SAMPLING SET SELECTION FOR GRAPH SIGNALS UNDER ARBITRARY SIGNAL PRIORS

Junya Hara[1] *and Yuichi Tanaka*[1,2]

[1]Tokyo University of Agriculture and Technology, Tokyo, Japan
[2]PRESTO, Japan Science and Technology Agency, Saitama, Japan

ABSTRACT

We propose a sampling set selection method for graph signals under arbitrary signal priors. Most approaches of graph signal sampling assume that signals are bandlimited. However, in practical situations, there exist many full-band graph signals like piecewise smooth/constant signals. Our sampling set selection method allows for arbitrary graph signal models as long as they are linear. This can be derived from a *generalized sampling* framework. In contrast to existing works, we focus on the *direct sum condition* between sampling and reconstruction subspaces where the direct sum condition plays a key role for the best possible recovery of sampled signals. We also design a fast sampling set selection algorithm based on the proposed method with the Neumann series approximation. In sampling and recovery experiments, we validate the effectiveness of the proposed method for several graph signal models.

***Index Terms*—** Generalized sampling, subspace prior, smoothness prior, stochastic prior, direct sum condition.

1. INTRODUCTION

Sampling of graph signals is one of the key topics in graph signal processing [1–7]. There are various promising applications of graph signal sampling including sensor placement [6], filter bank designs [8–11], traffic monitoring [12] and semi-supervised learning [13,14]. Most of these works are motivated by building sampling theory on graphs as an analog of the Shannon-Nyquist theorem [15, 16] and its extension [17, 18].

Many studies of graph signal sampling only focus on bandlimited graph signal models [1–3, 3–6, 19]. However, the bandlimited setting is *one of the possible subspaces of deterministic graph signals*. Many other full-band graph signal models are also useful for applications. For example, we often encounter piecewise constant/smooth graph signals [20] that are full-band in graph Fourier domain. Other full-band graph signal subspaces are also under consideration like periodic graph spectrum signals [21]. Unfortunately, sampling for arbitrary signal subspaces has not been studied without a few exceptions [21, 22].

In this paper, we propose a sampling set selection (SSS) algorithm for graph signals under arbitrary graph signal models. It is known that graph signal sampling (and recovery) with arbitrary signal models is well represented by a *generalized sampling* framework [7, 17, 18]. In generalized sampling, a correction operator is placed between sampling and reconstruction operators to compensate signal subspaces. An important condition to design the optimal correction operator is called the *direct sum* (DS) condition. This can be represented with two subspaces: The sampled signal subspace and the reconstruction subspace. Roughly speaking, we seek the optimal sampling set such that the two subspaces are maximally close to each other in the proposed SSS method.

The proposed SSS algorithm is deterministic and selects vertices with a greedy algorithm. For fast implementation, we also develop the sampling set search along with the Neumann series approximation for the matrix inversion in the objective function. In the recovery experiments, we demonstrate that the proposed SSS exhibits the lower MSE than existing SSS algorithms for various signal models.

Notation: Bold lower and upper cases represent a vector and matrix, respectively. We denote an ℓ_2 norm by $\|\cdot\|$. $\mathbf{A}_{\mathcal{X}\mathcal{Y}}$ and $\mathbf{A}_{\mathcal{X}}$ denote submatrices of $\mathbf{A}$ indexed by $\mathcal{X}$ and $\mathcal{Y}$, and $\mathcal{X}$ and $\mathcal{X}$, respectively. $\mathbf{A}_{xy}$ denotes the (x,y)th element of $\mathbf{A}$. $\mathbf{A}^\mathsf{T}$ denotes the transpose of $\mathbf{A}$. $\mathcal{M}^c$ denotes the complement set of $\mathcal{M}$. $|\mathcal{M}|$ represents the cardinality of the set $\mathcal{M}$.

