ROBUST SIGNAL PROCESSING OVER SIMPLICIAL COMPLEXES

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ABSTRACT

The goal of this paper is to investigate the impact of perturbations of topological descriptors, such as graphs and simplicial complexes, on the robustness of filters acting on signals observed over such domains. Given a nominal graph that may undergo small perturbations of its edges, we design robust FIR filters using approximate closed form expressions for the perturbed eigendecomposition of the Laplacian matrix associated with the nominal graph. Then, we extend the analysis to simplicial complexes and show how the perturbation of a few triangles affect the homology class of the simplicial complex. Our small perturbation analysis of a second order simplicial complex yields approximate closed form expressions of the high order Laplacian eigenvalue/eigenvector perturbation, which are useful for the design of robust FIR filters acting on solenoidal signals. Numerical results assess the accuracy of the derived analysis and the effectiveness of the proposed method.

Index Terms— Simplicial FIR filters, graph perturbation, topological signal processing.

1. INTRODUCTION

Topological signal processing (TSP) has been recently proposed as a powerful framework providing tools for the analysis and processing of data defined over simplicial complexes (SCs) [1–5]. TSP is a generalization of graph signal processing (GSP) [6], [7] to analyze signals defined over sets of any order, not only vertices, capturing higher-order relationships in the observed data. Practical applications of TSP range from geometric deep machine learning to brain, biological, information and social networks, to name a few. However, in many applications the topology associated with data is not static and may undergo perturbations in an unknown manner. Examples of perturbed networks can be found in a plethora of applications such as in wireless communications networks where some links may drop because of random blocking or fading [8], in brain networks where the interaction among different regions of the brain changes over time [9], [10], or in biological networks, where temporal transitions of the network topology describing protein–protein and protein–DNA interactions are observed [11].

Our goal in this paper is to investigate the impact of perturbations of both graphs and simplicial complexes on the design of finite impulse response (FIR) filters acting on signals defined over such domains. FIR filters have been largely investigated in the field of GSP [12], [13], [14]. The stability to perturbations of graph FIR filters has been studied in previous works as [15], [16]. The filtering of edge signals defined over simplicial complexes has been addressed in [17], [18], [19]. In [19] the authors consider finite impulse response filters to process signals defined over simplicial complexes.

In this work we investigate how the perturbation of topological domains, such as graphs and simplicial complexes, affects the robustness of FIR filters acting on signals defined over nodes and edges. The proposed approach hinges on the small perturbation analysis of the graph Laplacian eigenpairs developed in [20]. We use first-order closed form expressions for the Laplacian matrix eigenvalues/eigenvectors pairs to design graph FIR filters that are more robust to topology uncertainties. Then, we extend the analysis to second order simplicial complexes perturbed by the removal/addition of a small percentage of triangles. We will show how such a perturbation affects the homology class of the SC by altering the dimension of the harmonic subspace according to the number of holes that are created (destroyed) in the domain. Then, to deal with such strong perturbations of the first-order Laplacian matrix, we weight every altered triangles with a small weight enforcing a small perturbation of the simplicial complex that makes possible to derive a closed form expression of the perturbed eigenpairs of the upper Laplacian. Numerical results validate the goodness of the proposed approach in filtering the observed signals.

2. SIGNAL PROCESSING OVER SIMPLICIAL COMPLEXES

In this section we introduce the fundamental tools to analyze signals defined over simplicial complexes [1], [2], [4], [5]. Given a finite set $\mathcal{V} = \{v_i\}_{i=0}^{N-1}$ of N vertices, a k-*simplex* σ_i^k is an unordered set of $k+1$ points in $\mathcal{V}$. A *face* of the k-simplex is a $(k-1)$-simplex. An *abstract simplicial complex* $\mathcal{X}$ is a finite collection of simplices that is closed under inclu-

