Classical Ising Model on a Planar Graph and Its Duality

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Abstract

This note review the systematic approach to do duality for the Ising model on a planar graph. Finally the free energy of the Ising model can be mapped to a free energy of a free Majorana model, which is basically calculating a Pfaffian. Numerically, this might be much easier to do.

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1 Set-up

Suppose we have a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is collection of vertices and $\mathcal{E}$ is the collection of links. We can define an Ising model with pairwise interaction on this graph by putting spins on the vertices and couplings on the links. The partition function is written as,

$$Z = e^{-F} = \sum_{[\sigma]} e^{-E[\sigma]}, \quad E[\sigma] = -\sum_{e\in\mathcal{E}} J_e \prod_{v\in\partial e} \sigma_v. \tag{1}$$

A generic planar graph can contain any units, like triangles, rectangles and so on. Some spins can even be coupled to an external Zeeman field h_v, which explicitly breaks the $\mathbb{Z}_2$ symmetry. However, noticing we have the following arguments:

- First of all, every planar graph can be triangulated by adding virtual edges with the Ising coupling $J_e = 0$ being simply zero.

- Then as for the external Zeeman field h_v, one can consider introducing a fictitious spin at infinity and coupling those spins to the fictitious spin with the coupling strength set by h_v. This effectively only doubles the partition function by its $\mathbb{Z}_2$ symmetry partner ($\sigma \to -\sigma$ doesn't change the partition function), which only brings a factor 2 to the partition function but does not affect the free energy calculation. E.g.

$$\sum_{[\sigma]} e^{\sigma_1\sigma_2 + h\sigma_1} \to \sum_{[\sigma],\tau} e^{\sigma_1\sigma_2 + h\sigma_1\tau} = 2\sum_{[\sigma]} e^{\sigma_1\sigma_2 + h\sigma_1}$$

Therefore in fact we only have to consider the $\mathbb{Z}_2$ symmetric Ising model on the triangulated planar graph.

2 Dual descriptions

Every triangulated planar graph has a dual trivalent graph $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$, where each vertex is connected to three bonds. Several simplest examples are shown in Fig. 1. Notice that when we construct the dual graph near the boundary, we have to add some addition bonds to make sure that in the dual graph each vertex is connected to three bonds. Therefore, on the dual graph, not every edge is intersecting with one edge on the original graph. [2] For each Ising model on a triangulated planar graph, we can perform some duality to translate it to an equivalent model on the dual lattice.

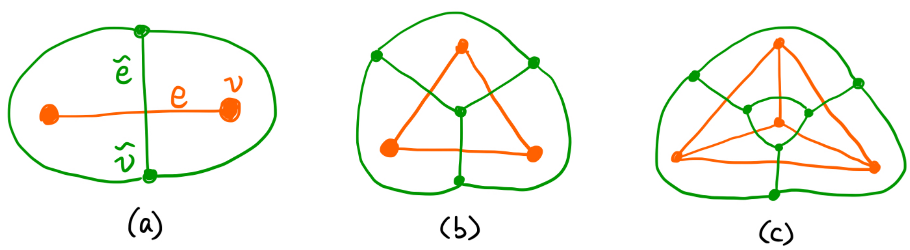

Figure 1: The orange lines and dots represent the original graph G and the green lines and dots represent the dual graph $\widetilde{G}$. (a) G is two lines with $\widetilde{G}$ is a closed graph. (b) G is a triangle while $\widetilde{G}$ contains a circle enclosing G. (c) G is a triangle with a vertex inside.

