COARRAY MANIFOLD SEPARATION IN THE SPHERICAL HARMONICS DOMAIN FOR ENHANCED SOURCE LOCALIZATION

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ABSTRACT

The order of a three-dimensional wavefield captured by a spherical array is limited by the number of sampling points i.e. the number of sensors in the array. This restricts the source localization performance of existing techniques for a spherical array. In this paper, we introduce the concept of difference coarray to spherical arrays and propose an algorithm which utilises the increased degrees-of-freedom (DOF) provided by the virtual coarray sensors to perform enhanced source localization. We make use of coarray manifold separation in the spherical harmonics domain to generate a Vandermonde structured coarray manifold matrix which allows us to propose a novel subspace-based algorithm, which we call the coarraySH-MUSIC. We also introduce a polynomial rooting version of our algorithm which does not rely on extensive grid searches. The proposed algorithms are evaluated using various simulated experiments on source localization.

Index Terms— Spherical Array, Difference Coarray, Source Localization, Spherical Harmonics, Spatial Smoothing, MUSIC

1. INTRODUCTION

The problem of estimating the direction-of-arrival (DOA) of signals is considered to be of great importance in multi-channel signal processing. Recently, using the concept of difference coarray, several algorithms have been developed to perform enhanced DOA estimation [1–6]. However, these works are limited to linear and planar arrays. Spherical arrays have the ability to estimate the azimuth and elevation angles with the same resolution and without any spatial ambiguity. Also, array processing in the spherical harmonics (SH) domain offer a powerful mathematical tool and has, recently, seen application in many areas [7–11]. Conventional subspace-based DOA estimation techniques like MUSIC [12] has also been extended into the spherical harmonics domain, called SH-MUSIC [13, 14]. In [15], a grid-search free DOA estimation algorithm, called SH-root-MUSIC, was proposed which utilized the Vandermonde structure of the spherical manifold in the SH domain for azimuth estimation. In this paper, we introduce the idea of difference coarray to spherical arrays and express the elements of the array covariance matrix as the signal received by the virtual sensors of the coarray. We explore the structure and geometry of the difference coarray of spherical arrays with I sensors and show that the coarray can provide an increased degree-of-freedom (DOF) of $\mathcal{O}(I^2)$ which enables enhanced DOA estimation. Then, we extend the manifold separation (MS) [16] technique to the coarray to expresses the coarray steering matrix in terms of a Vandermonde structured matrix. As the signal model of a coarray is a single snapshot model, the Vandermonde structure enables us to perform a spatial smoothing type operation to restore the rank of the coarray covariance matrix. This allows us to propose a novel subspace-based algorithm, which we call the coarraySH-MUSIC. We have also introduced the polynomial rooting version of our algorithm called the coarraySH-rootMUSIC. Finally, we have conducted extensive numerical simulations to verify the effectiveness and usefulness of the proposed methods.

Notations: $(.)^T$, $(.)^*$ and $(.)^H$ denotes the transpose, the conjugate and the conjugate transpose operator, respectively. $\mathbb{E}[.]$, $\odot$, $\otimes$ and $\|.\|_F$ represents the expectation operator, the Khatri-Rao product, the Kronecker product and the Frobenius norm, respectively. $\mathbf{I}_N$ denotes an identity matrix of size N, $(.)'$ represents the first derivative.

2. SPHERICAL ARRAY SIGNAL MODEL

Let us consider a spherical array with I sensors located on the surface of a sphere of radius r at elevation and azimuth angles of (θ_i, ϕ_i), $\{i = 1, 2, \ldots, I\}$. The elevation angle is measured down from the z-axis and the azimuth angle is measured counter-clockwise from the x-axis. The sphere is centered at the origin. The position vectors of the sensors are given by $\mathbf{r}_i = [r\sin\theta_i\cos\phi_i, r\sin\theta_i\sin\phi_i, r\cos\theta_i]^T$. Consider that a wavefield of L uncorrelated plane-waves impinge on the array with wavenumber k. The direction-of-arrival (DOA) of the l^{th} source is given by (ϑ_l, φ_l), $\{l = 1, 2, \ldots, L\}$. Now, the phase difference between the complex envelope of the l^{th} signal received at the origin and at sensor i is given by $\psi_{l,i} = e^{-j\mathbf{k}_l^T\mathbf{r}_i} = e^{-jkr(\sin\vartheta_l\sin\theta_i\cos(\varphi_l-\phi_i)+\cos\vartheta_l\cos\theta_i)}$ where $\mathbf{k}_l = -k[\sin\vartheta_l\cos\varphi_l, r\sin\vartheta_l\sin\varphi_l, r\cos\vartheta_l]^T$ is the wavevector corresponding to the l^{th} source pointing towards its DOA which is in the opposite direction of the wave propagation.

