Note on AdS_2 Spacetime

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Abstract

Note on AdS_2 spacetime. Talk about different choices of coordinate.

Version: December 21, 2018

Contents

AdS spacetime has been more and more popular because of the holographic duality. This gravity itself is also interesting. Its most important feature is that it has a boundary[1] (or two boundaries for AdS_2) so it serves as a container of gravity. This brings it many interesting properties. For example, (2+1)D Minkowski space doesn't allow a black hole solution but we can have a black hole in AdS_3, so called BTZ black hole. Another example is of course holographic duality.

Here we only talk about AdS_2, which is supposed to be the simplest member of this gorgeous family. Just as his brothers, it can be defined as a sphere embedded in a Minkowski-type spacetime $M^{1,2}$, which is illustrated in Fig.1

$$-T^2 - W^2 + X^2 = -1, \quad ds^2 = -dT^2 - dW^2 + dX^2. \tag{1}$$

The difference is AdS_2 has two boundaries while others have one boundary.

1 Symmetry of the spacetime

From the embedding coordinate, we can see that there are three "rotation" symmetries of the manifold. One is the $SO(2)$ rotation on $T-W$ plane, the other two are the $SO(1,1)$ boost on $T-X$ and $W-X$ planes. So The isometry group for AdS_2 is $SO(1,2)$, which is

[1]The boundary may still sit at the infinity in our coordinate but it takes finite time for a light ray to reach that point.

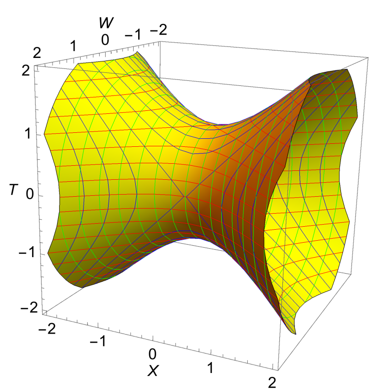

Figure 1: AdS_2 manifold plotted in a Euclidean space $T-W-X$.

isomorphic to $SL(2,R)$ (see Appendix.B for a proof using fundamental representation of $SO(1,2)$).

The three symmetry generators correspond to three "rotations":

- $T-W$ rotation, $g_1 = T\partial_W - W\partial_T$;
- $T-X$ boost, $g_2 = T\partial_X + W\partial_T$;
- $W-X$ boost, $g_3 = W\partial_X + X\partial_W$.

One can compute their commutators, which are the same as the Lie algebra of $SL(2,R)$,

$$[g_1, g_2] = -g_3, \quad [g_2, g_3] = g_1, \quad [g_3, g_1] = -g_2. \tag{2}$$

In the following, we are going to find coordinates which have static metric. So the construction there will, to a large extend, be motivated by choosing the time to lie along the Killing vector of one of these three symmetries.

2 Different coordinates

There are many ways to parametrize this manifold [1, 2]. They are not all equivalent. Some can only cover part of the whole manifold. This section will focus on Lorentzian AdS_2. Euclidean AdS_2 is very similar in many aspects and the discussion for that is left in Appendix.C.

2.1 Global coordinates

These three coordinates below cover the whole AdS_2 spacetime (strictly speaking, the unwinded AdS_2). All of them use the $T-W$ rotation to find a time-like Killing vector and time is just the rotational angle. If we fix a spatial point, the spatial slide will be a circle as demonstrated in Fig.2

(1) Spherical coordinate First, we can use the $T-W$ rotation to find a coordinate. Define $T = R\cos t, W = R\sin t, X = r$. So $R^2 = 1 + r^2$. Here t is just the rotation angle of $T-W$. So ∂_t must be a time-like Killing vector. The metric is,

$$ds^2 = -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2. \tag{3}$$

Now time is defined on a circle. To preserve causality, we unwind the time direction and extend it to $(-\infty, \infty)$. The spatial boundaries are at $r = -\infty$ and $r = \infty$.

This coordinate is useful to study black hole in AdS_4. For example, one can show that in $(3+1)D$, there is a Schwartzchild-AdS metric,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad , f(r) = 1 - \frac{C}{r} + k^2 r^2, \tag{4}$$

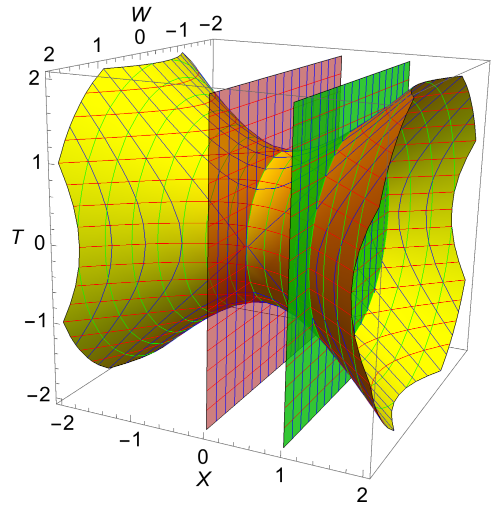

Figure 2: Illustrating global coordinate in embedding coordinate. To get the spatial slides for global coordinates, we just use planes perpendicular to X axis to cut the manifold.

satisfying the Einstein equation $R_{\mu\nu} = -\Lambda g_{\mu\nu}$, where $3k^2 = \Lambda$, i.e. determined by the AdS length and C is determined by the mass of black hole. It takes Schwartzchild form at short distance and has a asymptotic AdS exterior region.

(2) Conformal coordinate We further define $r = \tan\psi$ to compactify the space. So the boundaries now are at $\psi = \pm\pi/2$. And metric is,

$$ds^2 = \frac{1}{\cos^2\psi}\left(-dt^2 + d\psi^2\right). \tag{5}$$

which is conformally flat.

(3) Hyperbolic coordinate We can also define $r = \sinh\rho$. So the boundaries are at $\rho = \pm\infty$. The metric is,

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2. \tag{6}$$

2.2 Coordinates that are not global

Now, we are going to use the $W - X$ boost symmetry (or $T - X$ boost, they give equivalent result) to define our time. The corresponding coordinates we get will only cover a patch of the full AdS_2. And some of them have an extended region.

