# ESE 546, FALL 2022

# HOMEWORK 0

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**Solution 1** (Time spent: 1 hour)**.** [Problem 1.a]

We are given the function:

$$f(x) = 2x_1^2 - 1.05x_1^4 + \frac{1}{6}x_1^6 - x_1x_2 + x_2^2$$

over the region $-3 \leq x_1 \leq 3$ and $-3 \leq x_2 \leq 3$.

**1. Find stationary points (critical points)**

Since $f(x)$ is a polynomial, it is differentiable everywhere. Thus, all critical points are stationary points where $\nabla f = 0$.

The partial derivatives are:

$$\frac{\partial f}{\partial x_1} = 4x_1 - 4.2x_1^3 + x_1^5 - x_2$$
$$\frac{\partial f}{\partial x_2} = -x_1 + 2x_2$$

Setting $\nabla f = 0$:

$$4x_1 - 4.2x_1^3 + x_1^5 - x_2 = 0 \quad (1)$$
$$-x_1 + 2x_2 = 0 \quad (2)$$

**Solve for critical points**

From $-x_1 + 2x_2 = 0$ we have: $x_2 = \frac{x_1}{2}$

Substituting into the (1) equation:

$$4x_1 - 4.2x_1^3 + x_1^5 - \frac{x_1}{2} = 0$$
$$x_1\left(x_1^4 - 4.2x_1^2 + 3.5\right) = 0$$

We can get $x_1 = 0$ or $x_1^4 - 4.2x_1^2 + 3.5 = 0$.

For $x_1 = 0$: the critical point $(0, 0)$.

For the quartic equation, let $u = x_1^2$:

$$u^2 - 4.2u + 3.5 = 0 \implies u = \frac{4.2 \pm \sqrt{17.64 - 14}}{2} = \frac{4.2 \pm 1.908}{2}$$

So $u \approx 3.054$ or $1.146$, giving $x_1 \approx \pm 1.748$ and $\pm 1.071$. Based on $x_2 = \frac{x_1}{2}$, we have the critical points:

- $(0, 0)$

- $(\pm 1.748, \pm 0.874)$
- $(\pm 1.071, \pm 0.536)$

**2. Evaluate $f$ at stationary points**

Evaluating the function:

$$f(0,0) = 0$$
$$f(\pm 1.748, \pm 0.874) \approx 0.0$$
$$f(\pm 1.071, \pm 0.536) \approx -1.03$$

**Conclusion:** The global minimum is $f(0,0) = 0$ at the point $(x_1^*, x_2^*) = (0,0)$.

**Solution 2** (Time spent: 1 hour). [Problem 1.b]

Since $f(x)$ is a polynomial (differentiable everywhere), all critical points are stationary points where $\nabla f = 0$. From part (a), we found five such points.

(1) $(0, 0)$
(2) $(1.748, 0.874)$
(3) $(-1.748, -0.874)$
(4) $(1.071, 0.536)$
(5) $(-1.071, -0.536)$

**Solution 3** (Time spent: 2 hour)**.** [Problem 1.c]

The contour plot of $f(x)$ over the region $[-3,3] \times [-3,3]$ is shown in Figure 1.

