RECURRENT DESIGN OF PROBING WAVEFORM FOR SPARSE BAYESIAN LEARNING BASED DOA ESTIMATION

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ABSTRACT

Direction-of-arrival (DOA) estimation can be represented as a sparse signal recovery problem and effectively solved by sparse Bayesian learning (SBL). For the DOA estimation in active sensing, the SBL-based estimation error is related to the transmitted probing waveform. Therefore, it is expected to improve the estimation by waveform optimization. In this paper, we propose a recurrent scheme of waveform design by sequentially leveraging on the previous-round SBL estimates. Within this scheme, we formulate the waveform design problem as a minimization of the SBL estimation variance, which is nonconvex and then solved by a majorization-minimization based algorithm. The simulations demonstrate the efficacy of the proposed design scheme in terms of avoiding incorrect detection and accelerating the DOA estimation convergence. Further, the results indicate that the waveform design is essentially a beampattern shaping methodology.

Index Terms— Sparse Bayesian learning (SBL), DOA, waveform design, majorization-minimization (MM)

1. INTRODUCTION

Direction-of-arrival (DOA) estimation has been a classical research topic in signal processing over the past four decades [1], and gained renewed research interest in the next generation of wireless communications, especially for the massive MIMO [2] and integrated sensing and communications [3, 4]. Sparse representation [5] has been introduced to interpret the DOA estimation as a sparse signal recovery problem. This approach exhibits several advantages over the conventional ones, such as robustness to noise and mild requirements on number of snapshots. Among the investigated sparse signal recovery techniques, sparse Bayesian learning (SBL) [6, 7] is one of the most attractive solutions, which includes the well known ℓ_1-norm minimization as a special case when the maximum a $posteriori$ (MAP) estimate is adopted with a Laplace signal prior [8]. Further, it was shown theoretically in [9] that the SBL can smooth out some bad local minima and is amenable to handle highly correlated dictionary matrices. Therefore, the SBL-based DOA estimation has been studied in many works [10–13].

In many applications such as radar and sonar, an active sensing system sends a probing waveform and receives the reflected echoes from the passive targets for the DOA estimation [14]. In these cases, the design of probing waveform plays a crucial role as it decides to a large degree how good the quality of the echoes will be. Specifically, from the perspective of SBL, the dictionary matrix is a function of the probing waveform, and consequently, the variance of the estimate will be related to the probing waveform. Therefore, an interesting question arises in whether we can improve the SBL-based DOA estimation via designing the probing waveform.

In this paper, we answer this question with our initial but positive results. To adapt to the SBL framework, we first propose an iterative scheme, in which the current probing waveform is designed based on the previous round SBL estimate. The waveform design problem is formulated as a minimization of the SBL estimation variance. For this nonconvex problem, we propose a novel and efficient algorithm based on majorization-minimization (MM) [15], where a closed form solution is obtained at each iteration and the monotonic convergence is guaranteed. Supported by the simulation results, the proposed recurrent design improves the SBL-based DOA performance sequentially, and can effectively avoid false peaks and accelerate the angle convergence.

Notation: We use capital bold face letters for matrices while keeping small bold and normal face letters for vectors scalars, respectively. The superscript ^{H} and ^{T} are denoted to the Hermitian and standard transpose, respectively. The $N \times N$ identity matrix is denoted by $\boldsymbol{I}_N$, and $\mathbb{C}^{m\times n}$ denotes an m by n dimensional complex space. The notations $\mathrm{Re}\,(\cdot)$, $\mathrm{diag}\,(\cdot)$ and $\mathrm{Tr}\,(\cdot)$ represent the real part, diagonalization, and trace, respectively. $\arg\,(\cdot)$ is used to denote the argument of a complex value. $\boldsymbol{A} \succeq \boldsymbol{0}$ means $\boldsymbol{A}$ is positive semidefinite.