(4) Poincaré coordinate If we define a lightcone coordinate for W, X, say $X^+ = W + X, X^- = W - X$, then the boost becomes a dilation of the lightcone coordinate, $T \to T, X^\pm \to e^{\pm\lambda}X^\pm$.

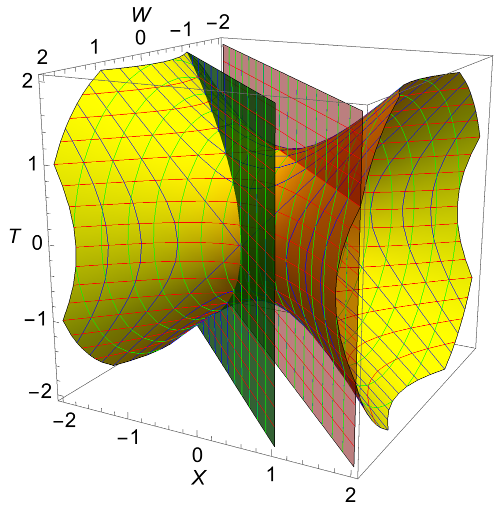

Figure 3: Illustrating Poincaré coordinate in embedding coordinate. Red plane corresponds to $w = \infty$ and green plane corresponds to $w = 1$. We can see that $w > 0$ covers half of the AdS_2. And the $w = \infty$ plane consists of two parallel null straight lines. The $w > 0$ planes give us parabolas.

Motivated by this, we define $T = \frac{t}{w}$, $W + X = \frac{1}{w}(-t^2 + w^2)$, $W - X = \frac{1}{w}$. The metric is,

$$ds^2 = \frac{1}{w^2}(-dt^2 + dw^2). \tag{7}$$

Sometimes people also define $w = 1/r$ and the corresponding metric is,

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} \tag{8}$$

If we compare it with the Eqn.3 we can see that they are the same at $r \to \infty$. So in Poincaré coordinate, the boundaries are at $r = \pm\infty$ or $w = \pm 0$. However, $r = 0$ is the infinite red-shift surface. So we can only use $r > 0$ or $r < 0$ at one time. In this sense, we say Poincaré coordinate only covers half of AdS_2. It's good to relate the Poincaré coordinate (t_P, w) to conformal coordinate (t, ψ),

$$t_P = \frac{\sin\psi + \sin t}{\cos^2\psi - \cos^2 t}\cos t, \quad w = \frac{\sin\psi + \sin t}{\cos^2\psi - \cos^2 t}\cos\psi. \tag{9}$$

To visualize the Poincaré coordinate, we fix w and look at the spatial slide in embedding coordinate, as illustrated in Fig.3

(5) Rindler coordinate A more straightforward way is just use the boost angle as our time. So we define $W = R\sinh t$, $X = R\cosh t$, $T = r$. The constrain of R and r is

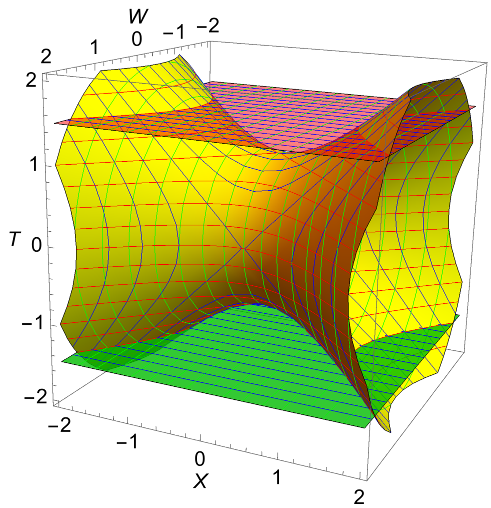

Figure 4: Illustrating Rindler coordinate in embedding coordinate. To get the spatial slides for Rindler coordinates, we just use planes perpendicular to T axis to cut the manifold. Red plane corresponds to $r > 1$ and green plan corresponds to $r < -1$.

$R^2 = r^2 - 1 > 0$, which means this coordinate transformation is legitimate only when $r^2 > 1$. And the metric is,

$$ds^2 = -(r^2 - 1)dt^2 + \frac{1}{r^2 - 1}dr^2. \tag{10}$$

This Rindler form can also be written in hyperbolic form: define $r = \cosh\rho$ and the metric becomes,

$$ds^2 = -\sinh^2\rho dt^2 + d\rho^2. \tag{11}$$

An intuitive way to understand this coordinate is to go back to the embedding coordinate. We still fix a spatial point and look at the spatial slide. This time it is two disconnected hyperbola, as illustrated in Fig.4. And the two hyperbola are separated by two lightcones at $r = 1$, so the two branches physically cannot talk to each other either.

(6) Kruskal-Szekeres coordinate Motivated by the Rindler form, we can write down a K-S coordinate for AdS_2. Rewrite the Rindler form as,

$$ds^2 = -(r^2 - 1)\left[dt^2 - \frac{dr^2}{(r^2 - 1)^2}\right] = -(r^2 - 1)dx^+dx^-.$$

where $dx^\pm = dt \pm \frac{1}{r^2-1}dr$. An integration tells us $x^\pm = t \pm \frac{1}{2}\log|\frac{r-1}{r+1}|$. However, in terms of $x^\pm$ the metric still has a coordinate singularity at $r^2 = 1$. So we define

$$U = e^{x^+} = \sqrt{\frac{r-1}{r+1}}e^t, \quad V = -e^{-x^-} = -\sqrt{\frac{r-1}{r+1}}e^{-t}, \quad r > 1, \tag{12}$$

which satisfies,

$$UV = -\frac{r-1}{r+1}, \quad U/V = -e^{2t}. \tag{13}$$

The metric is,

$$ds^2 = -\frac{4dUdV}{(1+UV)^2} \tag{14}$$

Understand the extended Kruskal coordinate. Is there a horizon or something?