We consider a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ represent sets of vertices and edges, respectively. The number of vertices is $N = |\mathcal{V}|$ unless otherwise specified. The adjacency matrix of $\mathcal{G}$ is denoted by $\boldsymbol{\mathcal{W}}$ where its (m,n)-element $[\boldsymbol{\mathcal{W}}]_{mn} \geq 0$ is the edge weight between the mth and nth vertices; $[\boldsymbol{\mathcal{W}}]_{mn} = 0$ for unconnected vertices. The degree matrix $\mathbf{D}$ is defined as $\mathbf{D} = \mathrm{diag}\,(d_0, d_1, \ldots, d_{N-1})$, where $d_m = \sum_n [\boldsymbol{\mathcal{W}}]_{mn}$ is the mth diagonal element. We use graph Laplacian $\mathbf{L} \coloneqq \mathbf{D} - \boldsymbol{\mathcal{W}}$ as a graph variation operator. A graph signal $\boldsymbol{x} \in \mathbb{R}^N$ is defined as a mapping from the vertex set to the set of real numbers, i.e., $x[n] : \mathcal{V} \to \mathbb{R}$.

The graph Fourier transform (GFT) of $\boldsymbol{x}$ is defined as

$$\hat{x}(\lambda_i) = \langle \boldsymbol{u}_i, \boldsymbol{x} \rangle = \sum_{n=0}^{N-1} u_i[n] x[n], \tag{1}$$

where $\boldsymbol{u}_i$ is the ith column of a unitary matrix $\mathbf{U}$ and it is obtained by the eigendecomposition of the graph Laplacian $\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\mathsf{T}$ with the eigenvalue matrix $\boldsymbol{\Lambda} = \mathrm{diag}\,(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})$. We refer to λ_i as a *graph frequency*.

2. GENERALIZED SAMPLING OF GRAPH SIGNALS

In this section, we briefly review generalized graph signal sampling [21, 23] because the proposed SSS algorithm is designed based on this framework.

2.1. Framework of Graph Signal Sampling

First of all, we define a vertex domain sampling operator as follows:

Definition 1 (Vertex domain sampling)**.** *Let* $\mathbf{I}_{\mathcal{M}\mathcal{V}} \in \{0,1\}^{K \times N}$ *be the submatrix of the identity matrix indexed by* $\mathcal{M} \subset \mathcal{V}$ $(|\mathcal{M}| = K)$

This work is supported in part by JST PRESTO (JPMJPR1935) and JSPS KAKENHI (20H02145).

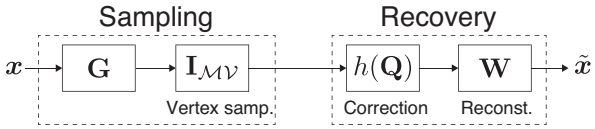


Fig. 1. Generalized sampling framework. The dotted left and right boxes are sampling and recovery phases, respectively.

and $\mathcal{V}$. The sampling operator is defined by

$$\mathbf{S}^\mathsf{T} \coloneqq \mathbf{I}_{\mathcal{MV}}\mathbf{G}, \tag{2}$$

where $\mathbf{G} \in \mathbb{R}^{N\times N}$ is an arbitrary graph filter. A sampled graph signal is thus given by $\boldsymbol{y} = \mathbf{S}^\mathsf{T}\boldsymbol{x}$.

Note that (2) is a natural extension of the sampling operator for the standard time domain sampling to the graph setting. In the graph setting, the reconstruction operator depends on the chosen sampling operator $\mathbf{S}^\mathsf{T}$ because $\mathbf{I}_{\mathcal{MV}}$ is not unique in general.

The sampling and recovery framework based on generalized sampling is illustrated in Fig. 1. The input signal $\boldsymbol{x}$ is sampled by $\mathbf{S}^\mathsf{T}$, and the sample is corrected by a *correction operator* $h(\mathbf{Q}) : \mathbb{R}^{N\times K} \mapsto \mathbb{R}^{K\times K}$. Note that $\mathbf{Q}$ plays the key role of this paper, which is derived from signal priors. Hereafter, we refer to $\mathbf{Q}$ as a *prior operator*. After correction, the corrected samples are reconstructed by a reconstruction operator $\mathbf{W} \in \mathbb{R}^{N\times K}$.[1] This framework explains many existing graph signal sampling, including sampling for bandlimited signals [7].

