Notes on two-dimensional conformal field theories

Ruihua Fan[1]

[1] *Department of Physics, Harvard University, Cambridge MA 02138, USA*

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Abstract

This is a note for a series of journal clubs about two-dimensional CFTs that I gave in 2018. Thank Shang, Jong-Yeon, Nat and Xue-Yang for many helpful questions. In particular, thank Yingfei for his patience of answering my questions.

Contents

1 Basics of conformal transformation

1.1 Conformal Transformation

Let's consider a generic manifold Σ without boundary. Given a choice of coordinate $\{x_\mu\}$, we have a corresponding metric $g_{\mu\nu}$. However, different choices of coordinate are just different gauges. So when we do a coordinate transformation from $\{x_\mu\}$ to another coordinate $\{y_\mu\}$, the line element is invariant, which leads to the transformation law of metric,

$$g'_{\mu\nu}(y) = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}. \tag{1}$$

We call it diffeomorphism.

Conformal transformation is a subset of diffeomorphism, where the transformed metric is related to the original metric by a scaling factor,

$$g'_{\mu\nu}(y) = \Omega(x)g_{\mu\nu}(x). \tag{2}$$

To study what kind of transformations satisfy this constraint, it is always good to start with an infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + \epsilon\xi^\mu$, $\Omega(x) = 1 + \epsilon f(x)$. Plug this into Eqn.1 we have,

$$g_{\mu\sigma}\partial_\rho\xi^\mu + g_{\rho\nu}\partial_\sigma\xi^\nu + f g_{\rho\sigma} = 0, \tag{3}$$

which is a highly nonlinear equation. To make life simply, from now on we only consider flat space and the conformal transformation on it. So the equation is reduced to

$$\eta_{\mu\sigma}\partial_\rho\xi^\mu + \eta_{\rho\nu}\partial_\sigma\xi^\nu + f\eta_{\rho\sigma} = 0, \tag{4}$$

We can contract $\eta^{\rho\sigma}$ to the LHS and get $f = -2(\partial\cdot\xi)/d$. Combined with Eqn.4, we get,

$$\partial^2\xi_\sigma = \frac{2-d}{d}\partial_\sigma(\partial\cdot\xi). \tag{5}$$

1.1.1 $d \geq 3$

We can multiply ∂_σ to both sides and get $\partial^2(\partial\cdot\xi) = 0$, which tells us ξ can at most be a quadratic function. Its generic form is[1]

$$\xi^\mu = a^\mu + b^\mu{}_\nu x^\mu + cx^\mu + d_\lambda(\eta^{\mu\nu}x^2 - 2x^\lambda x^\mu). \tag{6}$$

Each term has its own physical meaning: (1) a^μ is translation; (2) $b^\mu{}_\nu$ is antisymmetric and describes rotation; (3) c is dilation; (4) $d_\lambda(\eta^{\mu\nu}x^2 - 2x^\lambda x^\mu)$ is a new term which corresponds to inversion+translation+inversion [1], called special conformal transformation[2].

So in higher dimension, we only have four types of conformal transformations, all of which are globally well-defined.

[1] The most general quadratic form should be $c^\mu{}_{\rho\sigma}x^\rho x^\sigma$. However Eqn.4 will put some other symmetry constraint which leads to the following form.

[2] If we consider inversion $x^\mu = e^2y^\mu/y^2$, then $ds^2 = \frac{e^4}{y^4}\eta_{\lambda\sigma}dy^\lambda dy^\sigma$, which indeed is a conformal transformation. However, we want an infinitesimal version of it. So we could consider the following sequence: $x^\mu \to x^\mu/x^2 \to \widetilde{x}^\mu = x^\mu/x^2 + d^\mu \to x'^\mu = \widetilde{x}^\mu/\widetilde{x}^2$. If $d = 0$, the two inversion will exactly cancel each other. So when $d^\mu \ll 1$, we can get an infinitesimal transformation: $x'^\mu = x^\mu + d_\lambda(\eta^{\mu\lambda}x^2 - 2x^\lambda x^\mu)$, which is exactly the same form as we got from the formal way. And it is clear that d_λ is the amount of translation sandwiched by two inversion.

1.1.2 $d = 2$

We have $\partial^2 \xi_\sigma = 0$. So ξ_σ is any analytical function, which means that there are at least infinite number of local conformal transformation. Because of the specialness of 2D, we can give another derivation. Under a conformal transformation $z^i \to \omega^j$, the metric transforms as $\widetilde{g}_{\mu\nu}(\omega) = \left(\frac{\partial z^\rho}{\partial \omega^\mu}\right)\left(\frac{\partial z^\sigma}{\partial \omega^\nu}\right)\eta_{\rho\sigma}(z) = \Lambda(z)\eta_{\mu\nu}(z)$, which gives us,

$$\left(\frac{\partial z^0}{\partial \omega^0}\right)\left(\frac{\partial z^0}{\partial \omega^0}\right) + \left(\frac{\partial z^1}{\partial \omega^0}\right)\left(\frac{\partial z^1}{\partial \omega^0}\right) = \left(\frac{\partial z^0}{\partial \omega^1}\right)\left(\frac{\partial z^0}{\partial \omega^1}\right) + \left(\frac{\partial z^1}{\partial \omega^1}\right)\left(\frac{\partial z^1}{\partial \omega^1}\right)$$
$$\left(\frac{\partial z^0}{\partial \omega^0}\right)\left(\frac{\partial z^0}{\partial \omega^1}\right) + \left(\frac{\partial z^1}{\partial \omega^0}\right)\left(\frac{\partial z^1}{\partial \omega^1}\right) = 0.$$

This is equivalent to the Cauchy-Riemann equation,

$$\frac{\partial z^1}{\partial \omega^0} = \pm \frac{\partial z^0}{\partial \omega^1}\frac{\partial z^0}{\partial \omega^0} = \mp \frac{\partial z^1}{\partial \omega^1} \tag{7}$$

If we use complex coordinate $z = z^0 + iz^1$, $\omega = \omega^0 + i\omega^1$, then it just tells us $z(\omega)$ and $\omega(z)$ has to be holomorphic (anti-holomorphic).

But all above is just local constraint. We now require the transformation is global well-defined, which maps the whole z plane to the whole ω plane and has a inverse. This first excludes branch cut and essential singularity. Because near a branch cut the mapping is not single valued thus not reversible. And essential singularity is also not invertible due to the Picard's great theorem. So the only possibility is,

$$\omega(z) = \frac{P(z)}{Q(z)}, \tag{8}$$

where $P(z)$ and $Q(z)$ are simply polynomials. What's more, they cannot have multiple zero points, otherwise it will map different z's to $\omega = 0$ or $\omega = \infty$ hence non-invertible. So the final result is,

$$\omega(z) = \frac{az + b}{cz + d}, \tag{9}$$

which is a Möbius transformation. Reversibility requires $ad - bc \neq 0$. The overall phases of a, b, c, d are not important so we can choose a convention $ad - bc = 1$. So in 2D, the global conformal transformation is isomorphic to PSL$(2, \mathbb{C})$. And the six generators of PSL$(2, \mathbb{C})$ (See Appendix.A) corresponds to different conformal transformations,

$$\sigma_z \to \text{rotation} \tag{10}$$
$$i\sigma_z \to \text{dilation} \tag{11}$$
$$(\sigma_x + i\sigma_y)/2 \to \text{translation along x} \tag{12}$$
$$i(\sigma_x + i\sigma_y)/2 \to \text{translation along y} \tag{13}$$
$$(\sigma_x - i\sigma_y)/2, i(\sigma_x - i\sigma_y)/2 \to \text{inversion or special conformal transformation} \tag{14}$$

where we adopt the convention that the finite transformation is $G = \exp(-i\sum_j c_j t_j)$.

So we see that on the level of global transformation, 2D has similar structure to the higher dimension. But we also have infinite number of local conformal transformations, which makes 2D easier to solve.

Another nice thing is that, once $\omega(z, \overline{z}) = \omega(z)$ and $\overline{\omega}(z, \overline{z}) = \overline{\omega}(\overline{z})$, the transformation is formally conformal. So we can effectively treat $\omega(z)$ and $\overline{\omega}(\overline{z})$ as two independent transformations, which gives us two sets of algebras.

1.1.3 $d = 1$

In this case, $ds^2 = d\tau^2$. So any continuous function is conformal. The conformal transformation is thus the same as all the diffeomorphism.

1.2 Field Theory with Conformal Invariance

Here we still use the conventional definition of field theory, which is described by a path integral. And we restrict our discussion in flat Euclidean space. So the path integral is,

$$Z = \int D\phi e^{-S[\phi]}, \quad S[\phi] = \int d^d x\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)). \tag{15}$$

Under a conformal transformation $x \to x'$, the fields also transform accordingly, $\phi(x) \to \phi'(x') = \mathcal{F}[\phi](x)$. So the transformed action is,

$$S'[\phi'] = \int d^d x'\, \mathcal{L}(\phi(x'), \partial'_\mu \phi'(x')) = \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}\left(\mathcal{F}[\phi](x), \frac{\partial x^\sigma}{\partial x'^\mu} \partial_\sigma \mathcal{F}[\phi](x) \right). \tag{16}$$

If $S'[\phi'] = S[\phi]$ and x, x' both take values over the whole space[3], then we say the theory is classically conformally invariant [2]. If there is no anomaly, then this invariance can be promoted to the quantum level. For concreteness, let's look at few examples:

Free Boson

$$S = \frac{g}{2} \int d^2 x \partial_\mu \varphi \partial^\mu \varphi = 2g \int dz d\overline{z}\, \partial_z \varphi \partial_{\overline{z}} \varphi \tag{17}$$

We require $z \to \omega(z)$, $\varphi(z, \overline{z}) \to \varphi'(\omega, \overline{\omega}) = \varphi(z, \overline{z})$, then it is conformal invariant.

Liouville Theory

$$S = \frac{g}{2} \int d^2 x \,(\partial_\mu \varphi \partial^\mu \varphi + \Lambda e^\varphi) \tag{18}$$

This looks similar to the above theory expect for an additional exponential term. Now we have to require

$$z \to \omega(z), \quad \varphi(z, \overline{z}) \to \varphi'(\omega, \overline{\omega}) = \varphi(z, \overline{z}) - \frac{d\omega}{dz} \frac{d\overline{\omega}}{dz}. \tag{19}$$

Then up to a total derivative coming from the kinetic term, it is conformally invariant.

Free Fermion

$$S = \frac{g}{2} \int d^2 x \overline{\Psi} \gamma^\mu \partial_\mu \Psi \tag{20}$$

where Ψ is a two component Dirac spinor $(\psi, \overline{\psi})^T$ and $\overline{\Psi} = \Psi^\dagger \gamma^0$. We choose the convention that $\gamma^0 = \sigma_x$, $\gamma^1 = \sigma_y$. We can also write the action in terms of the two components as,

$$S = g \int d^2 x (\overline{\psi} \partial \overline{\psi} + \psi \overline{\partial} \psi) \tag{21}$$

[3]Without the second condition, the transformed theory may be defined on a different geometry. So they are not the same theory and may give different correlation functions in general.

Its classical solution is $\partial\overline{\psi} = 0$ and $\overline{\partial}\psi = 0$ which means $\psi(\overline{\psi})$ is holomorphic(anti-holomorphic). This inspires the transformation law to be,

$$z \to \omega(z), \quad \psi \to \psi'(\omega, \overline{\omega}) = \left(\frac{d\omega}{dz}\right)^{-1/2} \psi(z, \overline{z}) \tag{22}$$

You can see that different fields generally transform in different ways under conformal transformation. There is a small subset of fields transform nicely, which are what we are interested in.

Quasi-Primary fields They are defined to transform in the following way under *global* conformal transformation,

$$z \to \omega(z) \quad A(z, \overline{z}) \to A'(\omega, \overline{\omega}) = \left(\frac{d\omega}{dz}\right)^{-h} \left(\frac{d\overline{\omega}}{d\overline{z}}\right)^{-\overline{h}} A(z, \overline{z}), \tag{23}$$

This type of fields can also be defined in higher dimension. The physical meaning of h and $\overline{h}$ are clear by checking some special cases:

- $\omega = \lambda z, \lambda \in \mathbb{R}$, which is a scaling transformation. Then $A' = \lambda^{-(h+\overline{h})}A$. So $h + \overline{h} = \Delta$ is the scaling dimension.
- $\omega = e^{i\theta}z$, $\overline{\omega} = e^{-i\theta}\overline{z}$, which is a rotation. Then $A' = e^{-i(h-h')\theta}A$. So $h - \overline{h} = s$ is the spin.

So for an operator with scaling dimension Δ and spin s, it has two independent conformal dimension $h = \frac{1}{2}(\Delta + s)$, $\overline{h} = \frac{1}{2}(\Delta - s)$.