3 Geodesics in AdS_2

3.1 Massive particles

We can study this problem by writing down the Christoffel symbol and solve the geodesic equation. However, there is a much more elegant method using the embedding coordinates [1]. To make equations more compact, we use $Y = (Y^{-1}, Y^0, Y^1)$ to denote the embedding coordinate. And the inner product is $Y \cdot W = g_{MN} Y^M W^N = -Y^{-1}W^{-1} - Y^0 W^0 + Y^1 W^1$.

Instead of extremizing $\int \sqrt{-g_{MN} Y^M Y^N}$, we can equivalently extremize the following action,

$$S = \int d\zeta \left(\frac{1}{2}(d_\zeta Y)^2 - \frac{\lambda}{2}(Y^2 + 1) \right) \tag{15}$$

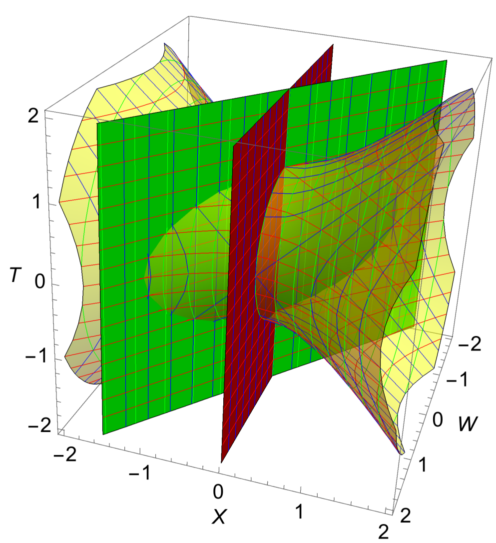

Figure 5: Illustrating geodesics for massive particles in embedding coordinate. The space-like vector we choose is $(0, \sinh\alpha, \cosh\alpha)$. Red plane is for $\alpha = 0$ and green plane is for $\alpha = 1$. The larger α is, the larger the velocity of the particle is.

where ζ is some parameter of the trajectory and λ is the Lagrangian multiplier giving us the constrain $Y^2 = -1$. Taking a derivative we can get $Y \cdot d_\zeta Y = 0$. Variation of the action w.r.t. Y^M gives,

$$\frac{\delta S}{\delta Y_M} = d_\zeta^2 Y^M + \lambda Y^M = 0. \tag{16}$$

So we have $d_\zeta Y \cdot d_\zeta^2 Y = 0$ and $\lambda = 1$ if ζ is the proper length. From these equations, we can find a conserved quantity,

$$J^{MN} = Y^M d_\zeta Y^N - d_\zeta Y^M Y^N. \tag{17}$$

Because J^{MN} is anti-symmetric, there are 3 conserved quantities, corresponding to the three killing vectors. For massive particles, its trajectory is time-like. So one can show that $J^2 = J^{MN} J_{MN} = -2(d_\zeta Y)^2 > 0$ is space-like. Due to the fact that Y has three components, one can write J^{MN} in a more compact way as,

$$Q_L = \frac{1}{2}\epsilon_{LMN} J^{MN} = \epsilon_{LMN} Y^M d_\zeta Y^N. \tag{18}$$

Now we have another way to write down the geodesics: (1) choose a space-like vector Q^L; (2) the trajectory is determined by the equation $Q \cdot Y = 0$ and Q is the associated $SL(2)$ charge. Some typical trajectories passing through the point $(1, 0, 0)$ are shown in Fig.5.

Remark Because all the points on the AdS_2 manifold can be connected by $SO(1, 2)$ transformation, once we know the massive geodesics through $(0, 0, 1)$, we can use $SO(1, 2)$ to get any other massive geodesics passing through any other points [2].

3.2 Massless particles

As we comment at the end of last subsection, if we're curious of the null geodesics going through an arbitrary point P, all we need to do is first find all the null geodesics passing through some special point say $(0, 0, 1)$, then use $SO(1, 2)$ to generate what we want.

The null geodesics through $(0, 0, 1)$ are two straight lines $T = 1, W = \pm X$, as shown in Fig.6

3.3 Geodesic distance

Using geodesics, we can define an important quantity: the geodesic distance between two points. learn this. Maybe it is easier to just do the calculation in Euclidean AdS. Focus on computing the geodesic distance between two points on the boundary of Poincaré disk.

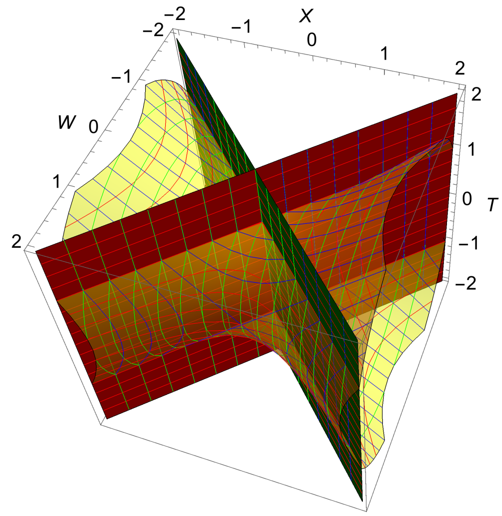

Figure 6: Illustrating geodesics for massless particles in embedding coordinate. The two planes can be got from $Q \cdot Y = 0$, with $Q = (0, \sinh\alpha, \cosh\alpha)$, $\alpha \to \pm\infty$. As we point out before, α represent the velocity of the particles. It now makes more sense that by tuning α to be infinity we get the trajectories for light.

4 Nearly AdS_2

After studying several properties of AdS_2, a natural question is how to generate such a geometry from some physical theories. A first attempt is of course Einstein gravity. However, due to Gauss-Bonnet theorem, (1+1)D Einstein gravity is topological, i.e. $S_{EH} + S_{GHK} = \text{const}$. So if we write down $S_{EH} + S_{GHK} + S_{cosmology}$, the equation of motion cannot give us an AdS_2 spacetime[2]. We need to look for something else. In the following, we will give two examples. In this section, we study the 2D dilaton gravity [3]. In the next section, we will look at classical Liouville theory briefly [4].

