Numerical Optimal Control

2nd semester 2022/2023, IST

Computational Project – Inverted Pendulum

Remarks: The report should be as short as possible, answering item by item. Direct answers are prefered, except when comments or justifications are required. Whenever necessary, graphical representations and/or tables should be included. The algorithms should be implemented in *Matlab* (Or Octave). The corresponding scripts should be attached with file extension .m. Use the format *function* whenever possible. While answering, clearly explain how to run the code in order to reproduce the results shown. The report and attachments should be sent by email.

Matlab can be obtained here:

`https://si.tecnico.ulisboa.pt/software/matlab`.

Report due on: April 23

1. An inverted pendulum is a classic problem in control theory and robotics. It involves a rigid pendulum that is mounted on a cart, where the objective is to keep the pendulum balanced in the upright position by moving the cart back and forth., see Figure 1. This problem is challenging because it requires precise control of the cart's position and velocity to maintain the pendulum's equilibrium.

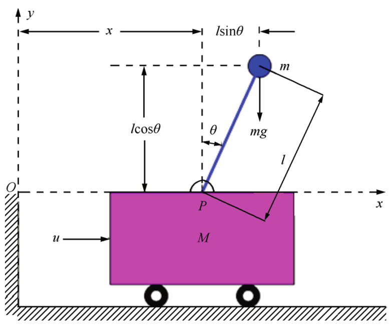

Fig. 1 Motor driven inverted pendulum-cart system

The inverted pendulum has many applications, including robotics, aerospace, and transportation. For example, the control of a helicopter's rotor blade is analogous to that of an inverted pendulum. The control of a self-balancing robot is also a popular application of the inverted pendulum problem.

The inverted pendulum problem can be solved using various control strategies, including classical control, modern control, and optimal control. The classical control approach involves designing a controller based on mathematical models of the system, while modern control techniques use advanced control algorithms to achieve better performance. Optimal control methods aim to minimize a certain cost function while maintaining the pendulum's balance.

As can be observed in the Figure 1, if we take the Cartesian axes with the origin O', we can deduce the equations of motion of the pendulum. We can apply both Newton's second law and the Lagrange's theorem (equations of energy).

Given the Lagrangian function, that is, the function $L(x, x', \theta, \theta')$ given by the difference of the kinetic energy of the system minus its potential energy.

$$L(x, x', \theta, \theta') := \frac{1}{2}(M+m)(x')^2 + mlx'\theta' \cos(\theta) + \frac{1}{2}ml^2|\theta'|^2 - mgl\cos(\theta).$$

Apply the Lagrange theorem:

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{\partial L}{\partial \theta'}\Big) = \dfrac{\partial L}{\partial \theta}, \\ \dfrac{d}{dt}\Big(\dfrac{\partial L}{\partial x'}\Big) = \dfrac{\partial L}{\partial x}, \end{cases}$$

in order to derive the following equations of motion:

$$\begin{cases} l\theta'' + x''\cos(\theta) = g\sin(\theta), \\ (M+m)x'' + ml\theta''\cos(\theta) - ml|\theta'|^2\sin(\theta) = 0. \end{cases} \tag{1}$$

where $M = 5$, $m = 1$, $l = 0.5$ and $g = 9.8$ in their corresponding units.

Solve the nonlinear problem by using a higher order method (such as Runge Kutta 4th order) and plot the solutions corresponding to different initial data, including $Y_0 = (0, 0, 0.15, 0)$.

2. Next, we can consider the control problem associated with the pendulum. In this case, we assume that our control u is an external force applied in the x direction on the cart. In this case, the equations would be modified using the second law of Newton, and we obtain:

$$\begin{cases} l\theta'' + x''\cos(\theta) = g\sin(\theta), \\ (M+m)x'' + ml\theta''\cos(\theta) - ml|\theta'|^2\sin(\theta) = u. \end{cases} \tag{2}$$

If we now introduce the vector of variables $Y = (y_1, y_2, y_3, y_4)^T = (x, x', \theta, \theta')$, the equations (2) can be written as a nonlinear ordinary differential system:

$$\frac{d}{dt}Y = F(Y) + G(Y)u, \tag{3}$$

To solve this nonlinear problem, we can use different methods, but ultimately they all involve linearization. Thus, if we linearize the previous system around 0 (since this is our equilibrium point), we can rewrite the system as follows:

$$\frac{d}{dt}Y = F(0) + DF(0)Y + G(0)u, \tag{4}$$

Find F, G, and the corresponding linearization of the model around the null state $Y = 0 \in R^4$. Provide numerical simulations to experimentally determine the valid time interval where it holds, from a qualitative (visual) perspective.

3. If we now consider the goal of controlling u in order to make the angle θ as close to 0 as possible, we can define the optimal control problem:

$$(P) \left\{ \begin{array}{l} \min J(Y, u) \\ \text{s.t. } (Y, u) \text{ solves } (4), \end{array} \right.$$

with

$$J(Y, u) := \frac{\alpha_1}{2} \int_0^T |y_3|^2 \, dt + \frac{\alpha_2}{2} |y_3(T)|^2 + \frac{\alpha_3}{2} \int_0^T |u|^2 \, dt.$$

Give an analytical derivation for the adjoint equation and for the gradient expression. Write the Steepest Descent Algorithm and implement it in Matlab to find an open loop control. Plot the uncontrolled states and compare them with the control and the controlled states.

4. Consider now the problem of

$$(P1) \left\{ \begin{array}{l} \min J(Y, u) \\ \text{s.t. } (Y, u) \text{ solves } (4), \end{array} \right.$$

with

$$J(Y, u) := \frac{\alpha_1}{2} \int_0^\infty |y_2|^2 + |y_3|^2 + |y_4|^2 \, dt + \frac{\alpha_2}{2} \int_0^\infty |u|^2 \, dt.$$

with $\alpha_1 = 1$, $\alpha_2 = 1.e - 6$.

Assume that you are looking for a feedback control of type $u = KY$. Determine K by solving the Algebraic Riccati equation associated with the $(P1)$. Solve the system and represent the results.

5. [Research] **Find an alternative strategy to improve the results obtained in question 3.**