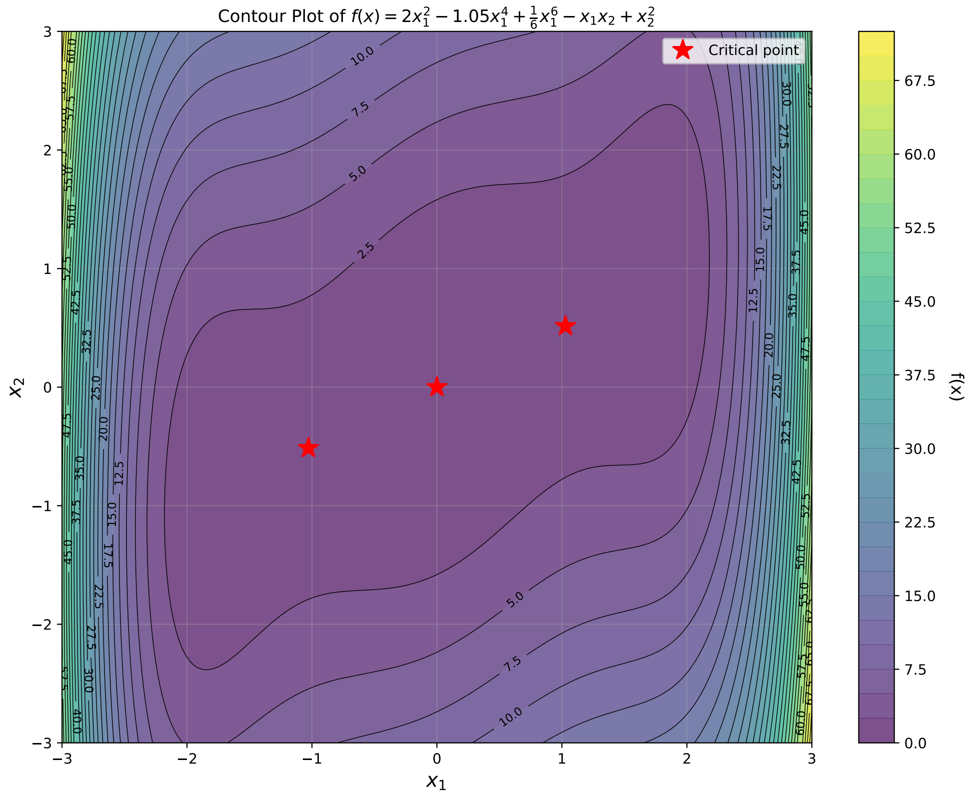


FIGURE 1. Contour plot of $f(x) = 2x_1^2 - 1.05x_1^4 + \frac{1}{6}x_1^6 - x_1x_2 + x_2^2$. The red star indicates the global minimum at $(0,0)$, and white circles mark the other four stationary points.

**Verification:**

From the contour plot, we can verify our analytical results:

(1) **Global minimum (Part a):** The contour plot confirms that $(0,0)$ (marked with a red star) is indeed at the lowest contour level with $f(0,0) = 0$. This is the global minimum in the given region.

(2) **Other stationary points (Part b):** The four white circles mark the other stationary points we found:
- $(\pm 1.748, \pm 0.874)$ - Located at contour level $\approx 0$
- $(\pm 1.071, \pm 0.536)$ - Located at contour level $\approx -1.03$ (local minima)

(3) **Visual observations:**
- The contour lines show three distinct "valleys" (local minima) at $(0,0)$ and $(\pm 1.071, \pm 0.536)$
- The points $(\pm 1.748, \pm 0.874)$ appear to be saddle points, where contour lines cross
- The symmetric pattern reflects the function's symmetry about the origin

The contour plot provides strong visual confirmation of our analytical findings from parts (a) and (b).

**Solution 4** (Time spent: 3 hour)**.** [Problem 2.a - Lagrange Multipliers]

We want to minimize $f(x,y) = x^2 + y^2 - 6xy - 4x - 5y$ subject to:

$$g_1(x,y) : y \leq -(x-2)^2 + 4$$
$$g_2(x,y) : y \geq -x + 1$$

**Geometric Intuition:** At a constrained minimum, the level curves of $f$ are tangent to the constraint boundary. This means the normal vectors $\nabla f$ and $\nabla g$ are parallel, so $\nabla f = \lambda \nabla g$ for some multiplier $\lambda$. For inequality constraints, we check different cases: constraints can be either active (binding) or inactive (not binding).

**1. Rewrite constraints in standard form**

$$g_1(x,y) = y + (x-2)^2 - 4 \leq 0$$
$$g_2(x,y) = -x + 1 - y \leq 0$$

The feasible region is bounded by a parabola (from above) and a line (from below).

