NOTE

Note on Coulomb Gas Formalism

Ruihua Fan[a]

[a] *Department of Physics, Harvard University*

Abstract: In this note, I will detail the coulomb gas formalism which is based on the 2D free boson model.

Contents

1 2D Free Boson

The Model We put the system on an infinite plane and use imaginary time. The action is written as,

$$S = \frac{1}{8\pi}\int d^2x \partial_\mu\varphi\partial^\mu\varphi = \frac{1}{8\pi}\int d^2x \left(\partial_z\varphi\partial^z\varphi + \partial_{\bar{z}}\varphi\partial^{\bar{z}}\varphi\right). \tag{1.1}$$

We can easily show that the classical action is invariant under the following global conformal transformation[1]:

$$\begin{aligned} z \to \omega(z), \quad \varphi'(w) = \varphi(z) \\ S[\varphi'] = S[\varphi] \end{aligned} \tag{1.2}$$

The two point function $K(\mathbf{x}) = \langle\varphi(\mathbf{x})\varphi(0)\rangle$ can be got by solving the equation of motion,

$$\partial_{\mathbf{x}}^2 K(\mathbf{x}) = -4\pi\delta^{(2)}(\mathbf{x}) \Rightarrow K(\mathbf{x}) = -2\log|\mathbf{x}| \tag{1.3}$$

[1]We need global transformation to preserve the geometry of the system.

The UV and IR divergence here can be regularized as following,

$$K(\mathbf{x}) = -\log m^2(\mathbf{x}^2 + a^2) \tag{1.4}$$

where m regularizes the IR[2] and a regularizes UV. Both of them are small.

Energy-Momentum Tensor Translation symmetry gives us the energy-momentum tensor:

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi\right) \tag{1.5}$$

In the conformal coordinates, it is written as,

$$T^z{}_z = \frac{\partial\mathcal{L}}{\partial(\partial_z\varphi)}\partial_z\varphi - \eta^z{}_z\mathcal{L} = 0 \tag{1.6}$$

$$T^z{}_{\overline{z}} = \frac{\partial\mathcal{L}}{\partial(\partial_z\varphi)}\partial_{\overline{z}}\varphi - \eta^z{}_{\overline{z}}\mathcal{L} = \frac{1}{4\pi}\partial^z\varphi\partial_{\overline{z}}\varphi \tag{1.7}$$

$$T^{\overline{z}}{}_{\overline{z}} = 0, \quad T^{\overline{z}}{}_z = \frac{1}{4\pi}\partial^{\overline{z}}\varphi\partial_z\varphi \tag{1.8}$$

The vanishing of $T^z{}_z$ and $T^{\overline{z}}{}_{\overline{z}}$ actually reflects the conformal invariance of the action, i.e. if we take an infinitesimal conformal transformation $z \to z + \epsilon(z), \overline{z} \to \overline{\epsilon}(\overline{z})$, then we have $\delta S \sim \int d^2x T^z{}_z\partial_z\epsilon + T^{\overline{z}}{}_{\overline{z}}\partial_{\overline{z}}\overline{\epsilon} = 0$. So $T^z{}_z = T^{\overline{z}}{}_{\overline{z}} = 0$ is a general feature.

The holomorphic and anti-holomorphic energy-momentum tensor is,

$$T = -2\pi T_{zz} = -\frac{1}{2}\partial_z\varphi\partial_z\varphi, \quad \overline{T} = -2\pi T_{\overline{z}\overline{z}} = -\frac{1}{2}\partial_{\overline{z}}\varphi\partial_{\overline{z}}\varphi \tag{1.9}$$

And the current conservation equation $\partial_\mu T^\mu_\nu = 0$ is now written as,

$$\partial_{\overline{z}}T = \partial_z\overline{T} = 0. \tag{1.10}$$

So $T(z)(\overline{T}(\overline{z}))$ is purely holomorphic(anti-holomorphic), which is also a general feature.

Internal Symmetry This action also has an internal symmetry $\varphi \to \varphi + a$, where a is a constant. This symmetry gives a Nöether current,

$$S' = \frac{1}{8\pi}\int d^2x(\partial_\mu\varphi + \partial_\mu a)(\partial^\mu\varphi + \partial^\mu a) = S + \frac{1}{4\pi}\int d^2x\partial_\mu\varphi\,\partial^\mu a + O(a^2)$$
$$\Rightarrow j^\mu = -\frac{1}{4\pi}\partial^\mu\varphi. \tag{1.11}$$

In the radial quantization scheme, it is more useful to define the holomorphic current $J(z) = i\partial\phi$. Then the corresponding charge is $Q = \dfrac{1}{2\pi i}\oint dzJ$. The charge of a field is defined as $[Q, A] = q_A A$.[3]

[2]Physically, it means beyond the length scale $1/m$, the correlator should be modified and decay maybe exponentially.

[3]This can be understood by making an analogy with the usual U(1) charge $[N, \psi^\dagger] = \psi^\dagger$.

Quantization We use radial quantization scheme. So we first work on a cylinder with spatially periodic boundary condition (still imaginary time) and then go to the Euclidean plane by using the conformal coordinates $z = e^{2\pi(\tau - ix)/L}$, $\overline{z} = e^{2\pi(\tau+ix)/L}$. Then we can write down the mode expansion of $\varphi(z, \overline{z})$:

$$\varphi(z,\overline{z}) = \varphi_0 - i\pi_0 \log(z\overline{z}) + i \sum_{n\neq 0} \frac{1}{n}(a_n z^{-n} + \overline{a}_n \overline{z}^{-n}) \tag{1.12}$$

$$i\partial\varphi(z) = \sum_n a_n z^{-n-1}, \quad a_0 \equiv \pi_0 \tag{1.13}$$

where we have $[a_n, a_m] = [\overline{a}_n, \overline{a}_m] = n\delta_{n+m,0}$ and $[a_n, \overline{a}_m] = 0$. So the vacuum is annihilated by $a_n, \overline{a}_n$ with $n > 0$. From this expansion we can also see that φ is not primary but $\partial\varphi$. It is also useful to decompose $\varphi(z, \overline{z})$ into its left-hand and right-hand parts as

$$\begin{aligned} \varphi(z,\overline{z}) \text{``}=\text{''} \phi(z) + \overline{\phi}(\overline{z}), \\ \phi(z) = \varphi_0 - i\pi_0 \log z + i\sum_{n\neq 0}\frac{1}{n} a_n z^{-n}. \end{aligned} \tag{1.14}$$

This decomposition is not faithful due to the double counting of the zero mode. But including the zero mode into $\phi(z)$ turns out to be crucial when we use it to define the left-hand vertex operator.

Using this we can also write down the mode expansion of energy-momentum tensor:

$$T(z) = -\frac{1}{2} :\partial\varphi\partial\varphi:= \frac{1}{2}\sum_{n,m} z^{-n-m-2} :a_n a_m: \tag{1.15}$$

which implies,

$$L_n = \frac{1}{2}\sum_m :a_{n-m}a_m:, \quad L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2 \tag{1.16}$$

Because φ_0 doesn't enter L_0, π_0 becomes a good quantum number, which can be used to label different vacuums.

