TRANSMIT BEAMFORMING WITH FIXED COVARIANCE FOR INTEGRATED MIMO RADAR AND MULTIUSER COMMUNICATIONS

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ABSTRACT

In this paper, we consider the design of a multiple-input multiple-output (MIMO) transmitter which simultaneously functions as a MIMO radar and a base station for downlink multiuser communications. In contrast to the previous designs which guarantee communication performance, we require the covariance of the transmit waveform to be equal to a given optimal covariance for MIMO radar, to guarantee the radar performance. With this constraint, we formulate and solve the signal-to-interference-plus-noise ratio (SINR) balancing problem for multiuser transmit beamforming via convex optimization. By numerical simulations, we first demonstrate the radar performance loss caused by previous designs, and then show the communication performance for our proposed design in terms of balanced SINR versus transmit signal-to-noise ratio.

Index Terms— Joint radar and communications, multiple-input multiple-output (MIMO), transmit beamforming, conic linear programming

1. INTRODUCTION

Joint radar and communications on a single platform is an emerging technique which can reduce cost, achieve spectrum sharing, and enhance performance via the cooperation of radar and communications [1,2]. Numerous schemes have been proposed in recent years to implement joint radar and communications, including multi-functional waveform design [3, 4], information embedding [5, 6], and joint transmit beamforming [2, 4, 7].

We focus on the joint transmit beamforming scheme, which achieves spatial multiplexing of radar and communications by forming multiple transmit beams towards the radar targets and communication receivers. Previous works based on joint transmit beamforming mainly consider the joint design of a multiple-input multiple-output (MIMO) radar and downlink multiuser communications. In particular, these works consider the optimization of MIMO radar performance, such as beam pattern mismatch [2, 8] and Cramér-Rao Bound [7], under individual signal-to-interference-plus-noise ratio (SINR) constraints at the communication receivers. Alternatively, some variants of the design [2, 4] simultaneously optimize the performance of radar and communications in the objective function. However, MIMO radars exhibit performance trade-off with multiuser communications in these works. In other words, to guarantee the SINRs at users, the achievable performance of MIMO radar is worse than the counterpart of a separate MIMO radar without considering communications. In high speed communication scenarios, the performance loss of MIMO radar can be significant to achieve high SINRs at the users [8].

Here, we consider a joint MIMO radar and multiuser communication system, in which radar is the primary function and communication is the secondary function. Under this scenario, the efficiency of MIMO radar needs to be first guaranteed and the performance loss of radar is not desired. The performance of MIMO radars highly depends on the covariance of the transmit waveform [9–11]. Therefore, we formulate transmitter optimizations for communications, under the constraint that the covariance of the transmit waveform is equal to the given optimal one for MIMO radar without communication. The proposed approach in [4] considers a similar approach, but constrains the instantaneous covariance and needs to optimize the instantaneous transmit waveform. Different from [4], we constrain the average covariance, and optimize the precoding matrices as in [2, 7, 8].

With this covariance constraint, we apply transmit beamforming to improve the SINRs at downlink users [12, 13]. In particular, we formulate the SINR balancing [14] problem for multiuser communications, which designs the precoding matrices to maximize the worst SINR at the users. We show that the problem can be reformulated as a linear conic optimization [15], and is solvable using convex optimization tools. The simulations show the benefit of our proposed design compared with previous design in [8] by the result that the latter causes uncontrollable radar beam pattern distortion under fad-

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ing channels. At the cost of guaranteeing radar performance, the balanced SINR by our design becomes constant when the transmit signal-to-noise ratio (SNR) is high. Nevertheless, an acceptable balanced SINR is achievable under high SNR if the number of users is controlled.

The rest of the paper is organized as follows. In Sec. 2, we provide the signal model and introduce the transmit covariance constraint. In Sec. 3, we formulate the SINR balancing problem for communications, and study numerical methods to solve it. We demonstrate the radar and communication performance of the proposed design and the previous trade-off design in [8] via simulations in Sec. 4. Sec. 5 concludes the paper.