In the spatial domain, the signal received by the I sensors of the spherical array can now be written as

$$\mathbf{p}(t) = \sum\nolimits_{l=1}^{L} s_l(t)\mathbf{v}(\vartheta_l, \varphi_l) + \mathbf{n}(t) = \mathbf{V}\mathbf{s}(t) + \mathbf{n}(t), \quad (1)$$

where $\mathbf{s}(t)$ is an $L \times 1$ vector of signal amplitudes and the array steering vector for source l, $\mathbf{v}(\vartheta_l, \varphi_l)$ of size $I \times 1$, is given as

$$\mathbf{v}(\vartheta_l, \varphi_l) = \left[e^{-j\mathbf{k}_l^T\mathbf{r}_1}, e^{-j\mathbf{k}_l^T\mathbf{r}_2}, \ldots, e^{-j\mathbf{k}_l^T\mathbf{r}_I}\right]^T. \quad (2)$$

In (1), $\mathbf{V} = [\mathbf{v}(\vartheta_1, \varphi_1), \mathbf{v}(\vartheta_2, \varphi_2), \ldots, \mathbf{v}(\vartheta_L, \varphi_L)] \in \mathbb{C}^{I\times L}$ is the array steering matrix and $\mathbf{n}(t) \in \mathbb{C}^{I\times 1}$ is a zero mean additive gaussian sensor noise (assumed to have a uniform power of σ_n^2 at each sensor) which is uncorrelated with any of the sources. Now, the covariance matrix of the received signal can be written as

$$\mathbf{R} = \mathbb{E}[\mathbf{p}\mathbf{p}^H] = \sum\nolimits_{l=1}^{L} \rho_l\mathbf{v}(\vartheta_l, \varphi_l)\mathbf{v}^H(\vartheta_l, \varphi_l) + \sigma_n^2\mathbf{I}_I, \quad (3)$$

$$= \mathbf{V}\mathbf{R}_\mathbf{s}\mathbf{V}^H + \mathbf{R}_\mathbf{n}, \quad (4)$$

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where $\mathbf{R_s} = \text{diag}(\rho_1, \rho_2, \ldots, \rho_L)$ and $\mathbf{R_n}$ are the signal and noise covariance matrices, respectively. From (2), the term $\mathbf{v}(\vartheta_l, \varphi_l)\mathbf{v}^H$ in the expression of $\mathbf{R}$ contains the pairwise differences of the sensor position vectors $(\mathbf{r}_u - \mathbf{r}_v)\,\forall\, u, v = 1, 2, \ldots, I$. This can also be seen from the $(u, v)^{th}$ element of $\mathbf{R}$ which according to (3) is given by

$$[\mathbf{R}]_{uv} = \sum\nolimits_{l=1}^{L} \rho_l e^{-j\mathbf{k}_l^T(\mathbf{r}_u - \mathbf{r}_v)} + \sigma_n^2 \delta(u - v), \tag{5}$$

