SPARSE ARRAY SOURCE ENUMERATION VIA COARRAY SUBSPACE OPTIMIZATION

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ABSTRACT

Linear sparse arrays, where the sensors are placed with non-uniform spacing, identify more source direction-of-arrivals (DOAs) than sensors. In contrast, uniform linear arrays (ULAs) can only find fewer source DOAs than sensors. To resolve these DOAs in practice, the sources have to be enumerated first. However, existing source enumerators are either designed for ULAs or computationally challenging for sparse arrays. This paper proposes sparse array source enumeration via coarray subspace optimization (SASE-CSO). The SASE-CSO algorithm first establishes multiple batches of array outputs. In each batch, we estimate the projection matrix onto the signal subspace on the difference coarray. Next, we propose the coarray subspace optimization (CSO) to combine these estimated projection matrices. The explicit relation between the optimal solution to the CSO and the estimated projection matrices reduces the complexity for implementation. Finally, the sources are enumerated from the most significant gap in the eigenvalues of the optimal solution to the CSO. The SASE-CSO can enumerate up to U sources, where U denotes the largest element in the central ULA segment in the difference coarray. Numerical examples demonstrate that the SASE-CSO increases the probability of detection for more sources than sensors and two closely-spaced sources with unequal powers.

Index Terms— Source enumeration, sparse arrays, coarray subspace optimization, difference coarrays.

1. INTRODUCTION

Sparse arrays have attracted considerable attention in array signal processing. With non-uniform separations of sensors, sparse arrays can resolve $\mathcal{O}(N^2)$ sources with uncorrelated amplitudes using only N elements. The reason is that the difference coarray (the set of differences between the locations of the elements) of a sparse array possesses $\mathcal{O}(N^2)$ distinct elements. The size of the difference coarray is much more than the number of antennas N. This concept has led to several notable designs of sparse arrays, such as the minimum redundancy array (MRA) [1], the nested array [2], and the coprime array [3]. Based on these arrays, quite a few coarray-based DOA estimators confirm the possibility of resolving more source DOAs than the number of antenna elements [2–10]. However, the existing coarray-based DOA estimators typically require perfect knowledge about the number of sources. This information is unavailable unless the number of sources is enumerated first.

In the past, a broad family of source enumerators was derived from information-theoretic criteria. These criteria include the Akaike information criterion (AIC) [11], the minimum description length (MDL) [12], and the Bayesian information criterion (BIC) [13]. Based on these criteria, several source enumerators were proposed in [14–17] and the references therein. In addition to information-theoretic criteria, other approaches such as subspace averaging [18,19], and signal subspace matching [20], are also utilized in source enumeration. However, due to the model assumptions, these methods are restricted to resolving fewer sources than sensors. Some source enumerators were shown to resolve more sources than sensors. These enumerators include the sequential noise-subspace equalization [21], detection and estimation with jackknifing [7], and source enumeration based on sparse reconstruction [9]. Some of these methods involve optimization associated with matrices with specific structures [21] or problems related to a dense grid [9]. These attributes make it challenging for implementation in systems with limited computational power. The work by Han and Nehorai [7] detects the number of sources based on the second order statistic of eigenvalues (SORTE) and jackknifing. This approach improves the detection performance compared to that without jackknifing, but the optimality of [7] remains to be studied.

This paper proposes sparse array source enumeration via coarray subspace optimization (SASE-CSO). Inspired by jackknifing [7], the SASE-CSO first establishes multiple batches of array output vectors. Next, in each batch, SASE-CSO sequentially estimates the autocorrelation vector on the difference coarray, an augmented Hermitian Toeplitz matrix, and a projection matrix onto the signal subspace related to the difference coarray. These projection matrices across all batches are combined through the *coarray subspace optimization* (CSO). We also explicitly relate the optimal solution to the CSO to the average of the projection matrices across all batches. Finally, the sources are enumerated according to the eigenvalues of the optimal solution of the CSO. The proposed SASE-CSO algorithm can enumerate more sources than sensors, is free from a dense grid, and can be viewed as a generalization of subspace averaging [18] to the difference coarray. Numerical examples demonstrate the improved performance of SASE-CSO in cases such as more sources than sensors and two closely-spaced sources with uneven source powers.