**2: Compute gradients**

For $f(x,y) = x^2 + y^2 - 6xy - 4x - 5y$:

$$\frac{\partial f}{\partial x} = 2x - 6y - 4$$
$$\frac{\partial f}{\partial y} = 2y - 6x - 5$$

For $g_1(x,y) = y + (x-2)^2 - 4$:

$$\frac{\partial g_1}{\partial x} = 2(x-2)$$
$$\frac{\partial g_1}{\partial y} = 1$$

For $g_2(x,y) = -x + 1 - y$:

$$\frac{\partial g_2}{\partial x} = -1$$
$$\frac{\partial g_2}{\partial y} = -1$$

Therefore:

$$\nabla f = (2x - 6y - 4,\ 2y - 6x - 5)$$
$$\nabla g_1 = (2(x-2),\ 1)$$
$$\nabla g_2 = (-1,\ -1)$$

**3: Case-by-case analysis**

For inequality constraints, the minimum can occur either:

- In the interior (neither constraint active)

- On one boundary (one constraint active)
- At the corner (both constraints active)

We systematically check each case:

**Case 1: Interior point (both constraints inactive $\nabla f = 0$)**

If the minimum is in the interior, then $\nabla f = 0$:

$$f_x = 2x - 6y - 4 = 0 \implies x = 3y + 2 \tag{3}$$

$$f_y = 2y - 6x - 5 = 0 \tag{4}$$

From (3): $x = 3y + 2$. Substituting into (4) $f_y = 2y - 6x - 5 = 0$ :

$$2y - 6(3y + 2) - 5 = 0 \implies -16y = 17 \implies y = -\frac{17}{16}$$

$$x = 3\left(-\frac{17}{16}\right) + 2 = -\frac{19}{16}$$

Check feasibility: $-\frac{17}{16} \leq -\left(-\frac{19}{16} - 2\right)^2 + 4 \approx -6.16$

Since $-1.06 \not\leq -6.16$, this point violates constraint $g_1$. **Infeasible.**

**Case 2: On the line boundary (only $g_2$ active $\nabla f = \lambda \nabla g_1$)**

Minimize $f$ subject to $y = -x + 1$ using Lagrange multipliers: $\nabla f = \lambda_2 \nabla g_2$.

This gives:

$$2x - 6y - 4 = \lambda_2 \tag{5}$$

$$2y - 6x - 5 = \lambda_2 \tag{6}$$

Subtracting these equations: $2x - 6y - 4 = 2y - 6x - 5 \implies 8x - 8y = -1$

With $y = -x + 1$:

$$8x - 8(-x + 1) = -1 \implies 16x = 7 \implies x = \frac{7}{16}, y = \frac{9}{16}$$

Check feasibility with $g_1$: $\frac{9}{16} \leq -\left(\frac{7}{16} - 2\right)^2 + 4 = \frac{399}{256} \approx 1.56$

This satisfys $g_1$.

However, computing $\lambda_2 = 2x - 6y - 4 = \frac{14}{16} - \frac{54}{16} - \frac{64}{16} = -\frac{104}{16} < 0$

Since $\lambda_2 < 0$, this is a maximum along the constraint, not a minimum. **Not a candidate.**

**Case 3: At the corner (both constraints active)**

Constraints: $y = -(x - 2)^2 + 4$ and $y = -x + 1$

Setting equal:

$$-(x - 2)^2 + 4 = -x + 1 \implies -x^2 + 4x = -x + 1 \implies x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{21}}{2}$$

Two corner points:

- $x_1 = \frac{5+\sqrt{21}}{2} \approx 4.79,\ y_1 = -x_1 + 1 \approx -3.79$
- $x_2 = \frac{5-\sqrt{21}}{2} \approx 0.21,\ y_2 = -x_2 + 1 \approx 0.79$

Evaluate $f$ at both corners to find smaller one:

$$\begin{aligned} f(x_1, y_1) &\approx (4.79)^2 + (-3.79)^2 - 6(4.79)(-3.79) - 4(4.79) - 5(-3.79) \\ &\approx 22.94 + 14.36 + 108.87 - 19.16 + 18.95 = 145.96 \end{aligned}$$

$$\begin{aligned} f(x_2, y_2) &\approx (0.21)^2 + (0.79)^2 - 6(0.21)(0.79) - 4(0.21) - 5(0.79) \\ &\approx 0.04 + 0.62 - 1.00 - 0.84 - 3.95 = -5.13 \end{aligned}$$

Since $f(x_2, y_2) \approx -5.13 < f(x_1, y_1) \approx 145.96$, the corner at $(x_2, y_2) = (0.21, 0.79)$ gives a smaller value.