2. SIGNAL MODEL

Consider a joint transmitter which simultaneously functions as a MIMO radar transmitter and a base station for downlink multiuser communications. In the transmitter, radar and communications share the transmit signal, as described in Sec. 2.1. Considering the radar performance, we introduce a transmit covariance constraint to the transmit signal in Sec. 2.2.

2.1. Shared transmit signal

The transmitter is equipped with a transmit array with M antennas and sends independent communication symbols to K users, where $K \leq M$. The average transmit power is P. The transmit signal $\boldsymbol{x}(n)$ for the shared transmit array is generated by the joint linear precoding scheme in [8]. In particular, $\boldsymbol{x}(n)$ is the sum of linear precoded radar waveforms and communication symbols, given by

$$\boldsymbol{x}(n) = \boldsymbol{W}_r \boldsymbol{s}(n) + \boldsymbol{W}_c \boldsymbol{c}(n), \; n = 0, \ldots, N-1, \tag{1}$$

where N is the number of samples, $\boldsymbol{s}(n) = [s_1(n), \ldots, s_M(n)]^T$ represents M orthogonal radar waveforms, and the $M \times M$ matrix $\boldsymbol{W}_r$ is the precoding matrix for radar [5]. The orthogonality of radar waveforms means that $(1/N)\mathbf{E}\left\{\sum_{n=0}^{N-1} \boldsymbol{s}(n)\boldsymbol{s}^H(n)\right\} = \boldsymbol{I}_M$. The K parallel communication symbols to the users are contained in $\boldsymbol{c}(n) = [c_1(n), \ldots, c_K(n)]^T$, precoded by the $M \times K$ matrix $\boldsymbol{W}_c$.

Following [2, 8, 13], we assume the following conditions:

(a) The communication symbols to different users are mutually independent, have zero mean, and are normalized to unit average power. Therefore, $\mathbf{E}(\boldsymbol{c}(n)\boldsymbol{c}^H(n)) = \boldsymbol{I}_M$.

(b) The radar waveforms and communication symbols are statistically independent.

Given $\{\boldsymbol{s}(n)\}$ and $\{\boldsymbol{c}(n)\}$, the design variables are $\boldsymbol{W}_r$ and $\boldsymbol{W}_c$.

2.2. Transmit covariance constraint for radar

The radar is monostatic so that the communication signals can also be used for target detection because they are completely known at the radar receiver. Unlike phased array radars, MIMO radars transmit independent or partially correlated signals from the array elements. The performance of MIMO radar highly depends on its transmit covariance [9–11]

$$\boldsymbol{R} = \mathbf{E}\left\{\frac{1}{N}\sum_{n=0}^{N-1} \boldsymbol{x}(n)\boldsymbol{x}^H(n)\right\}. \tag{2}$$

Substituting (1) into (2), $\boldsymbol{R}$ is given by

$$\boldsymbol{R} = \boldsymbol{W}_r \boldsymbol{W}_r^H + \boldsymbol{W}_c \boldsymbol{W}_c^H. \tag{3}$$

Given the average transmit power P, $\boldsymbol{R}$ should obey $\mathrm{tr}(\boldsymbol{R}) = P$. To guarantee the radar performance, in a MIMO radar without communications, the transmit covariance is optimized, yielding $\boldsymbol{R}_o$, under certain power constraints as in [9, 10, 16].

In the joint design considered in this paper, $\boldsymbol{W}_c$ and $\boldsymbol{W}_r$ are constrained so that the obtained $\boldsymbol{R}$ in (3) equals $\boldsymbol{R}_o$. Hence, the joint radar and communications system achieves the optimal radar performance as an independent radar system without communications. In the following, we design $\boldsymbol{W}_c$ and $\boldsymbol{W}_r$ to optimize the communication performance under this constraint. This approach is different from existing precoder design methods in [2, 7, 8] where they sacrifice the radar performance to achieve the desired SINRs for communications. Particularly in their methods, $\boldsymbol{W}_c$ and $\boldsymbol{W}_r$ are constrained to meet the minimum requirements on SINRs, and are optimized to improve the radar performance.