The outline of this paper is as follows. Section 2 reviews the coarray-based array processing for sparse arrays. Section 3 proposes the subspace-averaged source enumeration through the coarray subspace optimization (SASE-CSO). Numerical examples are demonstrated in Section 4, while Section 5 concludes this paper.

2. PRELIMINARIES

We consider D monochromatic, narrowband, far-field sources with common wavelength λ illuminating an N-element linear array whose elements are located at $n\lambda/2$. The index n belongs to an integer-valued set $\mathbb{S}$. The DOA of the i-th source

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θ_i satisfies $-\pi/2 \leq \theta_i \leq \pi/2$ for $i \in [\![1, D]\!]$. The notation $[\![a, b]\!] = \{a, a+1, \ldots, b\}$ represents the set of integers between two integers a and b. With these notions, the output of the array $\mathbb{S}$ is modeled as [22]

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}(\theta_i) + \mathbf{n}_{\mathbb{S}} \quad \in \mathbb{C}^N, \tag{1}$$

where A_i is the complex amplitude of the i-th source and $\mathbf{n}_{\mathbb{S}}$ denotes the noise term. The steering vector for the i-th source is denoted by $\mathbf{v}_{\mathbb{S}}(\theta_i)$. The entry of $\mathbf{v}_{\mathbb{S}}(\theta_i)$ associated with $n \in \mathbb{S}$ is represented by $\langle \mathbf{v}_{\mathbb{S}}(\theta_i) \rangle_n \triangleq e^{\jmath \pi n \sin \theta_i}$ where $\langle \mathbf{v}_{\mathbb{S}}(\theta_i) \rangle_n$ is the bracket notation [10] and $\jmath = \sqrt{-1}$ is the imaginary unit. The quantities $\{A_i\}_{i=1}^{D}$ and $\mathbf{n}_{\mathbb{S}}$ are assumed to satisfy $\mathbb{E}[A_i] = 0$, $\mathbb{E}[\mathbf{n}_{\mathbb{S}}] = \mathbf{0}$, $\mathbb{E}[A_i A_j^*] = p_i \delta_{i,j}$, $\mathbb{E}[A_i \mathbf{n}_{\mathbb{S}}^H] = \mathbf{0}^T$, and $\mathbb{E}[\mathbf{n}_{\mathbb{S}} \mathbf{n}_{\mathbb{S}}^H] = p_n \mathbf{I}_N$. Here $\mathbb{E}[\cdot]$ is the expectation operator, p_i is the power of the i-th source, p_n is the noise power, and $\delta_{i,j}$ is the Kronecker delta function. The matrix $\mathbf{I}_N$ is the N-by-N identity matrix.

Based on these assumptions, the covariance of $\mathbf{x}_{\mathbb{S}}$ becomes

$$\mathbf{R}_{\mathbb{S}} = \mathbb{E}\left[\mathbf{x}_{\mathbb{S}} \mathbf{x}_{\mathbb{S}}^H\right] = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{S}}(\theta_i) \mathbf{v}_{\mathbb{S}}^H(\theta_i) + p_n \mathbf{I}_N. \tag{2}$$

The covariance matrix $\mathbf{R}_{\mathbb{S}}$ can be converted to the difference-coarray domain. We first define the difference coarray [2, 3, 10, 23].

Definition 1. The difference coarray of an array $\mathbb{S}$ is defined as $\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$.

Next, we define the mapping matrix $\mathbf{J}$ between the physical array $\mathbb{S}$ and the difference coarray $\mathbb{D}$ [23]. To begin with, for a number m belonging to the difference coarray $\mathbb{D}$, we define the N-by-N matrix $\mathbf{I}(m)$ whose entries satisfy $\langle \mathbf{I}(m) \rangle_{n_1, n_2} = \delta_{n_1 - n_2, m}$. Based on the matrix $\mathbf{I}(m)$, the column of the mapping matrix $\mathbf{J}$ associated with $m \in \mathbb{D}$ is $\mathrm{vec}(\mathbf{I}(m))$, where $\mathrm{vec}(\cdot)$ denote the vectorization operation. By definition, the size of the mapping matrix $\mathbf{J}$ is N^2 by $|\mathbb{D}|$, where $|\mathbb{A}|$ denotes the cardinality of a set $\mathbb{A}$.

