Free Bose Gases

Free Fermion

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General Formulation & Sommerfeld Expansion

Consider free Fermi gas in a general dimension with density of states $g(\epsilon)$, the grand canonical partition function can be written as,

$$\log Z = \int d\epsilon\, g(\epsilon) \log\left(1 + e^{-\beta(\epsilon-\mu)}\right) \tag{48}$$

The particle number is,

$$N = \frac{1}{\beta} \frac{\partial \log Z}{\partial \mu} = \int_0^\infty d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1}\,. \tag{49}$$

The energy is,

$$E = -\frac{\partial \log Z}{\partial \beta} + \mu N = \int_0^\infty d\epsilon \frac{\epsilon\, g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1}. \tag{50}$$

■ Sommerfeld Expansion

Many of the physical quantities are of the following form,

$$I(\beta, \mu) = \int_0^\infty d\epsilon \frac{f(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1}. \tag{51}$$

Its generic behavior is complicated. However, at very low temperature, the most important contribution to the integral only comes from the region around μ controlled by $k_B T$. Therefore let's introduce $\epsilon - \mu = k_B T z$ to rewrite the integral as,

$$I = k_B T \int_{-\mu/T}^\infty dz \frac{f(\mu + k_B T z)}{e^z + 1} = k_B T \int_0^{\mu/T} dz \frac{f(\mu - k_B T z)}{e^{-z} + 1} + k_B T \int_0^\infty dz \frac{f(\mu + k_B T z)}{e^z + 1}$$

Notice that,

$$\frac{1}{e^{-z} + 1} = 1 - \frac{1}{e^z + 1},$$

we can rewrite the integral as,

$$I = k_B T \int_0^{\mu/T} dz\, f(\mu - k_B T z) - k_B T \int_0^{\mu/T} dz \frac{f(\mu - k_B T z)}{e^z + 1} + k_B T \int_0^\infty dz \frac{f(\mu + k_B T z)}{e^z + 1}.$$

In the second term, since $\mu / T >> 1$ and the whole integrand is decaying exponentially at the scale $z \sim 1$ for a regular $f(x)$, we can replace the upper limit with ∞ and have,

$$I = \int_0^\mu \mathrm{d}\epsilon\, f(\epsilon) + k_B\, T \int_0^\infty \mathrm{dz}\, \frac{f(\mu + k_B\, T\, z) - f(\mu - k_B\, T\, z)}{e^z + 1}. \tag{52}$$

Now we can do a Taylor expansion to the numerator of the second term and get,

$$\begin{aligned} I &= \int_0^\mu \mathrm{d}\epsilon\, f(\epsilon) + 2\,(k_B\, T)^2\, f'(\mu) \int_0^\infty \mathrm{dz}\, \frac{z}{e^z+1} + \frac{(k_B\, T)^4}{3}\, f'''(\mu) \int_0^\infty \mathrm{dz}\, \frac{z^3}{e^z+1} \\ &= \int_0^\mu \mathrm{d}\epsilon\, f(\epsilon) + \frac{\pi^2}{6}\,(k_B\, T)^2\, f'(\mu) + \frac{7\,\pi^4}{360}\,(k_B\, T)^4\, f'''(\mu)\,. \end{aligned} \tag{53}$$

This is the Sommerfeld expansion formula.

Low Temperature C_V for 1D Free Fermion

Let's consider non-relativistic free Fermi gas in a line with length L_x. The dispersion is $\epsilon = \hbar^2\, k^2 / 2\, m$. Therefore the density of state is,

$$D_1(\epsilon)\, \mathrm{d}\epsilon = \frac{2\, \mathrm{dk}}{(2\,\pi)/L} = \frac{L}{2\,\pi\,\hbar}\,(2\, m)^{1/2}\, \frac{\mathrm{d}\epsilon}{\sqrt{\epsilon}} = \alpha\, \epsilon^{-1/2}\, \mathrm{d}\epsilon, \tag{54}$$

where we introduce $\alpha = \frac{L}{2\,\pi\,\hbar}\,(2\, m)^{1/2}$ to keep formula simple. The partition function is,

$$\log Z_{1\,D} = \int_0^\infty \mathrm{d}\epsilon\, D_1(\epsilon)\, \log\!\left(1 + e^{-\beta(\epsilon-\mu)}\right). \tag{55}$$

