PARTIALLY RELAXED ORTHOGONAL LEAST SQUARES WEIGHTED SUBSPACE FITTING DIRECTION-OF-ARRIVAL ESTIMATION

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ABSTRACT

The Partial Relaxation framework has recently been introduced to address the Direction-of-Arrival (DOA) estimation problem [1–3]. DOA estimators under the Partial Relaxation (PR) framework are computationally efficient while preserving excellent DOA estimation accuracy. This is achieved by keeping the structure of the signal from the desired direction unchanged while relaxing the structure of the signals from the remaining undesired directions. This type of relaxation allows to compute closed-form estimates for the undesired signal part and improves the accuracy of the DOA estimates compared to conventional spectral-search methods like, e.g. MUSIC. Following a similar approach as in [4] the PR framework is combined with the Orthogonal Least Squares (OLS) technique of [5]. A novel DOA estimator is proposed that is based on Partially-Relaxed Weighted Subspace Fitting (PR-WSF) in which the DOAs are iteratively estimated. Thereby, one DOA is estimated per iteration, while accounting for both the signal contributions under the previously-determined DOAs, with full signal structure, as well as the remaining DOAs with relaxed structure. Moreover, an efficient implementation of the Partially-Relaxed Orthogonal Least Squares Weighted Subspace Fitting (PR-OLS-WSF) method is proposed that provides similar computational cost as the MUSIC algorithm. Simulation results show that the proposed PR-OLS-WSF estimator provides excellent performance especially in difficult scenarios with low Signal-to-Noise-Ratio (SNR) and closely spaced sources.

1. INTRODUCTION

Due to the vast variety of use cases DOA estimation is among the most relevant research topics in sensor array processing. Among other areas, DOA estimation finds its application in e.g. wireless communication, radar, sonar, radio astronomy, geophysics and biomedical engineering [6–9]. More recent developments are focused on increasing the resolution capabilities and the estimation accuracy of DOA estimators, as well as on improving the computational efficiency and robustness of the underlying algorithms.

Multi-source DOA estimation criteria like e.g. conventional Deterministic Maximum Likelihood (DML) [6], Weighted Subspace Fitting (WSF) [10] and Covariance Fitting (CF) [11] exhibit high estimation accuracy [12–14] but suffer from a high computational cost. To reduce the computational complexity, a common approach is to apply the single-source approximation [15, 16]. In the single-source approximation it is assumed that only one source is present in the measurements although a superposition of contributions from multiple sources may be received. Thereby, the influence and dependence between the sources is not taken into account which allows to come up with cost functions that admit simple one-dimensional spectral search. However, the DOA estimation performance of single-source approximations is generally degraded as compared to the full multi-dimensional search that is otherwise required [12].

This work was supported in part by the DFG PRIDE Project PE 2080/2-1 and the German Academic Exchange Service.

In order to overcome the shortcomings of the single-source approximation problem the PR approach was developed [1–3] which accounts for the existence of multiple sources. Hence, the dependence between the sources is considered. However, while the structure of the desired direction stays unchanged the structure of the remaining directions is relaxed. A closed-form expression for the relaxed signal part is determined and substituted back, resulting in a concentrated cost function that considers signals from all directions within the field of view of the sensor and still admits simple one-dimensional spectral search. This procedure allows to reduce the computational complexity in comparison to the multi-source DOA methods while maintaining high estimation accuracy. Furthermore, estimators under the PR framework enjoy great versatility as they are applicable to any array geometry.

In this paper, we propose to extend the PR approach based on the OLS technique in [5]. The idea is to iteratively estimate the DOA as well as the corresponding transmitted signals and to subtract them from the residual signal. However instead of using the conventional beamformer in each iteration which is the case in conventional OLS, the more sophisticated PR-WSF is used. Hence, in contrast to conventional OLS, not only the previously estimated DOAs but also the remaining DOAs (which have not been estimated yet) are taken into account. Therefore, for every DOA a one-dimensional spectral grid search is conducted taking into account the structured signal part of the previously estimated DOAs and the relaxed signal part of the remaining sources. Similarly as in [4] where the OLS technique was first applied to the Partially-Relaxed Deterministic Maximum Likelihood (PR-DML) estimator we introduce an OLS version of the PR-WSF method that provides excellent threshold performance that is similar to the one of PR-OLS-DML in [5]. Additionally, a computational efficient implementation of the PR-OLS-WSF algorithm is proposed that is based on a rank-one modification and allows to estimate the DOAs at a significantly reduced cost compared to the PR-OLS-DML technique.

This paper is structured as follows. In Section 2 the signal model is introduced. The PR approach and its application to the WSF method is presented in Section 3. Furthermore, the PR-OLS-WSF DOA estimator is introduced in Section 4 followed by simulation results in Section 5. Finally, Section 6 concludes this paper.