As a result, the reconstructed graph signal is represented as follows:

$$\tilde{\boldsymbol{x}} = \mathbf{W}h(\mathbf{Q})\boldsymbol{y} = \mathbf{W}h(\mathbf{Q})\mathbf{S}^\mathsf{T}\boldsymbol{x}. \tag{3}$$

In this scenario, the recovery problem turns out to be seeking the best possible $h(\mathbf{Q})$ based on signal priors.

In the following, we introduce representative signal models and recovery methods corresponding to the models (i.e., designs of $h(\mathbf{Q})$ and $\mathbf{W}$).

2.2. Graph Signal Models and Signal Recovery

In this paper, we consider three representative signal priors:

Subspace prior [21]: The generation subspace $\mathcal{A}$ spanned by $\mathbf{A}$ is known, i.e., $\mathcal{A} = \{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{x} = \mathbf{A}\boldsymbol{d}\}$, where $\boldsymbol{d}$ is an expansion coefficient. The well-known bandlimited setting is categorized into this prior.

Smoothnes prior [21]: The signal energy that is measured by the smoothness measuring function (i.e., high-pass filter) $\mathbf{V}$ is bounded by $\sigma \in \mathbb{R}_+$, i.e., $\mathcal{V} = \{\boldsymbol{x} \in \mathbb{R}^N : \|\mathbf{V}\boldsymbol{x}\| \le \sigma\}$.

Stochastic prior [23]: The covariances of graph signals and noise, $\mathbf{\Gamma}_x$ and $\mathbf{\Gamma}_\eta$, are known.

We will describe the recovery problem for each prior.

Suppose that $\mathbf{S}^\mathsf{T}$ is given. The correction operator $h(\mathbf{Q})$ is often designed based on three well-known criteria as follows.

Least squares (LS) criterion: The LS criterion seeks the signal that minimizes the error-in-sample, i.e.,

$$\tilde{\boldsymbol{x}} = \underset{\boldsymbol{x}\in\mathcal{T}}{\arg\min} \|\mathbf{S}^\mathsf{T}\boldsymbol{x} - \boldsymbol{y}\|^2. \tag{4}$$

[1]For simplicity, we suppose that $\mathbf{W}^\mathsf{T}\mathbf{W}$ is invertible.

Table 1. Filter designs for generalized graph signal sampling. SS, SM and ST denote subspace, smoothness, and stochastic priors, respectively. The following definitions are used: $\mathbf{P} = (\mathbf{W}^\mathsf{T}\mathbf{W})^{-1}\mathbf{W}^\mathsf{T}\mathbf{Q}$, $\widehat{\mathbf{W}} = \mathbf{W}(\mathbf{W}^\mathsf{T}\mathbf{V}^\mathsf{T}\mathbf{V}\mathbf{W})^{-1}\mathbf{W}^\mathsf{T}\mathbf{S}$, $\widetilde{\mathbf{W}} = (\mathbf{V}^\mathsf{T}\mathbf{V})^{-1}\mathbf{S}$ and $\overline{\mathbf{W}} = \mathbf{\Gamma}_x\mathbf{S}$. LS and MX solutions coincide with each other for the unconstrained case.

Prior	Criteria	Predefined		Unconstrained		
		$\mathbf{Q}$	$h(\mathbf{Q})$	$\mathbf{Q}$	$h(\mathbf{Q})$	$\mathbf{W}$
SS	LS	$\mathbf{W}$	$(\mathbf{S}^\mathsf{T}\mathbf{Q})^{-1}$	$\mathbf{A}$	$(\mathbf{S}^\mathsf{T}\mathbf{Q})^{-1}$	$\mathbf{A}$
	MX	$\mathbf{A}$	$\mathbf{P}(\mathbf{S}^\mathsf{T}\mathbf{Q})^{-1}$			
SM	LS	$\widehat{\mathbf{W}}$	$(\mathbf{S}^\mathsf{T}\mathbf{Q})^{-1}$	$\widetilde{\mathbf{W}}$	$(\mathbf{S}^\mathsf{T}\mathbf{Q})^{-1}$	$\widetilde{\mathbf{W}}$
	MX	$\widetilde{\mathbf{W}}$	$\mathbf{P}(\mathbf{S}^\mathsf{T}\mathbf{Q})^{-1}$			
ST	MMSE	$\overline{\mathbf{W}}$	$\mathbf{P}(\mathbf{S}^\mathsf{T}\mathbf{Q}+\mathbf{\Gamma}_\eta)^{-1}$	$\overline{\mathbf{W}}$	$(\mathbf{S}^\mathsf{T}\mathbf{Q}+\mathbf{\Gamma}_\eta)^{-1}$	$\overline{\mathbf{W}}$

where $\mathcal{T}$ is the set of signals under consideration.