According to the covariance matrix $\mathbf{R}_{\mathbb{S}}$ in (2) and the matrix $\mathbf{J}$, we define the autocorrelation vector $\mathbf{x}_{\mathbb{D}}$ on the difference coarray by [2, 3, 10, 23]

$$\mathbf{x}_{\mathbb{D}} \triangleq \mathbf{J}^{\dagger} \mathrm{vec}(\mathbf{R}_{\mathbb{S}}) = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{D}}(\theta_i) + p_n \mathbf{e}_0. \tag{3}$$

The notation $(\cdot)^{\dagger}$ represents the pseudo-inverse of a matrix. $\mathbf{v}_{\mathbb{D}}(\theta_i)$ corresponds to the steering vector on the difference coarray $\mathbb{D}$. The column vector $\mathbf{e}_0$ satisfies $\langle \mathbf{e}_0 \rangle_m = \delta_{m,0}$. The autocorrelation vector $\mathbf{x}_{\mathbb{D}}$ was utilized in the coarray-based MUSIC algorithm for resolving more source DOAs than sensors [2–5, 10]. However, these algorithms assume D to be known.

We review the second order statistic of eigenvalues (SORTE) algorithm [24] for source enumeration [7, 8] next. For real numbers $s_1, s_2, \ldots, s_L$ with $s_1 \geq s_2 \geq \cdots \geq s_L \geq 0$, the gap between adjacent numbers is defined as $\nabla s_l \triangleq s_l - s_{l+1}$, where $l \in [\![1, L-1]\!]$. The sample variance of $\nabla s_l, \nabla s_{l+1}, \ldots, \nabla s_{L-1}$ is defined as

$$V_l \triangleq \frac{1}{L-l} \sum_{q=l}^{L-1} (\nabla s_q - \overline{s})^2, \tag{4}$$

where $\overline{s} \triangleq \frac{1}{L-l} \sum_{q=l}^{L-1} \nabla s_q$. With these notations, we define the objective function as

$$\mathrm{SORTE}(l; s_1, \ldots, s_L) \triangleq \begin{cases} \frac{V_{l+1}}{V_l} & \text{if } V_l \neq 0, \\ \infty & \text{if } V_l = 0. \end{cases} \tag{5}$$

The SORTE algorithm solves the following optimization problem

$$\widehat{G} \triangleq \underset{l \in [\![1, L-1]\!]}{\arg\max} \ \mathrm{SORTE}(l; s_1, \ldots, s_L). \tag{6}$$

The optimizer $\widehat{G}$ tends to find the location of the significant gap in $s_1, s_2, \ldots, s_L$. This property is widely used in source enumerators [7, 8].

3. SOURCE ENUMERATION VIA COARRAY SUBSPACE OPTIMIZATION

In this section, we propose sparse array source enumeration via coarray subspace optimization (SASE-CSO). In SASE-CSO, the covariance matrix $\mathbf{R}_{\mathbb{S}}$ is estimated from multiple batches of the *finite snapshots*. Each estimate of $\mathbf{R}_{\mathbb{S}}$ is next converted to a Hermitian Toeplitz matrix associated with the difference coarray, leading to a projection matrix of the estimated signal subspace. These projection matrices are then optimized via a convex problem. Finally, the number of sources is inferred from the eigenvalues of the optimized projection matrix.

In the finite-snapshot scenario, the output of the array $\mathbb{S}$ has K snapshots $\widetilde{\mathbf{x}}_{\mathbb{S}}(k)$ for $k \in [\![1, K]\!]$. Next, we establish L batches of the array outputs from these K snapshots. These batches are specified by the index sets $\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_L$ with the properties

$$\mathbb{T}_\ell \subseteq [\![1, K]\!], \qquad |\mathbb{T}_\ell| = T, \tag{7}$$

for $\ell \in [\![1, L]\!]$ and $1 \leq T \leq K$. The elements in $\mathbb{T}_\ell$ denote the indices of the snapshots that are utilized to estimate the covariance matrix. Based on this concept, the covariance matrix of the array output, associated with $\mathbb{T}_\ell$ is estimated by

$$\widehat{\mathbf{R}}_{\mathbb{S}}(\ell) \triangleq \frac{1}{T} \sum_{k \in \mathbb{T}_\ell} \widetilde{\mathbf{x}}_{\mathbb{S}}(k) \widetilde{\mathbf{x}}_{\mathbb{S}}^H(k). \tag{8}$$

