PERFECT RECONSTRUCTION OF CLASSES OF NON-BANDLIMITED SIGNALS FROM PROJECTIONS WITH UNKNOWN ANGLES

Renke Wang *Roxana Alexandru* *Pier Luigi Dragotti*

Imperial College London
email: renke.wang19@imperial.ac.uk, roxana.alexandru12@imperial.ac.uk, p.dragotti@imperial.ac.uk

ABSTRACT

In this paper, we consider the 2D tomography problem for a finite number of point sources, where the line integral projections are taken at unknown angles. We address the problem of recovering the point sources and estimating the projection angles. Using the property of the Radon transform of a point source, which is a signal with Finite Rate of Innovation, we retrieve the projections using the annihilating filter method. The reconstruction method we propose is then able to unveil the 2D geometric information of the projection angles, as well as the locations of the point sources. Finally, we extend the approach to planar polygons.

***Index Terms*—** 2D tomography, point source and angle reconstruction, finite rate of innovation (FRI), sampling theory

1. INTRODUCTION

Much of the current research in tomography has focused on the problem of reconstructing an object from a set of its line integral projections at known projection angles. The state of the art approaches are mainly based on solving a regularized least square problem [1, 2] where the structure of the object is imposed by the regularization. Other methods [3, 4] make use of the finite complexity of the underlying object model.

Yet there are instances of real-world tomographic applications in which the projection angles are unknown. Examples include modelling crystal-structured molecules using X-ray crystallography [5], characterizing atomic structures using cryo-electron microscopy [6], source estimation in radio astronomy [7, 8], unknown patient motion in computational tomography [9].

The state of the art approaches in 2D tomography from unknown projection angles can be divided into two main categories. The authors in [10] have considered the problem of extracting rotation-invariant features from the projection data without estimating the projection angles. While other works [11, 12, 13, 14] retrieve the projection angles from properly processed projection data and then invert the operation of projection to recover the 2D objects. However, the retrieval of projection angles either relies on large numbers of realizations or exhaustive search.

In this paper, we consider the 2D tomographic problem of estimating a point source model from samples of projections taken along lines at *unknown* angles. The starting point of the sampling window along each projection is also *unknown*. We assume that the point sources lie within a disc region with a known radius, which is a typical assumption in tomography settings [5, 6, 9], and show that we can perfectly estimate the locations of the point sources, as well as the directions of the projection lines, up to an orthogonal transformation.

The main idea of our proposed framework is to make use of the fact that the Radon transform [15] of a point source model is a stream of Diracs, which is a signal with finite rate of innovation (FRI) [16]. This allows us to estimate the unknown parameters of the Diracs, which correspond to the projections of the sources onto the projection lines, using the annihilating filter method [16]. Using a similar approach as in [17, 18, 19], the 1D parameters are then used to build a rank-deficient matrix, which can be factorized to estimate the direction vectors of the projection lines and the locations of the sources up to an orthogonal transformation [19].

This paper is organized as follows. In Section 2, we introduce the problem formulation and assumptions. In Section 3, we present our estimation algorithm which is composed of 1) estimation of the direction vectors, 2) estimation of the unknown shifts in the sampling window, and 3) source localization. We then validate our method with experiments done with synthetic data in Section 4 and conclude in Section 5.

2. PROBLEM FORMULATION

We consider the following point source model:

$$g(x, y) = \sum_{k=1}^{K} a_k \delta(x - x_k, y - y_k), \tag{1}$$

or equivalently,

$$g(\mathbf{x}) = \sum_{k=1}^{K} a_k \delta(\mathbf{x} - \mathbf{S}_k), \tag{2}$$

where:

a_k = amplitude of the point source k,

x_k, y_k = coordinates of the point source k in $\mathbb{R}_2$ space,

$\mathbf{S}_k = [x_k, y_k]^T$.

We assume that the geometric center of the point sources is at the origin: $\sum_{k=1}^{K} \mathbf{S}_k = \mathbf{0}$, and that all points lie within a plane disk of known radius R centered at the origin.

Denote the Radon transform of $g(x, y)$ with $R_g(p, \theta)$:

$$R_g(p, \theta) = \iint_{\mathbb{D}^2} g(x, y)\delta(p - x\cos\theta - y\sin\theta)dxdy, \quad (3)$$

where $\mathbb{D}^2$ is the disk region in $\mathbb{R}^2$. The straight line $L_{p,\theta}$ over which the integral is taken is defined as $L_{p,\theta} = \delta(p - x\cos\theta - y\sin\theta)$, where θ is the angle between $L_{p,\theta}$ and the horizontal x axis, and p is the radial distance between $L_{p,\theta}$ and the origin. Given our point source model $g(\mathbf{x})$, the projections $R_g(p, \theta)$ are 1D streams of Diracs with a support of maximum length $2R$.