2. SIGNAL MODEL

Consider N narrowband source signals that impinge on an arbitrary array with M sensors. The DOAs are denoted by $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_N]^{\mathrm{T}}$ and it is assumed that $N < M$. The DOAs are supposed to lie within the the field of view Θ of the sensor array, i.e. $\theta_n \in \Theta$ for $n = 1, \ldots, N$. Furthermore, the full-rank steering matrix $\boldsymbol{A}(\boldsymbol{\theta}) \in \mathbb{C}^{M \times N}$ is defined as

$$\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_N)], \tag{1}$$

where $\boldsymbol{a}(\theta_n)$ denotes the sensor array response of the n-th source. At time instant t the received baseband signal at the sensor array is denoted by $\boldsymbol{x}(t) \in \mathbb{C}^{M \times 1}$ and given by

$$\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t) \quad \text{with} \quad t = 1, \ldots, T, \tag{2}$$

where $\boldsymbol{s}(t) = [s_1(t), \ldots, s_N(t)]^{\mathrm{T}} \in \mathbb{C}^{N\times 1}$ describes the transmitted source signal and $\boldsymbol{n}(t) = [n_1(t), \ldots, n_M(t)]^{\mathrm{T}} \in \mathbb{C}^{M\times 1}$ denotes the additive noise at the sensors. It is assumed that the source signals are not fully coherent and that the number of sources N is known. Furthermore, supposing the source signals and the noise to be uncorrelated, the covariance matrix $\boldsymbol{R} \in \mathbb{C}^{M\times M}$ of the received signal in (2) is defined as

$$\boldsymbol{R} = \mathbb{E}\{\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{H}}(t)\} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{R}_{\mathrm{s}}\boldsymbol{A}^{\mathrm{H}}(\boldsymbol{\theta}) + \sigma_{\mathrm{n}}{}^2\boldsymbol{I}_M, \tag{3}$$

where $\boldsymbol{R}_{\mathrm{s}} = \mathbb{E}\{\boldsymbol{s}(t)\boldsymbol{s}^{\mathrm{H}}(t)\} \in \mathbb{C}^{N\times N}$ denotes the covariance of the transmitted signal and $\sigma_{\mathrm{n}}{}^2\boldsymbol{I}_M = \mathbb{E}\{\boldsymbol{n}(t)\boldsymbol{n}^{\mathrm{H}}(t)\} \in \mathbb{C}^{M\times M}$ is the noise covariance.

Since the true covariance matrix $\boldsymbol{R}$ in (3) is unavailable in practice it is estimated using the sample covariance matrix

$$\hat{\boldsymbol{R}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}(t)\boldsymbol{x}(t)^{\mathrm{H}}, \tag{4}$$

which is based on T snapshots of the received signal $\boldsymbol{x}(t)$ in (2). Furthermore, the eigenvalue decomposition of the sample covariance matrix $\hat{\boldsymbol{R}}$ in (4) is given by

$$\hat{\boldsymbol{R}} = \hat{\boldsymbol{U}}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{U}}^{\mathrm{H}} = \hat{\boldsymbol{U}}_{\mathrm{s}}\hat{\boldsymbol{\Lambda}}_{\mathrm{s}}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}} + \hat{\boldsymbol{U}}_{\mathrm{n}}\hat{\boldsymbol{\Lambda}}_{\mathrm{n}}\hat{\boldsymbol{U}}_{\mathrm{n}}^{\mathrm{H}}, \tag{5}$$

where $\hat{\boldsymbol{U}}_{\mathrm{s}} \in \mathbb{C}^{M\times N}$ refers to the signal subspace containing the N-principal eigenvectors of $\hat{\boldsymbol{R}}$ and $\hat{\boldsymbol{\Lambda}}_{\mathrm{s}} \in \mathbb{C}^{N\times N}$ is a diagonal matrix comprising the associated N-largest eigenvalues $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_N\}$. In the same manner $\hat{\boldsymbol{U}}_{\mathrm{n}} \in \mathbb{C}^{M\times(M-N)}$ refers to the noise subspace containing the noise eigenvectors. The corresponding $(M-N)$-noise eigenvalues $\{\hat{\lambda}_{N+1}, \ldots, \hat{\lambda}_M\}$ appear in $\hat{\boldsymbol{\Lambda}}_{\mathrm{n}} \in \mathbb{C}^{(M-N)\times(M-N)}$.

For the sake of simplicity, from now on $\boldsymbol{a}$ will be written instead of $\boldsymbol{a}(\theta)$ and $\boldsymbol{A}$ instead of $\boldsymbol{A}(\boldsymbol{\theta})$.

