Analytical Structure of Two-Point Functions

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Abstract

In this note, we discuss the Lehmann representation of various kinds of two-point functions and how to do analytical continuation.

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1 Definition of two-point functions: Convention

Let's take a generic interacting theory. We choose two operators A and B, which are both bosonic or fermionic. We will be interested in the following four kinds of two-point functions, which are defined with additional phase factors to be consistent with condensed matter literatures [1]:

- Imaginary time-ordered Green's function,

$$G^{\tau}(\tau) = -\mathcal{T}\left\langle A(\tau)B(0)\right\rangle_{\beta} \tag{1}$$
$$= -\frac{1}{Z}\Bigg(\theta(\tau)\operatorname{Tr}\left[e^{-\beta H}A(\tau)B(0)\right] + \zeta\theta(-\tau)\operatorname{Tr}\left[e^{-\beta H}B(0)A(\tau)\right]\Bigg),$$

where $A(\tau) = e^{\tau H}Ae^{-\tau H}$, $Z = \operatorname{Tr} e^{\beta H}$ and $-\beta < \tau < \beta$. We introduce $\zeta = 1$ if A, B are bosonic and $\zeta = -1$ if A, B are fermionic. Inspection of positive and negative time shows this function acquires the following periodicity properties,

$$G^{\tau}(\tau) = \zeta G^{\tau}(\tau+\beta),\ -\beta < \tau < 0, \tag{2}$$

to prove which we have to use the cyclic property of trace of operators. $G^{\tau}(\tau)$ outside the domain $[-\beta, \beta]$ can be defined by analytical continuation which preserves this periodicity property.

- Real time-ordered Greens' function,

$$G^{T}(t) = -i\mathcal{T}\left\langle A(t)B(0)\right\rangle_{\beta} \tag{3}$$
$$= -\frac{i}{Z}\Bigg(\theta(t)\operatorname{Tr}[e^{-\beta H}A(t)B(0)] + \zeta\theta(-t)\operatorname{Tr}[e^{-\beta H}B(0)A(t)]\Bigg).$$

where $A(t) = e^{iHt}Ae^{-iHt}$ is the Heisenburg operator and $-\infty < t < \infty$.

- Retarded/Advanced Green's function,

$$G^{R}(t) = -i\theta(t)\left\langle [A(t), B(0)]_{\zeta}\right\rangle_{\beta} = \frac{-i\theta(t)}{Z}\operatorname{Tr}\left[e^{-\beta H}[A(t), B(0)]_{\zeta}\right], \tag{4}$$
$$G^{A}(t) = i\theta(-t)\left\langle [A(t), B(0)]_{\zeta}\right\rangle_{\beta} = \frac{i\theta(-t)}{Z}\operatorname{Tr}\left[e^{-\beta H}[A(t), B(0)]_{\zeta}\right], \tag{5}$$

where $[\bullet, \bullet]_{\zeta}$ represents commutator if $\zeta = 1$ and anti-commutator if $\zeta = -1$.

- (Generalized) Wightman Green's function,

$$G^{W}(t) = \frac{1}{Z}\operatorname{Tr}\left[e^{-(\beta-s)H}A(t)e^{-sH}B(0)\right]. \tag{6}$$

where $0 < s < \beta$ and $-\infty < t < \infty$. When $s = \beta/2$ this goes back to the Wightman function we have seen in the discussion of chaos [2]; when $s = 0, \beta$ this becomes the $G^{>/<}(t)$ we have seen in the Keldysh formalism [3].

2 Spectral Decomposition: Lehmann representation

In this section, we will insert resolution of identity to rewrite the above expressions in terms of the complete set of basis. Then we go to the frequency domain to get the *Lehmann representation*. The relation among this two-point functions will be clear in this representation.