We are interested in the heat capacity per unit length in particular at the low temperature,

$$C_V = \frac{1}{L}\, \frac{\partial E}{\partial T},\quad T << \mathrm{T}_F. \tag{56}$$

Therefore we can use the Sommerfeld expansion. However, since we are fixing the particle number, we first need to replace μ with N and T. Using the number equation,

$$N = \int_0^\infty \mathrm{d}\epsilon\, \frac{D_1(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \approx \int_0^\mu \mathrm{d}\epsilon\, D_1(\epsilon) + \frac{\pi^2}{6}\,(k_B\, T)^2\, D_1{}'(\mu). \tag{57}$$

By definition, N is related to the Fermi energy by,

$$N = \int_0^{\epsilon_F} \mathrm{d}\epsilon\, D_1(\epsilon) = 2\,\alpha\, \sqrt{\epsilon_F}\,.$$

Therefore we have,

$$0 = \int_{\epsilon_F}^\mu \mathrm{d}\epsilon\, D_1(\epsilon) + \frac{\pi^2}{6}\,(k_B\, T)^2\, D_1{}'(\mu)\,.$$

To the lowest order of temperature, we have,

$$\mu \approx \epsilon_F - \frac{\pi^2}{6}\,(k_B\, T)^2\, \frac{D_1{}'(\epsilon_F)}{D_1(\epsilon_F)} = \epsilon_F + \frac{\pi^2}{12}\, k_B\, T \left(\frac{k_B\, T}{\epsilon_F}\right). \tag{58}$$

Plug this into the definition of the energy, we have,

$$\begin{aligned} E &= \int_0^\infty \mathrm{d}\epsilon\, \frac{\epsilon\, D_1(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \approx \int_0^\mu \mathrm{d}\epsilon\, \epsilon\, D_1(\epsilon) + \frac{\pi^2}{6}\,(k_B\, T)^2\, (\epsilon\, D_1)'(\mu) \\ &\approx \int_0^{\epsilon_F} \mathrm{d}\epsilon\, \epsilon\, D_1(\epsilon) + \int_{\epsilon_F}^\mu \mathrm{d}\epsilon\, \epsilon\, D_1(\epsilon) + \frac{\pi^2}{6}\,(k_B\, T)^2\, (\epsilon\, D_1)'(\epsilon_F) \end{aligned}$$

$$= \alpha \frac{2\,\epsilon_F{}^{3/2}}{3} \left(\left(\frac{\mu}{\epsilon_F}\right)^{3/2} + \frac{\pi^2}{8} \left(\frac{k_B\,T}{\epsilon_F}\right)^2 \right)$$

$$\approx \alpha \frac{2\,\epsilon_F{}^{3/2}}{3} \left(1 + \frac{\pi^2}{4} \left(\frac{k_B\,T}{\epsilon_F}\right)^2\right)$$

Thus, the heat capacity is,

$$C_V = \frac{1}{L} \frac{\partial E}{\partial T} = \frac{\alpha}{L} \frac{\pi^2}{3} k_B \left(\frac{k_B\,T}{\epsilon_F{}^{1/2}}\right) = \frac{1}{2\,\pi\,\hbar} (2\,m)^{1/2} \frac{\pi^2}{3} k_B \left(\frac{k_B\,T}{\epsilon_F{}^{1/2}}\right) \tag{59}$$

Notice that $\epsilon_F = \frac{\hbar^2\,k_F{}^2}{2\,m}$, $v_F = \hbar\,k_F / m$, we can rewrite it as

$$C_V = \frac{1}{2\,\pi\,\hbar} (2\,m) \frac{\pi^2}{3} k_B \left(\frac{k_B\,T}{\hbar\,k_F}\right) = \frac{\pi}{3} k_B \left(\frac{k_B\,T}{\hbar\,v_F}\right), \tag{60}$$

which is perfectly dual to the result for the 1D photon result, if we replace v_F with the speed of sound c. This implies that the low energy physics for a 1D Fermi is the same as a 1D phonon, which is the bosonization.

