Numerical Approximation of Solutions to Obstacle Problems

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Abstract

The main goal of this article is to develop numerical methods for approximating solutions to elliptic and parabolic obstacle problems, using finite difference and finite element methods. The article deals with an elliptic obstacle problem for an elastic membrane, introducing Lagrange multipliers and implementing numerical methods for the approximation of its solution. In the context of financial mathematics, the Black-Scholes model for the valuation of American options was considered and treated as a parabolic obstacle problem. Finite difference and finite element methods were implemented to solve numerically this obstacle problem.
Keywords: Obstacle Problem; Finite Element Method; Finite Differences; Elastic Membrane; Black-Scholes Model.

1. Introduction

Free boundary problems (FBPs) are partial differential equations (PDEs) that involve unknown boundaries, requiring the determination of both the solution and the boundary itself. An important example is the Obstacle Problem, where the boundary is limited by an obstacle, represented by a function that imposes restrictions on the solution.

In the Obstacle Problem, the interaction between the solution and the obstacle is governed by a complementary condition, creating a contact region with an unknown free boundary. The Elastic Membrane is a classic example used to study these problems due to its relative simplicity and historical importance.

Another significant example is in finance, specifically the Black-Scholes model for American options. This problem involves time, adding another dimension to the problem, and is relevant to the study of parabolic operators due to its simple structure and practical application.

The analytical resolution of PDEs associated with the obstacle problem is often unfeasible. Numerical methods, such as the Finite Element Method (FEM) and the Finite Difference Method (FDM), are essential for obtaining accurate approximations to the solutions.

2. Elliptic Obstacle Problem: Elastic Membrane

A homogeneous membrane can be taken as a thin plate that offers no resistance to bending, acting only in tension (internal to the membrane). Occupying a domain $\Omega \subset \mathbb{R}^2$, in the Oxy plane in 2D. It is supposed to be equally stretched in all directions by a uniform tension, and loaded by a force $f : \Omega \to \mathbb{R}$, normally uniformly distributed. The vertical position u of the membrane on the boundary $\partial\Omega$ is fixed, but it is not necessarily constant, so we impose: $u = h$ on $\partial\Omega$.

Consider now the problem of finding the equilibrium position of the membrane, i.e. u, constrained so that it is forced to lie above an object/body which we will call an obstacle and represent by a function $g : \Omega \to \mathbb{R}$, defined in Ω, which verifies $g \leq u$ on Ω and verifies $g \leq h = u|_{\partial\Omega}$ in $\partial\Omega$ (cf. [11], [15]).

In accordance with [5] and [16], we now introduce the set of admissible solutions to the problem:

$$K = \{v \in V : v \geq g,\, x \in \Omega\} \tag{1}$$

where, for simplicity, we consider $V = \{v \in H^1(\Omega) : v = 0,\, x \in \partial\Omega\} = H_0^1(\Omega)$ which is a Hilbert space endowed with the inner product $(\cdot,\cdot)_V$, and the norm $\|\cdot\|_V$, induced by this inner product. The final potential energy of the membrane is given by: $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega f\, u\, dx$. In the equilibrium position the *principle of minimum potential energy* applies, and is stated in the form: $\exists u \in K : E(u) \leq$

$E(v),\ \forall v \in K$. Which allows us to formulate the **minimization problem**:

$$\begin{cases} \text{Find } u \in K \text{ such that:} \\ E(u) := \min_{v \in K} E(v) \end{cases} \tag{2}$$

The **variational problem** (in integral form), can be seen as finding $u \in K$:

$$\int_\Omega \nabla u \cdot \nabla (v-u)\, dx \geq \int_\Omega f(v-u)\, dx,\ \forall v \in K \tag{3}$$

It can be proved that the solution to the minimization problem (2), is also the solution to the variational problem (3), and vice versa (see [1]), so the two are equivalent.

Theorem 2.1 (Existence and Uniqueness)**.** *The problem* (3) *admits a unique solution* $u \in K$.
Proof. *See [12].*

Let's now divide our domain, Ω, into **contact domain** $\Omega_C = \{x \in \Omega : u(x) = g(x)\}$ and into **contact-free domain** $\Omega_f = \{x \in \Omega : u(x) > g(x)\}$. Furthermore, let $\Gamma = \partial\Omega_C \cap \partial\Omega_f$ be the boundary separating the two distinct regions Ω_C and Ω_f, which will be called the **free boundary**. Note that these sets do not necessarily have to be connected.
Assuming the solution u is sufficiently regular, we can define the **Boundary-value problem**: find $u \in H^2(\Omega) \cap C(\bar{\Omega})$, with $f \in L^2(\Omega)$ and $g \in H^1(\Omega) \cap C(\bar{\Omega})$, so putting all this information together allows us to rewrite the problem in a *strong formulation* (or classical):

$$\begin{cases} -\Delta u - f \geq 0 & \text{, in } \Omega \\ u \geq g & \text{, in } \Omega \\ (u-g)(-\Delta u - f) = 0 & \text{, in } \Omega \\ u = 0 & \text{, on } \partial\Omega \\ u = g & \text{, on } \Gamma \\ \left[\!\left[\frac{\partial u}{\partial \vec{n}} \right]\!\right] = 0 & \text{, on } \Gamma \end{cases} \tag{4}$$

The third equation in (4) is the complementary condition which says that one of the first two inequalities must be an equality.

The last two equations in (4) reinforce the continuity of the solution u and its normal derivative along the free boundary Γ. As $-\Delta u - f = 0$ on $x \in \Omega_f$, and $u = g$ on $x \in \Omega_C$, we can see that the second derivatives of u jump along the free boundary, so we cannot guarantee that the second derivatives of u will be continuous. The solution, in general, will be no more regular than $C^{1,1}(\Omega)$, i.e. its first derivatives are Lipschitz continuous. And the solution may belong to $H^2(\Omega)$ but not $H^3(\Omega)$. Regularity will always depend on the regularity of the obstacle and the boundary conditions.

