SL(2, $\mathbb{R}$) and the Harmonic Oscillator

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Abstract

In this note, we use $\mathfrak{sl}(2, \mathbb{R})$ algebra to solve the harmonic oscillator. The total Hilbert space of the harmonic oscillator can be decomposed into a direct sum of two representations of the $\mathfrak{sl}(2, \mathbb{R})$. This way of solving it gives us a lot insight of studying Floquet driving oscillator, i.e. the Mathieu oscillator.

Contents

1 Introduction

Classic Solution The harmonic oscillator is the simplest quantum mechanical model that can be solved exactly. The Hamiltonian writes,

$$H = \frac{p^2}{2} + \frac{x^2}{2}. \tag{1}$$

One can solve it by introducing the ladder operators,

$$a = \frac{p - ix}{\sqrt{2}}, \quad a^\dagger = \frac{p + ix}{\sqrt{2}}, \tag{2}$$

which has the commutation relation,

$$[a, a^\dagger] = i[p, x] = 1. \tag{3}$$

In terms of $a, a^\dagger$, the Hamiltonian can be written as,

$$H = a^\dagger a + 1/2, \tag{4}$$

which has a simple commutator with $a, a^\dagger$,

$$[H, a] = -a, \quad [H, a^\dagger] = a^\dagger. \tag{5}$$

Therefore, for a spectrum bounded from below, we know that the ground state must be annihilated by a and the whole spectrum is generated by keeping applying $a^\dagger$,

$$a\left|0\right\rangle = 0,\ \left|n\right\rangle = \frac{1}{\sqrt{n+1}} a^\dagger \left|0\right\rangle,\ E_n = n + \frac{1}{2}. \tag{6}$$

Except for using this method, there is actually a hidden $\mathfrak{sl}(2, \mathbb{R})$ algebra behind this problem, that can also be used to solve the spectrum. Although this method seems complicated and unnecessary, it turns out to be useful for other purposes as we will see in a moment.

2 The $\mathrm{SL}(2)$ structure

2.1 The $\mathfrak{sl}(2, \mathbb{R})$ Algebra

The reason that we have a $\mathrm{SL}(2, \mathbb{R})$ structure in this model is because the Hamiltonian is quadratic,[1]. Due to this property, its time dependent solution can be written as,

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}, \tag{7}$$

where c_{ij} are time dependent functions. Hermiticity requires c_{ij} to be all real and the EoM $\dot{x} = p$ constrains $c_{21} = \dot{c_{11}}$, $c_{22} = \dot{c_{12}}$. The equal time commutator doesn't depend on time. Thus we have,

$$[x(t), p(t)] = i \det(c_{ij}) = i \Rightarrow \det c_{ij} = 1, \tag{8}$$

which says the time evolution is an $\mathrm{SL}(2, \mathbb{R})$ transformation. [1] We can choose the three generators as σ^y, $i\sigma_z$ and $i\sigma_x$, which corresponds to the following three operators,

$$J_0 = \frac{p^2 + x^2}{4},\ J_1 = \frac{p^2 - x^2}{4},\ J_2 = \frac{xp + px}{4}. \tag{9}$$

[1]The discussion here doesn't require the Hamiltonian to be static.

J_0 is the Hamiltonian for a static harmonic oscillator that gives us a confined motion. J_1 contains a potential that decreases with x which will pull the particle towards infinity. J_2 doesn't have clear meaning but also generates an unbounded motion in the phase space. One can verify that they satisfy the $\mathfrak{sl}(2)$ algebra,

$$[J_1, J_2] = -iJ_0,\ [J_2, J_0] = iJ_1,\ [J_0, J_1] = iJ_2. \tag{10}$$

It is useful to contrast it with the familiar $\mathfrak{su}(2)$ algebra,

$$[s_x, s_y] = is_z,\ [s_y, s_z] = is_x,\ [s_z, s_x] = is_y, \tag{11}$$

which has a "+" sign in the first equation. Therefore, $\mathfrak{sl}(2,\mathbb{R})$ is like $\mathfrak{su}(2)$ but with Wick rotated $s_{x,y}$, i.e. $J_{1,2}$ generates hyperbolic/non-compact rotations, J_0 generates the Euclidean/compact rotation. That's why $\mathrm{SL}(2,\mathbb{R})$ is a non-compact group. From this comparison with the $\mathfrak{su}(2)$, we can easily write down the Casimir operator,

$$J^2 = J_0^2 - J_1^2 - J_2^2 = J_0^2 - \frac{J_+J_- + J_-J_+}{2} = J_0^2 - J_0 - J_+J_- = J_0^2 + J_0 - J_-J_+. \tag{12}$$

The $J_\pm$ are defined below in the discussion on representations. If we write J^2 in term of p and x, one can show that $J^2 = -3/16$.