where $\delta(u - v)$ is the Kronecker Delta function which is equal to 1 if $u = v$ and 0 otherwise. So, the second-order statistics of the signal captured by the array depends on the pairwise differences of the sensor locations. This leads us to define the difference coarray of the spherical array as a virtual array whose sensor position vectors are given by the set $\mathbf{D} = \{(\mathbf{r}_u - \mathbf{r}_v),\ \forall\, u, v = 1, 2, \ldots, I\}$. There are I zero vectors in $\mathbf{D}$ corresponding to the self differences (when $u = v$). So, the maximum cardinality of $\mathbf{D}$ is $|\mathbf{D}|_{\text{max}} = I(I - 1) + 1$. There may be other redundancies in $\mathbf{D}$ as well due to the possibility of there being the same difference between two distinct pairs of sensors. However, the thing to note is that the number of virtual coarray sensors (i.e. the cadinality of $\mathbf{D}$ also known as the DOF of the coarray) is always of the order of $\mathcal{O}(I^2)$. So, instead of applying DOA estimation methods to the actual array, we can apply them to the coarray and utilize its increased DOF. Fig. 1 shows the difference coarray of two spherical array configuration.

3. COARRAY SPHERICAL SIGNAL MODEL

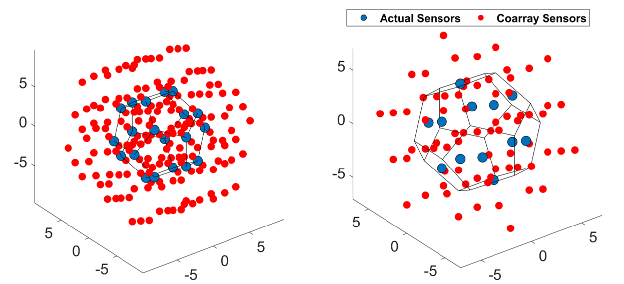
Before proposing DOA estimation techniques which uses the difference coarray, let us first establish the coarray signal model. Since, the elements of $\mathbf{R}$ depend on the coarray, vectorizing it will give us the coarray signal model as

$$\begin{aligned}\mathbf{z} = \text{vec}(\mathbf{R}) &= \text{vec}\left(\sum\nolimits_{l=1}^{L} \rho_l \mathbf{v}(\vartheta_l, \varphi_l)\mathbf{v}^H(\vartheta_l, \varphi_l) + \sigma_n^2 \mathbf{I}_I\right)\\ &= (\mathbf{V}^* \odot \mathbf{V})\boldsymbol{\rho} + \sigma_n^2 \text{vec}(\mathbf{I}_N)\\ &= \bar{\mathbf{V}}\boldsymbol{\rho} + \sigma_n^2 \mathbf{i}.\end{aligned} \tag{6}$$

where $\boldsymbol{\rho} = [\rho_1, \rho_2, \ldots, \rho_L]$ is the vector of signal powers and $\mathbf{i} = \text{vec}(\mathbf{I}_I)$. The matrix $\bar{\mathbf{V}} = [\bar{\mathbf{v}}(\vartheta_1, \varphi_1), \bar{\mathbf{v}}(\vartheta_2, \varphi_2), \ldots, \bar{\mathbf{v}}(\vartheta_L, \varphi_L)]$ represents the coarray steering matrix with the steering vector corresponding to the l^{th} source given by

$$\bar{\mathbf{v}}(\vartheta_l, \varphi_l) = \mathbf{v}^*(\vartheta_l, \varphi_l) \otimes \mathbf{v}(\vartheta_l, \varphi_l). \tag{7}$$

From (7), we can see that the size of $\bar{\mathbf{v}}(\vartheta_l, \varphi_l)$ (after removing the redundant differences) is equal to the DOF of the coarray and the actual array sensor positions have been replaced by the virtual coarray sensors. Compared to (1), $\mathbf{z}$ in (6) can be seen as the signal received by the difference coarray with the equivalent signal given by $\boldsymbol{\rho}$ and the noise by $\sigma_n^2\mathbf{i}$. However, the drawback of the coarray signal model in (6) is that it is a single snapshot model. This means that the coarray covariance matrix $[\mathbf{z}\mathbf{z}^H]$ is of rank 1. Conventional subspace based DOA estimation methods like MUSIC rely on the eigenvalue decomposition (EVD) of the covariance matrix to extract the noise subspace. So, these techniques cannot be directly applied to the coarray model. To alleviate this issue, in the next section, utilizing spherical harmonics, we propose a spatial smoothing type operation to recover the rank of the coarray covariance matrix.