We want a AdS_2 bulk geometry. The easiest way to realize this is to introduce a Lagrangian multiplier to pin the scalar curvature at a negative value. So we get the following (Euclidean) action, which is called Jackiw-Teitelboim theory,

$$I[g, \phi] = -\frac{1}{16\pi G}\left[\int d^2\sqrt{g}\phi(R+2) + 2\int_{bdy}\phi_b K\right]. \tag{19}$$

In two dimension, G and ϕ are dimensionless. And boundary integral is necessary for a well-defined variation problem with Dirichlet boundary condition. The equations of

[2]The fact that (1+1)D Einstein gravity is purely topological also implies it doesn't permit finite energy density. This is related to a similar dilemma in (0+1)D CFT. For a CFT1 $S = \int d\tau \mathcal{L}[O, \partial O]$, conformal invariance tells us transformation $\tau \to \widetilde{\tau}, O \to \widetilde{O}$ keeps the action invariant. We can also think about this transformation as couple the original system to a $g_{\tau\tau} = d\widetilde{\tau}/d\tau$. So this invariance actually implies $T_{\tau\tau} = 0$, which means CFT1 cannot support finite energy states.

motion for ϕ and $g_{\mu\nu}$ are,

$$\frac{\delta I}{\delta \phi} \propto R + 2 = 0, \tag{20}$$

$$-\frac{2}{\sqrt{g}}\frac{\delta I}{\delta g_{\mu\nu}} = \frac{1}{8\pi G}(D_\mu D_\nu \phi - g_{\mu\nu}\nabla^2\phi + g_{\mu\nu}\phi) = 0. \tag{21}$$

The first equation, as we said before, completely fixes the bulk geometry to be an AdS_2. The second equation[3] fixes the form of the bulk solution for ϕ. The only residual degrees of freedom live on the boundary: how to cut out the boundary curve and assign the boundary value for ϕ field.

4.1 Boundary curve and reparamatrization symmetry

Let's do the discussion using Poincaré coordinate which describes the Poincaré disk. The boundary curve follows the trajectory $(t(u), w(u))$, which we always assume to be very close to the boundary of Poincaré disk, i.e. $w \ll 1, w' \ll 1$. And $t \to \pm\infty$ are identified as the same point. However, given a curve, there are many ways to parametrize it. To get rid of this redundancy, we fix the induced metric of this curve,

$$g_{uu} = \frac{t'^2 + w'^2}{w^2} = \frac{1}{\epsilon^2}, \epsilon \ll 1 \Rightarrow w = \epsilon t' + O(\epsilon^3). \tag{22}$$

However, given two different trajectories, we still need to be careful whether they give different boundary conditions. This is because the interior geometry has a $SL(2,R)$ symmetry. If we do a $SL(2,R)$ transformation to the whole space, the physics is still the same. Thus if two boundary curves are related by a $SL(2,R)$ transformation, we should identify them as the same boundary condition. Generically such a $SL(2,R)$ acts on the boundary in a complicated way. But when $\epsilon \ll 1$, it is simply reduced to a $SL(2,R)$ for $t(u)$ with the same coefficients as the bulk $SL(2,R)$ [4],

$$t(u) \to \widetilde{t}(u) = \frac{at+b}{ct+d}, ad - bc = 1, \tag{23}$$

Therefore, the full set of different geometries is composed of different $t(u)$ up to the above $SL(2,R)$ transformation.

We now want to a situation where the bulk geometry arises as a low energy limit of a UV theory, which lives on the boundary. And we'd like to identify the parameter u as the

[3] To solve the second equation, we can contract it with $g^{\mu\nu}$ and get $\nabla^2\phi - 2\phi = 0$. E.g. In Poincaré coordinate, we have $\nabla^2 = w^2\partial_t^2 + w^2\partial_w^2$, which gives us a general solution $\phi = \dfrac{\alpha + \gamma t + \delta(t^2 + w^2)}{w}$.

[4] As we discussed in Appendix.C, the $SL(2,R)$ in the bulk is $z \to az + b/cz + d$, where $z = t + iw$. To see how it acts on the boundary, we can check the infinitesimal transformation. An infinitesimal translation in z induces a translation in t, so it is for scaling. For SCT, $t \to t - ct^2 + O(\epsilon^2)$. In the limit, $\epsilon \ll 1$, it can be approximated as a SCT for t.

boundary time. Because the UV theory lives on a closed curve, which implies it is a finite temperature system with a definite temperature. To be consistent with this point, when we cut out the boundary curve $(t(u), w(u))$, we will assume $u \in [0, \beta]$. We also need our UV theory to have a definite high energy cut-off (for SYK, it is the coupling J). In the bulk, this requires ϵ to be fixed, which means we consider the boundary curve with a fix total proper length $L = \beta/\epsilon$ (for SYK , this corresponds to a fixed βJ).[5]

Remark Before we move on to the discussion of dilaton field, let's give a short remark. If there is no dilaton field, then those different boundary conditions will give us the same energy. And different boundary curves are related by a reparametrization $\widetilde{t} = \widetilde{t}(t)$ [6]. So our action has a full reparametrization symmetry. However, once we pick up a boundary curve, the symmetry is *spontaneously broken* to $SL(2, R)$. All of these resemble what happens in the SYK model.

4.2 Boundary condition for ϕ and Schwarzian action

Now we talk about the boundary condition for dilaton field and show that the corresponding effective action is a Schwarzian.

Recall our discussion before, the solution for ϕ in the bulk talks the following for,

$$\phi = \frac{\alpha + \gamma t + \delta(t^2 + w^2)}{w} = Z \cdot Y \tag{24}$$

where α, γ, δ are three arbitrary constant. Y is the embedding coordinate and Z is an arbitrary vector. This bulk solution tell us the following things:

- Eqn.24 is a natural solution from the symmetry perspective. The problem has a $SO(2,1)$ symmetry. So in term of embedding coordinate, if $\phi(Y)$ is a solution, $\phi(R.Y)$ is also a solution (R is a $SO(2,1)$ rotation). This implies $\phi(Y)$ has to be a scalar function when expanded w.r.t Y. Eqn.24 is the simplest non-trivial form satisfying this condition.

- Understand how the physical constraint on ϕ restricts the choice of α, β, γ.