2.2 Representation and the Spectrum

Let's construct the representation that diagonalize J_0 (the Hamiltonian). A generic $\mathrm{SL}(2,\mathbb{R})$ algebra can have many representations. Here, since we know the spectrum for a physical system is bounded from below, it must correspond to some highest weight representation.

Therefore let's recombine our generators in the following way,

$$J_+ = J_1 + iJ_2 = \frac{p^2 - x^2 + i(xp + px)}{4}, \tag{13}$$
$$J_- = J_1 - iJ_2 = \frac{p^2 - x^2 - i(xp + px)}{4}, \tag{14}$$

so that the commutation relation becomes,[2]

$$[J_0, J_+] = J_+,\ [J_0, J_-] = -J_-,\ [J_+, J_-] = -2J_0. \tag{16}$$

Just as the $\mathfrak{su}(2)$ case, $J_\pm$ act like a raising/lowering operator. Thus we construct the highest weight representation that is diagonal in J_0 and the space is spanned by,

$$|m, j\rangle\,, \quad \text{with } J_- |m_0\rangle = 0,\ m - m_0 = 0, 1, 2, 3.... \tag{17}$$

[2] In the context of CFT, people usually call this basis of generators with different names,

$$J_0 \to D\,(\text{dilation}),\ J_+ \to P\,(\text{translation}),\ J_- \to K\,(\text{SCT}). \tag{15}$$

For the state $|m_0\rangle$, the Casimir gives us,

$$J^2 |m_0\rangle = m_0(m_0 - 1) |m_0\rangle = -\frac{3}{16} |m\rangle , \tag{18}$$

which requires $m_0 = 1/4$ or $m_0 = 3/4$. On the other hand, we need to make sure that the states built from $|m_0\rangle$ should all have a non-negative norm,

$$\langle m, j|J_- J_+|m, j\rangle = m(m + 1) - j(j + 1) = m(m + 1) + 3/16 \geq 0, \tag{19}$$
$$\langle m, j|J_+ J_-|m, j\rangle = m(m - 1) - j(j + 1) = m(m - 1) + 3/16 \geq 0, \tag{20}$$

which is automatically satisfied by the value of our m_0. Therefore, both are allowed.

We can introduce the $\mathfrak{sl}(2)$ spin $J^2 = j(j-1)$, then $J^2 = -3/16$ requires $j = 1/4$ or $j = 3/4$. Now instead of using m_0, the two distinct unitary representations can be labeled by $j = 1/4$ and $j = 3/4$, their direct sum forms the full Hilbert space.

To make a connection to the picture using $a, a^\dagger$, we can just rewrite the $\mathfrak{sl}(2, \mathbb{R})$ generators in terms of $a, a^\dagger$,

$$J_0 = \frac{1}{2} a^\dagger a + \frac{1}{4},\ J_+ = \frac{1}{2} \left(a^\dagger\right)^2 ,\ J_- = \frac{1}{2} a^2. \tag{21}$$

Therefore, J_0 is still measuring the number of bosons. The subspace with even number of bosons corresponds to the representation $j = 1/4$ and the subspace with odd number of bosons corresponds to the representation $j = 3/4$.

3 Floquet oscillator - general discussion

It seems totally unnecessary to develop this SL(2) algebra if we just want to solve the static harmonic oscillator. However, it turns out to be very useful and provide us many insights when we study a Floquet driven oscillator.

By a Floquet driven oscillator, we mean an harmonic oscillator that is subjected to a periodic harmonic driving force,

$$H(t) = H_0 + V(t) = \frac{p^2}{2} + \frac{x^2}{2} + f(t)\frac{x^2}{2}, \tag{22}$$

where $f(t) = f(t + T)$ is a periodic function. One of the most famous example is the Mathieu oscillator, where $f(t) = h \cos(\Omega t)$.

