NOTE

$\mathrm{SL}(2, \mathbb{R})$ and Its Unitary Representations

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Abstract: In this note, I will discuss some properties of the $\mathrm{SL}(2, R)$ group and its unitary representation. We will see that all the irreducible unitary representations are infinite dimension. Some of them form a continuous spectrum while others form a discrete spectrum.

Contents

1 Introduction

$\mathrm{SL}(2,\mathbb{R})$ is defined to be a group of all the 2×2 real matrices with unit determinant,

$$\mathfrak{S} = \left\{ g \in \mathfrak{S} \,\middle|\, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ ad - bc = 1 \right\}, \tag{1.1}$$

It has a cousin $\mathfrak{P} = \mathrm{PSL}(2,\mathbb{R}) = \mathfrak{S}/\mathbb{Z}_2$ and they are often used without much distinction. It has a wide application in many mathematics and physics problems:

- It is the group of conformal mappings from the upper half plane (UHP) to itself. Because of Cayley mapping, it can also be thought of the group of conformal mapping from the unit disk to itself.

- It double covers the Lorentz group in $(2+1)D$. They are locally homeomorphic and share the same Lie algebra.

- It is very useful in the study of some 1D systems with conformal symmetry such as conformal quantum mechanics, SYK models and Schwarzian theory. It's also important for understanding 2D CFTs.

- It is the isometry group of AdS_2 space, which makes it important to understand the kinematics and holography of AdS_2.

From the group theory view point, it is also interesting on its own:

- It is one of the simplest *non-compact* group which *cannot* be fully casted by an exponential form.
- However, it is still *connected but not simply connected.* As a manifold, it is tpologically equivalent to *the product of a circle with an Euclidean plane.*
- Its Lie algebra $\mathfrak{L}$ has three generators $\Lambda_0, \Lambda_1, \Lambda_2$, whose commutators are very similar to $\mathfrak{su}(2)$. But *all the unitary irreducible representations have an infinite dimension.*
- Depending on the eigenvalues of Λ_0, the unitary irreps can be used to construct unitary irreps of $\mathfrak{S}$ or $\mathfrak{P}$ or the universal covering group of $\mathfrak{S}$, which is denoted as $\widetilde{\mathfrak{S}}$.

In this note we will talk about some of its properties and applications in physics.

In the following will use shorthand notations for different groups and their Lie algebra: SL_2 or $\mathfrak{S}$ means $\mathrm{SL}(2,\mathbb{R})$; PSL_2 or $\mathfrak{P}$ means $\mathrm{PSL}(2,\mathbb{R})$; $\tilde{}$ means universal covering group; $\mathfrak{L}$ means there Lie algebra.

2 Double covering Lorentz group

3 The group manifold of $\mathrm{SL}(2,\mathbb{R})$

In this section, we will mainly talk about the topology and invariant metric of the group manifold. We will also make few comments on its relation with AdS_2 spacetime.

Roughly speaking, Lie group is a continuous group whose elements can change smoothly. This definition already requires us to think of the group as a differentiable manifold[1] with each point corresponding to a group element. Just like how we study normal manifolds, it is convenient to parametrize the Lie group/manifold by several continuous real variables. The way to visualize the group manifold is to check how the definition of the group constrains those variables and gives us a hyper-surface. Our discussion here will simply follow this recipe.

3.1 Topology

We use the fundamental representation of SL_2 to parametrize it. There are two different ways to proceed.

Method 1: Direct Method A generic 2×2 real matrix has four real parameters and can be written in the following form,

$$M = \begin{pmatrix} x+w & y+z \\ -y+z & x-w \end{pmatrix}. \tag{3.1}$$

Now we impose the SL_2 constraint $\det M = 1$, which under parametrization becomes,

$$x^2+y^2-z^2-w^2 = 1, \text{ or } x^2+y^2-z^2 = 1+w^2. \tag{3.2}$$

[1] Otherwise, it is meaningless to say the group elements can change smoothly or two group elements are close to each other.