3. SINR BALANCING FOR DOWNLINK MULTIUSER COMMUNICATIONS

With the radar constraint, we formulate SINR balancing for downlink multiuser communication into an optimization with respect to $\boldsymbol{W}_c$ and $\boldsymbol{W}_r$, and then show that it can be reformulated into a solvable conic linear program.

3.1. Problem formulation

For downlink multiuser communication, transmit beamforming is performed to increase the signal power at intended users and reduce interference to non-intended users [13, 17]. Here, a vector Gaussian broadcast channel (GBC) [18] is considered in which each user is equipped with a single receive antenna. The channel is denoted by a $K \times M$ matrix $\boldsymbol{H}$. The channel output of the GBC is given by [8]

$$\boldsymbol{r}(n) = \boldsymbol{H}\boldsymbol{x}(n) + \boldsymbol{v}(n) = \boldsymbol{H}\boldsymbol{W}_c\boldsymbol{c}(n) + \boldsymbol{H}\boldsymbol{W}_r\boldsymbol{s}(n) + \boldsymbol{v}(n). \tag{4}$$

Here, the k-th element of $\boldsymbol{r}(n)$ represents the received signal at the k-th user, and $\boldsymbol{v}(n)$ is complex additive white Gaussian noise (AWGN) whose covariance is $\sigma^2 \boldsymbol{I}_K$. For convenience, we let $\sigma^2 = 1$ in the sequel.

In (4), each user receives the mixture of its own signals, the interference from other users, the radar signal and the noise. Let $\boldsymbol{F} = \boldsymbol{H}\boldsymbol{W}_c$ and $\boldsymbol{G} = \boldsymbol{H}\boldsymbol{W}_r$. By transmit beamforming, the achieved SINR at the k-th user is [2, 8]

$$\mathrm{SINR}_k = \frac{|\boldsymbol{F}_{k,k}|^2}{\sum_{i\neq k}|\boldsymbol{F}_{k,i}|^2 + \sum_{i=1}^{M}|\boldsymbol{G}_{k,i}|^2 + 1}, \tag{5}$$

for $k = 1, \ldots, K$.

The goal of SINR balancing is to optimize the worst SINR at the users. With the transmit covariance constraint, we let $\gamma = \min_{1\leq k\leq K} \mathrm{SINR}_k$ be the balanced SINR, yielding the optimization problem:

$$\max_{\boldsymbol{W}_c,\boldsymbol{W}_r,\gamma} \gamma, \text{ s.t. } \boldsymbol{R}_o = \boldsymbol{W}_r\boldsymbol{W}_r^H + \boldsymbol{W}_c\boldsymbol{W}_c^H, \tag{6a}$$

$$\mathrm{SINR}_k \geq \gamma,\ k = 1, \ldots, K. \tag{6b}$$

The solvers for (6) are discussed in the following.

3.2. Solution via conic linear programming

In this section, we show that (6) can be solved via an equivalent conic linear programming.

First, with the covariance constraint, the sum power of the desired signal and the interference at the k-th user should be equal to $[\boldsymbol{R}_h]_{k,k}$, where $\boldsymbol{R}_h = \boldsymbol{H}\boldsymbol{R}_o\boldsymbol{H}^H$, i.e.

$$\sum_{i=1}^{K}|\boldsymbol{F}_{k,i}|^2 + \sum_{i=1}^{M}|\boldsymbol{G}_{k,i}|^2 = [\boldsymbol{R}_h]_{k,k}. \tag{7}$$

Substituting (7) into (5), the SINR constraints in (6b) can be simplified to

$$|\boldsymbol{F}_{k,k}| \geq \sqrt{\frac{\gamma}{1+\gamma}}s_k,\ k = 1, \ldots, K, \tag{8}$$

where $s_k = ([\boldsymbol{R}_h]_{k,k} + 1)^{1/2}$.