3.1 Algebraic approach

If we just write our Hamiltonian in this way, it is hard to predict what are the possible phases for the dynamics. Now, let's try to rewrite it in terms of the SL(2, $\mathbb{R}$) generators and we have,

$$H(t) = 2J_0 + f(t)(J_0 - J_1). \tag{23}$$

The Floquet operator and the effective Hamiltonian is,

$$F = \mathcal{T} \exp\left(-i \int_0^T dt H(t)\right) = e^{-iH_{\text{eff}}}. \tag{24}$$

Since $H(t)$ is a linear combination of $\mathrm{SL}(2,\mathbb{R})$ generators, the whole Floquet operator can be thought of a composition of many infinitesimal $\mathrm{SL}(2,\mathbb{R})$ transformation. Therefore H_{eff} can also be written as a linear combination of $\mathrm{SL}(2,\mathbb{R})$ generators,

$$H_{\text{eff}} = \text{const} + \phi\, (\widehat{n}_f \cdot \widehat{J}), \tag{25}$$

Here we choose $\widehat{n}_f$ to be a unit vector with respect to the metric $(1,-1,-1)$ representing the rotational axis and $\phi \in \mathbb{R}$ is the rotation angle. We also need to add a const to make sure F can be written as an exponential form. Since a constant doesn't affect dynamics, we can simply drop it in our following discussion.

Now we can use the effective Hamiltonian to discussion the stroboscopic dynamics. Although it is usually hard to solve the exact form of H_{eff}, we can already say a lot simply by using its formal expression. Depending on the sign of $|\widehat{n}_f|^2$, the effective Hamiltonian falls into three different classes:

- <u>Elliptic</u> $|\widehat{n}_f|^2 = 1$. In this case, when F acts on (x,p), the c_{ij} matrix (see Eq. (7) for definition) is an elliptic matrix, which means that the diagonal elements of its Jordan normal form are some pure phases. Therefore $x(t)$ and $p(t)$ only wind around in the phase space. This corresponds to a non-heating dynamics.

- <u>Hyperbolic</u> $|\widehat{n}_f|^2 = -1$. In this case, F generates a hyperbolic c_{ij} matrix, which means one of the diagonal elements of its Jordan normal form is larger than 1 and the other is smaller than 1. The vector associated to the larger eigenvalue is an unstable direction for the dynamics. $x(t)$ and $p(t)$ will quickly flow toward infinity along that direction. This corresponds to a heating dynamics and it happens exponentially fast.

- <u>Parabolic</u> $|\widehat{n}_f|^2 = 0$. In this case F generates a parabolic c_{ij} matrix, whose Jordan normal form is simply,

$$C = P \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} P^{-1}, \quad b \in \mathbb{R}. \tag{26}$$

  The off-diagonal element will increase linearly with the driving cycles, i.e. for C^N, $b \to Nb$. Therefore it also gives us an unstable direction and $(x(t), p(t))$ will flow along that direction to infinity but in a linear fashion.

3.2 Spectrum of the effective Hamiltonian

Another different and useful viewpoint is to study the spectrum of the effective Hamiltonian. For example, in the elliptic class, H_{eff} has a discrete spectrum with bounded eigen functions. For example $H_0 = 2J_0$, which is in this elliptic class, has a discrete spectrum $n + 1/2$ with all exponentially localized eigen functions. In the Hyperbolic case, the spectrum of H_{eff} resembles that of a tachyon, which actually is not well-defined in a basis of normalizable wavefunctions [2, 3]. For example $H' = 2J_1 = p^2/2 - x^2/2$, which is in this hyperbolic class, is a harmonic oscillator but with an imaginary frequency. Therefore, its eigenvalues are $i(n + 1/2)$ with unnormalizable eigenfunctions $\psi(x) \sim e^{ix^2/2}$. That is why the dynamics of this system is so unstable. In the parabolic case, H_{eff} has a well-defined but continuous spectrum.

Let's make these statements more precise. To solve the spectrum of the effective Hamiltonian, we can rewrite it in terms of a and $a^\dagger$. To be consistent with the convention in superfluid literatures, we introduce new symbols to replacing $\widehat{n}$ and write the Hamiltonian as,

$$H_{\text{eff}} = \epsilon_0 a^\dagger a + \frac{\Delta_x}{2}\left[(a^\dagger)^2 + a^2\right] - i\frac{\Delta_y}{2}\left[(a^\dagger)^2 - a^2\right], \tag{27}$$

$$\epsilon_0 = n_{f,0},\ \Delta_x = n_{f,1},\ \Delta_y = n_{f,2}. \tag{28}$$

where we already drop all the unimportant constant pieces. $\Delta_x - i\Delta_y = \Delta e^{i\theta}$ is the complex pairing. It is convenient to hide the dependence on the phase parameter θ into a redefinition of the $a, a^\dagger$,

$$a \to a e^{i\theta/2}, \quad a^\dagger \to a^\dagger e^{-i\theta/2}. \tag{29}$$

