Yang's Conjecture

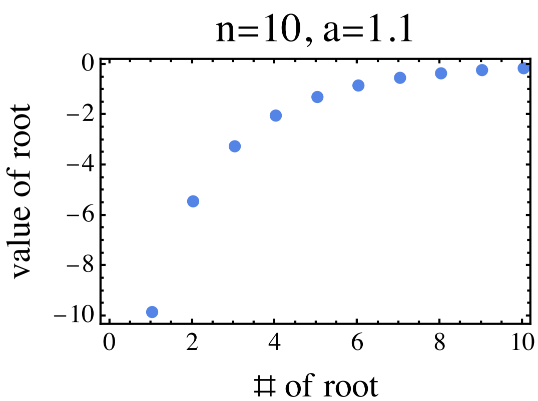
Conjecture: Consider the following polynomial

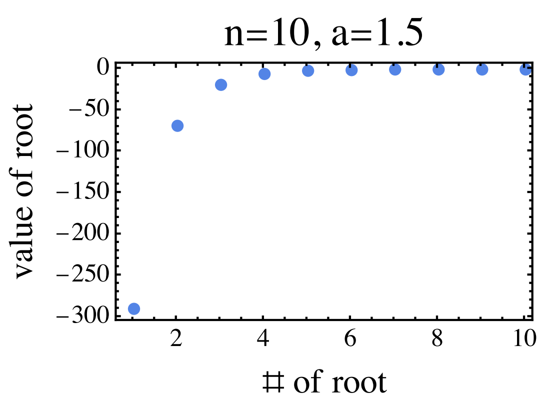
$$P_n(z) = \sum_{k=0}^{n} \binom{n}{k} z^k \, a^{k(n-k)} . \tag{1}$$

For $a \geq 1$, all the roots are real and smaller than 0.

Numerical Verifications for $a > 1$

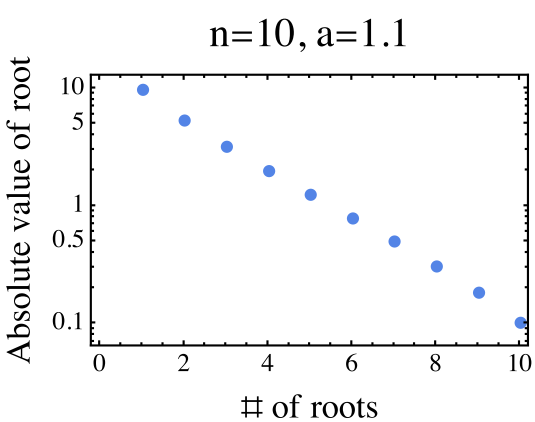
For $a > 1$, the roots are indeed real and smaller than 0.

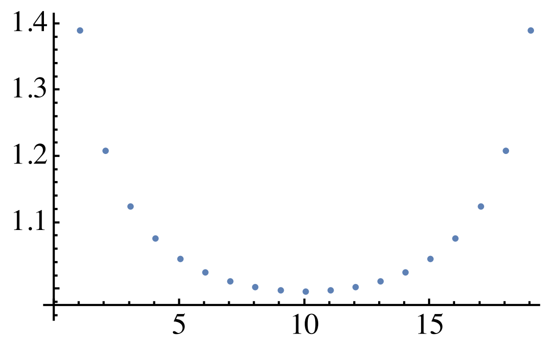




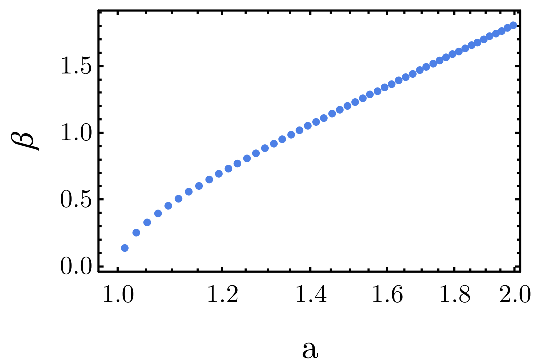
Their absolute values are decreasing exponentially, i.e. if we order the roots and use j to label the j-th root, then we have

$$|r_j| \propto e^{-\beta j}, \;\; \beta > 0 \tag{2}$$

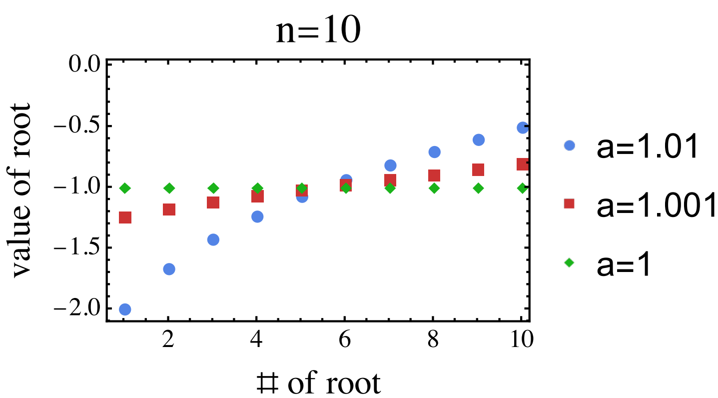




The number β increases with a logarithmically when a is large. And β becomes 0 when a approaches 1. These are demonstrated below.

