LOCALIZING MORE SOURCES THAN SENSORS IN PRESENCE OF COHERENT SOURCES

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ABSTRACT

DOA estimation with sparse linear arrays has been extensively studied, with an emphasis on localizing more sources than sensors. A critical assumption in previous studies however is that the sources are all uncorrelated. In this paper, we present an algorithm that is shown to be able to localize more sources than sensors in presence of correlated or coherent sources without the knowledge of the source coherence structure. Our algorithm is generalized from our recently proposed rank-constrained ADMM approach to maximum likelihood estimation for uncorrelated sources with a uniform linear array.

***Index Terms*—** Direction-of-arrival estimation, maximum likelihood estimation, rank-constrained ADMM, sparse linear array, more sources than sensors, coherent sources.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation using outputs of a sensor array has been an active research topic for decades [1]. Given a fixed number of antennas/sensors, one always wants to detect more sources by properly deploying them. A sparse linear array (SLA) is usually adopted since it possesses good properties of a uniform linear array (ULA) and has enhanced degrees of freedom. Since the maximum number of resolvable sources is determined by the aperture of the co-array that can be larger than the number of sensors, more sources than sensors can thus be localized with a proper SLA.

The research on array geometry design dates back to [2], where the minimum redundant array (MRA) is proposed that is redundant in the sense that its co-array is a ULA and minimizes such redundancy to achieve the largest aperture. Due to lack of close-form solution to the geometry when the number of sensors is large, several specialized SLAs, e.g., coprime arrays [3], nested and super nested arrays [4], have been proposed in the last decade, which are easy to design and maintain the capability of localizing $\mathcal{O}\left(M^2\right)$ sources with M sensors.

To localize more sources than sensors, various DOA estimation techniques have been proposed with general or specific SLAs. In these methods, the augmented covariance matrix regarding the co-array is usually estimated with different techniques such as covariance fitting [5, 6], spatial smoothing [7] and sparse representation [8], from which the DOAs are finally estimated with a subspace method; see [9, 10] for good reviews on this topic and a great number of references therein. In theory, the Cramér-Rao bound (CRB), which provides a lower bound on the accuracy of unbiased estimators, are studied to show the number of resolvable sources with an SLA; see [11, 12].

The aforementioned algorithms and theory show that more sources than sensors can be estimated with an SLA; however, a critical assumption therein is that the sources are all uncorrelated. We note that algorithms are available to deal with a mixture of uncorrelated and coherent signals when the source coherence structure is known [13–16].

In this paper, we investigate whether it is possible to localize more sources than sensors in presence of correlated or coherent sources given the total number of sources. To this end, inspired by our recent work [17], we first propose a sophisticatedly designed maximum likelihood estimation (MLE) algorithm, which is applicable to an arbitrary SLA, for DOA estimation of uncorrelated sources. Its efficiency for uncorrelated sources is validated via numerical simulations. Motivated by empirical robustness of the MLE to correlated sources (see, e.g., [18]), we apply the proposed algorithm to DOA estimation of a mixture of uncorrelated and coherent sources (without assuming detailed coherence structure). Extensive numerical simulations are provided showing that more sources than sensors can be localized with a practical algorithm even in presence of coherent sources.

Notations: For integer N, $[N]$ is defined as the set $\{1,\ldots,N\}$. For vector $\boldsymbol{x}$, $\operatorname{diag}(\boldsymbol{x})$ denotes a diagonal matrix with $\boldsymbol{x}$ on the diagonal. The jth entry of vector $\boldsymbol{x}$ is x_j. For matrix $\boldsymbol{A}$, $\boldsymbol{A}^T$, $\boldsymbol{A}^H$, $|\boldsymbol{A}|$, $\boldsymbol{A}^{-1}$, $\operatorname{rank}(\boldsymbol{A})$, $\operatorname{tr}(\boldsymbol{A})$ and $\|\boldsymbol{A}\|_{\mathrm{F}}$ denote the matrix transpose, conjugate transpose, determinant, inverse, rank, trace and Frobenius norm of $\boldsymbol{A}$, respectively. $\boldsymbol{A} \geq 0$ means that $\boldsymbol{A}$ is Hermitian positive semidefinite.