In terms of the newly defined variables, we can rewrite it as a following BdG-type Hamiltonian, again without worrying about the constant piece,

$$H_{\text{eff}} = \frac{1}{2}\begin{pmatrix} a^\dagger & a \end{pmatrix}\begin{pmatrix} \epsilon_0 & \Delta \\ \Delta & \epsilon_0 \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{30}$$

The criterion for the SL(2) classification now becomes the determinant of the single particle Hamiltonian,

$$|\widehat{n}_f|^2 = \det h = \epsilon_0^2 - \Delta^2. \tag{31}$$

To diagonalize this Hamiltonian, we can transform our basis to define a new bosonic mode,

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \quad |u|^2 - |v|^2 = 1. \tag{32}$$

The constraint on u and v is needed to guarantee the boson commutator $[b, b^\dagger] = 1$. Now we can plug this relation into the effective Hamiltonian and get,

$$H_{\text{eff}} = \frac{1}{2}\begin{pmatrix} b^\dagger & b \end{pmatrix}\begin{pmatrix} (|u|^2+|v|^2)\epsilon_0 - (uv^* + u^*v)\Delta & (u^2+v^2)\Delta - 2uv\epsilon_0 \\ c.c. & (|u|^2+|v|^2)\epsilon_0 - (uv^* + u^*v)\Delta \end{pmatrix}\begin{pmatrix} b \\ b^\dagger \end{pmatrix}.$$

We require the single particle matrix to be diagonal so we have our gap equation,

$$\frac{\Delta}{\epsilon_0} = \frac{2uv}{u^2+v^2}. \tag{33}$$

When solving this gap equation with the constraint, it will be simpler to reparametrize u and v as $u \sim \cosh\theta, v \sim \sinh\theta$. In general they can still be complex numbers. Therefore, we will use the following parametrization,

$$u = e^{i\alpha}\cosh\theta, \quad v = e^{i\beta}\sinh\theta, \quad \alpha, \beta, \theta \in \mathbb{R}. \tag{34}$$

We can rewrite the gap equation in terms of α, β and θ as,

$$\frac{\Delta}{\epsilon_0} = \frac{2}{e^{i(\alpha-\beta)}\coth\theta + e^{i(\beta-\alpha)}\tanh\theta}. \tag{35}$$

Since $\tanh\theta < 1$, the only way to guarantee the denominator of the right hand side to be real is to choose $\alpha - \beta = 0$ or π. Therefore, the gap equation is reduced to,

$$\frac{\Delta}{\epsilon_0} = \pm \tanh 2\theta. \tag{36}$$

Physically there is no difference between $\pm$. Without loss of generality, we can choose the '+' sign, i.e. $\alpha = \beta = 0$ and assume $\Delta > 0$ in the following discussion.

If we just naively solve the equation, we will find that

$$H_{\text{eff}} = E b^\dagger b,\ E = \sqrt{\epsilon_0^2 - \Delta^2}, \quad \cosh^2\theta = \frac{1}{2}\left(\frac{\epsilon_0}{\sqrt{\epsilon_0^2 - \Delta^2}} + 1\right). \tag{37}$$

Now let's discuss this solution in different cases:

- Elliptic $\Delta < \epsilon_0$. The solution is well-defined, which gives us an evenly spaced gapped spectrum. Therefore we expect the dynamics to be bounded and have many revivals.
- Hyperbolic $\Delta > \epsilon_0$. If we naively extend our solution to this regime, we will find the energy becomes imaginary, which implies a tachyonic mode. Therefore we expect the dynamics to be very unstable. Actually, in this case Eq. (36) doesn't have solution, or the Hamiltonian doesn't have a solution in a Hilbert space with normalizable basis.
- Parabolic $\Delta = \epsilon_0$. In this case, the spectrum becomes gapless and the transformation also becomes singular.

Remark The above discussion is purely algebraic therefore can be generalized to other kinds of Floquet oscillators once the Hamiltonian is quadratic.

The EoM for the quantum oscillator is the same as the classical case, thus it also applies to the discussion of classical Floquet oscillators. That explains why we can do swinging.