2 Vertex Operators

An important kind of primary fields is the vertex operator,

$$V_\alpha(z,\overline{z}) =: \exp\left(i\sqrt{2}\alpha\varphi(z,\overline{z})\right) := \sum_{n=0} \frac{i(\sqrt{2}\alpha)^n}{n!} :\varphi(z,\overline{z})^n: \tag{2.1}$$

with the conformal dimension $h_\alpha = \alpha^2$ and the charge $q = \alpha$. The $\sqrt{2}$ is included for the later convenience in the discussion of Coulomb gas. Here the normal ordering means moving all the annihilation operators to the right. So in terms of mode expansion, it can be written as,

$$V_\alpha(z,\overline{z}) = \exp\left(i\sqrt{2}\alpha\varphi_0 + \sqrt{2}\alpha\sum_{n>0}\frac{1}{n}(a_{-n}z^{-n} + \overline{a}_{-n}\overline{z}^{-n})\right)\exp\left(\sqrt{2}\alpha\pi_0 - \sqrt{2}\alpha\sum_{n>0}\frac{1}{n}(a_n z^{-n} + \overline{a}_n\overline{z}^{-n})\right) \tag{2.2}$$

This operator is useful in two ways: (1) build up the Hilbert space; (2) its correlation functions are the basis of the Coulomb gas formalism.

Build up Hilbert space We said that the eigenvalue of π_0 can label different vacuums $\pi_0 |h\rangle = h |h\rangle$, $a_n |h\rangle = \bar{a}_n |h\rangle = 0, n > 0$. If we know how to find $|h\rangle$, then the whole Hilbert space can be built upon the single-parameter set $\{|h\rangle\}$.

Noticing the commutator $[a_n, V_\alpha(z,\bar{z})] = \sqrt{2}\alpha z^n V_\alpha(z,\bar{z})$ and the fact $V_\alpha(z,\bar{z})$ is unitary, we can show that $|h = \sqrt{2}\alpha\rangle = V_\alpha(0)\,|0\rangle$ where $|0\rangle$ is the absolute vacuum.

Correlation Functions We are interested in the correlation among a string of vertex operators $\langle V_{\alpha_1}(z_1,\bar{z}_1) V_{\alpha_2}(z_2,\bar{z}_2) ... V_{\alpha_n}(z_n,\bar{z}_n)\rangle$. We will only list results here while leave the details to the Appendix. A,

$$\langle V_{\alpha_1}(z_1,\bar{z}_1) V_{\alpha_2}(z_2,\bar{z}_2) ... V_{\alpha_n}(z_n,\bar{z}_n)\rangle = \prod_{i<j} |z_i - z_j|^{4\alpha_i\alpha_j}, \tag{2.3}$$

$$\sum_i \alpha_i = 0, \quad \leftarrow \quad \text{neutrality condition} \tag{2.4}$$

Remarks: (1) The power law behavior comes from those $n \neq 0$ modes and coincide with the primary nature of vertex operators. (2) The neutrality condition is due to the internal symmetry or the zero mode, which are related to each other.

We can also define a left-hand vertex operator $V_\alpha(z) =: e^{i\sqrt{2}\alpha\phi(z)}:$, whose correlation function only contains the holomorphic part:

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) ... V_{\alpha_n}(z_n)\rangle = \prod_{i<j} (z_i - z_j)^{2\alpha_i\alpha_j}, \tag{2.5}$$

$$\sum_i \alpha_i = 0, \quad \leftarrow \quad \text{neutrality condition.} \tag{2.6}$$

From now on we will only deal with the left-hand component and will come back to the full correlator when talking about real physical properties.

Problem Up to now, this is just mathematics. Then, we want to fit the real physical systems at criticality into this framework. So we have to identity physical observables having definite scaling dimension with vertex operator having the same scaling dimension. But we encounter a problem. V_α and $V_{-\alpha}$ have the same conformal dimension so they should correspond to the same physical operator $\mathcal{O}_\alpha$ and it's tempting to write down $\langle \mathcal{O}_\alpha \mathcal{O}_\alpha\rangle = \langle V_\alpha V_\alpha\rangle = \langle V_\alpha V_{-\alpha}\rangle$. However, this physically reasonable equation is not correct in the current setting. The same dilemma also happens for the higher order correlators.

To resolve this problem, we have to introduce the *screening operators*. But it can only screen integer charge, which means that we can only deal with a physical system whose observables have scaling dimension n^2. To increase the flexibility, we need add background charge to the Coulomb gas.

3 Background Charge

Adding background charge is the basic of Coulomb gas formalism. It is to effectively put a charge at the infinity as a background which changes the neutrality condition. This modification will also change the properties of many operators and central charge. There are two different but effectively equivalent ways to implement this.

3.1 Modify the Correlator

We define the new correlator in this way:

$$\langle\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)...V_{\alpha_n}(z_n)\rangle\rangle \equiv \lim_{R\to\infty} R^{8\alpha_0^2}\,\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)...V_{\alpha_n}(z_n)V_{-2\alpha_0}(R)\rangle \tag{3.1}$$

where we insert a vertex operator with charge $-2\alpha_0$ and conformal dimension $4\alpha_0^2$ at the infinity. And the $R^{8\alpha_0^2}$ prefactor is added to guarantee a proper limit. Thus the neutrality condition is modified to be,

$$\sum_j \alpha_j = 2\alpha_0. \tag{3.2}$$

Now the non-vanishing two-point function is $\langle\langle V_\alpha(z)V_{2\alpha_0-\alpha}(\omega)\rangle\rangle = 1/(z-\omega)^{2\alpha(\alpha-2\alpha_0)}$. So we want to identify V_α and $V_{2\alpha_0-\alpha}$ with the same physical observable, which requires them to have the same conformal dimension $h = \alpha^2 - 2\alpha_0\alpha$. To realize this, we have to modify the energy-momentum tensor correspondingly by adding a $\partial^2\varphi$ term[4]. Solving the OPE equation,

$$\begin{aligned} T(z)V_\alpha(\omega) &= \left(-\frac{1}{2} : \partial\varphi\partial\varphi : + A : \partial^2\varphi :\right) V_\alpha(\omega) \\ &= \frac{\alpha(2\alpha_0-\alpha)}{(z-\omega)^2}V_\alpha(\omega) + \frac{1}{z-\omega}\partial_\omega V_\alpha(\omega) \end{aligned}$$

we can fix the coefficient to be $A = i\sqrt{2}\alpha_0$. With the extra term, $\partial\varphi$ is no longer primary and the central charge is also changed to $c = 1 - 24\alpha_0^2$.

3.2 Modify the Action

From the discussion in Appendix.A, we know the neutrality condition comes from the internal symmetry and corresponding Ward Identity. So to modify it, we have to kill this symmetry. This can be done by directly coupling φ to a new field. But we also want this new field to give a constant term as background charge when we derive the new Ward Identity for $\varphi \to \varphi + a$[5]. Above requirements can be satisfied by the following action:

$$S = \frac{1}{8\pi}\int d^2x\sqrt{g}\,(\partial_\mu\varphi\partial^\mu\varphi + 2\gamma\varphi R) = S^{(0)} + S^{(1)}, \tag{3.3}$$

[4] Apart from $\partial\varphi\partial\varphi$, this is the only term we can write down with a second order derivative

[5] although this is not a symmetry anymore we still can talk about the Ward Identity corresponding to this type of change of field

where R is the scalar curvature and we choose a closed manifold which is topologically equivalent to a 2-sphere this time so that we have $\int d^2x\sqrt{g}R = 8\pi$. However, we are interested in the correlations and physical properties on the plane. So $\sqrt{g} \to 1$ and $R \to 0$ in a large region around the original point, they deviate from a flat space only when away from the original point.