Primary fields In 2D, we also have local conformal transformation. So if the field also transforms in the same way under *local* conformal, it is called a primary field, which is unique to 2D.

1.3 Restrictions on Correlation Functions

For simplicity, let's first consider a 2pt correlation function between quasi-primary fields. And we work in a flat Euclidean space.

$$G(z_1, \overline{z}_1, z_2, \overline{z}_2) = \langle \phi_1(z_1, \overline{z}_1)\phi_2(z_2, \overline{z}_2)\rangle = \frac{1}{Z}\int D\phi\phi_1(z_1, \overline{z}_1)\phi_2(z_2, \overline{z}_2)e^{-S[\phi]}. \tag{24}$$

After a *global* conformal transformation, the new correlation function is,

$$\frac{1}{Z}\int D\phi'\phi'_1(z'_1, \overline{z}'_1)\phi'_2(z'_2, \overline{z}'_2)e^{-S'[\phi']}. \tag{25}$$

Because the geometry doesn't change, the new path integral doesn't change except for the position of the operators. Thus we can write the new correlator in the same functional form as the original one $G(z'_1, \overline{z}'_1, z'_2, \overline{z}'_2)$. By using the conformal invariance of the action and the definition of quasi-primary, we have [2],

$$G(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2) = \prod_{j=1}^{2} \left(\frac{d\omega}{dz}\right)_j^{-h_j} \left(\frac{d\overline{\omega}}{d\overline{z}}\right)_j^{-\overline{h}_j} G(z_1, \overline{z}_1, z_2, \overline{z}_2). \tag{26}$$

And this formula is easily generalized to multiple point functions.[4]

As is argued before, we can effectively treat $\omega(z)$ and $\overline{\omega}(\overline{z})$ as two independent transformation. So by choosing $\omega(z) = z + \epsilon z^n, n = 0, 1, 2$ and $\overline{\omega}(\overline{z}) = \overline{z}$, we have the following differential equation,

$$\begin{aligned} \sum_{j=1}^{2} \partial_{z_j} G = 0, \\ \sum_{j=1}^{2} [z_j \partial_{z_j} + h] G = 0, \\ \sum_{j=1}^{2} [z_j^2 \partial_{z_j} + 2h z_j] G = 0. \end{aligned} \tag{28}$$

By replacing z with $\overline{z}$, we get the counterpart for anti-holomorphic components. As a result, the correlation function shows a nice separation between holomorphic and anti-holomorphic part:

- 2pt function:

$$G(z_{12}, \overline{z}_{12}) = \frac{1}{z_{12}^{2h} \, \overline{z}_{12}^{2\overline{h}}} \tag{29}$$

  and it vanishes when two operators have different conformal dimensions. Or we can understand it as two different operators cannot fuse into identity channel.

- 3pt function:

$$G(z_{12}, z_{13}, z_{23}) = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} \, z_{13}^{h_1+h_3-h_2} \, z_{23}^{h_2+h_3-h_1} \times c.c.}, \tag{30}$$

  where C_{123} is a dynamical data and can only be determined by conformal algebra. And each exponent of z_{ij} can be understood as ϕ_i and ϕ_j fuse to ϕ_k so that the OPE coefficient must be $z_{ij}^{h_i+h_j-h_k}$ to balance the conformal dimension.

- 4pt function:

$$G(z_{ij}, \overline{z}_{ij}) = \prod_{i<j}^{4} \frac{1}{z_{ij}^{\mu_{ij}} \times c.c.} f(\eta, \overline{\eta}) \tag{31}$$

  where $\mu_{ij} = h_i + h_j - h/3$, $h = \sum_j h_j$ and η is the cross ratio.

Remark: Such a nice separation stems from the fact that we are living *an infinite plane* so that we have $\mathrm{PSL}(2, \mathbb{C})$ as our global transformation group, which has six generators giving six independent constraints, three for holomorphic, three for anti-holomorphic[5].

If we instead live on *the upper half plane*, to keep the geometry invariant, the global conformal transformation is reduced to $\mathrm{PSL}(2, \mathbb{R})$ which only has three generators. As a result, we cannot

[4]We can also use it for local transformation when ϕ_j are primary fields. However, because the transformation changes the geometry, the transformed correlation function may have different functional form. We have to modify Eqn.26 to be

$$G_\Sigma(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2) = \prod_{j=1}^{2} \left(\frac{d\omega}{dz}\right)_j^{-h_j} \left(\frac{d\overline{\omega}}{d\overline{z}}\right)_j^{-\overline{h}_j} G_{plane}(z_1, \overline{z}_1, z_2, \overline{z}_2), \tag{27}$$

where we add subscript to indicate the different geometries and functional form.

[5]Or we can say, $\omega(z) = (az + b)/(cz + d)$ has complex numbers as parameters. So we can treat a, $\overline{a}$ independently, which means $\omega(z)$ and $\overline{\omega}(\overline{z})$ are independent.

choose $\omega(z)$ and $\overline{\omega}(\overline{z})$ independently but have to transform them at the same time, i.e. $z \to z + \epsilon z^n$, $\overline{z} \to \overline{z} + \epsilon \overline{z}^n$, $\epsilon \in \mathbb{R}$. And the constraint differential equations are reduced to the following three [3],

$$\begin{aligned} \sum_{j=1}^{n} [\partial_{z_j} + \partial_{\overline{z}_j}] G^{(n)} = 0, \\ \sum_{j=1}^{n} [(z_j \partial_{z_j} + h_j) + (\overline{z}_j \partial_{\overline{z}_j} + \overline{h}_j)] G^{(n)} = 0, \\ \sum_{j=1}^{n} [(z_j^2 \partial_{z_j} + 2h_j z_j) + (\overline{z}_j^2 \partial_{\overline{z}_j} + 2\overline{h}_j \overline{z}_j)] G^{(n)} = 0. \end{aligned} \tag{32}$$

We group those terms in a certain way to inspire an analogy with Eqn.28. If we redefine $z_{j+n} = \overline{z}_j$ and $h_{j+n} = \overline{h}_j$, then Eqn.32 for an n-pt function look exactly the same as Eqn.28 but for a 2n-pt function.

This analogy tells us that to calculate n-pt function on an upper half plane, an equivalent way is to add some fictitious image operator $\widetilde{\phi}_{j+n}$ with image conformal dimension $h_{j+n} = \overline{h}_j$ sitting at $z_{j+n} = \overline{z}_j$, the mirror-image position of ϕ_j and consider the holomorphic part of the correlation between these 2n operators. So called *image charge method*.

2 Ward identity and Virasoro algebra

2.1 Properties of $T_{\mu\nu}$

We assume we have quantumly well-defined energy momentum tensor that satisfies the following three *operator* equation,

$$\text{translation} \Rightarrow \partial_\mu \mathrm{T}^{\mu\nu} = 0, \tag{33}$$

$$\text{rotation} \Rightarrow \mathrm{T}^{\mu\nu} - \mathrm{T}^{\nu\mu} = 0, \tag{34}$$

$$\text{dilatation} \Rightarrow \mathrm{T}^{\mu}{}_{\mu} = 0. \tag{35}$$

If we rewrite them in the $z, \overline{z}$ basis, then rotation symmetry requires $T^{z\overline{z}} = T^{\overline{z}z}$ and dilatation requires $T^z{}_z + T^{\overline{z}}{}_{\overline{z}} = 0$, which are combined to yield $T^{z\overline{z}} = T^{\overline{z}z} = 0$. We can plug this into the translation symmetry requirement and get, $\partial_z T^{zz} = 0$, $\partial_{\overline{z}} T^{\overline{z}\overline{z}} = 0$. This encourages us to define,

$$T(z) = -2\pi T_{zz} = -\frac{\pi}{2} T^{\overline{z}\overline{z}}, \quad T(\overline{z}) = -2\pi T_{\overline{z}\overline{z}} = -\frac{\pi}{2} T^{zz}, \tag{36}$$

which are holomorphic and anti-holomorphic respectively.

Because we can think of $T_{\mu\nu}$ defined as $\delta S = \int d^2x T_{\mu\nu} \partial_\mu \epsilon_\nu$, T has scaling dimension $\Delta = 2$. Then we consider a pure rotation $z \to e^{i\theta} z, \overline{z} \to e^{-i\theta} \overline{z}$, under which $T(z)$ transforms as $T(z) \to T_{\alpha\beta} \frac{\partial x^\alpha}{\partial z'} \frac{\partial x^\beta}{\partial z'} = e^{-2i\theta} T(z)$. So it also carries spin $s = 2$. In summary $T(z)$ has a definite conformal weight $(2, 0)$.

2.2 Conformal Ward Identity

Because $T^{\mu\nu}$ is coupled to the change of coordinate, it can be also used to generate the variation of fields under coordinate transformation, which is stated by the following Ward Identity,

$$\delta_{\epsilon_\mu} \langle X \rangle = \int_M d^2x \partial_\mu \langle T^{\mu\nu}(x) \epsilon_\nu(x) X \rangle, \tag{37}$$

where X is a string of some arbitrary operators and $\epsilon(x)$ is any coordinate transformation. M has to cover the points where operators sit. $\delta_\epsilon X = X'(z) - X(z)$.

The R.H.S. is a total derivative. Thus we can use Stoke's theorem (See Appendix.B) to write it as a surface integral and we again write the answer in the $z, \overline{z}$ basis,

$$\delta_{\epsilon,\overline{\epsilon}} \langle X \rangle = \frac{i}{2} \int_C -dz \, \langle T^{\overline{z}\overline{z}} \epsilon_{\overline{z}} X \rangle + d\overline{z} \, \langle T^{zz} \epsilon_z X \rangle \,, \tag{38}$$

where we have used the fact that $T^{z\overline{z}} = 0$. Here z goes along a counterclockwise contour on the z plane and $\overline{z}$ goes along a clockwise contour (on the same plane). We define $\epsilon = \epsilon^z = 2\epsilon_{\overline{z}}$ and use the $T, \overline{T}$ defined above to rewrite it as,

$$\delta_{\epsilon,\overline{\epsilon}} \langle X \rangle = \frac{-1}{2\pi i} \oint_C dz \, \epsilon(z) \, \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_{\overline{C}} d\overline{z} \, \overline{\epsilon}(\overline{z}) \, \langle \overline{T}(\overline{z}) X \rangle \,. \tag{39}$$

Because $\epsilon(z)$ and $\overline{\epsilon}(\overline{z})$ are independent of each other, we can write them separately,

$$\delta_\epsilon \langle X \rangle = \frac{-1}{2\pi i} \oint_C dz \, \epsilon(z) \, \langle T(z) X \rangle \,, \quad \delta_{\overline{\epsilon}} \langle X \rangle = \frac{1}{2\pi i} \oint_{\overline{C}} d\overline{z} \, \overline{\epsilon}(\overline{z}) \, \langle \overline{T}(\overline{z}) X \rangle \,. \tag{40}$$

Remarks If our theory is defined on the upper half plane, then $z, \overline{z}$ are not separated and we have to use Eqn.39, where C, $\overline{C}$ are on their own upper half plane. Things can be simplified by treating anti-holomorphic as the image of holomorphic component, which lives on the lower half $z-$plane.

We first require $T_{xy} = 0$ along the boundary (real axis), which leads to $T(z) = \overline{T}(\overline{z})$ on the real axis[6]. So we can define an extension of $T(z)$ to the lower half plane such that $T(z^*) = \overline{T}(\overline{z}), z \in \mathbb{H}^+$, i.e. $\overline{T}(\overline{z})$ is nothing but $T(z)$ evaluating at $\overline{z}$ on the lower half plane. Then we can rewrite the second term of Eqn.39 on the $z-$plane, where $\overline{C}$ becomes a clockwise contour on the lower half plane, as shown in Fig.1. By contour deformation, we can write the final answer as a single contour integral over z on the whole plane,

$$\delta_\epsilon \langle X \rangle = \frac{-1}{2\pi i} \oint_C dz \, \epsilon(z) \, \langle T(z) \phi_1(z_1, z_1^*) \phi_2(z_2, z_2^*) ... \rangle \,. \tag{41}$$

2.3 OPE of T

Now we can use Eqn.40 and what we have known(assumed) about quasi-primary and primary fields to study the OPE between $T(z)$ and those fields. The anti-holomorphic part shows similar result but everything with a bar.

Let's restrict X to be a string of *quasi-primary* operators and only consider global conformal transformation $\epsilon(z) = \sum_{n=-1}^{1} c_n z^{n+1}$, $\overline{\epsilon}(\overline{z}) = 0$.

