RATIONAL ARRAYS FOR DOA ESTIMATION

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ABSTRACT

Linear arrays used in array processing usually have sensor positions $r_i\lambda/2$ where λ is the wavelength of the impinging signals and r_i are integers. This paper considers rational arrays, where r_i are rational numbers. In particular, sparse rational arrays such as coprime rational arrays are introduced. In order to do this, some rational extensions of integer number theoretic concepts such as greatest common divisor and coprime numbers are required, which are introduced as well. The advantages of rational arrays are demonstrated with the help of rational coprime arrays. For example, they improve the accuracy of DOA estimation when the sensors have to be distributed with a fixed aperture constraint.

Index Terms— Rational arrays, non-integer arrays, identifiability, direction of arrival (DOA) estimation, MUSIC.

1. INTRODUCTION

In this paper, we introduce rational linear arrays, which have sensor positions $r_i\lambda/2$ where λ is the wavelength of the impinging monochromatic signals, and r_i are rational numbers, that is, $r_i = P_i/Q_i$ where P_i and Q_i are integers. These are in general non-integer arrays and include integer arrays ($r_i =$ integers) as special cases. Most of the literature on array processing has in the past focussed on integer arrays. This includes the uniform linear array (ULA) and sparse arrays such as the nested arrays [1], coprime arrays [2], and minimum redundancy arrays [3]. While non-integer arrays have been mentioned in the literature in the past (as elaborated below), there has not been a detailed study of such arrays.

Systematic development of rational arrays provides a more general framework and can offer some advantages compared to integer arrays. We will see that with a limited number of sensors and fixed available aperture, rational arrays have more flexibility and offer better performance (reduced DOA estimation error). In particular, we will consider *rational coprime arrays* and show that for a fixed aperture they have better performance than other integer-array alternatives. A unique advantage of rational coprime arrays is that they can entirely fill up the available aperture by sparsely placing the sensors and still provide an identifiability guarantee with MUSIC.

In order to introduce rational coprime arrays, it becomes necessary to extend number theoretic concepts such as the gcd (greatest common divisor), to the case of rational numbers. While some information on this is available in online sources [4, 5], we introduce these systematically here for completeness. We also introduce the notion of coprimality for rational numbers (Sec. 2.2).

One important consideration with these arrays is the unique identifiability of sources without ambiguity, using subspace-based algorithms such as MUSIC [6]. The well-known conditions for this in terms of the rank of the augmented array manifold matrix [7] are applicable to non-integer arrays as well. In particular, for the rational coprime array, we will present generalizations of some identifiability results from the integer case given in [8, 9]. Insights on detecting and resolving manifold ambiguities for general linear arrays can also be found in [10] and [11] respectively. In this paper, we do not consider the coarray domain; only direct MUSIC in the element-space domain is considered, even though the rational arrays are sparse. Coarrays for rational arrays will be discussed in future work [25].

Past work on non-integer arrays. Non-integer arrays do arise in practice when the sensor positions are optimized to achieve a certain objective. For example, [12, 13] propose to optimize the sensor locations in order to suppress the sidelobe level in beamforming. Similarly, [14] optimizes sensor locations to suppress interferences, whereas [15] develops a method to achieve a desired beampattern by appropriately choosing sensor locations. The resulting optimal arrays in these cases are non-integer arrays. In practice, one has to choose a rational approximation to the optimal sensor locations.

To perform DOA estimation with such arbitrary non-integer arrays, there are some methods in the literature that approximate the array manifold of any given array with a linear transformation of the array manifold of a virtual ULA. This includes techniques like array interpolation [16] and manifold separation [17], that allow root-MUSIC like techniques to be applied for DOA estimation. Fourier domain MUSIC methods [18] for DOA estimation can also be applied to arbitrary non-integer arrays in 2D. However, there has been no clear account of when such methods produce unambiguous DOA estimates.