We now derive the new Ward Identity for a string of vertex operators. Under $\varphi \to \varphi + a$,

$$\delta S^{(1)} = \frac{1}{8\pi}\int d^2x\sqrt{g}2\gamma Ra$$

So the Ward Identity $\langle X\delta S\rangle = \langle \delta X\rangle$ is written as,

$$\int d^2x\sqrt{g}a(x)\left\langle X\left(-\frac{1}{4\pi}\partial^\mu\partial_\mu\varphi + \frac{\gamma}{4\pi}R\right)\right\rangle = \int d^2x\sqrt{g}a(x)i\sqrt{2}\sum_{k=1}^{n}\alpha_k\delta^{(2)}(x-x_k)\langle X\rangle$$

We take $a(x)$ to be a constant over the whole space so that we can finish the integrals and get,

$$\begin{aligned} i\sqrt{2}\langle X\rangle\sum_{k=1}^{n}\alpha_k &= \frac{i}{4\pi}\oint dz\,\langle\partial\varphi X\rangle - \frac{i}{4\pi}\oint d\bar{z}\,\langle\bar{\partial}\varphi X\rangle + 2\gamma\langle X\rangle \\ \Rightarrow \sum_{k=1}^{n}\alpha_k &= -i\sqrt{2}\gamma \end{aligned} \tag{3.4}$$

$g = 1$ is implicitly used here to use the Stokes theorem. By setting $\gamma = i\sqrt{2}\alpha_0$, we get the desired new neutrality condition.

This additional term will also alter the energy-momentum tensor,

$$T(z) = -\frac{1}{2} : \partial\varphi\partial\varphi : + i\sqrt{2}\alpha_0 : \partial^2\varphi : \tag{3.5}$$

the derivation is left in Appendix.B. With the modification, V_α is now dual to $V_{2\alpha_0-\alpha}$ with conformal dimension $h_\alpha = \alpha^2 - 2\alpha_0\alpha$. The central charge is $c = 1 - 24\alpha_0^2$. The same result as we get before.

Here, all conclusions just follow the mathematics and we don't need any assumption apart from the flat space limit. While in the previous subsection, we get these from an observation over two-point functions and a physical requirement.

4 Screening Operator

Now we come back to the dilemma mentioned at the end of Sec.2 : V_α and $V_{2\alpha_0-\alpha}$ should be two physically equivalent operators but mathematically they are not, i.e. $\langle V_\alpha V_\alpha\rangle \neq \langle V_\alpha V_{2\alpha_0-\alpha}\rangle$. So if we want to use the framework to describe a physical system, we have to resolve this problem. The way out is to introduce the screening operator.

As the term suggests, by adding it into the correlation function, this operator can screening some charge to help satisfy the neutrality condition so it must carry charge, i.e. composed of vertex operators. On the other hand, we don't want to change the conformal properties of the correlator, so the conformal dimension of screening operators must be zero. Thus the simplest choice is

$$Q_{\pm} = \oint dz V_{\alpha_{\pm}}(z), \quad \alpha_{\pm} = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \tag{4.1}$$

$\alpha_{\pm}$ is chosen in order for $h_{\alpha_{\pm}} = \alpha_{\pm}^2 - 2\alpha_{\pm}\alpha_0 = 1$. The following two formula of $\alpha_{\pm}$ are more useful for later usage,

$$\alpha_+ + \alpha_- = 2\alpha_0, \quad \alpha_+\alpha_- = -1. \tag{4.2}$$

The contour in the definition of $Q_{\pm}$ is not fixed but determined by other operators in the correlator and some physical requirement. We will see examples in the next section.

Now given the background charge, we can classify the operators into two sets: physical and unphysical:

- **physical** For a vertex operator V_α (and its dual $V_{2\alpha_0-\alpha}$), if we can choose an appropriate number of screening operators such that we have,

$$\langle V_\alpha V_\alpha Q_+^r Q_-^s \rangle = \langle V_\alpha V_{2\alpha_0 - \alpha} \rangle \tag{4.3}$$

  then we can use it to represent a physical observable. This requirement gives a constraint on α via the neutrality condition,

$$\begin{aligned} &2\alpha + r\alpha_+ + s\alpha_- = 2\alpha_0 = \alpha_+ + \alpha_-. \\ &\Rightarrow \alpha_{r,s} = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_- \end{aligned} \tag{4.4}$$

  $V_{2\alpha_0-\alpha}$ has index $(-r, -s)$. Both of them have conformal dimension,

$$h_{r,s} = \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 - \alpha_0^2 \tag{4.5}$$

- **unphysical** Those operators that don't satisfy this requirement Eqn.4.3 are unphysical operators. When use this formalism to solve a physical problem, we only consider the physical operator subset.

Relation to Mininal Models Although the physical requirement has already selected a small amount of operators, we still have to impose conditions on the central charge to realize a conformal model containing a finite number of primary operators.

One way is to require

$$p'\alpha_+ + p\alpha_- = 0 \tag{4.6}$$

which combined with Eqn.4.2 fixes $\alpha_{\pm}$ and α_0 to be,

$$\alpha_+ = \sqrt{p/p'}, \ \alpha_- = \sqrt{p'/p}, \ \alpha_0 = \frac{p-p'}{2\sqrt{pp'}}. \tag{4.7}$$

The central charge $c = 1 - 24\alpha_0^2$ and conformal dimension are also fixed to be,

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{4.8}$$

$$h_{r,s} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'} \tag{4.9}$$

which formally coincide with the minimal model. We can also prove that Eqn.4.6 can also restrict r, s to lie within $1 \leqslant r < p'$, $1 \leqslant s < p$ (see Sec.9.2 and Appendix.9.B in [1]).

Up to now, we have shown how Coulomb gas formalism can naturally lead to minimal model.

5 Examples of Computing 4pt Correlators

Now let's consider 4pt functions of primary fields. Unlike the 2pt function, conformal invariance at most only restricts the 4pt functions to the following form,

$$\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\phi(z_4,\overline{z}_4)\rangle = f(\eta,\overline{\eta})\prod_{i<j} z_{ij}^{\mu_{ij}}\overline{z}_{ij}^{\overline{\mu}_{ij}}, \tag{5.1}$$

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad \mu_{ij} = \frac{1}{3}(\sum_{k=1}^{4} h_k) - h_i - h_j. \tag{5.2}$$

We also need extra dynamical data to fix $f(\eta,\overline{\eta})$.

One method is to use the information about null states to write down a differential equation of 4pt function (as a canonical example, see Sec.5.2 in [2]).

Coulomb gas formalism provides another approach. Under this scheme, we will rewrite the 4pt function of physical observables as a correlation function of vertex operators with some screening operators,

$$\begin{aligned}&\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\phi(z_4,\overline{z}_4)\rangle \\ =&\langle V_{\alpha_1}(z_1,\overline{z}_1)V_{\alpha_2}(z_2,\overline{z}_2)V_{\alpha_3}(z_3,\overline{z}_3)V_{\alpha_4}(z_4,\overline{z}_4)Q_+^m Q_-^n \overline{Q}_+^{\overline{m}}\overline{Q}_-^{\overline{n}}\rangle\end{aligned} \tag{5.3}$$

so that the R.H.S gives us an integral formula. By evaluating the integral and comparing the result with the L.H.S, we can also determine $f(\eta,\overline{\eta})$. Below we will show some canonical examples.