2.1. Formulation with Lagrange Multipliers

Following [8] and [7], we introduce a non-negative Lagrange multiplier, $\lambda : \Omega \to \mathbb{R}$, which allows us to rewrite the obstacle problem as:

$$\begin{cases} \Delta u - \lambda = f & \text{, in } \Omega \\ u - g \geq 0 & \text{, in } \Omega \\ \lambda \geq 0 & \text{, in } \Omega \\ (u-g)\lambda = 0 & \text{, in } \Omega \\ u = 0 & \text{, on } \partial\Omega \end{cases} \tag{5}$$

The Lagrange multiplier belongs to the dual space of $V = H_0^1(\Omega)$, i.e. $\lambda \in Q = H^{-1}(\Omega) = V'$ with norm

$$\|\xi\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle v, \xi \rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)}}{\|v\|_{H_0^1(\Omega)}}$$

where $\langle \cdot, \cdot \rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)} : V \times Q \to \mathbb{R}$ is the inner product of the dual, or just the dual product. In order to obtain the weak formulation (variational formulation) let's start by considering the equality $\Delta u - \lambda = f$, multiply by a test function and integrate by parts, obtaining: $(\nabla u, \nabla v) - \langle v, \lambda \rangle = (f, v)\ ,\forall v \in V$.

Considering the inequality $u-g \geq 0$, multiply by $\mu \in \Lambda$ and integrate, obtaining: $\langle u-g, \mu\rangle \geq 0 \ \ , \forall \mu \in \Lambda$. where

$$\Lambda = \{\mu \in Q : \langle v, \mu\rangle \geq 0 \ , \ \forall v \in V \ , \ v \geq 0 \ , \ \text{a.e. in } \Omega\} \tag{6}$$

Thus the variational formulation becomes: find the pair $(u, \lambda) \in V \times \Lambda$ such that:

$$\begin{cases} (\nabla u, \nabla v) - \langle v, \lambda\rangle = (f, v) & , \forall v \in V \\ \langle u-g, \mu-\lambda\rangle \geq 0 & , \forall \mu \in \Lambda \end{cases} \tag{7}$$

The existence of a unique solution $(u, \lambda) \in V \times \Lambda$ for the mixed problem (7) and the equivalence between the normal variational formulation and the variational formulation with Lagrange multipliers are proved by Haslinger, Hlaváček, and Nečas (c.f [9]).

3. Parabolic Obstacle Problem: Black-Scholes Model for American Options

In this section, we will analyse the Black-Scholes model (cf. [2]) for the pricing of American options. A derivative is a financial contract whose value derives from the value of other underlying assets. A contract starting at $t = 0$, also defines an expiry date for the contract, at $t = T > 0$, which is called *maturity.*

3.1. Call Options and Put Options

Options are a specific type of derivative. These options are sold by one party, the *writer/seller*, to the other party, the *buyer/holder*, for a certain price, called the *premium*, which is imposed by the writer. After paying the premium, the holder acquires the right, but not the obligation, to buy (*call option*) or sell (*put option*) the underlying asset at a set price, which we will represent by K, and is called the *strike price.* We define $S = S(t) = S_t$ as the market price of a certain underlying asset over time; and $S(t_0) = S_0$ as the price of the underlying asset at time t_0, i.e. at the start of the contract.

The problem, and our objective, is to value (or price) these options, or to obtain the *fair price of the option*, represented by $V(S_0, t_0)$.

3.2. Final value problem: Maturity

The function $V(S, t)$ returns the value of the option for any asset price $S \geq 0$ at any time $t \in [0, T]$. This function will depend on several factors, such as the strike price K, the maturity time T, the *interest rate* represented by r, the *volatility* represented by σ.

The value of the option at maturity is called the *payoff*: $V(S_T, T)$, which represents what we will receive at the end of the contract. We therefore impose the condition of final value, for call options and put options respectively:

$$V_C(S_T, T) = \max\{S_T - K; 0\} \ \ ; \ \ V_P(S_T, T) = \max\{K - S_T; 0\} \tag{8}$$

3.3. Black-Scholes Model

Since r and σ are known constants of our problem, let's consider the following second-order linear differential operator associated with the Black-Scholes model given by:

$$\mathcal{L}_{BS}(V) := \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \tag{9}$$

For ease of notation, we will call the function $g(S, t)$ the payoff, given by:

$$g_C(S, t) = \max\{S - K; 0\} \ \ ; \ \ g_P(S, t) = \max\{K - S; 0\} \tag{10}$$

3.4. Black-Scholes Model for American Options

In American Options, the holder has the right to exercise the option at any time during the life of the contract ($0 \leq t \leq T$). In order for there to be no chance of arbitrage, their value $V(S, t)$ must be greater than or equal to $g(S, t)$. Thus, the first general condition for American options is: $V(S, t) \geq g(S, t)$.

We thus encounter a Free Boundary Problem, as each time step presents a specific value of S that dictates two possible actions: either the holder retains the option, or they choose to exercise it. The critical value of S that distinguishes these two actions is the optimal exercise price $S_f(t)$. However, it is clear that the prices are optimal at the free boundary S_f. We impose:

$$\begin{cases} V_C(S_f(t),t) = S_f(t) - K \\ \frac{\partial V_C}{\partial S}(S_f(t),t) = 1 \end{cases} \tag{11}$$

$$\begin{cases} V_P(S_f(t),t) = K - S_f(t) \\ \frac{\partial V_P}{\partial S}(S_f(t),t) = -1 \end{cases} \tag{12}$$

The boundary conditions through a function h, given by:

$$h_c(S,t) = \begin{cases} 0, & \text{on } \{0\} \times (0,T) \\ S - K, & \text{on } \{\infty\} \times (0,T) \end{cases} \quad ; \quad h_p(S,t) = \begin{cases} K, & \text{on } \{0\} \times (0,T) \\ 0, & \text{on } \{\infty\} \times (0,T) \end{cases}$$

Therefore, the problem of valuing American options can be formulated as the following parabolic obstacle problem and seen as a free boundary problem, where there is an unknown boundary that depends on the time to exercise the option:

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{L}_{BS} V \leq 0 & \text{, in } (0,\infty) \times (0,T) \\ (g - V) \leq 0 & \text{, in } (0,\infty) \times (0,T) \\ \left(\frac{\partial V}{\partial t} + \mathcal{L}_{BS} V\right)(g - V) = 0 & \text{, in } (0,\infty) \times (0,T) \\ V = g & \text{, on } (0,\infty) \times \{t = T\} \\ V = h & \text{, on } \partial\{(0,\infty)\} \times (0,T) \end{cases} \tag{13}$$

the existence and uniqueness of the solution of this strong formulation are proved in [13]. It is possible to prove that the weak formulation of the problem, (13), has a single weak solution, cf. [3]. And as with the elliptic obstacle problem, the regularity will not be greater than $C^{1,1}$, cf. [14], and is limited by the regularity of the obstacle.

