Note on CFT approach to the topological entanglement

Ruihua Fan[1]

[1]*Department of Physics, Harvard University, Cambridge MA 02138, USA*

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Abstract

In this note, we discuss how to use the CFT technique to compute the topological entanglement, including the entanglement entropy and negativity. Some useful references are [1–3]

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1 Introduction and motivation

Topological entanglement entropy is one defining feature of two-dimensional topological ordered states. However, it does not applies nicely for mixed states. A tentative substitute is the entanglement negativity. Here we want to compute the entanglement negativity defined as

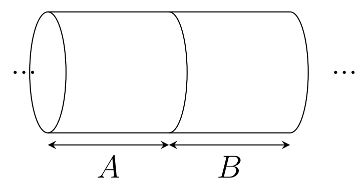
$$E_N = \log \|\rho^{T_A}\|_1 \tag{1}$$

Its Renyi versions are

$$R_N^{(n)} = \frac{1}{2-n} \log \frac{\mathrm{Tr}(\rho^{T_A})^n}{\mathrm{Tr}\,\rho^n} \tag{2}$$

The entanglement negativity is recovered by taking the limit $E_N = \lim_{n\to 1} R_N^{(n)}$ (with even n).

Let us assume the 2+1-dimensional topological order possesses edge states described by a CFT. We consider a cylinder geometry and bipartition the system into two parts A and B as shown below



The ground state $|G\rangle$ can belong to different topological sectors a, which are in one-to-one correspondence of the anyon labels. We want to consider the following two types of states

$$\sum_a \psi_a |G_a\rangle\,, \quad \sum_a p_a |G_a\rangle\langle G_a| \tag{3}$$

Because of the orthogonality between different sectors, topological entanglement entropy cannot distinguish them. We have to use better quantities. As we will see, negativity, as a better entanglement measure for mixed state, does tell them apart.

2 Preliminary

We want to calculate the entanglement by using the CFT approach. The first thing is to write our state $|G_a\rangle$ in terms of the CFT data. In this subsection, we will explain how this can be done. As a bi-product, we can also explain the relation between the edge Hamiltonian and entanglement Hamiltonian, which was first found by Li and Haldane that for certain topological phases (fractional quantum Hall, non-interacting topological insulators and the Kitaev honeycomb model, etc), to certain generality.

Let us consider the cylinder geometry and bipartition as shown above. The (physical) Hamiltonian can be written in the following form

$$H = H_A + H_B + H_{AB} \tag{4}$$

where H_A, H_B denote the Hamiltonians in disconnected regions A and B, and they are coupled by the term H_{AB}. Now we consider a one-parameter family of Hamiltonians

$$H(\lambda) = H_A + H_B + \lambda H_{AB} \tag{5}$$

which interpolates the Hamiltonian of the two decoupled cylinders A, B and the full Hamiltonian. Let us examine a few limits:

- Without the coupling H_{AB}, each subsystem supports a gapless edge that is described by CFTs.
- With the coupling, the whole system becomes gapped everywhere, whose gap is approximately the same as the bulk gap of H_A, H_B.
- When the coupling λ is small enough such that it does not affect the bulk states described by H_A and H_B, the main effect of λH_{AB} is to induce an inter-edge coupling between the two edge states. The problem is then reduced to what happens to a non-chiral CFT with the perturbation λH_{AB}.

The problem at interest is the system described by $H(\lambda = 1)$. If the perturbation is relevant, the edges will be gapped for arbitrarily small coupling. In this case, the two limits $0 < \lambda \ll 1$ and $\lambda = 1$ are expected to be adiabatically connected in the sense that the ground state wavefunctions in the two limits share the same universal properties such as the topological entanglement. In the following, we assume a relevant coupling[1] and study the entanglement properties of the Hamiltonian $H(\lambda)$ for small values of λ.