In (8), the reformulated SINR constraint only involves $\boldsymbol{F}$ but not $\boldsymbol{G}$. Similarly, the covariance constraint can also be rewritten as a constraint on $\boldsymbol{F}$, as stated by Theorem 1.

Theorem 1. *Given $\boldsymbol{F} \in \mathbb{C}^{K\times K}$, there exists $\boldsymbol{W}_c, \boldsymbol{W}_r$ that obey*

$$\boldsymbol{F} = \boldsymbol{H}\boldsymbol{W}_c,\ \boldsymbol{R}_o = \boldsymbol{W}_r\boldsymbol{W}_r^H + \boldsymbol{W}_c\boldsymbol{W}_c^H, \tag{9}$$

if and only if $\boldsymbol{F}\boldsymbol{F}^H \preceq \boldsymbol{R}_h$.

Proof. The necessity is obvious. To prove that the condition $\boldsymbol{F}\boldsymbol{F}^H \preceq \boldsymbol{R}_h$ is sufficient, we construct $\boldsymbol{W}_c, \boldsymbol{W}_r$ by

$$\boldsymbol{W}_c = \boldsymbol{R}_o^{1/2}(\boldsymbol{H}\boldsymbol{R}_o^{1/2})^\dagger \boldsymbol{F},\ \boldsymbol{W}_r = (\boldsymbol{R}_o - \boldsymbol{W}_c\boldsymbol{W}_c^H)^{1/2}, \tag{10}$$

where $(\cdot)^\dagger$ is the Moore-Penrose inverse [19] and $(\cdot)^{1/2}$ is the matrix square root. It can be verified that the constructed $\boldsymbol{W}_c, \boldsymbol{W}_r$ obey (9).

We first verify that $\boldsymbol{H}\boldsymbol{W}_c = \boldsymbol{F}_c$. Since $\boldsymbol{F}\boldsymbol{F}^H \preceq \boldsymbol{R}_h$, one has $\boldsymbol{F} \in C(\boldsymbol{H}\boldsymbol{R}_o^{1/2})$ [20], where $C(\cdot)$ represents the column space of a matrix. As a corollary, $\boldsymbol{H}\boldsymbol{R}_o^{1/2}(\boldsymbol{H}\boldsymbol{R}_o^{1/2})^\dagger\boldsymbol{F} = \boldsymbol{F}$, and thus

$$\boldsymbol{H}\boldsymbol{W}_c = \boldsymbol{H}\boldsymbol{R}_o^{1/2}(\boldsymbol{H}\boldsymbol{R}_o^{1/2})^\dagger\boldsymbol{F} = \boldsymbol{F}.$$

Then, we show that $\boldsymbol{R}_o \succeq \boldsymbol{W}_c\boldsymbol{W}_c^H$. Since $\boldsymbol{F}\boldsymbol{F}^H \preceq \boldsymbol{R}_h$, it follows that

$$\begin{aligned}\boldsymbol{W}_c\boldsymbol{W}_c^H &= \boldsymbol{R}_o^{1/2}(\boldsymbol{H}\boldsymbol{R}_o^{1/2})^\dagger\boldsymbol{F}\boldsymbol{F}^H(\boldsymbol{R}_o^{1/2}\boldsymbol{H}^H)^\dagger\boldsymbol{R}_o^{1/2} \\ &\preceq \boldsymbol{R}_o^{1/2}(\boldsymbol{H}\boldsymbol{R}_o^{1/2})^\dagger\boldsymbol{R}_h(\boldsymbol{R}_o^{1/2}\boldsymbol{H}^H)^\dagger\boldsymbol{R}_o^{1/2} \preceq \boldsymbol{R}_o^{1/2}\boldsymbol{R}_o^{1/2} = \boldsymbol{R}_o.\end{aligned}$$

where we use the inequality $(\boldsymbol{H}\boldsymbol{R}_o^{1/2})^\dagger\boldsymbol{R}_h(\boldsymbol{R}_o^{1/2}\boldsymbol{H}^H)^\dagger \preceq \boldsymbol{I}_K$. Therefore, it is valid to compute $\boldsymbol{W}_r$ via (10), and the second equation in (9) should hold if $\boldsymbol{W}_r$ is given by (10), completing the proof. □