- Once we write down a solution for ϕ, the bulk symmetry is broken from $SL(2, R)$ to a $U(1)$ rotation w.r.t. the vector Z. This assumes that the Z rotation is a symmetry for the bulk geometry. What if this is wrong? Does it means that we lose all the symmetries?

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[5] High energy cut-off for the boundary CFT corresponds to short distance cut-off in the bulk gravity. Such a thing is general in AdS/CFT correspondence.

[6] In the limit $\epsilon \to 0$, $w = \epsilon t'$. So such a reparametrization doesn't change the induced metric g_{uu}.

- ϕ is diverging near the boundary. So we define the following boundary condition to subtract the divergence,

$$\phi_{bdy} = \frac{\phi_r(u)}{\epsilon}, \tag{25}$$

where $\phi_r(u)$ is the renormalized dilaton field, which still remains finite in the limit $\epsilon \to 0$. Of course, assigning this boundary condition also restrict the choice of boundary curves in order to make the bulk solution for ϕ consistent with its boundary condition.

Now let's calculate the effective action to see the effect of dilaton field. Due to the equation of motion of ϕ, the first term in Eqn.19 vanishes. So all the dynamics is encoded to the second term,

$$I_{JT} = -\frac{1}{8\pi G}\int \frac{du}{\epsilon}\frac{\phi_r(u)}{\epsilon}K(u). \tag{26}$$

The extrinsic curvature is given by,

$$K = \frac{t'(t'^2 + z'^2 + zz'') - zz't''}{(t'^2 + z'^2)^{3/2}} = 1 + \epsilon^2\,\mathrm{Sch}(t,u), \tag{27}$$

$$\mathrm{Sch}(t,u) = -\frac{1}{2}\left(\frac{t''}{t'}\right)^2 + \left(\frac{t''}{t'}\right)'. \tag{28}$$

Plugging this into I_{JT} we get,

$$I_{eff} = -\frac{1}{8\pi G}\int du\phi_r(u)\,\mathrm{Sch}(t,u) - \frac{1}{8\pi G}\int \frac{du}{\epsilon}\frac{\phi_r(u)}{\epsilon}. \tag{29}$$

With a fixed boundary condition, the second term is just a number. So all the effect is described by the Schwarzian action. The time-dependence of $\phi_r(u)$ can be absorbed into a redefinition of boundary time $d\widetilde{u} = \frac{\overline{\phi}_r}{\phi_r(u)}du$, where $\overline{\phi}_r$ is a constant that fixes the range for $\widetilde{u}$ to be $[0,\beta]$. Then by applying the chain rule for Schwarzian, we will get,

$$\begin{aligned}
\int du\phi_r(u)\,\mathrm{Sch}(t,u) &= \int du\phi_r(u)\left[\mathrm{Sch}(\widetilde{u},u) + \left(\frac{d\widetilde{u}}{du}\right)^2\mathrm{Sch}(t,\widetilde{u})\right] \\
&= \int du\phi_r(u)\,\mathrm{Sch}(\widetilde{u},u) + \int d\widetilde{u}\phi_r(u)\left(\frac{d\widetilde{u}}{du}\right)\mathrm{Sch}(t,\widetilde{u}) \\
&= \int du\left[-\frac{1}{2}\left(\frac{\phi_r'}{\phi_r}\right) - \phi_r'\right]' + \overline{\phi}_r\int d\widetilde{u}\,\mathrm{Sch}(t,\widetilde{u}).
\end{aligned}$$

Because $\phi_r(u)$ lives on a closed curve, the total derivative term should vanish and we finally get a Schwarzian w.r.t. the new boundary time $\widetilde{u}$. However, the induced metric is also changed to a time-dependent one. But if one doesn't care about this and only consider the effective action, this result tells us it is equivalent to choose the following more convenient boundary condition,

$$g_{uu} = \frac{1}{\epsilon^2}, \quad \phi_{bdy} = \frac{\overline{\phi}_r}{\epsilon}. \tag{30}$$

Remark Before we consider dilaton, different choices of boundary curves are equivalent, which means the problem has a full reparametrization symmetry. However, the introducing of dilaton renders the action to be a Schwarzian, which reduces the symmetry to $SL(2, R)$. This is an *explicit symmetry breaking*, in contrary to the spontaneous symmetry breaking we comment before. This pattern also resembles the explicit symmetry breaking of conformal symmetry in SYK model.

4.3 Geometric Interpretation for the Schwarzian

The appearance of Schwarzian has a simple geometric interpretation if we assume $\phi_r(u) = \phi_r$ is a constant [6]. Gauss-Bonnet theorem tells us $\int R + 2\int_{bdy} K = 4\pi$. So under the constrain $R = -2$, the action can be written as,

$$I_{JT} = -\frac{\phi_r/\epsilon}{8\pi G}\int_{bdy} K = \frac{\phi_r/\epsilon}{16\pi G}\left(-4\pi + \int d^2x\sqrt{g}R\right) \rightarrow I_{eff} = -\frac{\phi_r/\epsilon}{8\pi G}\int d^2x\sqrt{g} \tag{31}$$

In the last step, we use the constrain $R = -2$ and drop the constant term. With the consideration of the boundary condition, the problem is now reduced to computing the area enclosed by a curve with a proper length $L = \beta/\epsilon$ in the hyperbolic space. It is easier to do this calculation in the real version of Eqn.57 [7],

$$ds^2 = \frac{4(dr^2 + r^2 d\tau^2)}{(1-r^2)^2}, 0 < r < 1.$$

The boundary curve is parametrized by $(r(u), \tau(u))$, where we also assume $r \sim 1, r' \ll 1$. We fix the induced metric in the same way as before, which also helps us to find the boundary curve satisfies[7],

$$g_{uu} = \frac{1}{\epsilon^2} = \frac{4(r'^2 + r^2\tau'^2)}{(1-r^2)^2} \Rightarrow r = 1 - \epsilon\tau' + \frac{\epsilon^2\tau'^2}{2}. \tag{32}$$

[7]It's easy to fix the leading order of r. We assume $r = 1 - \epsilon\tau' + \epsilon^2 f$ to the next order. Then the boundary condition becomes, $1 = \frac{\tau'^2 - 2\epsilon\tau'^3}{(\tau' - \epsilon f - \epsilon\tau'^2/2)^2}$. By matching two hand sides to the order ϵ, we get the answer.