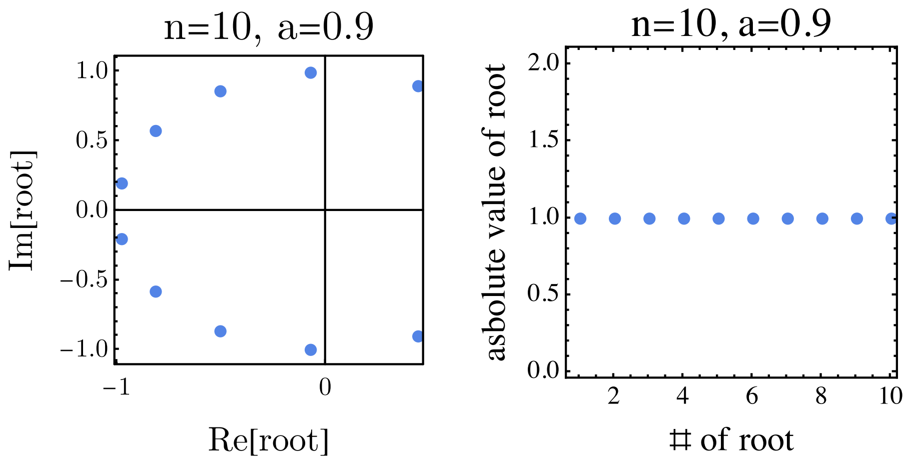


Right at $a = 1$, all the roots become 1.



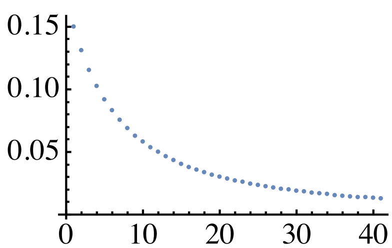
▫ Numerical Verification for $a < 1$

For $a < 1$, all the roots become complex numbers with the unit absolute value.

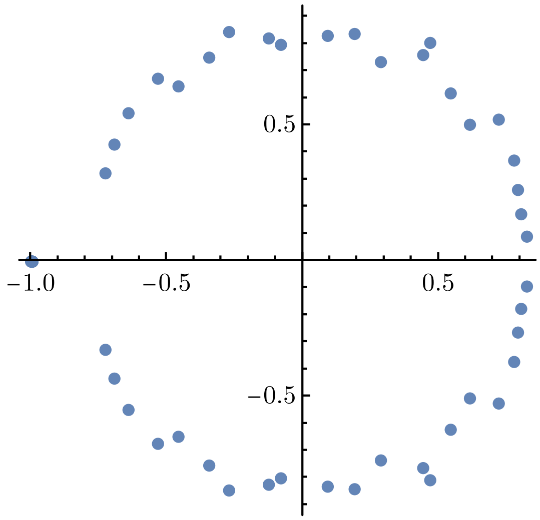


■ Fix z and solve a

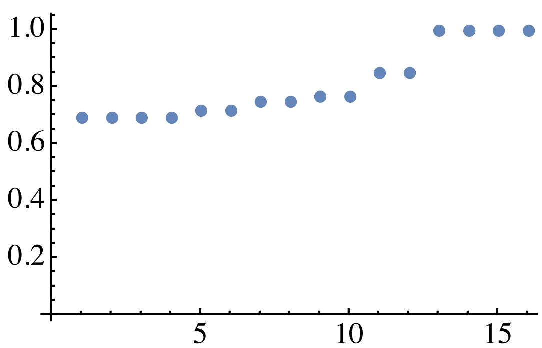
```
Block[{n, a, z = 1., P, roots},
 P[n_, a_, z_] := Sum[Binomial[n, k] z^k a^(k (n-k)), {k, 0, n}];
 ListPlot[Table[roots = a /. NSolve[P[n, a, z] == 0, a];
   Abs@Im[SortBy[roots, Re[#] &][[-1]]], {n, 10, 50}]]
]
```



```
Block[{n = 14, a, z = 1., P, roots},
 P[n_, a_, z_] := Sum[Binomial[n, k] z^k a^(k (n-k)), {k, 0, n}];
 roots = a /. NSolve[P[n, a, z] == 0, a];
 ComplexListPlot[roots, PlotMarkers → {Automatic, 7}, ImageSize → 250]
]
```




```
Block[{n = 8, a, z = 1., P, roots},
 P[n_, a_, z_] := Sum[Binomial[n, k] z^k a^(k (n-k)), {k, 0, n}];
 roots = a /. NSolve[P[n, a, z] == 0, a];
 ListPlot[Sort@Abs@roots,
  PlotMarkers → {Automatic, 7}, AxesOrigin → {0, 0}, ImageSize → 250]
]
```



Series$\left[n \,\text{Log}[n] - \frac{n+s}{2} \,\text{Log}\left[\frac{n+s}{2}\right] - \frac{n-s}{2} \,\text{Log}\left[\frac{n-s}{2}\right], \{s, 0, 2\}\right]$

$n \,\text{Log}[2] - \frac{s^2}{2\,n} + O[s]^3$