A NOTE ON TOTALLY SYMMETRIC EQUI-ISOCLINIC TIGHT FUSION FRAMES

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ABSTRACT

Consider the fundamental problem of arranging r-dimensional subspaces of $\mathbb{R}^d$ in such a way that maximizes the minimum distance between unit vectors in different subspaces. It is well known that equi-isoclinic tight fusion frames (EITFFs) are optimal for this packing problem, but such ensembles are notoriously hard to construct. In this paper, we present a novel construction of EITFFs that are *totally symmetric*: any permutation of the subspaces can be realized by an orthogonal transformation of $\mathbb{R}^d$.

Index Terms— Grassmannian codes, frame theory

1. INTRODUCTION

Motivated by design problems in communication [16, 12] and compressed sensing [5, 6], we are interested in arrangements of subspaces that are well spread apart. The *minimum distance* between subspaces $U, V \leq \mathbb{R}^d$ is defined in terms of the unit vectors they contain:

$$\delta(U,V) := \min_{\substack{u\in U\\ \|u\|=1}} \min_{\substack{v\in V\\ \|v\|=1}} \|u-v\|.$$

Given (n,d,r), we seek r-dimensional subspaces $\{W_j\}_{j=1}^n$ of $\mathbb{R}^d$ that maximize the smallest pairwise minimum distance.

This problem has a convenient basis-dependent formulation. For each $j \in \{1,\ldots,n\}$, select an orthonormal basis for W_j, expressed as columns of a matrix $\Phi_j \in \mathbb{R}^{d\times r}$. Then $P_j := \Phi_j\Phi_j^* \in \mathbb{R}^{d\times d}$ gives the orthogonal projection of $\mathbb{R}^d$ onto W_j, and for any $i \in \{1,\ldots,n\}$, the orthogonal projection of W_j into W_i is represented in the chosen bases by $Q_{ij} := \Phi_i^*\Phi_j \in \mathbb{R}^{r\times r}$. Conveniently, it holds that

$$\delta(W_i,W_j) = \sqrt{2-2\|Q_{ij}\|_2},$$

where $\|\cdot\|_2$ denotes spectral norm. Thus, the smallest pairwise minimum distance is captured by the *spectral coherence*

$$\mu_s := \max_{i\neq j} \|Q_{ij}\|_2.$$

In particular, we seek subspaces that minimize μ_s. To this end, it suffices to achieve equality in a lower bound, such as the following generalization of the Welch bound [18, 9]:

$$\mu_s \geq \sqrt{\frac{nr-d}{d(n-1)}}. \tag{1}$$

Equality in (1) occurs precisely when $\{W_j\}_{j=1}^n$ is an *equi-isoclinic tight fusion frame*, which we define below in terms of the matrix representation $\Phi := [\Phi_1 \cdots \Phi_n] \in \mathbb{R}^{d\times nr}$.

Definition 1. We say $\{W_j\}_{j=1}^n$ is a (d,n,r)-*equi-isoclinic tight fusion frame*, abbreviated (d,n,r)-*EITFF*, if

(i) $\{W_j\}_{j=1}^n$ is *equi-isoclinic*, i.e., $Q_{ij}Q_{ij}^* = \mu_s^2 I_r$ for every $i,j \in \{1,\ldots,n\}$ with $i \neq j$; and

(ii) $\{W_j\}_{j=1}^n$ is a *tight fusion frame*, i.e., $\Phi\Phi^* = \frac{nr}{d} I_d$.

By achieving equality in (1), EITFFs are optimal subspace packings in terms of the minimum distance. EITFFs with $r = 1$ are known as *equiangular tight frames (ETFs)*, which have received the most attention in recent literature [11]. Research in the case where $r > 1$ initially focused on the weaker notion of *equichordal tight fusion frames* [4, 2, 10], possibly due to the comparative difficulty in constructing EITFFs. EITFF parameters occur in trivial infinite families, since tensoring a (d,n,r)-EITFF with a $k\times k$ orthogonal matrix produces a (dk,n,rk)-EITFF for every $k \geq 1$. Furthermore, every (d,n,r)-EITFF produces a $(nr-d,d,r)$-EITFF via *Naimark complementation*. Recently, a few constructions were also found for EITFFs with parameters $(8,6,3)$ and $(12,15,3)$ [13], and for $(d,n,2)$-EITFFs with d odd [7, 1, 8]. Little else is known, as summarized in Theorem 5.2 of [10].