- translation, $\epsilon(z) = \epsilon$. LHS $= -\epsilon \sum_i \partial_{z_i} \langle X \rangle$. For the R.H.S. to reproduce the same result, we have to have,

$$T(z)X = \sum_i \frac{1}{z - z_i} \partial_{z_i} X + \text{higher singularities}$$

- rotation+dilation, $\epsilon(z) = \epsilon z$. LHS $= -\epsilon \sum_i z_i \partial_{z_i} \langle X \rangle - \sum_i \epsilon h_i \langle X \rangle$, which implies,

$$T(z)X = \sum_i \frac{1}{z - z_i} \partial_{z_i} X + \frac{h_i}{(z - z_i)^2} \langle X \rangle + \text{higher singularities}$$

[6] $T_{zz} = T_{\alpha\beta} \frac{\partial x^\alpha}{\partial z} \frac{\partial x^\beta}{\partial z} = \frac{1}{4}(T_{xx} - T_{yy}) = T_{\overline{z}\overline{z}}$.

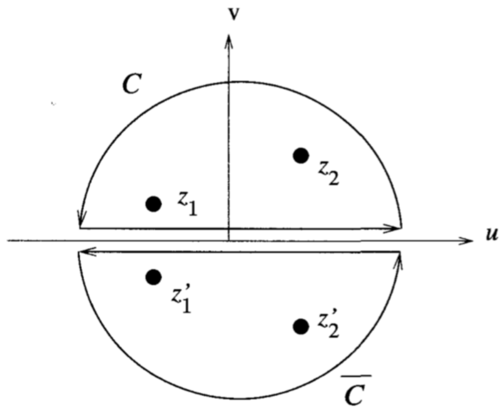


Figure 1: Schematics for the Ward Identity of boundary CFT. z_1 and z_2 are arbitrary two operators. We deform C and $\overline{C}$ to have an overlap on the real axis. Their integrals cancel each other due to the boundary condition we choose for $T(z)$.

- SCT, $\epsilon(z) = \epsilon z^2$. It doesn't bring more constraints.

So the OPE between $T(z)$ and a *quasi-primary* field is,

$$T(z)\phi(w) = \frac{\partial_w \phi(w)}{z-w} + \frac{h\phi(w)}{(z-w)^2} + \text{higher singularities} \tag{42}$$

We can see that (1) anti-holomorphic component doesn't enter this OPE, which is because we live on the whole plane; (2) those higher singularities cannot be determined solely by conformal invariance. Their form depends on the choice of operator and details of the theory; (3) to match its transformation under global conformal, the higher order singularities start from $1/(z-w)^4$, for example, TT OPE as shown below.[7]

However, for *primary* fields, we can choose $\epsilon(z)$ to be any locally analytical function, $\epsilon(z) = \sum c_n z^{n+1}$. This tells us its OPE with T doesn't have any higher order singularity,

$$T(z)\phi(w) = \frac{1}{z-w}\partial_w \phi(w) + \frac{h}{(z-w)^2}\phi(w). \tag{43}$$

From now on, we can use the OPE relation Eqn.42 and Eqn.43 as the definition of quasi-primary and primary fields. Their transformation law is specified by Eqn.40.

Remark Again, if we live on the upper half plane, holomorphic is always accompanied with anti-holomorphic. In contrast with Eqn.43, both components will enter the $T\phi$ OPE,

$$T(z)\phi(w,\overline{w}) = \left[\frac{1}{z-w}\partial_w + \frac{h}{(z-w)^2} + \frac{1}{z-\overline{w}}\partial_{\overline{w}} + \frac{\overline{h}}{(z-\overline{w})^2}\right]\phi(w,\overline{w}). \tag{44}$$

[7] There are also fields whose OPE with $T(z)$ starts from $1/(z-w)^3$. For example, assuming a primary ϕ, then for $\partial\phi(w)$, we have

$$T\partial\phi = \frac{1}{(z-w)}\partial^2\phi + \frac{h+1}{(z-w)^2}\partial\phi + \frac{2h}{(z-w)^3}\phi.$$

Descendant fields in general belongs to this category ($\partial\phi = L_{-1}\phi$ is the level-1 descendant of ϕ). One thing to notice is our previous argument for two point function only applies for quasi-primary fields. Thus for two (non quasi-primary) descendant fields, even though they have different conformal weight, their two point correlation function can be non-zero [4].

However, we can decompose a physical field $\phi(w, \overline{w})$ into a holomorphic field $\phi(w)$ and its image $\phi(\overline{w})$, which is also holomorphic, with conformal dimension $\overline{h}$. Then it is mathematically equivalent to talk about the OPE with these two fictitious fields once we always remember to put the constraint $\overline{z} = z^*$.

T(z)T(w) OPE An important example is the OPE of T with its self, which will naturally appears when talking about the algebraic structure of conformal transformation.

Because $T(z)$ has a conformal weight $(2, 0)$, its OPE should contain the following two terms,

$$T(z)T(w) = \frac{\partial_w T(w)}{z - w} + \frac{2T(w)}{(z - w)^2} + \sum_{n>2} \frac{O_n(w)}{(z - w)^n}$$

where O_n is some operator with conformal dimension $h[O_n] = 4 - n$ to balance h of the two sides. If we restrict us to a unitary CFT, then all the operators has to have non-negative conformal dimension. So the series has to terminate at $n = 4$.

Furthermore, we require $T(z)$ is a bosonic operator thus we must have $T(z)T(w) = T(w)T(z)$.[8] By writing down $T(w)T(z)$ and expanding operators $O(z)$ around w, we have,

$$\begin{aligned} T(w)T(z) &= \frac{\partial_z T(z)}{w - z} + \frac{2T(z)}{(w - z)^2} + \frac{O_3(z)}{(w - z)^3} + \frac{c/2}{(w - z)^4} \\ &= \frac{\partial_w T(w)}{z - w} + \frac{2T(w)}{(z - w)^2} + \frac{O_3(w) + (z - w)\partial_w O_3(w) + ...}{(w - z)^3} + \frac{c/2}{(z - w)^4}. \end{aligned}$$

So to satisfy the constrain, the only way out is $O_3 = 0$. Our final result is,

$$T(z)T(w) = \frac{\partial_w T(w)}{z - w} + \frac{2T(w)}{(z - w)^2} + \frac{c/2}{(z - w)^4}. \tag{45}$$

where c is called *central charge*. $T(w)$ is thus a quasi-primary operator.

2.4 Transformation Law for T

We can use Eqn.45 to calculate how T transforms under conformal transformation,

$$\delta_\epsilon T(z) = T'(z) - T(z) = -\frac{c}{12}\partial_z^3 \epsilon - 2\partial_z \epsilon T(z) - \epsilon \partial_z T(z), \tag{46}$$

or its finite transformation version,

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{w, z\}\right], \tag{47}$$

where $\{w, z\}$ is the Schwarzian derivative and $T'(z)$ is the expectation value on another plane, which may have different geometry.

As an application, we use this equation to get the vacuum energy of a system living on the cylinder. $w = \tau + i\sigma = \frac{L}{2\pi} \log z$ maps a plane onto a cylinder with circumference L. We can treat it as a zero temperature system with finite spatial size and periodic boundary condition. τ is the time direction and x is the spatial direction. If we require $T_{plane} = 0$, then we have,

$$T_{cyl}(w) = -\frac{c\pi^2}{6L^2}. \tag{48}$$

[8]Notice that all the OPE should be understood as inserted inside a correlator. So for bosonic operator the order doesn't matter..

We now apply an infinitesimal scaling $\tau \to (1+\epsilon)\tau$, where $\epsilon = \delta L/L$. Then the free energy will change as,

$$\delta F = \int d^2x \partial_\mu \epsilon_\nu T^{\mu\nu} = \int d^2x \left(\frac{\delta L}{L}\right) T^{\tau\tau} = \delta L \int d\tau (T_{ww} + T_{\overline{ww}}) \tag{49}$$

If we only care about the free energy per unit length, we can drop the $d\tau$ integral and get[9],

$$\delta f = \frac{\delta L}{L^2}\frac{(c+\overline{c})\pi}{12} \Rightarrow f = -\frac{(c+\overline{c})\pi}{12L}. \tag{50}$$

The integral constant is fixed by the requirement $f(L \to \infty) = 0$ because the system goes back to a plane in this limit. Our system is in zero temperature so the ground state energy is also given by f.

Similarly we can calculate the free energy for a finite temperature but infinite system. This time, we choose the conformal mapping $w = \sigma + i\tau = \frac{\beta}{2\pi}\log z$, where β is the inverse temperature. So we have $T_{cyl}(w) = -c\pi^2/6\beta^2$. This time we choose coordinate transformation $\tau \to (1+\epsilon)\tau$, where $\epsilon = \delta\beta/\beta$. So we have,

$$\delta F = \int d^2x \left(\frac{\delta \beta}{\beta}\right) T^{\tau\tau} = -\delta\beta \int d\sigma (T_{ww} + T_{\overline{ww}}). \tag{51}$$

So the change of free energy density is,

$$\delta f = -\frac{\delta\beta}{\beta^2}\frac{(c+\overline{c})\pi}{12} \Rightarrow f = \frac{(c+\overline{c})\pi}{12\beta}. \tag{52}$$

A puzzling thing is that, if we choose $\sigma \to (1+\epsilon)\sigma$ as the coordinate transformation instead, we will get an answer with opposite sign. Why is $\tau \to (1+\epsilon)\tau$ so special? Why we cannot choose other transformations?

There is another way to directly calculate the ground state energy, which is by doing a Wick rotation to go back to Minkowski space. Then the energy is the spatial integral of $T_{tt} = -T_{\tau\tau}$. So we have,

$$E = \int d\sigma T_{tt} = -\int d\sigma (T_{ww} + T_{\overline{ww}}) = \frac{1}{2\pi}\int d\sigma (T + \overline{T}) = -\frac{(c+\overline{c})\pi}{12L}. \tag{53}$$

For free boson, $c = \overline{c} = 1$, $E = -\pi/6L$, which coincides with the direct calculation.

3 Radial Quantization

General comments on quantization If we only care about correlation function, path integral is enough. However, we want to add more constraints to our theory, such as that the Hilbert space is self-consistently defined (i.e. it only has non-negative norm states), the evolution is unitary, the operator content is finite. These constraints are easier to study with quantization.

Quantization has two benefits: (1) it promotes the fields to operators so that we can naturally define algebras on them; (2) by studying the representation of the algebras, we can define a Hilbert space, on which we can easily put unitarity constraint[10].

To do quantization, we have to choose a direction to be the time and other orthogonal directions as the space. The Hilbert space lives on the spatial slides, on which we also define operators and their

[9] Remember that $T(z) = -2\pi T_{zz}$.

[10] Here unitarity has two meanings: (1) the representation is unitary, i.e. it only contains non-negative-norm states; (2) the time evolution is unitary, i.e. the Hamiltonian is Hermitian. The first is the requirement of defining an inner product and should always be satisfied. The second is a physical requirement, which may not be satisfied.

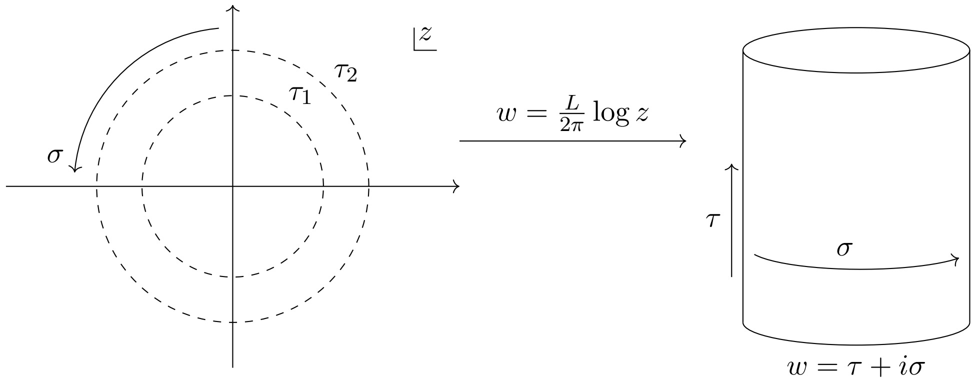


Figure 2: Mapping from plane to a cylinder. The radial direction on the plane becomes the longitudinal direction on the cylinder.

algebra (so called *equal-time commutator*). The evolution is defined by the transfer matrix along time direction.

In Minkowski path integral, the choice of time is restricted. And different choices are connected by a unitary transformation (i.e. a boost). In Euclidean path integral, the time direction can be arbitrarily defined. In CFT, the most convenient choice is *radial quantization*.

3.1 Quantization procedure

Radial quantization is done on a plane. It is to choose the radial direction as our time and the concentric circles as the spatial slides. A physical way to understand radial quantization is to map the plane onto a cylinder via, $w = \tau + i\sigma = \frac{L}{2\pi} \log z$, as shown in Fig.2. Then the radial quantization becomes the canonical quantization on the cylinder. We will frequently use this mapping to get physical intuition.