4 Example 1: A baby version of the Floquet CFT

The Mathieu oscillator has continuous dependence on time, which is hard to deal with technically. Inspired by the study of Floquet CFT [4], here we choose a much simplified Bang-Bang controlled protocol, which is still enough to show all the essential features,

$$H(t) = f(t)\frac{p^2}{2} + \frac{x^2}{2}, \quad f(t) = \begin{cases} 0 & 0 < t < T_1 \\ 1 & T_1 < t < T_1 + T_0 \end{cases}. \tag{38}$$

The lesson we learned from the last section is to write everything in terms of the $\mathrm{SL}(2,\mathbb{R})$ generators and study the Floquet operator,

$$F = e^{-iT_0 J_0} e^{-iT_1(J_0 - J_1)} = e^{-iH_{\text{eff}}}. \tag{39}$$

As we argued above, the effective Hamiltonian should be linear superposition of $\mathrm{SL}(2,\mathbb{R})$ generators up to an unimportant constant,

$$H_{\text{eff}} = \text{const} + \phi\,(\widehat{n} \cdot \widehat{J}). \tag{40}$$

Notice that, this relation can be completely determined by using the commutation relations of J_i thus is independent of the representation. Therefore, we can choose any representation we like to do the calculation first and then write the result in the form of Eq. (40). The simplest choice is the matrix representation, where the three generators are,

$$J_0 = \frac{i}{2}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \; J_1 = \frac{i}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \; J_2 = \frac{i}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \tag{41}$$

This representation is to $\mathfrak{sl}(2,\mathbb{R})$ just as the Pauli matrix is to $\mathfrak{su}(2)$. One can show that it have many simple algebraic properties. The most important one to us is the formula for $e^{i\alpha(\widehat{n}\cdot\widehat{J})}$,

$$e^{i\alpha(\widehat{n}\cdot\widehat{J})} = \begin{cases} \cos\frac{\alpha}{2} + i(\widehat{n}\cdot 2\widehat{J})\sin\frac{\alpha}{2} & |\widehat{n}|^2 = 1 \\ 1 + i\alpha(\widehat{n}\cdot\widehat{J}) & |\widehat{n}|^2 = 0 \\ \cosh\frac{\alpha}{2} + i(\widehat{n}\cdot 2\widehat{J})\sinh\frac{\alpha}{2} & |\widehat{n}|^2 = -1 \end{cases}. \tag{42}$$

where the norm is with respect to the metric $(1,-1,-1)$. Now by applying the formula, we can write the Floquet operator as,

$$F = \left(\cos\frac{T_0}{2} - \frac{T_1}{2}\sin\frac{T_0}{2}\right) - i\left(T_1\cos\frac{T_0}{2} + 2\sin\frac{T_0}{2}\right)J_0 + iT_1\cos\frac{T_0}{2}J_1 + iT_1\sin\frac{T_0}{2}J_2. \tag{43}$$

One can refer to the Appendix for all the calculation details. Now we can compare it with the Eq. (42) to write down the formula of the effective Hamiltonian. Though it is a complicated expression, the class of H_{eff} can be determined in a simple way. If we look at Eq. (42) carefully, we can notice that $\operatorname{Tr} e^{i\alpha(\widehat{n}\cdot\widehat{J})}$ takes different value for the three classes. Therefore the class of H_{eff} can be read out from the value of,

$$\Delta = \left(\cos\frac{T_0}{2} - \frac{T_1}{2}\sin\frac{T_0}{2}\right)^2. \tag{44}$$

Depending on whether Δ is larger or smaller than 1, the system gives a heating nor non-heating dynamics. The phase diagram is shown in Fig. 1, which is the same as that of the Floquet CFT studied in [4].