Minimax (MX) criterion: The MX criterion corresponds to the reconstruction that minimizes the worst-case error, i.e.,

$$\tilde{\boldsymbol{x}} = \underset{\boldsymbol{x}}{\arg\min} \max_{\tilde{\boldsymbol{x}}\in\mathcal{T},\mathbf{S}^\mathsf{T}\tilde{\boldsymbol{x}}=\boldsymbol{y}} \|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|^2. \tag{5}$$

The LS and MX criteria are used for signals under subspace and smoothness priors.

Minimum MSE (MMSE) criterion: When $\boldsymbol{x}$ is a random process under the stochastic prior, LS and MX may be ill-posed. Instead, the MMSE criterion seeks the signal that minimizes the MSE, i.e.,

$$\tilde{\boldsymbol{x}} = \underset{\tilde{\boldsymbol{x}}}{\arg\min}\, \mathbb{E}[\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|^2]. \tag{6}$$

Along with the design of $h(\mathbf{Q})$, we may consider the following two cases for $\mathbf{W}$: 1) $\mathbf{W}$ is determined before sampling and recovery (predefined case), or 2) $\mathbf{W}$ is optimally chosen (unconstrained case). Table 1 summarizes $\mathbf{Q}$, $h(\mathbf{Q})$, and $\mathbf{W}$ (for the unconstrained case) for the three priors. Their detailed derivations are omitted and can be found in [7, 17, 24].

In the following, we describe the important condition for the best possible recovery, i.e., requirements for $h(\mathbf{Q})$. In fact, all the recovery conditions in Table 1 are explained by the characteristics of $\mathbf{S}$ and $\mathbf{Q}$.

2.3. Direct Sum Condition

In any priors and recovery criteria mentioned above, $h(\mathbf{Q})$ contains matrix inversion (see Table 1). Hence, for the best possible recovery, we need to consider whether the corresponding matrix including $\mathbf{Q}$ is invertible. This condition is represented as the following DS condition[2]:

Definition 2 (Direct sum condition [17])**.** *Let $\mathcal{S}$ and $\mathcal{Q}$ be the range spaces of $\mathbf{S}$ and $\mathbf{Q}$, respectively. When $\mathcal{S}^\perp$ (the orthogonal complement of $\mathcal{S}$) and $\mathcal{Q}$ span the entire Euclidean spaces $\mathbb{R}^N$ and intersect only at the origin, the direct sum condition is satisfied, i.e., $\mathbb{R}^N = \mathcal{Q}\oplus\mathcal{S}^\perp$. If so, $\mathbf{S}^\mathsf{T}\mathbf{Q}$ is invertible. This condition is equivalent that any singular value of $\mathbf{S}^\mathsf{T}\mathbf{Q}$ is nonzero, i.e., $\sigma_i(\mathbf{S}^\mathsf{T}\mathbf{Q}) \neq 0$ for all i.*

[2]When the DS condition is not satisfied, the inverse in $h(\mathbf{Q})$ can be replaced by the Moore–Penrose pseudo inverse, which corresponds to least squares or minimal norm solutions.

In the next section, we formulate an optimization problem for SSS based on the DS condition in Definition 2. Note that our problem does not depend on the choice of $\mathbf{Q}$. Therefore, the proposed SSS is applicable for arbitrary signal priors as long as $\mathbf{Q}$ is appropriately chosen.

3. PROPOSED SAMPLING SET SELECTION

In this section, we introduce the proposed SSS algorithm for arbitrary graph signal priors. As mentioned in the previous section, the recovery performance essentially depends on the correction filter $h(\mathbf{Q})$, and it is based on the DS condition. Therefore, we consider to select vertices so that the vertex subset maximally satisfies the DS condition.