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sion of faces, i.e., if $\sigma_i \in \mathcal{X}$, then all faces of σ_i also belong to $\mathcal{X}$. The order of a simplex is one less than its cardinality. The order of a complex is the order of its highest order simplex. If an abstract simplicial complex is embedded into an Euclidean space, a vertex is a 0-dimensional simplex, a line segment has dimension 1, a triangle is a simplex of order 2 and so on. A graph is simply a simplicial complex of order one. The structure of a simplicial complex is captured by the neighborhood relations of its subsets: Two simplices of order k, $\sigma_i^k, \sigma_j^k \in \mathcal{X}$, are *upper adjacent* in $\mathcal{X}$, if they are both faces of a simplex of order $k+1$; they are *lower adjacent* in $\mathcal{X}$, if they share a common face of order $k-1$. Given an orientation of all simplices, the structure of a simplicial complex $\mathcal{X}$ of dimension K is fully described by the set of its incidence matrices $\mathbf{B}_k$, $k = 1, \ldots, K$, with entries $B_k(i,j) = 0$ if σ_i^{k-1} is not a face of σ_j^k, and $B_k(i,j) = 1$ (or -1), if σ_i^{k-1} is a face of σ_j^k and its orientation is coherent (or not) with the orientation of σ_j^k. The structure of a K-order simplicial complex is fully described by its higher order combinatorial Laplacian matrices [21], [22] defined as

$$\mathbf{L}_0 = \mathbf{B}_1\mathbf{B}_1^T, \mathbf{L}_k = \mathbf{B}_k^T\mathbf{B}_k + \mathbf{B}_{k+1}\mathbf{B}_{k+1}^T,\ k = 1, \ldots, K-1, \\ \mathbf{L}_K = \mathbf{B}_K^T\mathbf{B}_K \tag{1}$$

where $\mathbf{L}_k^d = \mathbf{B}_k^T\mathbf{B}_k$ and $\mathbf{L}_k^u = \mathbf{B}_{k+1}\mathbf{B}_{k+1}^T$ are the lower and upper Laplacians, expressing the lower and upper adjacency of the k-order cells, respectively. Note that $\mathbf{L}_0 = \mathbf{B}_1\mathbf{B}_1^T$ represents the graph Laplacian. Since, by construction $\mathbf{B}_k\mathbf{B}_{k+1} = \mathbf{0}, \forall k$, it is easy to prove that the eigenvectors associated with the nonzero eigenvalues of $\mathbf{L}_k^d$ are orthogonal to the eigenvectors associated with the nonzero eigenvalues of $\mathbf{L}_k^u$. A second order simplicial complex is denoted as $\mathcal{X} = \{\mathcal{V}, \mathcal{E}, \mathcal{T}\}$ where $\mathcal{V}$, $\mathcal{E}$, $\mathcal{T}$ denote the set of 0, 1 and 2-simplices, i.e. vertices, edges and triangles, respectively. The two incidence matrices describing the connectivity of the complex are $\mathbf{B}_1 \in \mathbb{R}^{N\times E}$ and $\mathbf{B}_2 \in \mathbb{R}^{E\times T}$, where $T = |\mathcal{T}|$, and the first-order Laplacian matrix is defined as $\mathbf{L}_1 = \mathbf{B}_1^T\mathbf{B}_1 + \mathbf{B}_2\mathbf{B}_2^T := \mathbf{L}_1^d + \mathbf{L}_1^u$.

Signals over a second order simplicial complex $\mathcal{X}$ are maps from the sets of vertex, edge, and triangles to the real domain and they are denoted as follows: $\boldsymbol{s}^0 : \mathcal{V} \to \mathbb{R}^N$, $\boldsymbol{s}^1 : \mathcal{E} \to \mathbb{R}^E$, and $\boldsymbol{s}^2 : \mathcal{T} \to \mathbb{R}^T$. A useful orthogonal basis to represent signals of order k, capturing the properties of the complex, is given by the eigenvectors of the higher order Laplacian $\mathbf{L}_k$. Denoting by $\mathbf{L}_k = \mathbf{U}_k\mathbf{\Lambda}_k\mathbf{U}_k^T$ the eigendecomposition of $\mathbf{L}_k$, where $\mathbf{U}_k$ is the eigenvectors matrix and $\mathbf{\Lambda}_k$ is a diagonal matrix with entries the eigenvalues λ_k^1 of $\mathbf{L}_k$, we may introduce, as in [1], the simplicial Fourier Transform of a k-th order signal as $\hat{\boldsymbol{s}}^k = \mathbf{U}_k^T\boldsymbol{s}^k$.