The reason for such a duality is that,

- The low energy/temperature physics is controlled by the modes near the Fermi surface. For those modes, their dispersion can be approximated as,

$$\epsilon = \frac{\hbar^2 (k_F + \delta \mathrm{k})^2}{2\,m} \approx \frac{\hbar^2\,k_F{}^2}{2\,m} + v_F\,\hbar\,\delta \mathrm{k} \tag{61}$$

which can be thought of as fermions with linear dispersion.

To test this, let's consider a gapless Fermi gas and compute its C_V to see whether it match with our result above.

Assume the the dispersion is $\epsilon = v_F\,\hbar\,k$ which yield the DoS,

$$D_1(\epsilon)\,\mathrm{d}\epsilon = \frac{2\,\mathrm{dk}}{(2\,\pi) / L} = \frac{L}{\pi\,\hbar\,v_F}\,\mathrm{d}\epsilon = \gamma\,\mathrm{d}\epsilon, \tag{62}$$

where we introduce $\gamma = \frac{L}{\pi\,\hbar\,v_F}$ to keep formula simple. Again, let's compute the number equation to replace μ with N and T. Introduce the dimensionless energy $x = \beta\,\epsilon$,

$$N = \int_0^\infty \mathrm{d}\epsilon \frac{D_1(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} = \gamma\,k_B\,T \int_0^\infty \frac{\mathrm{d}\epsilon}{e^{-\beta\mu}\,e^x + 1} = \gamma\,k_B\,T \log\left(1 + e^{\beta\,\mu}\right). \tag{63}$$

At the low temperature limit, we can drop 1 comparing with $e^{\beta\mu}$ and have,

$$\mu \approx N / \gamma = \epsilon_F \tag{64}$$

The energy is,

$$E = \gamma \int_0^\infty \mathrm{d}\epsilon \frac{\epsilon}{e^{\beta(\epsilon-\mu)} + 1} \approx \gamma \int_0^\mu \mathrm{d}\epsilon\,\epsilon + \frac{\pi^2}{6} (k_B\,T)^2 (\gamma\,\epsilon)'\,(\mu) = \frac{\gamma}{2} \left(\frac{N}{\gamma}\right)^2 + \frac{\pi^2}{6} (k_B\,T)^2\,\gamma$$

The heat capacity is,

$$C_V = \frac{1}{L} \frac{\partial E}{\partial T} = \frac{\gamma}{L} \frac{\pi^2}{3} k_B{}^2\,T = \frac{\pi}{3} k_B \frac{k_B\,T}{\hbar\,v_F}, \tag{65}$$

which indeed is the same as what we found for the non-relativistic fermion.

- When we fix the particle number, all the excitations are particle-hole excitations. For 1D fermions with linear dispersion, a particle-hole excitation, which is bosonic, also has linear dispersion, which can be thought of a 1D phonon.

Heat Capacity for 2D Dirac Fermion and Phonon

Since we know that we have this beautiful mapping between fermion and boson in 1D, a natural question is what happens if we move to 2D. Now we cannot compare non-relativistic fermion with phonon. Because we know the C_V for non-relativistic fermion is still proportional to T while the C_V for phonon is proportional to T^2.

2D Dirac Fermion

Therefore, let's try the 2D Dirac fermion instead,

$$H = \hbar\, v_F \sum_k \Psi^\dagger(k)\,(k_x\,\sigma^x + k_y\,\sigma^y)\,\Psi(k)\,, \tag{66}$$

where $\Psi = (\psi_R, \psi_L)^T$ is a two component complex fermion field. The dispersion relation is,

$$\epsilon = \pm\hbar\, v_F\, \sqrt{{k_x}^2 + {k_y}^2}$$

and the DoS is,

$$D(\epsilon) = \frac{2\,\pi\,k\,\mathrm{dk}}{(2\,\pi)^2 / A} = \frac{A}{2\,\pi\,(\hbar\,v_F)^2}\,|\epsilon|\,\mathrm{d}\epsilon\,. \tag{67}$$

At $T = 0$, we fill all the negative energy states therefore $\epsilon_F = 0$. The the **particle-hole symmetry** of the system will guarantee the chemical potential at finite temperature still be zero. The reason is that, the probability for a state of energy $\mu + \delta$ to be occupied is

$$P[n(\mu+\delta) = 1] = f(\mu+\delta) = \frac{1}{e^{\beta\,\delta} + 1}$$

The probability for a state of energy $\mu - \delta$ to be unoccupied is,

$$P[n(\mu-\delta) = 0] = 1 - f(\mu-\delta) = 1 - \frac{1}{e^{-\beta\,\delta} + 1} = \frac{1}{e^{\beta\,\delta} + 1}\,.$$