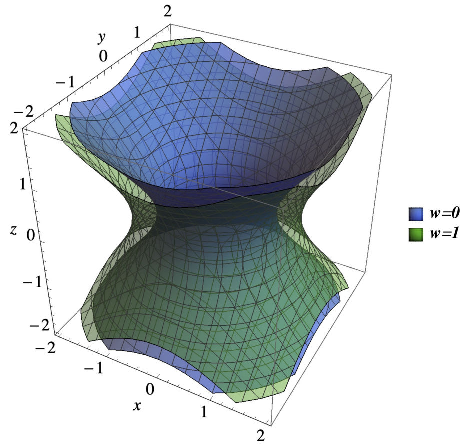

Figure 1. The chimney-shaped Hyperbolic surface given w.

We can think of w as a parameter. Given a w, (x, y, z) describes a chimney-shaped hyperbolic surface, see Fig.1, which is topologically $\mathbb{R} \times S^1$. As we vary w, the chimney becomes larger and we get a 3D hyper-surface, whose topology becomes $\mathbb{R}^2 \times S^1$. From this, we can also conclude that SL_2 is non-compact.

Method 2: Polar decomposition Instead of directly looking at each $M \in \mathrm{SL}_2$, we can decompose it into a product of simpler pieces to study. The basic idea is that:

- Prove that each $M \in \mathrm{SL}(2, R)$ has a *polar decomposition* $M = OR$, where $O \in O(2)$ and R is symmetric positive definite.
- Since $\det M = 1$, we will have $\det O = \det R = 1$. Thus $O \in \mathrm{SO}(2)$, which is homeomorphic to a circle and the set of R to an Euclidean plane. Therefore, the topology of $\mathrm{SL}(2, R)$ is the product of a circle and a plane.

Now we present the details:

- $M \in \mathrm{SL}(2, \mathbb{R})$ is invertible so all the eigenvalues of M are nonzero. Thus for any real vector x, $x^T M^T M x > 0$, i.e. $M^T M$ is positive definite. And we can define its square root $R = \sqrt{M^T M}$.
- By definition, R is symmetric. $M^T M$ is positive definite means R only has positive eigenvalues thus also positive definite.
- For any vector x, we have $x^T (MR^{-1})^T (MR^{-1}) x = x^T R^{-1} M^T M R^{-1} x = x^T x$. Therefore $MR^{-1} = O$ is an orthogonal matrix and we prove the polar decomposition $M = OR$. The manifold for $\mathrm{SL}(2, R)$ can be thought of as the product of the manifold of the set of O and R.
- Because $\det M = 1$ and $\det R > 0$, we must have $\det O = \det R = 1$. So $O \in SO(2)$ whose topology is a circle. For R, because it is real symmetry, its general form is

$R = w\mathbb{1} + x\sigma_x + z\sigma_z$. To make it positive definite, we need $w > \sqrt{x^2+z^2}$. Thus its parameter space can be described by,

$$\det R = w^2 - x^2 - z^2 = 1,\ w > 0, \tag{3.3}$$

which is a branch of hyperbolic surface. It is topologically an Euclidean plane. We get the same answer as before.

Since this method makes the decomposition clearer, we will continue use it in the following discussion.

As we see, each method will be valid for both SL_2 and PSL_2. Therefore, these two cousins has the same topology. The difference is just how to get S^1. This is clearer from the second method for SL_2 it's by identifying α with $\alpha + 2\pi$; for PSL_2 by identifying α with $\alpha + \pi$.

Remarks on connectivity From the topology, we can see that $\mathfrak{G}$ is connected but not simply connected, i.e. $\pi_1(\mathfrak{G}) = \mathbb{Z}$. However, this doesn't means that each $g \in \mathfrak{G}$ can be written in an exponential form[2]. For example, let's take a group element[1],

$$G = \begin{pmatrix} -4 & 0 \\ 0 & -1/4 \end{pmatrix}. \tag{3.4}$$

Assume it can be written as $G = e^x$, where $x = \sum_{l=1}^{3} w_l g_l$ is a linear superposition of the three real generators. Then we have $G = (e^{x/2})^2$. Since $e^x/2$ is also a real matrix, its eigenvalues must be two real numbers or $a \pm bi$, which means the eigenvalues of e^x should be two positive real numbers or two conjugate complex number. But the eigenvalues of G are $-4, -1/4$, which leads to a contradiction.

Remarks on universal covering From the second method, we can clearly see that, in the decomposition $S^1 \times \mathbb{R}^2$, the S^1 is generated by a compact rotation while the $\mathbb{R}^2$ is generated by two non-compact group actions. The universal covering group $\widetilde{\mathfrak{G}}$ is simply replacing S^1 by its universal covering $\mathbb{R}^1$. To construct the universal covering, we can choose the representation of $\mathfrak{L}$ such that the 2π compact rotation cannot be identified with the identity, which will be discussed in detail later.