3. PARTIALLY-RELAXED WEIGHTED SUBSPACE FITTING

Generally, the DOA estimation problem is defined as follows [2]

$$\left\{\hat{\boldsymbol{A}}\right\} = \underset{\boldsymbol{A}\in\mathcal{A}_N}{\arg\min}\, f(\boldsymbol{A}, \boldsymbol{X}), \tag{6}$$

where $f(\cdot)$ denotes the general cost function of the estimation criteria. The N-source array manifold $\mathcal{A}_N$, which is highly structured and non-convex, is specified as

$$\mathcal{A}_N = \{\boldsymbol{A}|\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_N)], \theta_1 < \cdots < \theta_N\}. \tag{7}$$

The cost function $f(\boldsymbol{A}, \boldsymbol{X})$ generally is multimodal, non-convex and exhibits multiple local extrema. To find the global minimizer of the estimation problem in (6) a multi-dimensional grid search is required, which causes high computational cost [10, 17, 18]. For example, the optimization problem for the WSF estimator is formulated in [7, 9, 10] as the following least-squares problem

$$\left\{\hat{\boldsymbol{A}}_{\mathrm{WSF}}, \hat{\boldsymbol{Y}}\right\} = \underset{\boldsymbol{A}\in\mathcal{A}_N,\,\boldsymbol{Y}}{\arg\min}\, \left\|\hat{\boldsymbol{U}}_{\mathrm{s}}\boldsymbol{W}^{1/2} - \boldsymbol{A}\boldsymbol{Y}\right\|_{\mathrm{F}}^2, \tag{8}$$

where $\boldsymbol{Y} \in \mathbb{C}^{N\times N}$ is a nuisance parameter and $\boldsymbol{W}$ a predefined positive definite weighting matrix of dimensions $N \times N$. Concentrating (8) with respect to $\boldsymbol{Y}$ leads to the WSF criterion

$$\left\{\hat{\boldsymbol{A}}_{\mathrm{WSF}}\right\} = \underset{\boldsymbol{A}\in\mathcal{A}_N}{\arg\min}\, \mathrm{tr}\left\{\boldsymbol{P}_{\boldsymbol{A}}^{\perp}\hat{\boldsymbol{U}}_{\mathrm{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right\}, \tag{9}$$

where $\boldsymbol{P}_{\boldsymbol{A}} = \boldsymbol{A}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{H}}$ denotes the projection matrix onto the subspace spanned by the columns of $\boldsymbol{A}$ and $\boldsymbol{P}_{\boldsymbol{A}}^{\perp} = \boldsymbol{I}_M - \boldsymbol{P}_{\boldsymbol{A}}$ denotes the corresponding orthogonal projection matrix. Furthermore, in [10] the weighting matrix is chosen as

$$\boldsymbol{W} = \hat{\tilde{\boldsymbol{\Lambda}}}{}^2\hat{\boldsymbol{\Lambda}}_{\mathrm{s}}^{-1}, \tag{10}$$

where $\hat{\tilde{\boldsymbol{\Lambda}}} = \hat{\boldsymbol{\Lambda}}_{\mathrm{s}} - \hat{\sigma}_{\mathrm{n}}^2\boldsymbol{I}_N$ and $\hat{\sigma}_{\mathrm{n}}^2 = \frac{1}{M-N}\sum_{k=N+1}^{M}\hat{\lambda}_k$ such that the Root-Mean-Squared-Error (RMSE) of the WSF method asymptotically approaches the stochastic Cramér-Rao Bound (CRB).

To relieve the high computational cost of (9) the PR approach was introduced which seeks for a sub-optimal solution for the optimization problem in (9) that still accounts for the existence of multiple sources. Therefore, the structure of the sensor array manifold $\mathcal{A}_N$ in (7) is partially relaxed and instead of searching for the steering matrix in the highly structured sensor array manifold $\mathcal{A}_N$, it is assumed that $\boldsymbol{A} \in \bar{\mathcal{A}}_N$ with partially relaxed sensor array manifold [2]

$$\bar{\mathcal{A}}_N = \left\{\boldsymbol{A}|\boldsymbol{A} = [\boldsymbol{a}, \boldsymbol{B}], \boldsymbol{a} \in \mathcal{A}_1, \boldsymbol{B} \in \mathbb{C}^{M\times(N-1)}\right\}. \tag{11}$$

It can be observed in (11) that $\bar{\mathcal{A}}_N$ preserves the structure of the first column $\boldsymbol{a}(\theta_1)$ of the steering matrix while relaxing the structure of the remaining $N-1$ sources $[\boldsymbol{a}(\theta_2), \ldots, \boldsymbol{a}(\theta_N)]$ to an arbitrary matrix $\boldsymbol{B} \in \mathbb{C}^{M\times(N-1)}$. In order to estimate the DOAs the null-spectrum (concentrated cost function) is computed by minimizing the cost function in (6) with respect to the relaxed signal part $\boldsymbol{B}$ and inserting the minimizer into the cost function. Afterwards, the field-of-view Θ of the sensor is sampled with $L \gg N$ directions $\vartheta \in \{\vartheta_1, \ldots, \vartheta_L\}$ and the null-spectrum is evaluated at all L grid points. The DOA estimates are given by the N directions at which the null-spectrum attains its N deepest local minima.

