term is the cosmological term with the cosmological constant Λ_c. A complete theory requires the path integral over both σ and $\widehat{g}$. In this discussion, we will fix $\widehat{g}$.

The classical limit is $\hbar \to 0$, where the physics is dominated by the classical Euler-Lagrangian equation

$$-\Delta_{\widehat{g}}\sigma + \widehat{R} + \Lambda e^\sigma = 0\,. \tag{124}$$

For a two-dimensional manifold, the Riemann curvatures of two conformally related metric $g = e^\sigma\widehat{g}$ are related by

$$\sqrt{g}R = \sqrt{\widehat{g}}\left(\widehat{R} - \Delta_{\widehat{g}}\sigma\right)\,. \tag{125}$$

Whence Eq. (124) can be rewritten as

$$R + \Lambda = 0\,, \tag{126}$$

which describes *surfaces with constant Riemann curvatures.*

Let us focus on the simplest case. The background metric is chosen to be flat $\widehat{g}_{\mu\nu} = \delta_{\mu\nu}$ such that the action becomes

$$S[\sigma] = \frac{1}{2\pi\hbar}\int d^2x\left(\frac{1}{2}(\partial\sigma)^2 + \Lambda e^\sigma\right) = \frac{1}{2\pi\hbar}\int dz d\bar{z}\left(2\partial_z\sigma\partial_{\bar{z}}\sigma + \Lambda e^\sigma\right)\,, \tag{127}$$

and the Euler-Lagrangian equation is simplified to

$$-4\partial_z\partial_{\overline{z}}\sigma + \Lambda e^{\sigma} = 0\,, \tag{128}$$

which is known as the *Liouville equation*. The conformal invariance property manifests by examining the stress-energy tensor

$$\begin{aligned} T^{\mu\nu} =& \frac{2\times 2\pi\hbar}{\sqrt{g}}\frac{\delta S[\sigma,\widehat{g}]}{\delta\widehat{g}_{\mu\nu}}\Big|_{\widehat{g}=\delta} \\ =& -\partial^\mu\sigma\partial^\nu\sigma + \delta^{\mu\nu}\left(\frac{1}{2}(\partial\sigma)^2 + \Lambda e^{\sigma}\right) + 2\left(\partial^\mu\partial^\nu\sigma - \delta^{\mu\nu}\partial^2\sigma\right)\,. \end{aligned} \tag{129}$$

In particular, the trace of the stress-energy tensor

$$T^\mu_\mu = 2\left(\Lambda e^{\sigma} - \Delta\sigma\right) \tag{130}$$

vanishes when σ satisfies the Liouville equation, which is the characterization of conformal invariance for a classical theory. As a result, we can define purely holomorphic and anti-holomorphic stress-energy tensor for the on-shell σ.

$$T(z) = T_{zz} = -\partial_z\sigma\partial_z\sigma + 2\partial_z^2\sigma\,,\quad \overline{T}(\overline{z}) = T_{\overline{z}\,\overline{z}} = -\partial_{\overline{z}}\sigma\partial_{\overline{z}}\sigma + 2\partial_{\overline{z}}^2\sigma\,. \tag{131}$$

To see the conformal invariance at the level of the action, a transformation $z \to w(z), \overline{z} \to \overline{w}(\overline{z})$ leaves the action Eq. (127) invariant (up to a boundary term) if the σ fields transforms as

$$\sigma(z,\overline{z}) \to \widetilde{\sigma}(w,\overline{w}) = \sigma(z,\overline{z}) - \log\frac{dw}{dz}\frac{d\overline{w}}{d\overline{z}}\,. \tag{132}$$

Accordingly, the exponentials $e^{\lambda\sigma}$ transforms as primary fields and the (anti-)holomorphic stress-energy tensor transforms as

$$T(z) \to \widetilde{T}(w) = -\partial_w\widetilde{\sigma}\partial_w\widetilde{\sigma} + 2\partial_w^2\widetilde{\sigma} = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + 2\{z,w\}\,. \tag{133}$$

D.2 Relation to Hill's equations

The relation between the Liouville equation and the Hill's equation appears when we study the (primary) field $e^{-\sigma/2}$. Straightforward calculation shows that

$$\left(-4\partial_z^2 + T(z)\right)e^{-\sigma/2} = 0\,,\quad \left(-4\partial_{\overline{z}}^2 + \overline{T}(\overline{z})\right)e^{-\sigma/2} = 0\,, \tag{134}$$

namely $e^{-\sigma/2}$ satisfies a homolomorphic and an anti-holomorphic Hill's equation.

This properties suggests the following method of solving the Liouville equation Eq. (128). Consider the Hill's equations

$$\left(-4\partial_z^2 + T(z)\right)\psi(z) = 0\,,\quad \left(-4\partial_{\overline{z}}^2 + \overline{T}(\overline{z})\right)\overline{\psi}(\overline{z}) = 0\,. \tag{135}$$

$T(z)$ and $\psi(z)$ are holomorphic functions satisfying the transformation rules under $z \to w$

$$\begin{aligned} \psi(z) \to \widetilde{\psi}(w) &= \left(\frac{dz}{dw}\right)^{-1/2} \psi(z) \\ T(z) \to \widetilde{T}(w) &= \left(\frac{\partial z}{\partial w}\right)^{2} T(z) + 2\{z, w\} \end{aligned} \tag{136}$$

such that Eq. (135) is invariant. Let us denote the two independent solutions by $(\psi_1, \psi_2)^T$, then the following combination

$$\sigma(z, \overline{z}) = -2 \log \left(\overline{\psi}(\overline{z}) \mathbf{\Lambda} \psi(z)\right) + \log 8 \tag{137}$$

solves the Liouville equation Eq. (128) provided $\det \mathbf{\Lambda} = \Lambda$. Here $\mathbf{\Lambda}$ is a constant two-by-two matrix. By now, we provide a motivation for the transformation laws Eq. (14) for the Hill's equation listed in the main text.

D.3 Monodromy of Hill's equations and the implication

In this context, Eq. (135) can also have monodromy phenomena when $T(z)$ has a canonical singularity. For example, if $T(z) = r/z^2$ at $z = 0$, then $\psi(z)$ can have elliptic, hyperbolic or parabolic type of monodromy when $r < 1$, $r > 1$ or $r = 1$ respectively. [This imposes conditions on the choice of $\mathbf{\Lambda}$ because $\sigma(z, \overline{z})$ has to be analytical and single-valued everywhere. A full discussion seems to require addition attention to the boundary effect. For example, $ds^2 = dz d\bar{z}$ is actually singular at the infinity. There might be conical singularity in the geometry. Need better understanding in the future.]

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