Note for section 8

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November 5, 2021

Abstract

In this note, we discuss the electric response of the (2+1)-dimensional rotor model using effective field theory. Some of the discussion follows [1, 2].

Contents

1 Linear response theory

In this section, we will review the linear response theory. This is a broad subject that can be applied to different fields of physics. We will first consider the general setup. Most conclusions follow from basic principles such as causality and positivity of dissipation and can be applied to both classical and quantum systems. Then we specialize to the electromagnetic response.

1.1 General response theory

Let us consider a system governed by the Hamiltonian H_0, which is initialized in some state. Usually, it is the ground state or thermal equilibrium state. But it can also be out-of-equilibrium state (e.g. laser). Then we perturb it by time-dependent external sources $f_i(t)$, i.e.

$$H(t) = H_0 - \sum_i f_i(t) X_i \tag{1}$$

where X_i is the (generalize) coordinate that is coupled to the source and i is a generic index (it can be a collective index as well). We will assume $f_i(t)$ is small such that we can use perturbation theory and write the response of the observable O_i to the source $f_j(t)$ as

$$\langle X_i(t)\rangle = \langle X_i\rangle_0 + \sum_j \int_{-\infty}^{+\infty} dt' \chi_{ij}(t,t') f_j(t') + \mathcal{O}(f^2)$$

At the zeroth order, $\langle X_i\rangle_0$ is the initial value and will be set to zero without loss of generality. At the linear order, we have $\chi_{ij}(t,t')$ which is called the *dynamical response function*. All the higher order terms will be ignored. Namely, we focus only on the *linear response regime*. We will assume time-translational invariance $\chi_{ij}(t,t') = \chi_{ij}(t-t')$ and have

$$\langle X_i(t)\rangle = \sum_j \int_{-\infty}^{+\infty} dt' \chi_{ij}(t-t') f_j(t') \tag{2}$$

It is also convenient to work in the frequency domain. Let us choose our convention of the Fourier transformation to be

$$f(t) = \int_{-\infty}^{+\infty} \frac{dw}{2\pi} e^{-iwt} f(w), \quad f(w) = \int_{-\infty}^{+\infty} dt\, e^{iwt} f(t)$$

Then, the integral in (2) becomes a simple product in the frequency domain

$$\langle X_i(w)\rangle = \sum_j \chi_{ij}(w) f_j(w) \tag{3}$$

Namely, in a time-translationally invariant system, the response must have the same frequency as the external source unless the non-linear responses are included.

Let us give two simple concrete examples before the abstract discussion on the properties of the response function.

Example 1: harmonic oscillator The first example is a classical system: harmonic oscillator with friction. The Newton equation is

$$m\ddot{x} + \eta\dot{x} + kx = f(t) \tag{4}$$

We can also represent it by a Hamiltonian where the friction can be modeled by a coupling to an environment. We apply a single frequency driving force $f \sim e^{-iwt}$, the oscillator also responses with the same frequency $x \sim e^{-iwt}$. Then we have

$$\chi(w) = \frac{1}{-mw^2 - i\eta w + k} \tag{5}$$

In the limit $\eta \to 0^+$, we have

$$\chi(w) = \frac{1}{2mw_0}\left(-\frac{1}{w - w_0 + i0^*} + \frac{1}{w + w_0 + i0^+}\right), \quad w_0 = \sqrt{\frac{k}{m}} \tag{6}$$

It is an analytic function away from the real axis and has two poles at $\pm w_0 - i0^*$. We can write its real and imaginary part separately as

$$\mathrm{Re}\,\chi(w) = \frac{1}{m}\frac{1}{w_0^2 - w^2}, \quad \mathrm{Im}\,\chi(w) = \frac{\pi}{2mw_0}\left(\delta(w - w_0) - \delta(w + w_0)\right) \tag{7}$$

Example 2: spin in a magnetic field We consider a spin-1/2 pointing along the y-direction and turn a magnetic field $\frac{B}{2}\sigma^x$ at $t = 0$. The magnetization along the z-direction is

$$\langle \sigma^z(t) \rangle = \begin{cases} 0 & t < 0 \\ \sin(tB) & t \geqslant 0 \end{cases} \tag{8}$$

We can expand it to the linear order and have

$$\langle \sigma^z(t) \rangle = Bt + \cdots, \quad t \geqslant 0$$

Thus the response function is $\chi(t) = t\theta(t)$, where $\theta(t)$ is the Heaviside theta function. The response is continuous in time but not smooth at $t = 0$. However, its frequency domain representation is still analytic

$$\chi(w) = -\frac{1}{w^2}, \quad \mathrm{Im}\,w > 0 \tag{9}$$

Causality and analyticity Although the integral in (2) is formally from $-\infty$ to $+\infty$, there is a *causality* constraint on the response function

$$\chi_{ij}(t - t') = 0, \quad \text{if} \quad t < t' \tag{10}$$

Namely, the response can depend only on the past but not the future. A corresponding statement of the response function in frequency domain is that

$$\chi_{ij}(w) \quad \text{is analytic on the upper-half plane } \mathrm{Im}\,w > 0 \tag{11}$$

The two statements are equivalent under some technical assumptions:

- Causality $\Rightarrow$ Analyticity. We need to assume the boundedness of $\chi(t)$ (bounded by a polynomial function of t). Then if $\operatorname{Im} w > 0$ and $\chi(t) < 0$ for $t < 0$, the following integral

$$\int_{-\infty}^{+\infty} \chi(t) e^{iwt} dt = \int_{0}^{+\infty} \chi(t) e^{iwt} dt$$

converges absolutely due to the exponential decay of e^{iwt}.

- Causality $\Leftarrow$ Analyticity. We need to assume $\chi(w)$ vanishes at infinity faster than $1/w$. If $t < 0$, we have[1]

$$\chi(t) = \int_{-\infty}^{\infty} \frac{dw}{2\pi} \chi(w) e^{-iwt} = \left(\int_{-\infty}^{\infty} + \int_{C_R} \right) \frac{dw}{2\pi} \chi(w) e^{-iwt}$$

where C_R denotes the infinite semi-circle on the upper-half plane at the infinity. Then, we can use analyticity and the residual theorem to deduce $\chi(t) = 0$.

Note that the analyticity of $\chi(w)$ holds only on the upper-half plane. It is allowed to have singularities, such as poles and branch cuts, on the real axis. These singularities have important physical consequences.

In practice, we are interested in the response function of real time t and real frequency w. However, they are normally hard to calculate due to the convergence issue. The analytical structure of $\chi(w)$ provides us with another approach. We can first compute the response function in imaginary time τ and imaginary frequency iw_n, which is generally easier, then perform analytical continuation from the upper-half plane $iw_n \to w + i0^+$.

If we give up time-translational invariance. We can still talk about the relation between causality and analyticity. In this case, we have to define $\chi_{ij}(w)$ by

$$\chi_{ij}(w) \equiv \int_{-\infty}^{+\infty} \chi_{ij}(t, 0) e^{iwt} dt$$

Then all the above discussion still follows.

Remark on analyticity in the time domain In the above discussion, the analyticity is imposed in the frequency domain. One might wonder whether we can require analyticity in the time domain as well. Let us make an attempt on this question by assuming quantum mechanics, i.e., $\langle X_i(t) \rangle$ is given by the unitary evolution

$$\langle X_i(t) \rangle = \operatorname{Tr} \left(\rho U(t)^\dagger X_i U(t) \right), \quad U(t) = \mathcal{T} e^{-i \int_0^t H(t) dt}$$

If $H(t)$ is a well-defined quantum system, its instantaneous spectrum must be *bounded from below*. Therefore, we are allowed to analytically continuation t to the lower-half plane.