Owing to (3), the autocorrelation vector on the difference coarray with respect to $\mathbb{T}_\ell$ is inferred by

$$\widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \triangleq \mathbf{J}^{\dagger} \mathrm{vec}(\widehat{\mathbf{R}}_{\mathbb{S}}(\ell)). \tag{9}$$

The estimated autocorrelation vector $\widehat{\mathbf{x}}_{\mathbb{D}}(\ell)$ enables us to establish a Hermitian Toeplitz matrix $\widehat{\mathbf{R}}(\ell)$ associated with $\mathbb{T}_\ell$. We denote U to be the largest non-negative integer in the central ULA segment of the difference coarray, i.e., the largest U so that $U \geq 0$ and $\{0, \pm 1, \ldots, \pm U\} \subseteq \mathbb{D}$. As a result, the Hermitian Toeplitz matrix $\widehat{\mathbf{R}}(\ell)$ is defined as [4, 10]

$$\widehat{\mathbf{R}}(\ell) \triangleq \begin{bmatrix} \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_0 & \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_{-1} & \ldots & \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_{-U} \\ \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_1 & \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_0 & \ldots & \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_{-U+1} \\ \vdots & \vdots & \ddots & \vdots \\ \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_U & \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_{U-1} & \ldots & \langle \widehat{\mathbf{x}}_{\mathbb{D}}(\ell) \rangle_0 \end{bmatrix}. \tag{10}$$

Next, we compute the eigenvalues and the eigenvectors of the Hermitian Toeplitz matrix $\widehat{\mathbf{R}}(\ell)$ to detect the number of sources in the ℓ-th batch. In what follows, the eigenvalues of $\widehat{\mathbf{R}}(\ell)$ are denoted by $\widehat{\lambda}_1(\ell), \widehat{\lambda}_2(\ell), \ldots, \widehat{\lambda}_{U+1}(\ell)$. The eigenvalues are sorted according to the magnitudes in the descending order, i.e., $\left|\widehat{\lambda}_1(\ell)\right| \geq \left|\widehat{\lambda}_2(\ell)\right| \geq \cdots \geq \left|\widehat{\lambda}_{U+1}(\ell)\right|$. According to these eigenvalues, the number of sources in the ℓ-th batch is detected by the SORTE algorithm

$$\widehat{D}(\ell) \triangleq \underset{d \in [\![1, U]\!]}{\arg\max} \ \mathrm{SORTE}\left(d; |\widehat{\lambda}_1(\ell)|, \ldots, |\widehat{\lambda}_{U+1}(\ell)|\right), \tag{11}$$

Algorithm 1 Sparse Array Source Enumeration via Coarray Subspace Optimization (SASE-CSO)