After defining time, all the correlation functions of fields now become time-ordered correlation of operators. Because time is just the radius, we call it *radial ordering* here. Now we want to use quantization to re-interpret the known results, i.e. those OPE relations and conformal Ward Identity.

The OPE relations must be understood as being inserted into correlation function. So if we want to interpret them as some operator equation, the operators have to be automatically radially ordered. So we have,

$$\mathcal{R}\{O_A(z)O_B(w)\} \sim \sum_C (z-w)^{-h_A-h_B+h_C} O_C(w), \tag{54}$$

$$\text{where, } \mathcal{R}\{O_A(z)O_B(w)\} = \begin{cases} O_A(z)O_B(w), & \text{if } |z| > |w|; \\ O_B(w)O_A(z), & \text{if } |w| > |z|. \end{cases} \tag{55}$$

$\mathcal{R}$ means radial ordering. In the following discussion, we will not write it explicitly but always implicitly assume radial ordering for all the operator equations.

Now we make a digression to discuss one type of operator that is of great importance, defined as,

$$A = \oint_C dz a(z), \tag{56}$$

where $a(z)$ is a local operator and the contour C is a circle centered at the origin. So we can think of A as a spatial integral of a local quantity at a certain time. It is useful to look at its *equal-time*

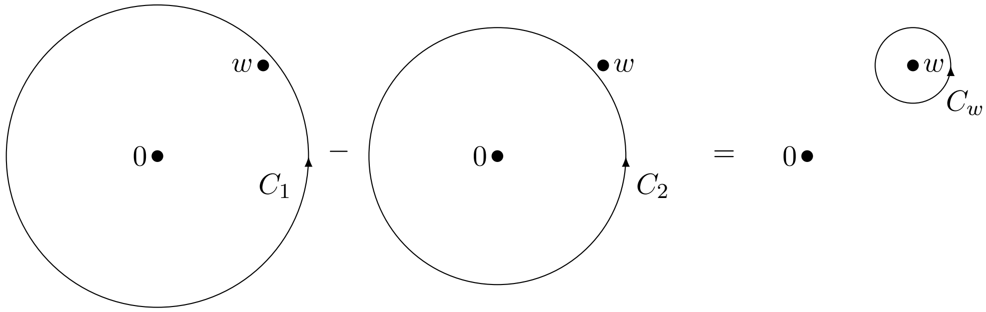


Figure 3: Deform the contour to satisfy the radial ordering. The possible singularities at the original point will cancel out by the subtraction. So the net effect is a contour integral around w.

commutator with another local operator,

$$[A, b(w)] = Ab(w) - b(w)A. \tag{57}$$

To know its value, we have to insert it into a correlation function while keep the order unchanged. So when inserting $Ab(w)$, we need to choose the radius of C to be $|w| + \epsilon$; when inserting $Ab(z)$ we need to choose the radius as $|w| - \epsilon$.[11] As shown in Fig.3. And we get,

$$[A, b(w)] = \oint_{C_1} dza(z)b(w) - b(w)\oint_{C_2} dza(z) = \oint_{C_w} dza(z)b(w). \tag{58}$$

The last form is what usually appears in the path integral formalism. Similarly, we can also calculate the commutator between A and $B = \oint_C dwb(w)$,

$$[A, B] = \oint_C dw \oint_{C_w} dza(z)b(w). \tag{59}$$

After this preparation, we're ready to see how to interpret the conformal Ward Identity,

$$\delta_\epsilon \phi(w) = \frac{-1}{2\pi i}\oint_{C_w} dz\epsilon(z)T(z)\phi(w). \tag{60}$$

The R.H.S is exactly what we see in Eqn.58. So the conformal Ward Identity can be interpreted as an equal-time commutator,

$$Q_\epsilon = \frac{1}{2\pi i}\oint_C dz\epsilon(z)T(z), \quad \delta_\epsilon\phi(w) = -[Q_\epsilon, \phi(w)]. \tag{61}$$

So Q_ϵ is the charge corresponding to conformal symmetry, which generates conformal transformation[12]. If we expand $\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}$, we can get,

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n, \quad L_n = \frac{1}{2\pi i}\oint_C dz z^{n+1}T(z). \tag{62}$$

L_n is the generators of local conformal transformation in the operator space,

$$[L_n, \phi(w)] = w^{n+1}\partial\phi(w) + (n+1)hw^n\phi(w). \tag{63}$$

[11] $\epsilon \to 0^+$ so that we can guarantee the radial ordering and don't affect other operators in the correlator.

[12] This is reminiscent to the normal case: once we have a conserved correct j_μ, its spatial integral $\int dxj_0$ is the charge generating corresponding symmetry transformation.

It's easy to see that L_{-1} generates translation, L_0 generates dilation and rotation, L_1 generates SCT. And L_n forms a mode expansion of $T(z)$,

$$T(z) = \sum_{z\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \tag{64}$$

which can be verified to be consistent with Eqn.62.

Remember that we also have the anti-holomorphic component, which gives another set of generators $\overline{L}_n$. By using Eqn.59, we can get the commutator among L_n, $\overline{L}_n$,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{65}$$

$$[L_n, \overline{L}_n] = 0, \tag{66}$$

$$[\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} + \frac{\overline{c}}{12}n(n^2-1)\delta_{n+m,0}. \tag{67}$$

which is called Virasoro algebra. This set of algebra applies for any CFT and our our general discussion of Hilbert space will be based on the representation of this algebra.

Remark on boundary CFT If we look at the CFT defined on the upper half plane, the Ward Identity only contains holomorphic component. So we don't have two commuting sets of algebra but only have L_n.

Now let's use the mapping between plane and cylinder to get more physical understanding of L_n. Recall that the energy-momentum tensor transforms as,

$$T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2 \left[T_{pl}z^2 - \frac{c}{24}\right] = \left(\frac{2\pi}{L}\right)^2 \left[\sum_{n\in\mathbb{Z}} L_n e^{-\frac{2\pi}{L}nw} - \frac{c}{24}\right]. \tag{68}$$

where we have plugged in Eqn.64 to get a mode expansion of T_{cyl}. The Hamiltonian on the cylinder is,

$$H = \frac{1}{2\pi}\int_0^L d\sigma(T+\overline{T}) = \frac{2\pi}{L}(L_0 + \overline{L}_0 - \frac{c+\overline{c}}{24}). \tag{69}$$

So we can see that L_0 on the plane is related to Hamiltonian on the cylinder. This makes sense because $L_0 + \overline{L}_0$ generates pure dilation on the plane, which is time translation on the cylinder. Thus, later when we find representation of the Virasoro algebra, we will choose the basis so that L_0 is diagonal because this basis corresponds to the energy eigenstates.

Apart from Hamiltonian, studying Hamiltonian density can also be helpful. In Minkowski spacetime, it is written as,

$$\mathcal{H} = \frac{2\pi}{L^2}\left[\sum_{n\in\mathbb{Z}} L_n e^{-\frac{2\pi}{L}n(it+i\sigma)} + \sum_{n\in\mathbb{Z}} \overline{L}_n e^{\frac{2\pi}{L}n(it+i\sigma)} - \frac{c+\overline{c}}{24}\right]. \tag{70}$$

If we require the time-evolution to be unitary, then $\mathcal{H}$ has to be Hermitian which leads to the constraint,

$$L_n^\dagger = L_{-n}, \quad \overline{L}_n^\dagger = \overline{L}_{-n}. \tag{71}$$

In particular, L_0 and $\overline{L}_0$ are Hermitian by themselves.

3.2 Primary States and Descendants

Now we study the representation of Virasoro algebra. As said above, we look for a basis where L_0 is diagonal. Due to $[L_0, L_n] = -nL_n$, L_n act as raising operators for $n < 0$ and lowering operators for $n > 0$. Because L_0 is related to the Hamiltonian and the energy of a physical system should have a bottom, we require the spectrum of L_0 is bounded from below. We call the state with the lowest eigenvalue a *primary state*,

$$L_0 \left|h\right\rangle = h \left|h\right\rangle , \quad L_n \left|h\right\rangle = 0, \text{ for } n > 0. \tag{72}$$

Because L_0 is Hermitian for a unitary theory, h has to be a real number. By acting $L_{-n}, n > 0$ on it, we can increase the eigenvalue. Those states are the *descendant states* of $\left|h\right\rangle$,

$$\left|h'\right\rangle = L_{-k_1} L_{-k_2} ... L_{-k_n} \left|h\right\rangle , \quad h' = h + k_1 + ... + k_n. \tag{73}$$

And we call the integer $N = k_1 + ... + k_n$ as the level of the descendant state $\left|h'\right\rangle$. States belonging to different levels are orthogonal to each other[13].

At a certain level, there can be many descendants, e.g.

Level	descendant states
0	$\left\|h\right\rangle$
1	$L_{-1} \left\|h\right\rangle$
2	$L_{-1}^2 \left\|h\right\rangle , L_{-2} \left\|h\right\rangle$
3	$L_{-1^3} \left\|h\right\rangle , L_{-1}L_{-2} \left\|h\right\rangle , L_{-3} \left\|h\right\rangle$

In the third level, we don't include $L_{-2}L_{-1} \left|h\right\rangle$ in the list. This is because its difference from $L_{-1}L_{-2} \left|h\right\rangle$ doesn't give us a new state.[14] This is true in general, i.e. changing the ordering or L_{-k} doesn't give us new states. So when writing down a descendant state Eqn.73, we implicitly choose a convention $1 \leq k_1 \leq ... \leq k_n$. In the most general case, the number of states at level N is the partition number $p(N)$, which is generated by the following function,

$$\frac{1}{\varphi(q)} = \prod_{n \in \mathbb{Z}^+} \frac{1}{1 - q^n} = \sum_{N \geqslant 0} p(N) q^N. \tag{74}$$

Verma Module The collection of a primary and all of its descendants form a representation of the Virasoro algebra, which is called a *Verma module* $\mathcal{V}(c, h)$. Similarly, $\{\overline{L}_n\}$ will also generates a set of Verma module $\overline{\mathcal{V}}(\overline{c}, \overline{h})$. And the whole Hilbert space is the direct sum of different Verma module,

$$\mathcal{H} = \sum_{h, \overline{h}} \mathcal{V}(c, h) \otimes \mathcal{V}(\overline{c}, \overline{h}). \tag{75}$$

The sum could be finite or infinite and in principle could contain several terms with the same conformal weight.

Because this is a Hilbert space, we have the definition of inner product. So we need to add another constraint that all the states have be have non-negative norm,

$$||L_{-n} \left|h\right\rangle ||^2 = (2nh + \frac{c}{12} n(n^2 - 1)) \left\langle h|h\right\rangle \geq 0, \tag{76}$$

where we have used Eqn.71. This leads to the following requirement,

$$h \geq 0, \quad c \geq 0. \tag{77}$$

[13] Eigenstates of a Hermitian operator are orthogonal if they have different eigenvalues.

[14] This is easy to verify using commutator $L_{-2}L_{-1} \left|h\right\rangle = L_{-1}L_{-2} \left|h\right\rangle - L_{-3} \left|h\right\rangle$.

Remark We give few comments on the results:

- The $h = 0$ state is called the *vacuum*. And for each CFT, we *assume that there is only one unique vacuum*. One can show that the vacuum is not only annihilated by L_0, L_1, but also L_{-1}, which means that the vacuum state is $SL(2,\mathbb{C})$ invariant.

- The $c = 0$ theory is completely trivial. First, for the descendants of $|0\rangle$, all of them have zero norm, so they are all zero. Second, a $h > 0$ primary states $|h\rangle$ cannot expand a unitary representation. This can be verified by studying the matrix of inner product of $L_{-n}^2 |h\rangle$, $L_{-2n}^2 |h\rangle$,

$$\begin{pmatrix} (4nh + 2n^2)2nh & 6n^2h \\ 6n^2h & 4nh \end{pmatrix}$$

  the determinant of which is $4n^3h^2(8h - 5n)$. So given h, it is negative for large enough n. This means that there is always a negative norm state. So the only allowed representation is $|0\rangle$ with $L_n |0\rangle = 0$ for all n, which is a trivial representation.

- Although there are $p(N)$ states at level N, it is not guaranteed that all these states are linearly independent. When c and h are chosen appropriately, it is possible that at certain level, a linear combination of states is zero, which is called a *null state*. E.g. at level 2, there are two states $L_{-1}^2 |h\rangle$ and $L_{-2} |h\rangle$. We assume the following state is a null state,

$$|\chi\rangle = (L_{-2} + aL_{-1}^2) |h\rangle = 0.$$

  So we have $L_1 |\chi\rangle = L_2 |\chi\rangle = 0$, which tells us,

$$a = -\frac{3}{2(2h+1)}, \quad c = \frac{2h(5-8h)}{2h+1}.$$

  For a general level n, the detection of null states is done by studying the matrix of inner products at that level. Its determinant is called *Kac determinant*. If Kac determinant is zero, there exist null states. And the eigenvector with zero eigenvalue gives that state. However, this null state may be a descendant of a null state at some lower level. Because the descendants of a null state are still null. Once we get a new null state $|\chi\rangle$ at level n, it will give $p(N-n)$ null states at level N. So the Kac determinant at level N at least contains a $p(N-n)$ order zero point.