Now, let us assume that $\langle X(t) \rangle$ is analytic on the lower-half plane. On the other hand, if the external source is turned on at $t = 0$, causality requires that $\langle X(t) \rangle$ must vanish on the negative real axis. As a result of "the edge of the wedge theorem", $\langle X(t) \rangle$ must vanish identically on the full plane. Therefore, we either have to turn on the source at the infinite past (adiabatic switch-on) or have to give up the analyticity in the time domain.

[1] There are special cases where this assumption fails. Rather, $\chi(w)$ approaches a constant $\chi(\infty)$. Some examples are the free electron gas and superfluid. In this case, we can assume that $\widetilde{\chi}(w) = \chi(w) - \chi(\infty)$ vanishes faster than $1/w$ and repeat the argument for $\widetilde{\chi}(w)$. Since $\chi(t)$ and $\widetilde{\chi}(t)$ only differs by a delta function $\delta(t)$ and have the same behavior at $t \neq 0$, our final conclusion still holds.

Dissipated power We then consider the energy dissipated into the system due to the response. The dissipated power is (force times velocity)

$$W = \sum_i f_i(t) \frac{d}{dt} \langle X_i(t) \rangle \tag{12}$$

For simplicity, let us assume that the external source only contains a single frequency $f(t) = f_w e^{-iwt} + c.c.$. Then we have

$$W = \sum_{i,j} f_{i,-w} f_{j,w} 2w \frac{\chi_{ij}(w) - \chi_{ji}(-w)}{2i} + \text{oscillating terms}$$

The oscillating terms will be ignored (we are interested in the time-average of the dissipated power). External sources are real functions, which requires $f_i(w)^* = f_i(-w)$. Physical observables are Hermitian operators, which requires $\langle X(w) \rangle^* = \langle X(-w) \rangle$ and $\chi_{ij}(w)^* = \chi_{ij}(-w)$. Therefore, we can rewrite the dissipated power as

$$W = 2w \sum_{ij} f_{i,w}^* f_{j,w} \chi''_{ij}(w), \quad \chi''_{ij}(w) \equiv \frac{\chi_{ij}(w) - \chi_{ji}(-w)}{2i} \tag{13}$$

where we have introduced the anti-Hermitian part of the response function $\chi''_{ij}(w)$. Similarly, we can also introduce the Hermitian part

$$\chi'_{ij}(w) \equiv \frac{\chi_{ij}(w) + \chi_{ji}(w)^*}{2}, \quad \chi''_{ij}(w) \equiv \frac{\chi_{ij}(w) - \chi_{ji}(-w)}{2i} \tag{14}$$

If the indices i, j are trivial, these are the real and imaginary part of the response function. If the system is initialized at the thermal equilibrium, it can only absorb energy. As a result, we have the following conclusion

$$\chi''_{ij}(w) \geqslant 0 \text{ at } w > 0 \tag{15}$$

One can check that the example of harmonic oscillator satisfies this property.

Being at the thermal equilibrium is an important assumption and can be broken. Laser is such an example. In quantum mechanics, $\chi''_{ij}(w)$ is related to the spectral function that counts the states in the Hilbert space. One can show that the positivity of $\chi''_{ij}(w)$ is related to whether higher-energy states have less population.

Kramers-Kronig relation The anti-Hermitian part $\chi''_{ij}(w)$ contains as much information as the full response function in the sense that we can reconstruct χ_{ij} from χ''_{ij}

$$\chi_{ij}(w) = \chi_{ij}(\infty) + \int_{-\infty}^{+\infty} \frac{d\xi}{\pi} \frac{\chi''_{ij}(\xi)}{\xi - w}, \quad \operatorname{Im} w > 0 \tag{16}$$

where the integral is along the real axis. This is called *Kramers-Kronig relation*, which relies mainly on the analyticity and is independent of the thermal equilibrium assumption. One additional technical assumption is that $\chi''_{ij}(w)$ vanishes at the infinity.

The proof is the following. Let us introduce $\widetilde{\chi}(w) = \chi(w) - \chi(\infty)$ to get rid of $\chi(\infty)$. Then we have

$$\int_{-\infty}^{+\infty} \frac{d\xi}{\pi} \frac{\chi''_{ij}(\xi)}{\xi - w} = \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi i} \frac{\widetilde{\chi}_{ij}(\xi)}{\xi - w} + \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi i} \frac{\widetilde{\chi}_{ji}(\xi)^*}{\xi - w}$$

For the first term, we can attach to it the large semi-circle on the upper-half plane and use the residual theorem to obtain $\widetilde{\chi}_{ij}(w)$. For the second term, we can first replace $\chi_{ji}(\xi)^*$ by $\chi_{ji}(\xi^*)^*$ (because χ is real) and regard it as an analytic function of ξ^* on the lower-half plane. Then we are free to attach to it the large semi-circle on the lower-half plane and show that it vanishes.

1.2 Electromagnetic response

Now, let us specialize to the electromagnetic (EM) response. It can be described in different and also equivalent ways. The main description we will use is the following one

$$\langle J^\mu(\boldsymbol{r}, t)\rangle = \int d\boldsymbol{r}' dt' \chi^{\mu\nu}(\boldsymbol{r}, t; \boldsymbol{r}', t') A_\mu(\boldsymbol{r}', t') \tag{17}$$

or in the momentum space (with time and spatial translation symmetry)

$$\langle J^\mu(\boldsymbol{q}, w)\rangle = \chi^{\mu\nu}(\boldsymbol{q}, w) A_\mu(\boldsymbol{q}, w) \tag{18}$$

Namely, the external force is the background electromagnetic field $A_\mu(x)$ and the generalized coordinates are the charge density and electric current. Since we use the covariant description, let us clarify our convention (we only write 2+1-d for simplicity)

$$g_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{array}{c} x^\mu = (t, \boldsymbol{x}),\ q^\mu = (w, \boldsymbol{q}),\ J^\mu = (\rho, \boldsymbol{J}) \\ A_\mu = (-\varphi, \boldsymbol{A}),\ F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & B_3 \\ E_2 & -B_2 & 0 \end{pmatrix} \end{array}$$

From now on, we will use Greek letters $\mu, \nu, \cdots$ for space-time components and save Latin letters $a, b, \cdots$ for spatial components.

Alternative description One can also choose the external force to be the electric field E. This will lead to some alternative descriptions of the EM response. For example, we can define the *dielectric function* $\epsilon_{ab}(\boldsymbol{q}, w)$ by

$$D_a(\boldsymbol{q}, w) = \epsilon_{ab}(\boldsymbol{q}, w) E_b(\boldsymbol{q}, w) \tag{19}$$

where D_a is the electric displacement and it is related to the electric field E_a and polarization P_a by $D_a = E_a + 4\pi P_a$. The dielectric function and response function are related to each other. Note that we can pick a gauge such that $E_a = -\partial_t A_a$. The polarization is related to the current by $J_a = \partial_t P_a$. Therefore, one can show

$$\epsilon_{ab}(\boldsymbol{q}, w) = \delta_{ab} + \frac{4\pi}{w^2} \chi_{ab}(\boldsymbol{q}, w) \tag{20}$$

Although the definition only involves electric field and the spatial components, it actually provides a complete description.