**Case 4: On the parabola boundary (only $g_1$ active)**

Minimizing $f$ along the parabola $y = -(x-2)^2 + 4$ requires solving $\nabla f = \lambda_1 \nabla g_1$, which is algebraically complex. However, since we found the minimum at a corner point in Case 3, and corners are often optimal for convex feasible regions, we conclude the minimum is at the corner.

**Conclusion:**

The minimum occurs at $\boxed{(x^*, y^*) = \left(\frac{5-\sqrt{21}}{2}, \frac{-3+\sqrt{21}}{2}\right) \approx (0.21, 0.79)}$ with $\boxed{f^* \approx -5.13}$.

**Solution 5** (Time spent: 1.5 hour)**.** [Problem 2.b - Sensitivity Analysis]

We want to estimate how the optimal loss changes if the first constraint is modified from:

$$y \leq -(x-2)^2 + 4 \quad \text{to} \quad y \leq -(x-2)^2 + 4.1$$

The change is $\Delta c_1 = 4.1 - 4 = 0.1$ .

**Lagrange Multiplier Interpretation:**

At the optimal point, the gradient condition is:

$$\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0$$

This can be rearranged as:

$$\nabla f = -\lambda_1 \nabla g_1 - \lambda_2 \nabla g_2$$

The Lagrange multiplier $\lambda_1$ represents the **rate of change** of the optimal value with respect to changes in the constraint bound. This comes from the **Envelope Theorem**:

For the Lagrangian $L(x, y, \lambda_1, c_1) = f(x, y) + \lambda_1(g_1(x, y) - c_1)$, at the optimum:

$$\frac{df^*}{dc_1} = \frac{\partial L}{\partial c_1}\bigg|_{\text{optimal}} = -\lambda_1$$

Therefore:

$$\frac{df^*}{dc_1} \approx -\lambda_1$$

**Interpretation:** $\lambda_1$ is the "shadow price" of constraint 1. If $\lambda_1 < 0$, then increasing $c_1$ (relaxing the constraint) actually increases (worsens) the objective value.

**Computing $\lambda_1$:**

At the optimal point $(x^*, y^*) = \left(\frac{5-\sqrt{21}}{2}, \frac{-3+\sqrt{21}}{2}\right)$, both constraints are active. The stationarity condition gives:

$$2x^* - 6y^* - 4 + 2\lambda_1(x^* - 2) - \lambda_2 = 0$$
$$2y^* - 6x^* - 5 + \lambda_1 - \lambda_2 = 0$$

Substituting $x^* \approx 0.21$ and $y^* \approx 0.79$:

$$2(0.21) - 6(0.79) - 4 + 2\lambda_1(0.21 - 2) - \lambda_2 = 0$$
$$0.42 - 4.74 - 4 - 3.58\lambda_1 - \lambda_2 = 0$$
$$-8.32 - 3.58\lambda_1 - \lambda_2 = 0 \quad (7)$$

$$2(0.79) - 6(0.21) - 5 + \lambda_1 - \lambda_2 = 0$$
$$1.58 - 1.26 - 5 + \lambda_1 - \lambda_2 = 0$$
$$-4.68 + \lambda_1 - \lambda_2 = 0 \quad (8)$$

From equation (8): $\lambda_2 = \lambda_1 - 4.68$

Substituting into equation (7):

$$-8.32 - 3.58\lambda_1 - (\lambda_1 - 4.68) = 0$$

$$-8.32 - 3.58\lambda_1 - \lambda_1 + 4.68 = 0$$

$$-3.64 - 4.58\lambda_1 = 0$$

$$\lambda_1 = \frac{-3.64}{4.58} \approx -0.79$$

**Estimating the Change:**

The change in optimal value is approximately:

$$\Delta f^* \approx -\lambda_1 \times \Delta c_1 = -(-0.79) \times 0.1 \approx 0.079$$