3.2 Invariant metric

After visualizing the group manifold, now we want to look for an *invariant metric* for it. We will adopt the result from the polar decomposition method above and parameterize the group elements by,

$$g(\alpha, w, x, z) = \mathrm{O}(\alpha) R(w,x,z) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} w+z & x \\ x & w-z \end{pmatrix}, \tag{3.5}$$

with w, z, x satisfying the SL_2 constraint. Therefore, the measure can also be decomposed into $ds^2 = ds_O^2 + ds_R^2$.

The measure for $O(\alpha)$ is straightforward, $ds_O^2 = d\alpha^2$

[2] The usual case we encounter is that a disconnected group cannot be fully casted by an exponential form, e.g. O(3).

4 Properties of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$

talk about generators, Killing form, Casmir, the form of generators in different situations

It has the same algebraic structure as the boost subgroup in $SO(3,1)$ group thus have three generators, which are:

$$l_0 = \frac{1}{2}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad l_1 = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad l_2 = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{4.1}$$

The finite group elements generated by them are,

$$e^{\phi l_0} = \begin{pmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}, \quad e^{\phi l_1} = \begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix}, \quad e^{\phi l_2} = \begin{pmatrix} \cosh\frac{\phi}{2} & \sinh\frac{\phi}{2} \\ \sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} \end{pmatrix}. \tag{4.2}$$

So we see that $e^{2\pi l_0}$ commutes with any group element. As a result, we require that $e^{2\pi l_0}$ has to be mapped to a factor times identity in any irreducible representation.

The generators satisfy the following commutation relation,

$$[l_0, l_1] = l_2, \quad [l_1, l_2] = -l_0, \quad [l_2, l_0] = l_1. \tag{4.3}$$

As physicists, we prefer to write a matrix element as $\exp(-i\sum_j \alpha_j t_j)$. So we can also define the complex generators as $t_j = i l_j$, whose commutators are,

$$[t_0, t_1] = it_2, \quad [t_1, t_2] = -it_0, \quad [t_2, t_0] = it_1 \tag{4.4}$$

It's cyclic property looks very similar to the $su(2)$ algebra. This inspires us to define the ladder operators: $t_\pm = t_1 + \pm it_2$, which gives the following commutators,

$$[t_0, t_+] = t_+, \quad [t_0, t_-] = -t_-, \quad [t_+, t_-] = -2t_0. \tag{4.5}$$

So, if we change the sign of t_+ or t_-, this is the same as $su(2)$. Using the knowledge of $su(2)$, we can easily write down the quadratic Casimir operator,

$$Q = \frac{1}{2}\{t_+, t_-\} - t_0^2 = t_1^2 + t_2^2 - t_0^2. \tag{4.6}$$

5 Unitary Irreducible Representations

Now we look for a Hilbert space $\mathcal{V}$ to implement a unitary representation, i.e. every group element $g \in G$ is mapped to a unitary matrix $\pi(g)$ acting on $\mathcal{V}$. So the complex generators t_j are mapped to corresponding Hermitian matrices denoted as H_j.

We assume $\mathcal{V}_g$ is an irreducible representation and use eigenstates of H_0 as a set of basis vectors. So we have,

$$H_0 f = \lambda f, \quad Qf = qf, \quad \text{for } f \in \mathcal{V}_q. \tag{5.1}$$

where λ and q are real numbers. And we have the requirement that $e^{-i2\pi H_0} \propto \mathbb{1}$, which restricts λ to be an integer or half integer.

From the commutation relations, we get,

$$\begin{aligned}
&H_0 H_+ f = (\lambda+1) H_+ f, \quad H_0 H_- f = (\lambda-1) H_- f; \\
&[H_+, H_-] f = -2\lambda f, \quad \{H_+, H_-\} f = 2(q+\lambda^2) f; \\
\Rightarrow &H_+ H_- f = [q+\lambda(\lambda-1)] f, \quad H_- H_+ f = [q+\lambda(\lambda+1)] f.
\end{aligned} \tag{5.2}$$