3. FILTERING OVER PERTURBED GRAPHS

Our goal in this section is to design robust finite impulse response (FIR) filters for graphs which undergo small perturbations of their edges, hinging on the small perturbation analysis developed in [20]. A graph filter $\mathbf{H}$ of order L is a local operator that may be efficiently parameterized as [12], [13]

$$\mathbf{H} = \sum_{n=0}^{L} a_n \mathbf{L}_0^n. \tag{2}$$

Denoting by $\widehat{\boldsymbol{x}}$ the Graph Fourier Transform (GFT) of an input (vertex) signal $\boldsymbol{x}$, the GFT of the output vector $\boldsymbol{y}$ is

$$\widehat{\boldsymbol{y}} = \sum_{n=0}^{L} a_n \mathbf{\Lambda}_0^n \widehat{\boldsymbol{x}} := \mathrm{diag}(\boldsymbol{h})\widehat{\boldsymbol{x}} \tag{3}$$

where $\mathrm{diag}(\boldsymbol{h})$ represents the desired frequency response of the filter. The filter coefficients $\{a_n\}_{n=0}^L$ can be computed as a solution of

$$\min_{\boldsymbol{a}\in\mathbb{R}^L} \quad \| \mathbf{h} - \mathbf{\Phi}\boldsymbol{a} \|_F^2 \tag{4}$$

with $\mathbf{\Phi} = [\mathbf{1}, \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_0^L]$, $\boldsymbol{\lambda}_0^k = \{(\lambda_i^0)^k\}_{i=1}^N$, $k = 1, \ldots, L$. The optimal least squares solution of (4) can be found in closed form as $\boldsymbol{a} = \mathbf{\Phi}^\dagger\boldsymbol{h}$. Let us now consider a perturbation of the graph domain produced by removing or adding a small number of edges. Let us write the perturbed Laplacian as $\tilde{\mathbf{L}}_0 := \mathbf{L}_0 + \Delta\mathbf{L}_0$, where $\Delta\mathbf{L}_0$ is a matrix representing a perturbation of the edges of $\mathcal{G}$ [23], [20]. Clearly, the perturbation of $\mathbf{L}_0$ induces a perturbation of its eigendecomposition, i.e. $\tilde{\mathbf{L}}_0 = \tilde{\mathbf{U}}_0\tilde{\mathbf{\Lambda}}_0\tilde{\mathbf{U}}_0^T$, where the columns of the perturbed eigenvectors matrix are $\tilde{\boldsymbol{u}}_i^0 = \boldsymbol{u}_i^0 + \delta\boldsymbol{u}_i^0$, $i = 1, \ldots, N$, while the diagonal entries of $\tilde{\mathbf{\Lambda}}_0$ $\tilde{\lambda}_i^0 = \lambda_i^0 + \delta\lambda_i^0$ are the associated perturbed eigenvalues. Let us denote with $\boldsymbol{b}_m^1 \in \mathbb{R}^N$, the m-th column of the incidence matrix $\mathbf{B}_1$. Then, if only the mth link is perturbed, the resulting perturbation $\Delta\mathbf{L}_{0,m}$ can be written as $\Delta\mathbf{L}_{0,m} = \sigma_m \boldsymbol{b}_m^1 \boldsymbol{b}_m^{1\,T}$, where $\sigma_m = 1$ if the edge m is added, or $\sigma_m = -1$ if edge m is removed. If the eigenvalues of $\mathbf{L}_0$ are all distinct, the first order analysis developed in [20] provides the following approximate expressions for the perturbation of the eigenvalues and eigenvectors of $\mathbf{L}_0$, induced by the alteration of edge m:

$$\begin{aligned}\delta\lambda_{i,m}^0 &= \boldsymbol{u}_i^{0\,T}\Delta\mathbf{L}_{0,m}\boldsymbol{u}_i^0 = \sigma_m \boldsymbol{u}_i^{0\,T}\boldsymbol{b}_m^1\boldsymbol{b}_m^{1\,T}\boldsymbol{u}_i^0 = \sigma_m q_{i,m} \\ \delta\boldsymbol{u}_{i,m}^0 &= \sigma_m \sum_{j=2, j\neq i}^{N} \frac{\boldsymbol{u}_j^{0\,T}\boldsymbol{b}_m^1\boldsymbol{b}_m^{1\,T}\boldsymbol{u}_i^0}{\lambda_i^0 - \lambda_j^0}\boldsymbol{u}_j^0 = \sigma_m \sum_{j=2, j\neq i}^{N} c_{ji}^{(m)}\boldsymbol{u}_j^0,\end{aligned} \tag{5}$$