Imaginary time-ordered Green's function By inserting resolution of identity, we can write it as,

$$G^\tau(\tau) = -\frac{1}{Z}\sum_{\alpha,\beta} A_{\alpha\beta}B_{\beta\alpha}e^{\tau E_{\alpha\beta}}\left(\theta(\tau)e^{-\beta E_\alpha} + \zeta\theta(-\tau)e^{-\beta E_\beta}\right).$$

Due to the periodic property of $G^\tau(\tau)$, it can be expanded into Matsubara frequency,

$$G^\tau(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau}G^\tau(\tau) = -\frac{1}{Z}\sum_{\alpha,\beta} A_{\alpha\beta}B_{\beta\alpha}e^{-\beta E_\alpha}\int_0^\beta d\tau e^{(i\omega_n+E_{\alpha\beta})\tau},$$

where ω_n is the bosonic or fermionic Matsubara frequency respectively. Notice that $e^{i\omega_n\beta} = \zeta$ we have,

$$G^\tau(i\omega_n) = \frac{1}{Z}\sum_{\alpha,\beta}\frac{A_{\alpha\beta}B_{\beta\alpha}}{i\omega_n + E_{\alpha\beta}}\left(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}\right). \tag{7}$$

Real time-ordered Green's function Using spectral decomposition, we can write it as,

$$G^T(t) = -i\frac{1}{Z}\sum_{\alpha,\beta} A_{\alpha\beta}B_{\beta\alpha}e^{itE_{\alpha\beta}}\left(\theta(t)e^{-\beta E_\alpha} + \zeta\theta(-t)e^{-\beta E_\beta}\right).$$

We can do the normal Fourier transformation,

$$\begin{aligned} G^T(\omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega t-\epsilon|t|}G^T(t) \\ &= -i\frac{1}{Z}\sum_{\alpha,\beta} A_{\alpha\beta}B_{\beta\alpha}\left(\int_0^\infty e^{(i\omega+iE_{\alpha\beta}-\epsilon)t}e^{-\beta E_\alpha} + \zeta\int_{-\infty}^0 e^{(i\omega+iE_{\alpha\beta}+\epsilon)t}e^{-\beta E_\beta}\right) \end{aligned}$$

where ϵ is to guarantee the convergence. Finally we have,

$$G^T(\omega) = \frac{1}{Z}\sum_{\alpha,\beta} A_{\alpha\beta}B_{\beta\alpha}\left(\frac{e^{-\beta E_\alpha}}{\omega + E_{\alpha\beta} + i\epsilon} - \zeta\frac{e^{-\beta E_\beta}}{\omega + E_{\alpha\beta} - i\epsilon}\right). \tag{8}$$

Retarded/Advanced Green's function In terms of the complete set of states, they can be written as,

$$\begin{aligned} G^R(t) &= \frac{-i\theta(t)}{Z}\sum_{\alpha,\beta} e^{itE_{\alpha\beta}}A_{\alpha\beta}B_{\alpha\beta}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}), \\ G^A(t) &= \frac{i\theta(-t)}{Z}\sum_{\alpha,\beta} e^{itE_{\alpha\beta}}A_{\alpha\beta}B_{\alpha\beta}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}). \end{aligned}$$

In frequency domain,

$$G^R(\omega) = \int_{-\infty}^{\infty} dt\, G^R(t) e^{i(\omega+i\epsilon)t} = \frac{1}{Z}\sum_{\alpha,\beta} \frac{A_{\alpha\beta}B_{\beta\alpha}}{\omega + E_{\alpha\beta} + i\epsilon}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}), \tag{9}$$

$$G^A(\omega) = \int_{-\infty}^{\infty} dt\, G^A(t) e^{i(\omega-i\epsilon)t} = \frac{1}{Z}\sum_{\alpha,\beta} \frac{A_{\alpha\beta}B_{\beta\alpha}}{\omega + E_{\alpha\beta} - i\epsilon}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}), \tag{10}$$

where ϵ is technically to make the integral convergent and physically to guarantee the causality.