Another complete description is to define the *conductivity function* $\sigma_{ab}(\boldsymbol{q}, w)$ by

$$J_a(\boldsymbol{q}, w) = \epsilon_{ab}(\boldsymbol{q}, w) E_b(\boldsymbol{q}, w) \tag{21}$$

which is related to the response function by

$$\sigma_{ab}(\boldsymbol{q}, w) = \frac{1}{iw} \chi_{ab}(\boldsymbol{q}, w) \tag{22}$$

The covariant description is over-complete because different components of $\chi_{\mu\nu}$ are not independent from each other. The constraints mainly come from the following two general principles:

- Current conservation. The current must be conserved $\partial_\mu J^\mu = 0$. Accordingly, we have $q_\mu \chi^{\mu\nu} A_\nu = 0$ for arbitrary A_μ, which requires

$$q_\mu \chi^{\mu\nu}(\boldsymbol{q}, w) = 0, \quad \text{or more explicitly} \quad -w\chi^{0\nu} + q_a \chi^{a\nu} = 0 \tag{23}$$

- Gauge invariance. A pure gauge $A_\mu = \partial_\mu \alpha$ cannot generate any physical response. Accordingly, we have $\chi^{\mu\nu} q_\nu \alpha = 0$ for arbitrary α, which requires

$$q_\nu \chi^{\mu\nu}(\boldsymbol{q}, w) = 0, \quad \text{or more explicitly} \quad -w\chi^{\mu 0} + q_a \chi^{\mu a} = 0 \tag{24}$$

Therefore, χ_{ab} determines all the other components $w, q \neq 0$. In particular, we have

$$\chi^{00} = \frac{q^a q^b}{w^2} \chi^{ab} \tag{25}$$

which is called the longitudinal response function (one has to distinguish it from the longitudinal component of χ_{ab} defined below) and is related to the compressibility.

For rotationally invariant systems, the indices of $\chi_{ab}(\boldsymbol{q}, w)$ can only come from q^a or δ_{ab}. Therefore we can decompose it into a longitudinal part and a transverse part

$$\chi_{ab}(\boldsymbol{q}, w) = \frac{q_a q_b}{q^2} \chi^{\parallel}(q, w) + \left(\delta_{ab} - \frac{q_a q_b}{q^2}\right) \chi^{\perp}(q, w) \tag{26}$$

where $q = |\boldsymbol{q}|$. We can see that $\chi^{\parallel}$ is proportional to χ^{00}. The two parts are generally independent from each other and have different meanings.

In order to understand the physical meaning of the electromagnetic response function, let us examine some limits to relate it to quantities that we are more familiar with. For the purpose of later discussion, we are mostly interested in two different limits:

- $w = 0, \boldsymbol{q} \to 0$. Namely, we consider the response to a static but non-uniform electromagnetic field.

  In this limit, the charge can locally redistribute in response to the electrostatic potential, i.e.

$$\langle J^0(\boldsymbol{q}) \rangle = \chi^{00}(\boldsymbol{q}, w = 0) A_0(\boldsymbol{q})$$

  Thus, $\chi^{00}(\boldsymbol{q}, w = 0)$ is the compressibility

$$\chi^{00}(\boldsymbol{q}, w = 0) = -\chi(\boldsymbol{q}) \tag{27}$$

  There can also be induced current due to the magnetic field. Note that a pure gauge $A_a(\boldsymbol{x}) = \partial_a \alpha(\boldsymbol{x})$ cannot generate any physical response, thus we must have

$$\chi^{\parallel}(q, w = 0) = 0 \tag{28}$$

And the spatial response can be written as

$$\langle J^a(\boldsymbol{q})\rangle = \chi^{\perp}(q, w=0)\left(\delta_{ab} - \frac{q_a q_b}{q^2}\right) A^a(\boldsymbol{q}) \tag{29}$$

In particular, the long-wavelength limit

$$D_s \equiv -\lim_{q\to 0} \chi^{\perp}(q, w=0) \tag{30}$$

is called the *superfluid weight*. This is because if $D_s \neq 0$, the system satisfies the *London equation* in the long-wavelength limit

$$\langle J^a(\boldsymbol{q})\rangle = -D_s\left(\delta_{ab} - \frac{q_a q_b}{q^2}\right) A^a(\boldsymbol{q}) = D_s A_a^{\perp}(\boldsymbol{q}) \tag{31}$$

which is the key to the Meissner effect and so on.

- $\boldsymbol{q} = 0, w \to 0$. Namely, we consider the response to a uniform but dynamic electromagnetic field.

  In this limit, $A_0(0, w)$ describes an electric potential that is homogeneous in space but changes with time. Because of charge conservation, the charge cannot globally fluctuate, therefore

  $$\chi^{00}(\boldsymbol{q}=0, w) = 0 \tag{32}$$

  The system is allowed to have a uniform electric current, which defines the AC conductivity, i.e.

  $$\sigma_{ab}(w) = \frac{1}{iw}\chi_{ab}(\boldsymbol{q}=0, w) \tag{33}$$

  The DC conductivity is defined by

  $$\sigma_{ab}^{DC} = \lim_{w\to 0}\sigma_{ab}(w) = \lim_{w\to 0}\frac{1}{iw}\chi_{ab}(\boldsymbol{q}=0, w) \tag{34}$$

  Let us write the longitudinal conductance as

  $$\sigma_{aa}(w) = D\delta(w) + \sigma_{aa}^{\mathrm{reg}}(w) \tag{35}$$

  and define the coefficient of the delta function as the *Drude weight* D. One motivation for this definition is to consider the infinite collision time limit of the Drude conductivity

  $$\lim_{\tau\to\infty}\frac{e^2 n\tau}{m}\frac{1}{1-iw\tau} = \frac{e^2 n}{m}\left(\pi\delta(w) + \frac{i}{w}\right)$$

  Thus, D measures the density of the charge carriers. In the absense of impurities, a metal will be characterized by a finite value of D (no resistence) and an insulator will have a D which vanishes in the thermodynamic limit.

The uniform limit $\boldsymbol{q} \to 0$ and the static limit $w \to 0$ of the response function $\chi^{ab}(\boldsymbol{q}, w)$ provide criteria for determining whether the system is insulating, metallic or superconducting

$$\text{superfluid: } \lim_{\boldsymbol{q}\to 0}\lim_{w\to 0}\chi^{ab}(\boldsymbol{q}, w), \quad \text{insulator or metal: } \lim_{w\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{aa}(\boldsymbol{q}, w)$$

The order of limit is essential. For example, taking the static limit first and the uniform limit then cannot give us the conductance. This is also reasonable on physical ground. In an actual experiment, the wavelength of the EM field is normally much larger than the system. Thus it can be regarded as uniform but still fluctuating. The two limits generally give independent results. For simple systems, the two limits are related as we will see in our example.

2 Rotor model

In this section, we study the electric response of the rotor model

$$H = \frac{U}{2}\sum_i (\widehat{n}_i - n_0)^2 - J\sum_{\langle ij\rangle}\cos(\widehat{\phi}_i - \widehat{\phi}_j) \tag{36}$$

It has insulator phases at $U \gg J$ and a superfluid phase at $U \ll J$. We assume that n_0 is an integer and can be safely ignored for our purpose here. We will mainly consider the zero-temperature case and comment on the superfluid phase at finite temperature only at the end.

In principle, the EM response depends on both the Maxwell equations with sources and the properties of systems at interest. Here, we only consider the second part by ignoring the dynamics of the EM field itself, hence, a background field. One way to obtain the response function is to calculate the current-current correlation function and apply Kobu formula. We will take a different approach, using the Euclidean effective action.