2. SBL BASED DOA ESTIMATION AND PROBLEM FORMULATION

2.1. System Model for Sparse DOA Estimation

Consider a co-located MIMO radar with N transmit and M receive antennas, illuminating K narrowband far-field sources with the DOAs being $\{\tilde{\theta}_k\}_{k=1}^{K}$. Then, the received signal can be written as

$$\boldsymbol{R}=\sum_{k=1}^{K}\boldsymbol{a}_r(\tilde{\theta}_k)x_k\boldsymbol{a}_t(\tilde{\theta}_k)^T\tilde{\boldsymbol{S}}+\boldsymbol{N}, \tag{1}$$

where x_k represents the complex amplitude, $\tilde{\boldsymbol{S}}\in\mathbb{C}^{N\times T}$ is the probing waveform matrix with snapshot number being T, $\boldsymbol{N}$ represents the Gaussian noise, and $\boldsymbol{a}_t(\tilde{\theta}_k)$ and $\boldsymbol{a}_r(\tilde{\theta}_k)$ are the transmitting and receiving steering vectors, respectively.

To cast the DOA estimation as a sparse representation problem, we define an L-element grid $\boldsymbol{\theta}=\{\theta_\ell\}_{\ell=1}^{L}$ that uniformly covers the angular domain. If the grid is fine enough such that all the true DOAs lie on the grid[1], (1) can be rewritten as

$$\boldsymbol{r}=\sum_{\ell=1}^{L}x_\ell\left(\boldsymbol{I}_T\otimes\left(\boldsymbol{a}_r\left(\theta_\ell\right)\boldsymbol{a}_t\left(\theta_\ell\right)^T\right)\right)\boldsymbol{s}+\boldsymbol{n} \tag{2}$$

with $\boldsymbol{s}=\text{vec}(\tilde{\boldsymbol{S}})$ and $\boldsymbol{n}=\text{vec}\left(\boldsymbol{N}\right)$, or equivalently,

$$\boldsymbol{r}=\boldsymbol{H}\boldsymbol{x}+\boldsymbol{n}, \tag{3}$$

where $\boldsymbol{x}=\left[x_1,\ldots,x_L\right]^T$, $\boldsymbol{H}=\boldsymbol{A}\left(\boldsymbol{I}_L\otimes\boldsymbol{s}\right)$ with

$$\boldsymbol{A}=\left[\boldsymbol{I}_T\otimes\left(\boldsymbol{a}_r\left(\theta_1\right)\boldsymbol{a}_t\left(\theta_1\right)^T\right),\ldots,\boldsymbol{I}_T\otimes\left(\boldsymbol{a}_r\left(\theta_L\right)\boldsymbol{a}_t\left(\theta_L\right)^T\right)\right]. \tag{4}$$

Note that there are only K ($\ll L$) nonzero elements in $\boldsymbol{x}$, i.e., the vector $\boldsymbol{x}$ is sparse. Thus, the problem is to recover the sparse $\boldsymbol{x}$ and hence the DOA by using (2).

Within the framework of SBL, $\boldsymbol{x}$ is assumed to follow a zero mean Gaussian prior distribution such that

$$p\left(\boldsymbol{x}|\boldsymbol{\alpha}\right)=\mathcal{CN}\left(\boldsymbol{x}|\boldsymbol{0},\text{diag}\left(\boldsymbol{\alpha}\right)\right), \tag{5}$$

where $\boldsymbol{\alpha}=\left[\alpha_1,\ldots,\alpha_L\right]^T$ with α_ℓ being the variance of x_ℓ (i.e. the power of the source from the ℓ-th angle) and modeled as independent inverse Gamma distribution

$$p\left(\boldsymbol{\alpha}\right)=\prod_{\ell=1}^{L}\mathcal{IG}\left(\alpha_\ell;a,b\right), \tag{6}$$

where the parameters a and b are fixed to small positive values (e.g. $a=b=10^{-10}$) to make these priors non-informative.