The null-spectrum of the PR-WSF DOA estimator is obtained by applying the PR approach to the conventional WSF estimator in (9) and yields [2]

$$\begin{aligned} f_{\text{PR-WSF}}(\vartheta) &= \min_{\boldsymbol{B}}\ \mathrm{tr}\left\{\boldsymbol{P}_{[\boldsymbol{a},\boldsymbol{B}]}^{\perp}\hat{\boldsymbol{U}}_{\mathrm{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right\} \\ &= \sum_{k=N}^{M}\lambda_k\left(\boldsymbol{P}_{\boldsymbol{a}(\vartheta)}^{\perp}\hat{\boldsymbol{U}}_{\mathrm{s}}\boldsymbol{W}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right), \end{aligned} \tag{12}$$

where $\vartheta \in \{\vartheta_1, \ldots, \vartheta_L\}$ and $\lambda_k(\cdot)$ denotes the k-th largest eigenvalue of the matrix argument. The PR-WSF DOA estimator shows identical asymptotic error performance as the MUSIC estimator [19] at improved threshold performance [2, 20]. Furthermore in [2, Section V] an efficient implementation of the PR-WSF algorithm is proposed that makes use of an algorithm that allows to iteratively determine the eigenvalues of a rank-one modified Hermitian matrix [21].

4. PARTIALLY RELAXED ORTHOGONAL LEAST SQUARES

In this section a novel DOA estimator is proposed that is based on the previously introduced PR-WSF method and the OLS technique in [5]. Instead of estimating the DOAs in one spectral sweep over the field-of-view Θ of the array as it is the case in conventional PR-WSF the idea is to iteratively estimate the N DOAs in N spectral sweeps thereby estimating one DOA at a time. In comparison to conventional OLS which only accounts for the $(k-1)$-previously estimated sources while estimating the k-th DOA, the idea is to also consider the $N-k$ remaining signals. However, instead of enforcing structure on the $N-k$ remaining signals, the structure is relaxed to some arbitrary matrix $\boldsymbol{B}^{(k)} \in \mathbb{C}^{M\times(N-k)}$. Hence, in the k-th iteration the steering matrix in (8) is separated into the relaxed part $\boldsymbol{B}^{(k)}$ and $\boldsymbol{A}^{(k)} = [\boldsymbol{A}^{(k-1)}, \boldsymbol{a}(\vartheta)]$ where $\boldsymbol{A}^{(k-1)} \in \mathcal{A}_{k-1}$ contains the $(k-1)$-previously estimated DOAs and $\boldsymbol{a}(\vartheta) \in \mathcal{A}_1$. Similarly, the nuisance parameter is separated into the part that belongs to the structured signal part, namely $\boldsymbol{Y}^{(k)} \in \mathbb{C}^{k\times N}$, and $\boldsymbol{E}^{(k)} \in \mathbb{C}^{(N-K)\times N}$ which belongs to the unstructured signal part

$\boldsymbol{B}^{(k)}$. In essence, the optimization in the k-th iteration can be expressed as the following nonlinear least-squares problem

$$p = \underset{\ell=1,\dots,L}{\arg\min} \min_{\boldsymbol{Y}^{(k)},\boldsymbol{B}^{(k)},\boldsymbol{E}^{(k)}} \left\| \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W}^{1/2} - \left[\boldsymbol{A}^{(k-1)}, \boldsymbol{a}(\vartheta_\ell)\right] \boldsymbol{Y}^{(k)} - \boldsymbol{B}^{(k)} \boldsymbol{E}^{(k)} \right\|_{\mathrm{F}}^2. \tag{13}$$