Require: The array geometry $\mathbb{S}$.
Require: Array output vectors $\widetilde{\mathbf{x}}_{\mathbb{S}}(k)$ for $k \in [\![1, K]\!]$.
Require: The batch size T with $T \leq K$.

```
for all ℓ ∈ [[1, L]] do
    Generate the index set T_ℓ satisfying (7).
    Estimate the covariance R̂_S(ℓ) using (8).
    Evaluate x̂_D(ℓ) according to (9).
    Construct the matrix R̂(ℓ) based on (10).
    Detect the number of sources D̂(ℓ) using (11).
    Estimate the signal subspace matrix Û_S(ℓ) using (12).
    Compute the projection matrix Π̂_S(ℓ) using (13).
end for
Compute Π̂_S according to (16).
Find the eigenvalues ν̂_d of Π̂_S for d ∈ [[1, U+1]].
Compute the eigenvalues of P̂* using (17).
Enumerate the number of sources D̂ using (18).
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where the objective function $\mathrm{SORTE}(\cdot; \cdots)$ is defined in (5).

The detected number of sources $\widehat{D}(\ell)$ in (11) helps to estimate the associated signal subspace. Denoting the orthonormal eigenvector associated with the eigenvalue $\widehat{\lambda}_i(\ell)$ by $\widehat{\mathbf{u}}_i(\ell)$, the orthonormal basis for the signal subspace is estimated by

$$\widehat{\mathbf{U}}_{\mathcal{S}}(\ell) \triangleq \begin{bmatrix} \widehat{\mathbf{u}}_1(\ell) & \widehat{\mathbf{u}}_2(\ell) & \dots & \widehat{\mathbf{u}}_{\widehat{D}(\ell)}(\ell) \end{bmatrix}. \tag{12}$$

Using the matrix $\widehat{\mathbf{U}}_{\mathcal{S}}(\ell)$, *the projection matrix onto the estimated signal subspace* in the ℓ-th batch is expressed as

$$\widehat{\mathbf{\Pi}}_{\mathcal{S}}(\ell) \triangleq \widehat{\mathbf{U}}_{\mathcal{S}}(\ell)\widehat{\mathbf{U}}_{\mathcal{S}}^H(\ell). \tag{13}$$

The projection matrix $\widehat{\mathbf{\Pi}}_{\mathcal{S}}(\ell)$ characterizes the signal subspace in the ℓ-th batch. The column space of $\widehat{\mathbf{\Pi}}_{\mathcal{S}}(\ell)$ spans the estimated signal subspace. The rank of $\widehat{\mathbf{\Pi}}_{\mathcal{S}}(\ell)$ carries the information of the detected number of sources.

Next, we aim to combine the detected number of sources and the estimated signal subspace across all L batches based on convex optimization. This step is done by the following convex optimization problem:

$$\min_{\mathbf{P}} \sum_{\ell=1}^{L} \left(\left\| \mathbf{P} - \widehat{\mathbf{\Pi}}_{\mathcal{S}}(\ell) \right\|_F^2 + \alpha \left\| \widehat{\mathbf{\Pi}}_{\mathcal{N}}(\ell)\mathbf{P} \right\|_F^2 \right), \tag{14}$$

where $\mathbf{P}$ is a $(U+1)$-by-$(U+1)$ complex matrix, $\|\cdot\|_F$ is the Frobenius norm of a matrix, and the regularization parameter α is a nonnegative number. The matrix $\widehat{\mathbf{\Pi}}_{\mathcal{N}}(\ell) \triangleq \mathbf{I} - \widehat{\mathbf{\Pi}}_{\mathcal{S}}(\ell)$ is the projection matrix onto the estimated noise subspace in the ℓ-th batch.

The rationale of (14) is as follows. The matrix $\mathbf{P}$ resembles the projection matrix onto the signal subspace in two ways. First, the matrix $\mathbf{P}$ is close to the matrix $\widehat{\mathbf{\Pi}}_{\mathcal{S}}(\ell)$. Second, due to the orthogonality between the signal subspace and the noise subspace, the iterated projection matrix $\widehat{\mathbf{\Pi}}_{\mathcal{N}}(\ell)\mathbf{P}$ is close to zero. Based on these concepts, a regularization parameter α is introduced to balance these two terms.