with $q_{i,m} = \boldsymbol{u}_i^{0\,T}\boldsymbol{b}_m^1\boldsymbol{b}_m^{1\,T}\boldsymbol{u}_i^0 = [u_i^0(v_{i_m}) - u_i^0(v_{f_m})]^2$, where v_{i_m},v_{f_m} are the endpoints of edge m, $c_{ji}^{(m)} = 0, \forall i = j$ and

$$c_{ji}^{(m)} = \frac{[u_j^0(v_{i_m}) - u_j^0(v_{f_m})][u_i^0(v_{i_m}) - u_i^0(v_{f_m})]}{\lambda_i^0 - \lambda_j^0},\ \forall i \neq j.$$

If a small percentage of edges is altered, the perturbation can be approximated as the sum of the perturbations, i.e. $\delta\lambda_i^0 = \sum_{m\in\mathcal{E}_p} \delta\lambda_{i,m}^0$ and $\delta\boldsymbol{u}_i^0 = \sum_{m\in\mathcal{E}_p} \delta\boldsymbol{u}_{i,m}^0$, where $\mathcal{E}_p$ denotes the set of perturbed edges. In matrix form, we can write the perturbed eigenvectors as $\tilde{\mathbf{U}}_0 = \mathbf{U}_0 + \Delta\mathbf{U}_0$ with

$$\Delta\mathbf{U}_0 = \sum_{m\in\mathcal{E}_p} \sigma_m \mathbf{U}_0\mathbf{C}_m \tag{6}$$

where $\mathbf{C}_m \in \mathbb{R}^{N\times N}$ is a matrix with entries $c_{ji}^{(m)}$. The above formulas are accurate, provided that the overall eigenvalue perturbation does not exceed the eigenvalue gap (see [20] for details). Assuming w.l.o.g. in our analysis that edge m is added with probability p_m and the random variables σ_m are i.i.d., we find

$$\begin{aligned} &E[\delta\lambda_i^0] = \textstyle\sum_{m\in\mathcal{E}_p} p_m q_{i,m}, \quad E[\Delta\mathbf{U}_0] = \displaystyle\sum_{m\in\mathcal{E}_p} p_m \mathbf{U}_0\mathbf{C}_m \\ &E[(\delta\lambda_i^0)^2] = \sum_{m\in\mathcal{E}_p} p_m (q_{i,m})^2 + \sum_{m,n,m\neq n\in\mathcal{E}_p} p_m p_n q_{i,m} q_{i,n}. \end{aligned} \tag{7}$$

In the presence of graph perturbations, the design of the coefficients $\tilde{\boldsymbol{a}}$ of a robust FIR filter reduces to the solution of the following mean squares problem

$$\min_{\tilde{\boldsymbol{a}}\in\mathbb{R}^L} \quad E[\| \tilde{\mathbf{h}} - \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{a}} \|_F^2] \tag{8}$$

where $\tilde{\boldsymbol{h}} = h(\tilde{\boldsymbol{\lambda}_0})$, $h(\lambda) : \mathbb{R} \to \mathbb{R}$ is the frequency response of the filter, $\tilde{\boldsymbol{\Phi}} = [\mathbf{1}, \tilde{\boldsymbol{\lambda}}_0, \ldots, \tilde{\boldsymbol{\lambda}}_0^L]$ and the expectation is taken with respect to the random variable σ_m representing the edge perturbations. The optimal solution to the problem in (8) can then be written as

$$\tilde{\boldsymbol{a}} = E[\tilde{\boldsymbol{\Phi}}^T\tilde{\boldsymbol{\Phi}}]^{-1} E[\tilde{\boldsymbol{\Phi}}^T\tilde{\mathbf{h}}]. \tag{9}$$