(a) Coarray DOF (|**D**|) = 195 (b) Coarray DOF (|**D**|) = 81

Fig. 1. Dodecahedron Spherical Array (radius = 5.12 cm and centered at the origin) with sensors (blue dots) on its a) 20 vertices and b) 12 faces and its difference coarray virtual sensors (red dots).

4. PROPOSED METHOD

4.1. Coarray Spherical Harmonics

Suppose there are a total of D coarray sensors i.e. $|\mathbf{D}| = D$. Then, $\bar{\mathbf{v}}(\vartheta_l, \varphi_l)$ is of size $D \times 1$ and its d^{th} element is given by $e^{-j\mathbf{k}_l^T\mathbf{r}_d}$ ($\mathbf{r}_d \in \mathbf{D}$). This may alternatively be written in the spherical harmonics domain as [17, 18]

$$e^{-j\mathbf{k}_l^T\mathbf{r}_d} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_n(kr)[Y_n^m(\vartheta_l, \varphi_l)]^* Y_n^m(\theta_d, \phi_d), \tag{8}$$

where Y_n^m is the spherical harmonic (SH) of order n and degree m and $b_n(kr)$ is the far-field mode strength and is given as

$$\begin{aligned} b_n(kr) &= 4\pi j^n j_n(kr), && \text{for open sphere} \quad (9)\\ &= 4\pi j^n\Big(j_n(kr) - \frac{j_n'(kr)}{h_n'(kr)}\Big), && \text{for rigid sphere} \quad (10)\end{aligned}$$

where $j_n(kr)$ and $h_n(kr)$ are the n^{th} order spherical Bessel function of the first kind and spherical Hankel function of the second kind, respectively. SH can be used to decompose square integrable functions defined on a unit sphere just as complex exponentials $e^{j\omega t}$ are used to decompose real periodic functions and is given as [17]

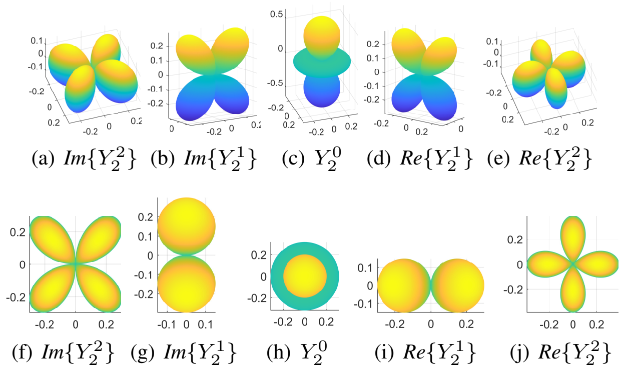
$$\begin{aligned} Y_n^m(\theta, \phi) &= \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(cos\theta)e^{jm\phi}, \quad (11)\\ &\qquad \forall\, 0 \le n \le \infty,\ 0 \le m \le n\\ &= (-1)^{|m|}Y_n^{|m|*}(\theta, \phi),\ \forall - n \le m < 0.\end{aligned}$$

Here, P_n^m is the associated Legendre function. Now, the spherical Bessel function $j_n(kr)$ decays exponentially when the order n exceeds its argument kr. Y_n^m do not increase exponentially as a function of n, so the first summation in (8) can accordingly be truncated to a finite value of $n = N$, which is called the array order. The number of modes that can be excited by an array of order N is thus $(N+1)^2$ and so, this is the minimum number of sensors required to capture the wavefield. Since, the coarray has more sensors, it captures higher orders of wavefield than the actual array which will be exploited for enhanced DOA estimation.