This result implies that for $\mu = 0$,

$$n(\epsilon) + n(-\epsilon) = \frac{1}{e^{\beta\,\epsilon} + 1} + \frac{1}{e^{-\beta\,\epsilon} + 1} = 1$$

doesn't depend on the temperature. (The reason that we can add them is because they share the same DoS, which is where particle-hole symmetry comes in) Adding up all such energies, we conclude that the total particle number is unchanged if μ stays at zero.

After figuring out the chemical potential to be $\mu = 0$, it is now easy to compute the energy. Let's use $+/-$ to denote the positive/negative energy states and introduce,

$$n_\pm(\epsilon) = \frac{1}{e^{\pm\beta\,\epsilon} + 1}\,. \tag{68}$$

Then we have,

$$\begin{aligned}E(T) - E(0) &= \sum_{\epsilon>0} [n_+(\epsilon)\,\epsilon + (1 - n_-(\epsilon)\,\epsilon)] = 2 \sum_{\epsilon>0} n_+(\epsilon)\,\epsilon \\ &= \frac{2\,A}{2\,\pi\,(\hbar\,v_F)^2} \int_0^\infty \mathrm{d}\epsilon\,\epsilon\,\frac{\epsilon}{e^{\beta\,\epsilon}+1} \\ &= \frac{2\,A}{2\,\pi}\left(\frac{k_B\,T}{\hbar\,v_F}\right)^2 k_B\,T \int_0^\infty \mathrm{dx}\,\frac{x^2}{e^{\beta\,x}+1} \\ &= 2\,\frac{3\,\zeta(3)}{2}\,k_B\,T\,\frac{A}{2\,\pi}\left(\frac{k_B\,T}{\hbar\,v_F}\right)^2\end{aligned}$$

where $\zeta(3) = \frac{2}{3}\int_0^\infty \mathrm{dx}\,\frac{x^2}{e^{\beta\,x}+1} \approx 1.202$. And the heat capacity is,

$$C_V = \frac{1}{A}\,\frac{\partial E}{\partial T} = k_B\,\frac{9\,\zeta(3)}{2\,\pi}\left(\frac{k_B\,T}{\hbar\,v_F}\right)^2. \tag{70}$$

The scaling form matches the phonon result.

2D Phonon

We consider a 2D phonon, which is a relativistic real scalar field described by the EoM,

$$\partial_t^{\,2}\,\phi = \hbar^2\,c^2\left(\partial_x^{\,2} + \partial_y^{\,2}\right)\phi\,. \tag{71}$$

The dispersion is,

$$\epsilon = \hbar\,c\,\sqrt{k_x^{\,2} + k_y^{\,2}} \tag{72}$$

The DoS is,

$$D(\epsilon) = \frac{2\,\pi\,k\,\mathrm{dk}}{(2\,\pi)^2 / A} = \frac{A}{2\,\pi\,(\hbar\,c)^2}\,\epsilon\,\mathrm{d}\epsilon \tag{73}$$

The partition function is,

$$\log Z = -\frac{A}{2\,\pi\,(\hbar\,c)^2}\int_0^\infty \mathrm{d}\epsilon\,\epsilon\,\log\left(1 - e^{-\beta\,\epsilon}\right). \tag{74}$$

Let's introduce the dimensionless energy $x = \beta\,\epsilon$ and write it as,

$$\log Z = -\frac{A}{2\,\pi}\left(\frac{k_B\,T}{\hbar\,c}\right)^2 \int_0^\infty \mathrm{dx}\,x\,\log(1 - e^{-x}) \;=\; \frac{A}{2\,\pi}\left(\frac{k_B\,T}{\hbar\,c}\right)^2 \zeta(3), \tag{75}$$

where $\zeta(3) = \int_0^\infty \mathrm{dx}\,x\,\log(1 - e^{-x})$. Therefore the energy is,

$$E = -\frac{\partial \log Z}{\partial \beta} = k_B\,T^2\,\frac{\partial \log Z}{\partial T} \;=\; \frac{A}{\pi}\,k_B\,T\left(\frac{k_B\,T}{\hbar\,c}\right)^2 \zeta(3). \tag{76}$$