The full Euclidean action is written via the minimal coupling, where A_τ, A_{ij} are background electromagnetic field pointing along the time and spatial direction respectively

$$S_E[\phi, A] = \int_0^\beta \frac{1}{2U}\sum_i (\partial_\tau\phi_i - A_{i,\tau})^2 - J\sum_{\langle ij\rangle}\cos(\phi_i - \phi_j + A_{ij}) \tag{37}$$

It is gauge invariant under the following transformation

$$\phi_i \to \phi_i + \alpha_i, \quad A_{i,\tau} \to A_{i,\tau} + \partial_\tau\alpha_i, \quad A_{ij} \to A_{ij} - \alpha_i + \alpha_j \tag{38}$$

The effective action of the background field can be obtained by integrating out all the dynamical fields, the ϕ field in this example

$$e^{-S_{\text{eff}}[A]} \equiv \int \mathcal{D}\phi e^{-S_E[\phi, A]} \tag{39}$$

where $S_{\text{eff}}[A]$ must be gauge invariant but does not have to be local in general. The current expectation value is given by the functional derivative of the effective action, i.e.

$$\langle J^\mu(\boldsymbol{q}, iw_n)\rangle = -(2\pi)^{d+1}\frac{\delta S_{\text{eff}}[A]}{\delta A_\mu(-\boldsymbol{q}, -iw_n)} \tag{40}$$

If $S_{\text{eff}}[A]$ is a quadratic in A, this will compute the response functions for us. However, this procedure yields response functions in Matsubara frequencies and we have to perform analytic continuation $iw_n \to w + i0^+$ to obtain the actual (physical) response functions. A priori, it is not obvious why this procedure gives the correct result, i.e. the evolution of an observable in real time after perturbing the thermal equilibrium state. This is related to the Kubo formula and analytical structure of general two-point correlation functions. We provide a justification in Appendix. A.

2.1 Insulating phase

In this subsection, we consider the insulating phase $U \gg J$. We will treat $J\cos(\phi_i - \phi_j + A_{ij})$ as a perturbation, the validity of which comes from choosing n_0 as an integer. We only consider the zero-temperature case. The behavior at the low temperature is expected to be the same if the temperature is smaller than the gap. Because the insulating phase is fully gapped, the effective action must take a *local* form. The gauge-invariant terms with lowest order derivatives are the Chern-Simons term and the Maxwell term. Other terms with higher order derivatives are suppressed by the gap $\Delta \sim U$. We will show that only the Maxwell term appears.

Let us separate the action into two parts $S_E = S_0 + \delta S$ with

$$S_0 = \int_0^\beta \frac{1}{2U}\sum_i (\partial_\tau \phi_i - A_\tau)^2, \quad \delta S = -J\int_0^\beta \sum_{\langle ij\rangle} \cos(\phi_i - \phi_j + A_{ij})$$

and do Taylor expansion in δS as follows

$$Z = \int \mathcal{D}\phi (1 - \delta S + \frac{1}{2}\delta S^2 + \cdots)e^{-S_0}$$

Note that different sites are decoupled in S_0, evaluate $\langle \delta S^n \rangle$ reduces to computing expectation value of operators for each site individually. By using the symmetry $\phi_i(\tau) \to \phi_i(\tau) + \alpha_i$ for a constant α_i, we can show that $\langle \delta S^{2n+1}\rangle = 0$ (assuming a *square* lattice). Then we have

$$Z = Z_0\left(1 + \frac{1}{2}\langle \delta S^2\rangle_0 + \frac{1}{24}\langle \delta S^4\rangle_0 + \cdots\right)$$

One can verify that $Z_0 = \int \mathcal{D}\phi e^{-S_0[\phi, A_\tau]}$ does not have any dependence on A_τ. Thus, the effective action can be written as the following series expansion

$$S_{\text{eff}}[A] = -\frac{1}{2}\langle \delta S^2\rangle_0 - \frac{1}{24}\left(\langle \delta S^4\rangle_0 - 3\langle \delta S^2\rangle_0^2\right) + \cdots \tag{41}$$

where higher order terms are suppressed by $1/U$ and will be ignored.

Evaluate $\langle \delta S^2\rangle_0$ We introduce $\phi_{ij} = \phi_i - \phi_j$ and write it as

$$\begin{aligned}\langle \delta S^2\rangle =& J^2 \sum_{\langle ij\rangle}\sum_{\langle kl\rangle}\int d\tau_1 \int d\tau_2 \left\langle \cos(\phi_{ij}(\tau_1) + A_{ij}(\tau_1))\cos(\phi_{kl}(\tau_2) + A_{kl}(\tau_2))\right\rangle_0 \\ =& J^2 \sum_{\langle ij\rangle}\int d\tau_1 \int d\tau_2 \left\langle \cos(\phi_{ij}(\tau_1) + A_{ij}(\tau_1))\cos(\phi_{ij}(\tau_2) + A_{ij}(\tau_2))\right\rangle_0\end{aligned}$$

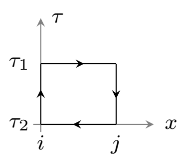
On the second line, we have used the fact that different sites are completely decoupled in S_0 and $\langle \cos\phi_{ij}\rangle_0 = 0$. We can expand each cosine function into two exponential functions and the non-zero term must be invariant under the shift $\phi_i(\tau) \to \phi_i(\tau) + \alpha_i$ for a constant α_i. Therefore we have

$$\langle \delta S^2\rangle_0 = \frac{J^2}{4}\sum_{\langle ij\rangle}\int d\tau_1 \int d\tau_2 e^{i(A_{ij}(\tau_1)-A_{ij}(\tau_2))}\left\langle e^{i(\phi_{ij}(\tau_1)-\phi_{ij}(\tau_2))}\right\rangle_0 + c.c$$

The correlation function can be evaluate by using the result in Appendix. C, then we have

$$\langle \delta S^2\rangle_0 = \frac{J^2}{2}\sum_{\langle ij\rangle}\int d\tau_1 \int d\tau_2 e^{-U|\tau_{12}|}\cos W_{\tau,x}(\tau_1,\tau_2,i,j) \tag{42}$$

where $W_{\tau,x}(\tau_1,\tau_2,i,j)$ is the Wilson loop in the $\tau - x$ plane

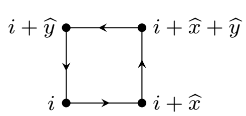


$$W_{\tau,x}(\tau_1,\tau_2,i,j) = \int_{\tau_2}^{\tau_1}(A_{i,\tau}(\tau) - A_{j,\tau}(\tau))d\tau + A_{ij}(\tau_1) - A_{ij}(\tau_2) \tag{43}$$

Note U is large, thus τ_{12} has to stay close to each other. We can do a large-U expansion, or equivalently, a derivative expansion in τ.[2] Up to a constant term, we have

$$\langle \delta S^2\rangle_0 = \frac{J^2}{U}\sum_{\langle ij\rangle}\int d\tau \cos\left(\frac{A_{i,\tau}(\tau) - A_{j,\tau}(\tau) + \partial_\tau A_{ij}(\tau)}{U} + \mathcal{O}(U^{-3})\right) \tag{44}$$

Evaluate $\langle \delta S^4\rangle_{0,c}$ The lowest order contribution comes from having four $\cos(\phi_{ij} + A_{ij})$ that form a closed loop, which is shown below. It generates Wilson loops in the $x - x$ plane and we have



$$\begin{aligned}\langle \delta S^4\rangle_{0,c} &= \left(\frac{J}{2}\right)^4 \times \frac{4!\times 40}{U^3}\sum_i\int d\tau \cos W_{x,x}(i) + \mathcal{O}(U^{-5}) \\ W_{x,x}(i) &= A_{i,i+\widehat{x}}(\tau) + A_{i+\widehat{x},i+\widehat{x}+\widehat{y}}(\tau) + A_{i+\widehat{x}+\widehat{y},i+\widehat{y}}(\tau) + A_{i+\widehat{y},i}(\tau)\end{aligned} \tag{45}$$

Here $W_{x,x}(i)$ is defined for a single plaquette with the lower left corner being i. If we choose a different lattice, such as the triangle or honeycomb lattice, this contribution will appear in different orders.