5.1 Second Order Correlator

Now let's consider the correlator

$$\langle \phi_{(m,n)}\phi_{(2,1)}\phi_{(2,1)}\phi_{(m,n)}\rangle = f(\eta,\overline{\eta})\prod_{i<j} z_{ij}^{\mu_{ij}}\overline{z}_{ij}^{\overline{\mu}_{ij}}, \tag{5.4}$$

which is identified as

$$\langle V_{m,n}V_{2,1}V_{2,1}V_{-m,-n}Q_+\overline{Q}_+\rangle\,. \tag{5.5}$$

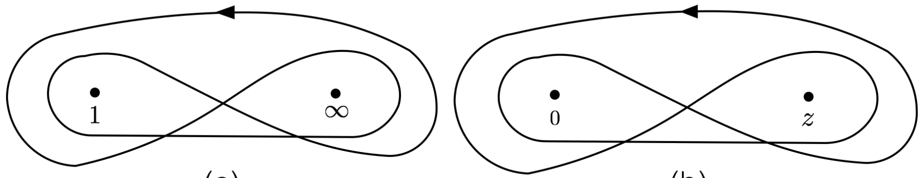

Figure 1. Two independent choices of integration contour.

For $V_{(1,2)}$, we need to insert Q_- instead.

We first look at its holomorphic part,

$$\langle V_{m,n} V_{2,1} V_{2,1} V_{-m,-n} Q_+ \rangle = \prod_{k<j}^{4} z_{kj}^{2\alpha_k \alpha_j} \oint d\omega \prod_{j=1}^{4} (\omega - z_j)^{2\alpha_+ \alpha_j}. \tag{5.6}$$

To simplify the calculation, we take a conformal transformation so that $z_1 \to 0$, $z_2 \to z$, $z_3 \to 1$, $z_4 \to \infty$, $\eta \to z$. And we have,

$$\begin{aligned} &\lim_{z_4 \to \infty} z_4^{2h_4} \langle V_{m,n}(0) V_{2,1}(z) V_{2,1}(1) V_{-m,-n}(z_4) Q_+ \rangle \\ =& z^{2\alpha_1 \alpha_2} (1-z)^{2\alpha_2 \alpha_3} \oint d\omega \, \omega^a (\omega - 1)^b (\omega - z)^c && (5.7) \\ =& z^{2\alpha_1 \alpha_2} (1-z)^{2\alpha_2 \alpha_3} I(a,b,c,z) && (5.8) \end{aligned}$$

where $a = 2\alpha_+ \alpha_1$, $b = 2\alpha_+ \alpha_3$, $c = 2\alpha_+ \alpha_2$. The integrand has four branching points $0, z, 1, \infty$. So if we choose an arbitrary contour, the result will depend on the choice of branch cut. However, the physics should be insensitive to the position of branch cut. To eliminate the unphysical sensitivity, we choose Pochhammer double contour to guarantee we goes back to the original Riemann surface. The two independent choices are shown in Fig.1. So we can write down two independent solutions for the integration:

$$\begin{aligned} I_1(a,b,c,z) =& \int_P^{(1,\infty)} d\omega \, \omega^a (\omega - 1)^b (\omega - z)^c \\ \overset{\omega = 1/t}{=}& \int_P^{(0,1)} dt \, t^{-a-b-c-2} (1-t)^b (1-zt)^c \\ =& \frac{\Gamma(-a-b-c-1)\Gamma(b+1)}{\Gamma(-a-c)} F(-c, -a-b-c-1; -a-c; z), && (5.9) \\ I_2(a,b,c,z) =& \int_P^{(0,z)} d\omega \, \omega^a (\omega - 1)^b (\omega - z)^c \\ \overset{\omega = zt}{=}& z^{1+a+c} \int_P^{(0,1)} dt \, t^a (1-t)^c (1-zt)^b \\ =& z^{1+a+c} \frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)} F(-b, a+1; a+c+2; z). && (5.10) \end{aligned}$$

where we're sloppy with the unimportant phase factors and drop the prefactors in the two final results[6]. When $a > -1, b > -1, c > -1, a+b+c < -1$, we can also replace the double contour integral with two line integrals[7] (we again neglect the phase factors and prefactors),

$$I_1(a,b,c,z) = \int_1^\infty d\omega\, \omega^a (\omega-1)^b (\omega-z)^c, \quad \text{when } b > -1, a+b+c < -1, \tag{5.11}$$

$$I_2(a,b,c,z) = \int_0^z d\omega\, \omega^a (\omega-1)^b (\omega-z)^c, \quad \text{when } a > -1, c > -1. \tag{5.12}$$

In the following part, we will keep these constraints on a, b, c and use this line integral representation.

Up to now, we only get the two independent solutions. Our goal is to solve the physical correlator Eqn.5.4 or Eqn.5.5. So we have to add the anti-holomorphic component and do a linear superposition. So the physical correlator is

$$|z|^{4\alpha_1\alpha 2}|1-z|^{4\alpha_2\alpha_3} \sum_{ij} X_{ij} I_i(z) \overline{I_j(z)} = |z|^{4\alpha_1\alpha 2}|1-z|^{4\alpha_2\alpha_3} G(z,\overline{z}) \tag{5.13}$$

The coefficients X_{ij} can be determined by monodromy invariance and conformal algebra.

Monodromy Invariance The left-hand or right-hand may have some singularities so that if we goes around the singularities, we may not go back to the original value. However, the physical correlator should be invariant under such an operation, which gives us constraints on X_{ij}.

In this problem, we have three singular points $z = 0, 1, \infty$. We will examine the monodromy around $z = 0$ and $z = 1$.[8]

Around $z = 0$, $F(\alpha, \beta, \gamma, z)$ can be Taylor expanded. So only the z^{1+a+c} factor in I_2 gives a non-trivial phase factor if we go around $z = 0$. The corresponding monodromy matrix is,

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(1+a+c)} \end{pmatrix} \tag{5.14}$$

The requires X_{ij} to be diagonal.

To solve the monodromy matrix around $z = 1$, it's easier to first express $I_j(z)$ in terms of $I_j(1-z)$: $I_i(z) = a_{ij} I_j(1-z)$. In this problem, we can simply look up the linear transformation formula of Hypergeometric function $F(\alpha, \beta, \gamma, z)$ to get the result (see Sec.4.8 of [3]). But for higher order 4pt correlation, we will encounter more complicated special function and this method is no longer applicable. A more generalizable method is

[6] The final coefficients will be determined by monodromy invariance. So it is not necessary to include all the prefactors here. The only useful information contained in the dropped prefactors is a, b, c cannot be integers otherwise the integral will vanish. See Sec.4.5 in [3] for more accurate results.

[7] In the Pochhammer double contour integral, we can deform the contour to be composed of several lines connecting the branching point and small circles around the branching points. These conditions actually guarantee the integral along small circles goes to zero if the radius of circles goes to zero. So that we can get rid of the circle integrals which leaves us only line integrals.

[8] Going around $z = \infty$ is equivalent to go around $z = 0$ and $a = 1$.