  E.g. $|h = 0\rangle$ gives a null state at level 1 ($L_{-1} |0\rangle = 0$). So we expect to see a order-1 zero point at $h = 0$ for the level-2 Kac determinant. Indeed, the matrix at level 2 is,

$$M_2(c,h) = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix} \tag{78}$$

  which gives a determinant,

$$\det M_2(c,h) = (h - h_{1,1})(h - h_{1,2})(h - h_{2,1}), \tag{79}$$

  where $h_{1,1} = 0$ comes from the level 1 null state and $h_{1,2}, h_{2,1} = \frac{5-c}{16} \mp \frac{1}{16}\sqrt{(1-c)(25-c)}$ correspond to the new null states.

3.3 Primary Fields and Descendants

A natural question is: apart from the vacuum state, how to find other primary states. As the name suggests, those states can be related to the corresponding primary field operators. Once we have a primary operator $\phi(z)$ of conformal weigh $(h, 0)$, we can use it to get a primary state $|h\rangle$,

$$|h\rangle = \lim_{z\to 0} \phi(z) |0\rangle . \tag{80}$$

One can verify this statement by using commutator Eqn.63,

$$\begin{aligned} &L_n\phi(w)|0\rangle = [L_n, \phi(w)]|0\rangle = (w^{n+1}\partial\phi(w) + (n+1)hw^n\phi(w))|0\rangle, n \geq 0 \\ &\Rightarrow L_0\phi(0)|0\rangle = h\phi(0)|0\rangle, \quad L_n\phi(0)|0\rangle = 0, n \geq 0, \end{aligned}$$

where we have used the fact $L_n |0\rangle$ for $n \geq 0$.

What's more, the reverse of the statement above is also correct, i.e. once we have a primary state, there is always a primary field, which implies a one-to-one correspondence. This is called *state-operator correspondence*, which will be discussed in the next section.

The form of descendant states $L_{-n}\phi(0)|0\rangle$ also inspires us to define a *descendant field*,

$$(L_{-n}\phi)(w) = \frac{1}{2\pi i}\oint_{C_w} dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w), n \geq 0, \tag{81}$$

which is quasi-primary with conformal weight $(h+n, 0)$. One can do some consistent check that

- its $w = 0$ limit does generate descendant state,

$$(L_{-n}\phi)(0)|0\rangle = \frac{1}{2\pi i}\oint_C dz \frac{1}{z^{n-1}} T(z)\phi(0)|0\rangle = L_{-n}\phi(0)|0\rangle .$$

- $L_n\phi(w)$ gives zero if $n > 0$ due to the $T\phi$ OPE, which is consistent with L_n annihilating primary states.

Actually, these descendant fields naturally appears in the $T\phi$ OPE,

$$T(z)\phi(w) = \frac{1}{(z-w)^2}(L_0\phi)(w) + \frac{1}{z-w}(L_{-1}\phi)(w) + (L_{-2}\phi)(w) + ...$$

So $L_0\phi = h\phi$, $L_{-1}\phi = \partial\phi$. Another simple example is the level-2 descendant of $\mathbb{I}$,

$$(L_{-2}\mathbb{I})(w) = \frac{1}{2\pi i}\oint_{C_w} dz \frac{1}{z-w} T(z)\mathbb{I} = T(w). \tag{82}$$

The correlation function between descendant fields are completely determined by the correlation between primaries. Suppose $\phi(w)$ and X are some primaries with a known correlation $\langle\phi(w)X\rangle$. Then $\langle(L_{-n}\phi)(w)X\rangle$ is given by,

$$\langle(L_{-n}\phi)(w)X\rangle = \mathcal{L}_{-n}\langle\phi(w)X\rangle, \quad \mathcal{L}_{-n} = \sum_i \frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}. \tag{83}$$

Conformal Family The collection of a primary field and its descendants are called a *conformal family* $[\phi]$.

As what we discussed for Verma module, there are also $p(N)$ descendants at a given level N. Those descendants may not be independent. If a linear combination gives zero, it is called a *null field*. The existence of null field not only restrict the choice of c, h, but also tells us the correlation function between primaries should satisfies a differential equation. This is because χ has zero correlation with other fields. If we know $\chi = \sum_i c_i L_{-k_i}\phi$, by Eqn.83, we can derive the correlation between primaries should satisfy,

$$\sum_i c_i \mathcal{L}_{-k_i} \langle \phi X \rangle = 0. \tag{84}$$

In a CFT, all the primaries and quasi-primaries belong to certain conformal family. So we can use conformal family to rewrite our OPE between any two quasi-primary fields,

$$\phi_i(z,\overline{z})\phi_j(w,\overline{w}) = \sum_p \sum_{k,\overline{k}} C_{ij,p}^{\{k,\overline{k}\}} z^{h_p-h_i-h_j+k} \overline{z}^{\overline{h}_p-\overline{h}_i-\overline{h}_j+\overline{k}} \phi_p^{\{k,\overline{k}\}}(w,\overline{w}), \tag{85}$$

where p labels primary field ϕ_p and $\phi^{\{k,\overline{k}\}} = L_{-k_1}..L_{-k_n}\overline{L}_{-\overline{k}_1}\overline{L}_{-\overline{k}_m}\phi_p$, $k = \sum_l k_l$ labels different states at level k. This is called *operator algebra*, which encodes all the information of correlation functions.

Because the correlation function of a descendant field $\phi_p^{\{k,\overline{k}\}}$ can be determinant by the correlation of its primary ϕ_p, we expect $C_{ij,p}^{\{k,\overline{k}\}}$ can be written as the following form,

$$C_{ij,p}^{\{k,\overline{k}\}} = C_{ij,p}\, \beta_{ij}^{p,\{k\}} \beta_{ij}^{p,\{\overline{k}\}}, \tag{86}$$

where $C_{ij,p}$ is the OPE coefficient for the primary ϕ_p, β's can be calculated using Eqn.83 or some equivalent methods.

So to solve a CFT, the necessary data are: central charge c, all the primary fields h and their OPE relation $C_{ij,p}$.

3.4 Operator State Correspondence

understand the operator-state correspondence in the future, lol.

4 Unitarity

If we only impose conformal symmetry without any other structure, then the only algebra we have is the *Virasoro algebra*,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{87}$$

Its representation is called *Verma Module* $V(c,h)$, which is composed of a highest weight state $|h\rangle$ and its descendants. The whole Hilbert space is constructed from the direct product and direct sum of different Verma modules,

$$\mathcal{H} = \sum_{h,\overline{h}} V(c,h) \otimes V(\overline{c},\overline{h}). \tag{88}$$

But the representations are not guaranteed to be unitary. We have already seen that by requiring $\langle h|L_n L_{-n}|h\rangle \geq 0$ we have $c > 0, h > 0$. However, this is just a necessary condition. At a certain level N, it is still possible to find a certain linear combination of descendant states which has a negative norm. Such a thing can be captured by the *Gram matrix*,

$$M_{ij} = \langle i|j\rangle\,, \quad |i\rangle \in \text{level N}. \tag{89}$$

If M_{ij} only has positive eigenvalues, then the representation is unitary. If it has zero eigenvalue or equivalently $\det M = 0$, then there exists singular(null) vector. If there is negative eigenvalue, then the representation is non-unitary. Detailed study gives the following results:

- $c > 1$, all possible CFTs are unitary.
- $c = 1$, unitary or not depends on the concrete theory. We have to do case-by-case study.
- $0 < c < 1$, most CFTs are non-unitary. The unitary ones have to satisfy the following conditions,

$$c = 1 - \frac{6}{m(m+1)}, \tag{90}$$

$$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \tag{91}$$

  where $m = 3, 4, 5...$ and $0 < r < m, 0 < s < m+1$ are two integers. And each theory only has $m(m-1)/2$ different primaries. So they belongs a minimal models.

In the following, we will detail the proof of these conclusions.

As we said before, to get the unitarity structure, we have to figure out when the Gram matrix is positive definite. However, studying M_{ij} for a general Verma Module $V(c, h)$ is very hard. We can start with looking at its determinant, called *Kac determinant*, which though only gives necessary conditions. The Kac determinant for level N is a polynomial for h and has the following form,

$$\det M_N(c, h) = \alpha_N \prod_{r,s=1}^{rs \leq N} (h - h_{r,s})^{p(N - rs)}, \tag{92}$$

where $\alpha_N > 0$ is not relevant to our discussion here and zero points $h_{r,s}$ indicates the existence of singular vector when h takes that value. And we justify this expression in two aspects:

- We label the zero points with subindex r, s which means this singular vector at most can first show up at level rs. We know that a singular vector at level rs will generates $p(N - rs)$ different singular vector at level N. That explains the power $p(N - rs)$.
- We can also check that this formula gives the correct order. The term with highest power comes from the product of diagonal terms of Gram matrix. Each diagonal term takes the form $|L_{-n_1} L_{-n_2} ... L_{n_k} |h\rangle|^2$, which gives a contribution h^k. So the order of $\det M_N(c, h)$ is $\nu_N = \sum_{n_1 + ... n_k = N} k$, the summation is taken over different partitions of N. However Eqn.92 suggests the order to be $\sum_{r,s} p(N - rs)$. So we must have,

$$\nu_N = \sum_{n_1 + ... n_k = N} k = \sum_{r,s=1}^{rs \leq N} p(N - rs). \tag{93}$$

  We save the proof of this identity in Appendix.D.

Without proof, we give the formula of $h_{r,s}$. It has a simple form if we parametrize the central charge with a complex number in the following way,

$$c = 1 - \frac{6}{m(m+1)}, \text{ or } m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25 - c}{1 - c}} \tag{94}$$

$$h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}. \tag{95}$$

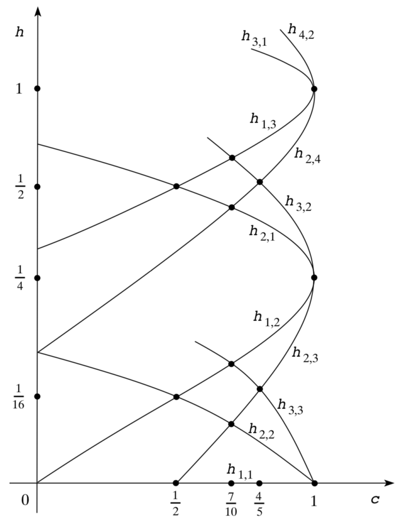


Figure 4: Vanishing curves for $c < 1$. One can see that $h_{r,s}(c)$ starts at $c = 1, \frac{(r-s)^2}{4}$ and has the asymptotic form $h_{r,s}(1-6\epsilon) = \frac{1}{4}[(r-s)^2 + (r^2 - s^2)\sqrt{\epsilon}]$ for $r \neq s$ and $h_{r,r}(1-6\epsilon) = \frac{\epsilon}{4}(r^2-1)$

The formula for $h_{r,s}$ has a symmetry $h_{r,s} = h_{m-r,m+1-s} = h_{m+r,m+1+s}$. The choice of branch for m is not important. One can show that $h_{p,q}(m_+) = h_{q,p}(m_-)$, so the choice of branches won't change the value of $\det M$. In the following discussion, we pick up the branch that is convenient to us.

Using these formula (so called Kac formula), we can now discuss the unitary property. (See [5] for more details)

- $c > 1, h > 0$. We pick up the "$-$" branch. One can prove that

  1. there is no zero point in this region (When $1 < c < 25$ m is complex. When $c > 25$, $-1 < m < 0$ which implies $h_{r,s} < 0$);
  2. all the eigenvalues are positive when h is large enough. This is because when $h \to \infty$, the eigenvalues are equal to the diagonal elements of the Gram matrix. (The corrections coming from the off-diagonal terms is of order $O(h^{-1})$ under perturbation theory) and the diagonal elements are all positive.

  So all the eigenvalues are positive in this region, which means the Gram matrix is positive definite. So all *the CFT constructed from Virasoro algebra in this region are unitary.*

- $c = 1, h > 0$, $h_{r,s} = (r-s)^2/4$. So there can be singular vectors but it doesn't forbid us to have unitary theory (like free boson). So in principle, we can have unitary CFT for $c \geq 1$.

- $c < 1$. We pick up the branch so that $m > 0$. For each (r, s) pair, there is a curve $h(c) = h_{r,s}(c)$ where $\det M = 0$ as shown in Fig.4. By using the fact that $\det M > 0$ at $c > 1$, we can show those point not sitting on these vanish curves have $\det M < 0$ thus giving non-unitary theories. For the points sitting on the vanishing curve, people further show that they are non-unitary. The only exceptional points are those crossing points of different vanishing curve. To touch the crossing points, we have to require m to be an integer and we get the unitary minimal models.