2.1 Kramer-Wannier duality

We can perform the Kramer-Wannier duality to make it to another Ising model on the dual graph with the reversed coupling strength. On the original graph, let's denote the two spins on one edge e by $\sigma_{e[v1]}$ and $\sigma_{e[v2]}$ and write the partition function as,

$$\begin{aligned} Z =& \sum_{[\sigma]} \prod_{e\in\mathcal{E}} e^{J_e \sigma_{e[v1]}\sigma_{e[v2]}} = \sum_{[\sigma]} \prod_{e\in\mathcal{E}} \left(\cosh J_e + \sigma_{e[v1]}\sigma_{e[v2]} \sinh J_e\right) \\ =& \sum_{[\sigma]} \prod_{e\in\mathcal{E}} \sum_{k_e=0,1} C_{k_e}(J_e) \left(\sigma_{e[v1]}\sigma_{e[v2]}\right)^{k_e}, \end{aligned}$$

where we introduce another variable $k_e = 0, 1$ living on the links and $C_{k_e=0/1} = \cosh J_e / \sinh J_e$. We can change the ordering of summation and product and rewrite the partition function as,

$$\begin{aligned} Z =& \sum_{[\sigma]} \sum_{[k]} \prod_{e\in\mathcal{E}} C_{k_e}(J_e) \prod_{v\in\mathcal{V}} (\sigma_v)^{\sum_{v\in\partial e} k_e} \\ =& \sum_{[k]} \prod_{e\in\mathcal{E}} C_{k_e}(J_e) \prod_{v\in\mathcal{V}} \sum_{\sigma_v=-1,1} (\sigma_v)^{\sum_{e,v\in\partial e} k_e}, \end{aligned}$$

where $\sum_{e,v\in\partial e}$ means given a vertex v we sum over the links that connect to that vertex. After the sum over σ, we get a model totally in terms of $[k]$,

$$Z = \sum_{[k]} \prod_{e\in\mathcal{E}} C_{k_e}(J_e) \prod_{v\in\mathcal{V}} 2\delta_{\mathbb{Z}_2} \left(\sum_{e,v\in\partial e} k_e \right). \tag{2}$$

Now we can translate it to an equivalent Ising model on the dual $\widetilde{\mathcal{G}}$ graph. Supposing e is intersecting with $\widetilde{e}$, then we can introduce new Ising spins on $\widetilde{\mathcal{V}}$ which are related to k_e by,

$$k_e = \frac{1 + \tau_{\widetilde{e}[\widetilde{v}1]}\tau_{\widetilde{e}[\widetilde{v}2]}}{2}. \tag{3}$$

When expressing in terms of τ's, the delta function constraints in Eq. (2) can be automatically satisfied. Further we notice that,

$$C_k(J) = (\cosh J)\left[1 + k(\tanh J - 1)\right] = (\cosh J) \exp\left[k \log \tanh J\right]. \tag{4}$$

So we can rewrite the partition function as,

$$Z = 2^{N_v - 1} \prod_{e\in\mathcal{E}} (\cosh J_e \sinh J_e) \sum_{[\tau]} \prod_{\widetilde{e}\in\widetilde{\mathcal{E}}} e^{\widetilde{J}_{\widetilde{e}} \tau_{\widetilde{e}[\widetilde{v}1]}\tau_{\widetilde{e}[\widetilde{v}2]}}, \tag{5}$$

$$\widetilde{J}_{\widetilde{e}} = \begin{cases} -\frac{1}{2}\log\tanh J_e & \text{if } \widetilde{e} \text{ intersects with } e \\ 0 & \text{if } \widetilde{e} \text{ doesn't intersect with } e \end{cases}.$$

We need the 2^{-1} factor to avoid double counting. We can see that when J_e is small/large, the dual coupling $\widetilde{J}_{\widetilde{e}}$ is large/small. So this is an example of strong-weak duality.

2.2 Loop Expansion

We can also map the Ising model to a loop model and further extend it to a dimer model, which finally can be mapped to a Majorana fermion model.

Loop Model The idea is to numerate the configurations by the domain walls. So instead of keeping track of the orientation of the spins, we can specify the position of domain walls we have:

- On the original graph: We first choose a reference state which can be all spins up or down, with the weight $e^{\sum_{e\in\mathcal{E}} J_e}$. The other states will have some spins flipped from the reference state, which corresponds to domain walls somewhere. Every segment of the domain wall will change the weight by e^{-2J_e}. And all the domain walls need to be closed or terminate at the boundary.