4.2. Coarray Manifold Separation and Spatial Smoothing

Substituting (8) in (7) and (6), the coarray steering matrix becomes

$$\bar{\mathbf{V}} = \mathbf{Y}(\Phi)\mathbf{B}(kr)\mathbf{Y}^H(\Psi), \tag{12}$$



(a) $Im\{Y_2^2\}$ (b) $Im\{Y_2^1\}$ (c) Y_2^0 (d) $Re\{Y_2^1\}$ (e) $Re\{Y_2^2\}$

(f) $Im\{Y_2^2\}$ (g) $Im\{Y_2^1\}$ (h) Y_2^0 (i) $Re\{Y_2^1\}$ (j) $Re\{Y_2^2\}$

Fig. 2. Plot of spherical harmonics for $n = 2$ with $Y_n^0(\theta, \phi)$, which is a real function, presented in the central column. *Im*{} and *Re*{} refers to imaginary and real parts of the spherical harmonics. Bottom row gives the top view.

where $\Phi = (\theta, \phi)$ and $\Psi = (\vartheta, \varphi)$. $Y(\Phi)$ is a $D \times (N+1)^2$ matrix whose d^{th} row is given by ($Y(\Psi)$ is defined in a similar manner)

$$\mathbf{y}(\Phi_d) = [Y_0^0(\Phi_d), Y_1^{-1}(\Phi_d), Y_1^0(\Phi_d), Y_1^1(\Phi_d), \dots, Y_N^N(\Phi_d)]$$

and $\mathbf{B}(kr) = \text{diag}(b_0(kr),\ b_1(kr),\ b_1(kr),\ b_1(kr), \dots, b_N(kr))$ is the $(N+1)^2 \times (N+1)^2$ matrix containing the information about the mode strengths. Now, assuming that the discrete sensors sampling the wavefield avoid spatial aliasing [19], the following orthogonality condition holds [20]

$$\mathbf{Y}^H(\Phi)\mathbf{\Gamma}\mathbf{Y}(\Phi) = \mathbf{I}_{(N+1)^2}, \tag{13}$$

where $\mathbf{\Gamma} = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_D)$ is a diagonal matrix composed of the the weights of the sampling scheme [20]. To perform a spatial smoothing type operation on the coarray signal model, we first express the coarray steering vector as having a Vandermonde structure. With that intent, multiplying both sides of (12) with $\mathbf{Y}^H(\Phi)\mathbf{\Gamma}$ and then by $\mathbf{B}^{-1}(kr)$, we get

$$\begin{aligned}\mathbf{B}^{-1}(kr)\mathbf{Y}^H(\Phi)\mathbf{\Gamma}\bar{\mathbf{V}} &= \mathbf{B}^{-1}(kr)\mathbf{Y}^H(\Phi)\mathbf{\Gamma}\mathbf{Y}(\Phi)\mathbf{B}(kr)\mathbf{Y}^H(\Psi)\\ &= \mathbf{B}^{-1}(kr)\mathbf{B}(kr)\mathbf{Y}^H(\Psi)\\ &= \mathbf{Y}^H(\Psi).\end{aligned} \tag{14}$$

Note that $\mathbf{Y}(\Phi)$ and $\mathbf{B}(kr)$ are constant for a given array configuration. Now, for a fixed elevation angle of ϑ_0, a column in $\mathbf{Y}^H(\Psi)$ can be written as

$$\begin{aligned}\mathbf{y}^H(\Psi) &= \mathbf{y}^H(\vartheta_0, \varphi)\\ &= [f_{00}, -f_{1(-1)}e^{j\varphi}, f_{10}, f_{11}e^{-j\varphi}, \dots, f_{NN}e^{-jN\varphi}]^T,\end{aligned}$$

where, $f_{nm} = \sqrt{\dfrac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}}P_n^{|m|}(\cos\vartheta_0)$.

If $\mathbf{F}(\vartheta_0) = \text{diag}(f_{00}, -f_{1(-1)}, f_{10}, f_{11}, \dots, f_{NN})$ and $\mathbf{h}(\varphi) = [1, e^{j\varphi}, 1, e^{-j\varphi}, \dots, e^{-jN\varphi}]^T$, then

$$\mathbf{y}^H(\vartheta_0, \varphi) = \mathbf{F}(\vartheta_0)\mathbf{h}(\varphi). \tag{15}$$