By neglecting perturbations of order higher than two and using a first-order Taylor approximation for $h(\lambda)$, we can derive the matrix $\mathbf{G}_1 = E[\tilde{\boldsymbol{\Phi}}^T\tilde{\boldsymbol{\Phi}}]$ and the vector $\boldsymbol{g}_1 = E[\tilde{\boldsymbol{\Phi}}^T\tilde{\mathbf{h}}]$ with entries

$$\begin{aligned} &G_1(i,j) = \textstyle\sum_{k=1}^N w(i+j-r_2, k), \\ &g_1(j,1) = \textstyle\sum_{k=1}^N \; h(\lambda_k^0) w(j-r_1,k) + \\ &d_\lambda h(\lambda_k^0)[(j-r_1)(\lambda_k^0)^{j-r_2} E[(\delta\lambda_k^0)^2] + (\lambda_k^0)^{j-r_1} E[\delta\lambda_k^0]] \end{aligned} \tag{10}$$

where $w(n_1,k) \;:=\; \sum_{n=n_1-2}^{n_1} \binom{n_1}{n} (\lambda_k^0)^n E[(\delta\lambda_k^0)^{n_1-n}]$, $r_1 = 1$ and $r_2 = 2$. In the above analysis we supposed that the eigenvalues are all distinct and the graphs undergoes small perturbations of its edges by preserving its connectivity. However, if the removed edges make the graph disconnected, the multiplicity of the zero eigenvalue becomes greater than one. We can address this case by following the approach proposed in [24], where instead of completely removing critical edges, a low weight is assigned to them so that the first eigenvalues of the nominal and perturbed graphs are very close. If the distance between the subspaces associated with the smallest eigenvalues of the nominal and perturbed graphs falling within a real interval keeps small, the above formulas are still

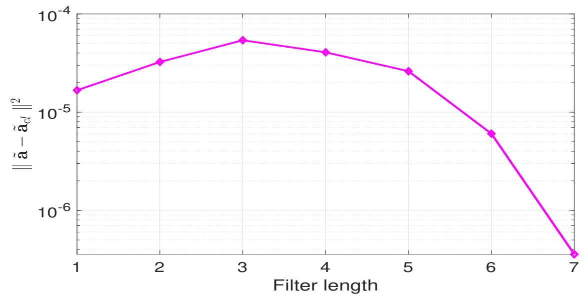

Fig. 1. Average mean squared error versus the filter length.

valid, as we will show in the ensuing section about perturbed simplices. As a numerical test, we considered a graph with two clusters, each composed of 20 nodes whose edges are perturbed with probability $p = 0.01$. In Fig. 1, we report the averaged error in the estimation of the coefficients of a FIR filter with exponential mask, assuming that the perturbation is perfectly known or by using the closed form solutions $\tilde{\boldsymbol{a}}_{cl}$ in (9), (10). It can be noted that the closed form solutions provide an accurate estimation of the filter coefficients for any filter length.

4. FILTERING OVER PERTURBED SIMPLICES

Now we extend the small perturbation analysis to simplicial complexes of order 2, to derive robust filters operating on edge signals. In this case, since the signals are defined over the edges of the complex, the matrix $\mathbf{B}_1$ must be perfectly known. However, there may be uncertainties about the matrix $\mathbf{B}_2$, i.e. about the presence of (filled) triangles. More specifically, if a triangle is removed from (or added to) the complex, the upper Laplacian $\mathbf{L}_1^u$ is perturbed by a term $t_m \boldsymbol{b}_m^2 \boldsymbol{b}_m^{2\,T}$, where $\boldsymbol{b}_m^2$ is the column m of $\mathbf{B}_2$ and $t_m = -1$ (or $t_m = 1$) if the m-th triangle is removed (or added). Let us assume $\mathbf{L}_1^u = \mathbf{U}_1^u \boldsymbol{\Lambda}_u \mathbf{U}_1^{u\,T}$ with $\mathbf{U}_1^u$ containing the eigenvectors associated with the non-zeros eigenvalues λ_i^1, $i = 1, \ldots, r_u$, where r_u is the rank of $\mathbf{L}_1^u$. This perturbation on the upper Laplacian induces a perturbation of its eigenvalues/eigenvectors that can be approximated as