In (13), the relaxed signal part is modeled by the low-rank unstructured matrix $\boldsymbol{K}^{(k)} = \boldsymbol{B}^{(k)}\boldsymbol{E}^{(k)} \in \mathbb{C}^{M\times N}$ with $\mathrm{rank}(\boldsymbol{K}^{(k)}) \leq N-k$. Moreover, the optimization problem in (13) can be seen as a generalized version of the PR-WSF estimator in (12) when the contribution of the previously-determined DOAs are taken into account. In order to concentrate the cost function in (13) we keep $\boldsymbol{K}^{(k)} = \boldsymbol{B}^{(k)}\boldsymbol{E}^{(k)}$ fixed and minimize with respect to $\boldsymbol{Y}^{(k)}$. The closed-form minimizer $\hat{\boldsymbol{Y}}^{(k)}$ is given by

$$\hat{\boldsymbol{Y}}^{(k)} = \left(\bar{\boldsymbol{A}}_\ell^{\mathrm{H}} \bar{\boldsymbol{A}}_\ell\right)^{-1} \bar{\boldsymbol{A}}_\ell^{\mathrm{H}} \left(\hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W}^{1/2} - \boldsymbol{K}^{(k)}\right), \tag{14}$$

where $\bar{\boldsymbol{A}}_\ell = [\boldsymbol{A}^{(k-1)}, \boldsymbol{a}(\vartheta_\ell)]$. Substituting the minimizer $\hat{\boldsymbol{Y}}^{(k)}$ in (14) back into (13) yields

$$p = \underset{\ell=1,\dots,L}{\arg\min} \min_{\mathrm{rank}(\boldsymbol{K}^{(k)})\leq N-k} \left\| \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_\ell} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W}^{1/2} - \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_\ell} \boldsymbol{K}^{(k)} \right\|_{\mathrm{F}}^2. \tag{15}$$

Furthermore, it is assumed that $M > N$ and since $\mathrm{rank}(\boldsymbol{K}^{(k)}) \leq N-k$ and $\mathrm{rank}(\boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_\ell}) = M-k$ it can be observed that

$$\mathrm{rank}\left(\boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_\ell} \boldsymbol{K}^{(k)}\right) \leq N-k.$$

Correspondingly, the inner optimization problem in (15) describes a classical low-rank approximation problem [22, 23] that is solved in closed-form by

$$\min_{\mathrm{rank}(\boldsymbol{K}^{(k)})\leq N-k} \left\| \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_\ell} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W}^{1/2} - \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_\ell} \boldsymbol{K}^{(k)} \right\|_{\mathrm{F}}^2 = \sum_{q=N-k+1}^{N} \sigma_q^2 \left(\boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_\ell} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W}^{1/2}\right)$$

where $\sigma_q(\cdot)$ denotes the q-th largest singular value of the matrix argument. Hence, the concentrated PR-OLS-WSF cost function in the k-th iteration yields

$$f^{(k)}_{\text{PR-OLS-WSF}}(\vartheta) = \sum_{q=N-k+1}^{N} \lambda_q \left(\boldsymbol{P}^{\perp}_{[\boldsymbol{A}^{(k-1)},\boldsymbol{a}(\vartheta)]} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W} \hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right), \tag{16}$$

where $\boldsymbol{W}$ is chosen as in (10). Finally, the DOA of the k-th source is obtained by performing a one-dimensional spectral sweep of $f^{(k)}_{\text{PR-OLS-WSF}}(\vartheta)$ in (16) over the field-of-view of the sensor. The PR-OLS-WSF DOA estimation technique is summarized in Algorithm 1. In the following an efficient implementation of the PR-OLS-WSF method is outlined.

4.1. Computational Aspects

In order to reduce the computational cost of the PR-OLS-WSF technique in Algorithm 1 the dimension of the eigenvalue decomposition in (17) is reduced from a $M \times M$ matrix to a $N \times N$ matrix by rewriting the concentrated cost function in (17) as follows

$$\begin{aligned} &\sum_{q=N-k+1}^{N} \lambda_q \left(\boldsymbol{P}^{\perp}_{[\boldsymbol{A}^{(k-1)},\boldsymbol{a}_\ell]} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W} \hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right) \\ &= \sum_{q=N-k+1}^{N} \lambda_q \left(\boldsymbol{W}^{1/2} \hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}} \boldsymbol{P}^{\perp}_{[\boldsymbol{A}^{(k-1)},\boldsymbol{a}_\ell]} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W}^{1/2}\right) \\ &= \sum_{q=N-k+1}^{N} \lambda_q \left(\boldsymbol{M}^{(k-1)} - \tilde{\boldsymbol{z}}_\ell^{(k-1)} \tilde{\boldsymbol{z}}_\ell^{(k-1)\,\mathrm{H}}\right), \end{aligned} \tag{18}$$

Algorithm 1 PR-OLS-WSF Algorithm

1: **Initialization:** Iteration index $k=0$, initial estimated DOA set $\Omega^{(0)} = \emptyset$ and initial empty steering matrix $\boldsymbol{A}^{(0)} = []$

