# OPERATOR FORMULATION FOR LINEAR TRANSFORMATIONS AND SIGNAL ESTIMATION IN THE JOINT SPATIAL-SLEPIAN DOMAIN

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**ABSTRACT**

We present an operator formulation for linear transformations in the joint spatial-Slepian domain, which is enabled by the spatial-Slepian transform. The operator is an integral type and is specified by the spatial-Slepian transformation kernel, which is also utilized in finding a matrix representation for the operator using the orthogonal basis of Wigner-$D$ functions. Conditions for the compactness and self-adjointness of the operator are presented and spectral analysis is carried out. The formulation is illustrated by choosing a Gaussian form for the spectral representation of the kernel to smooth out the noise in a bandlimited Earth topography map.

***Index Terms*—** 2-sphere, $\mathbb{SO}(3)$ rotation group, spatial-spectral concentration, spatial-Slepian transform, Wigner-$D$ functions.

## 1. INTRODUCTION

Representation of signals plays a pivotal role in the development of signal processing frameworks to analyze the underlying information contained in them. The mathematics for signal representation, in turn, depends greatly on the type of signal under consideration. This work is concerned with the representation of signals that have angular dependence and are naturally defined on the surface of the sphere. Aptly called the spherical signals, these find applications in various fields of interest in science and engineering such as geophysics [1, 2], cosmology [3–5], planetary sciences [6, 7], antenna theory [8], wireless communication [9], medical imaging [10], computer graphics [11] and acoustics [12,13]. Spherical signals are commonly represented by functions called spherical harmonics, which are defined globally on the surface of the sphere. Such a representation reveals global spectral characteristics of the signal and may not be suitable for signal analysis over local spherical regions of interest.

Another useful representation for spherical signals is yielded by the Slepian spatial-spectral concenrtation problem, which results in signals that are spatially optimally localized within a given region on the sphere. Although these so called Slepian functions serve as an alternative to the representation provided by spherical harmonics, their optimally localized spatial profile makes them suitable for probing localized signal content on the sphere. Motivated by this idea, a joint spatial-Slepian domain representation of spherical signals has been proposed in [14] using spatially well-optimally localized Slepian functions, which has further been generalized in [15] by the development of a framework for linear transformations and signal estimation in the joint spatial-Slepian domain. The transformations, as well as the estimates, have been shown to be completely specified by a spatial-Slepian transformation kernel, which, in turn, allows for a more general formulation of the framework, through a linear transformation operator, that has not been done before. As summarized below, definition of such an operator and a detailed analysis of its properties is precisely the focus of this work.

1. We associate an integral operator with the spatial-Slepian transformation kernel to formalize the framework of linear transformations and signal estimation in terms of a spatial-Slepian transformation operator.
2. Matrix representation of the spatial-Slepian transformation operator is formulated, which is further used to establish its compactness and self-adjointness.
3. Finally, spectral theory of the operator is presented, in which the spatial-Slepian transformation kernel is expressed in terms of the eigenvalues of the operator.

We present these developments in Section 3 of this correspondence and provide illustrations by specializing the form of the kernel to a Gaussian function, for denoising of a bandlimited Earth topography map through spectral smoothing, in Section 4, before concluding in Section 5. However, we begin with a necessary review of the fundamentals of signal analysis on the sphere and $\mathbb{SO}(3)$ rotation group in the next section.

## 2. MATHEMATICAL PRELIMINARIES

### 2.1. The Hilbert Space $L^2(\mathbb{S}^2)$ and $L^2(\mathbb{SO}(3))$

Surface of the unit 2-sphere, or just sphere, is defined as the set $\mathbb{S}^2 \triangleq \{\hat{\boldsymbol{x}} \in \mathbb{R}^3 : |\hat{\boldsymbol{x}}| = 1\}$, in which $|\cdot|$ denotes the Euclidean norm, $\hat{\boldsymbol{x}}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}}$ is a point on the sphere parameterized by the angles colatitude, denoted by $\theta \in [0, \pi]$ and measured from the positive $z$-axis, and longitude, denoted by $\phi \in [0, 2\pi)$ and measured from the positive $x$-axis in the $x$-$y$ plane, and $(\cdot)^{\mathrm{T}}$ represents the vector transpose operation. Then, the set of square-integrable and complex-valued functions on the sphere, represented by $f(\hat{\boldsymbol{x}})$, $g(\hat{\boldsymbol{x}})$, form a Hilbert space $L^2(\mathbb{S}^2)$, which is equipped with the following inner product [16]

$$\langle f, g \rangle_{\mathbb{S}^2} \triangleq \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{g(\hat{\boldsymbol{x}})} ds(\hat{\boldsymbol{x}}), \quad \int_{\mathbb{S}^2} \equiv \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi}, \tag{1}$$

where $\overline{(\cdot)}$ denotes the complex conjugate operation and $ds(\hat{\boldsymbol{x}}) \triangleq \sin\theta d\theta d\phi$. The inner product in (1) induces norm of the function $f(\hat{\boldsymbol{x}})$ as $\|f\|_{\mathbb{S}^2} \triangleq \sqrt{\langle f, f \rangle_{\mathbb{S}^2}}$, whose energy is given by $\langle f, f \rangle_{\mathbb{S}^2}$. Finite energy functions are commonly referred to as signals.