5 Conformal blocks

Due to the Virasoro algebra, conformal invariance becomes very powerful in a 2D CFT. However, in general this symmetry itself is still not adequate. We need more dynamical input. The necessary data to completely determine a CFT are:

- central charge c, which describes the degrees of freedom;
- all the primary fields and their conformal weights $(h, \bar{h})$. The choice for this data may *not* be independent of the value of c. For example, if we require unitarity and $c < 1$, then the possible choice is completely fixed by c. And we will take the following normalization for our primaries:

$$\langle \phi_i(z_i, \bar{z}_i)\phi_j(z_j, \bar{z}_j)\rangle = \frac{\delta_{ij}}{z_{ij}^{h_i+h_j}\bar{z}_{ij}^{\bar{h}_i+\bar{h}_j}}. \tag{96}$$

- OPE coefficient C_{ij}^k for the primaries, which is defined as,

$$\phi_i(z)\phi_j(0) = C_{ij}^k \phi_k(0) + \text{ descendant of } \phi_k \tag{97}$$

  The knowledge of C_{ij}^k guarantees us to solve the coefficients for all the following terms and the OPE coefficients for descendants. We will take about this in detail below.

Remarks If we just assign some arbitrary numbers to these quantities, the theory we get is in general not self-consistent. So when assigning values for those quantities, we also need to remember checking the consistency conditions: compatibility of OPEs, associativity of OPEs (which is equivalent to crossing symmetry), modular invariance if we want to put it on a torus, and so on. In the following, we will always assume that the data we use are self-consistent.

Sometimes, the value of c and h can automatically take care of the value of C_{ij}^k. For example, in a minimal model, each primary field $\phi_{(r,s)}$ has a null descendant field at level rs. This information combined with monodromy and boundary conditions can determine the four-point functions, which in turn tells us all the C_{ij}^k's. (It is quite surprising how c, h can automatically give us a self-consistent OPE relation. I guess the answer is related to using truncating operator algebra to determine the minimal models. Need more understanding.)

5.1 Operator Algebra

The information needed to write down all the correlation functions is called *operator algebra*. The most essential part of it is the OPE coefficients for the primary fields C_{ij}^k we defined before. Now, we are going to show that how it determines the correlation functions. We will show it in four steps: (1) it determines the three point functions for primaries, (2) it determines the OPE between primary fields, which determines the correlation between primaries, (3) the correlation between primaries determines the correlation between descendants, (4) from which we can get the OPE for descendants.

5.1.1 Three point functions

Based on conformal invariance, the three point function is determined up to a coefficient[15],

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}}. \tag{98}$$

[15]For simplicity, we just drop the anti-holomorphic component when it is not important to our discussion.

By sending $z_3 \to \infty$, we have,

$$\langle h_3|\phi_1(1)|h_2\rangle = \lim_{z_3\to\infty} z_3^{2h_3} \langle \phi_3(z_3)\phi_1(1)\phi_2(0)\rangle = (-1)^{2h_3+2\bar{h}_3} C_{123}. \tag{99}$$

By using the OPE, this quantity can be computed in another way. We first do the OPE between ϕ_1 and ϕ_2,

$$\phi_1(z)\phi_2(0) = \sum_{p,K} C_{12}^{p,\{K\}} z^{h_p-h_1-h_2+K} \phi_p^{\{K\}}(0),$$

then compute the 2pt function between ϕ_3 and the results. However, different primaries are orthogonal at 2pt function level so we only need to consider the conformal family $[\phi_3]$. In the limit $z_3 \to \infty$, only the most non-singular term is preserved and we have

$$\lim_{z_3\to\infty} z_3^{2h_3} \langle \phi_3(z_3)\phi_1(1)\phi_2(0)\rangle = C_{12}^3. \tag{100}$$

So the first OPE coefficient for primaries is exactly the coefficient for 3pt functions up to a phase factor,

$$C_{12}^3 = (-1)^{2h_3+2\bar{h}_3} C_{123}, \tag{101}$$

which proves C_{12}^3 determines the three point functions for primaries. In a three point function, the order of ϕ_1 and ϕ_2 is not important. So this result also implies $C_{12}^3 = C_{21}^3$.

5.1.2 Complete OPE for primaries

The OPE between two primaries is an infinite series, which not only contains primaries but also their descendants. However, because the descendants are built by applying Virasoro generators on the primaries, the coefficients for descendants can be completely determined by the coefficients for the corresponding primary. And we will expect the following form,

$$C_{12}^{p,\{K,\overline{K}\}} = C_{12}^p \beta_{12}^{p,\{K\}} \overline{\beta}_{12}^{p,\{\overline{K}\}}. \tag{102}$$

where β's are just some functions of c and h. We will now prove this statement by explicitly constructing β's from conformal invariance.

In terms of operator-state correspondence, we can write the OPE between two primaries ϕ_1 and ϕ_2 in the following way,

$$\phi_1(z)\,|h_2\rangle = \sum_p C_{12}^p z^{h_p-h_1-h_2} \sum_{\{K\}} \beta_{12}^{p,\{K\}} z^K L_{-\{K\}}\,|h_p\rangle \tag{103}$$

where $K = \sum_i k_i$ is the level and $\{K\}$ means the collection of all possible inequivalent choices. For the later convenience, we define the summation of all the level-K states as

$$|K, h_p\rangle = \sum_{\{N\},N=K} \beta_{12}^{p,\{N\}} L_{-\{N\}}\,|h_p\rangle\,. \tag{104}$$

Now, we apply $L_n, n > 0$ on both sides of Eqn.103. This is equivalent to saying they should transform in the same way under an infinitesimal conformal transformation. One the one hand, we have,

$$L_n\phi_1(z)\,|h_2\rangle = \sum_p C_{12}^p z^{h_p-h_1-h_2} \sum_K z^K\,|K, h_p\rangle\,.$$

On the other hand, because ϕ's are primary, we have,

$$
\begin{aligned}
L_n\phi_1(z)\left|h_2\right\rangle &= [L_n, \phi_1(z)]\left|h_2\right\rangle = \left(z^{n+1}\partial_z + (n+1)h_1 z^n\right)\phi_1(z)\left|h_2\right\rangle \\
&= \sum_p C_{12}^p z^{h_p-h_1-h_2}\sum_K z^{n+K}[h_p + nh_1 - h_2 + K]\left|K, h_p\right\rangle .
\end{aligned}
$$

These two different approaches should yield the same answer, which requires the following relation,

$$
L_n\left|n+K, h_p\right\rangle = [h_p + nh_1 - h_2 + K]\left|K, h_p\right\rangle . \tag{105}
$$

One can use this relation recursively combined with Virasoro algebra to get all the β's. We will work out the results for the first two levels.

Level-1 Take $n = 1, K = 0$, we have,

$$
\begin{aligned}
L_1\left|1, h_p\right\rangle = (h_p + h_1 - h_2)\left|h_p\right\rangle = \beta_{12}^{p,\{1\}}L_1L_{-1}\left|h_p\right\rangle &= \beta_{12}^{p,\{1\}}2h_p\left|h_p\right\rangle . \\
\Rightarrow \beta_{12}^{p,\{1\}} &= \frac{h_p + h_1 - h_2}{2h_p}.
\end{aligned} \tag{106}
$$

One may notice that the result is asymmetric to $1, 2$. It actually doesn't have to be because $\phi_1(z)\phi_2(0)$ $\phi_2(z)\phi_1(0)$ are just two different things. *The fusion rule has to be symmetric but the OPE coefficients don't.*

Level-2 There are two descendants at level-2, $\left|2, h_p\right\rangle = \beta_{12}^{p,\{1,1\}}L_{-1}^2\left|h_p\right\rangle + \beta_{12}^{p,\{2\}}L_{-2}\left|h_p\right\rangle$. One can either apply L_1^2 or L_2 to reduce it to $\left|h_p\right\rangle$. By using Eqn.105, we have,

$$
\begin{pmatrix} L_1^2 \\ L_2 \end{pmatrix}\left|2, h_p\right\rangle = \begin{pmatrix} (h_p + h_1 - h_2 + 1)(h_p + h_1 - h_2) \\ 2h_1 - h_2 + h_p \end{pmatrix}\left|h_p\right\rangle .
$$

By using the Virasoro algebra, we have,

$$
\begin{pmatrix} L_1^2 \\ L_2 \end{pmatrix}\left|2, h_p\right\rangle = \begin{pmatrix} L_1^2 \\ L_2 \end{pmatrix}\begin{pmatrix} L_{-1}^2 & L_{-2} \end{pmatrix}\left|h_p\right\rangle\begin{pmatrix} \beta_{12}^{p,\{1,1\}} \\ \beta_{12}^{p,\{2\}} \end{pmatrix} = \begin{pmatrix} 4h_p(1+2h_p) & 6h_p \\ 6h_p & \frac{c}{2} + 4h_p \end{pmatrix}\left|h_p\right\rangle\begin{pmatrix} \beta_{12}^{p,\{1,1\}} \\ \beta_{12}^{p,\{2\}} \end{pmatrix} .
$$

The equivalence between this two results gives us the following equations,

$$
\begin{pmatrix} 4h_p(1+2h_p) & 6h_p \\ 6h_p & \frac{c}{2} + 4h_p \end{pmatrix}\begin{pmatrix} \beta_{12}^{p,\{1,1\}} \\ \beta_{12}^{p,\{2\}} \end{pmatrix} = \begin{pmatrix} (h_p + h_1 - h_2 + 1)(h_p + h_1 - h_2) \\ 2h_1 - h_2 + h_p \end{pmatrix} \tag{107}
$$

Notice that matrix on the right-hand side is just the Gram matrix, which can be degenerate. For now, we will assume it to be non-degenerate and use concrete examples to discuss the degenerate case later. Under this assumption, it is easy to get the solutions but we won't list it here. It is straightforward to generalize the calculation above to higher orders, which is also programmable. During the whole process, we only use the conformal symmetry for ϕ and Virasoro algebra. So no matter at which level, β will only depend on c and h.

Examples Before we move on to the next step, it is good to see some concrete examples:

- Free boson $c = 1$. Let's first see a non-degenerate case and $h_1 = h_2$. We choose vertex operator $V_\alpha(z,\overline{z}) =: e^{i\sqrt{2}\alpha\varphi(z,\overline{z})}:$, whose conformal weight is $h = \alpha^2$.[16] Due to the symmetry $\varphi \to \varphi + a$, there is only one fusion channel, $V_\alpha \times V_\alpha = V_{2\alpha}$. Plugging the central charge and conformal weight into the formula above, we get,

$$\frac{V_\alpha(z)V_\alpha(0)}{z^{2\alpha^2}} = V_{2\alpha} + \frac{z}{2}L_{-1}V_{2\alpha} + \frac{z^2}{1-16\alpha^2}\left[\frac{1-8\alpha^2}{4}L_{-1}^2 - 2\alpha^2 L_{-2}\right]V_{2\alpha} + ... \tag{108}$$

- Free Majorana fermion $c = 1/2$. Now let's see a degenerate case and $h_1 = h_2$. We choose $\psi(z)$ whose conformal weight is $h = 1/2$ and look at the identity channel. From the two point function $\langle\psi(z)\psi(0)\rangle = 1/z$, we know that $C^0_{1/2,1/2} = 1$. This time the formula for $\beta_{12}^{p,\{1\}}$ becomes singular. However, $L_{-1}\mathbb{I}$ is a null vector, which doesn't have any non-vanishing correlation with other fields. So we can just choose $\beta_{12}^{p,\{1\}} = 0$. The same thing applies for the coefficient of any descendant of $L_{-1}\mathbb{I}$. So we can choose $\beta_{12}^{p,\{1,1\}} = 0$ at level-2. Although, Eqn.107 looks singular, it still gives us a solution up to an arbitrary component of null vector. Finally, we will get,

$$\psi(z)\psi(0) = \frac{1}{z}\mathbb{I} + 2zL_{-2}\mathbb{I} + ... \tag{109}$$

- Ising CFT $c = 1/2$. This time let's look at a slightly more complicated example. We have two primaries ϵ and σ whose conformal weights are $h_\epsilon = 1/2$, $h_\sigma = 1/16$ and fusion rules are $\epsilon\times\epsilon = \mathbb{I}$, $\sigma\times\sigma = \mathbb{I} + \sigma$, $\sigma\times\epsilon = \sigma$.