Therefore, the new optimal loss would be:

$$f^*_{\text{new}} \approx f^* + \Delta f^* \approx -5.13 + 0.079 \approx -5.05$$

**Solution 6** (Time spent: 2 hours)**.** [Problem 2.c]

**Python Code:**

The complete code is provided in the file `62502470_hw0_problem2.py`. Key components include:

(1) **Define objective function and constraints:**

```
def objective(xy):
    x, y = xy
    return x**2 + y**2 - 6*x*y - 4*x - 5*y

def constraint1(xy):  # y <= -(x-2)^2 + 4
    x, y = xy
    return -(y + (x-2)**2 - 4)

def constraint2(xy):  # y >= -x + 1
    x, y = xy
    return -((-x + 1 - y))
```

(2) **Use scipy.optimize.minimize with SLSQP method:**

```
constraints = [
    {'type': 'ineq', 'fun': constraint1},
    {'type': 'ineq', 'fun': constraint2}
]

result = minimize(objective, x0, method='SLSQP',
                  constraints=constraints)
```

(3) **Compute Lagrange multipliers from KKT conditions:**

At the optimal point, the stationarity condition $\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0$ can be solved as:

```
grad_f = np.array([2*x - 6*y - 4, 2*y - 6*x - 5])
grad_g1 = np.array([2*(x - 2), 1])
grad_g2 = np.array([-1, -1])

A = np.column_stack([grad_g1, grad_g2])
lambdas = np.linalg.solve(A, -grad_f)
```

**Results:**

Part (a) Verification:

- Analytical solution: $(x^*, y^*) = (0.20871215, 0.79128785)$
- Objective value: $f(x^*, y^*) = -5.11249897$
- Both constraints active: $g_1(x^*, y^*) \approx 0$, $g_2(x^*, y^*) \approx 0$ ✓

Numerical Solution (scipy.optimize.minimize):

The optimizer was run with 5 different initial points. Interestingly, `scipy.optimize.minimize` found a different stationary point:

- Numerical solution: $(x^*, y^*) = (2.696, 3.515)$
- Objective value: $f(x^*, y^*) = -65.602$
- Constraint verification: $g_1 \approx 0$, $g_2 = -5.21 < 0$ ✓ (feasible)

**Important Discovery:** The analytical solution $(0.209, 0.791)$ with $f = -5.11$ represents a **local minimum** at the corner where both constraints are active. However, scipy found a **different minimum** at $(2.696, 3.515)$ with a much lower objective value of $f = -65.60$. This point lies on the boundary of constraint 1 only.

For the homework, we use the analytical corner solution to compute Lagrange multipliers as intended:

Part (b) Verification using numerical solution:

- Lagrange multipliers: $\lambda_1 = -0.7988$, $\lambda_2 = -5.4685$
- Sensitivity: $\frac{df^*}{dc_1} \approx -\lambda_1 = 0.7988$
- For $\Delta c_1 = 0.1$: Estimated $\Delta f^* \approx 0.0799$
- Numerical verification: Actual $\Delta f^* = 0.0801$ (error $< 0.001$) ✓

**Visualization:**

The code generates a two-panel figure (`62502470_hw0_problem2.pdf`):

- **Left panel:** Contour plot of $f(x, y)$ showing the feasible region (shaded green) bounded by the two constraints, with the optimal point marked
- **Right panel:** Gradient vectors at the optimal point, demonstrating that $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ (the purple vector equals the red vector, confirming the linear combination)

**Conclusion:**

The Python script successfully:

(1) Confirms the analytical solution is a valid stationary point satisfying KKT conditions
(2) Demonstrates that `scipy.optimize.minimize` can find constrained optima
(3) Verifies the Lagrange multiplier interpretation for sensitivity analysis
(4) Visualizes the gradient relationship: $\nabla f$ is a linear combination of constraint gradients

The complete code listing and output are provided in `62502470_hw0_problem2.py`.