2. PROBLEM FORMULATION

This paper studies DOA estimation of narrowband far-field sources with an SLA. Let $\Omega \subset [N]$ denotes the SLA, of size $M = |\Omega| \leq N$, that is a subset of a virtual N-

element ULA. Suppose that K sources from directions $\{\theta_k \in [-90°, 90°)\}_{k=1}^{K}$ impinges on the SLA. Then, the array outputs at every snapshot $l \in [L]$ compose an $M \times 1$ vector $\boldsymbol{y}_{\Omega}(l)$ that can be written as [1]:

$$\boldsymbol{y}_{\Omega}(l) = \sum_{k=1}^{K} \boldsymbol{a}_{\Omega}\left(\theta_k\right) x_k(l) + \boldsymbol{e}(l) = \boldsymbol{A}_{\Omega}(\boldsymbol{\theta})\boldsymbol{x}(l) + \boldsymbol{e}(l), \tag{1}$$

where $\boldsymbol{A}_{\Omega}(\boldsymbol{\theta})$ denotes the $M \times K$ steering matrix of which each column $\boldsymbol{a}_{\Omega}\left(\theta_k\right)$ is an $M \times 1$ steering vector, $\boldsymbol{x}(l) \in \mathbb{C}^K$ denotes the $K \times 1$ source signal at snapshot l, and $\boldsymbol{e}(l)$ is the measurement noise. Since the SLA Ω is a subset of a virtual N-element ULA, the vector $\boldsymbol{y}_{\Omega}(l)$ is a subvector of an $N \times 1$ vector $\boldsymbol{y}(l)$ that is composed of outputs of the ULA. Let $\boldsymbol{\Gamma}$ denote the row selection matrix such that $\boldsymbol{y}_{\Omega} = \boldsymbol{\Gamma}\boldsymbol{y}$. It then follows that $\boldsymbol{a}_{\Omega}(\theta) = \boldsymbol{\Gamma}\boldsymbol{a}(\theta)$ and $\boldsymbol{A}_{\Omega}(\boldsymbol{\theta}) = \boldsymbol{\Gamma}\boldsymbol{A}(\boldsymbol{\theta})$, where $\boldsymbol{A}(\boldsymbol{\theta})$ and $\boldsymbol{a}\left(\theta_k\right)$ are the steering matrix and steering vector of the ULA, with

$$\boldsymbol{a}(\theta) = [1, e^{i2\pi d \sin\theta}, \ldots, e^{i(M-1)2\pi d \sin\theta}]^T, \tag{2}$$

where d denotes the ratio of inter-sensor distance to signal wavelength that is taken to be $1/2$ as usual. Our objective is to estimate $\{\theta_k\}_{k=1}^{K}$ from $\{\boldsymbol{y}_{\Omega}\left(l\right)\}_{l=1}^{L}$ with possibly $K \geq M$ and in presence of correlated or coherent sources.

3. PROPOSED ALGORITHM

3.1. Previous Algorithm for ULAs

A sophisticatedly designed iterative algorithm is proposed in our recent work [17] to solve the MLE problem. The algorithm is derived for independent Gaussian distributed sources, while it is shown to be robust to correlated and even coherent sources due to good properties of the MLE and outperform existing approaches in many scenarios. In particular, we assume that the sources and noise are independent and both are spatially and temporally independent, circular complex Gaussian distributed with

$$\mathbb{E}\boldsymbol{x}(l)\boldsymbol{x}^H(l) = \boldsymbol{P} \triangleq \text{diag}\left(\boldsymbol{p}\right), \tag{3}$$

$$\mathbb{E}\boldsymbol{e}(l)\boldsymbol{e}^H(l) = \sigma\boldsymbol{I}, \tag{4}$$

where the sources powers $p_k > 0$ and noise power $\sigma \geq 0$. It follows that the (stochastic) MLE regarding $\{\boldsymbol{\theta}, \boldsymbol{p}, \sigma\}$ attempts to solve the optimization problem:

$$\min_{\boldsymbol{\theta}, \boldsymbol{p}, \sigma} \ln|\boldsymbol{R}| + \text{tr}\left(\boldsymbol{R}^{-1}\widehat{\boldsymbol{R}}\right), \tag{5}$$

where

$$\boldsymbol{R} = \mathbb{E}\boldsymbol{y}(l)\boldsymbol{y}^H(l) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{P}\boldsymbol{A}^H(\boldsymbol{\theta}) + \sigma\boldsymbol{I}, \tag{6}$$

$$\widehat{\boldsymbol{R}} = \frac{1}{L}\sum_{l=1}^{L} \boldsymbol{y}(l)\boldsymbol{y}^H(l) \tag{7}$$

denote the data and sample covariance matrices, respectively. The above problem is difficult to solve since 1) $\boldsymbol{R}$ is a highly nonlinear function of the DOAs and 2) $\ln|\boldsymbol{R}|$ is concave on the positive semidefinite cone. To resolve these problems, the following techniques are adopted in [17].