To find the optimal solution to (14), we set the complex gradient of the objective function to zero. The optimal solution $\widehat{\mathbf{P}}_{\mathrm{opt}}$ is

$$\widehat{\mathbf{P}}_{\mathrm{opt}} = \left(\mathbf{I} + \alpha \left(\mathbf{I} - \widehat{\mathbf{\Pi}}_{\mathcal{S}} \right) \right)^{-1} \widehat{\mathbf{\Pi}}_{\mathcal{S}}, \tag{15}$$

where the averaged projection matrix $\widehat{\mathbf{\Pi}}_{\mathcal{S}}$ is defined as

$$\widehat{\mathbf{\Pi}}_{\mathcal{S}} = \frac{1}{L} \sum_{\ell=1}^{L} \widehat{\mathbf{\Pi}}_{\mathcal{S}}(\ell). \tag{16}$$

Note that the optimal solution $\widehat{\mathbf{P}}_{\mathrm{opt}}$ can be viewed as a generalization of subspace averaging [18]. When $\alpha = 0$, the optimal solution $\widehat{\mathbf{P}}_{\mathrm{opt}}$ reduces to $\widehat{\mathbf{\Pi}}_{\mathcal{S}}$. However, the parameter α in (14) balances the first term for the closeness to the signal subspace, and the second term for the orthogonality between the signal subspace and the noise subspace.

The optimal solution $\widehat{\mathbf{P}}_{\mathrm{opt}}$ can be utilized to enumerate the number of sources. In what follows, the eigenvalues of $\widehat{\mathbf{P}}_{\mathrm{opt}}$ are denoted by $\widehat{\mu}_d$ for $d \in [\![1, U+1]\!]$. We also denote the eigenvalues of $\widehat{\mathbf{\Pi}}_{\mathcal{S}}$ by $\widehat{\nu}_d$ for $d \in [\![1, U+1]\!]$. With these notations, we present the following results.

Property 1. For $\alpha \geq 0$, the matrix $\widehat{\mathbf{P}}_{\mathrm{opt}}$ has these properties.

1. The eigenvalues of $\widehat{\mathbf{P}}_{\mathrm{opt}}$ are related to those of $\widehat{\mathbf{\Pi}}_{\mathcal{S}}$ by

$$\widehat{\mu}_d = \frac{\widehat{\nu}_d}{1 + \alpha (1 - \widehat{\nu}_d)}. \tag{17}$$

2. The eigenvalues satisfy $0 \leq \widehat{\mu}_d \leq 1$.
3. The matrix $\widehat{\mathbf{P}}_{\mathrm{opt}}$ is positive semidefinite.
4. The matrices $\widehat{\mathbf{P}}_{\mathrm{opt}}$ and $\widehat{\mathbf{\Pi}}_{\mathcal{S}}$ share the same rank.

Proof. According to (16), the averaged projection matrix $\widehat{\mathbf{\Pi}}_{\mathcal{S}}$ is an Hermitian matrix. Let the eigen-decomposition of the averaged projection matrix $\widehat{\mathbf{\Pi}}_{\mathcal{S}}$ be $\mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$, where the columns of $\mathbf{V}$ contain the orthonormal eigenvectors, and $\mathbf{\Lambda}$ is a diagonal matrix whose diagonal elements are the eigenvalues $\widehat{\nu}_d$ for $d \in [\![1, U+1]\!]$. Substituting $\widehat{\mathbf{\Pi}}_{\mathcal{S}} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$ into (15) leads to $\widehat{\mathbf{P}}_{\mathrm{opt}} = \mathbf{V}\mathfrak{D}\mathbf{V}^H$, where the matrix $\mathfrak{D} = \left(\mathbf{I} + \alpha\left(\mathbf{I} - \mathbf{\Lambda}\right)\right)^{-1} \mathbf{\Lambda}$. Since $\mathbf{\Lambda}$ is a diagonal matrix, the matrix $\mathfrak{D}$ is also a diagonal matrix where the d-th diagonal element is $\widehat{\mu}_d = \widehat{\nu}_d / \left(1 + \alpha\left(1 - \widehat{\nu}_d\right)\right)$, which proves Property 1-1.

Property 1-2 is shown as follows. Based on the results in [18], we have $0 \leq \widehat{\nu}_d \leq 1$ for $d \in [\![1, U+1]\!]$. Next, we consider a continuous function $f(\nu) = \nu/(1 + \alpha(1 - \nu))$ for $\nu \in [0, 1]$. The derivative of $f(\nu)$ is $f'(\nu) = (1+\alpha)/(1+\alpha(1-\nu))^2$, which is positive over the interval $[0, 1]$. As a result, $f(\nu)$ is a strictly increasing function over $[0, 1]$. Furthermore, $f(0) = 0$ and $f(1) = 1$. These arguments prove Property 1-2.

