Global quench in CFTs

Ruihua Fan

Department of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract

In this note, we will discuss the global quench problem in the context of conformal field theories.

Contents

---

1 Introduction

Quench refers to the problem that the Hamiltonian is changed suddenly. A typical protocol is: (1) the system is first prepared in a pure state $|0\rangle$, which is usually chosen as the ground state of some gapped Hamiltonian H_0 therefore short-range entangled; (2) the system is evolved with another Hamiltonian $H \neq H_0$. The difference between H and H_0 can be either global or local, which is usually referred as global quench and local quench, respectively.

In this note, we focus on the global quench, a.k.a, H differs from H_0 globally. For example, H_0 is a gapped Hamiltonian while H is gappless. Because $|0\rangle$, though the ground state of H_0, can be a mixture of highly excited states of H. Therefore, a precise description of the quench dynamics involves the full spectrum of H. This makes the problem generally hard. In the context of conformal field theory, we have some tricks that makes the problem tractable [1].

Before going into the technical details, let us think of what to expect physically. Since $|0\rangle$ is not an eigenstate of H, generic measurement will depend on time. However, when H describes a strongly interacting system, we expect thermalization to happen:

- after a certain time scale τ_{th}, any local measurement (e.g. 1pt and 2pt correlation functions) gives the same result as a thermal state, the temperature of which is fixed by the energy;
- The initial state $|0\rangle$ is short-range entangled while a thermalized state has extensive entanglement, which implies the entanglement entropy of a subsystem to grow and saturates the volume-law value.

In the following, we will follow the Calabrese-Cardy prescription [1] to compute correlation functions and entanglement entropies to verify these expectations. The main results are summarized here:

- Assumption: replace the initial state SRE $|0\rangle$ with a conformally invariant boundary state up to an imaginary time evolution, i.e. $|0\rangle \propto e^{-H\tau_0}|B\rangle$.
- The energy density of this state is $\langle T_{tt}(x)\rangle = \dfrac{\pi c}{24(2\tau_0)^2}$. This implies a reparametrization $\tau_0 = \beta/4$, which will be useful when compare it with a thermal state.
- For a nonconserved scalar primary, its one point function, in the long-time regime ($t \gg \beta$), exponentially decays $\langle \mathcal{O}(x,t)\rangle \propto e^{-\Delta\frac{2\pi t}{\beta}}$. Δ is the scaling dimension.
- For a single scalar primary, its two point function saturates to $\langle \mathcal{O}(x,t)\mathcal{O}(0,t)\rangle \propto e^{-\Delta\frac{2\pi x}{\beta}}$ after $t > x/2$, which is the same as the two-point function for a thermal state at temperature β.
- The entanglement entropy for a subsystem of length l will linearly grow and saturate to volume law after $t > l/2$. If we ignore the subleading correction, we have $S_A(l) \propto cl/\beta$. Since the energy density is proportional to β^{-2}, this implies $S_A \propto \sqrt{E}$.

2 Correlation functions

In this section, let us discuss the local measurement and the (equal-time) correlation functions.

For example, the equal time two point function is

$$\langle 0|e^{iHt}\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)e^{-iHt}|0\rangle\,.$$

$|0\rangle$ is the initial state, which can be a short-range entangled state, and H is the CFT Hamiltonian. Since a field theory is more easily formulated in the Euclidean space, it is more tractable to start from an imaginary-time 2pt function

$$\langle 0|e^{H\tau_2}\mathcal{O}_1(x_1)e^{-H(\tau_2-\tau_1)}\mathcal{O}_2(x_2)e^{-H\tau_1}|0\rangle\,,$$

and perform an analytical continuation $\tau_1, \tau_2 \to it$. The imaginary-time 2pt function is defined on a strip with finite width. However, after the analytical continuation, the width of the strip becomes zero, which can cause singularity.