Re-parameterization: To remove the nonlinearity regarding the DOAs, an $N \times N$ positive semidefinite rank-K Toeplitz matrix is introduced and the data covariance matrix in (6) is re-parameterized as:

$$\boldsymbol{R} = \mathcal{T}\boldsymbol{t} + \sigma\boldsymbol{I}, \quad \mathcal{T}\boldsymbol{t} \geq \boldsymbol{0}, \quad \text{rank}\left(\mathcal{T}\boldsymbol{t}\right) = K, \tag{8}$$

where $\mathcal{T}\boldsymbol{t} = (t_{i-j})$ with $\boldsymbol{t} = [t_{1-N}, \ldots, t_{N-1}]^T$ and $t_{-j} = \bar{t}_j, j = 0, \ldots, N-1$. Note that $\{\boldsymbol{\theta}, \boldsymbol{p}\}$ can be uniquely retrieved from the Vandermonde decomposition of the $\mathcal{T}\boldsymbol{t}$ given above [1]. By doing so, $\boldsymbol{R}$ becomes a linear function of the new parameters $\{\boldsymbol{t}, \sigma\}$ where the challenging rank constraint has been extensively studied.

Majorization minimization: A majorization minimization (MM) strategy is exploited in [17] to resolve the concavity, as in [19]. In particular, at the jth iteration of the MM, we minimize the linearized (and majorized) objective function:

$$\text{tr}\left(\boldsymbol{R}_{j-1}^{-1}\boldsymbol{R}\right) + \text{tr}\left(\boldsymbol{R}^{-1}\widehat{\boldsymbol{R}}\right) \tag{9}$$

where $\boldsymbol{R}_{j-1}$ denotes the last iterate of $\boldsymbol{R}$. Substituting (8) into (9), we obtain the problem to solve at the jth iteration as:

$$\begin{aligned} &\min_{\boldsymbol{t}, \sigma} \text{tr}(\boldsymbol{R}_{j-1}^{-1}(\mathcal{T}\boldsymbol{t} + \sigma\boldsymbol{I})) + \text{tr}\left((\mathcal{T}\boldsymbol{t} + \sigma\boldsymbol{I})^{-1}\widehat{\boldsymbol{R}}\right), \\ &\text{subject to } \mathcal{T}\boldsymbol{t} \geq \boldsymbol{0}, \ \text{rank}\left(\mathcal{T}\boldsymbol{t}\right) \leq K. \end{aligned} \tag{10}$$

Rank-constrained reformulation: Let $\boldsymbol{W} = \boldsymbol{R}_{j-1}^{-1}$ and $\widehat{\boldsymbol{Y}}$ be such that

$$\widehat{\boldsymbol{R}} = \widehat{\boldsymbol{Y}}\widehat{\boldsymbol{Y}}^H. \tag{11}$$

The objective function at the jth iteration becomes:

$$\text{tr}(\boldsymbol{W}(\mathcal{T}\boldsymbol{t} + \sigma\boldsymbol{I})) + \text{tr}\left(\widehat{\boldsymbol{Y}}^H(\mathcal{T}\boldsymbol{t} + \sigma\boldsymbol{I})^{-1}\widehat{\boldsymbol{Y}}\right). \tag{12}$$