As $\boldsymbol{n}$ is the Gaussian noise, we have

$$p\left(\boldsymbol{n}\right)=\prod_{m=1}^{M}\mathcal{CN}\left(n_m|0,\beta^{-1}\right), \tag{7}$$

where β follows a Gamma distribution $p\left(\beta\right)=\mathcal{G}\left(\beta;a,b\right)$.

Based on the above probabilistic assumptions and applying the variational Bayesian inference (VBI) [17], we have

$$p\left(\boldsymbol{x}|\boldsymbol{r},\boldsymbol{\alpha},\beta\right)=\mathcal{CN}\left(\boldsymbol{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}\right), \tag{8}$$

[1]For the case of "off the grid", the true DOAs can be well approximated by the linearization using the grid angles [16].

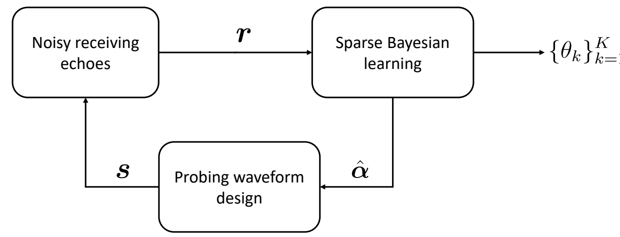

Fig. 1: Diagram of the recurrent waveform design in SBL-based DOA estimation.

where $\boldsymbol{\mu}=\hat{\beta}\boldsymbol{\Sigma}\boldsymbol{H}^H\boldsymbol{r}$ and

$$\boldsymbol{\Sigma}=\left(\hat{\beta}\boldsymbol{H}^H\boldsymbol{H}+\text{diag}\left(\hat{\boldsymbol{\alpha}}\right)^{-1}\right)^{-1}. \tag{9}$$

The parameters $\hat{\boldsymbol{\alpha}}=\left[\hat{\alpha}_1,\ldots,\hat{\alpha}_L\right]^T$ and $\hat{\beta}$ denote the expectations of $\boldsymbol{\alpha}$ and β, respectively, and iteratively update by

$$\begin{cases}\hat{\beta}=\frac{a+M}{\|\boldsymbol{r}-\boldsymbol{H}\boldsymbol{\mu}\|^2+\text{Tr}(\boldsymbol{H}\boldsymbol{\Sigma}\boldsymbol{H}^H)+b}\\ \hat{\alpha}_\ell=\frac{b+[\boldsymbol{\mu}\boldsymbol{\mu}^H+\boldsymbol{\Sigma}]_{\ell,\ell}}{a+1}, \qquad \forall\ell=1,\ldots,L,\end{cases} \tag{10}$$

of which the derivation follows [17] directly. Numerically, the estimated $\hat{\alpha}_\ell$ in SBL can be very close to zero. In the context of DOA estimation, it indicates that the corresponding angle θ_ℓ is not the DOA.

2.2. Problem Formulation for Waveform Design

The estimated $\hat{\boldsymbol{\alpha}}$ is the prior knowledge of $\boldsymbol{\alpha}$ to some degree for the next-round DOA estimation. Additionally, (9) indicates that the estimation error depends on the probing waveform since $\boldsymbol{H}$ is a function of $\boldsymbol{s}$. Therefore, by leveraging the prior $\hat{\boldsymbol{\alpha}}$, we can design the waveform $\boldsymbol{s}$ to improve the next estimation. For simplicity, we focus on exploiting $\hat{\boldsymbol{\alpha}}$ by assuming a known β without loss of generality. Fig. 1 sketches the diagram of the recurrent waveform design in the SBL-based DOA estimation. Accordingly, the waveform design problem is formulated as

$$\begin{aligned}\underset{\boldsymbol{s}}{\text{mininize}}\quad & \text{Tr}\left[\left(\beta\boldsymbol{H}^H\boldsymbol{H}+\boldsymbol{\Lambda}^{-1}\right)^{-1}\right]\\ \text{subject to}\quad & |s_n|=1,\forall n=1,\ldots,NT,\end{aligned} \tag{11}$$