(Generalized) Wightman function Using spectral decomposition, we can write it as,

$$G^W(t) = \frac{1}{Z}\sum_{\alpha,\beta} e^{-\beta E_\alpha} e^{(s+it)E_{\alpha\beta}} A_{\alpha\beta} B_{\alpha\beta}$$

In frequency domain, we have,

$$G^W(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} G^W(t) = \frac{2\pi}{Z}\sum_{\alpha,\beta} e^{-\beta E_\alpha + s E_{\alpha\beta}} A_{\alpha\beta} B_{\beta\alpha} \delta(\omega + E_{\alpha\beta}) \tag{11}$$

3 Spectral function and Hilbert transformation

All of these functions share a very similar structure. To relate them with each other, we need to define an auxiliary function, the *spectral function*,

$$A(\omega) = -2\,\mathrm{Im}\, G^R(\omega) = i(G^R(\omega) - G^A(\omega)). \tag{12}$$

Recalling the formula $\dfrac{1}{x \pm i\epsilon} = P\dfrac{1}{x} \mp i\pi\delta(x)$, we have,

$$A(\omega) = \frac{2\pi}{Z}\sum_{\alpha,\beta} A_{\alpha\beta}B_{\beta\alpha}(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta})\delta(\omega + E_{\alpha\beta}). \tag{13}$$

Physically, it probes the spectrum of the system, as will be shown in the next section. One can verify that it satisfies a sum rule,

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\omega) = \langle [A, B]_\zeta \rangle_\beta \, . \tag{14}$$

Now, many functions can be written in term of the spectral function via a Hilbert transformation:

$$G(z) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{A(\xi)}{z - \xi}, \tag{15}$$

which defines a master function $G(z)$ with a complex argument z. Its evaluation at the imaginary axis and real axis gives us G^τ and $G^{R/A}$ respectively,

$$G(z \to i\omega_n) = G^\tau(i\omega_n), \tag{16}$$

$$G(z \to \omega \pm i\epsilon) = G^{R/A}(i\omega). \tag{17}$$

As for the real time-ordered function, we can only write its real and imaginary part by $A(\omega)$ separately.

$$\mathrm{Re}\, G^T(\omega) = \frac{1}{Z} \sum_{\alpha,\beta} A_{\alpha\beta} B_{\beta\alpha} \left(e^{-\beta E_\alpha} - \zeta e^{-\beta E_\beta}\right) P \frac{1}{\omega + E_{\alpha\beta}},$$

$$\mathrm{Im}\, G^T(\omega) = -\frac{\pi}{Z} \sum_{\alpha,\beta} A_{\alpha\beta} B_{\beta\alpha} \left(e^{-\beta E_\alpha} + \zeta e^{-\beta E_\beta}\right) \delta(\omega + E_{\alpha\beta}).$$

Therefore we have,

$$G^T(\omega) = P \int \frac{dz}{2\pi} \frac{A(z)}{\omega - z} - \frac{i}{2} A(\omega) \frac{1 + \zeta e^{-\beta\omega}}{1 - \zeta e^{-\beta\omega}}. \tag{18}$$

The last one, (generalized) Wightman Green's function, can also be written in terms of $A(\omega)$ in a simple way,

$$G^W(\omega) = A(\omega) \frac{e^{-s\omega}}{1 - \zeta e^{-\beta\omega}}. \tag{19}$$

4 Examples

To get some physical intuition of these quantities, it is helpful to look at a few examples.

4.1 (Non-relativistic) Free fermion

One of the simplest example is a system composed of decoupled fermionic excitations,

$$H = \sum_k E_k c_k^\dagger c_k, \tag{20}$$

where k just labels the fermionic modes and doesn't have to be momentum. We choose $A = c_k$ and $B = c_k^\dagger$. Straightforward calculation shows that,

- Imaginary time-order Green's function,

$$G^\tau(\tau) = -\left[\theta(\tau) e^{-E_k \tau}(1 - n_F) - \theta(-\tau) e^{-E_k \tau} n_F\right], \tag{21}$$

$$G^\tau(i\omega_n) = \frac{1}{i\omega_n - E_k}, \tag{22}$$

where $n_F(E_k) = \dfrac{1}{e^{\beta E_k} + 1}$ is the Fermi-Dirac distribution and $\omega_n = (2n+1)\dfrac{\pi}{\beta}$.