In this paper, we consider a very special family of subspace arrangements, namely, those for which any permutation of the subspaces can be realized by applying a member

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of the orthogonal group $\mathrm{O}(d)$. We call such ensembles *totally symmetric*. As we will soon see, many new EITFFs arise by restricting the search to totally symmetric subspaces. In the next section, we discuss some preliminaries before leveraging the representation theory of the symmetric group to construct several new examples of EITFFs.

2. PRELIMINARIES

Given a positive integer n, we write $\lambda \vdash n$ to denote that λ is a *partition* of n, i.e., a monotonically decreasing sequence of nonnegative integers that sum to n. Such sequences necessarily terminate, so we typically express λ by listing the positive terms in the sequence. For example, $\lambda = (5, 4, 1)$ is a partition of 10. It is customary to illustrate a partition of n as a *Young diagram*. The following Young diagram represents the above partition of 10:

(2)

One may fill the cells of a Young diagram with distinct members of $\{1, \ldots, n\}$ to produce a *Young tableau*. A Young tableau is said to be *standard* if the numbers are arranged in increasing order when reading down and to the right, e.g.:

1	2	6	8	9
3	5	7	10	
4				

(3)

The partition $\lambda \vdash n$ that underlies a tableau T is called the *shape* of T, and we let $\mathrm{Tab}(\lambda)$ denote the set of standard tableaux of shape λ. The number $d_\lambda := |\mathrm{Tab}(\lambda)|$ of such tableaux is given by the *hook length formula* [14]. To be explicit, given the Young diagram of a partition $\lambda \vdash n$, we associate to each cell (i, j) the set of cells (a, b) in the diagram with $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The size of this set is the *hook length* $h_\lambda(i, j)$. For example, the following gives the hook lengths of each cell in the diagram in (2):

7	5	4	3	1
5	3	2	1	
1				

With this notation, we may state the hook length formula:

$$d_\lambda = \frac{n!}{\prod h_\lambda(i, j)},$$

where the product is over the cells (i, j) in the Young diagram of λ. For example, $d_{(5,4,1)} = 228$.

We are interested in the representation theory of the symmetric group S_n. Each conjugacy class of S_n consists of the permutations with a fixed multiset of cycle lengths, which corresponds to a partition of n. It follows that the irreducible representations of S_n are in one-to-one correspondence with the partitions of n, and in fact, each partition determines an irreducible representation by virtue of *Young's orthogonal form* [17]. We denote this representation by $\rho_\lambda \colon S_n \to \mathrm{O}(S^\lambda)$. The representation space S^λ has an orthonormal basis that we denote by $\{w_T^\lambda\}_{T \in \mathrm{Tab}(\lambda)}$, i.e., S^λ has dimension d_λ.

3. MAIN RESULT

We start by introducing some nonstandard notation. Given $\lambda \vdash n$ and $T \in \mathrm{Tab}(\lambda)$, we write $c(T) := j - i$, where (i, j) indexes the cell containing n in T. For example, if T is the tableau in (3), then since $n = 10$ appears in the cell with indices $(2, 4)$, we have $c(T) = 2$. Next,

$$C(\lambda) := \{c(T) : T \in \mathrm{Tab}(\lambda)\}.$$

For example, consider the Young diagram λ in (2). Then for every $T \in \mathrm{Tab}(\lambda)$, $n = 10$ must appear in a cell with indices in $\{(1, 5), (2, 4), (3, 1)\}$, and so $C(\lambda) = \{4, 2, -2\}$.

Given $\lambda \vdash n - 1$, we denote

$$\lambda + 1 := \{\mu \vdash n : \mu(i) \geq \lambda(i)\ \forall i\}.$$

In words, $\lambda + 1$ is the set of Young diagrams obtained by adding a single cell to λ. For example, $(5, 4, 1) + 1 = \{(6, 4, 1), (5, 5, 1), (5, 4, 2), (5, 4, 1, 1)\}$. Every $\mu \in \lambda + 1$ determines an embedding $\iota_{\lambda,\mu} \colon \mathrm{Tab}(\lambda) \to \mathrm{Tab}(\mu)$ that completes the input tableau by putting n in the new cell. We write $c(\lambda, \mu) := j - i$, where (i, j) indexes this new cell.