Note that the entanglement structure of $|G\rangle$ is complicated by the multi-party entanglement, which makes it hard to write down an expression for $|G\rangle$ that is able to include all the universal entanglement properties. Here, we are only interested in the bi-partite entanglement. Therefore, it suffices to replacing the original problem by the 1+1-dimensional problem of coupled edge systems

$$H_{\text{edge}}(\lambda) = H_L + H_R + \lambda H_{\text{int}}$$

and further *assume* that the structure of $|G\rangle$ can be extracted by understanding its reduced density matrix

$$\rho_L = \text{Tr}_R\left(|G\rangle\langle G|\right)$$

Here, $H_{L/R}$ denote the left/right edge, λH_{int} is a relevant coupling and $|G\rangle$ is also the ground state of $H_{\text{edge}}(\lambda)$. The reduced density matrix ρ_L is uniquely determined by all the equal-time correlation functions of operators which are supported solely on the left edge

$$\text{Tr}_L\left(e^{-itH_L}\rho_L e^{itH_L} O_{L,1} O_{L,2} \cdots O_{L,n}\right)$$

Let us rewrite the above expression in terms of the full wavefunction $|G\rangle$ as

$$\langle G| e^{it(H_L+H_R)} O_{L,1} O_{L,2} \cdots O_{L,n} e^{-it(H_L+H_R)} |G\rangle$$

which can be interpreted as a quantum quench problem: the initial state is the ground state $|G\rangle$ of $H_{\text{edge}}(\lambda_0)$ with $\lambda \neq 0$, at time $t = 0$ the coupling λ_0 is suddenly switched off so that the left and right edges evolve independently for $t \geqslant 0$. [in order to motivate B states, we have to do time evolution to make connection to the quench problem. however, the density matrix can be characterized by the correlation function just at the $t = 0$ time slice, it is not clear why we have to consider time-evolution.]

[1] This assumption is more natural for chiral edges because it is easier to have a relevant backscattering term.

This connection to the quench problem provides a way to approximate $|G\rangle$. Because H_{AB} is a relevant perturbation, the initial state $|G\rangle$ is short-range correlated. In quench problems, such short-range entangled states can be replace by one of the conformally invariant boundary states, i.e.

$$|G\rangle \approx \frac{1}{\sqrt{Z}} e^{-\tau_0 (H_L + H_R)} |B\rangle \tag{6}$$

where $\tau_0 > 0$ is the extrapolation length and controls the correlation length of the state $|G\rangle$, Z is a normalization factor. It is believed that this replacement captures the long-distance behavior of the correlation functions. [Similar phenomena was also studied in the coupled SYK model.]

The construction of the conformally invariant boundary states depends on properties of the CFT. We will always assume rational CFTs and discuss two situations separately:

- Chiral edges, namely, H_L and H_R describe a chiral and anti-chiral CFT which combines into a single full-fledge CFT. Examples are the edge of fractional quantum Hall. In this case, the conformally invariant boundary states are known to be finite linear superposition of Ishibashi states

$$|B_a\rangle = \sum_{n=0}^{\infty} \sum_{j=1}^{d_a(n)} |k_a(n), j\rangle_L \otimes \overline{|k_a(n), j\rangle}_R \tag{7}$$

  where a denotes the topological sector in the bulk (primary field in the CFT), $k_a(n) = 2\pi(h_a + n)/l$ denotes the level of descendant, j labels the non-vanishing orthogonal states at a given level n. The Ishibashi states are examples of maximally entangled states.

  The state $|G_a\rangle$ can be written as

$$|G_a\rangle \approx \sum_{n=0}^{\infty} \frac{e^{-\tau_0 (2k_a(n) - \frac{\pi(c_L + c_R)}{12l})}}{\sqrt{Z_a}} \sum_{j=1}^{d_a(n)} |k_a(n), j\rangle_L \otimes \overline{|k_a(n), j\rangle}_R \tag{8}$$

  where $c_L = c_R$. It resembles the thermal-field-double state. The corresponding reduced density matrix is

$$\rho_{L,a} \approx \sum_{n=0}^{\infty} \frac{e^{-4\tau_0 (k_a(n) - \frac{\pi c_L}{12l})}}{Z_a} \sum_{j=1}^{d_a(n)} |k_a(n), j\rangle_L \langle k_a(n), j|_L = \frac{1}{Z_a} P_a e^{-4\tau_0 H_L} P_a \tag{9}$$

  Here P_a is the projection operator onto the corresponding sector of the CFT. The normalization factor Z_a is a chiral character χ_a with modular parameter $\tau = 4\tau_0/l$

$$Z_a = \sum_{n=0}^{\infty} \sum_{j=1}^{d_a(n)} e^{-4\tau_0 \left(k_a(n) - \frac{\pi c_L}{12l}\right)} = \chi_a(\tau) \tag{10}$$

- Non-chiral edge, namely, $H_{L/R}$ is a full-fledge CFT on its own. In this case, there are multiple choices for the conformally invariant boundary states. The requirement is that it has to contain entanglement between the two edges, which rules out choices like $|B_a\rangle_L \otimes |B_a\rangle_R$, where $|B_a\rangle_{L/R}$ is the conformally invariant boundary states for the left/right edge.