Suppose also that we have access to a set of J projections along the lines l_j at unknown view angles θ_j as shown in Fig. 1, with $l_j \perp L(p, \theta_j)$ and $j = 1, ..., J$.

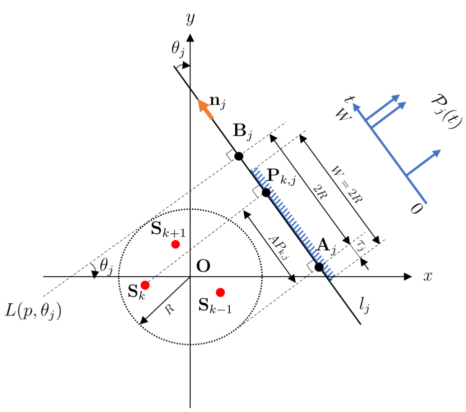


Fig. 1: The unknown point source model $g(\mathbf{x})$. The projections are taken along lines at unknown angles θ_j. The sampling window along direction vector $\mathbf{n}_j$ of length W is indicated with the blue stripes.

Based on our assumptions, the projection region of the disc region onto the line l_j is the line segment $\mathbf{A}_j\mathbf{B}_j = 2R\mathbf{n}_j$, where $\mathbf{n}_j$ is the unit direction vector of line l_j equal to $\mathbf{n}_j = [-\sin\theta_j, \cos\theta_j]^T$, $\|\mathbf{n}_j\| = 1$. We denote the foot point of the projection of $\mathbf{S}_k$ onto line l_j with $\mathbf{P}_{k,j}$, which is located within the line segment $\mathbf{A}_j\mathbf{B}_j$, given we assume all sources are located within the disc region of radius R centered at the origin (see also Fig. 1).

We further assume that the projection on line l_j is sampled along direction $\mathbf{n}_j$ and that the starting point is in the vicinity of $\mathbf{A}_j$ with an unknown shift τ_j. Since the projection is within a line segment of length $2R$, the sampling window is set to $W = 2R$ and it contains projections of all the point sources. Thus, the Radon transform $R_g(p, \theta_j)$ within the sampling window W can be expressed as a stream of Diracs:

$$\mathcal{P}_j(t) = \sum_{k=1}^{K} a_{k,j}\delta(t - AP_{k,j} - \tau_j), \ t \in [0, W], \quad (4)$$

where $AP_{k,j}$ is the geometric distance between $\mathbf{A}_j$ and $\mathbf{P}_{k,j}$. From the geometric relation, we can derive that:

$$AP_{k,j} = \|\mathbf{A}_j\mathbf{P}_{k,j}\| = (\mathbf{S}_k - \mathbf{A}_j)^T\mathbf{n}_j \overset{(a)}{=} R + \mathbf{S}_k^T\mathbf{n}_j, \quad (5)$$

where (a) follows from the fact that $\mathbf{A}_j^T\mathbf{n}_j = -R$, as evident from Fig. 1. Taking into account the unknown shift τ_j, the locations of the Diracs are given by:

$$AP'_{k,j} = AP_{k,j} + \tau_j. \quad (6)$$

Suppose the signal $\mathcal{P}_j(t)$ is filtered with the sampling kernel $\varphi(t)$ and uniformly sampled. The uniform samples can be expressed as:

$$\begin{aligned}\mathcal{P}_j[n] = \mathcal{P}_j(nT) &= \langle\sum_{k=1}^{K} a_{k,j}\delta(t - AP'_{k,j}), \ \varphi(t - nT)\rangle \\ &= \langle\sum_{k=1}^{K} a_{k,j}\delta(t - R - \tau_j - \mathbf{S}_k^T\mathbf{n}_j), \ \varphi(t - nT)\rangle,\end{aligned} \quad (7)$$

where $n \in \mathbb{N}$ and T is the uniform sampling period. The problem we want to solve is the simultaneous estimation of the locations $\mathbf{S}_k$ and the direction vectors $\mathbf{n}_j$ from the measurements in Eq. (7).