We have thus reformulated (6) into the following optimization with respect to $\boldsymbol{F}, \gamma$:

$$\max_{\boldsymbol{F},\gamma} \gamma, \text{ s.t. } \boldsymbol{F}\boldsymbol{F}^H \preceq \boldsymbol{R}_h, \tag{11a}$$

$$|\boldsymbol{F}_{k,k}| \geq \sqrt{\frac{\gamma}{1+\gamma}}s_k,\ k = 1, \ldots, K. \tag{11b}$$

From the Schur complement [20], the constraint in (11a) is equivalent to a convex semidefinite constraint. Note that for a feasible $\boldsymbol{F}$ of (11), rotating its k-th column by a scalar phase factor $e^{j\theta_k}$ does not violate feasibility [13, 21]. Therefore, we only need to consider $\boldsymbol{F}$ with real diagonal elements. Introducing a new variable $t = \sqrt{\gamma/(1+\gamma)}$, (11) is equivalent to the following:

$$\max_{\boldsymbol{F},t} t, \text{ s.t. } \boldsymbol{F}\boldsymbol{F}^H \preceq \boldsymbol{R}_h, \tag{12a}$$

$$\Re\{\boldsymbol{F}_{k,k}\} \geq ts_k,\ k = 1, \ldots, K. \tag{12b}$$

This is now solvable by linear conic programming.

In addition to solving (12) with optimization software, we also propose a fast iteration method to solve its dual in [22]. After solving (12), the optimum of the original problem in (6) can be computed by (10).

4. NUMERICAL RESULTS

We performed numerical simulations to demonstrate the performance of radar and communications under the transmit covariance constraint from radar.

In the simulations, the transmit array is a uniform linear array with equal antenna spacing. The antenna spacing is half of the wavelength, and the number of transmit antennas is $M = 10$. For MIMO radar, $\boldsymbol{R}_o$ is given by $\boldsymbol{R}_o = P\boldsymbol{S}_o$, where P is the transmit power and $\boldsymbol{S}_o$ is the power normalized covariance. Two design goals for MIMO radar are considered, resulting in two different $\boldsymbol{S}_o$. In the first, the radar transmits orthogonal waveforms and forms an omni-directional beam

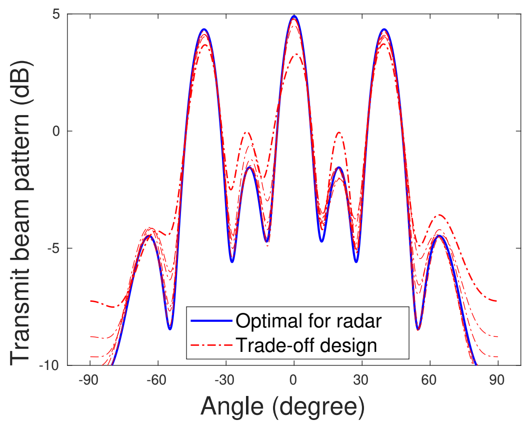

Fig. 1. The optimal radar transmit beam pattern, and the obtained transmit beam pattern under given SINR threshold Γ for communication users, where $K = 4$, $P = 100$, and $\Gamma = 12$dB.