Making use of the identity in [20, Lemma 5], the problem turns to be minimization of

$$\text{tr}(\boldsymbol{W}(\mathcal{T}\boldsymbol{t} + \sigma\boldsymbol{I})) + \text{tr}\left(\boldsymbol{Z}^H\left[\mathcal{T}\boldsymbol{t}\right]^{-1}\boldsymbol{Z}\right) + \sigma^{-1}\|\widehat{\boldsymbol{Y}} - \boldsymbol{Z}\|_{\text{F}}^2 \tag{13}$$

with respect to $(\boldsymbol{t}, \sigma, \boldsymbol{Z})$. By computing σ explicitly as

$$\sigma^*(\boldsymbol{Z}) = \frac{1}{\sqrt{\text{tr}(\boldsymbol{W})}}\|\widehat{\boldsymbol{Y}} - \boldsymbol{Z}\|_{\text{F}}, \tag{14}$$

the objective function becomes

$$\text{tr}(\boldsymbol{W}\mathcal{T}\boldsymbol{t}) + \text{tr}\left(\boldsymbol{Z}^H\left[\mathcal{T}\boldsymbol{t}\right]^{-1}\boldsymbol{Z}\right) + 2\sqrt{\text{tr}(\boldsymbol{W})}\|\widehat{\boldsymbol{Y}} - \boldsymbol{Z}\|_{\text{F}}. \tag{15}$$

Consequently, the problem to solve is formulated as:

$$\begin{aligned}&\min_{\boldsymbol{t},\boldsymbol{X},\boldsymbol{Z}} \operatorname{tr}(\boldsymbol{W}\mathcal{T}\boldsymbol{t}) + \operatorname{tr}(\boldsymbol{X}) + 2\sqrt{\operatorname{tr}(\boldsymbol{W})}\|\widehat{\boldsymbol{Y}} - \boldsymbol{Z}\|_{\mathrm{F}},\\ &\text{subject to } \begin{bmatrix}\boldsymbol{X} & \boldsymbol{Z}^H\\ \boldsymbol{Z} & \mathcal{T}\boldsymbol{t}\end{bmatrix} \geq \boldsymbol{0}, \operatorname{rank}(\mathcal{T}\boldsymbol{t}) \leq K,\end{aligned} \tag{16}$$

and further equivalently as:

$$\begin{aligned}&\min_{\boldsymbol{t},\boldsymbol{X},\boldsymbol{Z}} \operatorname{tr}(\boldsymbol{W}\mathcal{T}\boldsymbol{t}) + \operatorname{tr}(\boldsymbol{X}) + 2\sqrt{\operatorname{tr}(\boldsymbol{W})}\|\widehat{\boldsymbol{Y}} - \boldsymbol{Z}\|_{\mathrm{F}},\\ &\text{subject to } \begin{bmatrix}\boldsymbol{X} & \boldsymbol{Z}^H\\ \boldsymbol{Z} & \mathcal{T}\boldsymbol{t}\end{bmatrix} \in \mathbb{S}_+^K,\end{aligned} \tag{17}$$

where $\mathbb{S}_+^K$ denotes the set of positive-semidefinite matrices of rank no greater than K upon which the projection can be computed using the truncated eigen-decomposition.

Rank-constrained ADMM: By introducing an auxiliary matrix $\boldsymbol{Q}$, the problem in (17) is written briefly as

$$\min_{\boldsymbol{x},\boldsymbol{Q}\in\mathbb{S}_+^K} f(\boldsymbol{x}), \text{ subject to } \boldsymbol{Q} = \mathcal{A}\boldsymbol{x}, \tag{18}$$

where $\boldsymbol{x} \triangleq (\boldsymbol{t},\boldsymbol{X},\boldsymbol{Z})$, $f(\boldsymbol{x}) \triangleq \operatorname{tr}(\boldsymbol{W}\mathcal{T}\boldsymbol{t}) + \operatorname{tr}(\boldsymbol{X}) + 2\sqrt{\operatorname{tr}(\boldsymbol{W})}\|\widehat{\boldsymbol{Y}} - \boldsymbol{Z}\|_{\mathrm{F}}$, $\mathcal{A}\boldsymbol{x} \triangleq \begin{bmatrix}\boldsymbol{X} & \boldsymbol{Z}^H\\ \boldsymbol{Z} & \mathcal{T}\boldsymbol{t}\end{bmatrix}$. Encouraged by great success of the alternating direction method of multipliers (ADMM) in solving nonconvex and especially rank-constrained problems [21–23] as well as by identifying $\boldsymbol{x}$, $\boldsymbol{Q}$ as two groups of variables, the problem in (18) is effectively solved in [17] within the ADMM framework, where $\boldsymbol{x}$, $\boldsymbol{Q}$ and a Lagrangian multiplier are updated alternatingly either in closed form or via the truncated eigen-decomposition, with a per-iteration computational complexity of $\mathcal{O}(N^3)$.