5 Example 2:

Let's try a more physical set-up, we can imagine putting a cloud of BEC into a magnetic trap and change between a weak and strong harmonic potential periodically. Correspondingly, the BEC can expand or will be squeezed, pictorially which is like to let the BEC breath. The question is whether the BEC will become more energetic by breathing. The Floquet Hamiltonian is,

$$H(t) = \frac{p^2}{2} + \frac{x^2}{2} + f(t)\frac{x^2}{2}, \quad f(t) = \begin{cases} 1 & 0 < t < T_1 \\ 0 & T_1 < t < T_1 + T_0 \end{cases}. \tag{45}$$

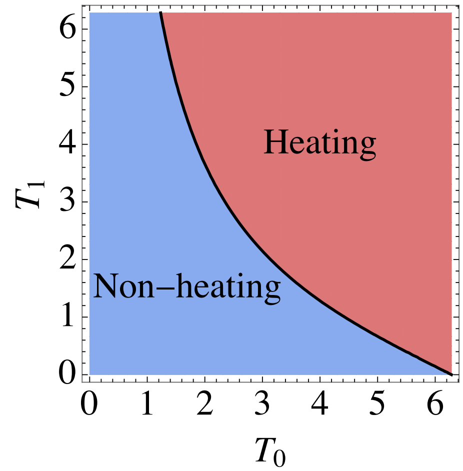

Figure 1: The phase diagram of the Floquet oscillator.

A Composition of two $\mathrm{SL}(2,\mathbb{R})$ transformations

In this section, we give a general discussion on how to write the composition of two $\mathrm{SL}(2,\mathbb{R})$ transformation into a single exponential form. We use $\widehat{J}$ for the $\mathrm{SL}(2,\mathbb{R})$ generators and $\widehat{n}$ for the rotation angle. Suppose the two rotations are around axis $\widehat{n}_{1,2}$ with rotation angle $\alpha_{1,2}$. We're looking for a way to write the composition as,

$$e^{i\alpha_2(\widehat{n}_2\cdot\widehat{J})}e^{i\alpha_1(\widehat{n}_1\cdot\widehat{J})}=e^{i\phi(\widehat{n}\cdot\widehat{J})}. \tag{46}$$

However, since $\mathrm{SL}(2,\mathbb{R})$ cannot be fully covered by the exponential form, strictly speaking we need to add some constant term on the right hand side.

Just as the fact that it is easier to use Pauli matrices to study the $SU(2)$ case, here it is easier to do the calculation if we use the matrix/defining representation. We list the representation of the three generators again,

$$J_0=\frac{i}{2}\begin{pmatrix} & -1\\ 1 & \end{pmatrix},\ J_1=\frac{i}{2}\begin{pmatrix} 1 & \\ & -1\end{pmatrix},\ J_2=\frac{i}{2}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}, \tag{47}$$

which generate a $SL(2,\mathbb{R})$ matrix through the exponential $\exp i\alpha_j J_j$. The nice property is that now they not only have the commutation relation but also anti-commute with each other, which simplifies many calculations.

Let's first try to expand a single exponential $e^{i\alpha(\widehat{n}\cdot\widehat{J})}$. Do a Taylor expansion,

$$e^{i\alpha(\widehat{n}\cdot\widehat{J})}=\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(i\frac{\alpha}{2}\right)^{2k}(\widehat{n}\cdot 2\widehat{J})^{2k}+\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(i\frac{\alpha}{2}\right)^{2k+1}(\widehat{n}\cdot 2\widehat{J})^{2k+1}. \tag{48}$$

One can show that due to the anti-commutation relations,

$$(\widehat{n}\cdot 2\widehat{J})^2=n_0^2-n_1^2-n_2^2=|n|^2, \tag{49}$$

where $|n|^2$ means the norm of $\widehat{n}$ in a Minkowski space with metric $g_{ij}=(1,-1,-1), i,j=0,1,2$. Therefore if we always normalize $\widehat{n}$ to a unit vector when it has a non-vanishing norm, we have three different cases,

- Elliptic $|n|^2 = 1$. In this case, the formula looks similar to the $SU(2)$ case,
$$e^{i\alpha(\widehat{n}\cdot\widehat{J})} = \cos\frac{\alpha}{2} + i(\widehat{n}\cdot 2\widehat{J})\sin\frac{\alpha}{2}. \tag{50}$$

- Parabolic $|n|^2 = 0$. In this case, all the higher order terms in the expansion vanish and we have,
$$e^{i\alpha(\widehat{n}\cdot\widehat{J})} = 1 + i\alpha(\widehat{n}\cdot\widehat{J}) \tag{51}$$

- Hyperbolic $|n|^2 = -1$. In this case, $(\widehat{n}\cdot 2\widehat{J})^{2k} = (-1)^k$ and we have,
$$e^{i\alpha(\widehat{n}\cdot\widehat{J})} = \cosh\frac{\alpha}{2} + i(\widehat{n}\cdot 2\widehat{J})\sinh\frac{\alpha}{2}. \tag{52}$$