Property 1-3 is a direct consequence of (13), (15), (16), and Property 1-2. Next, we prove Property 1-4. Due to (17), we have $\widehat{\nu}_d = 0$ if and only if $\widehat{\mu}_d = 0$ for $\alpha \geq 0$. Therefore, $\widehat{\mathbf{P}}_{\mathrm{opt}}$ and $\widehat{\mathbf{\Pi}}_{\mathcal{S}}$ have the same rank. □

The matrix $\widehat{\mathbf{P}}_{\mathrm{opt}}$ can be viewed as an *approximated projection matrix onto the signal subspace*, with respect to the optimality in (14). However, $\widehat{\mathbf{P}}_{\mathrm{opt}}$ is not necessarily idempotent in the finite-snapshot scenario.

For a projection matrix associated with the subspace $\mathcal{S}$, the eigenvalues must be 0 or 1. This property can be utilized to infer the dimension of subspace $\mathcal{S}$ by detecting the largest gap in the eigenvalues of a projection matrix. Owing to this concept, the number of sources is then determined by the maximum of the gaps between the eigenvalues of the matrix $\widehat{\mathbf{P}}_{\mathrm{opt}}$:

$$\widehat{D} \triangleq \underset{d \in [\![1, U]\!]}{\arg\max} \; (\widehat{\mu}_d - \widehat{\mu}_{d+1}). \tag{18}$$

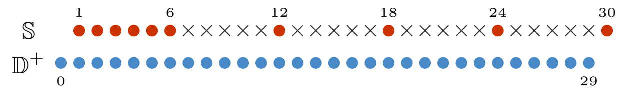

Fig. 1. Illustrations of the physical array $\mathbb{S}$ in red bullets, and the non-negative part of the difference coarray $\mathbb{D}^+$ in blue bullets. Empty space is represented by crosses. The array geometry is the nested array with $N_1 = N_2 = 5$. There are 10 sensors in this array.

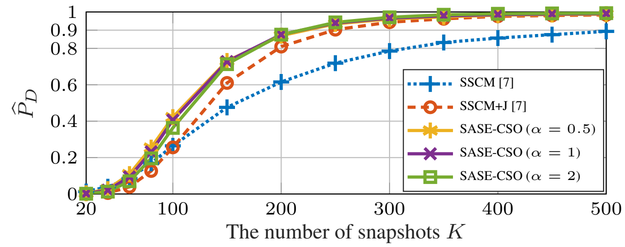

Fig. 2. The probability of detection as a function of the number of snapshots K, where "SASE-CSO" denotes the proposed SASE-CSO algorithm. There are $D = 14$ sources and 10 sensors.

Due to (18), the SASE-CSO enumerates at most U sources. For sparse arrays such as MRAs [1], nested arrays [2], and coprime arrays [3], the parameter U is greater than the number of sensors. For clarity, the steps in SASE-CSO are summarized in Algorithm 1.

4. NUMERICAL EXAMPLES

In this section, we study the performance of source enumeration for the nested array [2] with $N_1 = N_2 = 5$. There are 10 sensors in this array. Fig. 1 depicts the array configurations $\mathbb{S}$ and the non-negative part of the difference coarray $\mathbb{D}^+ \triangleq \{m : m \geq 0, m \in \mathbb{D}\}$. Based on the difference coarray, the parameter U is 29.

In our examples, we consider three source enumerators: SSCM [7], SSCM+J [7], and SASE-CSO. The first method, denoted by SSCM, utilizes the spatially smoothed covariance matrix over all snapshots and SORTE to infer the number of sources [7]. In the second approach, denoted by SSCM+J, the array output is randomly divided into several batches (jackknifing). In each batch, the number of sources is detected by SSCM. Finally, the number of sources is determined according to the mode of the detected number of sources in all the batches. We select $T = \lfloor 0.75K \rfloor$ and 20 batches in SSCM+J, where $\lfloor \cdot \rfloor$ is the floor function. The third method is the SASE-CSO algorithm with $T = \lfloor 0.75K \rfloor$, $L = 20$, and $\alpha = 0.5, 1, 2$. The index sets $\mathbb{T}_\ell$ are identical to the batches in SSCM+J. Denoting $\widehat{D}_r$ for $r \in [\![1, R]\!]$ as the detected number of sources in the r-th Monte-Carlo trial, the probability of detection is estimated by $\widehat{P}_D = |\{r : \widehat{D}_r = D, r = 1, 2, \ldots, R\}|/R$. We set $R = 10^4$ in all these experiments.