The heat capacity is,

$$C_V = \frac{1}{A}\,\frac{\partial E}{\partial T} = k_B\,\frac{3\,\zeta(3)}{\pi}\left(\frac{k_B\,T}{\hbar\,c}\right)^2 \tag{77}$$

Interpretation of the results

From the calculation above, we can see that $C_{V,\,\mathrm{fermion}} = \frac{3}{2}\,C_{V,\,\mathrm{boson}} > C_{V,\,\mathrm{boson}}$. Therefore, if we simply use free bosons, we cannot reproduce the same amount of degrees of freedom as free fermions. As a result, if we want to do a faithful bosonization, we need to introduce more degrees of freedom (**the gauge field**) on the boson side.

Stoner Ferromagnetism

Iron is a metal as well as a ferromagnetic material. How can we model such a phenomena: itinerant ferromagnetism? Stoner provided a very simple model that shows one possibility.

Let's consider a gas of electrons. Since they carry charge, we need to include the Coulomb interaction between them. This makes the problem impossible to be solved. However, we can try to extract the most important features and simplify it.

The Coulomb repulsion favors wavefunctions to be anti-symmetric in position space. If electrons have aligned spins, their spatial wavefunction has to be anti-symmetric thus there will be less interaction. Therefore as a very crude approximation, the interaction can be effective described by an interaction that favors spin aligned,

$$H_{\text{int}} = \frac{\alpha}{V} N_\uparrow N_\downarrow, \tag{78}$$

where $N_{\uparrow/\downarrow}$ is the number of electrons with spin up/down.

Since we fix the particle number, $N_\uparrow + N_\downarrow = N$, $N_\uparrow - N_\downarrow = 2 S_z$, we can rewrite the interaction as,

$$H_{\text{int}} = \frac{\alpha}{V} \frac{(N + 2 S_z)}{2} \frac{(N - 2 S_z)}{2} = \frac{\alpha}{V} \left(\frac{N^2}{4} - S_z{}^2\right) = \alpha\, V \left(\frac{n^2}{4} - s_z{}^2\right). \tag{79}$$

where we introduce the particle number density n and spin density s_z. Its clear that it favors a state with maximized S_z.

Now let's calculate the kinetic term. For one spin orientation, we have,

$$N_s = V \int \frac{d^3 k}{(2\pi)^3} = \frac{V k_{F,s}{}^3}{6\pi^2} \Rightarrow k_{F,s} = \left(6\pi^2 \frac{N_s}{V}\right)^{1/3}, \quad s = \uparrow \text{ or } \downarrow .$$

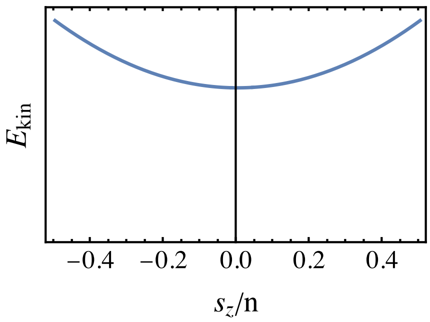
The kinetic energy is,

$$E_{\text{kin},s} = V \int \frac{d^3 k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} = V \frac{\hbar^2}{2m} \frac{4\pi}{(2\pi)^3} \frac{k_{F,s}{}^5}{5} = V \frac{\hbar^2}{2m} \frac{(6\pi^2)^{5/3}}{10\pi^2} n_s{}^{5/3}.$$

Thus the total kinetic energy is,

$$\frac{E_{\text{kin}}}{V} = \frac{\hbar^2}{2m} \frac{(6\pi^2)^{5/3}}{10\pi^2} \left(n_\uparrow{}^{5/3} + n_\downarrow{}^{5/3}\right) = \frac{\hbar^2}{2m} \frac{(6\pi^2)^{5/3}}{10\pi^2} \left(\left(\frac{n}{2} + s_z\right)^{5/3} + \left(\frac{n}{2} - s_z\right)^{5/3}\right). \tag{80}$$

We can plot this function and find that it favors a state without any magnetisation.