The Hilbert space $L^2(\mathbb{S}^2)$ is spanned by a complete set of orthonormal basis functions called spherical harmonics, which are denoted by $Y_\ell^m(\hat{\boldsymbol{x}})$ for integer degree $\ell \geq 0$ and integer order $|m| \leq$

$\ell$ [16]. Completeness of spherical harmonics enables the following expansion for a signal $f \in L^2(\mathbb{S}^2)$

$$f(\hat{\boldsymbol{x}}) = \sum_{\ell,m}^{\infty} (f)_\ell^m Y_\ell^m(\hat{\boldsymbol{x}}), \quad \sum_{\ell,m}^{\infty} \equiv \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}, \tag{2}$$

where $(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle_{\mathbb{S}^2}$ is the spherical harmonic (spectral) coefficient of degree $\ell$ and order $m$ and constitutes the spectral domain representation of the signal $f(\hat{\boldsymbol{x}})$. Signal $f(\hat{\boldsymbol{x}})$ is considered bandlimited to degree $L$ if $(f)_\ell^m = 0$ for $\ell \geq L$, $|m| \leq \ell$.

A very common operation on the surface of the sphere is the rotation of signals, which is specified by the Euler angles $\omega \in [0, 2\pi)$ around $z$-axis, $\vartheta \in [0, \pi]$ around $y$-axis and $\varphi \in [0, 2\pi)$ around $z$-axis in the right-handed $zyz$ convention. Designating the 3-tuple of the Euler angles as $\rho$, i.e., $\rho \triangleq (\varphi, \vartheta, \omega)$, rotation of a signal $f(\hat{\boldsymbol{x}})$, bandlimited to degree $L_f$, is given by the following action of the rotation operator $\mathcal{D}(\rho)$ on $f(\hat{\boldsymbol{x}})$ [16]

$$(\mathcal{D}(\rho) f)(\hat{\boldsymbol{x}}) = \sum_{\ell,m}^{L_f - 1} \left( \sum_{m'=-\ell}^{\ell} D_{m,m'}^\ell(\rho) (f)_\ell^{m'} \right) Y_\ell^m(\hat{\boldsymbol{x}}), \tag{3}$$

where $D_{m,m'}^\ell(\rho)$, for integer degree $\ell \geq 0$ and integer orders $|m|, |m'| \leq \ell$, are called Wigner-$D$ functions which satisfy [16]

$$\left\langle D_{m,m'}^\ell, D_{q,q'}^p \right\rangle_{\mathbb{SO}(3)} = c_\ell \delta_{\ell,p} \delta_{m,q} \delta_{m',q'}, \quad c_\ell \triangleq \left( \frac{8\pi^2}{2\ell + 1} \right). \tag{4}$$

Defining rotations on the sphere takes us into the regime of $\mathbb{SO}(3)$, which is a group of proper rotations on the sphere. The set of square-integrable and complex-valued functions on $\mathbb{SO}(3)$, represented by $f(\rho)$, $g(\rho)$, constitute the Hilbert space $L^2(\mathbb{SO}(3))$, which is equipped with the following inner product

$$\langle f, g \rangle_{\mathbb{SO}(3)} \triangleq \int_{\mathbb{SO}(3)} f(\rho) \overline{g(\rho)} d\rho, \quad \int_{\mathbb{SO}(3)} \equiv \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \int_{\omega=0}^{2\pi}, \tag{5}$$

where $d\rho \triangleq d\varphi \sin\vartheta d\vartheta d\omega$. Inner product in (5) induces norm of the function $f(\rho)$ as $\|f\|_{\mathbb{SO}(3)} \triangleq \sqrt{\langle f, f \rangle_{\mathbb{SO}(3)}}$ and its energy is given by $\langle f, f \rangle_{\mathbb{SO}(3)}$.