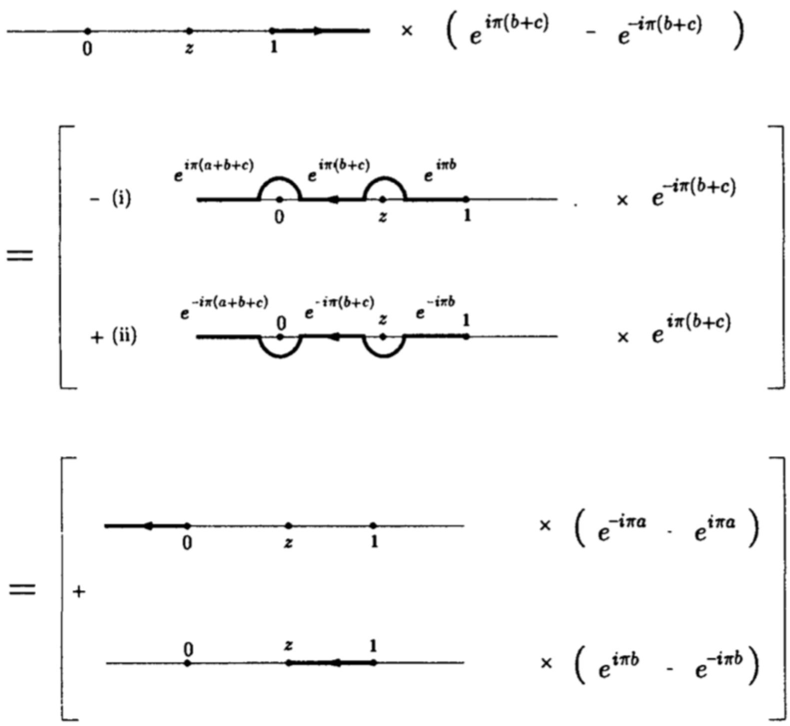

Figure 2. Deform the integral line of $I_1(z)$ to express it in terms of $I_j(1-z)$.

to use the line integral formula of I_j and deform the contour. E.g. for I_1 we can deform the contour in two different ways and attach them with two different phase factors, as shown by Fig.2. Comments on several subtleties are followed:

- Here we have to first require z to be real. However, the final result is analytic to z so we can continue z to be a complex number.
- Because of the constraints $a > -1, b > -1, c > -1, a+b+c < -1$, we don't have to worry about the integral at infinity or along the small circles around $0, z, 1$.
- No matter deforming the contour from the upper or lower half plane, we always choose $\arg\omega = \arg(\omega - z) = \arg(\omega - 1) = 0$ when doing the integral $\int_1^{+\infty} d\omega$ in I_1. For I_2, we define it to be $I_2(z) = \int_0^z d\omega\omega^a(1-\omega)^b(z-\omega)^c$ and $\arg\omega = \arg(1-\omega)\arg(z-\omega) = 0$. Only with these choices, can we get the desired phase factors and the final result.

For I_2, we can do a similar deformation. The relation is found to be,

$$I_1(a,b,c;z) = \frac{s(a)}{s(b+c)} I_1(b,a,c;1-z) - \frac{s(c)}{s(b+c)} I_2(b,a,c;1-z) \tag{5.15}$$

$$I_2(a,b,c;z) = -\frac{s(a+b+c)}{s(b+c)} I_1(b,a,c;1-z) - \frac{s(b)}{s(b+c)} I_2(b,a,c;1-z) \tag{5.16}$$

where $s(a) = \sin\pi a$. Although we use the line integral form which requires a, b, c to meet some constraints, the final result, as is some analytical function of a, b, c (see Sec.9.3 in

[4]), can be analytically continued to a larger region. The monodromy matrix for $I_j(1-z)$ around $z=1$ is also diagonal. So the invariance requires $\sum_i X_i a_{ij} a_{ik}$ is diagonal with respect to j,k, which fixes X_1/X_2. Up to an overall normalization factor A, we can write the final result as,

$$G(z,\bar{z}) = A\left[\frac{s(b)s(a+b+c)}{s(a+c)}|I_1(z)|^2 + \frac{s(a)s(c)}{s(a+c)}|I_2(z)|^2\right] \tag{5.17}$$

By comparing the general form Eqn.5.4 and the result we get for Eqn.5.5 to fix $f(\eta,\bar{\eta})$ as,

$$f(z,\bar{z}) = |z|^{4\alpha_1\alpha_2-2\mu_{12}}|1-z|^{4\alpha_2\alpha_3-2\mu_{23}}G(z,\bar{z}). \tag{5.18}$$

Plug this into Eqn.5.4 and we get the 4pt function in a general coordinate system,

$$\text{4pt} = \left[\prod_{i<j} z_{ij}^{\mu_{ij}}\eta^{2\alpha_1\alpha_2-\mu_{12}}(1-\eta)^{2\alpha_2\alpha_3-\mu_{23}} \times c.c.\right] G(\eta,\bar{\eta}) \tag{5.19}$$

Conformal Algebra Now we use conformal algebra to fix the overall factor A. We assume the following chiral OPE,

$$\phi_{(r_1,s_1)}(z_1)\phi_{(r_2,s_2)}(z_2) = \sum_{r,s}\frac{C^{(r,s)}_{(r_1,s_1),(r_2,s_2)}}{z_{12}^{h_{r_1,s_1}+h_{r_2,s_2}-h_{r,s}}}\phi_{(r,s)}(z_2) \tag{5.20}$$

for $\phi_1\phi_2$ and $\phi_3\phi_4$. Then using the OPE above, the 4pt function in the $z_1\to z_2$ and $z_3\to z_4$ limit can be written as,

$$\text{4pt} = \sum_{r,s}\frac{1}{z_{24}^{h_{r,s}}}\frac{C^{(r,s)}_{(r_1,s_1),(r_2,s_2)}}{z_{12}^{h_{r_1,s_1}+h_{r_2,s_2}-h_{r,s}}}\frac{C^{(r,s)}_{(r_3,s_3),(r_4,s_4)}}{z_{34}^{h_{r_3,s_3}+h_{r_4,s_4}-h_{r,s}}} \times c.c \tag{5.21}$$

Then we compute the 4pt function in the same limit again but using the result Eqn.5.19 we got from Coulomb gas formalism. In this limit $\eta\to z_{12}z_{34}/z_{24}^2\to 0$. So we have,

$$\begin{aligned}\text{4pt} &= \left[z_{12}^{2\alpha_1\alpha_2}z_{34}^{2\alpha_1\alpha_2}z_{24}^{6\mu_{12}-4\alpha_1\alpha_2}\times c.c\right] G(\eta,\bar{\eta})\big|_{\eta\to 0}\\ &= \left[z_{12}^{2\alpha_1\alpha_2}z_{34}^{2\alpha_1\alpha_2}z_{24}^{6\mu_{12}-4\alpha_1\alpha_2}\times c.c\right] A\left[\frac{s(b)s(a+b+c)}{s(b+c)}N_1^2 + \frac{s(a)s(c)}{s(a+c)}N_2^2\left|\frac{z_{12}z_{34}}{z_{24}^2}\right|^{2(1+a+c)}\right]\end{aligned}$$

where we have used the fact $\mu_{12}=\mu_{34}=\mu_{13}=\mu_{24}$. For the special value of α_2 in our problem, we have the following formula,

$$\begin{aligned}2\alpha_1\alpha_2 = 2\alpha_{r_1,s_1}\alpha_{2,1} =& h_{r_1+1,s_1} - h_{r_1,s_1} - h_{2,1}\\ 6\mu_{12} - 4\alpha_1\alpha_2 =& -2h_{r_1+1,s_1}\\ 1+a+c =& h_{r_1-1,s_1} - h_{r_1+1,s_1}\end{aligned}$$

which leads to,

$$
\begin{aligned}
4\text{pt} =& A\frac{s(b)s(a+b+c)}{s(b+c)}N_1^2\frac{1}{z_{24}^{2h_{r_1+1,s_1}}z_{12}^{h_{r_1,s_1}+h_{2,1}-h_{r_1+1,s_1}}z_{34}^{h_{r_1,s_1}+h_{2,1}-h_{r_1+1,s_1}}}\times c.c.\\
&+A\frac{s(a)s(c)}{s(a+c)}N_2^2\frac{1}{z_{24}^{2h_{r_1-1,s_1}}z_{12}^{h_{r_1,s_1}+h_{2,1}-h_{r_1-1,s_1}}z_{34}^{h_{r_1,s_1}+h_{2,1}-h_{r_1-1,s_1}}}\times c.c.
\end{aligned}
\tag{5.22}
$$