3. RECONSTRUCTION METHOD

Assume that we have to estimate the unknown point source model $g(\mathbf{x})$ defined by Eq. (2) with $K \geq 4$ sources located within a disc region of known radius R around their geometric center. In addition, the projections are taken along $J \geq K+1$ unknown angles, and the number of sources and projections K and J are assumed to be known. Samples of the set of projections $\{\mathcal{P}_j[n]\}_{j=1}^{J}$ are taken uniformly with an exponential reproducing or polynomial reproducing sampling kernel with a sampling window of length $W = 2R$ along the direction vector $\mathbf{n}_j$ on line l_j. The starting point of the window is unknown and deviates from $\mathbf{A}_j$ by an unknown shift τ_j, which is sufficiently small. Under the above assumptions, we can state the following result:

Proposition 1. *Let $g(\mathbf{x}) \in \mathbb{R}^2$ contain $K \geq 4$ non-colinear point sources. If there are $J \geq K+1$ projections taken along different unknown angles, the point sources and the direction vectors of the projection lines can be estimated up to an orthogonal transformation.*

Proof. In order to localize the sources and retrieve the direction vectors, we need to first estimate the parameters $AP'_{k,j}$

from the measurements in Eq. (7). Under the condition that the sampling kernel is able to reproduce polynomials or exponentials, the parameters, as well as the amplitudes $a_{k,j}$ of the Diracs can be perfectly retrieved using techniques in the area of finite rate of innovation [16].

If the estimated parameters $AP'_{k,j}$ are paired correctly, that is, if we can determine whether two parameters $AP'_{k,j}$ and $AP'_{k',i}$ on two different projections correspond to the same point source, i.e. $k = k'$, we can obtain the difference $\Omega_{k,j} = AP'_{k,j} - AP'_{k+1,j}$ for each projection line j and form the matrix $\mathbf{\Omega}$:

$$\mathbf{\Omega} = \begin{bmatrix} (\mathbf{S}_2 - \mathbf{S}_1)^T \mathbf{n}_1 & \dots & (\mathbf{S}_2 - \mathbf{S}_1)^T \mathbf{n}_J \\ (\mathbf{S}_3 - \mathbf{S}_2)^T \mathbf{n}_1 & \dots & (\mathbf{S}_3 - \mathbf{S}_2)^T \mathbf{n}_J \\ \vdots & \ddots & \vdots \\ (\mathbf{S}_K - \mathbf{S}_{K-1})^T \mathbf{n}_1 & \dots & (\mathbf{S}_K - \mathbf{S}_{K-1})^T \mathbf{n}_J \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \mathbf{S}_2 - \mathbf{S}_1 \\ \mathbf{S}_3 - \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_K - \mathbf{S}_{K-1} \end{bmatrix}^T}_{(K-1)\times 2} \underbrace{\begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \dots & \mathbf{n}_J \end{bmatrix}}_{2\times J} := \mathbf{SC}. \quad (8)$$

Please note that we will present an efficient way to do the pairing in Appendix A. Under the assumptions that there are $K \geq 4$ non-colinear point sources $\mathbf{S}_k$ and $J \geq 3$ different projection angles θ_j, the rank of $\mathbf{\Omega}$ satisfies rank($\mathbf{\Omega}$)= 2. Thus, we can exploit the rank deficiency property to pair the parameters across different projection directions.

Inspired by [17], we first perform the singular value decomposition of $\mathbf{\Omega}$: $\mathbf{\Omega} = \tilde{\mathbf{U}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{V}}^T$, where the first two singular values may be non-zero while the other diagonal entries of $\tilde{\mathbf{\Sigma}}$ are zero. Therefore, we denote with $\mathbf{U}$ the first two columns of $\tilde{\mathbf{U}}$, $\mathbf{\Sigma}$ the 2×2 diagonal matrix, and $\mathbf{V}^T$ the first two rows of $\tilde{\mathbf{V}}^T$. Then, the matrix $\tilde{\mathbf{C}}$ can be estimated with the following factorisation:

$$\tilde{\mathbf{C}} = \mathbf{U}^T \mathbf{\Omega} = \mathbf{\Sigma}\mathbf{V}^T. \quad (9)$$