The Hilbert space $L^2(\mathbb{SO}(3))$ is spanned by Wigner-$D$ functions, which constitute a complete set of orthogonal basis functions and enable the following representation for a signal $f \in L^2(\mathbb{SO}(3))$

$$f(\rho) = \sum_{\ell,m,m'}^{\infty} (f)_{m,m'}^\ell D_{m,m'}^\ell(\rho), \quad \sum_{\ell,m,m'}^{\infty} \equiv \sum_{\ell,m}^{\infty} \sum_{m'=-\ell}^{\ell}, \tag{6}$$

where $f_{m,m'}^\ell \triangleq 1/c_\ell \left\langle f, D_{m,m'}^\ell \right\rangle_{\mathbb{SO}(3)}$ is the $\mathbb{SO}(3)$ harmonic (spectral) coefficient of degree $\ell$ and orders $m, m'$ that encodes the spectral information of the signal $f(\rho)$. Signal $f(\rho)$ is considered bandlimited to degree $L$ if $(f)_{m,m'}^\ell = 0$ for $\ell \geq L$, $|m|, |m'| \leq \ell$.

### 2.2. The Hilbert Space $\ell^2$ and $\mathbb{C}^N$

The set of square-summable sequences of complex numbers, of the form $\mathfrak{a} = (a_1, a_2, \ldots) = \{a_k\}_{k=1}^\infty$, $\mathfrak{b} = (b_1, b_2, \ldots) = \{b_k\}_{k=1}^\infty$, forms a Hilbert space $\ell^2$, in which the inner product between two sequences, $\mathfrak{a}$, $\mathfrak{b}$, is defined as [16]

$$\langle \mathfrak{a}, \mathfrak{b} \rangle_{\ell^2} = \sum_{k=1}^{\infty} a_k \overline{b_k}. \tag{7}$$

The Hilbert space $\ell^2$ is spanned by a complete set of orthonormal sequences, given by $e_k = \{\delta_{k,n}\}_{n=1}^\infty, k = 1, 2, \ldots$, where $\delta_{k,n}$ is the Kronecker delta function, such that any sequence $\mathfrak{a} \in \ell^2$ can be expressed as

$$\mathfrak{a} = \sum_{k=1}^{\infty} a_k e_k. \tag{8}$$

Since the sequences in $\ell^2$ have infinitely-many countable components, $\ell^2$ is said to be infinite-dimensional. On the other hand, Hilbert space of sequences of complex numbers having finitely many countable components, such as $\mathfrak{a} = \{a_k\}_{k=1}^N$, $\mathfrak{b} = \{b_k\}_{k=1}^N$, is of finite dimension $N$ and is denoted by $\mathbb{C}^N$, which is spanned by $N$ orthonormal sequences of the form $e_k = \{\delta_{k,n}\}_{n=1}^N, k = 1, 2, \ldots, N$. Then, any sequence $\mathfrak{a} \in \mathbb{C}^N$ can be expressed through (8) by truncating the sum over index $k$ at $N$.

### 2.3. Linear Transformation in the Joint Spatial-Slepian Domain

The joint spatial-Slepian domain representation of a signal $f \in L^2(\mathbb{S}^2)$, bandlimited to degree $L_f$, is enabled by the following spatial-Slepian transform [14]

$$F_{g_\alpha}(\rho) \triangleq \langle f, \mathcal{D}(\rho) g_\alpha \rangle_{\mathbb{S}^2} = \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}}) \overline{\left(\mathcal{D}(\rho) g_\alpha\right)(\hat{\boldsymbol{x}})} ds(\hat{\boldsymbol{x}}), \tag{9}$$

for a Slepian scale $\alpha \in [1, N_R]$. Here, $g_\alpha(\hat{\boldsymbol{x}})$ is a Slepian function at Slepian scale $\alpha$, which is assumed bandlimited to degree $L_S$, $F_{g_\alpha}(\rho)$ is the joint spatial-Slepian domain representation of $f(\hat{\boldsymbol{x}})$ called the spatial-Slepian coefficient at Slepian scale $\alpha$, $\mathcal{D}(\rho) \equiv \mathcal{D}(\varphi, \vartheta, \omega)$ is the rotation operator and $N_R$ is the spherical Shannon number given by [17]

$$N_R = \frac{L_S^2}{4\pi} \int_R ds(\hat{\boldsymbol{x}}), \tag{10}$$

where $R \subset \mathbb{S}^2$ is the region which is considered for the solution of spatial-spectral concentration problem. Using expansion of signals in (2), along with (3) and orthonormality of spherical harmonics on the sphere, spatial-Slepian coefficients in (9) can be rewritten as[1]

$$F_{g_\alpha}(\rho) = \sum_{\ell,m}^{L_f - 1} \underbrace{\sum_{m'=\ell}^{\ell} \overline{(g_\alpha)_\ell^{m'}} \, \overline{D_{m,m'}^\ell(\rho)}}_{\psi_{\alpha,\ell m}(\rho)} (f)_\ell^m, \tag{11}$$

from which we observe that

$$(F_{g_\alpha})_{m,m'}^\ell = \frac{1}{c_\ell} \left\langle F_{g_\alpha}, \overline{D_{m,m'}^\ell} \right\rangle_{\mathbb{SO}(3)} = (f)_\ell^m \overline{(g_\alpha)_\ell^{m'}}, \tag{12}$$

that can be inverted to get the spectral coefficients of the signal $f(\hat{\boldsymbol{x}})$.