4. Discretization Methods for Elliptic Obstacle Problem

Consider the system problem arising from the system (4). The domain is $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$ and the problem data is given by:

$$\begin{cases} g(r) = \begin{cases} \sqrt{1 - r^2} \ , & r < b \\ c_1 r + c_2 \ , & c.c. \end{cases} \\ f(x,y) = -1 \\ u = 0 \ \text{, on } \partial\Omega \end{cases} \tag{14}$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the origin, and c_1 and c_2 are chosen such that the obstacle is $g \in C^1(\Omega)$, and the constant $b = 0.9$. The analytical solution is given by:

$$u(r) = \frac{\left(-a^2 + 4g(a) + r^2\right) \cdot \log(2) + \left(4 - r^2\right) \cdot \log(a) + \left(a^2 - 4 - 4g(a)\right) \cdot \log(r)}{4(\log(2) - \log(a))} \tag{15}$$

where $a \approx 0.8294$.

4.1. Discretization with Finite Differences

We started by making a regular discretization of a domain containing Ω. The points in the interior of Ω were defined as: $\mathbf{P}_I$, with a total of N_I points. To mark the boundary points, all the horizontal and vertical parallel lines passing through the interior discretization points were extended and intersected with the $\partial\Omega$ edge, forming the set: $\mathbf{P}_B$, with a total of N_B boundary points.

We define the points of the discretization, with a total of $N = N_I + N_B$ ordered points, as a set: $\mathbf{P} = \mathbf{P}_I \cup \mathbf{P}_B = \{(x_1,y_1); (x_2,y_2); \ldots; (x_N,y_N)\}$.

Consider $\mathbf{i}_I$ to be the set of indices that index all the points in the interior of the set $\mathbf{P}$, that is: $\mathbf{P}(\mathbf{i}_I) = \mathbf{P}_I$, and consider $\mathbf{i}_B$ to be the set of indices that index all the points on the frontier of the set $\mathbf{P}$, i.e.: $\mathbf{P}(\mathbf{i}_B) = \mathbf{P}_B$.

For any point, $p_i = (x_i, y_i) \in \mathbf{P}$, denote the solution evaluated at that point by $u(p_i) = u(x_i, y_i)$, and its approximation by u_i.

We use second-order centred finite differences, to discretize the Laplacian. This leads to vectors: $-\mathbf{A}\mathbf{u} - \mathbf{f}$, where the matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ is sparse with each row i being associated with the solution value at the point $p_i = (x_i, y_i)$. However, note that the solution u at the boundary points is already known, therefore we can remove from the system all the rows that don't correspond to the interior of the domain (the notation being ":", which represents all the rows or all the columns):

$$-\mathbf{A}(\mathbf{i}_I,:)\mathbf{u} - \mathbf{f}(\mathbf{i}_I) = -\left[\mathbf{A}(\mathbf{i}_I,\mathbf{i}_I)\mathbf{u}(\mathbf{i}_I) + \mathbf{A}(\mathbf{i}_I,\mathbf{i}_B)\mathbf{u}(\mathbf{i}_B)\right] - \mathbf{f}(\mathbf{i}_I).$$

This allows us to write the **complementarity problem** for the obstacle problem:

$$\begin{cases} \mathbf{A}(\mathbf{i}_I,\mathbf{i}_I)\mathbf{u}(\mathbf{i}_I) + \mathbf{A}(\mathbf{i}_I,\mathbf{i}_B)\mathbf{u}(\mathbf{i}_B) + \mathbf{f}(\mathbf{i}_I) \leq \mathbf{0} \\ \mathbf{g}(\mathbf{i}_I) - \mathbf{u}(\mathbf{i}_I) \leq \mathbf{0} \\ (\mathbf{g}(\mathbf{i}_I) - \mathbf{u}(\mathbf{i}_I))^T(\mathbf{A}(\mathbf{i}_I,\mathbf{i}_I)\mathbf{u}(\mathbf{i}_I) + \mathbf{A}(\mathbf{i}_I,\mathbf{i}_B)\mathbf{u}(\mathbf{i}_B) + \mathbf{f}(\mathbf{i}_I)) = 0 \end{cases} \tag{16}$$

which is equivalent to the expression: $\max\{\mathbf{A}(\mathbf{i}_I,\mathbf{i}_I)\mathbf{u}(\mathbf{i}) + \mathbf{A}(\mathbf{i}_I,\mathbf{i}_N) + \mathbf{f}(\mathbf{i}_I); \mathbf{g}(\mathbf{i}_I) - \mathbf{u}(\mathbf{i})\} = \mathbf{0}$, which can be solved with the Semi-smooth Newton Method (SSNM) (see [4]).

4.2. **Finite Element Discretization**
Here, we will consider the formulation (5), written with a Lagrange multiplier. The finite element spaces we will consider are based on a regular triangulation, denoted by $\mathcal{C}_h$ of Ω, which from now on we will assume to be polygonal. We denote by $\mathcal{E}_h$ the internal edges of Ω. And the finite element subspaces are: $V_h \subset V$ and $Q_h \subset Q$. We also define $\Lambda_h = \{\mu_h \in Q_h : \mu_h \geq 0 \ x \in \Omega\} \subset \Lambda$.

For the correct formulation of the methods, we will use two methods that add stability to the algorithms we will derive, these are the *mixed method* and the *stabilised method* (cf. [8], [7] and [6]).