- Real time-ordered Green's function,

$$G^T(t) = -i\left[\theta(t)e^{-iE_kt}(1-n_F) - \theta(-t)e^{-iE_kt}n_F\right], \tag{23}$$

$$G^T(\omega) = \frac{1-n_F}{\omega - E_k + i\epsilon} + \frac{n_F}{\omega - E_k - i\epsilon}, \tag{24}$$

where the first term can be interpreted as particle propagation while the second term describes hole propagation. $(1-n_F(E_k))$ and $n_F(E_k)$ are just the probabilities to create a particle or hole at that energy respectively.

- Retarded Green's function,

$$G^R(t) = -i\theta(t)e^{-iE_kt}, \quad G^R(\omega) = \frac{1}{\omega - E_k + i\epsilon}. \tag{25}$$

Therefore, the spectral function is,

$$A(\omega) = 2\pi\delta(\omega - E_k), \tag{26}$$

which has a delta-function peak at $\omega = E_k$.

- Wightman function,

$$G^W(t) = (1-n_F)e^{-iE_kt - E_ks}, \quad G^W(\omega) = 2\pi e^{-E_ks}(1-n_F)\delta(\omega - E_k). \tag{27}$$

One can verify that it satisfies the relation we derived above.

4.2 (Non-relativistic) Free boson

Another simplest example is a system composed of free bosonic modes,

$$H = \sum_k E_k b_k^\dagger b_k, \tag{28}$$

where k is just a label. We choose $A = b_k$ and $B = b_k^\dagger$, then we have,

- Imaginary time-ordered Green's function,

$$G^\tau(\tau) = -\left[\theta(\tau)e^{-E_k\tau}(1+n_B) + \theta(-\tau)e^{-E_k\tau}n_B\right], \tag{29}$$

$$G^\tau(i\omega_n) = \frac{1}{i\omega_n - E_k}, \tag{30}$$

where $n_B = \dfrac{1}{e^{\beta E_k} - 1}$ is the Bose distribution and $\omega_n = \dfrac{2\pi n}{\beta}$.

- Real time-ordered Green's function,

$$G^T(t) = -i\left[\theta(t)e^{-iE_kt}(1+n_B) + \theta(-t)e^{-iE_kt}n_B\right], \tag{31}$$

$$G^T(\omega) = \frac{1+n_B}{\omega - E_k + i\epsilon} + \frac{n_B}{\omega - E_k - i\epsilon}. \tag{32}$$

- Retarded Green's function,

$$G^R(t) = -i\theta(t)e^{-iE_k t}, \quad G^R(\omega) = \frac{1}{\omega - E_k + \imath\epsilon}. \tag{33}$$

Therefore, the spectral function is,

$$A(\omega) = 2\pi\delta(\omega - E_k), \tag{34}$$

which is the same as that of the free fermion. Because this two systems have the same single-particle spectrum.

- Wightman function,

$$G^W(t) = (1 + n_B)e^{-iE_k t - E_k s}, \quad G^W(\omega) = 2\pi e^{-E_k s}(1 + n_B)\delta(\omega - E_k). \tag{35}$$

4.3 Fermi Polaron as an example of Fermi liquid

Let's take spinful fermions with contact s-wave interaction,

$$H = \sum_{k,\sigma} \epsilon_{k,\sigma} c^\dagger_{k,\sigma} c_{k,\sigma} + \frac{g}{V} \sum_{k,k',p} c^\dagger_{-k'+p,\uparrow} c^\dagger_{k'+p,\downarrow} c_{k+p,\downarrow} c_{-k+p,\uparrow}, \tag{36}$$

where $\epsilon_k = k^2/2m$ and g satisfies the following renormalization relation,

$$\frac{1}{g} + \frac{1}{V}\sum_k \frac{1}{2\epsilon_k} = \frac{1}{4\pi a_s}, \tag{37}$$

where a_s is the s-wave scattering length.