$$\begin{aligned} \delta\lambda_{i,m}^1 &= t_m \boldsymbol{u}_i^{1\,T} \boldsymbol{b}_m^2 \boldsymbol{b}_m^{2\,T} \boldsymbol{u}_i^1 = t_m \left(\sum_{l=1}^3 b_{m_l}^2 u_i^1(m_l)\right)^2 = t_m q_{i,m}^1 \\ \delta\boldsymbol{u}_{i,m}^1 &= t_m \sum_{j=1, j\neq i}^{r_u} \frac{\boldsymbol{u}_j^{1\,T} \boldsymbol{b}_m^2 \boldsymbol{b}_m^{2\,T} \boldsymbol{u}_i^1}{\lambda_i^1 - \lambda_j^1} \boldsymbol{u}_j^1 \end{aligned} \tag{11}$$

where m_l are the indices of $\boldsymbol{b}_m^2$ associated with the edges of the triangle m, $q_{i,m}^1 = \left(\sum_{l=1}^3 b_{m_l}^2 u_i^1(m_l)\right)^2$ represents the square of the circulation (curl) of $\boldsymbol{u}_i^1$ along the triangle m. We can observe from the formulas in (11) that the eigenvector $\boldsymbol{u}_i^1$ does not perturb the eigenpairs if its curl along

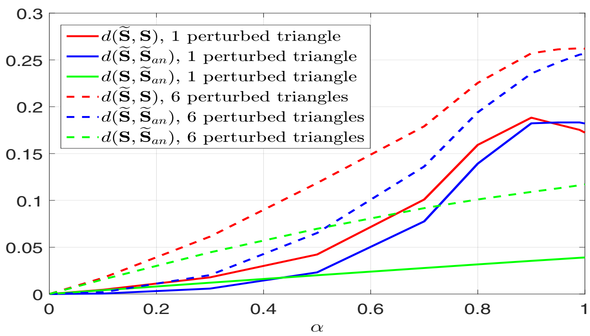

Fig. 2. Subspace distance versus α.

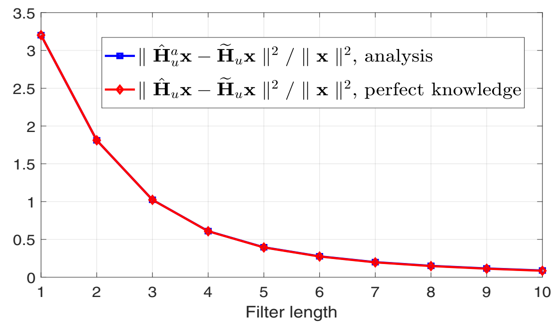

Fig. 3. Filtered signals error versus the filter length.

the altered triangle is zero. Furthermore, for some particular topologies, the perturbation keeps the eigenpairs unaltered by affecting only the dimension of $\ker(\mathbf{L}_1)$ and the multiplicity of the eigenvalues. This behaviour depends on the number N_2 of 2-simplicies (triangles) lower adjacent with each triangle. For instance, if the complex contains a triangle m that shares only one of its vertices (but no sides) with another triangle, the upper Laplacian exhibits a block diagonal structure with a block matrix given by a rank-one 3×3 matrix associated with the triangle m. Then, due to the block structure of $\mathbf{L}_1^u$, if we remove the triangle m, the perturbed Laplacian $\tilde{\mathbf{L}}_1^u$ preserves the same eigenpairs of $\mathbf{L}_1^u$, but the multiplicity of the non-zero eigenvalue associated with the rank one matrix is reduced by one.