4.2.1 Mixed method

The mixed variational formulation for the problem is as follows: find $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that:

$$\mathcal{B}(u_h, \lambda_h; v_h, \mu_h - \lambda_h) \leq \mathcal{L}(v_h, \mu_h - \lambda_h)\ ,\ \forall (v_h, \mu_h) \in V_h \times \Lambda_h$$

We will use the **bubble functions** technique to define a family of stable finite element pairs. With $b_K \in P_3(K) \cap H_0^1(K)$ we denote the bubble function scaled to have a maximum value of 1, and we define

$$B_{l+1}(K) = \left\{ z \in H_0^1(K) : z = b_K w\ ,\ w \in \tilde{P}_{l-2}(K) \right\}$$

where $\tilde{P}_{l-2}(K)$ denotes the space of homogeneous polynomials of degree $l-2$. Let $k \geq 1$ be the degree of the finite element spaces defined by

$$V_h = \begin{cases} \{v_h \in V : v_h|_K \in P_1(K) \oplus B_3(K)\ ,\ \forall K \in \mathcal{C}_h\} \text{ for } k = 1 \\ \{v_h \in V : v_h|_K \in P_k(K) \oplus B_{k+1}(K)\ ,\ \forall K \in \mathcal{C}_h\} \text{ for } k \geq 2 \end{cases} \tag{17}$$

And be:

$$Q_h = \begin{cases} \{\xi_h \in Q : \xi_h|_K \in P_0(K)\ ,\ \forall K \in \mathcal{C}_h\} \text{ for } k = 1 \\ \{\xi_h \in Q : \xi_h|_K \in P_{k-2}(K)\ ,\ \forall K \in \mathcal{C}_h\} \text{ for } k \geq 2 \end{cases} \tag{18}$$

The discrete weak formulation of our problem is described as: find the pair $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that:

$$\begin{cases} (\nabla u_h, \nabla v_h) - \langle v_h, \lambda_h \rangle = (f, v_h) & ,\forall v_h \in V_h \\ \langle u_h - g, \mu_h \rangle \geq 0 & ,\forall \mu_h \in \Lambda_h \\ \langle u_h - g, \lambda_h \rangle = 0 & ,\forall \mu_h \in \Lambda_h \end{cases} \tag{19}$$

We consider the case of lower order elements, i.e. with $1 \leq k \leq 3$. Let ξ_j, $j \in \{1, \ldots, M\}$, be the nodal Lagrange bases for Q_h. And we have

$$\Lambda_h = \left\{ \mu_h = \sum_{j=1}^{M} \mu_i \xi_j \ : \mu_j \geq 0\, , \forall \xi_j \text{ basis of } Q_h \right\}$$

Let Ψ_j, $j \in \{1, \ldots, N\}$, be the basis of the space V_h. And so: $\mu_h = \sum_{j=1}^{M} \mu_j \xi_j$ and $u_h = \sum_{j=1}^{N} u_j \Psi_j$ substituting in system (19) we get the following matrix system (complementary problem):

$$
\begin{cases}
\mathbf{A}\mathbf{u} - \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{f} \\
\mathbf{B}\mathbf{u} \geq \mathbf{g} \\
\boldsymbol{\lambda}^T(\mathbf{B}\mathbf{u} - \mathbf{g}) = 0 \\
\boldsymbol{\lambda} \geq \mathbf{0}
\end{cases}
\qquad (20) \qquad
\begin{aligned}
\mathbf{A} \in \mathbb{R}^{N\times N}, &\quad (\mathbf{A})_{ij} = (\nabla\varphi_i, \nabla\varphi_j) \\
\mathbf{B} \in \mathbb{R}^{M\times N}, &\quad (\mathbf{B})_{ij} = (\xi_i, \varphi_j) \\
\mathbf{f} \in \mathbb{R}^{N}, &\quad (\mathbf{f})_{i} = (f, \varphi_i) \\
\mathbf{u} \in \mathbb{R}^{N}, &\quad (\mathbf{u})_{i} = u_i \\
\mathbf{g} \in \mathbb{R}^{M}, &\quad (\mathbf{g})_{i} = (g, \xi_i) \\
\boldsymbol{\lambda} \in \mathbb{R}^{M}, &\quad (\boldsymbol{\lambda})_{i} = \lambda_i
\end{aligned}
$$

the last three constraints of the system (20) can be rewritten into a single non-linear equation $\min\{\boldsymbol{\lambda}; \mathbf{B}\mathbf{u} - \mathbf{g}\} = \mathbf{0}$, and with a few manipulations we get the following system:

$$
\begin{cases}
\mathbf{A}\mathbf{u} - \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{f} \\
\boldsymbol{\lambda} - \max\{\mathbf{0}, \boldsymbol{\lambda} + c(\mathbf{g} - \mathbf{B}\mathbf{u})\} = \mathbf{0}
\end{cases}
\tag{21}
$$

with any $c > 0$. Following [10], applying the semismooth Newton method to the system above, we arrive at the iterative solution algorithm: *Primal-dual active set method for the mixed problem*, described below:

(i) Initialize: $k = 0$, $\boldsymbol{\lambda}^{(0)} = \mathbf{0}$, and solve $\mathbf{A}\mathbf{u}^{(0)} = \mathbf{f}$;

(ii) For $k \geq 0$, calculate: $\widetilde{\boldsymbol{\lambda}}^{(k)} = \boldsymbol{\lambda}^{(k)} + c(\mathbf{g} - \mathbf{B}\mathbf{u}^{(k)})$, and initialize *Active-Set* and *Inactive-Set*: $\mathcal{A}_k = \left\{a : \widetilde{\boldsymbol{\lambda}}_a^{(k)} > 0\right\}$, $\mathcal{I}_k = \left\{i : \widetilde{\boldsymbol{\lambda}}_i^{(k)} \leq 0\right\}$ are sets of indices, such that $\widetilde{\boldsymbol{\lambda}}^{(k)}(\mathbf{a}^k) > 0$, $\widetilde{\boldsymbol{\lambda}}^{(k)}(\mathbf{i}^k) \leq 0$.