One possible choice is the following

$$|\widetilde{B}_a\rangle = \sum_{n=0}^{\infty}\sum_{j=1}^{d_a(n)}\sum_{n'=0}^{\infty}\sum_{j'=1}^{d_a(n')} |k_a(n),j\rangle_L \, \overline{|k_a(n'),j'\rangle}_L \otimes |k_a(n'),j'\rangle_R \, \overline{|k_a(n),j\rangle}_R \tag{11}$$

where a denotes the topological sector, $\widetilde{B}$ means that it is not the usual conformally invariant boundary state. The momentum is chosen appropriately such that it satisfies $(T-\overline{T})\,|\widetilde{B}_a\rangle = 0$ and contains entanglement between the left and right edges simultaneously. It reduces to the chiral case if we remove the anti-holomoprhic component from the left edge and holomorphic one from the right edge.

The state $|\widetilde{G}_a\rangle$ can be written as

$$\begin{aligned} |\widetilde{G}_a\rangle =& \frac{1}{\sqrt{\widetilde{Z}_a}} \sum_{n=0}^{\infty}\sum_{j=1}^{d_a(n)}\sum_{n'=0}^{\infty}\sum_{j'=1}^{d_a(n')} e^{-2\tau_0\left(k_a(n)+k_a(n')-\frac{\pi(c+\overline{c})}{12l}\right)} \\ & |k_a(n),j\rangle_L \, \overline{|k_a(n'),j'\rangle}_L \otimes |k_a(n'),j'\rangle_R \, \overline{|k_a(n),j\rangle}_R \end{aligned} \tag{12}$$

where $c,\overline{c}$ denotes the holomorphic and anti-holomorphic central charge one one edge. The reduced density matrix can also be written in terms of the edge Hamiltonian

$$\begin{aligned} \widetilde{\rho}_{L,a} =& \frac{1}{\widetilde{Z}_a} \sum_{n=0}^{\infty}\sum_{j=1}^{d_a(n)}\sum_{n'=0}^{\infty}\sum_{j'=1}^{d_a(n')} e^{-4\tau_0\left(k_a(n)+k_a(n')-\frac{\pi(c+\overline{c})}{12l}\right)} \\ & |k_a(n),j\rangle_L \, \overline{|k_a(n'),j'\rangle}_L \, \langle k_a(n'),j'|_L \, \overline{\langle k_a(n),j|}_L \\ =& \frac{1}{\widetilde{Z}_a} P_a e^{-4\tau_0 H_L} P_a \end{aligned} \tag{13}$$

In this case, the normalization factor $\widetilde{Z}_a$ is the product of chiral and anti-chiral character with modular parameter $\tau = 4\tau_0/l$

$$\widetilde{Z}_a = \sum_{n=0}^{\infty}\sum_{j=1}^{d_a(n)} e^{-4\tau_0\left(k_a(n)-\frac{\pi c_L}{12l}\right)} \sum_{n'=0}^{\infty}\sum_{j'=1}^{d_a(n')} e^{-4\tau_0\left(k_a(n')-\frac{\pi \overline{c}_L}{12l}\right)} = \chi_a(\tau)\overline{\chi}_a(\overline{\tau}) \tag{14}$$

Another possible choice is

$$\sum_{n=0}^{\infty}\sum_{j=1}^{d_a(n)} |k_a(n),j\rangle_L \, \overline{|k_a(n),j\rangle}_L \otimes |k_a(n),j\rangle_R \, \overline{|k_a(n),j\rangle}_R$$

However, this choices is inconsistent with the lattice calculation. For example, one can compute the topological entanglement entropy or modular-$\mathcal{S}$ matrix and show that this choice yields different answers.

Let us summarize and conclude this subsection with a few comments:

- We have shown that how to write the state $|G_a\rangle$ in terms of CFT data in both the chiral and non-chiral case. Let us remind ourselves the assumptions we have made.