Note that, differently from graphs, where only the addition (removal) of some critical edges may alter the kernel dimension of the Laplacian, for simplicial complexes adding (removing) any triangle decreases (increases) the dimension of the kernel of $\mathbf{L}_1$. Then, in general, by removing (adding) triangles with $N_2 > 0$ the perturbation may be large. To circumvent this problem, we can generalize the approach proposed in [24] for graphs to simplicial complexes by assuming that the addition or removal of a triangle is controlled by a real coefficient $0 \leq \alpha \leq 1$, which makes it possible to keep the homology of the complex approximately unaltered. To evaluate this behavior, we consider w.l.o.g. a simplicial complex with an empty kernel, and we (approximately) remove a small number of triangles by assigning them a small positive weight α. In such a case, the perturbation of the upper Laplacian is given by $\Delta \mathbf{L}_1^u = -\sum_{n \in \mathcal{T}_p} \alpha \boldsymbol{b}_n^2 \boldsymbol{b}_n^{2\,T}$ with $\mathcal{T}_p$ being the set of perturbed triangles. Then, we consider the subspaces spanned by the eigenvectors of the nominal upper Laplacian $\mathbf{L}_1^u$, the perturbed one $\tilde{\mathbf{L}}_1^u$ and the upper Laplacian $\tilde{\mathbf{L}}_{1,a}^u$ derived using the formulas in (11) under a small perturbation assumption. Specifically, we denote by $\mathbf{S}$, $\widetilde{\mathbf{S}}$ and $\widetilde{\mathbf{S}}_{an}$ the subspaces spanned by the eigenvectors associated with the non-zero eigenvalues of $\mathbf{L}_1^u$, $\tilde{\mathbf{L}}_1^u$ and $\tilde{\mathbf{L}}_{1,a}^u$ which are lower than a maximum value λ_{max}^1. Then, we consider the distance $d(\mathbf{A}, \mathbf{P}) = \| \sin(\boldsymbol{\theta}) \|_F \, / c$ between the subspaces spanned by the orthonormal matrices $\mathbf{A}, \mathbf{P}$, where $\theta_i = \text{acos}(\sigma_i)$ with σ_i the singular values of the matrix $\mathbf{A}^T \mathbf{P}$ and $c > 0$ normalizes the distance to one. In Fig. 2 we report the distances between the considered subspaces, averaged over 100 random simplicial complexes, versus the weight α, by removing a different number of triangles. Note that, as α gets near to 1, i.e. when the homology of the complex is near to be altered, the subspaces distance remains very small so that the formulas in (5) can still be used by tolerating a small approximation error. Then, we may filter the solenoidal component by finding the coefficients $\tilde{\boldsymbol{a}}_u = \{\tilde{a}_n^u\}_{n=1}^L$ of the FIR filter $\hat{\mathbf{H}}_u = \sum_{n=1}^L \tilde{a}_n^u (\tilde{\mathbf{L}}_1^u)^n$ as solution of the mean squares problem

$$\min_{\tilde{\boldsymbol{a}}_u \in \mathbb{R}^L} \quad E[\| \, \tilde{\mathbf{h}}_u - \tilde{\mathbf{\Phi}}_u \tilde{\boldsymbol{a}}_u \, \|_F^2] \tag{12}$$

with $\tilde{\mathbf{h}}_u = h(\tilde{\boldsymbol{\lambda}}_u)$, $\tilde{\mathbf{\Phi}}_u = [\tilde{\boldsymbol{\lambda}}_u, \ldots, \tilde{\boldsymbol{\lambda}}_u^L]$, where the constant vector $\mathbf{1}$ in the matrix $\tilde{\mathbf{\Phi}}_u$ is omitted to leave out the harmonic component from the filtering [25]. Then, we can still use the formulas in (9) and (10) with $r_1 = 0$ and $r_2 = 1$, using the eigenpairs of the nominal $\mathbf{L}_1^u$ and weighing each removed (added) triangles with a small coefficient α. We tested our robust design, starting from a nominal simplicial complex having all its $T = 80$ triangles filled, and then we removed some triangles, with probability $p = 0.01$, from a subset $\mathcal{T}_p$ composed of three triangles in order to give rise to perturbations of the solenoidal and harmonic subspaces. In Fig. 3 we report the error of the solenoidal filter output with respect to the ideal desired filter $\widetilde{\mathbf{H}}_u$ with known perturbation, versus the filter length. We found the FIR filter in the cases where the perturbed filter $\hat{\mathbf{H}}_u^a$ is derived using the formulas in (10) assuming a perturbation weight $\alpha = 1$ or assuming that the perturbation is perfectly known, $\hat{\mathbf{H}}_u$. It can be noticed that the proposed method provides the same performance of the case where the perturbation is perfectly known.

In conclusion, in this work we analyzed the impact of small perturbations of graphs or simplicial complexes topologies on the design of FIR filters operating on signals residing over such domains. Furthermore, we showed how to deal with perturbations altering the dimension of the signal subspaces.

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