(iii) $\boldsymbol{\lambda}^{(k+1)}(\mathbf{i}^k) = \mathbf{0}$, and solve

$$
\begin{bmatrix} \mathbf{A} & -\mathbf{B}[\mathbf{a}^k, :]^T \\ \mathbf{B}[\mathbf{a}^k, :] & \mathbf{0} \end{bmatrix}
\begin{bmatrix} \mathbf{u}^{(k+1)} \\ \boldsymbol{\lambda}^{(k+1)}(\mathbf{a}^k) \end{bmatrix}
=
\begin{bmatrix} \mathbf{f} \\ \mathbf{g}(\mathbf{a}^k) \end{bmatrix}
$$

(iv) check stop condition: if $\left\|\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\lambda}^{(k)}\right\| \leq \mathrm{TOL}$ stop; otherwise increment $k = k+1$ and return to point (ii).

4.2.2 Stabilized method

From the Stokes problem, we know that the use of bubble functions can be avoided by adding *stabilisation terms.* Let $U = V \times Q$, define the bilinear form $\mathcal{B} : U \times U \to \mathbb{R}$ and the linear form, $\mathcal{L} : U \to \mathbb{R}$, as:

$$
\mathcal{B}(w, \xi; v, \mu) = (\nabla w, \nabla v) - \langle v, \xi\rangle - \langle w, \mu\rangle \ \ ; \ \ \mathcal{L}(v, u) = (f, v) - \langle g, \mu\rangle
$$

Let's now introduce the bilinear and linear form $\mathcal{S}_h$ and $\mathcal{F}_h$, by:

$$
\mathcal{S}_h(w, \xi; v, \mu) = \sum_{K\in\mathcal{C}_h} h_K^2(-\Delta w - \xi, -\Delta v - \mu)_K \ \ ; \ \ \mathcal{F}_h(v, \mu) = \sum_{K\in\mathcal{C}_h} h_K^2(f, -\Delta v - \mu)_K
$$

and then we define the forms $\mathcal{B}_h$ and $\mathcal{L}_h$ by:

$$
\mathcal{B}_h(w, \xi; v, \mu) = \mathcal{B}(w, \xi; v, \mu) - \alpha\mathcal{S}_h(w, \xi; v, \mu) \ \ ; \ \ \mathcal{L}_h(v, \mu) = \mathcal{L}(v, \mu) - \alpha\mathcal{F}_h(v, \mu)
$$

where $\alpha > 0$ is a stabilisation parameter. Noting that with the assumption, $f \in L^2(\Omega)$, we get, $\Delta u + \lambda \in L^2(\Omega)$, therefore, it holds that: $\mathcal{S}_h(u, \lambda; v_h, \mu_h) = \mathcal{F}_h(v_h, \mu_h)$ $,\forall(v_h, \mu_h) \in V_h \times \Lambda_h$. In this method, the problem becomes: find the pair $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that

$$
\mathcal{B}_h(u_h, \lambda_h; v_h, \mu_h - \lambda_h) \leq \mathcal{L}_h(v_h, \mu_h - \lambda_h) \ \ ,\forall(v_h, \mu_h) \in V_h \times \Lambda_h. \tag{22}
$$

The discrete weak formulation of our problem is described as: find the pair $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that:

$$
\begin{cases}
(\nabla u_h, \nabla v_h) - \langle\lambda_h, v_h\rangle - \alpha\sum_{K\in\mathcal{C}_h} h_K^2(\Delta u_h + \lambda_h, \Delta v_h)_K = (f, v_h) + \alpha\sum_{K\in\mathcal{C}_h} h_K^2(f, -\Delta v_h)_K \\
\langle u_h - g, \mu_h - \lambda_h\rangle + \alpha\sum_{K\in\mathcal{C}_h} h_K^2(\Delta u_h + \lambda_h + f, \mu_h - \lambda_h)_K \geq 0
\end{cases}
\tag{23}
$$

valid for $\forall(v_h,\mu_h)\in V_h\times\Lambda_h$. For the stabilized method, we define the finite element spaces as:

$$V_h=\{v_h\in V: v_h|_K\in P_k(K)\ ,\ \forall K\in\mathcal{C}_h\} \tag{24}$$

and

$$Q_h=\begin{cases}\{\xi_h\in Q:\xi_h|_K\in P_0(K)\ ,\ \forall K\in\mathcal{C}_h\}\text{ para }k=1\\ \{\xi_h\in Q:\xi_h|_K\in P_{k-2}(K)\ ,\ \forall K\in\mathcal{C}_h\}\text{ para }k\geq 2\end{cases} \tag{25}$$

Through steps similar to those in the mixed case, we arrive at the algebraic system:

$$\begin{cases}\mathbf{A}_\alpha\mathbf{u}-\mathbf{B}_\alpha^T\boldsymbol{\lambda}=\mathbf{f}_\alpha\\ \mathbf{B}_\alpha\mathbf{u}+\mathbf{C}_\alpha\boldsymbol{\lambda}\geq\mathbf{g}_\alpha\\ \boldsymbol{\lambda}^T(\mathbf{B}_\alpha\mathbf{u}+\mathbf{C}_\alpha\boldsymbol{\lambda}-\mathbf{g}_\alpha)=0\\ \boldsymbol{\lambda}\geq\mathbf{0}\end{cases} \tag{26}$$

$$\begin{aligned}\mathbf{A}_\alpha\in\mathbb{R}^{N\times N},&\quad(\mathbf{A}_\alpha)_{ij}=(\nabla\varphi_i,\nabla\varphi_j)-\alpha\sum_{K\in C_h}h_K^2(\Delta\varphi_i,\Delta\varphi_j)_K\\ \mathbf{B}_\alpha\in\mathbb{R}^{M\times N},&\quad(\mathbf{B}_\alpha)_{ij}=(\xi_i,\varphi_j)+\alpha\sum_{K\in C_h}h_K^2(\xi_i,\Delta\varphi_j)_K\\ \mathbf{C}_\alpha\in\mathbb{R}^{M\times M},&\quad(\mathbf{C}_\alpha)_{ij}=\sum_{K\in C_h}h_K^2(\xi_i,\xi_j)_K\\ \mathbf{f}_\alpha\in\mathbb{R}^{N},&\quad(\mathbf{f}_\alpha)_{i}=(f,\varphi_i)+\alpha\sum_{K\in C_h}h_K^2(f,\Delta\varphi_i)_K\\ \mathbf{g}_\alpha\in\mathbb{R}^{M},&\quad(\mathbf{g}_\alpha)_{i}=(g,\xi_i)-\alpha\sum_{K\in C_h}h_K^2(f,\xi_i)_K\end{aligned}$$

arriving at the iterative solution algorithm: *Primal-dual active set method for the stabilized problem*, described below:

(i) Initialize: $k=0$, $\boldsymbol{\lambda}^{(0)}=\mathbf{0}$, solve $\mathbf{A}\mathbf{u}^{(0)}=\mathbf{f}$ and then $\boldsymbol{\lambda}^{(0)}=\max\{\mathbf{C}_\alpha^{-1}(\mathbf{g}_\alpha-\mathbf{B}_\alpha\mathbf{u}^{(0)});\mathbf{0}\}$;

(ii) For $k\geq 0$, initialize *Active-Set* and *Inactive-Set*: $\mathcal{A}_k=\left\{a:\boldsymbol{\lambda}_a^{(k)}>0\right\}$, $\mathcal{I}_k=\left\{i:\boldsymbol{\lambda}_i^{(k)}\leq 0\right\}$ are sets of indices, such that $\boldsymbol{\lambda}^{(k)}(\mathbf{a}^k)>0$, $\boldsymbol{\lambda}^{(k)}(\mathbf{i}^k)\leq 0$.

(iii) To get $\mathbf{u}^{(k+1)}$ solve:

$$(\mathbf{A}_\alpha+\mathbf{B}_\alpha[\mathbf{a}^k,:]^T\mathbf{C}_\alpha^{-1}[\mathbf{a}^k,\mathbf{a}^k]\mathbf{B}_\alpha[\mathbf{a}^k,:])\mathbf{u}^{(k+1)}=\mathbf{f}_\alpha+\mathbf{B}_\alpha[\mathbf{a}^k,:]^T\mathbf{C}_\alpha^{-1}[\mathbf{a}^k,\mathbf{a}^k]\mathbf{g}_\alpha$$

(iv) Get $\boldsymbol{\lambda}^{(k+1)}=\max\{\mathbf{C}_\alpha^{-1}(\mathbf{g}_\alpha-\mathbf{B}_\alpha\mathbf{u}^{(k+1)});\mathbf{0}\}$

(v) check stop condition: if $\left\|\boldsymbol{\lambda}^{(k+1)}-\boldsymbol{\lambda}^{(k)}\right\|\leq$ TOL stop; otherwise increment $k=k+1$ and return to point (ii).

4.3. Numerical Results

For the Finite Difference Discretisation method and SSNM implementation, Table 1 shows, for different values of h, the errors in the norm of the maximum between the numerical solution and the analytical solution, a column with the DOF (degrees of freedom), a column with the number of iterations, and finally a column with the running time of the algorithm.

h_{max}	DOF	iter	$\|e_h\|_\infty$	time (s)
0.8	16	4	7.6336e-02	0.001767
0.4	69	5	2.2057e-02	0.001698
0.2	305	9	1.5153e-02	0.006905
0.1	1245	15	4.2743e-03	0.069570
0.05	5013	29	2.0576e-03	4.644646
0.025	20069	54	3.5914e-04	282.017176

Table 1: Discretization with finite differences, and resolution of the LCP with SSNM, with tolerance $TOL=10^{-12}$

For the mixed method, $\mathbb{P}_1$-bubble/$\mathbb{P}_0$ elements were used, i.e. first-order Lagrange elements enriched with bubble functions, and zero-degree polynomials for the basis that make up λ_h. For the stabilized method, $\mathbb{P}_1$-$\mathbb{P}_0$ elements were used. We present the Tables 2 and 3, where for different meshes, we have a column with the DOF, a column for the number of iterations, and three columns for the u_h error, one in the norm of L^2, another in the norm of H^1, and another in the norm of the maximum.

h_{max}	DOF	iter	$\|e_h\|_{L^2}$	$\|e_h\|_{H^1}$	$\|e_h\|_{\infty}$	time (s)
0.8	189	5	2.2430e-01	1.1999	1.7276e-01	0.015605
0.4	874	7	5.3611e-02	6.0700e-01	4.6393e-02	0.014635
0.2	3546	10	1.5437e-02	4.3343e-01	1.7621e-02	0.031837
0.1	18325	15	3.4128e-03	3.8091e-01	4.7721e-03	0.375181
0.05	76514	23	4.5392e-04	3.7121e-01	2.0865e-03	1.095855
0.025	333090	39	1.2857e-04	3.6651e-01	3.3101e-04	68.359351

Table 2: Numerical results for the mixed method, with $TOL = 10^{-32}$

h_{max}	DOF	iter	$\|e_h\|_{L^2}$	$\|e_h\|_{H^1}$	$\|e_h\|_{\infty}$	time (s)
0.8	77	4	2.8823e-01	9.7053e-01	9.8367e-02	0.011604
0.4	366	7	6.5068e-02	5.4807e-01	3.2832e-02	0.016358
0.2	1502	9	1.7144e-02	4.2510e-01	1.0823e-02	0.021827
0.1	7813	14	3.0908e-03	3.7831e-01	2.5104e-03	0.153261
0.05	32710	22	7.7914e-04	3.6860e-01	6.6608e-04	1.006936
0.025	142582	39	1.9219e-04	3.6621e-01	1.6308e-04	8.954543

Table 3: Numerical results for the stabilised method, with $TOL = 10^{-32}$

Although the h_{max} values are the same in the Tables 1, 2 and 3, it is crucial to note that the discretization geometry is different for FDM and FEM. Analysing the three methods shows that they all converge to the exact solution with mesh refinement, as evidenced by the consistent reduction in errors. In conclusion, the stabilized method is generally the most efficient in terms of the use of computational resources, offering a good combination of accuracy and execution time. The mixed method may be favoured when accuracy is extremely critical and computational resources are available. The finite difference method is suitable for fast solutions on coarse meshes, but eventually becomes less practical for fine meshes due to the exponential increase in DOF and execution time.