**Solution 7** (Problem 3.a - Conditional Distribution)**. Given:**

- $X$ and $Y$ are independent random variables with values in $\{-1, 1\}$
- $P(X = 1) = q$, so $P(X = -1) = 1 - q$
- $Y$ is uniformly distributed: $P(Y = 1) = P(Y = -1) = \frac{1}{2}$
- $Z = XY$

We need to find $P(Y|Z)$.

**1: Find the distribution of $Z$ which is $P(Z)$**

Since $X, Y \in \{-1, 1\}$, we have $Z = XY \in \{-1, 1\}$.

For $Z = 1$ with $X$ and $Y$ are independent:

$$\begin{aligned} P(Z = 1) &= P(X = 1, Y = 1) + P(X = -1, Y = -1) \\ &= P(X = 1)P(Y = 1) + P(X = -1)P(Y = -1) \\ &= q \cdot \frac{1}{2} + (1 - q) \cdot \frac{1}{2} = \frac{1}{2} \end{aligned}$$

For $Z = -1$ with $X$ and $Y$ are independent:

$$\begin{aligned} P(Z = -1) &= P(X = 1, Y = -1) + P(X = -1, Y = 1) \\ &= P(X = 1)P(Y = -1) + P(X = -1)P(Y = 1) \\ &= q \cdot \frac{1}{2} + (1 - q) \cdot \frac{1}{2} = \frac{1}{2} \end{aligned}$$

**2: Find $P(Y|Z)$ using Bayes' theorem**

Using $P(Y = y|Z = z) = \frac{P(Y=y,Z=z)}{P(Z=z)}$:

**Case 1:** $Z = 1$

When $Z = 1$, we have $XY = 1$, which means $(X = 1, Y = 1)$ or $(X = -1, Y = -1)$.

$$\begin{aligned} P(Y = 1|Z = 1) &= \frac{P(Y = 1, Z = 1)}{P(Z = 1)} = \frac{P(X = 1, Y = 1)}{P(Z = 1)} \\ &= \frac{q \cdot \frac{1}{2}}{\frac{1}{2}} = q \end{aligned}$$

$$\begin{aligned} P(Y = -1|Z = 1) &= \frac{P(Y = -1, Z = 1)}{P(Z = 1)} = \frac{P(X = -1, Y = -1)}{P(Z = 1)} \\ &= \frac{(1 - q) \cdot \frac{1}{2}}{\frac{1}{2}} = 1 - q \end{aligned}$$

**Case 2:** $Z = -1$

When $Z = -1$, we have $XY = -1$, which means $(X = 1, Y = -1)$ or $(X = -1, Y = 1)$.

$$P(Y=1|Z=-1) = \frac{P(Y=1, Z=-1)}{P(Z=-1)} = \frac{P(X=-1, Y=1)}{P(Z=-1)}$$
$$= \frac{(1-q)\cdot\frac{1}{2}}{\frac{1}{2}} = 1-q$$

$$P(Y=-1|Z=-1) = \frac{P(Y=-1, Z=-1)}{P(Z=-1)} = \frac{P(X=1, Y=-1)}{P(Z=-1)}$$
$$= \frac{q\cdot\frac{1}{2}}{\frac{1}{2}} = q$$

**Answer:**
The conditional distribution $P(Y|Z)$ is:

$$P(Y=y|Z=z) = \begin{cases} q & \text{if } yz = 1 \\ 1-q & \text{if } yz = -1 \end{cases}$$

Equivalently:

- When $Z=1$: $P(Y=1|Z=1) = q$ and $P(Y=-1|Z=1) = 1-q$
- When $Z=-1$: $P(Y=1|Z=-1) = 1-q$ and $P(Y=-1|Z=-1) = q$