A generalized linear transformation in the joint spatial-Slepian domain has been proposed in [15] as

$$\nu_{g_\alpha}(\rho) \triangleq \sum_{\beta=1}^{N_R} \int_{\mathbb{SO}(3)} \zeta_{\alpha,\beta}(\rho, \rho_1) F_{g_\beta}(\rho_1) \, d\rho_1, \tag{13}$$

where $\zeta_{\alpha,\beta}(\rho, \rho_1)$ is the spatial-Slepian transformation kernel, $F_{g_\beta}(\rho)$ is the spatial-Slepian coefficient at Slepian scale $\beta$ and $\nu_{g_\alpha} \in L^2(\mathbb{SO}(3))$ is the modified spatial-Slepian representation of the signal $f(\hat{\boldsymbol{x}})$ at Slepian scale $\alpha$.

## 3. SPATIAL-SLEPIAN TRANSFORMATION OPERATOR

Considering $\zeta_{\alpha,\beta}(\rho, \rho_1)$ as a function of two variables on $\mathbb{SO}(3) \times \mathbb{SO}(3)$, we assume

$$\kappa_{\alpha,\beta}^2 \triangleq \int_{\mathbb{SO}(3) \times \mathbb{SO}(3)} |\zeta_{\alpha,\beta}(\rho, \rho_1)|^2 \, d\rho d\rho_1 < \infty, \quad \forall\, \alpha, \beta \in [1, N_R], \tag{14}$$

[1] We assume that $L_f = L_S$. For details, please refer to [14].

so that $\zeta_{\alpha,\beta}(\rho,\rho_1) \in L^2(\mathbb{SO}(3)\times\mathbb{SO}(3))$ for fixed Slepian scales $\alpha$, $\beta$. Then, for every $F_{g_\beta} \in L^2(\mathbb{SO}(3))$, we define the spatial-Slepian transformation operator $\mathfrak{S}_{\alpha,\beta}$ as

$$(\mathfrak{S}_{\alpha,\beta} F_{g_\beta})(\rho) = \int_{\mathbb{SO}(3)} \zeta_{\alpha,\beta}(\rho,\rho_1) F_{g_\beta}(\rho_1)\, d\rho_1, \tag{15}$$

which gives the modified spatial-Slepian representation as

$$\nu_{g_\alpha}(\rho) = \sum_{\beta=1}^{N_R} (\mathfrak{S}_{\alpha,\beta} F_{g_\beta})(\rho) = \sum_{\beta=1}^{N_R} \int_{\mathbb{SO}(3)} \zeta_{\alpha,\beta}(\rho,\rho_1) F_{g_\beta}(\rho_1) d\rho_1. \tag{16}$$

We observe that the spatial-Slepian transformation operator is bounded on $L^2(\mathbb{SO}(3))$ by $\kappa_{\alpha,\beta}$, i.e.,

$$\begin{aligned}
&\left\|(\mathfrak{S}_{\alpha,\beta} F_{g_\beta})\right\|^2_{\mathbb{SO}(3)} = \int_{\mathbb{SO}(3)} \left| \int_{\mathbb{SO}(3)} \zeta_{\alpha,\beta}(\rho,\rho_1) F_{g_\beta}(\rho_1)\, d\rho_1 \right|^2 d\rho \\
&\le \int_{\mathbb{SO}(3)} \int_{\mathbb{SO}(3)} |\zeta_{\alpha,\beta}(\rho,\rho_1)|^2\, d\rho d\rho_1 \int_{\mathbb{SO}(3)} \left|F_{g_\beta}(\rho_1)\right|^2 d\rho_1 \\
&= \kappa^2_{\alpha,\beta} \left\|F_{g_\beta}\right\|^2_{\mathbb{SO}(3)} \\
&\Rightarrow \|\mathfrak{S}_{\alpha,\beta}\|_{\mathrm{op}} \triangleq \sup_{\left\|F_{g_\beta}\right\|_{\mathbb{SO}(3)}=1} \left\|(\mathfrak{S}_{\alpha,\beta} F_{g_\beta})\right\|_{\mathbb{SO}(3)} \le \kappa_{\alpha,\beta}. \qquad (17)
\end{aligned}$$