5. Discretization of the Parabolic Obstacle Problem

The domain where we approximate the solution is $[0,T]\times[0,\infty]$. So that we can calculate our numerical approximation in a feasible way, we introduce an artificial limit $S_{\max}$ ($S_{\max}\approx\infty$), which allows us to have a finite domain, given by: $[0,T]\times[0,S_{\max}]$.

5.1. Finite Difference Method for Black Scholes

The space $[0,T]\times[0,S_{\max}]$ is approximated by a rectangular grid, where the intervals $[0,T]$ and $[0,S_{max}]$, are divided into N_t+1 and M_S+1 subintervals, of equal length h_t and h_S, respectively. Where the function $V(S,t)=V(m\cdot\Delta S, n\cdot\Delta t)\approx V_m^n$, for $n\in\{0,1,\dots,N_t+1\}$ and $m\in\{0,1,\dots,M_S+1\}$, is the Option Value. Let's start by discretising the derivatives relative to the variable S, where we use the centred difference approximation for the first and second order partial derivatives of $\frac{\partial}{\partial t}V+\mathcal{L}_{BS}V$:

$$\frac{\partial}{\partial t}V_m(t)+\frac{1}{2}\sigma^2S_m^2\frac{V_{m+1}(t)-2V_m(t)+V_{m-1}(t)}{h_S^2}+rS_m\frac{V_{m+1}(t)-V_{m-1}(t)}{2h_S}-rV_m(t)+O(h_S^2)$$

Putting together all the differential operations relating to the interior of the domain $[0,S_{max}]$, i.e. for $m=1,\dots,M_S$, in a total of M_S lines, these operators discretised in space can be seen in matrix form as:

$$\frac{\partial}{\partial t}\mathbf{V}_r(t)+\mathbf{L}\mathbf{V}(t)=\frac{\partial}{\partial t}\mathbf{V}_r(t)+\mathbf{L}_r\mathbf{V}_r(t)+\mathbf{r}(t) \tag{27}$$

The vector $\mathbf{V}(t)\in\mathbb{R}^{M_S+2}$, is the solution vector composed of all the values $V_m(t)$ with $m=0,\dots,M_S+1$ at each instant t. The vector $\mathbf{V}_r(t)\in\mathbb{R}^{M_S}$, on the other hand, is composed of only the interior points of the domain $V_m(t)$ with $m=1,\dots,M_S$. The matrix $\mathbf{L}\in\mathbf{R}^{(M_S)\times(M_S+2)}$, is a non-square tridiagonal matrix resulting from the discretisation. The $\mathbf{L}_r\in\mathbb{R}^{M_S\times M_S}$ matrix is a square matrix, which results from eliminating the first and last columns of the $\mathbf{L}$ matrix. The vector $\mathbf{r}(t)\in\mathbb{R}^{M_S}$ results from adding the first and last columns of the matrix $\mathbf{L}$ and multiplying them by $V_0(t)$ and $V_{M_S+1}(t)$ respectively.

5.2. Finite Element Method for Black Scholes

Let $V=H^1(\Omega)$ be a Hilbert space, with $\Omega=(0,S_{max})$. Assuming $u\in L^2(0,T;V)$ is such that $u_t\in L^2(0,T;V')$, we multiply the operator $u_t+\mathcal{L}_{BS}u$ by a test function $v\in V$, and integrate in Ω:

$$\int_\Omega vu_t\,dx+\int_\Omega v\mathcal{L}_{BS}u\,dx=P_d[u_t,v]+B[u,v] \tag{28}$$

where $B[.,.]$ is a bilinear operator:

$$B[u,v]=-\frac{1}{2}\sigma^2\int_\Omega x^2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx+(r-\sigma^2)\int_\Omega x\frac{\partial u}{\partial x}v\,dx-r\int_\Omega uv\,dx+\frac{1}{2}\sigma^2\int_{\partial\Omega}x^2u_xv\,dx \tag{29}$$

and the duality product, i.e. the usual identification between an element $v\in V$, with an element $u_t\in V'$, of the dual:

$$P_d[u_t,v]=\langle u_t,v\rangle_{V'\times V}=\int_\Omega u_tv\,dx \tag{30}$$

By discretising the domain $\bar{\Omega} = [0, S_{\max}]$ into a total of $M_x + 2$ points. We are left with a total of $M_x + 1$ intervals, of the form: $I_m = [x_m, x_{m+1}],\ m = 0, \ldots, M_x$. Take $\mathcal{I}_h$ as the mesh of the domain $\bar{\Omega}$, and it is therefore the finite family of elements I_m, which satisfies: $\bar{\Omega} = \bigcup_{I \in \mathcal{I}_h} I$, where $I_i \cap I_j = \emptyset\ ,\forall I_i, I_j \in \mathcal{I}_h\ , I_i \neq I_j$. We say $\{I, \mathbb{P}_k(I), \Sigma_I\}$ is a finite Lagrange element of degree k, where $I \in \mathcal{I}_h$, and Σ_I is the set of interpolation nodes of I and $\mathbb{P}_k(I)$ is the vector space of polynomial functions of degree less than or equal to k in I. Let's define V_h, as a finite-dimensional vector subspace of $V = H^1(\Omega)$, by:

$$V_h = \left\{ v_h \in C^0(\overline{\Omega}) : v_h|_I \in \mathbb{P}_1(I),\ \forall I \in \mathcal{I}_h \right\} \subset H^1(\Omega) = V$$

In this case, the number of global basis will coincide with the number of nodes in the discretization. Thus the solution $u_h \in V_h$, formed by the linear combination of basis $\{\phi_0, \ldots, \phi_{M_S+1}\}$, i.e.: $u_h = \sum_{j=0}^{Ms+1} \phi_j(x) U_j(t)$. It follows that $\frac{d}{dt} u_h = \sum_{j=0}^{Ms+1} \phi_j(x) U_j^{'}(t)$. The test functions, $v_h \in V_h$, are defined by: $v_h = \sum_{i=0}^{Ms+1} \phi_i(x) \alpha_i$ where the coefficients α do not depend on t.

