JOINT MODEL ORDER ESTIMATION FOR MULTIPLE TENSORS WITH A COUPLED MODE AND APPLICATIONS TO THE JOINT DECOMPOSITION OF EEG, MEG MAGNETOMETER, AND GRADIOMETER TENSORS

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ABSTRACT

The efficient estimation of an approximate model order is essential for applications with multidimensional data if the observed low-rank data is corrupted by additive noise. Certain signal processing applications such as biomedical studies, where the data are collected simultaneously through heterogeneous sensors, share some common features, i.e., coupled factors among multiple tensors. The exploitation of this coupling can lead to a better model order estimation, especially in case of low SNRs. In this paper, we extend the rank estimation techniques, designed for a single tensor, to noise-corrupted coupled low-rank tensors that share one of their factor matrices. To this end, we consider the joint effect of the global eigenvalues (calculated from the coupled HOSVD) and exploit the exponential behavior of the resulting coupled global eigenvalues. We show that the proposed method outperforms the classical criteria and can be successfully applied to EEG, MEG Magnetometer, and Gradiometer measurements. Our real data simulation results show that the estimated rank is highly reliable in terms of dominant components extraction.

Index Terms— Coupled tensor decomposition, coupled model order estimation, data fusion, joint EEG-MEG analysis.

1. INTRODUCTION

Multidimensional processing is widely used in different applications, for example, image processing and completion, pattern recognition, blind source separation, channel modeling in wireless communications, the estimation of MIMO channels parameters, and many more. Tensor decompositions allow processing the data in a more efficient way by exploiting their multidimensional structure [1]. Tensors can be decomposed or factorized into components or factor matrices. The most widely used types of tensor decompositions are the Canonical Polyadic Decomposition (CPD) [2] and the Higher Order Singular Value Decomposition (HOSVD) also known as Multi-Linear Singular Value Decomposition (MLSVD) [3].

In practical applications of biomedical signal processing, the data can be recorded simultaneously from many sensors, for example, in Electroencephalograms (ECGs), in Magnetoencephalograms (MEGs), and in gradiometer measurements (GRAD) [4]. If the resulting tensors have one of the factor matrices in common, the computation of the coupled CPD is beneficial, for instance, to achieve a better feature extraction.

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A coupled matrix and tensor factorization approach is considered in [5] for capturing the internal latent features from heterogeneous data. In [6] and [7], the authors have extended the Semi-Algebraic framework to approximate the CPD via Simultaneous matrix diagonalization (SECSI) for coupled tensors. The CPD allows to extract the minimum number of rank-one components that often have a meaningful physical interpretation.

However, the observed data is often corrupted by noise, and the rank of the noise-corrupted low-rank tensors (or model order) is unknown. A method based on the maximum likelihood estimation procedure (AIC) was proposed in [8], while the authors in [9] presented the Minimum Description Length (MDL) criterion to estimate the model order. Wax and Kailath implemented the AIC and MDL criteria to detect the number of signals in a multi-channel time series.

The authors in [10] introduced the Exponential Fitting Test (EFT) for model order estimation based on the profile of the eigenvalues that represent the noise and are computed from the covariance matrix of the observations. It was shown that the profile of the noise eigenvalues fits an exponential law. In [11] and [12], the concept of the Global EigenValues (GEVs) was introduced. Moreover, the presented Modified Exponential Fitting Test (M-EFT) based on the profile of the GEVs significantly increases the accuracy of the model order estimation. Furthermore, the authors showed how classical techniques such as AIC and MDL can be extended to the multidimensional case using the GEVs. Several methods for the rank estimation of noise-corrupted low-rank CPD data have been presented in [13], [14], [15], etc.

In [16] a novel method named LineAr Regression of Global Eigenvalues (LaRGE) based on the concept of GEVs was introduced. The authors proposed to approximate the noise GEVs on a logarithmic scale by a straight line and to use the deviation from this linear regression to separate the signal and noise GEVs. The number of the signal GEVs corresponds to the model order of the multidimensional data. LaRGE provides an excellent performance as compared to classical techniques.

