Note on Average Null Energy Condition

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Abstract

Note on average null energy condition. Use (1+1)D free boson as an example to show, how it is satisfied in general and when it will be violated.

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Einstein gravity theory tells us the metric is determined by the matters. However, it doesn't tell us anything about the matter. Energy conditions are some constraints on the matters [1]. Average null energy condition is one of them and stated in the following form,

$$\int dx^+ T_{++} \geq 0, \quad \int dx^- T_{--} \geq 0. \tag{1}$$

It is also related to causality. Here we're not going to talk about this AVEG generally. Instead, the purpose of this note is very modest: we are going to use free boson as an example to show how it is satisfied and how to break it.

We choose the following action [3],

$$S = \frac{g}{2} \int d^2x (\partial_t \varphi)^2 - (\partial_x \varphi)^2, \tag{2}$$

which boundary condition $\varphi(x) = \varphi(x+L)$. Under canonical quantization, the field operator can be expanded as,

$$\varphi(x,t) = \varphi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n} \left(a_n e^{2\pi i n(x-t)/L} - \overline{a}_{-n} e^{2\pi i (x+t)/L} \right) \tag{3}$$

φ_0 and π_0 describe the center of mass motion. $a_n, \overline{a}_n$ are the left and right movers moving along the lightcone direction with finite momentum, which satisfy a set of commutators,

$$[a_n, a_m] = n\delta_{n+m}, \quad [\overline{a}_n, \overline{a}_m] = n\delta_{n+m}, \quad [a_n, \overline{a}_m] = 0. \tag{4}$$

And in the following discussion, for simplicity, we will use lightcone coordinate,

$$x^+ = x + t, \quad x^- = x - t, \tag{5}$$

where the energy-momentum tensor has a simpler form,

$$T_{--} = g\partial_-\varphi\partial_-\varphi, \quad T_{++} = g\partial_+\varphi\partial_+\varphi, \quad T_{-+} = 0. \tag{6}$$

1 Perturb the vacuum

First, we want to use a non-local unitary to perturb the vacuum state, then measure the energy-momentum tensor, i,e.

$$\langle T_{--}\rangle_p = \langle 0|e^{-i\lambda\varphi_L\varphi_R}T_{--}(x)e^{i\lambda\varphi_L\varphi_R}|0\rangle = -i\lambda \langle 0|[\varphi_L\varphi_R, T_{--}(x)]|0\rangle . \tag{7}$$

where $\varphi_L = \varphi(-x_0, 0), \varphi_R = \varphi(x_0, 0)$, which can be thought of as two sources of left and right movers. In the last step, we do a expansion w.r.t. λ and use $\langle T_{--}(x)\rangle = 0$. So up to this order, we have

$$\langle T_{--}\rangle_p = -i\lambda\Big(\langle 0|[\varphi_L, \partial_-\varphi]\varphi_R\partial_-\varphi|0\rangle + \langle 0|\partial_-\varphi[\varphi_L, \partial_-\varphi]\varphi_R|0\rangle + \tag{8}$$

$$\langle 0|\varphi_L[\varphi_R, \partial_-\varphi]\partial_-\varphi|0\rangle + \langle 0|\varphi_L\partial_-\varphi[\varphi_R, \partial_-\varphi]|0\rangle \Big). \tag{9}$$

Using the mode expansion formula, one can show that (See Appendix.A),

$$[\varphi(x_1, t_1), \partial_-\varphi(x_2, t_2)] = -\frac{i}{2g}\delta(x_1^- - x_2^-), \tag{10}$$

$$\langle\varphi(x_1, t_1)\partial_-\varphi(x_2, t_2)\rangle + \langle\partial_-\varphi(x_2, t_2), \varphi(x_1, t_1)\rangle = \frac{1}{2\pi g}P\frac{1}{x_1^- - x_2^-}. \tag{11}$$

So that we have

$$\langle T_{--}(x)\rangle_p = \frac{-\lambda}{4\pi g^2}\frac{\delta(x_L^- - x^-)}{x_R^- - x_L^-} - \frac{-\lambda}{4\pi g^2}\frac{\delta(x_R^- - x^-)}{x_L^- - x_R^-}. \tag{12}$$

Because $x_R^- x_L^- = 2x_0 > 0$, the two terms have opposite signs thus cancel each other, which preserves the ANEC.

2 Change the Hamiltonian

Another scenario is to change the Hamiltonian by adding a time-dependent non-local term, which breaks Lorentz symmetry and locality,

$$H = H_0 + \delta(t)\lambda\varphi_L\varphi_R. \tag{13}$$

where φ_L and φ_R follow the same definition above. The Heisenberg operators evolving under this new Hamiltonian is written as $\widetilde{O}(x)$. Now we have,

$$\begin{aligned} t<0, \quad & \langle \widetilde{T}_{--}(x)\rangle = \langle T_{--}(x)\rangle = 0\,; & (14)\\ t>0, \quad & \langle \widetilde{T}_{--}(x)\rangle = \langle 0|e^{-i\lambda\varphi_L\varphi_R}T_{--}(x)e^{i\lambda\varphi_L\varphi_R}|0\rangle\,. & (15) \end{aligned}$$

So this time, ANEC is violated for x^- slides between $-x_0$ and x_0.