3.1. Problem Formulation

First, we define the *quality* of sampling sets through the DS condition. Suppose that the prior operator $\mathbf{Q}$ (Table 1) is given. Since the DS condition is satisfied if all singular values of $\mathbf{S}^\mathsf{T}\mathbf{Q}$ are nonzero, it is natural to consider to maximize $|\det(\mathbf{S}^\mathsf{T}\mathbf{Q})|$ as a measure of the sampling set quality because $|\det(\mathbf{S}^\mathsf{T}\mathbf{Q})| = \prod_{i=1}^{i=N} \sigma_i(\mathbf{S}^\mathsf{T}\mathbf{Q})$ [25].

Let us define $\mathbf{Z}$ as follows.

$$\mathbf{Z} \coloneqq \mathbf{G}^\mathsf{T}\mathbf{Q}\mathbf{Q}^\mathsf{T}\mathbf{G}. \tag{7}$$

With $\mathbf{Z}$, we consider the following problem as a sampling set selection:

$$\mathcal{M}^* = \arg\max_{\mathcal{M}\subset\mathcal{V}} \det(\mathbf{Z}_\mathcal{M}). \tag{8}$$

Since $\det(\mathbf{Z}_\mathcal{M}) = \det(\mathbf{S}^\mathsf{T}\mathbf{Q}\mathbf{Q}^\mathsf{T}\mathbf{S}) = |\det(\mathbf{S}^\mathsf{T}\mathbf{Q})|^2$, (8) considers the DS condition directly. It is different from the other approaches based on the bandlimited assumption [1,5]. Note that this cost function is applicable for any signal model (including the bandlimited one).

The direct maximization of (8) is combinatorial and is practically intractable. Therefore, we use a greedy algorithm like those used in the previous SSS methods. The greedy selection is formulated as follows.

$$y^* = \arg\max_{y\in\mathcal{M}^c} \det(\mathbf{Z}_{\mathcal{M}\cup\{y\}}). \tag{9}$$

That is, we select a vertex y^* one by one.

Here, we consider the geometric relationship between our objective function and the DS condition. The principal angle between $\mathcal{S}$ and $\mathcal{Q}$ is defined by [26][3]

$$\cos^2(\mathcal{S},\mathcal{Q}) \coloneqq \frac{\det(\mathbf{S}^\mathsf{T}\mathbf{Q}\mathbf{Q}^\mathsf{T}\mathbf{S})}{\det(\mathbf{S}^\mathsf{T}\mathbf{S})\det(\mathbf{Q}^\mathsf{T}\mathbf{Q})}. \tag{10}$$

The DS condition in Definition 2 can be expressed by $\cos^2(\mathcal{S}^\perp,\mathcal{Q}) \neq 1$. Since $\cos^2(\mathcal{S}^\perp,\mathcal{Q}) = 1-\cos^2(\mathcal{S},\mathcal{Q})$, the DS condition is satisfied if $\mathcal{S}$ is close to $\mathcal{Q}$. In other words, the maximization of $\det(\mathbf{Z}_\mathcal{M})$ implies the maximum separation between $\mathcal{S}^\perp$ and $\mathcal{Q}$.

3.2. Proposed Algorithm

The determinant term in (9) requires $O(|\mathcal{M}|!)$ computational complexity in the worst case if we perform to compute it straightforwardly [25]. Therefore, we rewrite (9) into a more efficient form.

[3]For simplicity, we suppose that $\mathbf{S}^\mathsf{T}\mathbf{S}$ and $\mathbf{Q}^\mathsf{T}\mathbf{Q}$ are invertible.