By repeating using the formulas above, we have,

$$||H_+^{j+1} f||^2 = [q+(\lambda+j)(\lambda+j+1)]||H_+^j f||^2 = \alpha_{j+1} ||H_+^j f||^2, \tag{5.3}$$

$$||H_-^{j+1} f||^2 = [q+(\lambda-j)(\lambda-j-1)]||H_-^j f||^2 = \beta_{j+1} ||H_+^j f||^2. \tag{5.4}$$

And unitarity requires $\alpha_{j+1}, \beta_{j+1} \geq 0$. Now, let's discuss all the possible situations [2, 3]:

Continuous Class No member of $\{H_+^j f\}$ and $\{H_-^j f\}$ vanishes. So we pick up a state f as reference, i.e. we have $H_0 f = \tau f$, $\tau = 0$ or $1/2$, and $||f|| = 1$. Now, the unitarity condition is $q > \tau(1-\tau)$.

- $\tau = 0$. Then $q > 0$, $\lambda = 0, \pm 1, \pm 2, ...$
- $\tau = 1/2$. Then $q > 1/4$, $\lambda = (1/2 \pm m)$, $m = 0, 1, 2...$ Sometimes, people write $q = -j(j+1)$, $j = -\frac{1}{2} + ik, k > 0$ and call j as SL$(2, \mathbb{R})$ spin.

Discrete Class We assume for example $H_+^j f = 0$ for some $j > 0$. (The case for H_- is similar.) If $j = h+1$ is the minimal integer for which this occurs, then we define the reference state (highest weight state) to be

$$\widetilde{f} = H_+^h f, \quad H_0 \widetilde{f} = k \widetilde{f}, \tag{5.5}$$

where λ is an integer or half integer. By using Eqn.5.2, we know $q = -\lambda(\lambda+1)$ only taking discrete values. Unitarity condition is $\beta_j \geq 0$ which leads to $\lambda \leq 0$.

- $\lambda = 0$, then we have $||H_- \widetilde{f}|| = 0$. So the only nonzero state is $\widetilde{f}$. This is a trivial representation.
- $\lambda < 0$, then $\beta_{j+1} = (1-2\lambda)(j+1)$ are all nonzero for $j \geq 0$. So this is also an infinite dimension representation. We define $\lambda = -k$, then $q = k(1-k)$, $k = 1, 2, 3...$ or $k = -1/2, -3/2....$

If we require $H_-^j f = 0$, then the spectrum of Casimir is the same while the sates are different.

6 Application in Schwarzian QM

The Schwazian action is written as,

$$S = -\frac{1}{2} \int dt \{f, t\}, \quad \{f, t\} = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2, \tag{6.1}$$

which has a PSL$(2,\mathbb{R})$ symmetry: $f \to \dfrac{af+b}{cf+d}$ leaves the action invariant. So if we do quantization and go to the Hamiltonian representation, we expect the corresponding Hilbert space forms a representation of PSL$(2,\mathbb{R})$ (or its Lie algebra).

We introduce another degree of freedom ϕ and their conjugate momenta ϕ_f, π_ϕ. The Hamiltonian is [5],

$$H = \pi_\phi^2 + \pi_f e^\phi \tag{6.2}$$

We now show that it indeed gives the Schwarzian action:

- In real time, the partition function is,

$$Z = \int D\phi Df D\pi_\phi D\pi_f \exp\left(i\int dt[\pi_f f' + \pi_\phi \phi' - \pi_\phi^2 - \pi_f e^\phi]\right) \tag{6.3}$$

  Integration over π_f gives a constraint $f' = e^\phi$ or $\phi = \log f'$. Then do the integration over ϕ and π_ϕ we get,

$$Z = \int Df e^{i\int dt L},\ L = \frac{1}{4}\left(\frac{f''}{f'}\right)^2 = \frac{1}{2}\left(\frac{1}{2}\left(\frac{f''}{f'}\right)^2 - \left(\frac{f''}{f'}\right)'\right) = -\frac{1}{2}\{f,t\} \tag{6.4}$$

  where we add a total derivative to the Lagrangian to get the Schwarzian.

- In imaginary time, the partition function is,

$$Z = \int D\phi Df D\pi_\phi D\pi_f \exp\left(\int d\tau[i\pi_f f' + i\pi_\phi \phi' - \pi_\phi^2 - \pi_f e^\phi]\right) \tag{6.5}$$

  We can still integrate over the auxiliary fields and get a constraint $f = -ie^\phi$ and the final action,

$$Z = \int Df e^{-\int d\tau L},\ L = \frac{1}{4}\left(\frac{f''}{f'}\right)^2 = -\frac{1}{2}\{f,\tau\} \tag{6.6}$$

  Again, we have to add a total derivative here.