- One is having a relevant coupling λH_{AB}. Now, let us understand what if we relax this assumption. Because the state is eventually gapped at $\lambda = 1$, the only possibility is that H_{AB} contains an exactly marginal component and an irrelevant component such that it continuously tunes the scaling dimension of the irrelevant term until it becomes relevant and drives a transition. However, the identification to the quench problem still works and our conclusion does not change.

- The other assumption is only considering bi-partite entanglement. This is important in making connection to the quench problem. Strictly speaking, we can only justify the final expression as a good approximation of the reduced density matrix and entanglement spectrum. It does not guarantee any reliable results if one wants to use use the full state $|G\rangle$ to compute other bi-partite entanglement quantity such as the negativity. [the result of negativity looks reasonable, so there might be a better understanding on the validity of the approximated wavefunction of the state.]

- The results establish the connection between the entanglement Hamiltonian and the edge Hamiltonian to a wide range of generality. We can also apply it to understand the modular-$\mathcal{T}$ transformation, the Dehn twist on the torus. To define a corresponding legitimate quantum operation, we have to find a translation operator that only acts near the cut. The translation defined in the UV (on the lattice scale) does not work because it acts on the entire system.

  What we have shown provides an answer to this question. For topological phases whose the low energy physics near the cut can be captured by the conformally invariant boundary states, thus we can use the translation defined for the CFT. This is, by definition, an operator in the IR and only acts near the cut. After the translation, the state returns to itself with an additional phase factor $e^{i2\pi h_a}$. Namely, we can extract the topological spin from the modular-$\mathcal{T}$.

3 Topological entanglement entropy

As a warm-up, let us apply the above conclusion to compute the von Neumann entropy and extract the topological part. For simplicity, let us first assume the subsystem A is in a certain topological sector a. Physically, it means that the subsystem A contain an anyon a. The reduced density matrix is $\rho_{L,a} = \frac{1}{Z_a} e^{-\beta H_a}$, where $\beta = \frac{1}{T}$ is a non-universal constant and the modular Hamiltonian is (defined in the sector a)

$$H_a = \frac{2\pi}{L}(L_0 + \overline{L}_0 - \frac{c + \bar{c}}{24})$$

Although we write both the homolomorphic and anti-holomorphic part, the edge is allowed to be chiral such that we can treat the chiral and non-chiral case in the same formalism. Then the von Neumann entropy can be regarded as the thermodynamic entropy

$$S_A = \frac{1}{T} \log Z_a, \quad Z_a = \mathrm{Tr}_a \, e^{-\beta H_a} \tag{15}$$

For chiral edges, the partition function Z_a is the chiral character. For non-chiral edge, it involves both holomorphic and anti-holomorphic part. can be evaluated by using the modular invariance. Let L denote the length of the edge, then Z_a is an Euclidean path-integral on a torus of temporal length β and spatial length L. After a modular-$\mathcal{S}$ transformation, we have

$$Z_a = \sum_b \mathcal{S}_a^b \widetilde{Z}_b \tag{16}$$

where $\widetilde{Z}_b$ is the partition function for the block b on a torus of temporal length L and spatial length β, where $\mathcal{S}$ is the modular-$\mathcal{S}$ matrix of the CFT. In the limit $L \to \infty$, the sum is dominated by the identity block and we find

$$\log Z_a \approx \log(\mathcal{S}_a^1 \widetilde{Z}_1) \approx \log \mathcal{S}_a^1 + \frac{\pi}{12}(c + \overline{c})\frac{L}{\beta} \tag{17}$$

where $c, \overline{c}$ are the holomorphic and anti-holomorphic central charges and $\mathcal{S}_a^1 = d_a/\mathcal{D}$ by the Verlinde formula. Abelian anyons have $d_a = 1$ and we have $S_A = \alpha L - \log \mathcal{D}$.