In the context of CFTs, this can be avoided by replacing $|0\rangle$ with a conformally invariant boundary state $|B\rangle$, i.e.

$$|0\rangle \propto e^{-H\tau_0} |B\rangle \,, \quad \big(T(x) - \overline{T}(x)\big) |B\rangle = 0. \tag{1}$$

The validity of this replacement relies on two assumptions:

1. $|0\rangle$ is a short range entangled state. This is because the right hand side is a short-range entangled state, the validity of this replacement at least requires $|0\rangle$ also to be SRE.

2. The long-time behavior after the quench is insensitive to the details of the initial state. If this is true, we can effectively describe $|0\rangle$ by a fixed-point state plus irrelevant perturbations. Each bulk CFT is supposed to have a particular allowed set of boundary states, each of which corresponds to a fixed point of the boundary RG, each with their own basins of attraction. For the minimal models, there is a finite number of these states, and their basins of attraction are supposed to contain the ground states of each possible non-critical hamiltonian H_0. Therefore generally we have

$$|0\rangle \propto \prod_k e^{\beta_k \int dx \Phi_k(x)} |B\rangle \tag{2}$$

where $\Phi_k(x)$ are irrelevant operators. From this point of view, Eq. (1) amounts to assuming T_{tt} is the only term in the above sum thus is just one special choice.

With this replacement, the real-time correlation function can be rewritten as

$$\frac{\langle B|e^{-H\tau_0}e^{iHt}\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}(x_3)...\mathcal{O}_n(x_n)e^{-iHt}e^{-H\tau_0}|B\rangle}{\langle B|e^{-2H\tau_0}|B\rangle}. \tag{3}$$

Since the conformal boundary state $|B\rangle$ is not normalizable, this expression is singular when $\tau_0 = 0$ which is consistent with what we said before. Its imaginary-time version is a n-point function on a strip with width $2\tau_0$ and conformally invariant boundary condition

$$\langle \mathcal{O}_1(x_1,\tau_1)\mathcal{O}_2(x_2,\tau_2)...\mathcal{O}_n(x_n,\tau_n)\rangle_{\text{strip}} \,, \quad 0 < \tau < 2\tau_0. \tag{4}$$

The analytical continuation $\tau_1 \to \epsilon_1 + it$, $\tau_2 \to \epsilon_2 + it$,...$\tau_n \to \epsilon_n + it$ brings it back to the real-time. $\epsilon_1 > \epsilon_2 > ... > \epsilon_n$ guarantees the operator ordering. Let us introduce the complex coordinate $w = x+i\tau$. Then we can go to the upper half-plane through the mapping $\xi = \exp \frac{\pi w}{2\tau_0}$, where we know how to compute the correlators.

2.1 Energy density

Let us first study the energy density $E(x,t)$. Due to the energy conservation and translational invaraince, the energy density must be a uniform constant. In the imaginary-time, we have[1]

$$E(x,t) = \langle 0|T_{tt}(x,t)|0\rangle \to \frac{-1}{2\pi}\left(\langle T(w)\rangle_{\text{strip}} + \langle \overline{T}(\overline{w})\rangle_{\text{strip}}\right). \tag{5}$$

[1]Since we define $w = x + i\tau$, $T_{\tau\tau} = \frac{\partial w}{\partial \tau}\frac{\partial w}{\partial \tau}T_{ww} + \frac{\partial \overline{w}}{\partial \tau}\frac{\partial \overline{w}}{\partial \tau}T_{\overline{w}\overline{w}} = -T_{ww} - T_{\overline{w}\overline{w}} = \frac{1}{2\pi}(T + \overline{T}) = -T_{tt}$.