A General formula of Quadratic Casimir

Casimir operators are multi-linear forms that commutes with every generators in the Lie algebra. Quadratic Casimir is the bi-linear one, which is used more often. And we will discuss how to find it from the structure constant.

A Lie algebra g is fully determined by the commutator and Jacob Identity,

$$[t_a, t_b] = i\sum_c f_{ab}{}^c t_c, \tag{A.1}$$

$$[t_a,[t_b,t_c]] + [t_b,[t_c,t_a]] + [t_c,[t_a,t_b]] = 0 \tag{A.2}$$

where $f_{ab}{}^c$ is anti-symmetric with respect to a, b.

If we choose a certain basis such that the representation satisfies $\mathrm{Tr}(t_a t_b) = \delta_{ab}$, then we have,

$$f_{ab}{}^{c} = -i\,\mathrm{Tr}([t_a, t_b]t_c) = -\,\mathrm{Tr}([t_a, t_c], t_b). \tag{A.3}$$

So $f_{ab}{}^{c}$ are fully anti-symmetric, which are usually written as f_{abc} for simplicity. And the quadratic Casimir operator is,

$$Q = \sum_a t_a t_a, \tag{A.4}$$

which looks like a inner product.

For a general basis, we need to figure out how different generators overlap and find the correct "inner product" [4]. To do this, we define an adjoint action $\mathrm{ad}_X : g \to g$, for every $X \in g$,

$$\mathrm{ad}_X Z = [X, Z], \quad Z \in g \tag{A.5}$$

And we can take a basis $\{X_\sigma\}$, $\mathrm{ad}_{X_\sigma} X_\rho = [X_\sigma, X_\rho] = C_{\sigma\rho}{}^{r} X_r$. By using the adjoint action, we can define a killing form,

$$g_{\lambda\sigma} = C_{\lambda\rho}{}^{\tau} C_{\sigma\tau}{}^{\rho} = g_{\sigma\lambda}. \tag{A.6}$$

For semi-simple Lie algebras, $g_{\lambda\sigma}$ is shown to be non-degenerate and can be used as a metric. We can define its inverse $g^{\sigma\lambda} g_{\lambda\mu} = \delta^{\sigma}_{\mu}$. We can also use it to raising or lowering indices. As an example we can define $C_{\nu\lambda\mu} = C_{\nu\lambda}{}^{\rho} g_{\rho\mu}$.

The quadratic Casimir operator is,

$$Q = g^{\sigma\lambda} X_\sigma X_\lambda. \tag{A.7}$$

We now justify it by computing its commutator with X_μ,

$$\begin{aligned}[Q, X_\mu] =& g^{\sigma\lambda}[X_\sigma X_\lambda, X_\mu] = g^{\sigma\lambda}(C_{\lambda\mu}{}^{\rho} X_\sigma X_\rho + C_{\sigma\mu}{}^{\rho} X_\rho X_\lambda) \\ =& g^{\sigma\lambda} g^{\rho\nu} C_{\nu\lambda\mu}\{X_\rho, X_\sigma\}.\end{aligned}$$

Because $C_{\nu\lambda\mu}$ is fully anti-symmetric[3], the formula above vanishes.

[3] Now we use Jacob Identity to prove anti-symmetry of $C_{\nu\lambda\mu}$. For simplicity, we use Latin letters here. By definition, we have

$$C_{ABD} = \sum_P C_{AB}{}^{P} g_{PD} = \sum C_{AB}{}^{P} C_{PQ}{}^{R} C_{DR}{}^{Q}.$$

The Jacob identity for t_a implies a Jacob identity for the structure constant C. By using this, we can permute the lower indices of the first two $C's$,

$$C_{ABD} = \sum(-C_{BQ}{}^{P} C_{PA}{}^{R} - C_{AB}{}^{P} C_{PB}{}^{R}) C_{DR}{}^{Q} = \sum C_{BQ}{}^{P} C_{AP}{}^{R} C_{DR}{}^{Q} - C_{AQ}{}^{P} C_{BP}{}^{R} C_{DR}{}^{Q},$$

which is fully anti-symmetric.

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