Example: non-chiral Ising As an example, let us consider the case where the bulk is in the same universality class of toric code (i.e. anyon contents are $\mathbb{I}, e, m, f = em$) and the edge is described by an (non-chiral) Ising CFT. The partition functions of the four sectors are

$$\begin{aligned} Z_{\mathbb{I}} =& \chi_0\overline{\chi}_0 + \chi_{1/2}\overline{\chi}_{1/2} \\ Z_e =& \chi_{1/16}\overline{\chi}_{1/16} \\ Z_m =& \chi_{1/16}\overline{\chi}_{1/16} \\ Z_f =& \chi_0\overline{\chi}_{1/2} + \chi_0\overline{\chi}_{1/2} \end{aligned} \qquad \begin{cases} \chi_0 =& \dfrac{1}{2}\left(\sqrt{\dfrac{\theta_3}{\eta}} + \sqrt{\dfrac{\theta_4}{\eta}}\right) \\ \chi_{1/2} =& \dfrac{1}{2}\left(\sqrt{\dfrac{\theta_3}{\eta}} - \sqrt{\dfrac{\theta_4}{\eta}}\right) \\ \chi_{1/16} =& \dfrac{1}{\sqrt{2}}\sqrt{\dfrac{\theta_2}{\eta}} \end{cases} \tag{18}$$

where θ_i is the theta function and η is the Dedekind eta-function, which has the following modular-S properties

$$\begin{gathered} \theta_2(-1/\tau) = (-i\tau)^{1/2}\theta_4(\tau), \quad \theta_3(-1/\tau) = (-i\tau)^{1/2}\theta_3(\tau), \\ \theta_4(-1/\tau) = (-i\tau)^{1/2}\theta_2(\tau), \quad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \end{gathered} \tag{19}$$

One can use this to compute the modular-$\mathcal{S}$ matrix. For example

$$\begin{aligned} Z_{\mathbb{I}}(-1/\tau) =& \frac{1}{2}(\chi_0 + \chi_{1/2})(\overline{\chi}_0 + \overline{\chi}_{1/2}) + \chi_{1/16}\overline{\chi}_{1/16} \\ =& \frac{1}{2}(Z_{\mathbb{I}} + Z_e + Z_m + Z_f) \end{aligned}$$

Similarly, we can get the full modular-$\mathcal{S}$ matrix

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{20}$$

where the columns from left to right (or rows from up to down) correspond to $\mathbb{I}, e, m, f$. Therefore, the topological entanglement entropy is $\log 2$, which is consistent with the lattice calculation.

Example: chiral Ising A related but different example is the Ising topological order with the anyon content $1, \varepsilon, \sigma$ and the fusion rule $\sigma \times \sigma = 1 + \varepsilon$. The edge is described by chiral CFTs and the corresponding partition functions are the chiral characteristics

$$Z_{\mathbb{I}} = \chi_0, \quad Z_{\varepsilon} = \chi_{1/2}, \quad Z_{\sigma} = \chi_{1/16}, \tag{21}$$

For example, a modular-S maps $Z_{\mathbb{I}}$ to

$$\begin{aligned} Z_{\mathbb{I}}(-1/\tau) =& \frac{1}{2}(\chi_0 + \chi_{1/2}) + \frac{1}{\sqrt{2}}\chi_{1/16}\overline{\chi}_{1/16} \\ =& \frac{1}{2}(Z_{\mathbb{I}} + Z_{\sigma} + \sqrt{2} Z_{\varepsilon}) \end{aligned}$$

The full modular-$\mathcal{S}$ matrix is

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \tag{22}$$

where the columns from left to right (or rows from up to down) correspond to $\mathbb{I}, \varepsilon, \sigma$. Thus, the subsystem A contains an σ anyon, the topological entanglement entropy is $\log \sqrt{2}$.

We can also consider more general cases where the bulk is not in any fixed topological sector. The bulk state can be either pure or mixed. Then we have

$$\rho_L = \sum_a p_a \rho_{L,a}, \quad \sum_a p_a = 1$$

To compute the von Neumann entropy for ρ_L, let us first consider the Renyi entropy $S^{(n)} = \frac{1}{1-n}\log \operatorname{Tr}\rho_L^n$. Note that different sectors are orthogonal, we have $\operatorname{Tr}\rho_L^n = \sum_a p_a^n \operatorname{Tr}^n_{L,a}$, and $\operatorname{Tr}\rho_{L,a}^n$ can be computed by using the modular properties

$$\begin{aligned} \operatorname{Tr}\rho_{L,a}^n =& \frac{1}{Z_a(\tau)^n}\operatorname{Tr} e^{-n\beta H_a} = \frac{1}{Z_a(\tau)^n} Z_a(n\tau), \quad \tau = \frac{\beta}{L} \\ =& \frac{\sum_b \mathcal{S}_a^{b'} Z_b(-1/n\tau)}{[\sum_{b'} \mathcal{S}_a^{b'} Z_{b'}(-1/\tau)]^n} \to \frac{\mathcal{S}_a^1 Z_1(1/n\tau)}{[S_a^1 Z_1(1/\tau)]^n} \\ =& \exp\left(\left(\frac{1}{n} - n\right)\frac{\pi L}{\beta}\frac{c+\overline{c}}{12}\right)(S_a^1)^{1-n} \end{aligned}$$