We can map the strip to the upper half-plane and have

$$\langle T(w)\rangle_{\text{strip}} = \left(\frac{\partial\xi}{\partial w}\right)^2 \langle T(\xi)\rangle_{\text{UHP}} + \frac{c}{12}\{\xi;w\}\,,\{\xi;w\} = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2. \tag{6}$$

Conformal invariance requires $\langle T(\xi)\rangle_{\text{UHP}} = 0$ [2] thus all of the nonzero contribution comes from the Schwarzian term which yields

$$E(x,t) = \frac{\pi c}{24(2\tau_0)^2} = \frac{\pi c}{6(4\tau_0)^2}. \tag{7}$$

This functional form is the same as the energy density for a thermal density matrix, which is

$$E(x) = \frac{\mathrm{Tr}[He^{-\beta H}]}{\mathrm{Tr}\, e^{-\beta H}} = \frac{\pi c}{6\beta^2}. \tag{8}$$

Therefore we can make an identification $\tau_0 = \beta/4$. In the following we will use β instead of τ_0. The state and the conformal mapping becomes

$$|0\rangle \propto e^{-\beta H/4}\,|B\rangle\ , \quad \xi = e^{2\pi w/\beta}. \tag{9}$$

Though just a reparametrization, it has physical meanings: the quenched state in the long time behaves similar to a thermal state at the temperature β (which implies thermalization).

2.2 Single point function

Now let us look at a single point function of a generic scalar primary field

$$\langle 0|e^{iHt}\mathcal{O}(x)e^{-iHt}|0\rangle \to \langle \mathcal{O}(x,\tau)\rangle_{\text{strip}}\,. \tag{10}$$

Using the conformal mapping, we have

$$\langle \mathcal{O}(x,\tau)\rangle_{\text{strip}} = \left(\frac{\partial\xi}{\partial w}\right)^h \left(\frac{\partial\bar{\xi}}{\partial\bar{w}}\right)^h \langle \mathcal{O}(\xi,\bar{\xi})\rangle_{\text{UHP}} = \left(\frac{2\pi}{\beta}\right)^{2h} \left(\xi\bar{\xi}\right)^h \frac{A_{\mathcal{O}}}{\left(\xi-\bar{\xi}\right)^{2h}}. \tag{11}$$

Plugging in $\xi = \bar{\xi}^* = e^{2\pi(x+i\tau)/\beta}$ and introducing $\Delta = 2h$ as the scaling dimension, we find

$$\langle \mathcal{O}(x,\tau)\rangle_{\text{strip}} = \left(\frac{2\pi}{\beta}\right)^{\Delta} \frac{A_{\mathcal{O}}}{\left(e^{i2\pi\tau/\beta} - e^{-i2\pi\tau/\beta}\right)^{\Delta}}. \tag{12}$$

Performing the analytic continuation $\tau \to \tau_0 + it$ and noticing that $e^{i2\pi\tau_0/\beta} = i$, we will have

$$\langle 0|e^{iHt}\mathcal{O}(x)e^{-iHt}|0\rangle = A_{\mathcal{O}}\left(\frac{2\pi}{\beta}\right)^{\Delta}\left(\frac{1}{2i\cosh\frac{2\pi t}{\beta}}\right)^{\Delta} \approx A_{\mathcal{O}}\left(\frac{2\pi}{\beta}\right)^{\Delta} e^{-\frac{2\pi\Delta}{\beta}t}. \tag{13}$$

Thus any scalar primary field exponentially decays in time to zero, which is also the value for a thermal state. The decay rate depends on $\beta = 4\tau_0$ and is not universal.

[2] Another way to say that is $T(\xi)$ is purely holomorphic therefore doesn't have any image.