where $\boldsymbol{H}=\boldsymbol{A}\left(\boldsymbol{I}_L\otimes\boldsymbol{s}\right)$, $\boldsymbol{\Lambda}=\text{Diag}\left(\hat{\boldsymbol{\alpha}}\right)$, and the unit-modulus constraint is widely considered in radar and communications for the power amplifier efficiency [18]. Note that $\hat{\boldsymbol{\alpha}}$ may be extremely close to zero causing numerical instabilities. To resolve this issue, we assume that apart from these small-value elements, there are Q significantly nonzero elements in $\hat{\boldsymbol{\alpha}}$ with the locations indexed by $\ell_q\in\{1,\ldots,L\},\forall q=1,\ldots,Q$. We replace the above $\boldsymbol{\Lambda}$ and $\boldsymbol{A}$ with

$$\boldsymbol{\Lambda}=\text{diag}\left(\left[\hat{\alpha}_{\ell_1},\ldots,\hat{\alpha}_{\ell_Q}\right]\right), \tag{12}$$

$$\boldsymbol{A}=\left[\boldsymbol{I}_T\otimes\left(\boldsymbol{a}_r\left(\theta_{\ell_1}\right)\boldsymbol{a}_t^T\left(\theta_{\ell_1}\right)\right),\ldots,\boldsymbol{I}_T\otimes\left(\boldsymbol{a}_r\left(\theta_{\ell_Q}\right)\boldsymbol{a}_t^T\left(\theta_{\ell_Q}\right)\right)\right], \tag{13}$$

respectively, and then problem (11) is still the same but well-defined. In fact, the estimate $\hat{\boldsymbol{\alpha}}$ will become sparser as the estimation round goes, and consequently, the dimension of the matrix $\beta\boldsymbol{H}^H\boldsymbol{H}+\boldsymbol{\Lambda}^{-1}$ will become smaller, which means that the estimation will gradually converge to some target angles and the waveform is targeting these angles.

3. PROPOSED WAVEFORM DESIGN ALGORITHM

As mentioned earlier, a MM based method is considered for waveform optimization. Within this MM framework, the key step is to construct the majorizer (i.e. a global upper bound of the objective function). By the matrix inversion lemma [19],

$$\mathrm{Tr}\left[\left(\beta\boldsymbol{H}^H\boldsymbol{H}+\boldsymbol{\Lambda}^{-1}\right)^{-1}\right]=\mathrm{Tr}\left[\boldsymbol{\Lambda}-\boldsymbol{\Lambda}\boldsymbol{H}^H\boldsymbol{G}^{-1}\boldsymbol{H}\boldsymbol{\Lambda}\right]. \tag{14}$$

where $\boldsymbol{G}=\boldsymbol{H}\boldsymbol{\Lambda}\boldsymbol{H}^H+\beta^{-1}\boldsymbol{I}\succ\boldsymbol{0}$. We let $f(\boldsymbol{H},\boldsymbol{G})=\mathrm{Tr}\left(\boldsymbol{\Lambda}-\boldsymbol{\Lambda}\boldsymbol{H}^H\boldsymbol{G}^{-1}\boldsymbol{H}\boldsymbol{\Lambda}\right)$ and have the following lemma, which provides an majorizer of $f(\boldsymbol{H},\boldsymbol{G})$.