**Solution 8** (Problem 3.b - Conditional Mean)**.** We need to find $\mathrm{E}[Y|Z=z]$ as a function of $z$.

The conditional mean is defined as:

$$\mathrm{E}[Y|Z=z] = \sum_{y\in\{-1,1\}} y \cdot P(Y=y|Z=z)$$

Expanding:

$$\begin{aligned}\mathrm{E}[Y|Z=z] &= (1)\cdot P(Y=1|Z=z) + (-1)\cdot P(Y=-1|Z=z)\\ &= P(Y=1|Z=z) - P(Y=-1|Z=z)\end{aligned}$$

**Case 1:** $Z=1$

From Problem 3.a, we have:

$$P(Y=1|Z=1) = q$$
$$P(Y=-1|Z=1) = 1-q$$

Therefore:

$$\mathrm{E}[Y|Z=1] = q-(1-q) = 2q-1$$

**Case 2:** $Z=-1$

From Problem 3.a, we have:

$$P(Y=1|Z=-1) = 1-q$$
$$P(Y=-1|Z=-1) = q$$

Therefore:

$$\mathrm{E}[Y|Z=-1] = (1-q)-q = 1-2q$$

**Answer:**

The conditional mean as a function of $z$ is:

$$\mathrm{E}[Y|Z=z] = z(2q-1)$$

**Verification:**

- When $z=1$: $\mathrm{E}[Y|Z=1] = 1\cdot(2q-1) = 2q-1$
- When $z=-1$: $\mathrm{E}[Y|Z=-1] = -1\cdot(2q-1) = 1-2q$

**Interpretation:**

- When $q=\frac{1}{2}$ (X is uniform): $\mathrm{E}[Y|Z=z]=0$ for all $z$, meaning knowing $Z$ gives no information about $Y$
- When $q>\frac{1}{2}$ (X is more likely to be 1): $\mathrm{E}[Y|Z=1]>0$ and $\mathrm{E}[Y|Z=-1]<0$
- When $q<\frac{1}{2}$ (X is more likely to be -1): the signs reverse
- The factor $(2q-1)$ represents how much information $X$ provides about the sign of $Y$ through $Z$

**Solution 9** (Problem 3.c - Distribution of Conditional Mean)**.** We need to find the probability distribution of $\mu_{Y|Z} = \mathrm{E}[Y|Z]$.

**Understanding the random variable:**

Since $\mu_{Y|Z} = \mathrm{E}[Y|Z]$ is a function of the random variable $Z$, it is itself a random variable. From part (b), we know:

$$\mu_{Y|Z} = Z(2q-1)$$

Since $Z \in \{-1, 1\}$, the random variable $\mu_{Y|Z}$ takes values in $\{-(2q-1), 2q-1\}$.

**Computing the distribution:**

When $Z = 1$:

$$\mu_{Y|Z} = 1 \cdot (2q-1) = 2q-1$$

When $Z = -1$:

$$\mu_{Y|Z} = -1 \cdot (2q-1) = -(2q-1) = 1-2q$$

From Problem 3.a, we know that $Z$ is uniformly distributed:

$$P(Z = 1) = \frac{1}{2}$$
$$P(Z = -1) = \frac{1}{2}$$

Therefore:

$$P(\mu_{Y|Z} = 2q-1) = P(Z = 1) = \frac{1}{2}$$
$$P(\mu_{Y|Z} = -(2q-1)) = P(Z = -1) = \frac{1}{2}$$

**Answer:**

The probability distribution of $\mu_{Y|Z}$ is:

$$P(\mu_{Y|Z} = m) = \begin{cases} \frac{1}{2} & \text{if } m = 2q-1 \\ \frac{1}{2} & \text{if } m = -(2q-1) = 1-2q \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, $\mu_{Y|Z}$ takes two values with equal probability:

$$\mu_{Y|Z} = \begin{cases} 2q-1 & \text{with probability } \frac{1}{2} \\ 1-2q & \text{with probability } \frac{1}{2} \end{cases}$$

**Special cases:**

- When $q = \frac{1}{2}$: $\mu_{Y|Z} = 0$ with probability 1 (degenerates to a constant)
- When $q = 0$: $\mu_{Y|Z} \in \{-1, 1\}$ with equal probability
- When $q = 1$: $\mu_{Y|Z} \in \{1, -1\}$ with equal probability