We can then prove that $\mathbf{Q}\tilde{\mathbf{C}} = \mathbf{C}$ where $\mathbf{Q}$ is a 2×2 matrix, using an approach similar to [19]. This approach exploits the fact that the direction vectors $\mathbf{n}_j$ have unit norm. Given that $\mathbf{n}_j = \mathbf{Q}\tilde{\mathbf{n}}_j$ and $\|\mathbf{n}_j\| = 1$, we have $\tilde{\mathbf{n}}_j^T \mathbf{Q}^T \mathbf{Q} \tilde{\mathbf{n}}_j = 1$, and $\mathbf{Q}^T\mathbf{Q}$ is a symmetric matrix:

$$\tilde{\mathbf{n}}_j^T \begin{bmatrix} m & p \\ p & n \end{bmatrix} \tilde{\mathbf{n}}_j = 1. \quad (10)$$

Using Eq. (10), we can obtain a system of linear equations, which we can solve to retrieve the unknowns m,n and p, provided we have $J \geq 3$ projection lines. Finally, once m,n and p are found, we perform the eigen decomposition of the matrix $\mathbf{Q}^T\mathbf{Q} = \mathbf{A}\Delta\mathbf{A}^T$, and $\mathbf{Q}$ is given by $\mathbf{Q} = \mathbf{B}(\mathbf{A}\mathbf{\Delta}^{\frac{1}{2}})^T$, where $\mathbf{B}$ is an arbitrary orthogonal matrix.

After obtaining the direction vectors, we can solve for the unknown shifts τ_j by summing the parameters $AP'_{k,j}$ along all directions, as follows:

$$\sum_{k=1}^{K} AP'_{k,j} = KR + (\sum_{k=1}^{K} \mathbf{S}_k^T)\mathbf{n}_j + K\tau_j \stackrel{(a)}{=} KR + K\tau_j, \quad (11)$$

where (a) follows from the assumption that the geometric center of the points is at the origin.

Once the direction vectors and unknown shifts are retrieved, $AP_{k,j}$ can be obtained using Eq. (6), and it is now possible to retrieve the locations of the point sources through the following linear system of equations given parameters $AP_{k,j}$ and the estimated direction vectors $\{\mathbf{n}_j\}_{j=1}^J$:

$$\begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_K \end{bmatrix}^T \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \dots & \mathbf{n}_J \end{bmatrix} = \mathbf{M}, \quad (12)$$

where $\mathbf{M}_{k,j} = AP_{k,j} - R$. Given that $\mathbf{B}$ is an arbitrary orthogonal matrix, the reconstruction of point sources and direction vectors will be up to an orthogonal transformation compared with the ground-truth. □

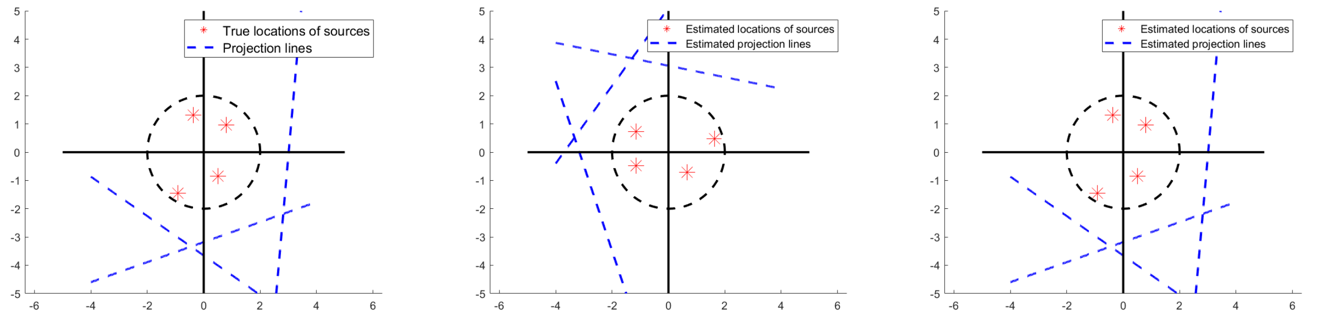
Remark 1. *Given any two direction vectors $\mathbf{n}_i$ and $\mathbf{n}_j$ are known, we can compute the orthogonal matrix $\mathbf{B}$ and estimate the locations of sources and the projection angles exactly.*

Remark 2. *The above estimation algorithm can also be extended to the estimation of the closures of planar polygons. The Radon transform of a planar polygon is piecewise linear, and its discontinuities correspond to the vertices of the polygon. The locations of discontinuities can be estimated using FRI related techniques [16], which are the same quantities $AP'_{k,j}$ as in the 2D point source model. Thus, using the presented estimation algorithm, we could entirely retrieve the vertices of the polygon up to an orthogonal transformation.*