- On the dual graph: Introduce the $\mathbb{Z}_2$ variable $l_{\widetilde{e}} = 0, 1$ on the edges $\widetilde{e}$, such that $l_{\widetilde{e}} = 1$ means domain wall going through the edge $\widetilde{e}$ and $l_{\widetilde{e}} = 0$ corresponds to no domain wall. A domain wall in the bulk corresponds to a loop with $l_e = 1$. A domain terminating at the boundary corresponds to a loop closing from the boundary additional edges, see the examples below.

- Assigning weight: If $\widetilde{e}$ intersects with some $e \in G$, we assign it a weight $w_{\widetilde{e}} = e^{-2J_e}$, meaning domain wall/loop has a energy cost. For those additional edges on the boundary of $\widetilde{G}$, which are not intersecting with any e, a loop can go through there without any energy cost (domain wall terminates at the boundary), we assign a trivial weight $w_{\widetilde{e}} = 1$ to them.

- Constraint on the dual graph: If we take a vertex $\widetilde{v}$ and sum over the three $l_{\widetilde{e}}$ that connect to this $\widetilde{v}$, we should get 0 or 2, meaning either no domain walls or one domain wall coming in then going out. [3]

Putting these things together, we can rewrite Eq. (1), which is called a loop model

$$
\begin{aligned}
Z =& Z_0 \sum_{[l]} \prod_{\widetilde{e}\in\widetilde{\mathcal{E}}} w_{\widetilde{e}}^{l_{\widetilde{e}}} \prod_{\widetilde{v}\in\widetilde{\mathcal{V}}} \delta_{\mathbb{Z}_2}\Big(\sum_{\widetilde{e}\in \mathrm{d}\widetilde{v}} l_{\widetilde{e}}\Big) \\
w_{\widetilde{e}} =& \begin{cases} e^{-2J_e} & \text{if } \widetilde{e} \text{ intersects with } e \\ 1 & \text{if } \widetilde{e} \text{ doesn't intersect with } e \end{cases},
\end{aligned}
\tag{6}
$$

The factor $Z_0 = e^{-F_0} = 2e^{\sum_{e\in\mathcal{E}} J_e}$ is the weight for the no-domain-wall state. The $\sum_{\widetilde{e}\in \mathrm{d}\widetilde{v}}$ inside the delta function $\delta_{\mathbb{Z}_2}$ is to sum over the three links connecting to the vertex $\widetilde{v}$, which imposes the close loop constraint.

Dimer Model The loop model can be further mapped to a dimer model. We first introduce a new variable $l'_{\widetilde{e}} = 1 - l_{\widetilde{e}}$. Then the partition function Eq. (6) can be rewritten as,

$$
Z = Z'_0 \sum_{[l']} \prod_{\widetilde{e}\in\widetilde{\mathcal{E}}} w_{\widetilde{e}}'^{l'_{\widetilde{e}}} \prod_{\widetilde{v}\in\widetilde{\mathcal{V}}} \delta_{\mathbb{Z}_2}\Big(1 + \sum_{\widetilde{e}\in \mathrm{d}\widetilde{v}} l'_{\widetilde{e}}\Big),
$$

where $Z'_0 = e^{-\sum J_e}$ and $w'_{\widetilde{e}} = e^{2J_e}$. Due to the delta constraint, for each vertex, only an odd number of links will have $l' = 1$.

Now we expand each trivalent site into a triangle, as shown in Fig. 2(b). Each thick edge is weighted by $w_{e'} = e^{2J_e}$. The remaining thin edges all share $w_{e'} = 1$. We can imagine there are some dimers living on the edges with the corresponding weight. The $[l']$ configuration can be replaced by the dimer configurations, in which each vertex has to be covered by one and only one dimer. Let Ω be the set of all dimer coverings[1] of the extended graph $\mathcal{G}'$ in Fig. 2(b), the partition function Eq. (6) becomes

$$
Z = Z_0 \sum_{M\in\Omega} \prod_{e'\in M} w_{e'}. \tag{7}
$$

[1] Also called perfect matchings in graph theory. A matching of a graph G is a subset of the edges such that no two edges in the subset share a vertex of G. A perfect matching is matching in which every vertex of the graph is incident to exactly one edge of the matching