The whole iterative algorithm, termed as MLE-ADMM, consists of the outer MM loop and the inner ADMM loop.

3.2. New Algorithm for SLAs

With an SLA, we also derive the rank-constrained ADMM algorithm for the MLE under the assumption of uncorrelated sources. In particular, under the same assumptions as previously, the MLE problem is given instead by:

$$\min_{\boldsymbol{\theta},\boldsymbol{p},\sigma} \ln|\boldsymbol{R}_\Omega| + \operatorname{tr}\left(\boldsymbol{R}_\Omega^{-1}\widehat{\boldsymbol{R}}_\Omega\right), \tag{19}$$

where

$$\boldsymbol{R}_\Omega = \mathbb{E}\boldsymbol{y}_\Omega(l)\boldsymbol{y}_\Omega^H(l) = \boldsymbol{\Gamma}\boldsymbol{R}\boldsymbol{\Gamma}^T, \tag{20}$$

$$\widehat{\boldsymbol{R}}_\Omega = \frac{1}{L}\sum_{l=1}^{L}\boldsymbol{y}_\Omega(l)\boldsymbol{y}_\Omega^H(l) = \boldsymbol{\Gamma}\widehat{\boldsymbol{R}}\boldsymbol{\Gamma}^T \tag{21}$$

denote the current data and sample covariance matrices, and $\boldsymbol{R}$ and $\widehat{\boldsymbol{R}}$ are defined as previously.

Re-parameterization: This step is simply accomplished by making use of (20) and the re-parameterization in (8).

Majorization minimization: The objective function at the jth iteration becomes in this case:

$$\begin{aligned}&\operatorname{tr}\left(\boldsymbol{R}_{\Omega,j-1}^{-1}\boldsymbol{R}_\Omega\right) + \operatorname{tr}\left(\boldsymbol{R}_\Omega^{-1}\widehat{\boldsymbol{R}}_\Omega\right)\\ &= \operatorname{tr}\left(\boldsymbol{\Gamma}^T\boldsymbol{R}_{\Omega,j-1}^{-1}\boldsymbol{\Gamma}\boldsymbol{R}\right) + \operatorname{tr}\left(\widehat{\boldsymbol{Y}}_\Omega\boldsymbol{R}_\Omega^{-1}\widehat{\boldsymbol{Y}}_\Omega^H\right)\\ &= \operatorname{tr}(\boldsymbol{W}\boldsymbol{R}) + \operatorname{tr}\left(\widehat{\boldsymbol{Y}}_\Omega\boldsymbol{R}_\Omega^{-1}\widehat{\boldsymbol{Y}}_\Omega^H\right),\end{aligned} \tag{22}$$

where (21) and (11) are used and $\boldsymbol{W} = \boldsymbol{\Gamma}^T\boldsymbol{R}_{\Omega,j-1}^{-1}\boldsymbol{\Gamma}$.

Rank-constrained reformulation: Making use of [20, Lemma 6], we obtain

$$\operatorname{tr}\left(\widehat{\boldsymbol{Y}}_\Omega\boldsymbol{R}_\Omega^{-1}\widehat{\boldsymbol{Y}}_\Omega^H\right) = \min_{\widehat{\boldsymbol{Y}}_{\overline{\Omega}}}\operatorname{tr}\left(\widehat{\boldsymbol{Y}}\boldsymbol{R}^{-1}\widehat{\boldsymbol{Y}}^H\right), \tag{23}$$

where the set $\overline{\Omega}$ denotes the complement of Ω. By substituting (23) and (8) into (22), the objective function becomes

$$\operatorname{tr}(\boldsymbol{W}(\mathcal{T}\boldsymbol{t}+\sigma\boldsymbol{I})) + \operatorname{tr}\left(\widehat{\boldsymbol{Y}}^H(\mathcal{T}\boldsymbol{t}+\sigma\boldsymbol{I})^{-1}\widehat{\boldsymbol{Y}}\right) \tag{24}$$