Suppose that $\mathrm{rank}(\mathbf{Z}) \geq K$ and $\det(\mathbf{Z}_\mathcal{M}) > 0$. By applying the Schur determinant formula [27] to (9), it can be rewritten as

$$\begin{aligned} y^* &= \arg\max_{y\in\mathcal{M}^c} \det(\mathbf{Z}_{\mathcal{M}\cup\{y\}}) \\ &= \arg\max_{y\in\mathcal{M}^c} \det(\mathbf{Z}_\mathcal{M})\cdot(\mathbf{Z}_{y,y} - \mathbf{Z}_{y,\mathcal{M}}(\mathbf{Z}_\mathcal{M})^{-1}\mathbf{Z}_{\mathcal{M},y}) \\ &= \arg\max_{y\in\mathcal{M}^c} \mathbf{Z}_{y,y} - \mathbf{Z}_{y,\mathcal{M}}(\mathbf{Z}_\mathcal{M})^{-1}\mathbf{Z}_{\mathcal{M},y}, \end{aligned} \tag{11}$$

where we omit the multiplication with $\det(\mathbf{Z}_\mathcal{M})$ in the third equivalence because it does not depend on y^*. For the first selection, i.e., $\mathcal{M}=\emptyset$, the vertex y maximizing $\mathbf{Z}_{y,y}$ is chosen because it immediately follows (9).

Still, (11) is computationally expensive due to the matrix inversion, which typically requires $O(|\mathcal{M}|^3)$ computational complexity. To alleviate this, we utilize the Neumann series for $(\mathbf{Z}_\mathcal{M})^{-1}$, i.e.,

$$(\mathbf{Z}_\mathcal{M})^{-1} = \alpha\sum_{k=0}^{\infty}(\mathbf{I}-\alpha\mathbf{Z}_\mathcal{M})^k, \tag{12}$$

where α is chosen such that $\|\mathbf{I}-\alpha\mathbf{Z}_\mathcal{M}\| \leq 1$. Let us define $\boldsymbol{\varepsilon}_y \coloneqq (\mathbf{Z}_\mathcal{M})^{-1}\mathbf{Z}_{\mathcal{M},y}$ in (11). Then, substitution of (12) into $\boldsymbol{\varepsilon}_y$ leads to

$$\boldsymbol{\varepsilon}_y = \alpha\sum_{k=0}^{\infty}(\mathbf{I}-\alpha\mathbf{Z}_\mathcal{M})^k\mathbf{Z}_{\mathcal{M},y} = \alpha\sum_{k=0}^{\infty}(\mathbf{I}-\alpha\mathbf{Z}_\mathcal{M})^k\mathbf{Z}_\mathcal{M}\boldsymbol{\delta}_y, \tag{13}$$

where $\boldsymbol{\delta}_y$ is the Kronecker delta centered at the vertex y.

In this paper, we truncate (13) until the mth term. We can update (13) by the following rule [17]:

$$\begin{aligned} \boldsymbol{\varepsilon}_y^{m+1} &= \alpha\sum_{k=0}^{m+1}(\mathbf{I}-\alpha\mathbf{Z}_\mathcal{M})^k\mathbf{Z}_\mathcal{M}\boldsymbol{\delta}_y \\ &= \boldsymbol{\varepsilon}_y^0 + (\mathbf{I}-\alpha\mathbf{Z}_\mathcal{M})\boldsymbol{\varepsilon}_y^m, \end{aligned} \tag{14}$$

where $\boldsymbol{\varepsilon}_y^0 = \alpha\mathbf{Z}_\mathcal{M}\boldsymbol{\delta}_y$. Since $\boldsymbol{\varepsilon}_y^\infty = \boldsymbol{\varepsilon}_y = (\mathbf{Z}_\mathcal{M})^{-1}\mathbf{Z}_{\mathcal{M},y}$, the error $\boldsymbol{\varepsilon}_y^0 - \alpha\mathbf{Z}_\mathcal{M}\boldsymbol{\varepsilon}_y^m$ converges to zero.