Effective action and continuum limit As a result, the effective action to the lowest order in $1/U$ can be written as

$$S_{\text{eff}}[A] = -\int d\tau \frac{J^2}{2U}\sum_{\langle ij\rangle}\cos\left(\frac{A_{i,\tau}(\tau) - A_{j,\tau}(\tau) + \partial_\tau A_{ij}(\tau)}{U}\right) + \frac{5J^4}{2U^3}\sum_i \cos W_{x,x}(i) \tag{46}$$

[2] We use the formula
$$\int_{-\infty}^{+\infty} e^{-U|\xi|}f(\xi)d\xi = \frac{2}{U}f(0) + \frac{2}{U^3}f''(0) + \cdots$$

Then we assume that the background fields are slowly varying in the spacetime and go to the continuum limit by using the following replacement rule

$$A_{i,\tau} \to A_\tau(x), \quad A_{i,i+\hat{x}} \to aA_x(x), \quad A_{i,i+\hat{y}} \to aA_y(x) \tag{47}$$

where a is the lattice constant. For slowly varying fields, we can expand the cosine function and get the following continuum action

$$S_{\text{eff}}[A] = \frac{J^2}{4U^3}\int d\tau d^2x \sum_{j=x,y}(\partial_j A_\tau - \partial_\tau A_j)^2 + 5(Ja)^2(\partial_x A_y - \partial_y A_x)^2 \tag{48}$$

This is nothing but the Maxwell action, where $c = \sqrt{5}Ja$ is the speed of light.

Electromagnetic response Taking the derivative of the effective action with respect to the background field, we have

$$\chi_{ab}(\boldsymbol{q}, iw) = -\frac{J^2}{2U^3}\left((c^2q^2 - (iw)^2)\delta_{ab} - c^2 q_a q_b\right) \tag{49}$$

Note that we have written iw explicitly to indicate that this is the response function in the imaginary frequency and we have to do analytical continuation $iw \to w + i0^+$

$$\begin{aligned}
\chi_{ab}(\boldsymbol{q}, w) =& -\frac{J^2}{2U^3}\left((c^2q^2 - w^2)\delta_{ab} - c^2 q_a q_b\right)\\
\chi^{00}(\boldsymbol{q}, w) =& \frac{J^2}{2U^3}q^2
\end{aligned} \tag{50}$$

Regardless of the order, both functions vanishes in the limit $\boldsymbol{q}, w \to 0$, which says the system has (1) zero superfluid density, (2) zero Drude weight and (3) is incompressible.

2.2 Superfluid phase at the zero temperature

In this subsection, we consider the superfluid phase $U \ll J$. In this limit, we are allowed to expand the cosine term and write the action in the continuum limit as

$$S_E[\phi] = \frac{J}{2}\int d\tau d^2x \left(\frac{1}{c^2}(\partial_\tau\phi - A_\tau)^2 + (\partial_a\phi - A_a)^2\right) \tag{51}$$

where $c = \sqrt{JU}a$ is the speed of light and a is the lattice spacing.

To compute the effective action, it is more convenient to first rescale the coordinate and field by $\tau \to \tau/c, A_\tau \to cA_\tau$ to absorb the speed of light. Then, we can do a Gaussian integral to get the effective action

$$S_{\text{eff}}[A] = \frac{J}{2}\int \frac{d^3k}{(2\pi)^3}A_\mu(k)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)A_\nu(-k) \tag{52}$$

It has two important features:

- This is not a local action due to the k^2 in the denominator, which is a manifestation of the gapless nature of the superfluid phase.

- The kernel $\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is a projector that projects out all the longitudinal components of the background field. This also naturally follows from the form of the action $(\partial_\mu \phi - A_\mu)^2$. Namely, any longitudinal component of A will be absorbed by $\partial \phi$.

Electromagnetic response The response function is given by (we need to rescale the coordinate and field back to get the speed of light)

$$\begin{aligned} \chi_{ab}(\boldsymbol{q}, w) =& - Jc\left(\delta_{ab} - \frac{c^2 q_a q_b}{c^2 q^2 - w^2}\right) \\ \chi^{00}(\boldsymbol{q}, w) =& Jc\frac{q^2}{c^2 q^2 - w^2} \end{aligned} \tag{53}$$

where we have already performed the analytical continuation $iw \to w + i0^+$. Now, let us examine the static and uniform limit:

- $w = 0$. In this limit, the longitudinal response function becomes a constant

  $$\chi^{00}(\boldsymbol{q}, w = 0) = \frac{J}{c} = \sqrt{\frac{J}{U}}\frac{1}{a} \tag{54}$$

  Thus the superfluid is compressible with a finite compressibility $K = a\sqrt{J/U}$. The current response is given by χ_{ab} and we have

  $$J_i(\boldsymbol{q}) = -Jc\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) A_j(\boldsymbol{q}) \tag{55}$$

  which is the famous *London equation*. Following the definition (30), we have a nonzero superfluid weight

  $$D_s = Jc = Kc^2 \tag{56}$$

- $\boldsymbol{q} = 0$. In this limit, the longitudinal response function vanishes identically, which is consistent with the charge conservation. The response function χ_{ab} is diagonal, thus there is not Hall conductivity. The longitudinal conductivity is

  $$\sigma(w) = \frac{-Jc}{iw} = \pi Jc\delta(w) + i\frac{Jc}{w} \tag{57}$$

  In the second step, we have used the formula $\frac{1}{x+i0^+} = \frac{1}{x} - i\pi\delta(x)$. Following the definition (35), we have a nonzero Drude weight

  $$D = \pi Jc \tag{58}$$

  Thus, the superfluid is good at conducting current.

In this case, the superfluid weight and Drude weight are proportional to each other $D = \pi D_s$. This is related to the fact that the transverse component of the response function $\chi^\perp$ has nice analytical property at low energies (simply a constant). Generally, they are independent from each other.

2.3 Superfluid phase at low temperature

In this subsection, we consider the superfluid phase at low temperature. If the temperature is too high, ϕ_i does not have to change slowly in space and the simple expansion of the cosine term breaks down. Thus, we assume that the temperature is nonzero but low enough to expand the cosine term. The action in the continuum limit is

$$S_E[\phi, A] = \frac{K}{2}\int_0^\beta d\tau d^2x\, \frac{1}{c}(\partial_\tau\phi - A_\tau)^2 + c(\partial_j\phi - A_j)^2, \quad K = \frac{\sqrt{J/U}}{a} \tag{59}$$

One can still apply the same procedure to calculate the EM response function. Actually, all the calculation follows without any modification[3], which implies that the EM response does not change at low enough temperature. Another characteristic of the superfluidity, which also persists at low temperature, is the *phase rigidity*. Let us consider the system of size $L_x \times L_y$ with a periodic boundary condition in y and a twisted boundary condition in x, i.e. $\phi(x + L_x) = \phi(x) + \Delta\phi$. This is equivalent to having periodic boundary condition in both directions with a uniform vector potential $A_x = \Delta\phi/L_x$. Such a twist costs nonzero free energy density, which is given by

$$e^{-\beta\Delta F} \equiv \frac{Z_{twist}}{Z}, \quad \frac{\Delta F}{L_xL_y} = \frac{Kc}{2}A_x^2 \tag{60}$$