Signal model. We consider the standard signal model in this paper, where D far-field monochromatic sources with wavelength λ impinge on the array

$$\mathbf{z} = [0 \quad r_1 \quad r_2 \quad \cdots \quad r_{m-1}] \tag{1}$$

from the directions $\theta_1, \theta_2, \ldots \theta_D$. Without loss of generality, we make a normalizing assumption $\lambda/2 = 1$ throughout the rest of the paper. The associated array manifold matrix of the array is given by

$$\mathbf{A} = [\mathbf{a}(\omega_1) \quad \mathbf{a}(\omega_2) \quad \ldots \quad \mathbf{a}(\omega_D)] \tag{2}$$

This work was supported by the Office of Naval Research grant N00014-21-1-2521, and the California Institute of Technology.

and the steering vector corresponding to direction ω is:

$$\mathbf{a}(\omega) = \begin{bmatrix} 1 & e^{j\omega r_1} & e^{j\omega r_2} & \ldots & e^{j\omega r_{m-1}} \end{bmatrix}^T \tag{3}$$

where $\omega = \pi \sin\theta$. Physically, θ lies in the range $[-90°, 90°)$, and thus ω belongs to the range $[-\pi, \pi)$. But since r_i may not be integers, we cannot replace $[-\pi, \pi)$ with $[0, 2\pi)$ or some such shift, as we do in digital filtering. See Sec. 3.3.

Scope and Outline. In this paper, we systematically develop the framework for rational arrays. In Sec. 2, we review some basic properties of rational numbers and extend the notions of gcd and coprimality to rational numbers. In Sec. 3, we introduce rational arrays and discuss the scenario where such arrays can be better suited. In Sec. 4 coprime rational arrays are proposed and an identifiability result with MUSIC is mentioned. In Sec. 5 we demonstrate the superiority of rational arrays over integer arrays in certain situations, by using DOA estimation examples. Finally, Sec. 6 concludes the paper and provides future directions.

Notations. (a, b) or $gcd(a, b)$ denotes the greatest common divisor (gcd) of two integers a and b, so $(a, b) = 1$ means that the integers are *coprime*. $lcm(a, b)$ denotes the least common multiple (lcm) of integers a and b. $\mathbb{Z}$ denotes the set of integers and $\mathbb{Z}^+$ denotes the set of positive integers. $\lfloor r \rfloor$ denotes the floor function for a real number r.

2. RATIONAL NUMBER THEORETIC CONCEPTS

In this section, we explain how some basic definitions and properties from integer number theory can be extended to rational numbers. A real number r is said to be rational if it can be expressed in the form $r = P/Q$, where $Q \neq 0$, and $P, Q \in \mathbb{Z}$. Furthermore, if $(P, Q) = 1$, P/Q is called "reduced form" or "lowest form" of r.

The notions of gcd, lcm, and coprime numbers are commonly used when dealing with integers. For integer arrays, gcd and coprimality of the sensor positions are useful in determining some properties of the array. For example, [8, 9] show that the necessary and sufficient condition for the invertibility of the steering vector of an integer array ($r_i \in \mathbb{Z}$) is given by $(r_1, r_2, \ldots, r_N) = 1$. Similarly, properties 1-5 from sec. VI of [2] pertaining to the coarrays of coprime arrays are based on Euclid's theorem for coprime numbers. In order to introduce rational coprime arrays (Sec. 4) and develop some similar identifiability results for rational arrays, we first extend the notions of gcd and coprimality to rationals. Even though gcd and lcm for rational numbers are discussed in some online sources [4, 5] we would like to establish more specific results that are useful in array processing.

2.1. GCD of rational numbers

The gcd for rationals can be defined analogously to integers:

Definition 1 (gcd of rationals)**.** The greatest common divisor of N positive rational numbers $r_1, r_2, \ldots r_N$ is defined as the largest possible rational number r such that

$$r_i = K_i \cdot r \quad \text{where,} \quad K_i \in \mathbb{Z}, \quad i = 1, 2, \ldots N.$$

It is denoted as $r = (r_1, r_2, \ldots r_N)$ or $gcd(r_1, r_2, \ldots r_N)$.

We can express the gcd in terms of numerators and denominators of the individual rational numbers as follows [25]:

Fact 1 (Formula for gcd of rationals)**.** Let $r_i = P_i/Q_i,\ i = 1, 2, \ldots N$ be positive rational numbers in their reduced form, i.e., $P_i, Q_i \in \mathbb{Z}^+, (P_i, Q_i) = 1\ \forall i$. Then,

$$gcd(r_1, r_2, \ldots r_N) = \frac{gcd(P_1, P_2, \ldots P_N)}{lcm(Q_1, Q_2, \ldots Q_N)} \tag{4}$$

2.2. Coprime rational numbers

Two integers r_1 and r_2 are said to be coprime if $(r_1, r_2) = 1$. For rational numbers if we use the same definition, then in view of Definition 1, we see that rational numbers cannot be coprime unless they happen to be integers. So, instead of using $(r_1, r_2) = 1$, we define two rational numbers to be coprime if $(r_1, r_2) \leq 1$. More generally:

Definition 2 (Coprime rationals)**.** N positive rational numbers $r_1, \cdots, r_N$ are said to be coprime if $(r_1, \cdots, r_N) \leq 1$.