(15) is referred to as coarray manifold separation. Substituting (15) in (14), for one column, we get

$$\mathbf{B}^{-1}(kr)\mathbf{Y}^H(\Phi)\mathbf{\Gamma}\bar{\mathbf{v}} = \mathbf{y}^H(\vartheta_0, \varphi) = \mathbf{F}(\vartheta_0)\mathbf{h}(\varphi), \tag{16}$$

which implies $\mathbf{F}^{-1}(\vartheta_0)\mathbf{B}^{-1}(kr)\mathbf{Y}^H(\Phi)\mathbf{\Gamma}\bar{\mathbf{v}} = \mathbf{h}(\varphi)$. Let $\mathbf{T}$ be a selection matrix that selects the rows of $\mathbf{h}$ corresponding to $m = \{-n, n\}$ for each n such that

$$\mathbf{Th}(\varphi) = [e^{-jN\varphi}, \dots, e^{-j\varphi}, 1, e^{j\varphi}, \dots, e^{jN\varphi}]^T = \bar{\mathbf{h}}(\varphi). \tag{17}$$

So, finally, $\mathbf{TF}^{-1}(\vartheta_0)\mathbf{B}^{-1}(kr)\mathbf{Y}^H(\Phi)\mathbf{\Gamma}\bar{\mathbf{v}} = \bar{\mathbf{h}}(\varphi)$ or

$$\mathbf{G}\bar{\mathbf{v}}(\vartheta_0, \varphi) = \bar{\mathbf{h}}(\varphi), \tag{18}$$

where $\mathbf{G} = \mathbf{TF}^{-1}\mathbf{B}^{-1}\mathbf{Y}^H(\Phi)\mathbf{\Gamma}$. Hence, by applying a linear transformation, the coarray steering vector can be transformed into a vector with Vandermonde structure. Also, from (18), a straight forward way of deriving the transformation matrix $\mathbf{G}$ is by solving the simple following least-squares (LS) problem [21,22]

$$\arg\min_{\mathbf{G}} \left\{ \sum_{\gamma=1}^{\Gamma} \|\mathbf{G}\bar{\mathbf{v}}(\vartheta_0, \varphi_\gamma) - \bar{\mathbf{h}}(\varphi_\gamma)\|_F^2 \right\}, \tag{19}$$

where the Γ ($>> L$) azimuth angles, φ_γ, are collected uniformly over $[0, 2\pi)$ to solve (19) and get as good an approximation as possible. Now, we propose a spatial smoothing type operation on the coarray signal model to recover the rank of the coarray covariance matrix. Multiplying both sides of (6) by $\mathbf{G}$, we get

$$\tilde{\mathbf{z}} = \mathbf{Gz} = \mathbf{G}\left(\bar{\mathbf{V}}\boldsymbol{\rho} + \sigma_n^2\mathbf{i}\right) = \mathbf{H}\boldsymbol{\rho} + \sigma_n^2\tilde{\mathbf{i}}, \tag{20}$$

where $\tilde{\mathbf{i}} = \mathbf{Gi}$. Note that $\tilde{\mathbf{z}}$ is of size $(2N+1)$. To perform spatial smoothing (SS) type operation [1], we divide this array into $(N+1)$ overlapping subarrays, each with $(N+1)$ elements such that the k^{th} subarray has elements at $\{(-k+1+n),\ \ n = 0, 1, \dots, N\}$. The k^{th} subarray corresponds to the $(N+2-k)^{th}$ to $(2N+1-k)^{th}$ rows of $\tilde{\mathbf{z}}$ denoted as $\tilde{\mathbf{z}}_k = \mathbf{H}_k\boldsymbol{\rho} + \sigma_n^2\tilde{\mathbf{i}}_k$. The covariance matrix of the k^{th} subarray can now be written as $\mathbf{R}_k = \tilde{\mathbf{z}}_k\tilde{\mathbf{z}}_k^H$. Taking the average of the covariance matrices $\mathbf{R}_k$ over all k gives us

$$\mathbf{R}_{\text{SS}} = \frac{1}{N+1}\sum_{k=1}^{N+1}\mathbf{R}_k \tag{21}$$

The matrix $\mathbf{R}_{\text{SS}}$ is referred to as the full rank spatially smoothed matrix which allows us to perform source localization of N sources.