However, the system does not have off-diagonal-long-range order even at an infinitesimal temperature. Let us show this by turning off the background EM fields and calculating the following equal-time two-point function

$$\langle e^{i\phi(\boldsymbol{r})}e^{-i\phi(0)}\rangle = \int \mathcal{D}e^{i\phi(\boldsymbol{r})}e^{-i\phi(0)}e^{-S_E[\phi]} \tag{61}$$

By doing the Gaussian integral in the momentum space, we find

$$\langle e^{i\phi(\boldsymbol{r})}e^{-i\phi(0)}\rangle = \exp\left[-\frac{1}{K}\frac{1}{\beta}\sum_{iw_n}\int\frac{d^2q}{(2\pi)^2}\frac{1-\cos(\boldsymbol{q}\cdot\boldsymbol{r})}{w_n^2+q^2}\right]$$

The sum over bosonic Matsubara frequency gives

$$\exp\left[-\frac{1}{K}\int\frac{d^2q}{(2\pi)^2}\frac{1-\cos(\boldsymbol{q}\cdot\boldsymbol{r}_{12})}{2q}\frac{\sinh\beta q}{\cosh\beta q - 1}\right]$$

If we take the zero temperature limit $\beta \to \infty$ first, it yields the off-diagonal-long-range order. For a finite β, this integral is hard to compute in general. But we are only interested in the long distance limit so that analytical results can be obtained and we have (details will be shown later)

$$\langle e^{i\phi(\boldsymbol{r})}e^{-\phi(0)}\rangle \propto r^{-\frac{T}{2\pi K}}, \quad \beta/r \ll 1 \tag{62}$$

Namely, the two-point function decays algebraically in their separation.

[3]One simply replace iw by iw_n for the response function in the imaginary time. But this will not affect the final response function in the real time.

Details on computing the integral Let us explain the details of the calculation for (62). After the summation over Matsubara frequency, we can further complete the angular integral and get

$$\exp\left[-\frac{1}{K}\int_0^{\Lambda}\frac{dq}{2\pi}\frac{1-J_0(qr)}{2}\frac{\sinh\beta q}{\cosh\beta q-1}\right]$$

where $r=|\boldsymbol{r}|$, Λ is the UV cutoff and J_0 is the Bessel function. It is more convenient to rewrite the expression as

$$\exp\left[-\frac{1}{K}\int_0^{\Lambda r}\frac{dq}{2\pi}\frac{1-J_0(q)}{2r}\frac{\sinh(\beta q/r)}{\cosh(\beta q/r)-1}\right]$$

We fix the temperature and take the long distance limit, i.e. $\beta/r\ll 1$. Thus, we can separate the integral into two parts

$$\int_0^{\Lambda r}=\int_0^{r/\beta}+\int_{r/\beta}^{\Lambda r}$$

In the first part, $\beta q/r$ is small and we can expand the sinh and cosh terms, i.e.

$$\begin{aligned}\int_0^{r/\beta}\frac{dq}{2\pi}\frac{1-J_0(q)}{2r}\frac{\sinh(\beta q/r)}{\cosh(\beta q/r)-1}&\approx\int_0^{r/\beta}\frac{dq}{2\pi}\frac{1-J_0(q)}{\beta q}\\&\approx\frac{1}{2\pi\beta}\left(\gamma_E+\log\frac{r}{2\beta}\right)+\mathcal{O}(r^{-1})\end{aligned}$$

In the second part, we can assume that $\beta q/r$ is large such that we can approximate the integral as follows to show that it is upper bounded by a constant term

$$\int_{r/\beta}^{\Lambda r}\frac{dq}{2\pi}\frac{1-J_0(q)}{2r}\frac{\sinh(\beta q/r)}{\cosh(\beta q/r)-1}\approx\int_{r/\beta}^{\Lambda r}\frac{dq}{2\pi}\frac{1-J_0(q)}{2r}\leqslant\frac{(\Lambda-\beta^{-1})}{4\pi}$$

Therefore, the whole integral is controlled by the $\log r$ term in the long distance limit

$$\int_0^{\Lambda r}\frac{dq}{2\pi}\frac{1-J_0(q)}{2r}\frac{\sinh(\beta q/r)}{\cosh(\beta q/r)-1}=\frac{1}{2\pi\beta}\log\frac{r}{2\beta}+\mathcal{O}(1)$$

One can verify this equation by explicitly compute and compare the two sides numerically and will find good agreement. This justifies the result in (62).

A Quantum theory of linear response

In this section, we consider the quantum linear response theory and prove the Kubo formula. Then we digress to discuss the analytic structure of general two-point functions, which will help us be prepared to justify (40).

A.1 Kubo formula

Recall that the Hamiltonian is written as $H(t)=H_0+V$ with $V=-\sum_i f_i(t)X_i$ and we solve the evolution using the first-order perturbation in the interaction picture. However, sometimes the full Hamiltonian $H(t)$ is quadratic in external source f_i, which can modify definition of the observable X_i. Therefore, let us be more careful and separate X_i into two terms

$$X_i = X_i^{(0)} + X_i^{(1)}[f]$$

where $X_i^{(0)}$ is the operator in the absence of the external source. Since we are only interested in the linear response, it suffices to using $V = -\sum_i f(t) X_i^{(0)}$. As a concrete example, when a system is coupled to an electromagnetic field, the electric current is modified by adding a diamagnetic term $O[A^a]$, i.e.

$$J^a = -\frac{\partial H}{\partial A} = j^a + O[A^a] \tag{63}$$

where j^a is proportional to the particle current.

For simplicity, let $|\Psi_0\rangle$ denote the initial state and the final result applies to a generic mixed state as well. The state at time t to the first order is

$$|\Psi(t)\rangle = |\Psi_0\rangle + |\delta\Psi(t)\rangle\,, \quad |\delta\Psi(t)\rangle = -i\int_{-\infty}^{t} V(t')dt'\,|\Psi_0\rangle \tag{64}$$

Accordingly, the evolution of the observable X_i to the first order is

$$\begin{aligned}\delta\langle X_i(t)\rangle &= \langle\Psi(t)|X_i(t)|\Psi(t)\rangle - \langle\Psi_0|X_i^{(0)}|\Psi_0\rangle \\ &= \langle\Psi_0|X_i(t)|\delta\Psi(t)\rangle + \langle\delta\Psi(t)|X_i(t)|\Psi_0\rangle + \langle\Psi_0|X_i^{(1)}[f(t)]|\Psi_0\rangle\end{aligned}$$

By plugging in the expression of $\delta\Psi$ and V, we arrive at the Kubo formula

$$\begin{aligned}\delta\langle X_i(t)\rangle &= \int_{-\infty}^{\infty} \chi_{ij}(t,t') f_j(t')dt', \\ \chi_{ij}(t,t') &= i\theta(t-t')\,\langle[X_i(t), X_j(t')]\rangle_0 + \frac{\delta\langle X_i^{(1)}[f(t)]\rangle_0}{\delta f_j(t')}\end{aligned} \tag{65}$$

where we recognize the retarded Green's function (note the minus sign)

$$G^R_{ij}(t,t') = -i\theta(t-t')\,\langle[X_i(t), X_j(t')]\rangle_0 \tag{66}$$

A.2 Justification of the Euclidean effective action approach

Now, let us justify the approach of calculating the response function using the Euclidean effective action. For concreteness, we will focus on the electromagnetic response though the formalism can be generalized to other cases as well.