### 3.1. Matrix Representation

Using the expansion for $F_{g_\beta} \in L^2(\mathbb{SO}(3))$, for fixed Slepian scales $\alpha, \beta$, in terms of Wigner-$D$ functions, the action of the spatial-Slepian transformation operator is given by

$$\begin{aligned}
(\mathfrak{S}_{\alpha,\beta} F_{g_\beta})(\rho) &= \mathfrak{S}_{\alpha,\beta} \left( \sum_{\ell,m,m'}^{L_f-1} (F_{g_\beta})^\ell_{m,m'} \overline{D^\ell_{m,m'}(\rho)} \right) \\
&= \sum_{\ell,m,m'}^{L_f-1} (F_{g_\beta})^\ell_{m,m'} (\mathfrak{S}_{\alpha,\beta} \overline{D^\ell_{m,m'}})(\rho) \\
&= \sum_{\ell,m,m'}^{L_f-1} (F_{g_\beta})^\ell_{m,m'} \left( \sum_{s,t,t'}^{L_v-1} \frac{1}{c_s} \mathfrak{s}^{stt',\ell mm'}_{\alpha,\beta} \overline{D^s_{t,t'}(\rho)} \right), \qquad (18)
\end{aligned}$$

where

$$\begin{aligned}
&\mathfrak{s}^{stt',\ell mm'}_{\alpha,\beta} = \left\langle \mathfrak{S}_{\alpha,\beta} \overline{D^\ell_{m,m'}}, \overline{D^s_{t,t'}} \right\rangle_{\mathbb{SO}(3)} \\
&= \int_{\mathbb{SO}(3)} \left( \int_{\mathbb{SO}(3)} \zeta_{\alpha,\beta}(\rho,\rho_1) \overline{D^\ell_{m,m'}(\rho_1)} d\rho_1 \right) D^s_{t,t'}(\rho) d\rho \\
&= \left\langle \zeta_{\alpha,\beta}, \mathfrak{E}^{stt',\ell mm'} \right\rangle_{\mathbb{SO}(3)\times\mathbb{SO}(3)}, \mathfrak{E}^{stt,\ell mm'} \triangleq \overline{D^s_{t,t'}} D^\ell_{m,m'} \qquad (19)
\end{aligned}$$

are called the spatial-Slepian transformation operator matrix elements, which quantify the projection of $\overline{D^\ell_{m,m'}(\rho)}$ onto $\overline{D^s_{t,t'}(\rho)}$. Here, $\mathfrak{E}^{stt,\ell mm'}$ represents a complete set of orthogonal basis functions in the space of square-integrable functions $L^2(\mathbb{SO}(3) \times \mathbb{SO}(3))$. Using the definition of spectral representation of spatial-Slepian coefficients, we can rewrite (18) as

$$\begin{aligned}
(\mathfrak{S}_{\alpha,\beta} F_{g_\beta})(\rho) = \int_{\mathbb{SO}(3)} \sum_{s,t,t'}^{L_v-1} \sum_{\ell,m,m'}^{L_f-1} \frac{1}{c_\ell} \frac{1}{c_s} \mathfrak{s}^{stt',\ell mm'}_{\alpha,\beta} \times \\
D^\ell_{m,m'}(\rho_1) \overline{D^s_{t,t'}(\rho)} F_{g_\beta}(\rho_1) d\rho_1,
\end{aligned}$$

which, upon comparison with (15), yields

$$\zeta_{\alpha,\beta}(\rho,\rho_1) = \sum_{\ell,m,m'}^{L_f-1} \sum_{s,t,t'}^{L_v-1} \frac{1}{c_\ell} \frac{1}{c_s} \mathfrak{s}^{stt',\ell mm'}_{\alpha,\beta} D^\ell_{m,m'}(\rho_1) \overline{D^s_{t,t'}(\rho)}. \tag{20}$$

The elements $\mathfrak{s}^{stt',\ell mm'}_{\alpha,\beta}$ are square-summable, as is shown by the following Parseval's relation

$$\int_{\mathbb{SO}(3)\times\mathbb{SO}(3)} |\zeta_{\alpha,\beta}(\rho,\rho_1)|^2\, d\rho d\rho_1 = \sum_{\ell,m,m'}^{L_f-1} \sum_{s,t,t'}^{L_v-1} \frac{1}{c_\ell} \frac{1}{c_s} \left| \mathfrak{s}^{stt',\ell mm'}_{\alpha,\beta} \right|^2 < \infty, \tag{21}$$

where we have used the expression in (20) and orthogonality of Wigner-$D$ functions on $\mathbb{SO}(3)$. We note that the expression in (19) relates the spatial-Slepian transformation operator $\mathfrak{S}_{\alpha,\beta}$ in $L^2(\mathbb{SO}(3))$ to the matrix operator $\mathfrak{s}$ in $\mathbb{C}^N$, having elements $\mathfrak{s}^{stt',\ell mm'}$, where

$$N = \sum_{\ell,m,m'}^{L_f-1} 1 \sum_{s,t,t'}^{L_v-1} 1 = \frac{L_f}{3}(4L_f^2 - 1) \frac{L_v}{3}(4L_v^2 - 1). \tag{22}$$