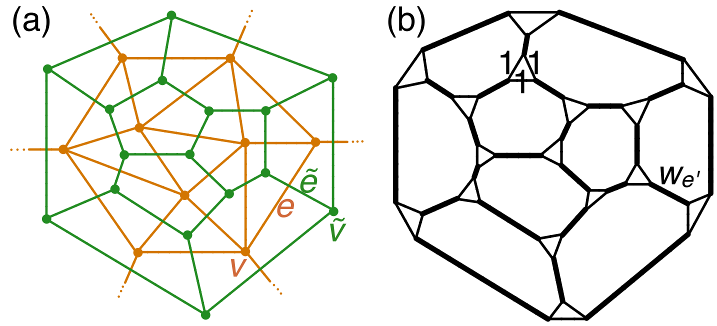

Figure 2: (a) The original graph (in orange) and its dual graph (in green). Each edge e in the original graph is dual to a unique edge $\widetilde{e}$ in the dual graph, such that e and $\widetilde{e}$ intersect. (b) The extended graph (star lattice) by expanding each site to a three sites in a triangle.

One can convince himself that the delta function constraint is equivalent to the perfect matching condition.

Examples One simplest example is to take two spins, i.e. G contains one edge and two vertex. Its partition function and its dual is shown in Fig. 3. We can explicitly see that different descriptions are equivalent. Due to the $\mathbb{Z}_2$ symmetry, the domain wall configurations cannot distinguish spin all up from all down state therefore it seem that we miss half of the states. However, in the loop model, we can use how the loop goes through the boundary edges to get that missed half states back.

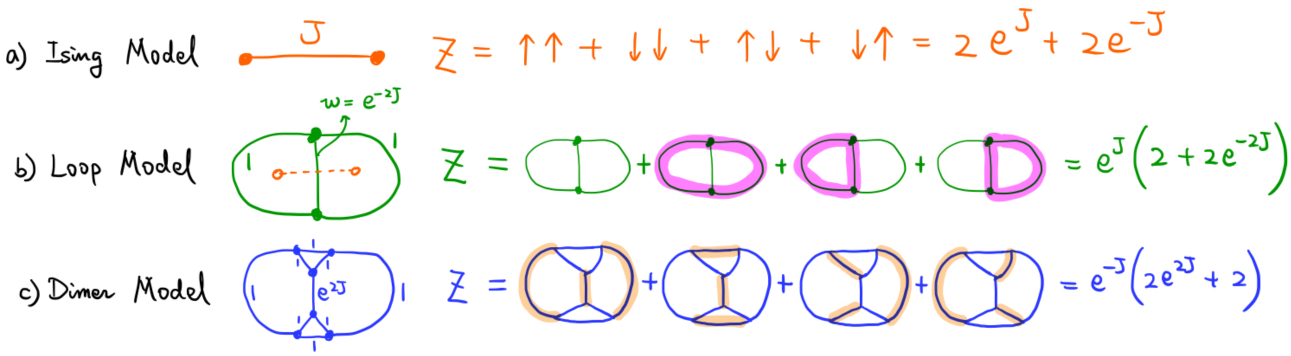

Figure 3: Partition functions in different models and their equivalence. Different terms are written in the appropriate order such that the n-th term in one summation gets mapped to the n-th term in the other summations.

2.3 Fermionic Description

The partition function of the dimer model Eq. (7) can be formulated as a path integral of free Majorana fermions living on the vertices, with the fermion spin structure specified by the Kasteleyn orientation. [5]

First we need to guarantee that we have even vertices. The graph in Fig. 1(a) satisfies. A triangulated graph can be thought of being built from many triangles. Let's start from one triangle, whose dual graph has 4 vertices. Each time we add one more triangle, it will increase two vertices in the dual graph. Therefore $\widetilde{G}$ always have even number of vertices so does G'.