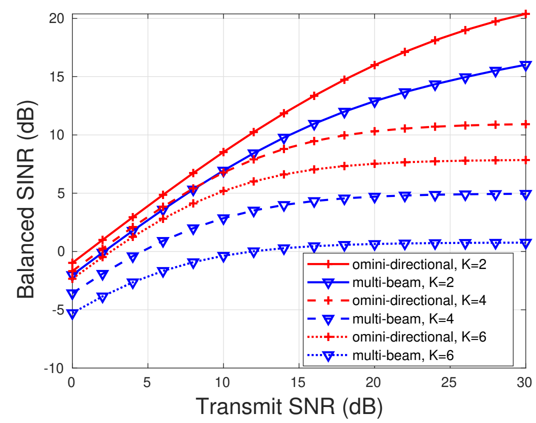

Fig. 2. Balanced SINR versus transmit SNR.

pattern with $\boldsymbol{S}_o = (1/M)\boldsymbol{I}_M$. In the second, one follows the beam pattern matching design in [9] to form multiple beams towards $-40°, 0°, 40°$ with a beam width of $10°$, where $\boldsymbol{S}_o$ is obtained by a semidefinite quadratic optimization.

For communications, the channel $\boldsymbol{H}$ obeys Rayleigh fading, namely the elements in $\boldsymbol{H}$ satisfy independent standard complex normal distributions. The noise power is $\sigma^2 = 1$.

We first show the potential drawback of the previous trade-off design in [8], which solves the beam pattern matching design mentioned above with additional constraints that the achieved SINRs at users are higher than a given threshold Γ. The obtained beam patterns with six realizations of $\boldsymbol{H}$ are displayed in Fig. 1. The optimal beam pattern in the radar-only case (blue line) is also given as a baseline. Compared to the optimal beam pattern for radar, the trade-off design may reduce the main-lobe power and lift the side-lobe power. Meanwhile, the obtained beam pattern may change when $\boldsymbol{H}$ changes. Under fading channels, it can be difficult to control the shape the beam pattern via the trade-off design, and the beam pattern mismatch can be notable under some channel realizations. Conversely, with the covariance constraint, the obtained transmit beam pattern by our design is always the same as the optimal one for radar, regardless of the fading of the channel.

We next show the communication performance with the covariance constraint. The balanced SINR versus transmit SNR for different $\boldsymbol{S}_o$ and K is displayed in Fig. 2. From Fig. 2, it is observed that the balanced SINR increases with transmit SNR, but the increment becomes slow when the SNR is high enough. The reason is that the interference cannot be effectively canceled via transmit beamforming with the transmit covariance constraint. To zero-forcing the interference, transmit beamforming requires $\boldsymbol{S}_h = \boldsymbol{H}\boldsymbol{S}_o\boldsymbol{H}^H$ to be a diagonal matrix [8], while this condition generally does not hold if $\boldsymbol{H}$ is Rayleigh fading. Despite this limitation from the covariance constraint, the communication performance can actually be further enhanced via multiuser interference elimination techniques such as dirty paper coding [23], see our work in [22].

When there are only a few users, the inter-user interference can be less serious, and an acceptable balanced SINR can be achieved under high SNR. With the increase of K, the balanced SINR may become very low, e.g. $K = 6$ for multi-beam patterns. Note that the rank of $\boldsymbol{S}_o$ is 4 for the multi-beam pattern. When K exceeds the rank, the received signals at the users become linearly dependent, making it impractical to send independent data streams to the users. In other words, the meaningful operating regime for multiuser transmit beamforming is $K \le \text{rank}(\boldsymbol{R}_o)$. For Rayleigh fading channels, the balanced SINR for the omni-directional pattern is better than that for multi-beam patterns, since its transmit covariance has a higher rank, and thus can provide more degrees of freedom for communications.

5. CONCLUSION

In this paper, we consider the transmit design of a joint MIMO radar and downlink multiuser communications system, in which the communication performance is optimized under a transmit covariance constraint from radar. In particular, we formulate the SINR balancing problem for downlink multiuser transmission, and proposed a method to solve this problem via convex optimization. In the simulations, it is observed that the balanced SINR becomes constant when the transmit SNR is high, since the interference cannot be fully eliminated with the covariance constraint. Nevertheless, the numerical results under fading channels indicate that an acceptable balanced SINR is achievable if the number of users is controlled and the radar forms an omni-directional beam rather than directional beams.

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