The area is,

$$
\begin{aligned}
A &= \int d^2x\sqrt{g} = \int_0^{2\pi} d\tau \int_0^{r(\tau)} dr \frac{4r}{(1-r^2)^2} \\
&= 2\int_0^{2\pi} d\tau \left(-1 + \frac{1}{1-r(\tau)^2}\right) = -4\pi + \frac{1}{\epsilon}\int_0^{2\pi} d\tau \frac{1}{(r'^2 + r^2\tau'^2)^{-1/2}}.
\end{aligned}
$$

In the last step we use the boundary condition Eqn.32. Expand the expression w.r.t. ϵ, we get,

$$
\begin{aligned}
A &= -4\pi + \frac{1}{\epsilon}\int_0^{2\pi} d\tau \left(\frac{1}{\tau'} + \epsilon + \frac{\tau'^4 - \tau''^2}{2\tau'^3}\epsilon^2\right) \\
&= -2\pi + L + \epsilon \int_0^{\beta} du \, \frac{1}{2}\left[\tau'^2 - \left(\frac{\tau''}{\tau'}\right)^2\right] \\
&= -2\pi + L + \epsilon \int_0^{\beta} du \, \mathrm{Sch}(\tan\frac{\tau}{2}, u) \\
\Rightarrow I_{eff} &= -\frac{\phi_r}{8\pi G}\int du \, \mathrm{Sch}(\tan\frac{\tau}{2}, u) + O(1/\epsilon)
\end{aligned}
$$

So we can see that the Schwarzian action describes the area enclosed by a curve with fixed length in hyperbolic space. The saddle point will be given by the configuration that has the largest area.

4.4 $SL(2,R)$ symmetry of Schwarzian

Under stand this part. (1) Use Noether theorem to compute the charges, (2) check their quadratic Casmir? not sure. Compare the result with the Hamiltonian.

What is the Hamiltonian if we require a translation symmetry w.r.t. u.

5 Classical Liouville theory

The Liouville action with a cosmology term is written as,

$$
S[\sigma, \widehat{g}] = \frac{1}{2\pi}\int d^2x \, \sqrt{\widehat{g}}\left[\frac{1}{2}\widehat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \sigma\widehat{R} + \Lambda e^{\sigma}\right] \tag{33}
$$

where $\widehat{g}_{\mu\nu}$, $\widehat{R}$ is the reference metric and curvature, which is fixed in our problem. σ is a conformal factor, also the dynamical field, relating reference metric to the physical metric is $g_{\mu\nu} = e^{\sigma}\widehat{g}_{\mu\nu}$. In 2D, the physical curvature can be written in terms of $\widehat{R}$ in a simple way[8],

$$\sqrt{g}R = \sqrt{\widehat{g}}(\widehat{R} - \Delta_{\widehat{g}}\sigma). \tag{34}$$

So the equation of motion for σ field is (of course, we need to add some appropriate boundary term to the action to get this bulk EoM),

$$e^{-\sigma}(\widehat{R} - \Delta_{\widehat{g}}\sigma) + \Lambda = R + \Lambda = 0. \tag{35}$$

A convenient choice is conformal gauge $d\widehat{s}^2 = dzd\overline{z}$ such that the equation of motion is simplified as the Liouville equation,

$$4\partial_z\partial_{\overline{z}}\sigma = \Lambda e^{\sigma}. \tag{36}$$

For negative curvature $\Lambda > 0$, Liouville equation yields a solution [4],

$$\sigma(z, \overline{z}) = -2\log(1 - z\overline{z}) + \log(8/\Lambda), \quad ds^2 = \frac{8}{\Lambda}\frac{dzd\overline{z}}{(1 - z\overline{z})^2}, \tag{37}$$

which is the Poincaré patch of Euclidean AdS_2. If we fix the boundary condition in the same way as we did for dilaton gravity, then we can find that the effective action is,

$$S_{eff} = \frac{1}{2\pi}\int d^2z[\Lambda e^{\sigma} + 2\partial_z\sigma\partial_{\overline{z}}\sigma] + \text{boundary terms}. \tag{38}$$

The first term counts the area so will give us a Schwarzian action. The second term will diverge as $\epsilon \to 0$ so we expect it to be canceled by the boundary term and other counter terms. So although we start from different theories, we end up with the same effective action which is the Schwarzian.

A Schwartzchild-AdS metric

We add negative dark energy to the (3+1)D universe. Then the EH action is written as,

$$S = \int d^4x\left(\frac{1}{16\pi G}R + \Lambda\right), \tag{39}$$

which gives the following Einstein equation,

$$R_{\mu\nu} = -8\pi G\Lambda g_{\mu\nu} = -\widetilde{\Lambda}g_{\mu\nu}. \tag{40}$$

[8]Using this relation, the action can be rewritten as,

$$S[\sigma, g] = \frac{1}{2\pi}\int d^2x\,\sqrt{g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \sigma R + \Lambda\right].$$

It looks similar to Jackiw-Teitelbom theory we write down before but they are not exactly the same.

We assume there is a static spherical-symmetric solution in the following form,

$$ds^2 = -A(r)dr^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{41}$$

Then one can calculate the Ricci tensor by brute force and show that each component of the Einstein equation is [1],

$$R_{tt} = \frac{A''}{2B} + \frac{B'}{rB} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) = -\widetilde{\Lambda} g_{tt}, \tag{42}$$

$$R_{rr} = \frac{A''}{2A} + \frac{B'}{rB} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) = -\widetilde{\Lambda} g_{rr}, \tag{43}$$

$$R_{\theta\theta} = 1 - \frac{1}{B} - \frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B}) = -\widetilde{\Lambda} g_{\theta\theta}, \tag{44}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} = -\widetilde{\Lambda} g_{\phi\phi}, \text{ which is automatically satisfied.} \tag{45}$$

Combining the first two equations, we get $R_{tt}/A + R_{tt}/B = 0$, which leads to $AB = 1$. Plugging this into the third equation, we can finally find,

$$A(r) = f(r) = 1 - \frac{C}{r} + k^2 r^2, \quad 3k^2 = \widetilde{\Lambda}, \tag{46}$$

with C undetermined.