### 3.2. Self-Adjointness of $\mathfrak{S}_{\alpha,\beta}$

If the spatial-Slepian transformation operator matrix elements are Hermitian, i.e., $\mathfrak{s}^{stt',\ell mm'}_{\alpha,\beta} = \overline{\mathfrak{s}^{\ell mm',stt'}_{\alpha,\beta}}$, then the spatial-Slepian transformation kernel is Hermitian, i.e.,

$$\begin{aligned}
\zeta_{\alpha,\beta}(\rho,\rho_1) &= \sum_{\ell,m,m'}^{L_f-1} \sum_{s,t,t'}^{L_v-1} \frac{1}{c_\ell} \frac{1}{c_s} \mathfrak{s}^{stt',\ell mm'}_{\alpha,\beta} D^\ell_{m,m'}(\rho_1) \overline{D^s_{t,t'}(\rho)} \\
&= \overline{\zeta_{\alpha,\beta}(\rho_1,\rho)}, \qquad (23)
\end{aligned}$$

and the spatial-Slepian transformation operator is self-adjoint, i.e.,

$$\begin{aligned}
&\left\langle \mathfrak{S}_{\alpha,\beta} F_{g_\beta}, H_{g_\gamma} \right\rangle_{\mathbb{SO}(3)} \\
&= \sum_{\ell,m,m'}^{L_f-1} (F_{g_\beta})^\ell_{m,m'} \sum_{p,q,q'}^{L_h-1} \overline{(H_{g_\gamma})^p_{q,q'}} \left\langle \mathfrak{S}_{\alpha,\beta} \overline{D^\ell_{m,m'}}, \overline{D^p_{q,q'}} \right\rangle_{\mathbb{SO}(3)} \\
&= \left\langle \sum_{\ell,m,m'}^{L_f-1} (F_{g_\beta})^\ell_{m,m'} \overline{D^\ell_{m,m'}}, \sum_{p,q,q'}^{L_h-1} (H_{g_\gamma})^p_{q,q'} \mathfrak{S}_{\alpha,\beta} \overline{D^p_{q,q'}} \right\rangle_{\mathbb{SO}(3)} \\
&= \left\langle F_{g_\beta}, \mathfrak{S}_{\alpha,\beta} H_{g_\gamma} \right\rangle_{\mathbb{SO}(3)}, \qquad (24)
\end{aligned}$$

where we have used the expression in (19) and the representations in (11), (18) and (23) to obtain the final equality.

### 3.3. Spectral Theory of $\mathfrak{S}_{\alpha,\beta}$

From Theorem 5.1 in [16] and (21), we conclude that spatial-Slepian transformation operator is *compact*. For self-adjoint and compact operators, their exists an eigen-decomposition

$$(\mathfrak{S}_{\alpha,\beta} \mathfrak{V}^p_{q,q'})(\rho) = \lambda^p_{q,q'} \mathfrak{V}^p_{q,q'}(\rho), \quad p \le L_v - 1, \tag{25}$$

where $\mathfrak{V}^p_{q,q'}(\rho)$ is an orthonormal function in $L^2(\mathbb{SO}(3))$[2], $\lambda^p_{q,q'}$ are real (Theorem 4.5 in [16]) and are given by

$$\lambda^p_{q,q'} = \left\langle \mathfrak{S}_{\alpha,\beta} \mathfrak{V}^p_{q,q'}, \mathfrak{V}^p_{q,q'} \right\rangle_{\mathbb{SO}(3)}. \tag{26}$$

By fixing $\rho_2$, we define

$$\zeta^{\rho_2}_{\alpha,\beta}(\rho) \triangleq \zeta_{\alpha,\beta}(\rho,\rho_2) \tag{27}$$

[2]For instance, we can define $\mathfrak{V}^p_{q,q'}(\rho) = \sqrt{1/c_p} D^p_{q,q'}(\rho)$.

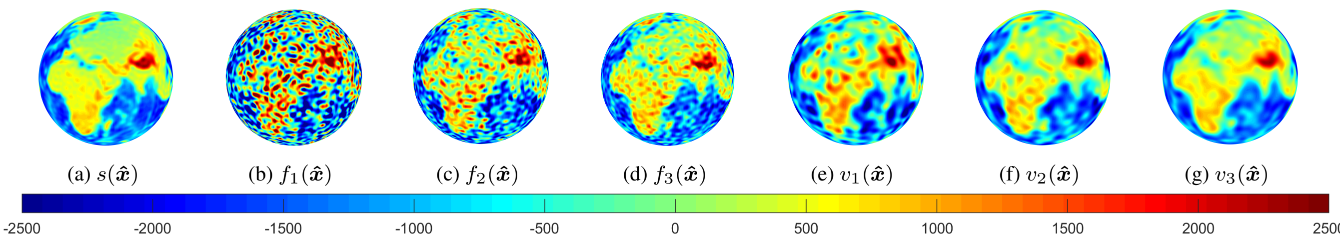


**Fig. 1**: (a) Earth topography map, (b)–(d) noise-contaminated maps having SNR of 0.04, 5.17 and 10.14 dBs respectively, (e)–(g) real Earth topography maps reconstructed through Gaussian smoothing, exhibiting SNR of 6.47, 10.48 and 13.59 dBs respectively.