Given $\lambda \vdash n - 1$ and $\mu \in \lambda + 1$, the embedding $\iota_{\lambda,\mu}$ induces a linear embedding $\tilde{\iota}_{\lambda,\mu} \colon S^\lambda \to S^\mu$, defined by

$$w_T^\lambda \mapsto w^\mu_{\iota_{\lambda,\mu}(T)}$$

and extending linearly. Note that ρ_μ determines an action of S_n on the Grassmannian $\mathrm{Gr}(S^\mu, d_\lambda)$, and one may show that the stabilizer of $\tilde{\iota}_{\lambda,\mu}(S^\lambda)$ is S_{n-1}, so its orbit has size n. By construction, these n subspaces are totally symmetric, and furthermore, they form a tight fusion frame. Our main result establishes when a careful combination of such configurations yields a totally symmetric *equi-isoclinic* tight fusion frame:

Theorem 2. *Select $\lambda \vdash n - 1$ and $\mu_1, \ldots, \mu_\ell \in \lambda + 1$ for which there exists $\alpha \geq 0$ such that*

$$\left| \sum_{t=1}^{\ell} \frac{d_{\mu_t}}{n d_\lambda} \cdot \frac{1}{c(\lambda, \mu_t) - k} \right| = \alpha \qquad \forall k \in C(\lambda).$$

Then there exists a totally symmetric equi-isoclinic tight fusion frame of n subspaces of $\mathbb{R}^d$ of dimension r, where

$$d := \sum_{t=1}^{\ell} d_{\mu_t}, \qquad r := d_\lambda.$$

Table 1. EITFF parameters with $\ell = 1$ and $n \leq 10$

λ	μ	d	n	r
$(2,2)$	$(3,2)$	5	5	2
$(3,1,1)$	$(3,2,1)$	16	6	6
$(3,3)$	$(4,3)$	14	7	5
$(4,1,1,1)$	$(4,2,1,1)$	90	8	20
$(4,4)$	$(5,4)$	42	9	14
$(4,2,2)$	$(4,3,2)$	168	9	56
$(3,3,3)$	$(4,3,3)$	210	10	42
$(5,1,1,1,1)$	$(5,2,1,1,1)$	448	10	70

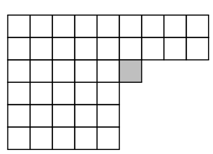
The proof of this result will appear in a forthcoming journal article. While the hypothesis in Theorem 2 is a complicated statement about $\lambda \vdash n-1$ and $\mu_1, \ldots, \mu_\ell \in \lambda + 1$, it is less complicated when the number ℓ of *layers* is small. In what follows, we illustrate our main result with a few examples that take $\ell \leq 3$.

3.1. Examples with one layer

Take $\ell = 1$ and denote $\mu := \mu_1$. Then the hypothesis in Theorem 2 reduces to

$$|c(\lambda, \mu) - k| = \text{const} \qquad \forall k \in C(\lambda),$$

which in turn implies $|C(\lambda)| \leq 2$. In fact, the condition trivially holds when $|C(\lambda)| = 1$, i.e., when $\lambda = (a, a, \ldots, a)$ for some a. Otherwise, if $|C(\lambda)| = 2$, then $c(\lambda, \mu)$ is the average of $C(\lambda)$. For example, the following λ and $\mu \in \lambda + 1$ satisfy this condition (μ's extra cell is shaded in gray):



Indeed, $C(\lambda) = \{7, -1\}$ and $c(\lambda, \mu) = 3$.

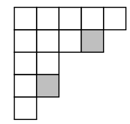
See Table 1 for EITFF parameters with $n \leq 10$ that arise in this way. Note that this table omits certain redundancies and trivialities. For example, transposing the diagrams of λ and μ gives an EITFF with the same parameters. Also, if $\mu = (n)$, then $d = r = 1$; if $\mu = (a, a, \ldots, a)$ for some a, then $d = r$; and if $\mu = (n-1, 1)$, then $r = 1$ and $d = n-1$. Parameters that appear in gray rows are new to the literature.