Now we compare this result with the one from OPE and we can get,

$$
A\frac{s(b)s(a+b+c)}{s(b+c)}N_1^2 = C^{(r_1+1,s_1)}_{(r_1,s_1),(2,1)}C^{(r_1+1,s_1)}_{(r_1,s_1),(2,1)} \tag{5.23}
$$

$$
A\frac{s(a)s(c)}{s(a+c)}N_2^2 = C^{(r_1-1,s_1)}_{(r_1,s_1),(2,1)}C^{(r_1-1,s_1)}_{(r_1,s_1),(2,1)} \tag{5.24}
$$

Are these two condition compatible?

5.2 An example: Ising Model

We use Ising model as a concrete example of the formal framework above. Ising model has $c = 1/2$ and three primary fields:

$$
\mathbb{1} \sim \phi_{(1,1)}, \quad \epsilon \sim \phi_{(2,1)}, \quad \sigma \sim \phi_{(1,2)}. \tag{5.25}
$$

So if use Coulomb gas to describe this system, we need to choose $\alpha_0 = 1/4\sqrt{3}$ and thus $\alpha_+ = 2/\sqrt{3}, \alpha_- = -\sqrt{3}/2$. And the physical field $\phi_{(m,n)}$ corresponds to the vertex operator $V_{(m,n)}$ or $V_{(-m,-n)}$.

Here we compute the 4pt function of the σ field,

$$
\langle \phi_{(1,2)}\phi_{(1,2)}\phi_{(1,2)}\phi_{(1,2)}\rangle = \langle V_{(1,2)}V_{(1,2)}V_{(1,2)}V_{(-1,-2)}Q_-\overline{Q}_-\rangle \,. \tag{5.26}
$$

So $\alpha_1 = \alpha_2 = \alpha_3 = -\alpha_-/2 = \sqrt{3}/4$ and $a = b = c = -3/4$. The two independent solutions are,

$$
\begin{aligned}
I_1(-\frac{3}{4},-\frac{3}{4},-\frac{3}{4},z) &= \frac{\Gamma(5/4)\Gamma(1/4)}{\Gamma(3/2)}F(\frac{3}{4},\frac{5}{4},\frac{3}{2},z)\\
&= \frac{\Gamma(5/4)\Gamma(1/4)}{\Gamma(3/2)}(1-z)^{-1/2}F(\frac{3}{4},\frac{1}{4},\frac{3}{2},z)\\
&= \frac{\Gamma(1/4)^2}{\sqrt{2\pi}}\left(\frac{1-\sqrt{1-z}}{z(1-z)}\right)^{1/2}
\end{aligned}
\tag{5.27}
$$

$$
\begin{aligned}
I_2(-\frac{3}{4},-\frac{3}{4},-\frac{3}{4},z) &= z^{-1/2}\frac{\Gamma(1/4)^2}{\Gamma(1/2)}F(\frac{3}{4},\frac{1}{4},\frac{1}{2},z)\\
&= z^{-1/2}\frac{\Gamma(1/4)^2}{\Gamma(1/2)}(1-z)^{-1/2}F(-\frac{1}{4},\frac{1}{4},\frac{1}{2},z)\\
&= \frac{\Gamma(1/4)^2}{\sqrt{2\pi}}\left(\frac{1+\sqrt{1-z}}{z(1-z)}\right)^{1/2}
\end{aligned}
\tag{5.28}
$$

where we have used the formula $F(a,b,c;z) = (1-z)^{c-a-b}F(c-a,c-b,c;z)$, $F(a-1/2,a,2a;z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a}$.

We now combine the left and right hand to write down,

$$G(z,\overline{z}) = A\frac{\Gamma(1/4)^4}{4\pi}\left|\frac{1}{z(1-z)}\right|\left(|1-\sqrt{1-z}|+|1+\sqrt{1-z}|\right). \tag{5.29}$$

The additional 1/2 comes from $s(b)s(a+b+c)$. So the $f(z,\overline{z})$ is (omit the prefactor first),

$$\begin{aligned}
f(z,\overline{z}) &= |z|^{4\alpha_1\alpha_2-2\mu_{12}}|1-z|^{4\alpha_2\alpha_3-2\mu_{23}}G(z,\overline{z}), \quad \mu_{12}=\mu_{23}=-\frac{1}{24} \\
&= \left[(z(1-z))^{5/12}\left(\frac{1}{z(1-z)}\right)^{1/2}\times c.c\right]\left(|1-\sqrt{1-z}|+|1+\sqrt{1-z}|\right) \\
&= \left[(z(1-z))^{-1/12}\times c.c\right]\left(|1-\sqrt{1-z}|+|1+\sqrt{1-z}|\right).
\end{aligned} \tag{5.30}$$

So the four point function determined up a prefactor is,

$$A\frac{\Gamma(1/4)^4}{4\pi}\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right|^{1/4}\left(|1-\sqrt{1-z}|+|1+\sqrt{1-z}|\right). \tag{5.31}$$

Now let's use the operator algebra to fix the prefactor A. The OPE of σ reads,

$$\sigma(z_1,\overline{z}_1)\sigma(z_2,\overline{z}_2) \sim \frac{1}{z_{12}^{1/8}\overline{z}_{12}^{1/8}} + ... \tag{5.32}$$

In the $1\to 2$, $3\to 4$, $z\to 0$ limit, the four point function should recover this OPE result,

$$4pt \to A\frac{\Gamma(1/4)^4}{2\pi}\left|\frac{1}{z_{12}z_{34}}\right|^{1/4} + ... \Rightarrow A = \frac{2\pi}{\Gamma(1/4)^4} \tag{5.33}$$

And the final result is,

$$\langle\sigma\sigma\sigma\sigma\rangle = \frac{1}{2}\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right|^{1/4}\left(|1-\sqrt{1-z}|+|1+\sqrt{1-z}|\right). \tag{5.34}$$

A Correlation Functions of Vertex Operators

Here we will use two different methods to compute the correlator, which are based on the Ex9.1 and Ex9.2 in [1]. Both methods will also lead to the neutrality condition but in different ways. Finally, we will give a more elegant argument on the neutrality condition from the symmetry viewpoint.

