Note on Average Null Energy Condition

Ruihua Fan

Department of Physics, Harvard University, Cambridge, MA 02138, USA

ruihuafan.phys@gmail.com

Abstract

Note on average null energy condition. Use (1+1)D free boson as an example to show, how it is satisfied in general and when it will be violated.

Version: June 25, 2018

Contents

Einstein gravity theory tells us the metric is determined by the matters. However, it doesn't tell us anything about the matter. Energy conditions are some constraints on the matters [1]. Average null energy condition is one of them and stated in the following form,

$$\int dx^+ T_{++} \geq 0, \quad \int dx^- T_{--} \geq 0. \tag{1}$$

It is also related to causality. Here we're not going to talk about this AVEG generally. Instead, the purpose of this note is very modest: we are going to use free boson as an example to show how it is satisfied and how to break it.

We choose the following action [3],

$$S = \frac{g}{2} \int d^2x (\partial_t \varphi)^2 - (\partial_x \varphi)^2, \tag{2}$$

which boundary condition $\varphi(x) = \varphi(x+L)$. Under canonical quantization, the field operator can be expanded as,

$$\varphi(x,t) = \varphi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi i n(x-t)/L} - \overline{a}_{-n}e^{2\pi i(x+t)/L}\right) \tag{3}$$

φ_0 and π_0 describe the center of mass motion. $a_n, \overline{a}_n$ are the left and right movers moving along the lightcone direction with finite momentum, which satisfy a set of commutators,

$$[a_n, a_m] = n\delta_{n+m}, \quad [\overline{a}_n, \overline{a}_m] = n\delta_{n+m}, \quad [a_n, \overline{a}_m] = 0. \tag{4}$$

And in the following discussion, for simplicity, we will use lightcone coordinate,

$$x^+ = x + t, \quad x^- = x - t, \tag{5}$$

where the energy-momentum tensor has a simpler form,

$$T_{--} = g\partial_-\varphi\partial_-\varphi, \quad T_{++} = g\partial_+\varphi\partial_+\varphi, \quad T_{-+} = 0. \tag{6}$$

1 Perturb the vacuum

First, we want to use a non-local unitary to perturb the vacuum state, then measure the energy-momentum tensor, i,e.

$$\langle T_{--}\rangle_p = \langle 0|e^{-i\lambda\varphi_L\varphi_R}T_{--}(x)e^{i\lambda\varphi_L\varphi_R}|0\rangle = -i\lambda\langle 0|[\varphi_L\varphi_R, T_{--}(x)]|0\rangle\,. \tag{7}$$

where $\varphi_L = \varphi(-x_0, 0), \varphi_R = \varphi(x_0, 0)$, which can be thought of as two sources of left and right movers. In the last step, we do a expansion w.r.t. λ and use $\langle T_{--}(x)\rangle = 0$. So up to this order, we have

$$\langle T_{--}\rangle_p = -i\lambda\Big(\langle 0|[\varphi_L, \partial_-\varphi]\varphi_R\partial_-\varphi|0\rangle + \langle 0|\partial_-\varphi[\varphi_L, \partial_-\varphi]\varphi_R|0\rangle + \tag{8}$$

$$\langle 0|\varphi_L[\varphi_R, \partial_-\varphi]\partial_-\varphi|0\rangle + \langle 0|\varphi_L\partial_-\varphi[\varphi_R, \partial_-\varphi]|0\rangle\Big). \tag{9}$$

Using the mode expansion formula, one can show that (See Appendix.A),

$$[\varphi(x_1,t_1), \partial_-\varphi(x_2,t_2)] = -\frac{i}{2g}\delta(x_1^- - x_2^-), \tag{10}$$

$$\langle\varphi(x_1,t_1)\partial_-\varphi(x_2,t_2)\rangle + \langle\partial_-\varphi(x_2,t_2), \varphi(x_1,t_1)\rangle = \frac{1}{2\pi g}P\frac{1}{x_1^- - x_2^-}. \tag{11}$$

So that we have

$$\langle T_{--}(x)\rangle_p = \frac{-\lambda}{4\pi g^2}\frac{\delta(x_L^- - x^-)}{x_R^- - x_L^-} - \frac{-\lambda}{4\pi g^2}\frac{\delta(x_R^- - x^-)}{x_L^- - x_R^-}. \tag{12}$$

Because $x_R^- x_L^- = 2x_0 > 0$, the two terms have opposite signs thus cancel each other, which preserves the ANEC.

2 Change the Hamiltonian

Another scenario is to change the Hamiltonian by adding a time-dependent non-local term, which breaks Lorentz symmetry and locality,

$$H = H_0 + \delta(t)\lambda\varphi_L\varphi_R. \tag{13}$$

where φ_L and φ_R follow the same definition above. The Heisenberg operators evolving under this new Hamiltonian is written as $\widetilde{O}(x)$. Now we have,

$$t<0, \quad \langle\widetilde{T}_{--}(x)\rangle = \langle T_{--}(x)\rangle = 0\,; \tag{14}$$

$$t>0, \quad \langle\widetilde{T}_{--}(x)\rangle = \langle 0|e^{-i\lambda\varphi_L\varphi_R}T_{--}(x)e^{i\lambda\varphi_L\varphi_R}|0\rangle\,. \tag{15}$$

So this time, ANEC is violated for x^- slides between $-x_0$ and x_0.