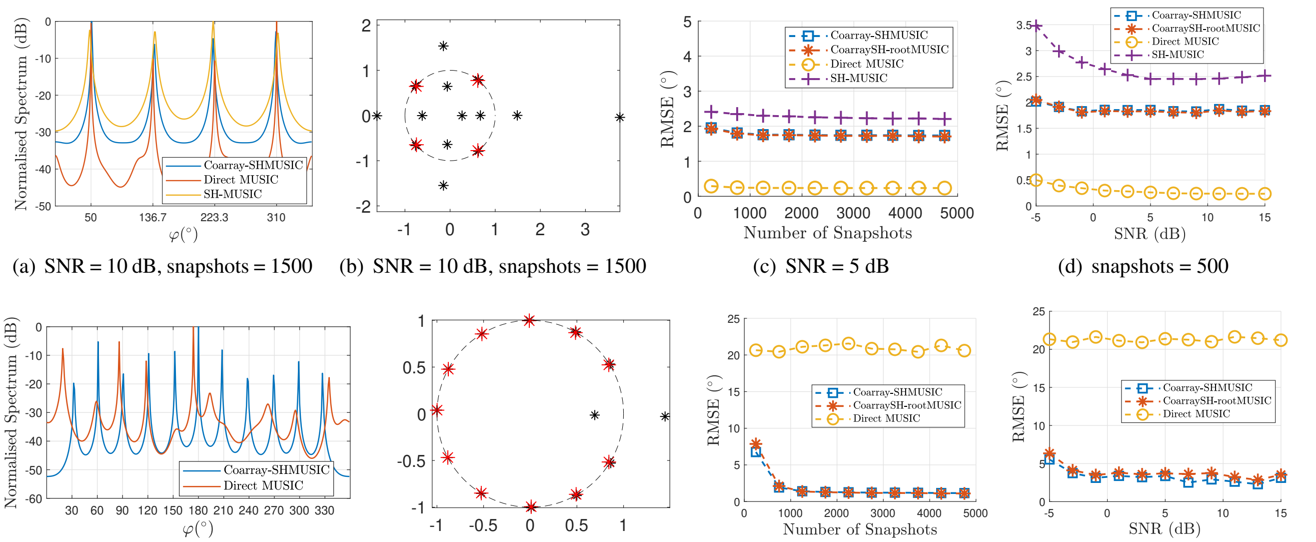
4.3. Proposed Subspace Based Approaches

We can apply subspace based techniques like MUSIC [12] to localize $L \leq N$ sources on the noise subspace of $\mathbf{R}_{\text{SS}}$. That MUSIC spectrum can be expressed as

$$S(\varphi) = \frac{1}{\left(\mathbf{h}_1^H(\varphi)\mathbf{E}_\eta\mathbf{E}_\eta^H\mathbf{h}_1(\varphi)\right)}, \tag{22}$$

where $\mathbf{E}_\eta$ is the $(N+1) \times (N+1-L)$ noise subspace of $\mathbf{R}_{\text{SS}}$ and $\mathbf{h}_1(\varphi)$ is the steering vector for azimuth angle φ corresponding to the first subarray. The spectrum $S(\varphi)$ gives peaks at the location of the DOAs $\{\varphi_1, \varphi_2, \dots, \varphi_L\}$ of the sources. We refer to (22) as coarraySH-MUSIC. The DOAs in (22) are found by performing an exhaustive search in the range $\phi \in [0, 2\pi)$. To avoid this grid search, polynomial rooting technique was introduced in [23] which can be extended to coarraySH-MUSIC. Since, $\mathbf{h}_1(\phi)$ has a Vandermonde structure, it can be written as $\mathbf{h}_1(\varphi) = [1 \quad z \quad z^2 \quad \cdots \quad z^N]^T$, where $z = e^{j\varphi}$. The cost function in the denominator of $S(\varphi)$ now assumes a polynomial of degree $2N$ given by

$$P(z) = \sum_{w=-N}^{N}\mathbf{C}_w z^w, \tag{23}$$



(a) SNR = 10 dB, snapshots = 1500 (b) SNR = 10 dB, snapshots = 1500 (c) SNR = 5 dB (d) snapshots = 500