The spatial response function χ_{ab} in the Euclidean time is defined as the second order derivative of the effective action

$$\chi_{ab}(\tau,\tau') \equiv -\frac{\delta^2 S_{\text{eff}}[A]}{\delta A_a(\tau)\delta A_b(\tau')}\bigg|_{A=0} = \frac{\delta^2 \log Z[A]}{\delta A_a(\tau)\delta A_b(\tau')}\bigg|_{A=0} \tag{67}$$

Plugging in the expression of $Z[A]$, we have

$$\chi_{ab}(\tau,\tau') = \frac{1}{Z}\int \mathcal{D}\phi \left(\frac{\delta S_E}{\delta A_a(\tau)}\frac{\delta S_E}{\delta A_b(\tau')} - \frac{\delta^2 S_E}{\delta A_a(\tau)\delta A_b(\tau')}\right) e^{-S_E[\phi,A]}\bigg|_{A=0}$$

Note that $j^a = -\frac{\delta S_E}{\delta A_a(\tau)}\Big|_{A=0}$ is the current in the absence of the background field and $-\frac{\delta^2 S_E}{\delta A_a(\tau)\delta A_b(\tau')}$ is the diamagnetic term. Namely, we can write $\chi_{ab}(\tau,\tau')$ as

$$\chi_{ab}(\tau,\tau') = \langle j^a(\tau)j^b(\tau')\rangle + \frac{\delta\langle J^a(\tau)\rangle}{\delta A_b(\tau')} \tag{68}$$

which exactly involves the time-ordered imaginary two-point function of j^a (up to a minus sign) and the diamagnetic term. By using the analytical structure of the two-point function, we know that (68) can be transformed to (65) under the analytical continuation in the frequency domain $iw_n \to w + i0^+$. This justifies the Euclidean effective action approach.

B Analytical structure of two-point functions

This section is a digression on analytical structure of two-point functions. Let us consider a generic theory and two operators A and B, both of which are either bosonic or fermionic. For the purpose of studying linear response theory, only the retarded and imaginary time-ordered Green's function will be relevant. We will discuss other types of two-point function for the sake of completeness.

B.1 Definition of the two-point functions

We will be interested in the following four kinds of two-point functions, which are defined with additional phase factors to be consistent with condensed matter literatures:

- Imaginary time-ordered Green's function,

$$\begin{aligned} G^\tau(\tau) &= -\mathcal{T}\langle A(\tau)B(0)\rangle_\beta \\ &= -\frac{1}{Z}\left(\theta(\tau)\operatorname{Tr}\left[e^{-\beta H}A(\tau)B(0)\right] + \zeta\theta(-\tau)\operatorname{Tr}\left[e^{-\beta H}B(0)A(\tau)\right]\right), \end{aligned} \tag{69}$$

  where $A(\tau) = e^{\tau H}Ae^{-\tau H}$, $Z = \operatorname{Tr} e^{\beta H}$ and $-\beta < \tau < \beta$, $\zeta = 1$ if A, B are bosonic and $\zeta = -1$ if A, B are fermionic. By using the cyclic property of trace of operators, we can show the following periodicity properties,

$$G^\tau(\tau) = \zeta G^\tau(\tau+\beta),\ -\beta < \tau < 0, \tag{70}$$

  $G^\tau(\tau)$ outside the domain $[-\beta,\beta]$ will be defined by analytical continuation which preserves this periodicity property.

- Real time-ordered Greens' function,

$$\begin{aligned} G^T(t) &= -i\mathcal{T}\langle A(t)B(0)\rangle_\beta \\ &= -\frac{i}{Z}\left(\theta(t)\operatorname{Tr}[e^{-\beta H}A(t)B(0)] + \zeta\theta(-t)\operatorname{Tr}[e^{-\beta H}B(0)A(t)]\right). \end{aligned} \tag{71}$$

  where $A(t) = e^{iHt}Ae^{-iHt}$ is the Heisenburg operator and $-\infty < t < \infty$.

- Retarded/Advanced Green's function,

$$
\begin{aligned}
G^R(t) &= -i\theta(t)\left\langle [A(t), B(0)]_\zeta \right\rangle_\beta = \frac{-i\theta(t)}{Z} \operatorname{Tr}\left[e^{-\beta H}[A(t), B(0)]_\zeta\right], \\
G^A(t) &= i\theta(-t)\left\langle [A(t), B(0)]_\zeta \right\rangle_\beta = \frac{i\theta(-t)}{Z} \operatorname{Tr}\left[e^{-\beta H}[A(t), B(0)]_\zeta\right],
\end{aligned}
\tag{72}
$$

  where $[\bullet, \bullet]_\zeta$ represents commutator if $\zeta = 1$ and anti-commutator if $\zeta = -1$. The retarded Green's function is used in the Kubo formula (65).

- (Generalized) Wightman Green's function,

$$
G^W(t) = \frac{1}{Z} \operatorname{Tr}\left[e^{-(\beta - s)H} A(t) e^{-sH} B(0)\right]. \tag{73}
$$

  where $0 < s < \beta$ and $-\infty < t < \infty$. When $s = \beta/2$ this goes back to the Wightman function widely used in the discussion of chaos; when $s = 0, \beta$ this becomes the $G^{>/<}(t)$ in the Keldysh formalism.

B.2 Lehmann representation

It turns out that these four types of two-point functions are intimately related to each other in the frequency domain. This will be manifested in the *Lehmann representation*, which we introduce below.

Imaginary time-ordered Green's function By inserting resolution of identity, we can write it as,

$$
G^\tau(\tau) = -\frac{1}{Z} \sum_{\alpha,\beta} A_{\alpha\beta} B_{\beta\alpha} e^{\tau E_{\alpha\beta}} \left(\theta(\tau) e^{-\beta E_\alpha} + \zeta\theta(-\tau) e^{-\beta E_\beta}\right).
$$

Due to the periodic property of $G^\tau(\tau)$, it can be expanded into Matsubara frequency,

$$
G^\tau(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} G^\tau(\tau) = -\frac{1}{Z} \sum_{\alpha,\beta} A_{\alpha\beta} B_{\beta\alpha} e^{-\beta E_\alpha} \int_0^\beta d\tau e^{(i\omega_n + E_{\alpha\beta})\tau},
$$

where ω_n is the bosonic or fermionic Matsubara frequency respectively. Notice that $e^{i\omega_n \beta} = \zeta$ we have,

$$
G^\tau(i\omega_n) = \frac{1}{Z} \sum_{\alpha,\beta} \frac{A_{\alpha\beta} B_{\beta\alpha}}{i\omega_n + E_{\alpha\beta}} \left(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}\right). \tag{74}
$$

Real time-ordered Green's function Using spectral decomposition, we can write it as,

$$
G^T(t) = -i\frac{1}{Z} \sum_{\alpha,\beta} A_{\alpha\beta} B_{\beta\alpha} e^{itE_{\alpha\beta}} \left(\theta(t) e^{-\beta E_\alpha} + \zeta\theta(-t) e^{-\beta E_\beta}\right).
$$

Its Fourier transformation is defined by,

$$\begin{aligned}
G^T(\omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega t-\epsilon|t|} G^T(t) \\
&= -i\frac{1}{Z}\sum_{\alpha,\beta} A_{\alpha\beta}B_{\beta\alpha}\left(\int_0^{\infty} e^{(i\omega+iE_{\alpha\beta}-\epsilon)t}e^{-\beta E_\alpha} + \zeta\int_{-\infty}^{0} e^{(i\omega+iE_{\alpha\beta}+\epsilon)t}e^{-\beta E_\beta}\right)
\end{aligned}$$

where ϵ is to guarantee the convergence. Finally we have,

$$G^T(\omega) = \frac{1}{Z}\sum_{\alpha,\beta} A_{\alpha\beta}B_{\beta\alpha}\left(\frac{e^{-\beta E_\alpha}}{\omega+E_{\alpha\beta}+i\epsilon} - \zeta\frac{e^{-\beta E_\beta}}{\omega+E_{\alpha\beta}-i\epsilon}\right). \tag{75}$$