We initialize the system to be composed of spin-up fermions, which, because they are non-interacting, simply fill the Fermi sea up to a certain Fermi momentum k_F. Now we insert a single spin-down fermion as an impurity. Its interaction with the background spin-up Fermi sea will dress up itself. Now we want to take a look at the two-point functions of this impurity fermion.

We first do the calculation in imaginary time,

$$G_\downarrow(\tau, k) = -\mathcal{T}\left\langle c_\uparrow(\tau, k) c^\dagger_\uparrow(0, k)\right\rangle_\beta. \tag{38}$$

Bare Green's function,

$$G^0_\sigma(p) = \frac{1}{iw - \xi_{p,\sigma}}, \tag{39}$$

where $\xi_{p,\sigma} = \epsilon_p - \mu_\sigma$, with $\mu_\uparrow = \mu_F = k_F^2/2m$ and $\mu_\downarrow = \mu = 0$.

Dyson equation in frequency domain, $p = (iw_n, \mathbf{p})$,

$$G(p) = G^0(p) + G^0(p)\Sigma(p)G(p). \tag{40}$$

Self-energy,

$$\Sigma(p) = \frac{1}{\beta V}\sum_q G^0_\uparrow(q)\Gamma(p+q), \tag{41}$$

where $G^0_\uparrow$ is the bare hole propagator.

Dressed vertex function,

$$\Gamma(q) = -\frac{1}{g^{-1}+\Pi(q)}, \tag{42}$$

where w_q is the bosonic Matsubara frequency.

Bubble function,

$$\begin{aligned}\Pi(q) =&\frac{1}{\beta V}\sum_k G^0_\uparrow(k)G^0_\downarrow(q-k) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\beta}\sum_{w_k}\frac{1}{iw_k-\xi_{k,\uparrow}}\frac{1}{iw_q-iw_k-\xi_{q-k,\downarrow}} \\ =&\int\frac{d^3k}{(2\pi)^3}\left[n_F(\xi_{k,\uparrow})+n_B(-\xi_{q+k,\downarrow})\right]\frac{1}{iw_q-\xi_{k,\uparrow}-\xi_{q-k,\downarrow}} \\ \to&\int\frac{d^3k}{(2\pi)^3}\frac{-\theta(k-k_F)}{iw_q-\xi_{k,\uparrow}-\xi_{q-k,\downarrow}}.\end{aligned} \tag{43}$$

Therefore, the self-energy function at zero temperature is,

$$\Sigma(p) = \frac{1}{V}\sum_q\int_{-\infty}^{\infty}\frac{dw_q}{2\pi}\frac{1}{iw_q-\xi_{q,\uparrow}}\frac{-1}{g^{-1}-\int\frac{d^3k}{(2\pi)^3}\frac{\theta(k-k_F)}{iw_p+iw_q-\xi_{k,\uparrow}-\xi_{p+q+k,\downarrow}}}. \tag{44}$$

To perform the integration on w_q, we need to know the analytical structure of the vertex function $\Gamma(q)$, basically compute the integral,

$$\begin{aligned}&\frac{1}{4\pi a_s}-\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_k}-\int\frac{d^3k}{(2\pi)^3}\frac{\theta(k-k_F)}{iw_q-\xi_{k,\uparrow}-\xi_{q-k,\downarrow}} \\ =&\frac{1}{4\pi a_s}+\int\frac{d^3k}{(2\pi)^3}\frac{\theta(k_F-k)}{iw_q-\xi_{k,\uparrow}-\xi_{q-k,\downarrow}}-\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{iw_q-\xi_{k,\uparrow}-\xi_{q-k,\downarrow}}+\frac{1}{2\epsilon_k}\right].\end{aligned} \tag{45}$$