Now, we have a competition between interaction energy and kinetic energy. Let's expand E_{kin} with respect to s_z to fourth order and write the total energy as,

$$\begin{aligned}\frac{E_{\text{tot}}}{V} &= \frac{\hbar^2}{2\,m}\,\frac{\left(6\,\pi^2\right)^{5/3}}{10\,\pi^2}\left(\frac{n^{5/3}}{2^{2/3}}+\frac{10\,\sqrt[3]{2}\,\,s_z{}^2}{9\,\sqrt[3]{n}}+\frac{40\,\sqrt[3]{2}\,\,s_z{}^4}{243\,n^{7/3}}\right)+\alpha\left(\frac{n^2}{4}-s_z{}^2\right)\\ &= E_0(n)+\left(\frac{\hbar^2}{2\,m}\,\frac{4\,\pi^{4/3}}{\sqrt{3}\,\,n^{1/3}}-\alpha\right)s_z{}^2+\frac{\hbar^2}{2\,m}\,\frac{16\,\pi^{4/3}}{27\,\sqrt[3]{3}\,\,n^{7/3}}\,s_z^4\end{aligned} \tag{81}$$

In particular, the coefficient of the second order term can change its sign when the interaction is strong enough and the system will favor a nonzero magnetisation. The criterion is,

$$\alpha > \alpha_c = \frac{\hbar^2}{2\,m}\,\frac{4\,\pi^{4/3}}{\sqrt{3}\,\,n^{1/3}}. \tag{82}$$

The magnetisation will have the following scaling behavior,

$$M \propto \begin{cases} 0 & \alpha < \alpha_c \\ \sqrt{\alpha-\alpha_c} & \alpha > \alpha_c \end{cases}. \tag{83}$$

▫ Formulate in terms of Fermi Hubbard Model

Another equivalent way to state this problem is to start from the Hubbard interaction and do mean field,

$$\begin{aligned}H_{\text{int}} &= U\sum_i n_{i,\uparrow}\,n_{i,\downarrow} = U\sum_i [\langle n_{i,\uparrow}\rangle+(n_{i,\uparrow}-\langle n_{i,\uparrow}\rangle)]\left[\langle n_{i,\downarrow}\rangle+\left(n_{i,\downarrow}-\langle n_{i,\downarrow}\rangle\right)\right]\\ &\approx U\sum_i\left[\langle n_{i,\uparrow}\rangle\langle n_{i,\downarrow}\rangle+\langle n_{i,\uparrow}\rangle\left(n_{i,\downarrow}-\langle n_{i,\downarrow}\rangle\right)+(n_{i,\uparrow}-\langle n_{i,\uparrow}\rangle)\langle n_{i,\downarrow}\rangle\right]\\ &= U\sum_i\left[\langle n_{i,\uparrow}\rangle\,n_{i,\downarrow}+n_{i,\uparrow}\langle n_{i,\downarrow}\rangle-\langle n_{i,\uparrow}\rangle\langle n_{i,\downarrow}\rangle\right]\end{aligned}$$

We assume the translation symmetry is not broken and have $\langle n_{i,\,s}\rangle = \frac{n}{2} \pm s_z$, therefore

$$\begin{aligned}H_{\text{int}} &= E_0+U\sum_i\left[\left(\frac{n}{2}+s_z\right)n_{i,\downarrow}+\left(\frac{n}{2}-s_z\right)n_{i,\uparrow}\right]\\ &= E_0+U\sum_k\left[\left(\frac{n}{2}+s_z\right)n_{k,\downarrow}+\left(\frac{n}{2}-s_z\right)n_{k,\uparrow}\right].\end{aligned} \tag{84}$$

The total energy is,

$$H_{\text{MF}} = E_0+\sum_k\left[\epsilon_k+U\left(\frac{n}{2}+s_z\right)\right]n_{k,\downarrow}+\sum_k\left[\epsilon_k+U\left(\frac{n}{2}-s_z\right)n_{k,\uparrow}\right]. \tag{85}$$

Reference

[1] Kardar, Statistical Physics of Particles

[2] Ashvin Vishwanath, Lectures on Topological Phases, Problem Set 4.

[3] Liujun Zou, private discussion on the dinner table