Then let's design the fermion path integral. Call the fermion on each vertex χ_i with a generic coupling matrix A_{ij}. The path integral can be written as,

$$\int \mathcal{D}[\chi] e^{-\frac{1}{2}\chi^T A \chi} = \int \mathcal{D}[\chi] \prod_{i<j} e^{-\chi_i A_{ij} \chi_j} = \int \mathcal{D}[\chi] \prod_{i<j} (1 - \chi_i A_{ij} \chi_j)\,. \tag{8}$$

If we expand the product, the terms that can survive the integral will contain every χ_i exactly once. If A_{ij} only contains nearest neighbor term, such terms drawn on the graph will exactly correspond to those perfect matchings of the graph. Basically, we just need to choose A_{ij} the same as the weight $w_{e'}$ on the graph G', then each term in the fermion path integral Eq. (8) will correspond to a term in the dimer partition function Eq. (7).

However, to make this correspondence exact, we also have to choose the sign of A_{ij} to avoid any additional minus sign. That means to place the fermion system on the graph $\mathcal{G}'$, each edge must be assigned an orientation, such that for every face (except possibly the external face) the number of edges on its perimeter oriented in a clockwise manner is odd, known as the clockwise-odd rule. Any orientation satisfying the clockwise-odd rule is a Kasteleyn orientation, which ensures all dimer configurations to be mapped to even fermion parity states. The Kasteleyn orientation can be assigned systematically on planar graphs by first choosing an arbitrary vertex in the graph and build a spanning tree from that vertex, then closing the loops respecting the clockwise-odd rule, as demonstrated in Fig. 4.

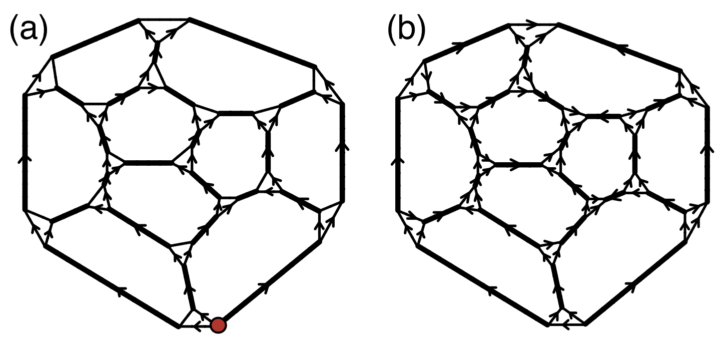

Figure 4: Systematic assignment of the Kasteleyn orientation on planar graph. (a) Start from an arbitrary vertex (mark by the red dot) and build a spanning tree. (b) Close the loops respecting the clockwise-odd rule.

With the Kasteleyn orientation assigned, we can construct the weighted adjacency matrix A of the graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, such that $\forall i, j \in \mathcal{V}'$: $A_{ij} = 0$ if $\langle ij \rangle$ is not an edge in $\mathcal{E}'$, $A_{ij} = w_{ij}$ if the orientation on edge $\langle ij \rangle$ runs from i to j, and $A_{ij} = -w_{ij}$ otherwise. The partition function can then be shown to be

$$Z = Z_0 \int \mathcal{D}[\chi] e^{-\frac{1}{2}\chi^\intercal A \chi} = Z_0 \operatorname{Pf} A. \tag{9}$$

So the free energy of the Ising model can be calculated from

$$F = F_0 - \ln \mathrm{Pf}\, A, \tag{10}$$

where $F_0 = \sum_{e\in\mathcal{E}} J_e$ and A is the adjacency matrix of the Kasteleyn oriented extended dual graph $\mathcal{G}'$.

3 Comments

We show how to get dualities between Ising models, loop models, dimer models and fermion models on a triangulated planar graph. For certain pairs of duality, some assumptions are not necessary:

- The Kramer-Wannier duality is quite general and can be performed on graphs that is not triangulated or even not planar. For example, we can do Kramer-Wannier duality on a square lattice, honeycomb lattice or cubic lattice.
- The loop expansion method is also quite general and actually doesn't rely on the graph to be triangulated. However, if we want to map it to a dimer model, we need a graph to have perfect matching. Then the triangulation assumption is needed because not every graph has its perfect matching.
- The mapping between the dimer model (on a graph with perfect matching) to the Majorana fermion is also generally true. It can be done on any graph that has perfect matching.

References

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