B Isomorphism between $SO(1,2)$ and $SL(2,R)$

The fundamental representation of the indefinite orthogonal group $SO(p,q)$ is a matrix group satisfying,

$$gA^T g = A^{-1}, \quad g = \mathrm{diag}(\underbrace{1,1,1,..}_{p}\underbrace{-1,-1...}_{q}) \tag{47}$$

Specialized to $SO(1,2)$, we have,

$$gA^T g = A^{-1}, \quad g = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}. \tag{48}$$

In the infinitesimal form $A = 1 + a$, $A^{-1} = 1 - a$, this definition will give us,

$$ga^T g = -a, \quad \text{or} \quad (ag)^T - ag. \tag{49}$$

So ag has to be an anti-symmetric matrix,

$$ag = \begin{pmatrix} & 1 & \\ -1 & & \\ & & \end{pmatrix}, \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix}, \begin{pmatrix} & & \\ & & 1 \\ & -1 & \end{pmatrix}.$$

Now we can write down the three generators of this group,

$$a_1 = \begin{pmatrix} & -1 & \\ -1 & & \\ & & \end{pmatrix}, \; a_2 = \begin{pmatrix} & & -1 \\ & & \\ -1 & & \end{pmatrix}, \; a_3 = \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix}. \tag{50}$$

Now it is straightforward to check the Lie algebra,

$$[a_1, a_2] = -a_3, \; [a_2, a_3] = -a_1, \; [a_3, a_1] = a_2, \tag{51}$$

which is the same as $SL(2, R)$.

C Hyperbolic space H^2 or Euclidean AdS_2

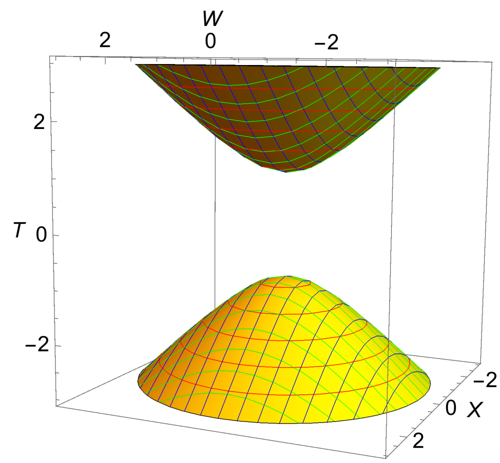


Figure 7: Plot of Euclidean AdS_2 in the embedding coordinate.

Sometimes, people also use Euclidean AdS_2 space to discuss some problem, which is the hyperbolic space H^2. Its definition in terms of embedding coordinate is,

$$-T^2 + W^2 + X^2 = -1, \quad ds^2 = -dT^2 + dW^2 + dX^2. \tag{52}$$

So it is a sphere in $(2+1)D$ Minkowski spacetime, or a negatively cured dS_2 space. This manifold has two disconnected pieces as shown in Fig.7. Its symmetry is $SO(2,1)$, which is composed of $T-W$ boost, $W-X$ rotation and $T-X$ boost. It also has different parametrization and results are just the metric we get previously but with a Wick a rotation.

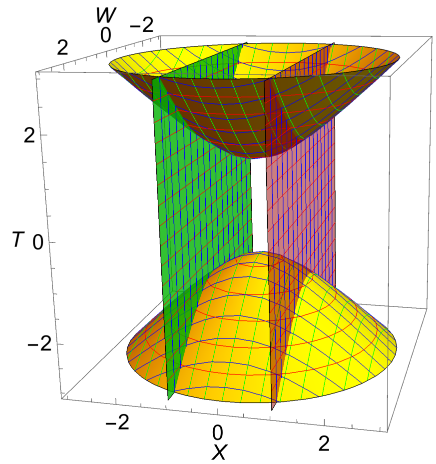


Figure 8: Time slides for global coordinates. Green plane corresponds to $r = -1$ and red plane corresponds to $r = 1$.

C.1 Global coordinate

We use the $T - W$ boost symmetry to get the time. So we define[9],

$$T = R\cosh\tau, W = R\sinh\tau, X = r, \quad ds^2 = (1+r^2)d\tau^2 + \frac{1}{1+r^2}dr^2. \tag{53}$$

If we compactify the r direction with $r = \tan\psi$, we get the conformal coordinate,

$$ds^2 = \frac{d\tau^2 + d\psi^2}{\cos^2\psi}. \tag{54}$$

The time slides are depicted in Fig.8, which are two disconnected hyperbola.

C.2 Local patches

Rindler coordinate We use the $W - X$ rotation to define our time Killing vector,

$$T = r, W = R\sin\tau, X = R\cos\tau, \quad ds^2 = (r^2-1)d\tau^2 + \frac{1}{r^2-1}dr^2, |r| > 1. \tag{55}$$

Now the imaginary time automatically has a period 2π and this coordinate, the spatial slides become some concentric circles as depicted in Fig.9. We can also write it in a hyperbolic form,

$$r = \cosh\rho, \quad ds^2 = \sinh^2\rho d\tau^2 + d\rho^2, \quad \rho > 0. \tag{56}$$

[9]One can compare the definition here and what we did before. They are the same up to a Wick rotation to time.

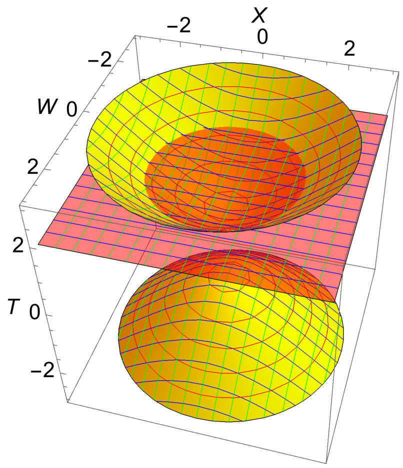


Figure 9: Spatial slides for Rindler coordinates. Red plane corresponds to $r = 1$.