The Renyi entropy is

$$S^{(n)} = \frac{1+n}{n}\frac{\pi L}{\beta}\frac{c+\overline{c}}{12} + \frac{1}{1-n}\log\sum_a p_a^n \left(\frac{d_a}{\mathcal{D}}\right)^{1-n} \tag{23}$$

Therefore, the von Neumann entropy is

$$S^{(n)} = \frac{\pi L}{\beta}\frac{c+\bar{c}}{6} - \log \mathcal{D} + \left(\sum_a p_a (\log d_a - \log p_a) \right) \tag{24}$$

A few comments are followed:

- If the subsystem is not in a definite sector, its von Neumann entropy always becomes larger by the amount $\sum_a p_a(\log d_a - \log p_a)$. One part is the classical uncertainty $-\sum_a p_a \log p_a$, the other part is associated to the quantum dimension of the anyon $\sum_a p_a \log d_a$.
- If the whole system is on a cylinder or torus such that the entangling surface ∂A can be a non-contractible loop, then different ground state wavefunctions generally leads to different topological sectors for the subsystem A. Therefore, this result means that the topological entanglement entropy has a state dependence.
- This entropy measurement cannot distinguish pure and mixed state.

4 Topological entanglement negativity

Let us list the two cases we are interested in:

1. The bulk state is in a coherent superposition of several topological sectors, i.e. $|G\rangle = \sum_a \psi_a |G_a\rangle$. Note that primary states in different sectors are orthogonal, we have

$$\rho_{L,\text{pure}} \approx \sum_a |\psi_a|^2 \rho_{L,a} \tag{25}$$

   where the subscript means that the reduced density matrix is obtained from a pure state.

2. The bulk state is in an incoherent superposition of several topological sectors, i.e. $\rho_G = \sum_a p_a \rho_{G_a}$, the reduced density matrix is

$$\rho_{L,\text{mixed}} \approx \sum_a p_a \rho_{L,a} \tag{26}$$

   where the subscript means that the reduced density matrix is obtained from a mixed state.

Apparently, the two cases cannot be distinguished from the reduced density matrix on L and we have to use better entanglement measures. In this subsection, we compute the negativity and try to see whether it can tell the difference between pure and mixed states. We follow our discussion in the first subsection and replace ρ by the conformally invariant boundary states near the edge.

For simplicity, we only consider chiral edges. The case of non-chiral edges is not expected to show essential difference. If the state is pure, we have

$$\rho_{pure} = \sum_{a,a'} \psi_a \psi_{a'}^* |G_a\rangle \langle G_{a'}| , \quad \sum_a |\psi_a|^2 = 1 \tag{27}$$

where $|G_a\rangle$ denotes the state in the sector a. Note that different topological sectors are orthogonal, the denominator can be computed easily $\mathrm{Tr}\,\rho^n = \sum_a |\psi_a|^{2n}$. To compute the numerator, recall that $|G_a\rangle$ is a normalized short-range entangled state obtained from the conformally invariant boundary state by an imaginary-time evolution

$$|G_a\rangle \approx \sum_{n=0}^{\infty} \frac{e^{-\frac{\beta}{4}(2k_a(n)-\frac{\pi(c+\bar{c})}{12l})}}{\sqrt{Z_a}} \sum_{j=1}^{d_a(n)} |k_a(n), j\rangle_L \otimes \overline{|k_a(n), j\rangle}_R$$

whose holomorphic and anti-holomorphic components denote the two edges respectively. The factor of $1/4$ in defining the imaginary time is chosen such that the normalization factor Z_a carries the modular parameter $\tau = \beta/L$. After performing partial transpose, we have

$$\rho^{T_A}_{pure} = \sum_{a,a'} \frac{\psi_a\psi_{a'}}{\sqrt{Z_a}\sqrt{Z_{a'}}} \sum_{n=0}^{\infty}\sum_{j=1}^{d_a(n)}\sum_{n'=0}^{\infty}\sum_{j'=1}^{d_a(n')} e^{-\frac{\beta}{4}(2k_a(n)-\frac{\pi(c+\bar{c})}{12l})} e^{-\frac{\beta}{4}(2k_{a'}(n')-\frac{\pi(c+\bar{c})}{12l})}$$
$$|k_{a'}(n'), j'\rangle_L \otimes \overline{|k_a(n), j\rangle}_R \langle k_a(n), j|_L \otimes \overline{\langle k_{a'}(n'), j'|}_R$$