Let's work out each integral in steps. The first integral is,

$$\begin{aligned}&\int\frac{d^3k}{(2\pi)^3}\frac{\theta(k_F-k)}{iw_q+\mu_F-q^2/2m-k^2/m+q\cdot k/m} \\ =&\int\frac{dkk^2}{(2\pi)^2}\int_{-1}^{1}d\cos\alpha\frac{\theta(k_F-k)}{iw_q+\mu_F-q^2/2m-k^2/m+qk\cos\alpha/m} \\ =&\int\frac{dkk^2}{(2\pi)^2}\int_{-1}^{1}d\cos\alpha\frac{m\theta(k_F-k)}{\Delta-k^2+qk\cos\alpha}\end{aligned}$$

where we introduce,

$$\Delta = imw_q + m\mu_F - q^2/2 = imw_q + k_F^2/2 - q^2/2 \tag{46}$$

to simplify the formula. Integration over $\cos\alpha$ yields,

$$\begin{aligned}
&\int_0^{k_F} \frac{dk}{(2\pi)^2}\frac{mk}{q}\log\frac{\Delta - k^2 + qk}{\Delta - k^2 - qk} \\
=&\frac{m}{4\pi^2}\left\{\frac{\sqrt{-q^2-4\Delta}}{2}\left(\arctan\left[\frac{2k_F - q}{\sqrt{-q^2-4\Delta}}\right] + \arctan\left[\frac{2k_F + q}{\sqrt{-q^2-4\Delta}}\right]\right)\right. \\
&\left. -k + \frac{2k_F^2 - q^2 - 2\Delta}{4q}\log\frac{k_F^2 + qk_F - \Delta}{k_F^2 - qk_F - \Delta}\right\},
\end{aligned}$$

where $-q^2 - 4\Delta = q^2 - 4m\mu_F - i4mw_q$. The second integral is,

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3}\frac{(iw_q - \xi_{k,\uparrow} - \xi_{q-k,\downarrow}) + 2\epsilon_k}{(iw_q - \xi_{k,\uparrow} - \xi_{q-k,\downarrow})2\epsilon_k} \\
=&\int \frac{d^3k}{(2\pi)^3} m\frac{\Delta + q\cdot k}{(\Delta - k^2 + q\cdot k)k^2} \\
=&\int_9^{\infty}\frac{dk}{(2\pi)^2}\int_{-1}^{1} d\cos\alpha\, m\frac{\Delta + qk\cos\alpha}{\Delta - k^2 + qk\cos\alpha} \\
=&\int_0^{\infty}\frac{dk}{(2\pi)^2} m\left[2 + \frac{k}{q}\log\frac{\Delta - k^2 + qk}{\Delta - k^2 - qk}\right] \\
=&\frac{m}{8\pi^2}\sqrt{-q^2-4\Delta}\left(\arctan\left[\frac{2k-q}{\sqrt{-q^2-4\Delta}}\right] + \arctan\left[\frac{2k+q}{\sqrt{-q^2-4\Delta}}\right]\right)\Bigg|_{k\to+\infty} \\
=&\frac{m}{4\pi^2}\sqrt{-q^2-4\Delta}\arctan\left[\frac{2k}{\sqrt{-q^2-4\Delta}}\right]\Bigg|_{k\to+\infty}
\end{aligned}$$

Continue...

5 Analytical continuation of Dyson Equation

Most of the time, it is easier to formulate field theories in the imaginary time. The time evolution is well controlled and we can easily write down the Dyson equation for the imaginary time two point functions. However, sometimes we are more interested in real time correlation functions, such as Retarded Green's function (relevant to linear response experiment) and spectral function (relevant to everything). Therefore we have a general question:

- *If we know the Dyson equation of imaginary two point functions, how to perform an analytical continuation to get the correct Dyson equation of the retarded two point function?*

Here, let's use the SYK model as an example to give an answer.