Lemma 1. *The function $f(\boldsymbol{H},\boldsymbol{G})$ is jointly concave on $\boldsymbol{H}$ and $\boldsymbol{G}$ and can be majorized by*

$$u(\boldsymbol{H},\boldsymbol{G};\boldsymbol{H}_k,\boldsymbol{G}_k)=\mathit{Tr}\left[\boldsymbol{P}_k^H\boldsymbol{G}\right]-2\mathit{Re}\left[\mathit{Tr}\left(\boldsymbol{Q}_k^H\boldsymbol{H}\right)\right]+\mathit{Tr}(\boldsymbol{\Lambda}) \tag{15}$$

where $\boldsymbol{P}_k=\boldsymbol{G}_k^{-1}\boldsymbol{H}_k\boldsymbol{\Lambda}^2\boldsymbol{H}_k^H\boldsymbol{G}_k^{-1}$ and $\boldsymbol{Q}_k=\boldsymbol{G}_k^{-1}\boldsymbol{H}_k\boldsymbol{\Lambda}^2$.

Proof. The function $g(\boldsymbol{H},\boldsymbol{G})=\boldsymbol{\Lambda}-\boldsymbol{\Lambda}\boldsymbol{H}^H\boldsymbol{G}^{-1}\boldsymbol{H}\boldsymbol{\Lambda}$ is jointly matrix concave on $\boldsymbol{H}$ and $\boldsymbol{G}\succ\boldsymbol{0}$ [34, pp. 108–111]. The function $\mathrm{Tr}(\cdot)$ is linear and nondecreasing. Thus, $f(\boldsymbol{H},\boldsymbol{G})=\mathrm{Tr}(g(\boldsymbol{H},\boldsymbol{G}))$ is jointly concave on $\boldsymbol{H}$ and $\boldsymbol{G}$. Further, its upper bound is given by

$$\begin{aligned}&u(\boldsymbol{H},\boldsymbol{G};\boldsymbol{H}_k,\boldsymbol{G}_k)\\=&\mathrm{Tr}\left[\boldsymbol{G}_k^{-1}\boldsymbol{H}_k\boldsymbol{\Lambda}^2\boldsymbol{H}_k^H\boldsymbol{G}_k^{-1}(\boldsymbol{G}-\boldsymbol{G}_k)\right]\\&-2\mathrm{Re}\left\{\mathrm{Tr}\left[\boldsymbol{\Lambda}^2\boldsymbol{H}_k^H\boldsymbol{G}_k^{-1}(\boldsymbol{H}-\boldsymbol{H}_k)\right]\right\}+f(\boldsymbol{H}_k,\boldsymbol{G}_k),\end{aligned} \tag{16}$$

which will reduce to (15) after some algebraic manipulations, left out for brevity here, thereby completing the proof. □

By applying the lemma, the majorized problem becomes

$$\begin{aligned}\underset{\boldsymbol{s}}{\text{mininize}}\quad&\mathrm{Tr}\left[\boldsymbol{P}_k^H\boldsymbol{G}\right]-2\mathrm{Re}\left[\mathrm{Tr}\left(\boldsymbol{Q}_k^H\boldsymbol{H}\right)\right]\\\text{subject to}\quad&|s_n|=1,\forall n=1,\ldots,NT,\end{aligned} \tag{17}$$

We define $\boldsymbol{S}=\boldsymbol{I}_Q\otimes\boldsymbol{s}$, and then the above objective function can be rewritten as

$$\begin{aligned}&\mathrm{Tr}\left[\boldsymbol{P}_k^H\boldsymbol{G}\right]-2\mathrm{Re}\left[\mathrm{Tr}\left(\boldsymbol{Q}_k^H\boldsymbol{H}\right)\right]\\=&\mathrm{Tr}\left(\boldsymbol{\Lambda}\boldsymbol{S}^H\boldsymbol{A}^H\boldsymbol{P}_k^H\boldsymbol{A}\boldsymbol{S}\right)-2\mathrm{Re}\left[\mathrm{Tr}\left(\boldsymbol{Q}_k^H\boldsymbol{A}\boldsymbol{S}\right)\right]+\beta^{-1}\mathrm{Tr}\left(\boldsymbol{P}_k^H\right)\\=&\mathrm{vec}(\boldsymbol{S})^H\left[\boldsymbol{\Lambda}\otimes\left(\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\right)\right]\mathrm{vec}(\boldsymbol{S})\\&-2\mathrm{Re}\left[\mathrm{Tr}\left(\boldsymbol{Q}_k^H\boldsymbol{A}\boldsymbol{S}\right)\right]+\beta^{-1}\mathrm{Tr}\left(\boldsymbol{P}_k^H\right).\end{aligned} \tag{18}$$