2: **for** $k = 1, \dots, N$ **do**

3: Find the index p such that

$$p = \underset{\ell=1,\dots,L}{\arg\min} \sum_{q=N-k+1}^{N} \lambda_q \left(\boldsymbol{P}^{\perp}_{\left[\boldsymbol{A}^{(k-1)},\boldsymbol{a}(\vartheta_\ell)\right]} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W} \hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}}\right) \tag{17}$$

4: Update the estimated DOA set and the corresponding estimated steering matrix

$$\Omega^{(k)} = \Omega^{(k-1)} \cup \{\vartheta_p\},$$
$$\boldsymbol{A}^{(k)} = \left[\boldsymbol{A}^{(k-1)}, \boldsymbol{a}(\vartheta_p)\right].$$

5: **end for**

6: **return** The estimated DOA set $\Omega^{(N)}$.

where we have used $\boldsymbol{P}^{\perp}_{[\boldsymbol{A}^{(k-1)},\boldsymbol{a}_\ell]} = \boldsymbol{P}^{\perp}_{\boldsymbol{A}^{(k-1)}} - \boldsymbol{P}_{\boldsymbol{P}^{\perp}_{\boldsymbol{A}^{(k-1)}}\boldsymbol{a}_\ell}$ and denoted $\boldsymbol{a}(\vartheta_\ell)$ by $\boldsymbol{a}_\ell$. Furthermore, it can be seen that the $N \times N$ matrix $\boldsymbol{M}^{(k-1)} = \boldsymbol{W}^{1/2}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}} \boldsymbol{P}^{\perp}_{\boldsymbol{A}^{(k-1)}} \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W}^{1/2}$ in (18) does not depend on the direction ℓ whereas the vector $\tilde{\boldsymbol{z}}_\ell^{(k-1)} = \boldsymbol{W}^{1/2}\hat{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}} \frac{\boldsymbol{P}^{\perp}_{\boldsymbol{A}^{(k-1)}}\boldsymbol{a}_\ell}{\|\boldsymbol{P}^{\perp}_{\boldsymbol{A}^{(k-1)}}\boldsymbol{a}_\ell\|}$ does. Thus, computing the eigenvalue decomposition of $\boldsymbol{M}^{(k-1)} = \boldsymbol{V}^{(k-1)}\boldsymbol{W}^{(k-1)}\boldsymbol{V}^{(k-1)\,\mathrm{H}}$ the cost function in (18) can equivalently be expressed as

$$\begin{aligned} &\sum_{q=N-k+1}^{N} \lambda_q \left(\boldsymbol{V}^{(k-1)}\boldsymbol{W}^{(k-1)}\boldsymbol{V}^{(k-1)\,\mathrm{H}} - \tilde{\boldsymbol{z}}_\ell^{(k-1)} \tilde{\boldsymbol{z}}_\ell^{(k-1)\,\mathrm{H}}\right) \\ &= \sum_{q=N-k+1}^{N} \lambda_q \left(\boldsymbol{W}^{(k-1)} - \boldsymbol{V}^{(k-1)\,\mathrm{H}} \tilde{\boldsymbol{z}}_\ell^{(k-1)} \tilde{\boldsymbol{z}}_\ell^{(k-1)\,\mathrm{H}} \boldsymbol{V}^{(k-1)}\right), \end{aligned} \tag{19}$$

where we have used the unitarity property $\boldsymbol{V}^{(k-1)\,\mathrm{H}}\boldsymbol{V}^{(k-1)} = \boldsymbol{I}_N$. The eigenvalues of the rank-one modified Hermitian matrix in (19) can efficiently be computed using the iterative method in [2, Theorem 1] (also see [21]). Additionally, also the updates for $\boldsymbol{V}^{(k)}$ and $\boldsymbol{W}^{(k)}$ can be efficiently computed by adopting the same iterative procedure. The computationally efficient implementation of the PR-OLS-WSF method is summarized in Algorithm 2. We remark that the proposed PR-OLS-WSF DOA estimation method is a nontrivial extension to the OLS procedure in [5]. In comparison to the convention OLS technique the proposed PR-OLS-WSF method also considers the "interference" from all the source signals which have not been estimated in the current DOA estimation step, however the array manifold structure of these signals is relaxed. Furthermore, the efficient implementation of PR-OLS-WSF in Algorithm 2 is of significantly reduced computational cost compared to the PR-OLS-DML method in [4].