regarding $\left(\boldsymbol{t},\sigma,\widehat{\boldsymbol{Y}}_{\overline{\Omega}}\right)$, which is exactly in the form of (12). Consequently, the same derivations as in the ULA case can be applied, yielding the following optimization problem:

$$\begin{aligned}&\min_{\boldsymbol{t},\boldsymbol{X},\widehat{\boldsymbol{Y}}_{\overline{\Omega}},\boldsymbol{Z}} \operatorname{tr}(\boldsymbol{W}\mathcal{T}\boldsymbol{t}) + \operatorname{tr}(\boldsymbol{X}) + 2\sqrt{\operatorname{tr}(\boldsymbol{W})}\|\widehat{\boldsymbol{Y}} - \boldsymbol{Z}\|_{\mathrm{F}},\\ &\text{subject to } \begin{bmatrix}\boldsymbol{X} & \boldsymbol{Z}^H\\ \boldsymbol{Z} & \mathcal{T}\boldsymbol{t}\end{bmatrix} \in \mathbb{S}_+^K,\end{aligned} \tag{25}$$

or equivalently,

$$\begin{aligned}&\min_{\boldsymbol{t},\boldsymbol{X},\boldsymbol{Z}} \operatorname{tr}(\boldsymbol{W}\mathcal{T}\boldsymbol{t}) + \operatorname{tr}(\boldsymbol{X}) + 2\sqrt{\operatorname{tr}(\boldsymbol{W})}\|\widehat{\boldsymbol{Y}}_\Omega - \boldsymbol{Z}_\Omega\|_{\mathrm{F}},\\ &\text{subject to } \begin{bmatrix}\boldsymbol{X} & \boldsymbol{Z}^H\\ \boldsymbol{Z} & \mathcal{T}\boldsymbol{t}\end{bmatrix} \in \mathbb{S}_+^K\end{aligned} \tag{26}$$

by noting that the solution $\widehat{\boldsymbol{Y}}_{\overline{\Omega}}^* = \boldsymbol{Z}_{\overline{\Omega}}$.

Rank-constrained ADMM: The only difference between (26) and (17) is the inclusion of the index set Ω in the term $\|\widehat{\boldsymbol{Y}} - \boldsymbol{Z}\|_{\mathrm{F}}$. Consequently, the only difference in the ADMM algorithm happens in the update rule of $\boldsymbol{Z}_{\overline{\Omega}}$. We will omit the detailed algorithm. Interested readers can consult [17].

Again, the overall algorithm consists of the outer MM loop and the inner ADMM loop and is termed as MLE-ADMM. It becomes exactly the algorithm in [17] of Ω is taken to be the ULA. The main computations in each inner iteration are dominated by an eigen-decomposition of a symmetric matrix of order $M + N$ that is affordable in array processing where the array size M and aperture N are typically small. We note that the MLE-ADMM algorithm are robust to correlated or coherent sources, although it is derived under the assumption of uncorrelated sources.

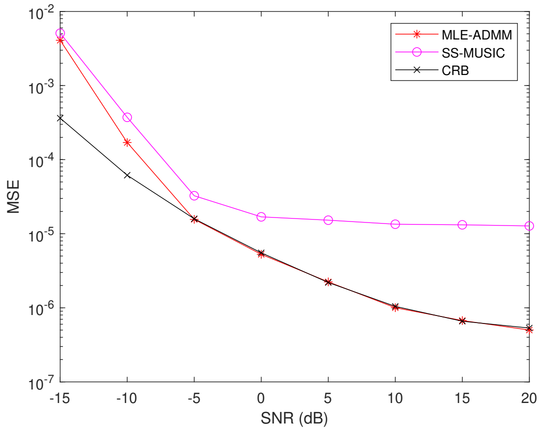

Fig. 1. MSE of DOA estimation for $K = 6$ uncorrelated sources with $\Omega = \{1, 2, 5, 8, 10\}$.