To the best of our knowledge, the notion of coprimality has not been extended to rational numbers before. With this definition, two or more numbers, either integers or rationals, are coprime if and only if they cannot be expressed as integer multiples of a number larger than unity. This unifying way of defining coprimality for integers and rationals motivates the above definition. Furthermore, we will see that rational arrays which are coprime in the above sense have some identifiability properties (Theorem 1), analogous to integer coprime arrays.

3. RATIONAL ARRAYS

Now, we consider the class of m element arrays with the sensors located at rational positions. Since arrays can be translated without changing their properties, we assume that the first sensor is located at $r = 0$. Thus the array has the form

$$\begin{aligned} &\mathbf{z} = [0 \quad r_1 \quad r_2 \quad \cdots \quad r_{m-1}], \qquad \text{where,} \\ &r_i = P_i/Q_i, \quad P_i, Q_i \in \mathbb{Z}^+, \text{and } (P_i, Q_i) = 1 \quad \forall i. \end{aligned} \tag{5}$$

In general, these arrays will be called *rational arrays* to contrast them with *integer arrays*, where r_i-s are all integers.

We now discuss how rational arrays can be useful when we have to place m sensors in a fixed aperture of length A. When $A = m - 1$, we can place m sensors to form a standard integer ULA ($r_i = i$). The array manifold matrix (Eq. (2)) of a ULA has a Vandermonde structure. This gives ULAs the capability to identify $m-1$ sources unambiguously with MUSIC algorithm [7, 19]. Furthermore, it also simplifies the application of other subspace based methods like root-MUSIC [20] and ESPRIT [21] for DOA estimation. When $A \neq m-1$, two cases are possible:

3.1. Case when aperture $A < m - 1$

When the available aperture is smaller than $m - 1$, the standard m-sensor integer ULA cannot be used. One may use a shorter ULA $\mathbf{z} = [0 \quad 1 \quad 2 \quad \cdots \quad \lfloor A \rfloor]$, that fits the available aperture by choosing not to use some sensors, but this restricts the number of sources that can be resolved to $\lfloor A \rfloor$. Instead, we can simply scale the standard integer ULA by a rational factor $r < 1$:

$$\mathbf{z} = [0 \quad r \quad 2r \quad \cdots \quad (m-1)r], \qquad r < 1 \tag{6}$$

This "dense" rational ULA still has a Vandermonde array manifold matrix and has no ambiguity for sources in

$[-90°, 90°)$. Thus with appropriately chosen $r < 1$, the array in (6) can satisfy the aperture constraint and still be used to identify $m-1$ sources. However such shrinking inevitably comes at the cost of increased DOA estimation error and Cramér-Rao bound (CRB) due to reduced aperture [7] compared to the standard m-sensor integer ULA. The mutual coupling between sensors also increases due to such shrinking.

3.2. Case when aperture $A > m-1$

For this complementary scenario, we cannot simply expand the m-sensor integer ULA by a factor $r > 1$, as this will create ambiguity. For example, two sources corresponding to ω_1 and ω_2 such that $\omega_1 - \omega_2 = 2\pi/r$ have the same array output and hence cannot be distinguished. Before proceeding further, let's consider two possible integer array alternatives that may be used in this case. We will see that some of the limitations of these integer arrays motivate the use of the rational coprime array proposed in the next section.

One option is to still use the standard m-sensors ULA. Although this can identify $m-1$ sources, the array does not utilize the entire available aperture. Utilizing the entire available aperture is often beneficial for the DOA estimation task, as a larger aperture is known to reduce the mean squared error in DOA estimates along with CRB [7].