Algorithm 1 SBL-Based Waveform Design Method

Input: Estimated $\hat{\boldsymbol{\alpha}}$ and initial waveform $\boldsymbol{s}_0$
Output: Designed waveform $\boldsymbol{s}$
construct $\boldsymbol{\Lambda}$ and $\boldsymbol{A}$ via (12) and (13)
set $k\leftarrow 0$
repeat
  $\boldsymbol{H}_k=\boldsymbol{A}(\boldsymbol{I}_Q\otimes\boldsymbol{s}_k)$
  $\boldsymbol{G}_k=\boldsymbol{H}_k\boldsymbol{\Lambda}\boldsymbol{H}_k^H+\beta^{-1}\boldsymbol{I}$
  $\boldsymbol{P}_k=\boldsymbol{G}_k^{-1}\boldsymbol{H}_k\boldsymbol{\Lambda}^2\boldsymbol{H}_k^H\boldsymbol{G}_k^{-1}$
  $\lambda=\lambda_{\max}\left(\boldsymbol{\Lambda}\otimes\left(\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\right)\right)$
  $\boldsymbol{Q}_k=\boldsymbol{G}_k^{-1}\boldsymbol{H}_k\boldsymbol{\Lambda}^2$
  $\boldsymbol{B}_k=\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\boldsymbol{S}_k\boldsymbol{\Lambda}-\lambda\boldsymbol{S}_k-\boldsymbol{A}^H\boldsymbol{Q}_k$
  $\boldsymbol{s}_{k+1}=-e^{j\arg\left(\sum_{i=1}^{Q}\boldsymbol{b}_i^k\right)}$
  $k\leftarrow k+1$
until convergence

Besides, we have $\mathrm{vec}(\boldsymbol{S})^H\mathrm{vec}(\boldsymbol{S})=\mathrm{Tr}(\boldsymbol{I}_QN)=QN$. Thus, problem (17) is equivalent to

$$\begin{aligned}\underset{\boldsymbol{s}}{\text{mininize}}\quad&\mathrm{vec}(\boldsymbol{S})^H\left[\boldsymbol{\Lambda}\otimes\left(\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\right)-\lambda\boldsymbol{I}\right]\mathrm{vec}(\boldsymbol{S})\\&-2\mathrm{Re}\left[\mathrm{Tr}\left(\boldsymbol{Q}_k^H\boldsymbol{A}\boldsymbol{S}\right)\right]\\\text{subject to}\quad&|s_n|=1,\forall n=1,\ldots,NT\\&\boldsymbol{S}=\boldsymbol{I}_Q\otimes\boldsymbol{s},\end{aligned} \tag{19}$$

where λ is a constant defined as

$$\lambda\triangleq\lambda_{\max}\left(\boldsymbol{\Lambda}\otimes\left(\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\right)\right)=\lambda_{\max}(\boldsymbol{\Lambda})\,\lambda_{\max}\left(\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\right), \tag{20}$$

and $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a matrix.