However, due to $\cosh \rho > 0$, this new coordinate only covers the upper half patch. Paralleling what we did before, we can develop a K-S like coordinate but in terms of complex variables [5],

$$z = e^{i\tau} \tanh \frac{\rho}{2}, \quad ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2}. \tag{57}$$

Because $|z| < 1$, the upper half patch is now represented in a unit disk on the complex plane, which is called the Poincaré disk. The spatial boundary is the $|z| = 1$ circle and the bottom of the upper patch is the center of the disk.

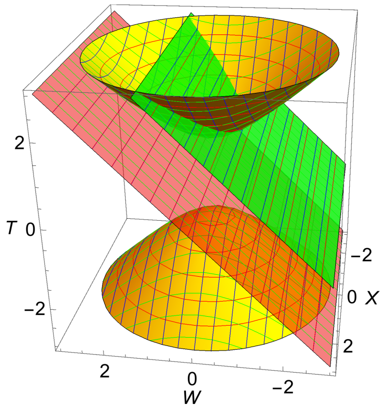
Poincaré coordinate To define a Poincaré coordinate, we have to use some scaling symmetry. So we go back to the $T - W$ boost. In the lightcone coordinate $W^{\pm} = T \pm W$, the boost acts like a scaling transformation $W^{\pm} \to e^{\pm\lambda} W^{\pm}$. So we define,

$$T + W = \frac{\tau^2 + w^2}{w}, T - W = \frac{1}{w}, X = \tau/w, \quad ds^2 = \frac{d\tau^2 + w^2}{w^2}. \tag{58}$$

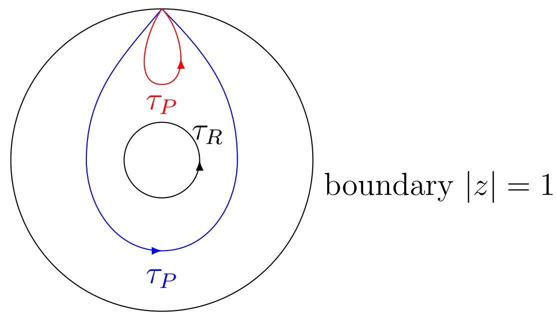
And the spatial slides are some parabola as depicted in Fig.10(a). And one can see that the $w > 0$ part only covers the upper half patch.

Because the Poincaré disk also describes the same patch, one can also try to draw the spatial slides of Poincaré coordinate in Poincaré disk. Notice that any $w =$ const slide extends to infinity which means it must intersect the boundary of Poincaré disk. We first need to figure out the intersecting point, i.e. how $\tau \to$ is mapped out. In this limit, we have,

$$W = R \sin \tau_R \approx \tau_P^2/2w > 0, \quad W/X = \tan \tau_R \approx \tau_P/2,$$



(a)



(b)

Figure 10: (a) Spatial slides for Poincaré coordinates. Red plane corresponds to $w = \infty$ and green plane corresponds to $w = 1/2$. When $\tau \to \pm\infty$, the curve will extend to infinity. (b) Present time slides for Rindler coordinate (black) and Poincaré coordinate(red and blue) in Poincaré disk. τ_P is the time for Poincaré and τ_R is the time for Rindler. Blue curve corresponds to $w < 1$ while red curve corresponds to $w > 1$.

which tells us $\tau_P = \pm\infty$ is mapped to the $\tau_R = \pi/2$ point on the boundary.[10] So in Poincaré disk, a $w =$ const curve in Poincaré coordinate is represented as a closed curve intersecting the boundary at $\tau_R = \pi/2$, as depicted in Fig.10(b). It encloses the original point when $w < 1$ and doesn't if $w > 1$.

C.3 $SL(2, R)$ symmetry of Poincaré coordinate

From the embedding coordinate, we can clearly see that there is a $SO(2, 1)$ symmetry. This symmetry is of course inherited by all these parametrized coordinates in some way. To see how this works, we do an explicit check in Poincaré coordinate. We take the following approach: we first identify the isometry group for Poincaré coordinate then see what these symmetry transformation corresponds to in embedding coordinate.

To make the symmetry manifested, we redefine the coordinate,

$$z = \tau + iw, \overline{z} = \tau - iw, \quad ds^2 = \frac{dzd\overline{z}}{(\mathrm{Im}\, z)^2}. \tag{59}$$

One can check that if we do a $SL(2, R)$ transformation to z, then the metric transforms in the following way,

$$z \to \widetilde{z} = \frac{az + b}{cz + d}, \; ad - bc = 1, \quad ds^2 = \frac{d\widetilde{z}d\overline{\widetilde{z}}}{(\mathrm{Im}\, \widetilde{z})^2}. \tag{60}$$

[10]We choose the $R > 0$ branch implicitly.

So $SL(2,R)$ of z is the isometry group in Poincaré coordinate, which has the same Lie algebra as $SO(2,1)$. So we get the correct answer.

Then let's explicitly show how the three generators for $SL(2,R)$ corresponds to different operations in $SO(2,1)$:

- Dilation: $z \to \widetilde{z} = a^2 z$, which is also a dilation in terms of τ, w. So it corresponds to the $T-W$ boost.

- Translation: $z \to \widetilde{z} = z + a$. One can work out how T, W, X transform,

$$\begin{cases} X \to X + aT - aW, \\ T \to T + aX, \\ W \to W + aX. \end{cases} \tag{61}$$

- SCT: $z \to \frac{z}{cz+1} \approx z - cz^2$. So τ, w transforms as $\tau \to \tau - c(\tau^2 - w^2)$, $w \to w - c2\tau w$. The embedding coordinates transform as,

$$\begin{cases} X \to X + cT + cW, \\ T \to T + cX, \\ W \to W - cX. \end{cases} \tag{62}$$

So we can combine the translation and SCT by fixing $c = b$ or $c = -b$, which can give us $T - X$ boost and $W - X$ rotation respectively.

D Gibbons-Hawking-York Term

To learn

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