5. SIMULATION RESULTS

In this section the DOA estimation performance of different estimators is compared to the stochastic CRB in [14]. The field-of-view Θ is uniformly sampled with $L = 1800$ directions and the simulations are conducted for $N_R = 1000$ Monte-Carlo runs. The RMSE is used as a measure to evaluate the DOA estimation accuracy and computed as

$$\mathrm{RMSE} = \sqrt{\frac{1}{N_R N} \sum_{r=1}^{N_R} \sum_{n=1}^{N} \left(\hat{\theta}_n^{(r)} - \theta_n\right)^2}$$

where the true DOAs $\boldsymbol{\theta} = [\theta_1, \dots, \theta_N]^{\mathrm{T}}$ and the DOA estimates in the r-th Monte-Carlo run $\hat{\boldsymbol{\theta}}^{(r)} = [\hat{\theta}_1^{(r)}, \cdots, \hat{\theta}_N^{(r)}]^{\mathrm{T}}$ are sorted in

Algorithm 2 Fast Implementation of the PR-OLS-WSF Algorithm

1: **Initialization:** Iteration index $k = 0$, initial diagonalizing matrix $\boldsymbol{V}^{(0)} = \boldsymbol{I}_N$, initial weighting matrix $\boldsymbol{W}^{(0)} = \boldsymbol{W}$ in (10), initial normalized dictionary $\bar{\boldsymbol{A}}^{(0)} = \left[\bar{\boldsymbol{a}}_1^{(0)}, \dots, \bar{\boldsymbol{a}}_L^{(0)}\right]$ (such that each column is unit-norm) and initial estimated DOA set $\Omega^{(0)} = \emptyset$

2: **for** $k = 1, \dots, N$ **do**

3: Find the index p such that

$$p = \underset{\ell=1,\dots,L}{\arg\min} \sum_{q=N-k+1}^{N} \lambda_q \left(\boldsymbol{W}^{(k-1)} - \boldsymbol{z}_\ell \boldsymbol{z}_\ell^{\mathrm{H}}\right)$$

with $\boldsymbol{z}_\ell = \boldsymbol{V}^{(k-1)\,\mathrm{H}} \tilde{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}} \bar{\boldsymbol{a}}_\ell^{(k-1)}$, where $\tilde{\boldsymbol{U}}_{\mathrm{s}} = \hat{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{W}^{1/2}$ using the rank-one modification problem in [2, Theorem 1] (also see [21]).

4: Compute the diagonalizing matrix $\boldsymbol{V}_{\mathrm{mod}}^{(k)}$ and the diagonal matrix $\boldsymbol{W}^{(k)}$ using the rank-one modification problem

$$\begin{aligned}&\boldsymbol{V}_{\mathrm{mod}}^{(k)} \boldsymbol{W}^{(k)} \boldsymbol{V}_{\mathrm{mod}}^{(k)\mathrm{H}} \\ &= \boldsymbol{W}^{(k-1)} - \boldsymbol{V}^{(k-1)\mathrm{H}} \tilde{\boldsymbol{U}}_{\mathrm{s}}^{\mathrm{H}} \bar{\boldsymbol{a}}_p^{(k-1)} \bar{\boldsymbol{a}}_p^{(k-1)\mathrm{H}} \tilde{\boldsymbol{U}}_{\mathrm{s}} \boldsymbol{V}^{(k-1)}\end{aligned}$$

5: Update the estimated DOA set $\boldsymbol{\Omega}^{(k)}$, the orthogonal basis $\boldsymbol{V}^{(k)}$ and the updated dictionary matrix $\bar{\boldsymbol{A}}^{(k)} = \left[\bar{\boldsymbol{a}}_1^{(k)}, \dots, \bar{\boldsymbol{a}}_L^{(k)}\right]$ as follows

$$\begin{aligned}\Omega^{(k)} &= \Omega^{(k-1)} \cup \{\vartheta_p\}, \\ \boldsymbol{V}^{(k)} &= \boldsymbol{V}^{(k-1)} \boldsymbol{V}_{\mathrm{mod}}^{(k)}, \\ \bar{\boldsymbol{A}}^{(k)} &= \bar{\boldsymbol{A}}^{(k-1)} - \frac{1}{\|\bar{\boldsymbol{a}}_p^{(k-1)}\|_2^2} \bar{\boldsymbol{a}}_p^{(k-1)} \bar{\boldsymbol{a}}_p^{(k-1)\mathrm{H}} \bar{\boldsymbol{A}}^{(k-1)}.\end{aligned}$$

6: Normalize each column of the updated dictionary matrix $\bar{\boldsymbol{A}}^{(k)}$ such that the norm of each column is one.

7: **end for**

8: **return** The estimated DOA set $\Omega^{(N)}$.

ascending order. In the simulations a Uniform Linear Array (ULA) with $M = 10$ antennas is used with antenna spacing equal to half of the wavelength. Furthermore, a scenario with $T = 40$ snapshots and $N = 4$ impinging uncorrelated signals from directions $\boldsymbol{\theta} = [0°, 3°, 45°, 50°]^{\mathrm{T}}$ is considered. The transmitted signals are assumed to be zero-mean with unit power and the SNR is calculated as $\mathrm{SNR} = 1/\sigma_{\mathrm{n}}^2$.