Alternatively, sparse integer ULAs with an extra coprime location sensor proposed in [8, 9] are useful to an extent. These arrays have the form:

$$\mathbf{z} = [0 \quad M \quad 2M \quad \cdots . \quad (m-2)M \quad | \quad N] \qquad (7)$$

where $(M, N) = 1$. The vertical separator in (7) is to visually separate the sparse ULA part of the array from the extra sensor at location N. Theorem 2 of [8, 9] shows that such an array can identify any $m-2$ sources with MUSIC, if no two sources are equivalent modulo $2\pi/M$ in the ω domain. One possible choice is to take M to be $\lfloor A/(m-2) \rfloor$ and N to be the largest integer coprime to M that is smaller than A. This choice of M and N utilizes the available aperture almost entirely. However, in addition to maximizing the array aperture, it might also be beneficial to distribute the sensors in a "spread-out" fashion. This is because "spreading-out" of sensors increases second-order and higher moments of sensor positions, which are also known to reduce CRB and mean squared error in DOA estimates [22, 23]. For some values of m and A, it might not be possible to achieve a "spread-out" positioning of sensors with array (7) because of integer constraint on M and N. For example, if $m = 52$ and the available aperture $A = 99$, only $M = 1$ is possible, and thus the sparse ULA part of (7) is concentrated in only half of the aperture. Even if $N = 99$ can be chosen in such a case to make use of the entire aperture, we will see in Sec. 5 that the rational counterpart of (7) proposed in (8) provides better DOA estimates. This is because the rational array (8) spreads out the ULA part of the array to span the entire aperture, instead of just moving one sensor to the extreme.

3.3. A caution about frequency interpretion of ω

The property $e^{j(\omega+2\pi)z} = e^{j\omega z}$ is valid if and only if z is an integer. Hence, when dealing with rational z (as in rational arrays), the naturally used concept in DSP about the equivalence of digital frequencies ω and $\omega \pm 2k\pi, k \in \mathbb{Z}$ no longer holds valid. This in particular also means that we cannot consider other shifted intervals of ω with length 2π, like $[0, 2\pi)$, as a substitute for the range $[-\pi, \pi)$. Thus in this paper, we plot the MUSIC spectrum with physical DOA variable θ on the abscissa instead of ω. MUSIC spectrum thus plotted for rational arrays does not always have the same value at angles $-90°$ and $90°$, unlike integer arrays.

4. RATIONAL COPRIME ARRAY

For the case considered in Sec. 3.2 where aperture $A > m-1$, we propose to use the following m-sensor rational array:

$$\mathbf{z} = [0 \quad r_1 \quad 2r_1 \quad \cdots . \quad (m-2)r_1 \quad | \quad r_2] \qquad (8)$$

where, r_1 and r_2 are rational coprime numbers, i.e. $(r_1, r_2) \leq 1$. The array is a "sparse" rational ULA with $m-1$ elements, with an extra element appended at r_2. While this array "spreads-out" the ULA part compared to its integer counterpart (7), we must still ensure that it can identify the DOAs unambiguously. It is well-known [7] that MUSIC algorithm can identify D sources corresponding to $\omega_1, \ldots, \omega_D$ unambiguously, i.e., without producing any false peaks, if and only if the augmented array manifold matrix of the array given by

$$\mathbf{A}_{aug} = [\mathbf{a}(\omega_1) \quad \mathbf{a}(\omega_2) \quad \ldots \quad \mathbf{a}(\omega_D) \quad | \quad \mathbf{a}(\omega)] \qquad (9)$$

is full column rank for all ω in the range $[-\pi, \pi)$. This condition ensures that no other steering vector $\mathbf{a}(\omega)$ lies in the column span of the steering vectors corresponding to the DOAs, and thus no false peaks are produced. A proof for this condition is given in [24]. It can be verified that the proof holds valid for *any* linear array, and in particular for the rational arrays under consideration. Using the rank condition for matrix in Eq. (9) we can arrive at an identifiability result similar to Theorem 2 of [8, 9]. We only state the identifiability result here and proof is deferred to [25] due to space constraints.

Theorem 1 (Identifiability with rational coprime array)**.** *Consider an m-sensor rational array as in Eq. (8), with $(r_1, r_2) \leq 1$. Let there be $n \leq m-2$ sources with DOAs $\theta_1, \theta_2, \ldots, \theta_n$. If the frequency variables $\omega_1, \omega_2, ..., \omega_n$, where $\omega_i = \pi \sin\theta_i$, are distinct modulo $2\pi/r_1$, then the MUSIC algorithm will identify the DOAs without producing any false peaks.*

Note that ω_i and ω_j are said to be distinct modulo $2\pi/r_1$ if $\omega_i - \omega_j \neq 2\pi k/r_1$ for any integer k. In practice, such distinctness is satisfied with probability one when the spatial distribution of sources is assumed to be uniform in $[-90°, 90°)$.