Thus, we have (n is an even number)

$$\begin{aligned}\mathrm{Tr}(\rho^{T_A}_{pure})^n &= \left(\sum_a \frac{|\psi_a|^n}{Z_a^{n/2}} Z_a(n\tau/2)\right)^2 \to \left(\sum_a |\psi_a|^n \frac{\mathcal{S}^1_a Z_1(-2/(n\tau))}{(\mathcal{S}^1_a Z_1(-1/\tau))^{n/2}}\right)^2 \\ &= \left(\exp\left((\frac{2}{n} - \frac{n}{2})\frac{L\pi c}{12\beta}\right) \sum_a |\psi_a|^n (\mathcal{S}^1_a)^{1-\frac{n}{2}}\right)^2\end{aligned}$$

The Renyi negativity is

$$R^{(n)}_N = \frac{2+n}{n}\frac{\pi L}{\beta}\frac{c}{12} + \frac{2}{2-n}\log\sum_a |\psi_a|^n (\mathcal{S}^1_a)^{1-\frac{n}{2}} - \frac{1}{2-n}\log\sum_a |\psi_a|^{2n} \tag{28}$$

The entanglement negativity is

$$\begin{aligned}E_{N,pure} =& \frac{3\pi L}{\beta}\frac{c}{12} + \log\left(\sum_a \sqrt{|\psi_a|^2 \mathcal{S}^1_a}\right)^2 \\ =& \frac{3\pi L}{\beta}\frac{c}{12} - \log\mathcal{D} + \log\left(\sum_a \sqrt{|\psi|_a d_a}\right)^2\end{aligned} \tag{29}$$

which coincides with the Renyi entropy with Renyi index $1/2$, as expected.

If the state is mixed, we have

$$\rho_{mixed} = \sum_a p_a |G_a\rangle\langle G_a|, \quad \sum_a p_a = 1 \tag{30}$$

The denominator is $\mathrm{Tr}\,\rho^n_{mixed} = \sum_a p^n_a$. After doing partial trace, we have

$$\begin{aligned}\mathrm{Tr}(\rho^{T_A}_{mixed})^n &= \sum_a p^n_a \frac{Z_a(n\tau/2)^2}{Z_a(\tau)^n} \to \sum_a p^n_a \frac{(\mathcal{S}^1_a Z_1(\frac{-2}{n\tau}))^2}{(\mathcal{S}^1_a Z_1(\frac{-1}{\tau}))^n} \\ &= \left(\exp\left((\frac{4}{n} - n)\frac{L\pi c}{12\beta}\right) \sum_a p^n_a (\mathcal{S}^1_a)^{2-n}\right)^2\end{aligned}$$

Thus, the entanglement negativity is

$$\begin{aligned}E_{N,mixed} =& \frac{3\pi L}{\beta}\frac{c}{12} + \log\sum_a p_a \mathcal{S}_a^1 \\ =& \frac{3\pi L}{\beta}\frac{c}{12} - \log\mathcal{D} + \log\sum_a p_a d_a\end{aligned} \tag{31}$$

From this result, we can see that the topological part is always nonzero because $\sum_a p_a d_a < \mathcal{D}$.

Let us compare the two cases by choosing $p_a = |\psi_a|^2$. Then we can see that the negativity for pure states is always larger (because $(\sum_i x_i^{1/2})^2 \geqslant \sum_i x_i$) and they differ by

$$E_{N,pure} - E_{N,mixed} = \log\left(1 + \frac{\sum_{a\neq b}|\psi_a||\psi_b|\sqrt{d_a d_b}}{\sum_a |\psi_a|^2 d_a}\right) \tag{32}$$

Thus, the universal part of the negativity can distinguish the pure and mixed state.

References

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