In the first simulation in Fig. 1 the RMSE is depicted for different SNR values. It can be observed that the conventional beamformer that is used in each iteration of the conventional OLS procedure is not able to resolve the closely spaced sources and therefore does not approach the CRB regardless of the SNR. The proposed PR-OLS-WSF method provides similar threshold performance as the existing PR-OLS-DML method and both methods approach the CRB in the high SNR regime. In this scenario both PR-OLS methods provide better RMSE performance than root-MUSIC which is only applicable to ULAs while the PR-OLS can be applied to any array geometry. Furthermore the proposed PR-OLS-WSF method provides in comparison to its conventional counterpart PR-WSF significantly improved RMSE performance.

In the second simulation in Fig. 2, the execution time of the DOA estimation algorithms with respect to the number of antennas M are depicted. It can be seen that the proposed fast implementation of the PR-OLS-WSF method in Algorithm 2 shows similar execution time than MUSIC and the fast implementation of PR-WSF in [2]. The improvements in execution time of Algorithm 2 compared to its trivial implementation in Algorithm 1 with Matlab function *eig* is remarkable. Although the PR-OLS-DML method provides slightly

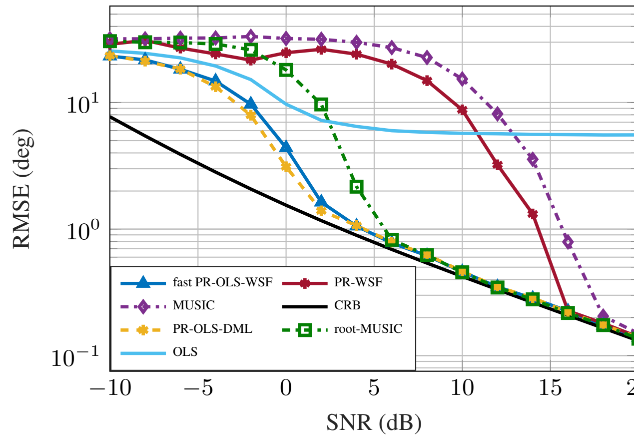

Fig. 1. Uncorrelated source signals, $N = 4$, $T = 40$.

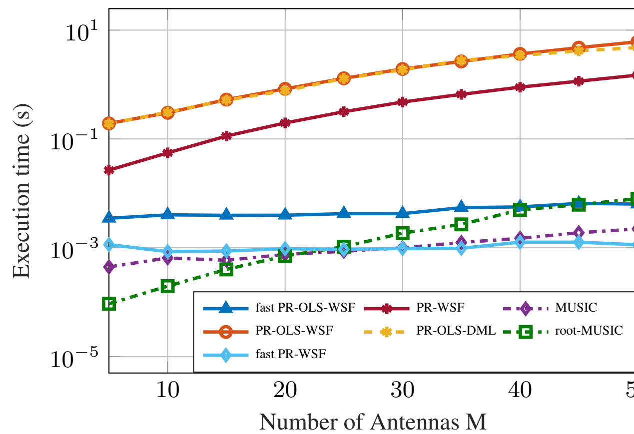

Fig. 2. SNR = 10 dB, $N = 4$, $T = 40$.

better RMSE performance than PR-OLS-WSF, the efficient implementation of PR-OLS-WSF provides significantly reduced execution time in comparison to PR-OLS-DML whose implementation has not been optimized yet. Hence, the PR-OLS-WSF method offers a great compromise between estimation accuracy and computational complexity.

6. CONCLUSION

The proposed PR-OLS-WSF DOA estimator provides a nontrivial extension to the class of PR-OLS estimators. The PR-OLS-WSF method estimates the DOAs in an iterative manner thereby estimating one DOA per iteration while performing a one-dimensional grid search over the field-of-view. In each iteration the PR-OLS-WSF estimator takes into account the previously-determined DOAs as well as the remaining "interfering" sources whose structure is relaxed. Furthermore, an efficient implementation of the PR-OLS-WSF algorithm is proposed. Simulation results show that PR-OLS-WSF exhibits excellent threshold performance at low computational cost that is comparable to MUSIC. Although PR-OLS-DML provides slightly better RMSE performance the proposed PR-OLS-WSF method offers significantly reduced execution time.

In the future we intend to optimize the implementation of the PR-OLS-DML method. Moreover, a combination of the PR-OLS technique with other estimators under the PR framework is of great interest.

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