2.3 Two point function

Finally let us look at the two point function of two scalar primaries

$$\langle 0|e^{iHt}\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)e^{-iHt}|0\rangle \to \langle \mathcal{O}_1(x_1,\tau_1)\mathcal{O}_2(x_2,\tau_2)\rangle_{\text{strip}}\,. \tag{14}$$

Because of the translational invariance, we can put x_2 at $x=0$ and x_1 at x. Using the conformal mapping, we have

$$\langle \mathcal{O}_1\mathcal{O}_2\rangle_{\text{strip}} = \left(\frac{\partial \xi_1}{\partial w_1}\right)^{h_1}\left(\frac{\partial \overline{\xi}_1}{\partial \overline{w}_1}\right)^{\overline{h}_1}\left(\frac{\partial \xi_2}{\partial w_2}\right)^{h_2}\left(\frac{\partial \overline{\xi}_2}{\partial \overline{w}_2}\right)^{\overline{h}_2}\langle \mathcal{O}_1(\xi_1,\overline{\xi}_1)\mathcal{O}_2(\xi_2,\overline{\xi}_2)\rangle_{\text{UHP}} \tag{15}$$

On the upper half-plane, we can use the mirror trick to convert it to a 4pt function. We assume the conformal dimension of $\mathcal{O}_1$ and $\mathcal{O}_2$ are h_1 and h_2 respectively so that we have

$$\langle \mathcal{O}_1(\xi_1,\overline{\xi}_1)\mathcal{O}_2(\xi_2,\overline{\xi}_2)\rangle_{\text{UHP}} = \frac{A_{\mathcal{O}_1}}{(\xi_1-\overline{\xi}_1)^{2h_1}}\frac{A_{\mathcal{O}_2}}{(\xi_2-\overline{\xi}_2)^{2h_2}}F(r), \quad r=\frac{\xi_{1\overline{1}}\xi_{2\overline{2}}}{\xi_{12}\xi_{\overline{12}}}. \tag{16}$$

$F(r)$ is a superposition of the chiral conformal blocks and the coefficients are determined by the boundary conditions. Two limits are particularly useful:

- $r\to\infty$, which corresponds to that $\mathcal{O}_1$ and $\mathcal{O}_2$ are deep in the bulk. In this limit, the boundary two-point function should reduce to the bulk two-point function, regardless of the conformal boundary condition. Combined with conformal invariance, this implies

$$F(r\to\infty)\propto \begin{cases} \left(\dfrac{1}{r}\right)^{-2h}, & \mathcal{O}_1=\mathcal{O}_2=\mathcal{O}; \\ \left(\dfrac{1}{r}\right)^{h_q-h_1-h_2}, & \mathcal{O}_1\neq\mathcal{O}_2. \end{cases} \tag{17}$$

- $r\to 0$, which corresponds to that $\mathcal{O}_1$ and $\mathcal{O}_2$ are close to the boundary (real axis). In this limit, the two-point function should factorize to the product of two one-point functions, which is exactly the prefactor in front of $F(r)$. This implies $F(r\to 0)=1$. If the one point function happens to be zero, we have $F(r\to 0)=\#r^{h_p}$.

Besides $\left(\frac{2\pi}{\beta}\right)^{\Delta_1+\Delta_2}$, the prefactor in front of $F(r)$ gives

$$\left(e^{\frac{i2\pi\tau_1}{\beta}}-e^{-\frac{i2\pi\tau_1}{\beta}}\right)^{-2h_1}\left(e^{\frac{i2\pi\tau_2}{\beta}}-e^{-\frac{i2\pi\tau_2}{\beta}}\right)^{-2h_2}\to\left(2i\cosh\frac{2\pi t}{\beta}\right)^{-2h_1-2h_2} \tag{18}$$

The cross ratio is

$$r=\frac{e^{\frac{2\pi x}{\beta}}\left(e^{\frac{i2\pi\tau_1}{\beta}}-e^{-\frac{i2\pi\tau_1}{\beta}}\right)\left(e^{\frac{i2\pi\tau_2}{\beta}}-e^{-\frac{i2\pi\tau_2}{\beta}}\right)}{\left(e^{\frac{2\pi(x+i\tau_1)}{\beta}}-e^{\frac{i2\pi\tau_2}{\beta}}\right)\left(e^{\frac{2\pi(x-i\tau_1)}{\beta}}-e^{-\frac{i2\pi\tau_2}{\beta}}\right)}\to\frac{e^{\frac{2\pi x}{\beta}}\left(2i\cosh\frac{2\pi t}{\beta}\right)^2}{\left(e^{\frac{2\pi x}{\beta}}-1\right)^2} \tag{19}$$