The first term of the objective function in problem (19) is concave. Applying the majorization technique again, we have

$$\begin{aligned}&\mathrm{vec}(\boldsymbol{S})^H\left[\boldsymbol{\Lambda}\otimes\left(\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\right)-\lambda\boldsymbol{I}\right]\mathrm{vec}(\boldsymbol{S})\\\leq&2\mathrm{Re}\left\{\mathrm{vec}(\boldsymbol{S}_k)^H\left[\boldsymbol{\Lambda}\otimes\left(\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\right)-\lambda\boldsymbol{I}\right]\mathrm{vec}(\boldsymbol{S})\right\}+\mathit{const}.\\=&2\mathrm{Re}\left\{\mathrm{Tr}\left(\left(\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\boldsymbol{S}_k\boldsymbol{\Lambda}-\lambda\boldsymbol{S}_k\right)^H\boldsymbol{S}\right)\right\}+\mathit{const}.,\end{aligned} \tag{21}$$

where $\mathit{const}.$ represents an unrelated constant term. Then, problem (19) can be further majorized to be

$$\begin{aligned}\underset{\boldsymbol{s},\boldsymbol{S}}{\text{mininize}}\quad&\mathrm{Re}\left[\mathrm{Tr}\left(\boldsymbol{B}_k^H\boldsymbol{S}\right)\right]\\\text{subject to}\quad&|s_n|=1,\forall n=1,\ldots,NT,\\&\boldsymbol{S}=\boldsymbol{I}_Q\otimes\boldsymbol{s},\end{aligned} \tag{22}$$

where $\boldsymbol{B}_k=\boldsymbol{A}^H\boldsymbol{P}_k\boldsymbol{A}\boldsymbol{S}_k\boldsymbol{\Lambda}-\lambda\boldsymbol{S}_k-\boldsymbol{A}^H\boldsymbol{Q}_k$. Problem (22) can be recast into

$$\begin{aligned}\underset{\boldsymbol{s}}{\text{mininize}}\quad&\mathrm{Re}\left[\left(\sum_{i=1}^{Q}\boldsymbol{b}_i^k\right)^H\boldsymbol{s}\right]\\\text{subject to}\quad&|s_n|=1,\forall i=1,\ldots,NT,\end{aligned} \tag{23}$$

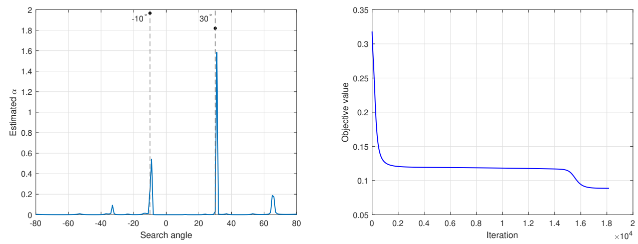

Fig. 2: Initial DOA estimation and convergence curve.

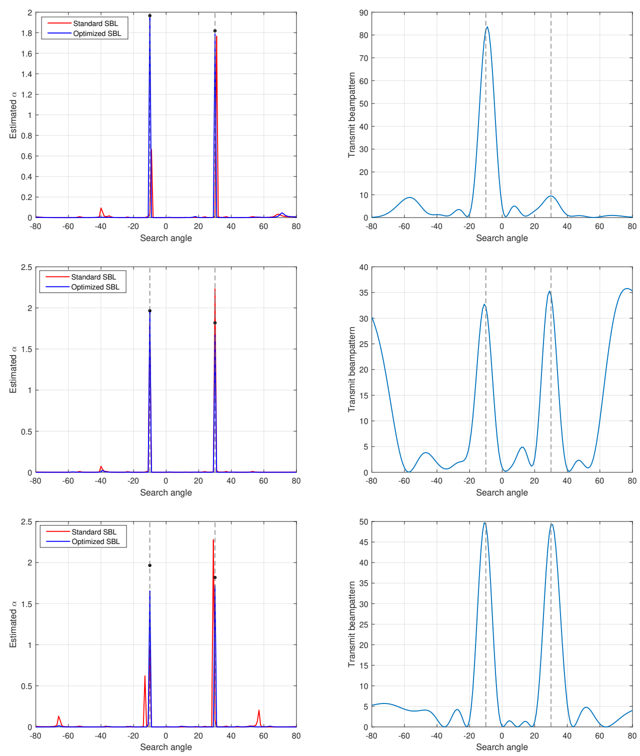

Fig. 3: DOA performance via the recurrent scheme.

where $\boldsymbol{b}_i^k$ consists of the $(i-1)NT+1 : iNT$ elements of the i-th column of $\boldsymbol{B}_k$. The closed form solution is

$$\boldsymbol{s} = -e^{j\arg\left(\sum_{i=1}^{Q}\boldsymbol{b}_i^k\right)}. \tag{24}$$

Finally, the proposed algorithm is summarized in Alg. 1.