4. NUMERICAL RESULTS

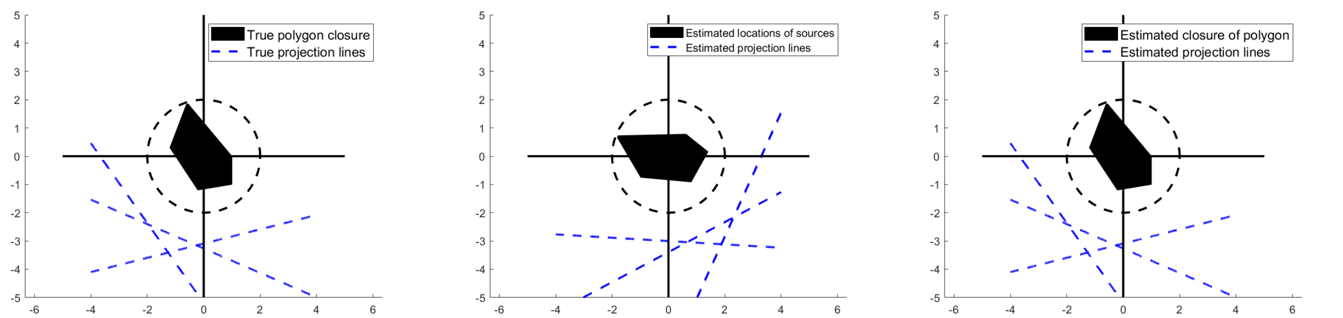
We consider the point source model $g(\mathbf{x})$ composed of $K = 4$ Diracs of unit weight within the disc region of radius $R = 2$, and projections of the model are taken at $J = 3$ different unknown angles. The samples are taken uniformly with Daubechies 8 scaling function as the sampling kernel at a sampling period $T = \frac{1}{2^6}$. We show that the locations of the sources and the projection angles can be retrieved using the method presented in Section 3.

Fig. 2 shows a visual example of the estimation results. When compared to the true locations and projection lines in Fig. 2 (a), the estimation of the locations of the point sources and the projection lines shown in Fig. 2 (b) are correct up to an orthogonal transformation. Given any two projections



(a) True locations of sources and projection lines. (b) Estimated locations and projection lines. (c) Estimated locations of sources and projection lines when two angles are known.

Fig. 2: Estimation of the locations of the point sources and the projection directions, using our proposed method.



(a) True closure of polygon and projection lines. (b) Estimated closure of polygon and projection lines. (c) Estimated closure of polygon and projection lines when two angles are known.

Fig. 3: Estimation of the polygon closure and projection directions, using our proposed method.

angles, the orthogonal transformation can be found exactly as shown in Fig. 2 (c). In Fig. 3 we show an example of the estimation of a polygon closure.

5. CONCLUSION

In this paper, we presented a method to estimate the unknown locations of multiple point sources from samples taken along projection lines at unknown angles. Under the assumption that the point sources are located within a disc region of known radius, the method can retrieve the locations, as well as the direction vectors of the projections up to an orthogonal transformation. In future work, we would like to further address the performance of the algorithm under noisy conditions and to extend the method to more complex 2D objects.

A. PARAMETER PAIRING ACROSS PROJECTIONS

After estimating the set of locations $\{AP'_{k=1,..,K,j}\}_{j=1}^{J}$ and amplitudes $\{a_{k=1,...,K,j}\}_{j=1}^{J}$ of the 1D Diracs of the projections using techniques in the area of finite rate of innovation, there are several clues that can help perform the pairing. One available choice is the amplitude $a_{k,j}$ of the Diracs, whose value should be invariant across projections. Another more robust approach is to exploit the rank-deficient property of the matrix Ω to perform the pairing in a combinatorial way. We first choose 3 projection angles $\theta_{i,j,k}$ and then randomly assign the sources on one of these projection lines to $\mathbf{S}_1, ..., \mathbf{S}_K$. To align the other two projections, there are at most $(A_K^K)^2$ permutations, where $A_n^m = \frac{n!}{(n-m)!}$. The matrix $\tilde{\boldsymbol{\Omega}}$ is formed as in Eq. (8). Using the rank-deficient property of the matrix $\tilde{\boldsymbol{\Omega}}$, the pairing is chosen to have the maximum ratio between the second and the third largest singular value. After aligning the first three projections, we fix any two of them, and form a new matrix $\tilde{\boldsymbol{\Omega}}$ using estimated parameters from a new projection angle and repeat for the rest of the projections. For each projection left, there are only $A_K^K = \frac{K!}{0!} = K!$ permutations.

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