Let us focus on the long-distance limit $x \gg \beta$ so that we can approximate

$$r \approx e^{-\frac{2\pi x}{\beta}} \left(2i \cosh \frac{2\pi t}{\beta}\right)^2 = -e^{-\frac{2\pi(x-2t)}{\beta}} \left(1 - e^{-\frac{4\pi t}{\beta}}\right)^2. \tag{20}$$

Therefore for the early-time regime $t < x/2$, the cross ratio is exponentially small. For the late-time regime, the cross ratio is exponentially large. Different time regimes correspond to different OPE limits for $F(r)$ and we can draw the following conclusions:

- Early time $\beta \ll t < x/2$ corresponding to $r \ll 1$, the two point function depends on time exponentially

$$\langle 0|\mathcal{O}_1(x,t)\mathcal{O}_2(0,t)|0\rangle \approx \left(\frac{2\pi}{\beta}\right)^{\Delta_1+\Delta_2} e^{-\frac{2\pi(\Delta_1+\Delta_2)}{\beta}t} e^{-\frac{2\pi h_p}{\beta}(x-2t)}. \tag{21}$$

  It is exponential decay when $2h_p < \Delta_1 + \Delta_2$.

- Late time $t > x/2$ corresponding to $r \gg 1$, the two point function becomes

$$\langle 0|\mathcal{O}_1(x,t)\mathcal{O}_2(0,t)|0\rangle \approx \begin{cases} e^{-\frac{2\pi\Delta}{\beta}x} & \mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O} \\ e^{-\frac{2\pi(h_1+h_2)}{\beta}x} e^{\frac{2\pi h_q}{\beta}(x-2t)} & \mathcal{O}_1 \neq \mathcal{O}_2 \end{cases} \tag{22}$$

  One can see that for the case $\mathcal{O}_1 = \mathcal{O}_2$, the two-point function saturates to the same value as the thermal state result after $t > x/2$. $t = x/2$ is also the time for two operators enters the lightcone of each other. For the other case $\mathcal{O}_1 \neq \mathcal{O}_2$, it keeps decaying to zero (to be consistent with the boundary condition), which is also the same value as the thermal state result.

- Intermediate regime $t \sim x/2$, the width of this time window is controlled by β.

Example: Ising CFT In the Ising CFT, we can compute the $\sigma\sigma$ correlation $\langle\sigma(x,t)\sigma(0,t)\rangle$. The two-point function on the upper half-plane can be written as

$$\langle\sigma_1\sigma_2\rangle_{\text{HUP}} = \frac{1}{(z_{1\bar{1}}z_{2\bar{2}})^{1/8}} r^{1/8} \left(\frac{r}{r-1}\right)^{1/8} F(1/r) \tag{23}$$

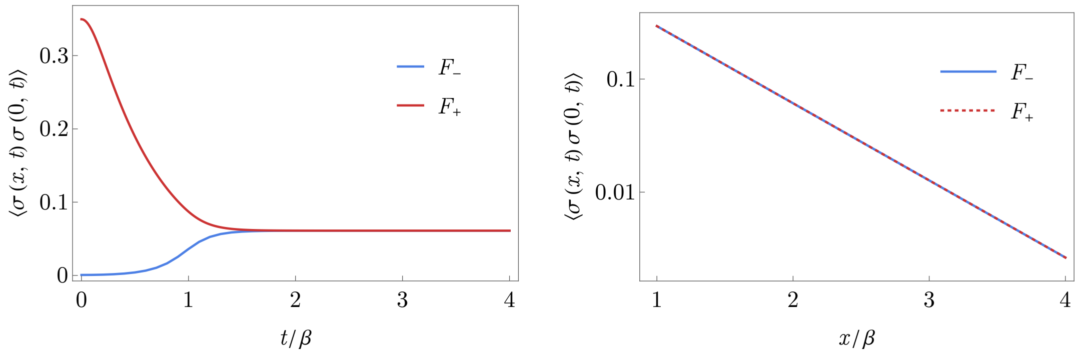
in which the conformal block has two different choices corresponding to different conformal boundary conditions