A.1 Functional Method

We can use functional integral to rewrite the correlator as,

$$\begin{aligned}
\langle V_{\alpha_1}(z_1,\overline{z}_1)V_{\alpha_2}(z_2,\overline{z}_2)...V_{\alpha_n}(z_n,\overline{z}_n)\rangle &= \frac{1}{Z}\int D\varphi \exp\left(-S+\int d^2x j(x)\varphi(x)\right) \\
&= \exp\left(\frac{1}{2}\int d^2x d^2y\, j(x)K(x-y)j(y)\right)
\end{aligned} \tag{A.1}$$

where $j(x) = i\sqrt{2}\sum_{k=1}^{n}\alpha_k\delta^{(2)}(x-x_k)$. To carry out the integral, we have to use the regularized $K(x-y)$.

$$
\begin{aligned}
\langle ... \rangle &= \exp\big\{ -2\sum_{k<j}\alpha_k\alpha_k K(x_k-k_j) - \sum_k \alpha_k^2 K(0)\big\} \\
&= \prod_{k<j}[m^2(x_k-x_j)^2+a^2]^{2\alpha_k\alpha_j}(ma)^{2\sum_k\alpha_k^2} \\
&= (ma)^{2(\alpha_1+...+\alpha_n)^2}\prod_{k<j}\left[1+\left(\frac{x_k-x_j}{a}\right)^2\right]^{2\alpha_k\alpha_j}
\end{aligned}
$$

If we take the conformal limit $m, a \to 0$, we will go back to the result we show in the text and the neutrality condition is necessary to have a nonzero result.

A.2 Mode Expansion Method

From the mode expansion, we see that $\varphi(z,\overline{z})$ has three mutually commuting parts: zero mode $\tilde{\varphi}(z,\overline{z}) = \varphi_0 - ia_0\log(z\overline{z})$, holomorphic part $\phi(z) = i\sum_{n\neq 0}\frac{1}{n}a_n z^{-n}$ and anti-holomorphic part $\overline{\phi}(\overline{z})$. So the vertex operator can also be divided into these three parts respectively,

$$
\begin{aligned}
V_\alpha(z,\overline{z}) &=: \exp\left(i\sqrt{2}\alpha\tilde{\varphi}(z,\overline{z})\right) :: \exp\left(i\sqrt{2}\alpha\phi(z)\right) :: \exp\left(i\sqrt{2}\alpha\overline{\phi}(\overline{z})\right) : \\
&=: \exp\left(i\sqrt{2}\alpha\tilde{\varphi}(z,\overline{z})\right) : V'_\alpha(z)\overline{V}'_\alpha(\overline{z}).
\end{aligned} \tag{A.2}
$$

where the normal order of zero mode means moving a_0 to the right and φ_0 to the left. Because these three parts are mutually commuting, we can compute the correlation for them separately.

We first look at the holomorphic part. And we need the following correlator $\langle\phi(z)\phi(\omega)\rangle$,

$$
\begin{aligned}
\langle\phi(z)\phi(\omega)\rangle &= \left\langle i\sum_{n\neq 0}\frac{1}{n}a_n z^{-n} i\sum_{m\neq 0}\frac{1}{m}a_M\omega^{-m}\right\rangle \\
&= \sum_{n>0}\frac{1}{n^2}\langle a_n a_{-n}\rangle\left(\frac{\omega}{z}\right)^n \\
&= -\log\left(1-\frac{\omega}{z}\right)
\end{aligned}
$$

So the correlation of holomorphic part is (for this formula see Appendix 6.A in [1]),

$$
\begin{aligned}
\langle V'_{\alpha_1}(z_1)V'_{\alpha_2}(z_2)...V'_{\alpha_n}(z_n)\rangle &= \exp\left(-2\sum_{k<j}\alpha_k\alpha_j\langle\phi(z_k)\phi(z_j)\rangle\right) \\
&= \prod_{k<j}(z_k-z_j)^{2\alpha_k\alpha_j}z_k^{-2\alpha_k\alpha_j}.
\end{aligned} \tag{A.3}
$$

By changing $z's$ to $\overline{z}$, we get the correlation for anti-holomorphic part.

Now consider the zero mode. For an arbitrary vacuum state $|\beta\rangle$, $a_0|\beta\rangle = \sqrt{2}\beta|\beta\rangle$. With the commutator

$$[a_0, e^{i\sqrt{2}\alpha\varphi_0}] = \sqrt{2}\alpha e^{i\sqrt{2}\alpha\varphi_0},$$

we have $a_0 e^{i\sqrt{2}\alpha\varphi_0}|\beta\rangle = (\alpha+\beta)e^{i\sqrt{2}\alpha\varphi_0}|\beta\rangle$. Because $e^{i\sqrt{2}\alpha\varphi_0}$ is unitary, $e^{i\sqrt{2}\alpha\varphi_0}|\beta\rangle = |\alpha+\beta\rangle$. We are now ready for the computation of correlator. Taking any arbitrary vacuum $|h\rangle$, we have,

$$\begin{aligned}
&\langle h| : e^{i\sqrt{2}\alpha_1\tilde{\varphi}_1} : ... : e^{i\sqrt{2}\alpha_n\tilde{\varphi}_n} : |h\rangle \\
=&\langle h| : e^{i\sqrt{2}\alpha_1\tilde{\varphi}_1} : ... |\alpha_n + h\rangle\, e^{\sqrt{2}\alpha_n h \log(z_n\bar{z}_n)} \\
=&\langle h| : e^{i\sqrt{2}\alpha_1\tilde{\varphi}_1} : ... |\alpha_{n+1} + \alpha_n + h\rangle\, e^{\sqrt{2}\alpha_{n-1}(\alpha_n+h)\log(z_{n-1}\bar{z}_{n-1})} e^{\sqrt{2}\alpha_n h \log(z_n\bar{z}_n)} \\
=&\delta_{\sum_i \alpha_i, 0} \exp\big[2\sum_{i<j}\alpha_i\alpha_j \log(z_i\bar{z}_i)\big] \exp\big[2h\sum_i \alpha_i \log(z_i\bar{z}_i)\big] \\
=&\delta_{\sum_i \alpha_i, 0} \exp\big[2\sum_{i<j}\alpha_i\alpha_j \log(z\bar{z})\big]
\end{aligned}$$

Combining these together, we get the final answer. From the derivation, we can clearly see that the non-zero modes give the power-law behavior and zero mode gives the neutrality condition. Because the zero mode kind of describes the field moves as a whole, so this argument is related to the symmetry argument below.

A.3 Symmetry Arguments on the Neutrality Condition

The action has an internal symmetry $\varphi \to \phi + a$ for constant a. This means that for any string operator $X(\varphi)$ we should have $\langle X(\varphi)\rangle = \langle X(\varphi + a)\rangle$. However, vertex operator will get an extra phase $e^{i\sqrt{2}\alpha a}$ under this transformation. So only when $\sum_k \alpha_k = 0$ does the accumulated phase factor vanish and can we have nonzero correlation function.

This argument can also be formulated in terms of Ward Identity $\langle X\delta S\rangle = \langle \delta X\rangle$. $\delta S = \int d^2x\, a(x)\partial_\mu j^\mu$. For vertex operators,

$$\delta X = \delta V_{\alpha_1} V_{\alpha_2} .. + V_{\alpha_1}\delta V_{\alpha_2} .. + .. = i\sqrt{2}\alpha_1 a(x_1) V_{\alpha_1} V_{\alpha_2} .. + ...$$

So we have,

$$-\frac{1}{4\pi}\partial_\mu \langle \partial^\mu \varphi X\rangle = i\sqrt{2}\sum_{k=1}\alpha_k \delta^{(2)}(x - x_k)\langle X\rangle \tag{A.4}$$

Integrate both side over the region containing all x_k's,

$$i\sqrt{2}\sum_{k=1}\alpha_k\langle X\rangle = -\frac{1}{4\pi}\oint d\xi_\mu \langle \partial^\mu\varphi X\rangle, d\xi_\mu = \epsilon_{\mu\rho}ds^\rho = (dy, -dx)$$

Rewrite it in the conformal coordinate[9],

$$i\sqrt{2}\sum_{k=1}\alpha_k\langle X\rangle = \frac{i}{4\pi}\oint dz\,\langle\partial\varphi X\rangle - \frac{i}{4\pi}\oint d\bar{z}\,\langle\bar{\partial}\varphi X\rangle \tag{A.5}$$

[9] $d\xi_\mu\partial^\mu\varphi = \epsilon_{\mu\rho}ds^\rho\partial^\mu\varphi = -\frac{i}{2}dz\partial^{\bar{z}}\varphi + \frac{i}{2}d\bar{z}\partial^z\varphi = -idz\partial_z\varphi + id\bar{z}\bar{\partial}\varphi$

Because the contour circulates around all the possible singular points, so the integral vanishes finally which yields the neutrality condition.