to observe that

$$\begin{aligned}\zeta^{\rho_2}_{\alpha,\beta}(\rho) &= \sum_{p,q,q'}^{L_v-1} \frac{1}{c_p} \left\langle \zeta^{\rho_2}_{\alpha,\beta}, \overline{D^p_{q,q'}} \right\rangle_{\mathbb{SO}(3)} \overline{D^p_{q,q'}(\rho)} \\ &= \sum_{p,q,q'}^{L_v-1} \frac{1}{c_p} \overline{\int_{\mathbb{SO}(3)} \zeta^{\rho_1}_{\alpha,\beta}(\rho_2)\, \overline{D^p_{q,q'}(\rho_1)} d\rho_1} \, \overline{D^p_{q,q'}(\rho)} \\ \zeta_{\alpha,\beta}(\rho,\rho_1) &= \sum_{p,q,q'}^{L_v-1} \frac{1}{c_p} \overline{\left(\mathfrak{S}_{\alpha,\beta} \overline{D^p_{q,q'}}\right)(\rho_1)}\, \overline{D^p_{q,q'}(\rho)} \\ &= \overline{\zeta_{\alpha,\beta}(\rho_1,\rho)} = \sum_{p,q,q'}^{L_v-1} \frac{1}{c_p} \left(\mathfrak{S}_{\alpha,\beta} \overline{D^p_{q,q'}}\right)(\rho) D^p_{q,q'}(\rho_1),\end{aligned} \tag{28}$$

where we have used Hermitian symmetry of the kernel in the second and penultimate expressions. Using the orthonormal sequence of functions in (25), we can show that

$$\begin{aligned}\zeta_{\alpha,\beta}(\rho,\rho_1) &= \sum_{p,q,q'}^{L_v-1} \left(\mathfrak{S}_{\alpha,\beta} \overline{\mathfrak{V}^p_{q,q'}}\right)(\rho) \mathfrak{V}^p_{q,q'}(\rho_1) \\ &= \sum_{p,q,q'}^{L_v-1} \lambda^p_{q,q'} \overline{\mathfrak{V}^p_{q,q'}(\rho)}\, \mathfrak{V}^p_{q,q'}(\rho_1).\end{aligned} \tag{29}$$

Therefore, the above spatial-Slepian transformation kernel expansion completely characterizes the spatial-Slepian transformation operator in terms of the eigenvalues and orthonormal eigenfunctions. The spatial-Slepian transformation operator matrix elements can now be rewritten in terms of the eigenfunctions as

$$\mathfrak{s}^{(\mathfrak{V})(stt',\ell mm')}_{\alpha,\beta} = \left\langle \mathfrak{S}_{\alpha,\beta} \overline{\mathfrak{V}^\ell_{m,m'}}, \mathfrak{V}^s_{t,t'} \right\rangle_{\mathbb{SO}(3)} = \lambda^\ell_{m,m'} \delta_{\ell,s} \delta_{m,t} \delta_{m',t'},$$

which shows that the operator matrix becomes diagonal. Moreover, since the trace of the spatial-Slepian transformation operator is given by the sum of the eigenvalues, we note from (29) that

$$\operatorname{trace}(\mathfrak{S}_{\alpha,\beta}) = \sum_{p,q,q'}^{L_v-1} \lambda^p_{q,q'} = \int_{\mathbb{SO}(3)} \zeta_{\alpha,\beta}(\rho,\rho) d\rho. \tag{30}$$

Upper bound of the norm of the spatial-Slepian transformation operator can also be expressed in terms of its eigenvalues as

$$\begin{aligned}\kappa_{\alpha,\beta} &= \sqrt{\int_{\mathbb{SO}(3)\times\mathbb{SO}(3)} \left|\zeta_{\alpha,\beta}(\rho,\rho_1)\right|^2 d\rho d\rho_1} \\ &= \sqrt{\sum_{p,q,q'}^{L_v-1} \lambda^p_{q,q'} \sum_{s,t,t'}^{L_v-1} \lambda^s_{t,t'} \int_{\mathbb{SO}(3)} \overline{\mathfrak{V}^p_{q,q'}(\rho)} \mathfrak{V}^s_{t,t'}(\rho) d\rho \times} \\ &\sqrt{\int_{\mathbb{SO}(3)} \mathfrak{V}^p_{q,q'}(\rho_1) \overline{\mathfrak{V}^s_{t,t'}(\rho_1)} d\rho_1} = \sqrt{\sum_{p,q,q'}^{L_v-1} \left(\lambda^p_{q,q'}\right)^2},\end{aligned} \tag{31}$$

where we have used orthonormality of $\mathfrak{V}^p_{q,q'}(\rho_1)$ on $\mathbb{SO}(3)$ rotation group.