3.2. Examples with two layers

Next, we report explicit examples of our construction with $\ell = 2$. First, consider

$$\begin{aligned} \lambda &:= (5,3,2,1,1), & d_\lambda &= 7700, \\ \mu_1 &:= (5,4,2,1,1), & d_{\mu_1} &= 21450, \\ \mu_2 &:= (5,3,2,2,1), & d_{\mu_2} &= 21450. \end{aligned}$$

The following diagram illustrates this example, where the extra cells from μ_1 and μ_2 are shaded in gray:



In this case, $n = 13$, $C(\lambda) = \{4, 1, -1, -4\}$, $c(\lambda, \mu_1) = 2$, and $c(\lambda, \mu_2) = -2$. Thanks to symmetry, we have $d_{\mu_1} = d_{\mu_2}$, and so the hypothesis in Theorem 2 reduces to

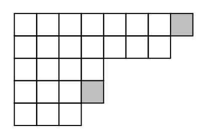
$$\left| \frac{1}{2-k} + \frac{1}{-2-k} \right| = \text{const} \qquad \forall k \in C(\lambda).$$

Indeed, this condition holds with $\text{const} = \frac{2}{3}$. As such, there exists a totally symmetric EITFF of 13 subspaces of $\mathbb{R}^{42900}$ of dimension 7700.

Next, we consider an example with asymmetric λ:

$$\begin{aligned} \lambda &:= (7,7,4,3,3), & d_\lambda &= 116032672, \\ \mu_1 &:= (8,7,4,3,3), & d_{\mu_1} &= 74959204320, \\ \mu_2 &:= (7,7,4,4,3), & d_{\mu_2} &= 41644002400. \end{aligned}$$

Here is an illustration:



In this case, $n = 25$, $C(\lambda) = \{5, 1, -2\}$, $c(\lambda, \mu_1) = 7$, and $c(\lambda, \mu_2) = 0$, and so

$$\sum_{t=1}^{\ell} \frac{d_{\mu_t}}{n d_\lambda} \cdot \frac{1}{c(\lambda, \mu_t) - k} = \frac{9}{35} \cdot \frac{1}{7-k} + \frac{1}{7} \cdot \frac{1}{-k},$$

which equals $\pm \frac{1}{10}$ for each $k \in \{5, 1, -2\}$. Overall, this example determines a totally symmetric EITFF consisting of 25 subspaces of a high-dimensional space.

3.3. Examples with three layers

Finally, we consider examples with $\ell = 3$ that take the form illustrated in Figure 1. Here, λ is a partition of

$$n - 1 = \sum_{\substack{i<j \\ i \text{ odd} \\ j \text{ even}}} a_i a_j$$

that is determined by $\{a_i\}_{i=1}^{12}$, while μ_1, μ_2, and μ_3 are three particular members of $\lambda + 1$. We will select $\{a_i\}_{i=1}^{12}$ in such a way that $\lambda, \mu_1, \mu_2, \mu_3$ together satisfy the hypothesis in Theorem 2. To this end, it is convenient to denote

$$s_{i,j} := a_i + \cdots + a_j$$

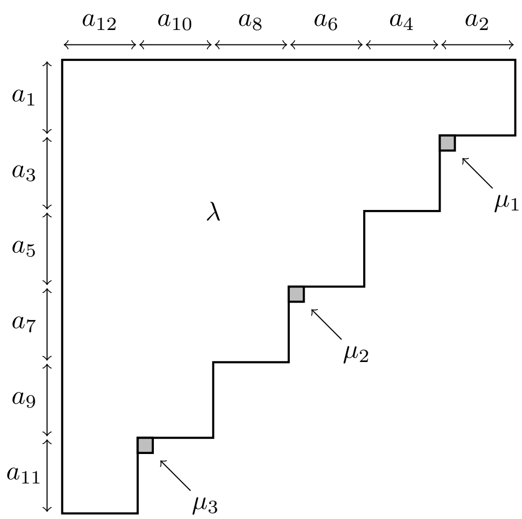


Fig. 1. Young diagram of λ with $\mu_1, \mu_2, \mu_3 \in \lambda + 1$.

for $i \leq j$. For example, $s_{1,2} = a_1 + a_2$ and $s_{1,1} = a_1$. In this notation, one may simplify telescoping products to obtain

$$\frac{d_{\mu_1}}{nd_\lambda} = \frac{s_{2,2}}{s_{1,2}} \cdot \frac{s_{3,3}}{s_{3,4}} \cdot \frac{s_{3,5}}{s_{3,6}} \cdot \frac{s_{3,7}}{s_{3,8}} \cdot \frac{s_{3,9}}{s_{3,10}} \cdot \frac{s_{3,11}}{s_{3,12}},$$
$$\frac{d_{\mu_2}}{nd_\lambda} = \frac{s_{2,6}}{s_{1,6}} \cdot \frac{s_{4,6}}{s_{3,6}} \cdot \frac{s_{6,6}}{s_{5,6}} \cdot \frac{s_{7,7}}{s_{7,8}} \cdot \frac{s_{7,9}}{s_{7,10}} \cdot \frac{s_{7,11}}{s_{7,12}},$$
$$\frac{d_{\mu_3}}{nd_\lambda} = \frac{s_{2,10}}{s_{1,10}} \cdot \frac{s_{4,10}}{s_{3,10}} \cdot \frac{s_{6,10}}{s_{5,10}} \cdot \frac{s_{8,10}}{s_{7,10}} \cdot \frac{s_{10,10}}{s_{9,10}} \cdot \frac{s_{11,11}}{s_{11,12}}.$$