For a fast convergence, we need to select an optimal α in (14) in each update. In particular, we seek the best step size at the $(m+1)$th iteration as follows:

$$\begin{aligned} \alpha_{m+1}^* &= \arg\min_\alpha \|\boldsymbol{\varepsilon}_y^{m+1} - \boldsymbol{\varepsilon}_y^m\|^2 \\ &= \arg\min_\alpha \|(\mathbf{I}-\alpha\mathbf{Z}_\mathcal{M})\boldsymbol{\varepsilon}_y^m\|^2. \end{aligned} \tag{15}$$

The solution of (15) results in

$$\alpha_{m+1}^* = \frac{(\boldsymbol{\varepsilon}_y^m)^*\mathbf{Z}_\mathcal{M}\boldsymbol{\varepsilon}_y^m}{\|\mathbf{Z}_\mathcal{M}\boldsymbol{\varepsilon}_y^m\|^2}. \tag{16}$$

In the initial step, α_0^* is arbitrarily set such that $\|\mathbf{I}-\alpha_0^*\mathbf{Z}_\mathcal{M}\| \leq 1$. While α is tuned in the adaptive manner, (14) converges to $(\mathbf{Z}_\mathcal{M})^{-1}\mathbf{Z}_{\mathcal{M},y}$ [17].

We present the proposed sampling set selection method in Algorithm 1. When $\mathbf{Z}$ is diagonalizable by $\mathbf{U}$, we can approximate $\mathbf{Z}$ by the Chebyshev polynomial approximation (CPA) without directly calculating $\mathbf{U}$ [28]. This could result in a further speed-up of the algorithm.

Algorithm 1: Greedy SSS for arbitrary signal prior

Input: $\mathbf{Z}, \mathcal{M} = \emptyset, K, m = 0$

while $|\mathcal{M}| < K$ **do**

  Compute $\boldsymbol{\varepsilon}_y^0 = \alpha_0^* \mathbf{Z}_{\mathcal{M}} \boldsymbol{\delta}_y$

  while $\|\boldsymbol{\varepsilon}_y^0 - \alpha_m^* \mathbf{Z}_{\mathcal{M}} \boldsymbol{\varepsilon}_y^m\| \geq \beta$ *for some* $\beta > 0$ **do**

    $\alpha_{m+1}^* \leftarrow \frac{(\boldsymbol{\varepsilon}_y^m)^* \mathbf{Z}_{\mathcal{M}} \boldsymbol{\varepsilon}_y^m}{\|\mathbf{Z}_{\mathcal{M}} \boldsymbol{\varepsilon}_y^m\|^2}$

    $\boldsymbol{\varepsilon}_y^{m+1} \leftarrow \boldsymbol{\varepsilon}_y^0 + (\mathbf{I} - \alpha_{m+1}^* \mathbf{Z}_{\mathcal{M}}) \boldsymbol{\varepsilon}_y^m$

    $m \leftarrow m + 1$

  $y^* \rightarrow \arg\max_{y \in \mathcal{M}^c} \mathbf{Z}_{yy} - \mathbf{Z}_{y,\mathcal{M}} \boldsymbol{\varepsilon}_y^m$

  $\mathcal{M} \leftarrow \mathcal{M} \cup \{y\}$

Output: $\mathcal{M}$

4. RECOVERY EXPERIMENTS

In this section, we demonstrate the effectiveness of the proposed method by sampling graph signals with several graph signal models. We perform recovery experiments on a random sensor graph with $N = 256$ for six graph signal models:

Subspace prior

- Bandlimited (BL) graph signals with the bandwidth $\mathcal{B} = \{1, \ldots, K/4\}$ [2].
- Periodic graph spectrum (PGS) signals with the generator response $A(\lambda) = \exp(-1.5\lambda/\lambda_{\max})$ [21].
- Piecewise constant (PWC) signals with five pieces [20].

Smoothness prior

- Gaussian Markov random field (GMRF) signals with the power spectrum $\Gamma_x(\lambda) \propto 1/(\lambda + \epsilon)$ [29].
- Piecewise linear (PWL) signals with the density $p = K/N$ [13].

Stochastic prior

- GMRF with $\Gamma_x(\lambda) \propto 1/(\lambda + \epsilon)$.
- Bandpass (BP) graph signals with power spectrum $\Gamma_x(\lambda) \propto \exp(\{(2\lambda - \lambda_{\max})/(\sqrt{\lambda + \epsilon})\}^2)$.