5.1 SYK as an example

For the SYK model with four fermion interaction, in the conformal limit, the imaginary-time two point function satisfies the following Dyson equations,

$$
\begin{aligned}
G^{-1}(iw_n) =& G_0^{-1}(iw_n) - \Sigma(iw_n), && (47)\\
\Sigma(\tau) =& J^2 G(\tau)^3, && (48)
\end{aligned}
$$

where iw_n is the fermion Matsubara frequency. The first equation can be rewritten as,

$$
\Sigma(iw_n) = G_0^{-1}(iw_n) - G^{-1}(iw_n). \tag{49}
$$

Both of the two terms on the RHS are analytical on the upper half complex plane. If we do the substitution $iw \to w + i\epsilon$, we will get the retarded function. Therefore, the LHS should also be analytical on the upper half complex plane and $iw \to w + i\epsilon$ yields the corresponding retarded self-energy $\Sigma_R(w)$, and we have,

$$
\Sigma_R(w) = G_{0R}^{-1}(w) - G_R^{-1}(w). \tag{50}
$$

Now let's consider the second equation and things become very subtle. We can proceed in two different ways. One is less confusing but we will see that they essentially have the same subtlety.

- We could write everything in frequency domain first,

$$
\begin{aligned}
\Sigma(iw_n) =& \int_0^\beta d\tau\, e^{iw_n\tau}\Sigma(\tau) = J^2 \int_0^\beta d\tau\, e^{iw_n\tau} G(\tau)^3\\
=& J^2 \int_0^\beta d\tau\, e^{iw_n\tau}\left[\frac{1}{\beta}\sum_{w_k} G(iw_k)e^{-iw_k\tau}\right]^3\\
=& \frac{J^2}{\beta^3}\sum_{w_1,w_2,w_3}\int_0^\beta d\tau\, e^{iw_n\tau - iw_1\tau - iw_2\tau - iw_3\tau} G(iw_1)G(iw_2)G(iw_3)\\
=& \frac{J^2}{\beta^2}\sum_{w_1,w_2} G(iw_1)G(iw_2)G(iw_n - iw_1 - iw_2). && (51)
\end{aligned}
$$

  To proceed, let's write $G(iw)$ in terms of the spectral function,

$$
G(iw) = \int \frac{dz}{2\pi}\frac{\rho(z)}{z - iw}. \tag{52}
$$

Therefore, we have,

$$\begin{aligned}\Sigma(iw_n) =& \frac{J^2}{\beta^2}\int\prod_{j=1}^{3}\frac{dz_j}{2\pi}\sum_{w_1,w_2}\frac{\rho(z_1)}{z_1-iw_1}\frac{\rho(z_2)}{z_2-iw_2}\frac{\rho(z_3)}{z_3-iw_n+iw_1+iw_2}\\ =& -J^2\int\prod_i\left(\frac{dz_i}{2\pi}A(z_i)\right)\frac{n_F(z_1)n_F(z_2)n_F(z_3)+n_F(-z_1)n_F(-z_2)n_F(-z_3)}{i\omega_n-z_1-z_2-z_3},\end{aligned} \tag{53}$$

which can be analytically continued without any obstruction.

- By using Eqn.52, we can first rewrite the imaginary time two point function as,

$$G(\tau) = \frac{1}{\beta}\sum_{w_n}e^{-iw_n\tau}G(iw_n) = \frac{1}{\beta}\sum_{w_n}\int\frac{d\xi}{2\pi}e^{-iw_n\tau}\frac{\rho(\xi)}{\xi-iw_n} \tag{54}$$

$$= \int\frac{d\xi}{2\pi}\oint\frac{dz}{2\pi i}\frac{e^{-z\tau}}{e^{-\beta z}+1}\frac{\rho(\xi)}{\xi-z} \tag{55}$$

$$= \int\frac{d\xi}{2\pi}e^{-\xi\tau}\frac{\rho(\xi)}{e^{-\beta\xi}+1}. \tag{56}$$

Then $\Sigma(iw_n)$ can be written as,

$$\begin{aligned}\Sigma(iw_n) &= \left[\int\frac{d\xi_j}{2\pi}\frac{\rho(\xi_j)}{e^{-\beta\xi_j}+1}\right]^3\int_0^\beta d\tau e^{i(w_n-\sum_{j=1}^3\xi_j)\tau}\\ &= \left[\int\frac{d\xi_j}{2\pi}\frac{\rho(\xi_j)}{e^{-\beta\xi_j}+1}\right]^3\frac{e^{(iw_n-\xi_1-\xi_2-\xi_3)\beta}-1}{iw_n-\xi_1-\xi_2-\xi_3}.\end{aligned} \tag{57}$$

Now we encounter a subtlety here: do we directly substitute iw_n with $w+i\epsilon$ or first take $e^{-iw_n\beta}=-1$ then do the substitution? The answer is that we have to choose the second way.