For simplicity, we impose symmetry on λ by forcing

$$a_i = a_{13-i} \qquad \forall i \in \{1, \ldots, 12\}.$$

This gives $c(\lambda, \mu_1) = s_{3,6}$, $c(\lambda, \mu_2) = 0$, $c(\lambda, \mu_3) = -s_{3,6}$, and $C(\lambda) = \{\pm s_{2,6}, \pm s_{4,6}, \pm s_{6,6}\}$.

Before proceeding, let's reformulate the hypothesis in Theorem 2. If we fix $k_0 \in C(\lambda)$, then there exists a sign pattern $\epsilon\colon C(\lambda) \to \{\pm 1\}$ such that

$$\sum_{t=1}^{\ell} \frac{d_{\mu_t}}{nd_\lambda} \cdot \frac{1}{c(\lambda, \mu_t) - k} = \epsilon(k) \cdot \sum_{t=1}^{\ell} \frac{d_{\mu_t}}{nd_\lambda} \cdot \frac{1}{c(\lambda, \mu_t) - k_0}$$

for every $k \in C(\lambda) \setminus \{k_0\}$. In our setting, each ϵ induces a system of 5 polynomial equations in $\{a_i\}_{i=1}^6$ (after clearing denominators and rearranging). Denoting these polynomials by $\{p_j^\epsilon\}_{j=1}^5$, our problem is then equivalent to finding ϵ for which there exists $\{a_i\}_{i=1}^6$ satisfying the constraints

$$p_j^\epsilon(a_1, \ldots, a_6) = 0 \qquad \forall j \in \{1, \ldots, 5\}, \tag{4}$$
$$a_i > 0 \qquad \forall i \in \{1, \ldots, 6\}, \tag{5}$$
$$a_i \in \mathbb{Z} \qquad \forall i \in \{1, \ldots, 6\}. \tag{6}$$

Notice that $\{p_j^\epsilon\}_{j=1}^5$ are homogeneous in our setting, and so we may relax constraint (6) to $a_i \in \mathbb{Q}$ and clear denominators.

Fix $k_0 := \max C(\lambda) = s_{2,6}$ and take ϵ to be alternating, i.e., $\epsilon(s_{2,6}) := 1$, $\epsilon(s_{4,6}) = -1$, etc. By a serendipitous application of cylindrical algebraic decomposition [3] in Mathematica, it suffices to take $t \in (\frac{\sqrt{241}-7}{16}, \frac{\sqrt{5}-1}{2})$ and

$$a_1 = -3 + \sqrt{\tfrac{9-18t-27t^2}{6-7t-8t^2}}, \qquad a_2 = 1, \qquad a_3 = 1 - t,$$
$$a_4 = t, \qquad a_5 = 1 + \tfrac{1-3t}{2-t}, \qquad a_6 = \tfrac{-1+3t}{2-t},$$

such that $t, a_1 \in \mathbb{Q}$. To achieve rationality, one may apply a birational transformation to an elliptic curve (à la Exercise 1.15 in [15]) and then apply group law–based algorithms available in Sage. For example, one may take $t = 87/163$. After clearing denominators in $\{a_i\}_{i=1}^6$, this delivers a totally symmetric EITFF of 7347412056953 subspaces in an extraordinarily high-dimensional space.

4. DISCUSSION

In this paper, we leveraged the representation theory of the symmetric group to obtain many examples of totally symmetric EITFFs. Apparently, the EITFF parameters grow very quickly with the number ℓ of layers in our construction. Some open questions remain. For each $\ell \in \mathbb{N}$, do there exist totally symmetric EITFFs with ℓ layers? Can one obtain explicit (block) restricted isometry constants for an infinite family of totally symmetric EITFFs? Can one leverage the representation theory of other groups to obtain additional EITFFs?

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