$$F_\pm(\eta) = \sqrt{\sqrt{1-\eta}+1} \pm \sqrt{\sqrt{1-\eta}-1}. \tag{24}$$

This way of writing makes it transparent that the two-point function reduces to the bulk two-point function in the limit $r \to -\infty$, regardless of the choice of $F_\pm$.

To probe the $r \sim 0$ limit, let us use the identity $\sqrt{A} \pm \sqrt{B} = \sqrt{A + B \pm 2\sqrt{AB}}$ to rewrite the conformal block as

$$F_\pm(1/r) = \sqrt{2}(-r)^{-1/4}\sqrt{\sqrt{1-r} \pm 1} \tag{25}$$



(a) $\langle\sigma(x,t)\sigma(0,t)\rangle$ as a function of time (b) $\langle\sigma(x,t)\sigma(0,t)\rangle$ as a function of distance

Figure 1: For (a) we choose $x/\beta = 3$ so the turning point is at $t/\beta = 3/2$. For (b) we choose $t = 4$ so that $t > x/2$.

The two-point function becomes

$$\langle\sigma_1\sigma_2\rangle_{\mathrm{HUP}} = \frac{\sqrt{2}}{(z_{1\bar{1}}z_{2\bar{2}})^{1/8}}\frac{\sqrt{\sqrt{1-r}\pm 1}}{(r-1)^{1/8}}. \tag{26}$$

It is now clear that in the limit $r \sim 0$, F_+ will give a finite result while F_- will give a vanishing result.

Therefore the late time value of $\langle\sigma(x,t)\sigma(0,t)\rangle$ will be independent of the conformal boundary condition and is equal to the two-point function in a thermal state. The initial behavior of $\langle\sigma(x,t)\sigma(0,t)\rangle$ is decay for F_+ and increasing for F_-. The whole curve is plotted in Fig. 1.

3 Entanglement entropy

In this section, let us look at the time evolution of the entanglement entropy of the subsystem $A = [0, x]$, whose reduced density matrix is denoted as ρ_A. We will follow the Calabrese-Cardy's prescription to compute the von Neumann entropy S_A. That is to say, we insert two twist operators $\mathcal{T}_n$ at x and 0 to compute the n-th Rényi entropy

$$S_A^{(n)} = \frac{1}{1-n}\log\mathrm{Tr}\,\rho_A^n, \tag{27}$$

whose $n \to 1$ limit gives the von Neumann entropy.

The twist operator $\mathcal{T}_n$ is a primary with conformal dimensions $h_n = \bar{h}_n = \frac{c}{24}(n - \frac{1}{n})$. Its two-point correlation function reproduces $\mathrm{Tr}\,\rho_A^n$

$$\mathrm{Tr}\,\rho_A^m = \langle 0|e^{iHt}\mathcal{T}_n(x)\mathcal{T}_n(0)e^{-iHt}|0\rangle\,. \tag{28}$$

Therefore we can just copy our result above, which yields

$$S_A(t) = \frac{c}{3}\log\beta + \begin{cases} \frac{2\pi c}{3\beta}t & t < x/2 \\ \frac{\pi c}{3\beta}x & t > x/2 \end{cases}. \tag{29}$$

Thus the entanglement entropy linearly grows and saturates to a volume-law.

References

[1] Pasquale Calabrese and John Cardy, *Quantum quenches in 1+1 dimensional conformal field theories*, *arXiv:* 1603.02889.

[2] John Cardy, *Quantum Quenches to a Critical Point in One Dimension: some further results*, *arXiv:* 1507.07266.