5. DOA ESTIMATION EXAMPLES WITH PROPOSED RATIONAL COPRIME ARRAYS

We now present simulation examples to show that it is indeed possible to unambiguously identify sources with the proposed rational coprime arrays and demonstrate the scenario where such arrays are better suited.

Consider $m = 8$ sensors and an available aperture $A = 10$. We consider three arrays under this setting:

$$\mathbf{z}_1 = [0 \quad 5/3 \quad 10/3 \quad 5 \quad 20/3 \quad 25/3 \quad 10 \quad | \quad 12/5]$$
$$\mathbf{z}_2 = [0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7]$$
$$\mathbf{z}_3 = [0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad | \quad 10]$$

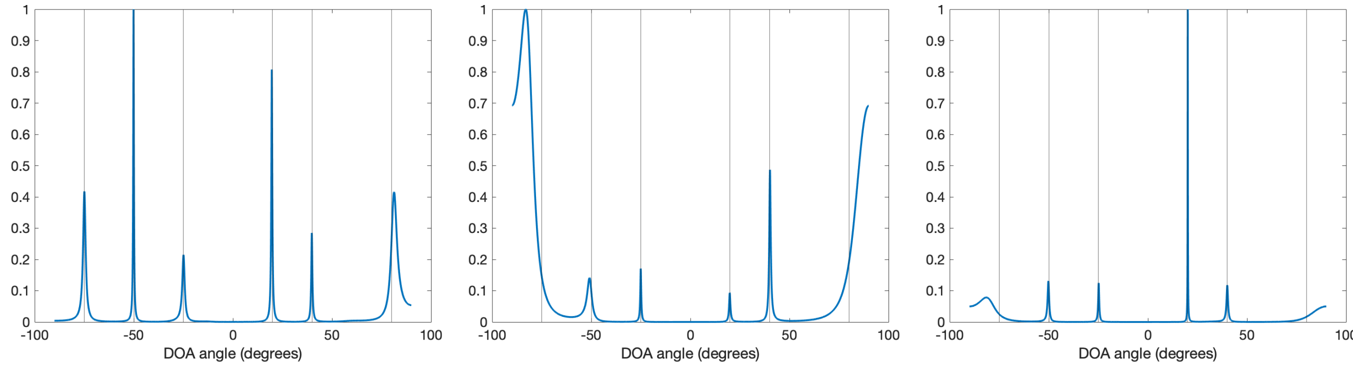

Fig. 1. MUSIC pseudospectrum for Example 1, obtained with a rational coprime array $\mathbf{z}_1$ (left), standard ULA $\mathbf{z}_2$ (middle), ULA with extra sensor $\mathbf{z}_3$ (right). True DOAs are shown in solid vertical lines.