  If we care about the $\sigma\times\sigma$ OPE. For the $\mathbb{I}$ channel, we just repeat what we did for fermion. For ϵ channel, at level-1, we get $\beta_{12}^{p,\{1\}} = 1/2$. At level-2, we have a degenerate Gram matrix,

$$\begin{pmatrix} 4 & 4 \\ 3 & 9/4 \end{pmatrix}\begin{pmatrix} \beta_{12}^{p,\{1,1\}} \\ \beta_{12}^{p,\{2\}} \end{pmatrix} = \begin{pmatrix} 3/4 \\ 9/16 \end{pmatrix} \Rightarrow \beta_{12}^{p,\{1,1\}} = 0, \beta_{12}^{p,\{2\}} = 1/4.$$

  The answer is again determined up to a null vector. So we get,

$$\sigma(z)\sigma(0) = \frac{1}{z^{1/8}}\left(\mathbb{I} + \frac{z^2}{4}L_{-2}\mathbb{I} + ...\right) + C^\epsilon_{\sigma\sigma}z^{3/8}\left(\epsilon + \frac{z}{2}L_{-1}\epsilon + \frac{z^2}{4}L_{-2}\epsilon + ...\right). \tag{110}$$

5.1.3 Correlation for descendants

Now we want to demonstrate that the correlation functions for primaries determine the correlation for descendants. Because the descendants are got from doing contour integral of $T(w)\phi(z)$, it is equivalent to prove that the following correlation,

$$\langle T(w_1)T(w_2)...T(w_M)\phi_1(z_1)\phi_2(z_2)...\phi_N(z_N)\rangle \tag{111}$$

is determined by $\langle\phi_1(z_1)\phi_2(z_2)...\phi_N(z_N)\rangle$, where ϕ's are primaries. We prove this statement by induction.

For $M = 1$, we know this is true. We assume this is also true for $M-1$ $T(w)$ insertions. Then for M $T(w)$ insertions, we define the following quantity,

$$f(w) = \langle T(w)T(w_2)...T(w_M)\phi_1(z_1)\phi_2(z_2)...\phi_N(z_N)\rangle\,.$$

[16] We follow the convention of the yellow book.

We further assume that $f(w)$ won't have singularities unless w meets other fields and $w = \infty$ is not an essential singularity. This means $f(w)$ is a meromorphic function, which has the following decomposition,

$$f(w) = \text{singularities} + \text{const.} \tag{112}$$

The constant is fixed by $f(w \to \infty)$. To probe that value, we can do a conformal transformation

$$w \to w' = 1/w, \quad f'(w') = \left(\frac{dw'}{dw}\right)^{-2} \# f(w) \propto f(w)/w^4$$

Because $f'(w')$ is regular at $w' = 0$, we get $f(w) = 0$ at $w = \infty$, which means $f(w)$ only has singular term. So we only need to consider the singular part of the OPE between $T(w)$ and other fields to get $f(w)$ from $\langle T(w_2)...T(w_M)\phi_1...\phi_N\rangle$,

$$\begin{aligned} f(w) = \Bigg(&\sum_{j=2}^{M} \left[\frac{2}{(w-w_j)^2} + \frac{1}{w-w_j}\partial_{w_j} + \frac{c/2}{(w-w_j)^4}\right] + \\ &\sum_{j=1}^{N} \left[\frac{h_j}{(w-z_j)^2} + \frac{1}{w-z_j}\partial_{z_j}\right]\Bigg) \langle T(w_2)...T(w_M)\phi_1...\phi_N\rangle\,. \end{aligned} \tag{113}$$

From this, one can also determine all the OPE between descendants. However, one still needs to make sure that all the OPE are self-consistent. (It's not very clear whether a set of self-consistent data for primary will give us a self-consistent OPE for descendants).

5.2 Four point functions

As we said before, the OPE can in principle determine all the correlation functions. It is especially instructive to look at the four point function for primaries. The structure we will get is very useful in other discussions.

5.2.1 Conformal blocks

We take four primary fields $\phi_j, j = 1,2,3,4$. Conformal invariance can fix the four point function of them to the following form,

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \prod_{ij} z_{ij}^{h/3-h_i-h_j} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} f(x,\bar{x}), \tag{114}$$

where $x = z_{12}z_{34}/z_{13}z_{24}$ is the cross ratio and $f(x,\bar{x})$ depends on the dynamical data. We will now get $f(x,\bar{x})$ by using the OPE. Because $f(x,\bar{x})$ only depends on the cross ratio, we can consider the limit $z_1 = \infty, z_2 = 1, z_3 = x, z_4 = 0$ and define the following quantity,

$$G^{12}_{34}(x,\bar{x}) = \lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1} \langle \phi_1(\infty)\phi_2(1)\phi_3(x)\phi_4(0)\rangle = \langle \phi_1|\phi_2(1)\phi_3(x)|\phi_4\rangle\,. \tag{115}$$

And $G^{12}_{34}(x,\bar{x}) = (1-x)^{h/3-h_2-h_3} x^{h/3-h_3-h_4} f(x,\bar{x})$. The OPE between ϕ_3 and ϕ_4,

$$\phi_3(x)\phi_4(0) = \sum_p C^p_{34} x^{h_p-h_3-h_4} \bar{x}^{\bar{h}_p-\bar{h}_3-\bar{h}_4} \Psi_p(x,\bar{x}|0,0) \tag{116}$$

where we define,

$$\Psi_p(x,\overline{x}|0,0) = \sum_{\{K\},\{\overline{K}\}} \beta_{34}^{p,\{K\}} \overline{\beta}_{34}^{p,\{\overline{K}\}} x^K \overline{x}^{\overline{K}} \phi_p^{\{K,\overline{K}\}}(0,0) \tag{117}$$

so that we can rewrite G_{34}^{12} as,

$$G_{34}^{12}(x,\overline{x}) = \sum_p C_{34}^p x^{h_p-h_3-h_4} \overline{x}^{\overline{h}_p-\overline{h}_3-\overline{h}_4} \langle h_1,\overline{h}_1|\phi_2(1,1)\Psi_p(x,\overline{x}|0,0)|0\rangle\,. \tag{118}$$

We notice that $\langle h_1|\phi_2(1,1)\Psi_p(x,\overline{x}|0,0)|0\rangle$ must be proportional to C_{12}^p because we can compute it by doing OPE between ϕ_1 and ϕ_2. Thus we can just extract this factor and write the result as,

$$G_{34}^{12}(x) = \sum_p C_{34}^p C_{12}^p A_{34}^{12}(p|x,\overline{x}), \tag{119}$$

where,

$$A_{34}^{12}(p|x,\overline{x}) = (C_{12}^p)^{-1} x^{h_p-h_3-h_4} \overline{x}^{\overline{h}_p-\overline{h}_3-\overline{h}_4} \langle h_1,\overline{h}_1|\phi_2(1,1)\Psi_p(x,\overline{x}|0,0)|0\rangle$$

only depends on c and h and can be completely determined by conformal symmetry. And from Eqn.117, we find that A_{34}^{12} can be factorized as,

$$A_{34}^{12}(p|x,\overline{x}) = \mathcal{F}_{34}^{12}(p|x)\overline{\mathcal{F}}_{34}^{12}(p|\overline{x}), \tag{120}$$

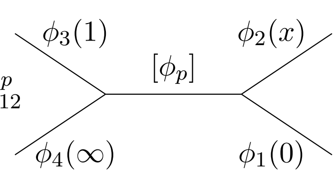
where $\mathcal{F}_{34}^{12}$ is an infinite power series[17],

$$\mathcal{F}_{34}^{12}(p|x) = x^{h_p-h_3-h_4} \sum_{\{K\}} \beta_{34}^{p,\{K\}} x^K \frac{\langle h_1|\phi_2(1)L_{-K}|h_p\rangle}{\langle h_1|\phi_2(1)|\phi_p\rangle}, \tag{121}$$

which is called *conformal blocks.* This quantity is one of the most important quantities in CFT. The G_{34}^{12} can be rewritten in terms of the conformal blocks as,

$$G_{34}^{12}(x) = \sum_p C_{34}^p C_{12}^p \mathcal{F}_{34}^{12}(p|x)\overline{\mathcal{F}}_{34}^{12}(p|\overline{x}). \tag{122}$$

An intuitive way to think about this result is to do ϕ_3, ϕ_4 OPE and ϕ_1, ϕ_2 OPE first and evaluate the correlation between their OPE results. Schematically, people use the following partial wave diagram to represent it,

$$G_{34}^{12}(x|\overline{x}) = \textstyle\sum_p C_{34}^p C_{12}^p$$


Inspired by the partial wave picture, it is appealing to try to rewrite our four point function in the following way,

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle = \sum_p \underbrace{z_{12}^{-h_1-h_2}\overline{z}_{12}^{-\overline{h}_1-\overline{h}_2}}_{\text{12 OPE}} \underbrace{z_{34}^{-h_3-h_4}\overline{z}_{34}^{-\overline{h}_3-\overline{h}_4}}_{\text{34 OPE}} \times \underbrace{F(z_j,\overline{z}_j,x,\overline{x})}_{\text{corrleation for fused channel } [p]} \;.$$

[17] Although it looks like we put $(C_{12}^p)^{-1}$ for both holomorphic and anti-holomorphic part, we don't double count. Because both the numerators for holo and anti-holo also contain C_{12}^p.

However, in general $F(z_j, \overline{z}_j, x, \overline{x})$ is very complicated and depends not only on cross ratio but also on each z_j. Only when $h_1 = h_2 = h_W$, $h_3 = h_4 = h_V$, F can be reduced to a function of cross ratio. And the four point function can be written in terms of the conformal blocks in a very nice form. The calculation is straightforward and left to the Appendix.F. Here we just show the result,

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \frac{1}{z_{12}^{2h_W} z_{34}^{2h_V}} \frac{1}{\overline{z}_{12}^{2\overline{h}_W} \overline{z}_{34}^{2\overline{h}_V}} \sum_p C_{12}^p C_{34}^p \widetilde{\mathcal{F}}_{34}^{12}(p|x) \overline{\widetilde{\mathcal{F}}_{34}^{12}}(p|\overline{x}), \tag{123}$$

where $\widetilde{\mathcal{F}}$ is a redefined conformal block,

$$\widetilde{\mathcal{F}}_{34}^{12}(p|x) = x^{2h_V} \mathcal{F}_{34}^{12}(p|x) = x^{h_p} \sum_{\{K\}} \beta_{34}^{p,\{K\}} x^K \frac{\langle h_1 | \phi_2(1) L_{-K} | h_p \rangle}{\langle h_1 | \phi_2(1) | \phi_p \rangle}. \tag{124}$$

Eqn.123 has a clear physical meaning: the prefactor represents the OPE; $\widetilde{F}(x)$ describes the correlation for fused channels and its power series at order x^{h_p+N} corresponds to the contribution from level-N descendant of primary ϕ_p. Sometimes, we will see people use this form of four-point function and conformal blocks in the literatures.

5.3 Examples

As an example, let's look at the conformal blocks in Ising model. The four point correlation for σ is,

$$\begin{aligned} \langle \sigma\sigma\sigma\sigma \rangle &= \frac{1}{2} \left| \frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{14}} \right|^{1/4} \left(\left|1 + \sqrt{1-z}\right| + \left|1 - \sqrt{1-z}\right| \right) \\ &= \frac{1}{|z_{12}|^{1/4} |z_{34}|^{1/4}} \frac{\left|1 + \sqrt{1-z}\right| + \left|1 - \sqrt{1-z}\right|}{2|1-z|^{1/4}}, \end{aligned}$$

where $|h(z)| = \sqrt{h(z)}\sqrt{h(\overline{z})}$. Recalling that the OPE coefficients are $C_{\sigma\sigma}^{\mathbb{I}} = 1$ and $C_{\sigma\sigma}^{\epsilon} = 1/2$, we can write down the conformal blocks,

$$\widetilde{\mathcal{F}}_{\sigma\sigma}(\mathbb{I}|x) = \frac{1}{(1-x)^{1/8}} \left(\frac{1 + \sqrt{1-x}}{2} \right)^{1/2} \propto 1 + ..., \widetilde{\mathcal{F}}_{\sigma\sigma}(\epsilon|x) = \frac{\sqrt{2}}{(1-x)^{1/8}} \left(1 - \sqrt{1-x}\right)^{1/2} \propto x^{1/2}(1 + ...). \tag{125}$$

We write down the first term of the series expansion w.r.t. x. One can see that the power equals to the conformal weight of the fusion channel[18]

As we can see that although the four point function as a physical quantity is analytical and single-valued, the conformal blocks doesn't have to be. They can have branch cuts or *monodromy*. So when we use conformal blocks to construct the four point functions, we need to organize $\mathcal{F}$ and $\overline{\mathcal{F}}$ in such a way that their non-analyticity cancels out.