We can use the associated charge Q defined by the conserved current to give another argument. The commutator of Q and the string of operators is

$$[Q, V_{\alpha_1} V_{\alpha_2} ... V_{\alpha_n}] = \left(\sum_i \alpha_i \right) V_{\alpha_1} V_{\alpha_2} ... V_{\alpha_n}$$

And we evaluate both sides on the vacuum. Because the charge of two vacuums are the same, we have,

$$\left(\sum_i \alpha_i \right) \langle V_{\alpha_1}(z_1, \overline{z}_1) V_{\alpha_2}(z_2, \overline{z}_2) ... V_{\alpha_n}(z_n, \overline{z}_n) \rangle = 0.$$

So only when $\sum_i \alpha_i = 0$ can we have nonzero correlation.

B Energy-Momentum Tensor of Eqn.3.3

We use the following general definition of the energy-momentum tensor,

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \Big|_{g \to 1}. \tag{B.1}$$

For our problem, we only have to compute the variation of $\delta S^{(1)}$, which can be divided into three parts:

$$\begin{aligned} \delta S^{(1)} &= \int d^2x (\delta \sqrt{g}) \varphi R + \int d^2x \sqrt{g} \varphi (\delta g^{\mu\nu}) R_{\mu\nu} + \int d^2x \sqrt{g} \varphi g^{\mu\nu} \delta R_{\mu\nu} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Because both I_1 and I_2 are proportional to the curvature which will be taken to be zero finally, these two terms don't make contribution. To compute I_3 we can first go to a locally flat coordinate where $\Gamma^\alpha_{\beta\gamma} = 0$ so that $R^\rho_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma}$. And $\delta R_{\mu\nu}$ is simply,

$$\delta R_{\mu\nu} = \partial_\sigma \delta \Gamma^\sigma_{\mu\nu} - \partial_\nu \delta \Gamma^\sigma_{\mu\sigma} = D_\sigma \delta \Gamma^\sigma_{\mu\nu} - D_\nu \delta \Gamma^\sigma_{\mu\sigma} \tag{B.2}$$

In the second equality, we change ordinary derivative ∂ to covariant derivative D. In the locally flat coordinate, there is no difference between ∂ and D thus this replacement is actually doing nothing. However, $\delta\Gamma$ is the difference between two connections thus is itself a vector (up to first order). So this equation continues to be correct in a general coordinate and is known as Palatini identity. Now we can write I_3 as,

$$\begin{aligned} I_3 &= \int d^2x \sqrt{g} \varphi g^{\mu\nu} \left(D_\sigma \delta \Gamma^\sigma_{\mu\nu} - D_\nu \delta \Gamma^\sigma_{\mu\sigma} \right) \\ &= \int d^2x \sqrt{g} \varphi D_\sigma \left(g^{\mu\nu} \delta \Gamma^\sigma_{\mu\nu} - g^{\mu\sigma} \delta \Gamma^\lambda_{\lambda\mu} \right) \end{aligned}$$

We plug in the following explicit expression of $\delta\Gamma$,

$$\delta\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(D_\mu \delta g_{\lambda\nu} + D_\mu \delta g_{\lambda\nu} - D_\nu \delta g_{\lambda\mu} \right), \tag{B.3}$$

and do integral by parts twice, we have[10]

$$I_3 = \int d^2x \sqrt{g}\, (D_\mu D_\sigma \varphi)\, (g^{\mu\nu} g^{\sigma\lambda} - g^{\mu\sigma} g^{\lambda\nu}) \delta g_{\lambda\nu}$$

Adding the prefactor, we finally get the EM tensor,

$$T^{\lambda\nu}_{(1)} = \frac{\gamma}{2\pi} (\partial_\mu \partial_\sigma \varphi) \left(\eta^{\mu\nu} \eta^{\sigma\lambda} - \eta^{\mu\sigma} g^{\lambda\nu} \right) \tag{B.4}$$

This is different from the result in the yellow book [1] but doesn't change the holomorhphic Em tensor. And the holomorphic EM tensor is,

$$T(z) = -2\pi T_{zz} = \gamma \partial_z \partial_z \varphi \tag{B.5}$$

C Integral Representation of Hypergeometric Function

The Hypergeometric equation

$$z(1-z)\omega'' + [\gamma - (\alpha + \beta + 1) z]\omega' - \alpha\beta\omega = 0 \tag{C.1}$$

is known to give a solution Hypergeometric function

$$F(\alpha, \beta, \gamma; z) = \sum_{k=0} \frac{(a)_n (b)_n}{(c)_n} z^n. \tag{C.2}$$

This result can be easily proved using series expansion method.

This equation can also be solved by Euler transformation

$$\omega(z) = \int_{\mathcal{C}} dt\, (z-t)^\mu \nu(t), \tag{C.3}$$

which yields a solution,

$$\omega(z) = A \int_{\mathcal{C}} dt\, t^{\alpha-\gamma} (1-t)^{\gamma-\beta-1} (z-t)^{-\alpha}. \tag{C.4}$$

However, we have to choose a proper contour to get rid of the boundary term,

$$Q = -A\alpha t^{\alpha-\gamma-1} (1-t)^{\gamma-\beta} (z-t)^{-\alpha-1}. \tag{C.5}$$

One of the choices is $\mathcal{C} = [1, +\infty]$ when $\mathrm{Re}\gamma > \mathrm{Re}\beta > 0$. So we have,

$$\omega(z) = A \int_1^{+\infty} dt\, t^{\alpha-\gamma} (1-t)^{\gamma-\beta-1} (z-t)^{-\alpha} \tag{C.6}$$

$$\overset{\omega=1/t}{=} A' \int_0^1 dt\, t^{\alpha-\gamma} (1-t)^{\gamma-\beta-1} (1-zt)^{-\alpha}, \tag{C.7}$$

when $\mathrm{Re}\gamma > \mathrm{Re}\beta > 0$.

If $\mathrm{Re}\gamma > \mathrm{Re}\beta > 0$ is not satisfied, we can see that the integral defined in this way will diverge by expanding $(1-zt)^{-\alpha}$. So we have to use Pochhammer contour.

[10] During the derivation, we have to use the compatible condition $D_\rho g^{\mu\nu} = 0$.

References

[1] P. D. Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*.

[2] P. Ginsparg, *Applied Conformal Field Theory*, *arXiv*:hep-th/9108028v1.

[3] Z. X. Wang, D. R. Guo, X. J. Xia, *Special Functions*.

[4] N. N. Lebedev, *Special Functions and Their Applications*