For the subspace priors, we set to $\boldsymbol{d} \sim \mathcal{N}(1, 1)$. The sampling ratio is set to $K = N/8$. Further, the spectral response of the sampling filter $\mathbf{G}$ is defined by

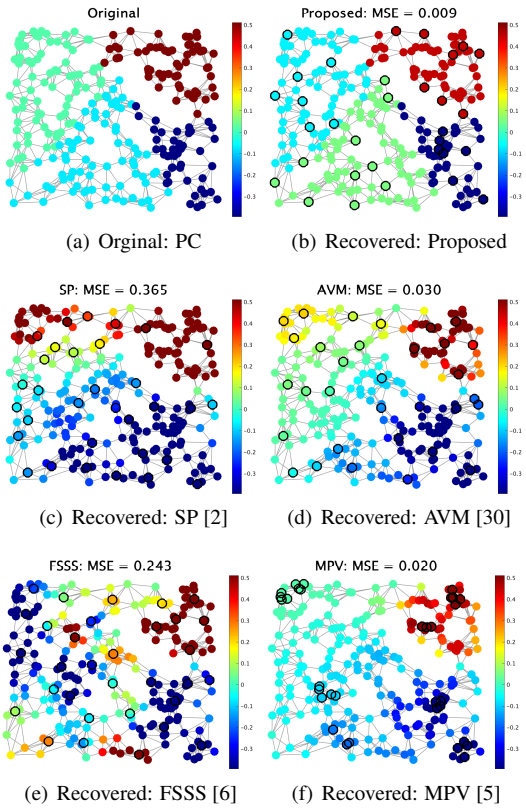
$$\hat{g}(\lambda) := \begin{cases} 1 & \lambda \leq 1 \\ 2 - 2\lambda/(\lambda_{\max} + \epsilon) & \lambda > 1. \end{cases} \tag{17}$$

Under the stochastic prior, noise conforms to $\boldsymbol{\eta} \sim \mathcal{N}(0, 0.2)$. We set $\epsilon = 0.1$.

The proposed method is compared with four existing methods [2, 5, 6, 30]. Table 2 summarizes averaged MSEs in decibels for 100 runs. Although we only show the results in the unconstrained case due to the limitation of space, the predefined case also presents the same tendency. It is observed that the proposed method outperforms the alternative methods for all signal models. In particular, the reconstructed signals with the subspace priors are perfect recovery in machine precision. While the existing approaches based on the bandlimited model show perfect recovery for bandlimited signals, they do not perfectly recover full-band signals like PGS and PWC signals. For the other priors, the proposed method also presents consistently lower reconstruction errors than the other methods.

Table 2. Average reconstruction MSEs (in decibels) for 100 independent runs.

Signal Model	Subspace			Smoothness		Stochastic	
	BL	PGS	PWC	GMRF	PWL	GMRF w/ noise	BP w/ noise
Proposed	**-733.9**	**-632.3**	**-684.5**	**-22.88**	**-18.12**	**-18.74**	**-25.43**
MPV [5]	-718.0	-12.57	-7.29	-18.13	-16.68	-17.64	-24.01
AVM [30]	-724.3	-12.88	-9.33	-18.66	-17.70	-18.49	-23.10
SP [2]	-718.9	-11.65	-6.88	-17.19	-14.81	-15.51	-21.45
FSSS [6]	-703.3	-14.26	-19.46	-21.98	-14.81	-13.89	-14.25



(a) Orginal: PC (b) Recovered: Proposed

(c) Recovered: SP [2] (d) Recovered: AVM [30]

(e) Recovered: FSSS [6] (f) Recovered: MPV [5]

Fig. 2. Examples of recovery for PWC graph signals on sensor graphs. Black circles represent selected vertices. Colors on vertices represent magnitudes of signal values.

While most existing methods rely on bandlimited models, this paper first presents perfect recovery with vertex domain sampling beyond the bandlimited signal model. Examples of recovered signals are visualized in Fig. 2 along with the selected sampling sets.

5. CONCLUSION

In this paper, a SSS algorithm for arbitrary graph signal models is proposed. In contrast to the existing approach, we directly focus on the DS condition for signal recovery based on the generalized sampling framework. We seek the optimal sampling set such that the DS condition is maximally satisfied. The proposed greedy SSS algorithm utilizes the Neumann series approximation for fast implementation. The recovery experiments demonstrate that the proposed method is effective for various graph signal models.

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