[18] In this example, the four point function splits into the two conformal blocks in the most straightforward ways. However, this is not alway true [4]. For example, we still look at the σ 4pt function but look for the $F_{13}^{24}(p|1/x)$ conformal blocks this time. By definition, we have,

$$G_{13}^{24}(1/x, 1/\overline{x}) = \frac{|1 + \sqrt{1-x}| + |1 - \sqrt{1-x}|}{2|1-x|^{1/4}} \tag{126}$$

From the general discussion on conformal blocks, we know that $F_{13}^{24}(p|1/x) \sim (1/x)^{h_p - h_1 - h_3}$ at the leading term. If we naively use $(1 \pm (1-x)^{1/2})^{1/2}/(1-x)^{1/8}$ as the conformal block as before, it will give an incorrect expansion. It turns out that the conformal blocks take the following form,

$$F_{13}^{24}(\mathbb{I}|x) = \tag{127}$$

$$F_{13}^{24}(\epsilon|x) = \tag{128}$$

where we choose the branch cut ... so that the anti-holomorphic part is...

5.4 Crossing Symmetry

There are different ways to do the OPE between these fields. And they should give us the physical same answer. For instance, we can do 14 OPE and 23 OPE by sending $z_1 \to \infty$, $z_2 \to 0$, $z_3 \to 1-x$, $z_4 \to 1$ without changing the cross ratio. So we will have,

$$\begin{aligned} G_{32}^{14}(1-x, 1-\overline{x}) &= \lim_{z_1, \overline{z}_1 \to \infty} z_1^{2h_1} z_1^{2\overline{h}_1} \langle \phi_1(\infty)\phi_4(1)\phi_3(1-x)\phi_2(0)\rangle \\ &= (1-x)^{h/3-h_2-h_3} x^{h/3-h_3-h_4} \times (\text{anti-holo}) f(x, \overline{x}). \end{aligned} \tag{129}$$

Or we can send $z_1 \to 0$, $z_2 \to 1$, $z_3 \to 1/x$, $z_4 \to \infty$, which defines,

$$\begin{aligned} G_{31}^{42}(1/x, 1/\overline{x}) &= \lim_{z_4, \overline{z}_4 \to \infty} z_4^{2h_4} \overline{z}_4^{2\overline{h}_4} \langle \phi_4(\infty)\phi_2(1)\phi_3(1/x)\phi_1(0)\rangle \\ &= (1-x)^{h/3-h_2-h_3} x^{-2h/3+h_1+h_2+2h_3} \times (\text{anti-holo}) f(x, \overline{x}). \end{aligned} \tag{130}$$

From these equations, we get the following condition for G_{ij}^{kl},

$$G_{34}^{12}(x, \overline{x}) = G_{32}^{14}(1-x, 1-\overline{x}) = \frac{1}{x^{2h_3}\overline{x}^{2\overline{h}_3}} G_{31}^{42}(1/x, 1/\overline{x}). \tag{131}$$

This is called *crossing symmetry*.

A Generators of $\mathrm{PSL}(2, \mathbb{C})$

A general $\mathrm{PSL}(2, \mathbb{C})$ matrix has four complex numbers with a constrain among them. So there are six independent real degrees of freedom corresponding to six generators. For an arbitrary group element $G \in \mathrm{PSL}(2, \mathbb{C})$, we can write it in terms of generators as $G = \exp(-i \sum_j \alpha_j t_j)$, where $\alpha_j \in \mathbb{R}$ and t_j is the generator. Let's use condition $\det G = 1$ for infinitesimal α_j, we can get $\mathrm{Tr}\, t_j = 0$, i.e. t_j only has to be traceless. So three Pauli matrices σ_j and their product with i exhaust all the generators.

We can also get this conclusion by studying the isomorphism between $\mathrm{PSL}(2, \mathbb{C})$ and Lorentz group $SO(3,1)$ [6]. The three rotations of $SO(3,1)$ corresponds to the $SU(2)$ subgroup of $\mathrm{PSL}(2, \mathbb{C})$, which has Pauli matrices as generator. The three boosts corresponds to the rest of $\mathrm{PSL}(2, \mathbb{C})$ whose generators can be got by doing a Wick rotation.

B Holomorphic Stoke's Theorem

If M is a simple-connected region and F^μ is a smooth vector field, then we have the following Stoke's theorem,

$$\int_M d^2x \partial_\mu F^\mu = \int_{\partial M} d\xi_\mu F^\mu = \int_{\partial M} dr^\nu \epsilon_{\mu\nu} F^\mu = \int_{\partial M} (dy F^x - dx F^y) \tag{132}$$

where $\xi_\mu = \epsilon_{\mu\nu} dr^\nu$ is the line element perpendicular to the boundary. For our purpose, we'd like to explicitly write it in the $z, \overline{z}$ basis. We already know that $dz = dx + idy$ and $F^z = F^x + iF^y$ so we only have to work out $\epsilon_{\mu\nu}$ in the $z, \overline{z}$ basis, which is

$$\epsilon_{z\overline{z}} = \epsilon_{\alpha\beta} \frac{\partial x^\alpha}{\partial z} \frac{\partial x^\beta}{\partial \overline{z}} = \frac{i}{2} = -\epsilon_{\overline{z}z}.$$

So we get the final answer

$$\int_M d^2x \partial_\mu F^\mu = \int_{\partial M} (dz \epsilon_{\overline{z}z} F^{\overline{z}} + d\overline{z} \epsilon_{z\overline{z}} F^z) = \frac{i}{2} \int_{\partial M} \left(-dz F^{\overline{z}} + d\overline{z} F^z\right). \tag{133}$$

where dz clearly goes counterclockwise on $z-$plane. As for $d\bar{z}$, if we view $\bar{z}$ as a normal complex number living on $z-$plane, then it goes clockwise. However we can also view it as an independent beast living on its own $\bar{z}-$ plane (or $\{x, -y\}$ plane if we want), then $\bar{z}$ goes counterclockwise while the Cauchy integral formula also gets a minus sign on this plane.

C Casimir Energy of Free Boson

In Minkowski spacetime, the action is,

$$S = \frac{g}{2} \int dt dx \left[(\partial_t \phi)^2 - (\partial_x \phi)^2 \right] \tag{134}$$

We compactify the spatial dimension with a circumference L. The field can be decomposed into Fourier mode,

$$\phi(x, t) = \sum_n e^{2\pi i n x / L} \phi_n(t) \tag{135}$$

where n are integers or half integers depending on using periodic or anti-periodic boundary condition. The Canonical Quantization gives its Hamiltonian as,

$$H = \frac{1}{2gL} \sum_n \left[\pi_n \pi_{-n} + (2\pi n g)^2 \phi_n \phi_{-n} \right]. \tag{136}$$

which are many independent harmonic oscillators. For the mode (π_n, ϕ_n), it has a zero energy $\omega_n/2 = |n|\pi/L$. So even in the absence of excitation, there is still a nonzero group state energy,

$$E = \frac{2\pi}{L} \sum_{n=1}^{\infty} n = \frac{2\pi}{L} \times \frac{-1}{12}, \quad \text{(periodic)}, \tag{137}$$

$$E = \frac{2\pi}{L} \sum_{n=0}^{\infty} (n + \frac{1}{2}) = \frac{2\pi}{L} \times \frac{1}{24}, \quad \text{(anti-periodic)} \tag{138}$$

where the factor of 2 comes from summing over $n < 0$ modes. The summation of $n + 1/2$ can be derived from sum of n,

$$\sum_{n=0}^{\infty} (n + \frac{1}{2}) = \frac{1}{2} \sum_{n=0} (2n + 1) = \frac{1}{2} \left(\sum_{n=0} n - \sum_{n=0} 2n \right) = -\frac{1}{2} \sum_{n=0} n = \frac{1}{24}.$$

D Proof of Eqn.93

Here we use number theory to give a proof of Eqn.93 [7]. We use $p(n, k)$ as the restricted partition, which counts the number of ways to partition n into exactly k parts. With this notation, the core part of Eqn.93 can be rewritten as,

$$\sum_{k=1}^{N} k p(N, k) = \sum_{r,s=1}^{rs \leq N} p(N - rs). \tag{139}$$

We will prove this using generating function, i.e. trying to prove the following identity,

$$\sum_{N=1}^{\infty} q^N \sum_{k=1}^{N} k p(N, k) = \sum_{N=1}^{\infty} q^N \sum_{r,s=1}^{rs \leq N} p(N - rs). \tag{140}$$

One can show that the generating function for $\sum_{r,s=1}^{rs\leq N} p(N-rs)$ is[19],

$$\sum_{N=1}^{\infty} q^N \sum_{r,s=1}^{rs\leq N} p(N-rs) = \sum_{n=0}^{\infty}\left(\sum_{r=1}^{\infty}\frac{q^r}{1-q^r}\right)p(n)q^n = \sum_{r=1}^{\infty}\left(\prod_{j=1}^{\infty}\frac{1}{1-q^j}\right)\frac{q^r}{1-q^r}. \tag{141}$$

where in the second equality, we interchange the order of summation and use the generating function for $p(n)$. The r-th term in the summation is,

$$\frac{q^r}{(1-q^r)^2}\prod_{j\neq r}^{\infty}\frac{1}{1-q^j} = \left(\sum_{t=1}^{\infty} tq^{rt}\right)\prod_{j\neq r}^{\infty}\frac{1}{1-q^j} \tag{142}$$

By doing Taylor expansion of $1/1-q^j$, we will have,

$$\left(\sum_{t=1}^{\infty} tq^{rt}\right)\prod_{j\neq r}^{\infty}(1+q^j+q^{2j}+...) = \sum_{\{n_i\}} tq^{rt+\{n_i\}} = \sum_{\lambda\in P} a_r(\lambda)q^{n_P}. \tag{143}$$

In the third formula, P is the set of all the partitions, $n_\lambda = rt+\{n_i\}$ is the number partitioned by λ, $a_r(\lambda)=t$ is the multiplicity of r in the partition λ. So after summing over r, for a given q^{n_λ}, its coefficient will count and sum the multiplicity of different r in a partition λ. So we will get,

$$\sum_{\lambda\in P} A(\lambda)q^{n_\lambda} \tag{144}$$

where $A(\lambda)$ is the number of parts in the partition λ. This is exactly the L.H.S. of Eqn.140.

E Restricted Partition

Here, we take a digression and discuss the restricted partition number $p(n,k)$ and its properties.

$p(n,k)$ is defined as the number of partitions of n into exactly k parts, or the number of partitions of n in which the largest part(s) is k. The equality of these two definition is easily understood with the Ferrers diagram. Given a partition, we can use Ferrers diagram to represent it. And a 90^0 rotation interchanges the number of parts and largest parts.

It satisfies the recurrence relation,

$$p(n,k) = p(n-k,k) + p(n-1,k-1). \tag{145}$$

We justify it as following. The partitions for $p(n,k)$ can be divided into two groups: one is partitions where the smallest part is 1, the other is partitions where the smallest part is larger than 1. For a partition in the first group, if we drop one 1, the rest gives us a partition of $n-1$ into $k-1$ parts. So the number of partitions in the first group is $p(n-1,k-1)$. For a partition in the second group, if we subtract every part by 1, this gives us a partition of $n-k$ into k parts. So the number of partitions of the second group is $p(n-k,k)$.

Any partition, in which the largest part is k, can be written as $n=\sum_{i=1}^{k} i\times n_i$. This inspires us to get the generating function for $p(n,k)$,

$$\sum_{n=1}^{\infty} p(n,k)x^n = x^k\prod_{j=1}^{k}\frac{1}{1-x^j}. \tag{146}$$

[19]To justify this generating function, we can expand $q^r/(1-q^r)$, which gives $\sum_{n=0}\sum_{r=1}\sum_{t=0} q^{r(t+1)}p(n)q^n = \sum_{n=0}\sum_{r,s=1} p(n)q^{n+rs} = \sum_{N=1}\sum_{r,s=1,rs\leq N} p(N-rs)q^N$.

F Proof of Eqn.123

We first recall the general form of four point function,

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \prod_{ij} z_{ij}^{h/3-h_i-h_j} \overline{z}_{ij}^{\overline{h}/3-\overline{h}_i-\overline{h}_j} f(x,\overline{x}),$$

where $f(x,\overline{x}) = (1-x)^{-h/3+h_2+h_3} x^{-h/3+h_3+h_4} G_{34}^{12}(x,\overline{x})$. We first extract a $z_{12}^{h_1+h_2} z_{34}^{h_3+h_4}$ and reduce the prefactor to,

$$\frac{1}{z_{12}^{h_1+h_2} z_{34}^{h_3+h_4}} x^{h/3}(1-x)^{h/3} z_{13}^{h_2+h_4} z_{24}^{h_1+h_3} z_{14}^{-h_1-h_4} z_{23}^{-h_2-h_3}.$$

Now we use the condition $h_1 = h_2 = h_W$, $h_3 = h_4 = h_V$, it can be further reduced to,

$$\frac{1}{z_{12}^{2h_W} z_{34}^{2h_V}} x^{h/3}(1-x)^{h/3-h_W-h_V}.$$

Combining them together we will get Eqn.123.

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