## 4. ANALYSIS

The developments in Section 3 are illustrated on a bandlimited Earth topography map[3] by using the following convolutive form of the spatial-Slepian transformation kernel

$$\begin{aligned}\zeta_{\alpha,\beta}(\rho,\rho_1) &\triangleq \zeta_\alpha(\rho\rho_1^{-1})\delta_{\alpha,\beta} \Rightarrow \mathfrak{S}_{\alpha,\beta} \equiv \mathfrak{S}_\alpha, \\ (\mathfrak{S}_\alpha F_{g_\alpha})(\rho) &= \int_{\mathbb{SO}(3)} \zeta_\alpha(\rho\rho_1^{-1}) F_{g_\alpha}(\rho_1) d\rho_1 \\ &= (\zeta_\alpha \circledast F_{g_\alpha})(\rho) = \nu_{g_\alpha}(\rho),\end{aligned} \tag{32}$$

where $\circledast$ represents convolution of signals on $\mathbb{SO}(3)$ [18] and (15), (16) have been employed in the second and third expressions. Evidently, such a choice leads to a kernel in the joint spatial-Slepian domain and turns the spatial-Slepian transformation operator into a convolution operator on $L^2(\mathbb{SO}(3))$ at a fixed Slepian scale $\alpha$. By choosing the spectral coefficients of the kernel to be

$$(\zeta_\alpha)^p_{q,q'} = \frac{1}{c_p} e^{-[p(p+1)+q]^2\alpha^2/L_f^4} \delta_{q,q'}, \quad p, |q|, |q'| \in [0, L_f), \tag{33}$$

the spectral representation of the modified signal, corresponding to $\nu_{g_\alpha}$ in (32), is given by [15]

$$(v)^q_p = \left(\sum_{\alpha=1}^{N_R} E_{p,\alpha}\right)^{-1} \sum_{\alpha=1}^{N_R} E_{p,\alpha}\, e^{\frac{-[p(p+1)+q]^2\alpha^2}{L_f^4}} (f)^q_p, \tag{34}$$

where $E_{p,\alpha} \triangleq \sum_{q=-p}^{p} \left|(g_\alpha)^q_p\right|^2$ is the energy per degree of $g_\alpha(\hat{\boldsymbol{x}})$. Hence, if $(f)^q_p$ represents the spectral information of the noisy Earth topography map, then $(v)^q_p$ is its spectral estimate obtained by Gaussian spectral smoothing through the kernel given in (33).

Fig. 1 shows the Earth topography map $s(\hat{\boldsymbol{x}})$, bandlimited to degree $L_f = 64$, which is contaminated by zero-mean, uncorrelated and anisotropic Gaussian noise at an SNR of $0.04, 5.17$ and $10.14$ dBs[4] to obtain noise-contaminated signals $f_1(\hat{\boldsymbol{x}})$, $f_2(\hat{\boldsymbol{x}})$ and $f_3(\hat{\boldsymbol{x}})$ respectively. The corresponding estimated maps $v_1(\hat{\boldsymbol{x}}), v_2(\hat{\boldsymbol{x}})$ and $v_3(\hat{\boldsymbol{x}})$ exhibit an SNR of $6.47, 10.48$ and $13.59$ dBs respectively.

## 5. CONCLUSION

We have introduced spatial-Slepian transformation operator for linear transformations in the joint spatial-Slepian domain, which is enabled by the spatial-Slepian transform. The operator has been specified by the spatial-Slepian transformation kernel and a matrix representation, using Wigner-$D$ functions, has been presented. Establishing the conditions for compactness and self-adjointness, the operator has further been described using an eigenbasis. By choosing a simple Gaussian form for the spectral representation of the spatial-Slepian transformation kernel, the framework has been employed to smooth out the noise in a bandlimited Earth topography map.

[3] http://geoweb.princeton.edu/people/simons/software.html

[4] For a signal $f(\hat{\boldsymbol{x}})$, $\mathrm{SNR}^f \triangleq 20\log\left(\|s(\hat{\boldsymbol{x}})\|_{\mathbb{S}^2} \,/\, \|f(\hat{\boldsymbol{x}}) - s(\hat{\boldsymbol{x}})\|_{\mathbb{S}^2}\right)$, where $s(\hat{\boldsymbol{x}})$ is the noise-free ground truth.

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