From this calculation, it seems that the coupling term $\varphi_L\varphi_R$ is not essential. If we only add φ_L or φ_R, the AVNE can also be violated.

A Details for the calculation

We first consider the commutator between two boson fields,

$$\begin{aligned} [\varphi(x_1,t_1),\varphi(x_2,t_2)] =& \frac{1}{gL}\Big([\pi_0 t_1,\varphi_0] + [\varphi_0,\pi_0 t_2]\Big)\\ &+ \frac{-1}{4\pi g}\left[\sum_{n\neq 0}\frac{1}{n}a_n e^{2\pi i n x_1^-/L}, \sum_{m\neq 0}\frac{1}{m}a_m e^{2\pi i m x_2^-/L}\right]\\ &+ \frac{-1}{4\pi g}\left[\sum_{n\neq 0}\frac{1}{n}\overline{a}_{-n} e^{2\pi i n x_1^+/L}, \sum_{m\neq 0}\frac{1}{m}\overline{a}_{-m} e^{2\pi i m x_2^+/L}\right]\\ =& \frac{-i}{gL}(t_1-t_2) + \frac{1}{4\pi g}\sum_{n\neq 0}\frac{1}{n}\left(e^{2\pi i n(x_1^- - x_2^-)/L} + e^{2\pi i n(x_1^+ - x_2^+)/L}\right). \end{aligned}$$

Using this, we have,

$$\begin{aligned} [\varphi(x_1,t_1),\partial_-\varphi(x_2,t_2)] =& \frac{-i}{2gL}\sum_{n=-\infty}^{\infty} e^{2\pi i n(x_1^- - x_2^-)/L}\\ =& \frac{-i}{2g}\sum_{k=-\infty}^{\infty}\delta((x_1^- - x_2^-) - kL) & (16)\\ \xrightarrow{L\to\infty}& \frac{-i}{2g}\delta(x_1^- - x_2^-). & (17) \end{aligned}$$

From the first to the second line, we use Poisson summation formula $\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{2\pi i k x/T} = \sum_{n=-\infty}^{\infty}\delta(x-nT)$.

Then let's compute the correlation functions,

$$\begin{aligned}
\langle\varphi(x_1,t_1)\partial_-\varphi(x_2,t_2)\rangle &= \langle 0|\left(\varphi_0+\frac{1}{gL}\pi_0 t_1\right)\frac{-\pi_0}{2gL}|0\rangle \\
&\quad + \langle 0|\frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}a_n e^{2\pi i n x_1^-/L}\frac{i}{\sqrt{4\pi g}}\sum_{m\neq 0}\frac{1}{m}\frac{2\pi i m}{L}a_m e^{2\pi i m x_2^-/L}|0\rangle \\
&= \langle 0|\left(\varphi_0+\frac{1}{gL}\pi_0 t_1\right)\frac{-\pi_0}{2gL}|0\rangle - \frac{i}{2gL}\sum_{n>0}e^{2\pi i n(x_1^- - x_2^-)/L}. \\
\langle\partial_-\varphi(x_2,t_2)\varphi(x_1,t_1)\rangle &= \langle 0|\frac{-\pi_0}{2gL}\left(\varphi_0+\frac{1}{gL}\pi_0 t_1\right)|0\rangle + \frac{i}{2gL}\sum_{n>0}e^{2\pi i n(x_2^- - x_1^-)/L}.
\end{aligned}$$

In the thermodynamic limit, we can drop the contribution from zero mode. For the $n>0$ modes, we use the formula

$$\frac{1}{L}\sum_{n=1}^{\infty}e^{2\pi i n x/L} = \int_0^\infty \frac{dk}{2\pi}e^{ikx} = \int_0^\infty \frac{dk}{2\pi}e^{ik(x+i\epsilon)} = \frac{1}{2\pi}\frac{i}{x+i\epsilon} = \frac{i}{2\pi}\left(P\frac{1}{x} - i\pi\delta(x)\right),$$

which leads to,

$$\langle\varphi(x_1,t_1)\partial_-\varphi(x_2,t_2)\rangle + \langle\partial_-\varphi(x_2,t_2)\varphi(x_1,t_1)\rangle = \frac{1}{2\pi g}P\frac{1}{x_1^- - x_2^-}. \tag{18}$$

All of these calculations can be easily extended to T_{++}.

References

[1] A. Zee, "Einstein gravity in a nutshell", *Princeton*.

[2] D. Stanford, "talk at the conference celebrating Hawking's birthday ".

[3] F. D. Francesco, *et.al.*, "Conformal Field theory", *Springer*.