Retarded/Advanced Green's function In terms of the complete set of states, they can be written as,

$$\begin{aligned}
G^R(t) &= \frac{-i\theta(t)}{Z}\sum_{\alpha,\beta} e^{itE_{\alpha\beta}}A_{\alpha\beta}B_{\alpha\beta}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}), \\
G^A(t) &= \frac{i\theta(-t)}{Z}\sum_{\alpha,\beta} e^{itE_{\alpha\beta}}A_{\alpha\beta}B_{\alpha\beta}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}).
\end{aligned}$$

In frequency domain,

$$\begin{aligned}
G^R(\omega) &= \int_{-\infty}^{\infty} dt\, G^R(t)e^{i(\omega+i\epsilon)t} = \frac{1}{Z}\sum_{\alpha,\beta}\frac{A_{\alpha\beta}B_{\beta\alpha}}{\omega+E_{\alpha\beta}+i\epsilon}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}), \\
G^A(\omega) &= \int_{-\infty}^{\infty} dt\, G^A(t)e^{i(\omega-i\epsilon)t} = \frac{1}{Z}\sum_{\alpha,\beta}\frac{A_{\alpha\beta}B_{\beta\alpha}}{\omega+E_{\alpha\beta}-i\epsilon}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}),
\end{aligned} \tag{76}$$

where ϵ is technically to make the integral convergent and physically to guarantee the causality. We can see that the retarded/advanced Green's function can be obtained from the imaginary time-ordered function by replacing $iw_n \to w \pm i\epsilon$. This is useful in the calculation of the response function.

(Generalized) Wightman function Using spectral decomposition, we can write it as,

$$G^W(t) = \frac{1}{Z}\sum_{\alpha,\beta} e^{-\beta E_\alpha}e^{(s+it)E_{\alpha\beta}}A_{\alpha\beta}B_{\alpha\beta}$$

In frequency domain, we have,

$$G^W(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}G^W(t) = \frac{2\pi}{Z}\sum_{\alpha,\beta} e^{-\beta E_\alpha + sE_{\alpha\beta}}A_{\alpha\beta}B_{\beta\alpha}\delta(\omega+E_{\alpha\beta}) \tag{77}$$

B.3 Spectral function and Hilbert transformation

These four functions share similar structures. All of them can be expressed in terms of the *spectral function*,

$$A(\omega) = -2\,\mathrm{Im}\, G^R(\omega) = i(G^R(\omega) - G^A(\omega)). \tag{78}$$

Recalling the formula $\frac{1}{x \pm i0^+} = \frac{1}{x} \mp i\pi\delta(x)$, we have,

$$A(\omega) = \frac{2\pi}{Z} \sum_{\alpha,\beta} A_{\alpha\beta} B_{\beta\alpha} (e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}) \delta(\omega + E_{\alpha\beta}). \tag{79}$$

Physically, it probes the spectrum of the system. One can verify that it satisfies a sum rule,

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\omega) = \langle [A, B]_\zeta \rangle_\beta \,. \tag{80}$$

Let us introduce the Hilbert transformation of the spectral function

$$G(z) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{A(\xi)}{z - \xi}, \tag{81}$$

which defines a master function $G(z)$ with a complex argument z. Its evaluation at the imaginary axis and real axis gives us G^τ and $G^{R/A}$ respectively,

$$G(z \to i\omega_n) = G^\tau(i\omega_n), \tag{82}$$

$$G(z \to \omega \pm i\epsilon) = G^{R/A}(i\omega). \tag{83}$$

The real time-ordered function has a more complicated expression in terms of $A(w)$[4]

$$G^T(\omega) = P \int \frac{dz}{2\pi} \frac{A(z)}{\omega - z} - \frac{i}{2} A(\omega) \frac{1 + \zeta e^{-\beta\omega}}{1 - \zeta e^{-\beta\omega}}. \tag{84}$$

The last one, (generalized) Wightman Green's function, can be written in terms of $A(\omega)$ in a simple way,

$$G^W(\omega) = A(\omega) \frac{e^{-s\omega}}{1 - \zeta e^{-\beta\omega}}. \tag{85}$$

C Correlation function for a particle on a ring

In this section, we consider the particle on a ring $\phi \equiv \phi + 2\pi$ and study the correlation function of vertex operator $e^{i\phi}$. The Euclidean action is

$$S_E[\phi] = \frac{1}{2U} \int_0^\beta d\tau (\partial_\tau \phi - A)^2 \tag{86}$$

[4]One can show it by writing down the real and imaginary part of $G^T(w)$ separately.

$$\mathrm{Re}\, G^T(\omega) = \frac{1}{Z} \sum_{\alpha,\beta} A_{\alpha\beta} B_{\beta\alpha} \left(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}\right) P \frac{1}{\omega + E_{\alpha\beta}},$$

$$\mathrm{Im}\, G^T(\omega) = -\frac{\pi}{Z} \sum_{\alpha,\beta} A_{\alpha\beta} B_{\beta\alpha} \left(e^{-\beta E_\alpha} + \zeta e^{-\beta E_\beta}\right) \delta(\omega + E_{\alpha\beta}).$$

and we want to compute the two-point function (higher-point functions are similar)

$$\langle e^{i\phi(\tau_1)} e^{-i\phi(\tau_2)} \rangle = \frac{1}{Z} \int \mathcal{D}\phi e^{i\phi(\tau_1)} e^{-i\phi(\tau_2)} e^{-S_E[\phi]} \tag{87}$$

For simplicity, we only consider the zero-temperature case.

We can treat the operator insertions as source terms and rewrite the two-point function in the frequency domain as

$$\frac{1}{Z} \int \mathcal{D}\phi \exp\left[-\frac{1}{2U} \int \frac{dw}{2\pi} (-iw\phi_w - A_w)(iw\phi_{-w} - A_{-w}) + i \int \frac{dw}{2\pi} (e^{-iw\tau_1} - e^{-iw\tau_2}) \phi_w \right]$$

and perform the Gaussian integral to obtain

$$\exp\left[-U \int \frac{dw}{2\pi} \frac{1 - \cos w\tau_{12}}{w^2} - \int \frac{dw}{2\pi} \frac{e^{-iw\tau_1} - e^{-iw\tau_2}}{w} A_w \right]$$

For the first term, the integral evaluates to

$$\int \frac{dw}{2\pi} \frac{1 - \cos w\tau_{12}}{w^2} = \frac{1}{2} |\tau_{12}|$$

For the second term, we note that it can be written as an integral in the time domain (one can check that the two sides have the same value at $\tau = 0$ and same derivative)

$$\int \frac{dw}{2\pi} \frac{e^{-iw\tau}}{w} A_w = -i \int_0^\tau A(\tau) + \int \frac{dw}{2\pi} \frac{A_w}{w}$$

Eventually, the two-point function takes the following form

$$\langle e^{i\phi(\tau_1)} e^{-i\phi(\tau_2)} \rangle = \exp\left[-\frac{U}{2} |\tau_{12}| + i \int_{\tau_2}^{\tau_1} A(\tau) d\tau \right] \tag{88}$$

Let us explain the physical meaning of the two terms:

- The first term is the energy cost due to the excitation created by $e^{\pm i\phi}$. At the zero temperature, the initial state is the ground state and $e^{\pm i\phi}$ maps it to the first excited state which has an energy cost $U/2$. This term will be modified at finite temperature.

- The second term is the Wilson line to guarantee that both sides match under the gauge transformation $\phi(\tau) \to \phi(\tau) + \alpha(\tau)$, $A \to A + \partial_\tau \alpha$. This term will not change even at finite temperature.

References

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