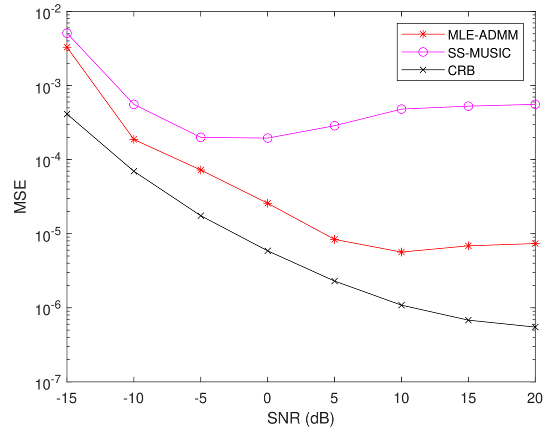

Fig. 2. MSE of DOA estimation for $K = 6$ sources with $\Omega = \{1, 2, 5, 8, 10\}$, where the first two are coherent.

4. NUMERICAL RESULTS

We present numerical results in this section to illustrate the performance of the proposed MLE-ADMM algorithm for DOA estimation with a SLA. Since each DOA θ_k is uniquely determined by a frequency parameter $f_k = \frac{1}{2}\sin\theta_k \in [-0.5, 0.5)$ and vice versa, we consider equivalently the estimation of $\{f_k\}$ for simplicity. Moreover, we consider an MRA with $\Omega = \{1, 2, 5, 8, 10\}$ where the array size $M = 5$ and aperture $N = 10$. We compare our algorithm with SS-MUSIC [7] that is implemented with forward-backward spatial smoothing and root-MUSIC.

In *Experiment 1*, we consider $K = 6$ uncorrelated sources with equal powers from directions $\{\theta_k\}$ such that $f_k = -0.43 + 0.98(k-1)/K$, $k \in [K]$. We fix the number of snapshots $L = 200$ and vary the signal-to-noise ratio (SNR) from -15 to 20 dB that is defined as the ratio of source power to noise power. The mean square error (MSE) of $\{f_k\}$ is averaged over all DOAs and 100 Monte Carlo runs. Our simulation results are presented in Fig. 1. It can be seen that the proposed algorithm outperforms SS-MUSIC in accuracy by nearly two orders of magnitude and matches the CRB, as expected, as the SNR exceeds -5dB.

In *Experiment 2*, we consider $K = 6$ sources with equal powers from the same directions but the first two are coherent. The CRB is computed in a standard routine with the knowledge that all but the first two sources are uncorrelated. Our simulation results are presented in Fig. 2. It is seen that the proposed algorithm can still estimate the DOA with satisfactory accuracy, showing that more sources than sensors can be localized even in presence of coherent sources without using the knowledge of coherence. It is also seen that a large gap exhibits between SS-MUSIC and MLE-ADMM.

In *Experiment 3*, we fix SNR$=$ 10dB and vary K from 3 to $N - 1 = 9$. For each K, the kth DOA satisfies $f_k = -0.43 + 0.98(k-1)/K$, $k = 1, \ldots, K$ so that they are sufficiently separated. The K sources remain to have equal powers and the first two are coherent. Our simulation results are presented in 3. It is seen that the error of the proposed algorithm has a same trend as the CRB and is constantly smaller than that of SS-MUSIC. We observed that SS-MUSIC cannot accurately localize the uncorrelated sources in this case and results in a seemingly strange error curve.

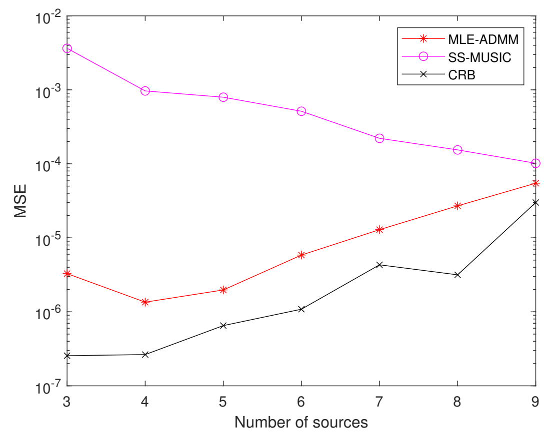

Fig. 3. MSE of DOA estimation for $K \in \{3, \ldots, 9\}$ sources with $\Omega = \{1, 2, 5, 8, 10\}$, where the first two are coherent.

5. CONCLUSION

In this paper, an effective MLE algorithm was proposed for DOA estimation with an arbitrary SLA. Numerical results were provided showing that more sources than sensors can be localized with the algorithm in presence of coherent sources without using the knowledge of coherence. Rigorous analysis of the presented results needs further investigation.

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