Discretising the expression (28), by substituting the functions v_h and u_h, successively choosing the test functions, made up of only one base $v_h = \phi_i(x),\ i = 0, 1, \ldots, M_s, M_s + 1$, we get:

$$P_d \left[\frac{du_h}{dt}, v_h \right] + B[u_h, v_h] = \sum_{j=0}^{M_x+1} U_j^{'}(t) \cdot P_d \left[\phi_j(x), \phi_i(x) \right] + \sum_{j=0}^{M_x+1} U_j(t) \cdot B \left[\phi_j(x), \phi_i(x) \right],\ \ i = 0, \ldots, M_s + 1$$

which can be seen in matrix form as

$$\mathbf{P}\mathbf{U}^{'}(t) + \mathbf{B}\mathbf{U}(t) \tag{31}$$

Where $\mathbf{B}, \mathbf{P} \in \mathbb{R}^{(M_S+2)\times(M_S+2)}$ are square tridiagonal matrices, such that $b_{ij} = B[\phi_j, \phi_i]$ and $p_{ij} = P_d[\phi_j, \phi_i]$.

5.3. Finite differences in time: θ-Scheme

In both cases of spatial discretization, FEM and FDM, let's assume that we have the following discretisation $\mathbf{P}_R\mathbf{U}^{'}(t) + \mathbf{B}_R\mathbf{U}(t)$, where the matrices $\mathbf{P}_R \in \mathbb{R}^{(M_S)\times(M_S+2)}$ and $\mathbf{B}_R \in \mathbb{R}^{(M_S)\times(M_S+2)}$ are known. The θ scheme consists of a convex combination between the explicit scheme and the implicit scheme, given $\theta \in [0, 1]$, then:

$$\begin{aligned} &\theta \left(\frac{1}{h_t} \mathbf{P_R}(\mathbf{U}^{n+1} - \mathbf{U}^n) + \mathbf{B}_R\mathbf{U}^n \right) + (1-\theta) \left(\frac{1}{h_t} \mathbf{P_R}(\mathbf{U}^{n+1} - \mathbf{U}^n) + \mathbf{B}_R\mathbf{U}^{n+1} \right) = \mathbf{E}_R\mathbf{U}^n + \mathbf{H}_R\mathbf{U}^{n+1} = \\ &= \mathbf{E}_r\mathbf{U}_r^n + \mathbf{r}_E^n + \mathbf{H}_R\mathbf{U}^{n+1} \end{aligned} \tag{32}$$

where the matrices $\mathbf{E}_R, \mathbf{H}_R \in \mathbb{R}^{(M_S)\times(M_S+2)}$ are given by: $\mathbf{E}_R = \left[\theta\mathbf{B}_R - \frac{1}{h_t}\mathbf{P}_R \right], \mathbf{H}_R = \left[(1-\theta)\mathbf{B}_R + \frac{1}{h_t}\mathbf{P}_R \right]$. The matrix $\mathbf{E}_r \in \mathbb{R}^{(M_S)\times(M_S)}$ is square, which results from cutting the first and last columns of the matrix $\mathbf{E}_R$. Multiply the first and last columns of $\mathbf{E}_R$ by $U_0(t)$ and $U_{M_S+1}(t)$ respectively, and add everything together to form the vector $\mathbf{r}_E(t) \in \mathbb{R}^{M_S}$.

Imposing the final value condition $U_m^{N_t+1} = g_m$ for $m = 0, \ldots, M_S + 1$, where in matrix form: $\mathbf{U}^{N_t+1} = \mathbf{g}$. And the boundary conditions $U_0^n = h_0^n$ and $U_{M_S+1}^n = h_{M_S+1}^n$ for $n = 0, \ldots, N_t + 1$. The discretization of the problem corresponds to finding $\mathbf{U}_r^n$, such that, from the complementarity problem we can write the following expression equivalent to our problem:

$$\mathbf{0} = \max\{\mathbf{E}_r\mathbf{U}_r^n + \mathbf{r}_E^n + \mathbf{H}_R\mathbf{U}^{n+1}; (\mathbf{g}_r - \mathbf{U}_r^n)\}\ ,\text{ for } n = N_t, \ldots, 1, 0 \tag{33}$$

Which will be solved with the Semi-smooth Newton Method (SSNM).

5.4. Numerical Results

The time implicit scheme ($\theta = 1$) was implemented for the two types of discretization done in space, one with finite $\mathbb{P}_1$-elements and the other with finite differences. We chose to implement the algorithms for put options, setting $T = 1$, $r = 0.06$, $\sigma = 0.3$, $K = 10$, $S_{max} = 200$, and a tolerance for the SSNM of $TOL = 10^{-12}$. In order to compare the two methods, we computed the relative errors between the 2 algorithms, $e_{\text{rela}} = |V_{\text{FEM}}(S_0, 0) - V_{\text{FDM}}(S_0, 0)|$ for different values of S_0, which we present in Table 4.

		(h_S, h_t)	$(h_S, h_t)/2$	$(h_S, h_t)/4$	$(h_S, h_t)/8$
S_0	6	0	0	0	0
	7	0	0	0	0
	8	1.3610e-03	4.3100e-04	1.1900e-04	3.1000e-05
	10	3.4100e-03	8.9000e-04	2.3600e-04	6.1000e-05
	12	1.8570e-03	5.0800e-04	1.3900e-04	3.6000e-05
	14	3.2300e-04	1.1000e-04	3.5000e-05	1.0000e-05
	16	2.2500e-04	5.1000e-05	1.0000e-05	2.0000e-06

Table 4: Relative errors between the two methods implemented for different values of S_0

The results obtained demonstrate the convergence of the finite difference and finite element methods for solving the American Options problem. As the discretization in space and time is refined, the values of $V(S_0, 0)$ for different S_0 converge consistently to the same results. The Table 4 shows that the relative errors between the two methods decrease significantly with mesh refinement, indicating that both methods converge to the same solution. This convergence, validated by the relative errors, confirms the accuracy and robustness of the implemented algorithms.

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