From this calculation, it seems that the coupling term $\varphi_L\varphi_R$ is not essential. If we only add φ_L or φ_R, the AVNE can also be violated.

A Details for the calculation

We first consider the commutator between two boson fields,

$$\begin{aligned}
[\varphi(x_1,t_1),\varphi(x_2,t_2)] =& \frac{1}{gL}\Big([\pi_0 t_1,\varphi_0] + [\varphi_0,\pi_0 t_2]\Big) \\
&+ \frac{-1}{4\pi g}\left[\sum_{n\neq 0}\frac{1}{n}a_n e^{2\pi i n x_1^-/L}, \sum_{m\neq 0}\frac{1}{m}a_m e^{2\pi i m x_2^-/L}\right] \\
&+ \frac{-1}{4\pi g}\left[\sum_{n\neq 0}\frac{1}{n}\overline{a}_{-n} e^{2\pi i n x_1^+/L}, \sum_{m\neq 0}\frac{1}{m}\overline{a}_{-m} e^{2\pi i m x_2^+/L}\right] \\
=& \frac{-i}{gL}(t_1-t_2) + \frac{1}{4\pi g}\sum_{n\neq 0}\frac{1}{n}\left(e^{2\pi i n(x_1^- - x_2^-)/L} + e^{2\pi i n(x_1^+ - x_2^+)/L}\right).
\end{aligned}$$

Using this, we have,

$$\begin{aligned}
[\varphi(x_1,t_1),\partial_-\varphi(x_2,t_2)] =& \frac{-i}{2gL}\sum_{n=-\infty}^{\infty} e^{2\pi i n(x_1^- - x_2^-)/L} \\
=& \frac{-i}{2g}\sum_{k=-\infty}^{\infty}\delta((x_1^- - x_2^-) - kL) \qquad (16) \\
\xrightarrow{L\to\infty}& \frac{-i}{2g}\delta(x_1^- - x_2^-). \qquad (17)
\end{aligned}$$

From the first to the second line, we use Poisson summation formula $\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{2\pi i k x/T} = \sum_{n=-\infty}^{\infty}\delta(x-nT)$.

Then let's compute the correlation functions,

$$\begin{aligned}\langle\varphi(x_1,t_1)\partial_-\varphi(x_2,t_2)\rangle &= \langle 0|\left(\varphi_0+\frac{1}{gL}\pi_0 t_1\right)\frac{-\pi_0}{2gL}|0\rangle \\ &\quad + \langle 0|\frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}a_n e^{2\pi i n x_1^-/L}\frac{i}{\sqrt{4\pi g}}\sum_{m\neq 0}\frac{1}{m}\frac{2\pi i m}{L}a_m e^{2\pi i m x_2^-/L}|0\rangle \\ &= \langle 0|\left(\varphi_0+\frac{1}{gL}\pi_0 t_1\right)\frac{-\pi_0}{2gL}|0\rangle - \frac{i}{2gL}\sum_{n>0}e^{2\pi i n(x_1^- - x_2^-)/L}. \\ \langle\partial_-\varphi(x_2,t_2)\varphi(x_1,t_1)\rangle &= \langle 0|\frac{-\pi_0}{2gL}\left(\varphi_0+\frac{1}{gL}\pi_0 t_1\right)|0\rangle + \frac{i}{2gL}\sum_{n>0}e^{2\pi i n(x_2^- - x_1^-)/L}.\end{aligned}$$

In the thermodynamic limit, we can drop the contribution from zero mode. For the $n>0$ modes, we use the formula

$$\frac{1}{L}\sum_{n=1}^{\infty}e^{2\pi i n x/L} = \int_0^\infty \frac{dk}{2\pi}e^{ikx} = \int_0^\infty \frac{dk}{2\pi}e^{ik(x+i\epsilon)} = \frac{1}{2\pi}\frac{i}{x+i\epsilon} = \frac{i}{2\pi}\left(P\frac{1}{x} - i\pi\delta(x)\right),$$

which leads to,

$$\langle\varphi(x_1,t_1)\partial_-\varphi(x_2,t_2)\rangle + \langle\partial_-\varphi(x_2,t_2)\varphi(x_1,t_1)\rangle = \frac{1}{2\pi g}P\frac{1}{x_1^- - x_2^-}. \tag{18}$$

All of these calculations can be easily extended to T_{++}.

References

[1] A. Zee, "Einstein gravity in a nutshell", *Princeton*.

[2] D. Stanford, "talk at the conference celebrating Hawking's birthday ".

[3] F. D. Francesco, *et.al.*, "Conformal Field theory", *Springer*.