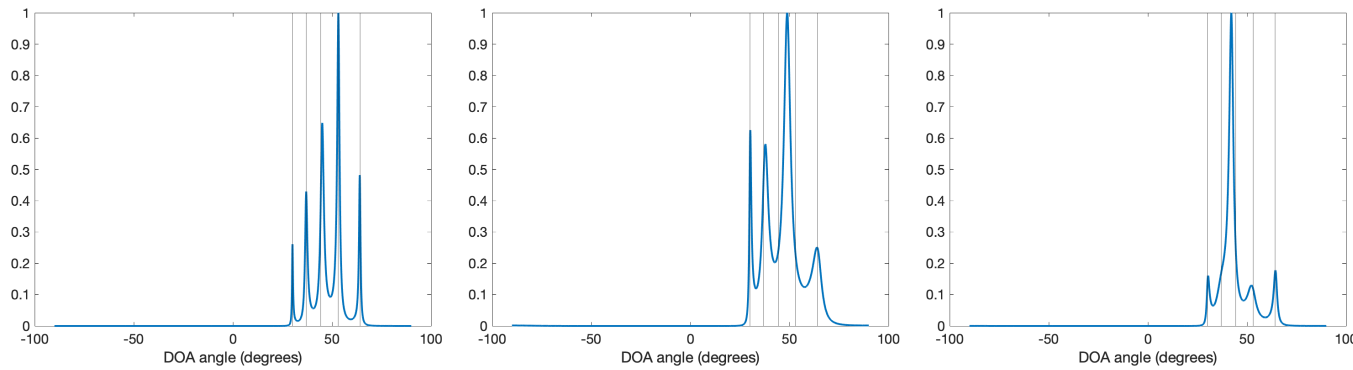

Fig. 2. MUSIC pseudospectrum for Example 2, obtained with a rational coprime array $\mathbf{z}_1$ (left), standard ULA $\mathbf{z}_2$ (middle), ULA with extra sensor $\mathbf{z}_3$ (right). True DOAs are shown in solid vertical lines.

Of the three arrays, $\mathbf{z}_1$ is a rationally coprime array with $r_1 = 5/3$ and $r_2 = 12/5$ in (8), and satisfies $(r_1, r_2) = 1/15 < 1$. Thus it can identify any 6 sources that are in accordance with the conditions in Theorem 1. $\mathbf{z}_2$ and $\mathbf{z}_3$ are two possible integer-array alternatives considered in Sec. 3.2. $\mathbf{z}_2$ is a standard 8-sensor integer ULA that can identify up to 7 sources unambiguously. However, it does not utilize the entire available aperture. $\mathbf{z}_3$ is a modified version of $\mathbf{z}_2$ to utilize the entire available aperture. Since it has a ULA part with 7 sensors, it can still unambiguously identify 6 sources. $\mathbf{z}_3$ can also be considered as an array of the form (7) with $M = 1$ and $N = 10$. Note that it is not possible to use $M \geq 2$ here, because that would exceed the aperture constraint. In order to utilize the entire available aperture, one might also use the 8 sensor sparse rational ULA of the form

$$\mathbf{z}_4 = [0 \quad 10/7 \quad 20/7 \quad 30/7 \quad 40/7 \quad 50/7 \quad 60/7 \quad 10]$$

But as mentioned before, it cannot distinguish between some pairs of sources, for example, the DOAs that correspond to ω_1 and ω_2 such that $\omega_2 - \omega_1 = 2\pi/(10/7) = 7\pi/5$.

Example 1. Now consider 6 uncorrelated sources with DOAs $-75°$, $-50°$, $-25°$, $20°$, $40°$, and $80°$ and SNR 0 dB impinging on these arrays. $K = 500$ snapshots are used. Fig. 1 shows the MUSIC pseudospectrum produced by the three arrays. It can be seen that both the integer arrays, namely the standard ULA $\mathbf{z}_2$ and its modified version $\mathbf{z}_3$, produce 6 peaks in the spectrum, but suffer from significant DOA estimation error for some of the DOAs. On the other hand, the proposed rational coprime array $\mathbf{z}_1$ produces 6 clear peaks in the spectrum that are close to the true DOAs.

Example 2. Next, we consider 5 uncorrelated sources with DOAs closely spaced compared to the above example: $30°, 36.86°, 44.42°, 53.13°,$ and $64.16°$. The SNR is 10 dB, and the same three arrays $\mathbf{z}_1, \mathbf{z}_2,$ and $\mathbf{z}_3$ are used with $K = 450$ snapshots. It can be seen from Fig. 2 that both the ULA $\mathbf{z}_2$ and modified ULA $\mathbf{z}_3$ are unable to clearly distinguish 5 sources. Their spectrum has four not so sharp peaks, that do not necessarily correspond to any of the five DOAs. On the other hand, the proposed rational coprime array $\mathbf{z}_1$ identifies all five sources with sharp peaks in the spectrum.

These examples make it clear that by distributing the sensors more "sparsely" in a given aperture, the rational coprime array can estimate the correct number of sources (Example 2) and provide better DOA estimation error (Example 1) than the integer-array alternatives.

6. CONCLUDING REMARKS

In this paper, we proposed rational arrays for DOA estimation. Extending the definition of coprime numbers to rationals allowed us to propose coprime rational arrays and an identifiability result for unambiguous DOA estimation with MUSIC. Theoretical improvements in MSE and CRB with such arrays compared to the integer alternatives will be analyzed in the future. Further results on non-integer arrays including a study of coarrays will be presented in [25].

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