Although not necessary for our purpose here, for the generic case, we also need a formula to compute $(\widehat{n}\cdot\widehat{J})(\widehat{\widetilde{n}}\cdot\widehat{J})$,
$$(\widehat{n}\cdot 2\widehat{J})(\widehat{\widetilde{n}}\cdot 2\widehat{J}) = \widehat{n}\cdot\widehat{\widetilde{n}} - i(\widehat{n}\times\widehat{\widetilde{n}})\cdot 2\widehat{J}, \tag{53}$$
where $(\widehat{n}\times\widehat{\widetilde{n}})_i = \epsilon_{ijk}n_j\widetilde{n}_k$ and the dot product is with respect to the Minkowski metric.

Now let's compute the composition we want, which we write using a slightly different notation here to keep the formula look nicer,
$$e^{i\alpha J_0}e^{i\beta(J_0-J_1)} \tag{54}$$
Working in the matrix representation and applying the formula above, we have,
$$\begin{aligned}
e^{i\alpha J_0}e^{i\beta(J_0-J_1)} &= \left(\cos\frac{\alpha}{2} + i2J_0\sin\frac{\alpha}{2}\right)(1+i\beta(J_0-J_1)) \\
&= \cos\frac{\alpha}{2} + i\beta\cos\frac{\alpha}{2}(J_0-J_1) + i2J_0\sin\frac{\alpha}{2} - 2\beta\sin\frac{\alpha}{2}J_0(J_0-J_1) \\
&= \left(\cos\frac{\alpha}{2} - \frac{\beta}{2}\sin\frac{\alpha}{2}\right) + i\left(\beta\cos\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\right)J_0 - i\beta\cos\frac{\alpha}{2}J_1 + i\beta\sin\frac{\alpha}{2}J_2.
\end{aligned}$$
Now let's try to identify it with $e^{i\phi(\widehat{n}\cdot\widehat{J})}$. However, depending on the value of $\Delta = \left(\cos\frac{\alpha}{2} - \frac{\beta}{2}\sin\frac{\alpha}{2}\right)^2$, the composite transformation falls into different classes, which yields different constant piece, therefore we have to deal with them separately:

- $\Delta < 1$, the composite transformation is Elliptic,
$$\cos\frac{\phi}{2} = \cos\frac{\alpha}{2} - \frac{\beta}{2}\sin\frac{\alpha}{2}, \tag{55}$$
$$2\sin\frac{\phi}{2}\widehat{n} = \left(\left(\beta\cos\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\right), -\beta\cos\frac{\alpha}{2}, \beta\sin\frac{\alpha}{2}\right). \tag{56}$$

- $\Delta = 1$, the composite transformation is parabolic;
$$\eta = \cos\frac{\alpha}{2} - \frac{\beta}{2}\sin\frac{\alpha}{2}, \tag{57}$$
$$\eta\phi\widehat{n} = \left(\left(\beta\cos\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\right), -\beta\cos\frac{\alpha}{2}, \beta\sin\frac{\alpha}{2}\right). \tag{58}$$
where $\eta = \pm$ to make sure that the composite transformation can be written as an exponential form.

- $\Delta > 1$, the composite transformation is hyperbolic.

$$\eta \cosh \frac{\phi}{2} = \cos \frac{\alpha}{2} - \frac{\beta}{2} \sin \frac{\alpha}{2}, \tag{59}$$

$$2\eta \sinh \frac{\phi}{2} \widehat{n} = \left(\left(\beta \cos \frac{\alpha}{2} + 2 \sin \frac{\alpha}{2} \right), -\beta \cos \frac{\alpha}{2}, \beta \sin \frac{\alpha}{2} \right), \tag{60}$$

where $\eta = \pm$ to make sure the composite transformation can still be written as an exponential form. $\eta = 1$ when $\cos \frac{\alpha}{2} - \frac{\beta}{2} \sin \frac{\alpha}{2} > 0$ and $\eta = -1$ when $\cos \frac{\alpha}{2} - \frac{\beta}{2} \sin \frac{\alpha}{2} < 0$.

References

[1] A.M.Perelomov and V.S.Popov, *Group-theoretical aspects of the variable frequency oscillator problem.*

[2] Private discussion with Yingfei Gu.

[3] Private discussion with Xueda Wen.

[4] Xueda Wen, Jieqiang Wu, *Floquet conformal field theory*, *arXiv:*1805.00031.