Actually, the first method also has this subtlety, if we substitute iw_n with $w+i\epsilon$ before the last step, we won't get the correct answer. Why do we have to choose this particular order instead of the other one?

5.2 Two theorems for analytical continuation

To explain the reason behind this, we need to take a digression and introduce the following two theorems:

Lemma *Suppose f and g are holomorphic in a region Ω and $f(z)=g(z)$ for all z in some non-empty open subset of Ω (or more generally for z in some sequence of distinct points with limit point in Ω). Then $f(z)=g(z)$ throughout Ω.* [4]

Notice that the condition "with limit point" is very important. For example, $\sin(\pi z)$ and $1/\Gamma(-z)$ are both 0 when $z = 1, 2, 3....$ However, both of they are not regular at the infinity, i.e., no limit point, thus based on the lemma we cannot claim they are identical. Obviously, they are indeed not equal.

The above lemma has an equivalent but more general version, which is summarized as the following theorem.

Carlson's theorem *Assume that f satisfies the following three conditions*[1]*:*

- *$f(z)$ is analytic in $\mathrm{Im}\, z > 0$, continuous in $\mathrm{Im}\, z \geq 0$, and satisfies $|f(z)| \leq Ce^{\tau|z|}$, for some C and τ;*
- *There exists $c < \pi$ such that $|f(y)| \leq Ce^{c|y|}, y \in \mathbb{R}$;*
- *$f(in) = 0$ for any non-negative integers n.*

Then $f(z)$ is identically zero on the upper half plane.

For example, let's look at the function $1/\Gamma(1 + iz)$. It vanishes at $z = in, n = 1, 2, 3...$ but not identically zero. It is easy to check that it violates the first condition,

$$\lim_{y\to\infty} \frac{1}{\Gamma(1-y)} \sim \frac{1}{\pi} e^{-y} y^y, \tag{58}$$

therefore the theorem doesn't apply to it.

5.3 Explanation and general comments

Now going back to our problem, what we are doing is exactly the same as what the two above lemma says.

The Dyson equation gives us two expressions for $\Sigma(z)$. One is

$$\Sigma(z) = G_0^{-1}(z) - G^{-1}(z),$$

which is analytic on the upper half plane. The other one is the complicated integral $I(z)$ and we know its value at the fermion Matsubara frequency,

$$I(iw_n) = \left[\int \frac{d\xi_j}{2\pi} \frac{\rho(\xi_j)}{e^{-\beta\xi_j} + 1}\right]^3 \frac{e^{(iw_n - \xi_1 - \xi_2 - \xi_3)\beta} - 1}{iw_n - \xi_1 - \xi_2 - \xi_3}.$$

We know that they are analytical on the upper half plane and match each other at $z = iw_n$. However, to really identify them with each other, we also need them to be both regular enough at infinity. If we choose the substitution $e^{iw_n\beta} \to e^{z\beta}$, $I(z)$ will violate the second

[1]Please refer to Wikipedia for the original version. Here we make a little modification for the sake of our discussion here.

condition of Carlson's theorem. Therefore we have to replace $e^{iw_n\beta}$ with -1 in Eqn.57 before any analytical continuation.

More generally, when we perform analytical continuation, we not only need the two functions to have the same value at a certain domain, but also need to require them to be regular enough at the infinity.

References

[1] Alexander Altland and Ben Simons, *Condensed Matter Field Theory*, *Cambridge Press.*

[2] Juan Maldacena and Douglas Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **94** 106002.

[3] Alex Kamenev, *Field theory of non-equilibrium systems*, *Cambridge Press.*

[4] E. M. Stein, *Complex Analysis*, Chapter 2, corollary 4.9.