(e) SNR = 10 dB, snapshots = 1500 (f) SNR = 10 dB, snapshots = 1500 (g) SNR = 10 dB (h) snapshots = 1500

Fig. 3. Plot of the various simulations performed. Top row is for the case of 4 sources and the bottom row is for the case of 11 sources. (a) and (e) show the estimated DOA spectrum of the algorithms, (b) and (f) plot the roots of the coarraySH-rootMUSIC (red star are the desired DOAs), & (c), (d), (g), (h) show the RMSE performance versus SNR and number of snapshots.

Table 1. Probability of detecting the correct number of sources

No. of sources	Methods	Number of Snapshots			
		100	500	1000	1500
5	CoarraySH	0.8580	0.8900	0.9080	0.9160
	Direct	0.9960	0.9980	1	1
7	CoarraySH	0.8760	1	1	1
	Direct	0.9120	0.9200	0.9380	0.9680
9	CoarraySH	0.8840	1	1	1
	Direct	0.3240	0.3900	0.4020	0.4080
11	CoarraySH	0.7400	0.9840	0.9960	1
	Direct	0.1880	0.1920	0.1960	0.200

where $\mathbf{C}_w$ is obtained mathematically. We can see that $P(z)$ has $2N$ roots and additionally if z is a root of the polynomial then $1/z^*$ is also a root (evident from the expression of z). Hence N roots are within the unit circle and N roots are outside. Of the N roots that lie within the unit circle, L roots closest to the unit circle would give us the DOAs. Thus, solving a polynomial allows us to find the DOAs which reduces complexity and eliminates off-grid effects. We refer to this technique as coarraySH-rootMUSIC.

5. SIMULATION RESULTS

For simulations, we consider a spherical array with $I = 12$ sensors on the faces of a regular dodecahedron with radius $r = 5.12$ cm as shown in Fig. 1(b). Since, $N = 2$ is the highest mode captured by the actual array, SH-MUSIC will be able to localize only 4 sources in the azimuth direction (as can be seen from Fig. 2(j)). We compare the performances of the various algorithm for 4 sources and the result is shown in the top row of Fig. 3. We can see that when number of sources is low, direct MUSIC performs the best. We also conducted simulations to evaluate the RMSE performance which is calculated as

$$\text{RMSE} = \sqrt{\frac{1}{ZL}\sum\nolimits_{z=1}^{Z}\sum\nolimits_{l=1}^{L}\left(\hat{\varphi}_{lz}-\varphi_l\right)^2}, \tag{24}$$

where $\hat{\varphi}_{lz}$ is the estimated DOA of the l^{th} source in the z^{th} Monte Carlo run and the total number of independent runs (Z) for each SNR and snapshot is set to 1000. For 4 sources, our algorithms perform better than SH-MUSIC and are comparable to the direct-MUSIC algorithm. The major advantage of our algorithms is when the number of sources increase. For Table 1, we use the method in [24] to calculate the probability of detection (we use stricter threshold for lower number of sources) and we can see that as the number of sources increases, direct MUSIC is not able to detect all the sources (the same can be seen in Fig. 3(e)). However, both our algorithms are not only able to detect much higher number of sources, but also give a much better RMSE performance. This is because of the utilization of the increased DOF of the coarrays by our proposed algorithms. Also, note that perfromance of the Coarray-SHMUSIC and CoarraySH-rootMUSIC are close togehter because we did not consider off-grid effects in our simulations.

6. CONCLUSION

In this paper, we extend the concept of difference coarray to spherical arrays and use the increased degrees-of-freedom (DOF) provided by the coarray to propose a novel DOA estimation algorithm capable of estimating more number of sources than existing algorithms. Comparison with other algorithms clearly illustrates the effectiveness and advantages of the proposed algorithm. In future work, a detailed theoretical analysis of the performance bounds of the proposed algorithm would be studied to gain further insights into its behaviour. Also, a 2D coarray spatial smoothing type operation will be studied to perform simultaneous elevation and azimuth estimation in the coarray domain.

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