Fig. 2 shows the probability of detection for more sources ($D = 14$) than sensors ($N = 10$). The DOA of the i-th source is $\theta_i = -65^\circ + 140^\circ \times (i-1)/(D-1)$ for $i \in [\![1, D]\!]$. The signal-to-noise ratio (SNR) is $p_i/p_n = 0$dB for $i \in [\![1, 7]\!]$ and $p_i/p_n = 3$dB for $i \in [\![8, 14]\!]$. It is assumed that $\{A_i\}_{i=1}^D$ and $\mathbf{n}_{\mathbb{S}}$ in (1) are jointly complex circularly-symmetric Gaussian distributed, We observe that the probability of detection for all these source enumerators increases with the number of snapshots K. To reach the condition that $\widehat{P}_D \geq 90\%$, the SASE-CSO with $\alpha = 0.5, 1, 2$ require fewer snapshots than SSCM and SSCM+J.

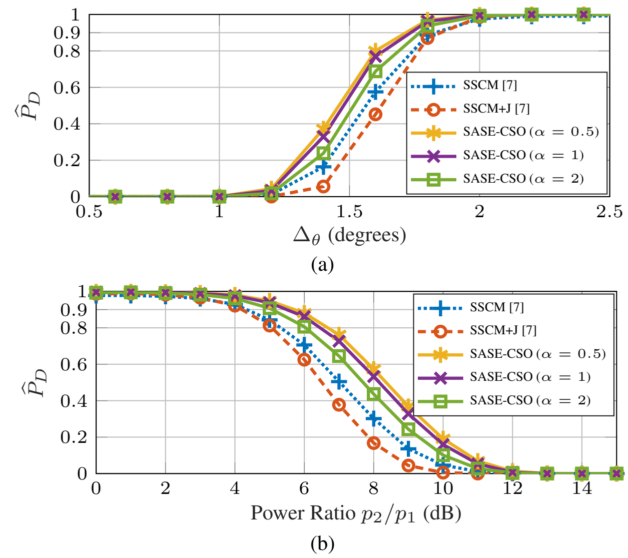

Fig. 3. The dependence of the probability of detection on (a) the separation of the DOAs Δ_θ and (b) the power ratio p_2/p_1.

Next, we consider two sources impinging on the nested array in Fig. 1. The first source has power p_1 and DOA 0°, and the second source has power p_2 and DOA Δ_θ. The SNR is $p_1/p_n = 0$ dB. The number of snapshots K is 200.

Fig. 3(a) plots $\widehat{P}_D$ against the separation Δ_θ for these sources. The two sources are equal-powered ($p_1 = p_2$). All these methods exhibit phase transition in the interval $1^\circ < \Delta_\theta < 2^\circ$. It is observed that SASE-CSO with $\alpha = 0.5$ owns the smallest Δ_θ for $\widehat{P}_D \geq 90\%$. This phenomenon results from the additional information of the signal subspace and the parameter α in SASE-CSO. On the contrary, SSCM and SSCM+J do not estimate the signal subspace.

Fig. 3(b) studies the detection performance against the power ratio p_2/p_1, where the DOA separation is $\Delta_\theta = 2^\circ$. We observe that the probability of detection deteriorates as the power ratio increases for all these enumerators. Furthermore, SASE-CSO with $\alpha = 0.5$ has the highest $\widehat{P}_D$ for a fixed power ratio. In addition, if p_2/p_1 is 6dB, the probability of detection exceeds 80% for SASE-CSO with $\alpha = 0.5, 1, 2$, but $\widehat{P}_D$ is lower than 80% for SSCM and SSCM+J. The experiments in Fig. 3 demonstrate that the SASE-CSO algorithm can enumerate two closely-spaced sources with unequal powers with higher detection accuracy than the methods in [7].

5. CONCLUDING REMARKS

This paper proposed the sparse array source enumeration via coarray subspace optimization (SASE-CSO). In the SASE-CSO, the array output vectors are divided into multiple batches of data. In each batch, we estimated the signal subspace of the Hermitian Toeplitz matrix associated with the difference coarray. Next, we deployed coarray subspace optimization to enumerate the number of sources. Numerical examples showed that the SASE-CSO algorithm enumerates more sources than sensors with sufficient snapshots. Furthermore, the SASE-CSO demonstrates improved performance for two closely-spaced sources with unequal powers.

In the future, it is of considerable interest to design the index sets in batches for the improved probability of detection. Future directions also include analyses of the detection performance and the computational complexity of the SASE-CSO.

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