4. NUMERICAL EXPERIMENTS

Consider a colocated MIMO radar with $N = 10$ transmit and $M = 10$ receive antennas and two targets with the DOAs being $\boldsymbol{\theta} = [-10°, 30°]$. The system noise $\boldsymbol{n}$ is generated from $\mathcal{CN}\left(\boldsymbol{0}, \sigma^2\boldsymbol{I}\right)$. We set SNR $= |x_k|^2/\sigma^2 = 2$ dB for both the two targets. The initial waveform is randomly generated and the resolution of the angle grid is $1°$.

Fig. 2 shows the initial estimated $\hat{\boldsymbol{\alpha}}$ by the standard SBL with a random waveform and the convergence plot of the proposed algorithm for this $\hat{\boldsymbol{\alpha}}$. We see that the initial estimation is not accurate and the algorithm guarantees the monotonicity.

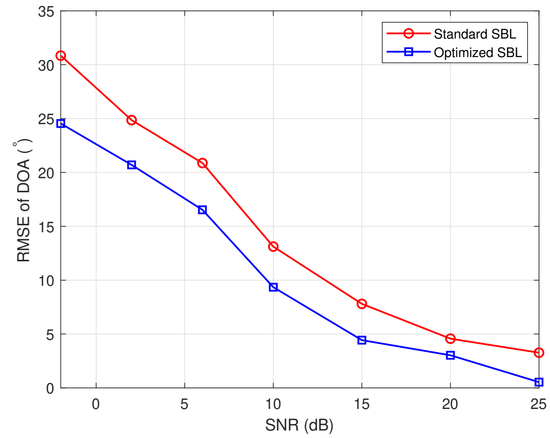

Fig. 4: RMSE of DOA estimation at different SNR levels.

Fig. 3 shows the recurrent process for 3 consecutive rounds with $T = 1$, in which the current-round waveform is designed using the previous-round estimated $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\alpha}}$ in Fig. 2 is the initialization for the whole rounds. As the round goes, the estimation accuracy of the optimized SBL is better than the standard one in both the angles and values of $\boldsymbol{\alpha}$. This superior performance can be understood intuitively by inspecting the transmit beampattern of the designed waveform, which gradually focuses its mainlobes towards the DOAs to improve the estimation with a higher SNR. Besides, the algorithm will converge faster as the round goes if the optimized waveform is used as the initial point for the next design.

The root mean-square error (RMSE) is defined as RMSE $= \sqrt{\frac{1}{I}\sum_{i=1}^{I}\|\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i\|_2^2}$, where $I = 200$, $\hat{\boldsymbol{\theta}}_i$ and $\boldsymbol{\theta}_i$ are the estimated and true DOAs of the i-th trail, respectively. Fig. 4 shows the average estimation accuracy at different SNR levels. While both average RMSEs improve as the SNR increases, the optimized SBL has smaller errors.

5. ACKNOWLEDGMENT

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6. CONCLUSIONS

In this paper, we aim to improve the SBL-based DOA estimation performance by optimizing the probing waveform. An recurrent scheme of waveform design is proposed, in which the current waveform is designed by leveraging the previous estimation. The design problem is formulated as a minimization of the SBL estimation variance and solved efficiently by an novel MM based algorithm, in which the closed form solution is found at each iteration with a guaranteed monotonic convergence. The simulation results show the efficacy of the proposed waveform design and indicate that the design is essentially shaping a desired beampattern gradually.

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