In this paper, we present a novel approach for the model order estimation of noise-corrupted coupled low-rank tensors. Instead of using the GEVs like in the LaRGE algorithm, we consider the Coupled GEVs (C-GEVs) as the geometric mean of the GEVs of coupled tensors. To this end, a novel robust method is presented in this paper that estimates the model order of coupled tensors jointly under the assumption that all noise-corrupted coupled low-rank tensors have the same rank or model order. We compare the proposed Coupled LaRGE (C-LaRGE) method with the AIC, MDL, 3D-AIC, 3D-MDL, and the original LaRGE method. Furthermore, we present the results of the model order estimation and low-rank decomposition of simultaneously recorded EEG and MEG Magnetometer and Gradiometer measurements during an Intermittent Photic Stimulation experiment [17].

The rest of the paper is organized as follows. Section 2 reviews the concept of the global eigenvalues and the LaRGE method. Section 3 describes the extension of the LaRGE method for the model order estimation of coupled tensors. Section 4 presents the numerical results and the synthetic data simulations. In Section 5, we apply the proposed C-LaRGE method to the joint model order estimation of simultaneously recorded biomedical data. Finally, Section 6 concludes the paper.

2. SEPARATE MODEL ORDER ESTIMATION

Before considering the coupled model order estimation, let us first summarize techniques that have been proposed for the model order estimation of a single noise-corrupted low-rank tensor. In this paper, we utilize the LaRGE technique proposed in [16], which exploits the multi-linear singular values obtained from the HOSVD and the exponential profile of the noise eigenvalues to estimate the proper CPD rank.

First, let us consider a single measurement tensor $\boldsymbol{\mathcal{X}}$ which is given by $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}_0 + \boldsymbol{\mathcal{N}} \in \mathbb{C}^{M_1 \times \ldots \times M_N}$, where $\boldsymbol{\mathcal{X}}_0 \in \mathbb{C}^{M_1 \times \ldots \times M_N}$ is the noiseless data of rank R, and $\boldsymbol{\mathcal{N}} \in \mathbb{C}^{M_1 \times \ldots \times M_N}$ is an additive noise tensor. The model order estimation algorithm starts with the computation of the HOSVD of $\boldsymbol{\mathcal{X}}$ given by

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{S}} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2 \times_3 \ldots \times_N \boldsymbol{U}_N, \tag{1}$$

where $\boldsymbol{\mathcal{S}} \in \mathbb{C}^{M_1 \times \ldots \times M_N}$ is a core tensor, and $\boldsymbol{U}_n \in \mathbb{C}^{M_n \times M_n}, n \in \{1, \ldots, N\}$ contains the left singular vectors of the n-mode unfoldings of $\boldsymbol{\mathcal{X}}$. In the next step, the GEVs [18, 11, 12] are computed as the product of the n-mode singular values $\sigma_i^{(n)}$

$$\tilde{\lambda}_i^{[G]} = \prod_{n=1}^{N} \left(\sigma_i^{(n)}\right)^2, i = 1, \ldots, M, \tag{2}$$

where $M = \min\{M_n\}, n \in \{1, \ldots, N\}$ is the smallest dimension in $\boldsymbol{\mathcal{X}}$. Following [12, 10], where authors have shown that the noise global eigenvalues obey an exponential profile, the further steps of the LaRGE technique are based on a linear approximation of the GEVs on a logarithmic scale, starting from the smallest noise global eigenvalue. The detection of a significant gap, compared to the predicted exponential profile, indicates that the smallest signal GEV has been found. The model order corresponds to the number of the signal GEVs. For more details on the LaRGE algorithm, we refer the reader to [16].

In the following section, we propose a new LaRGE-based algorithm that allows estimating the CPD rank of multiple coupled tensors with one common mode. Moreover, we show that with the proposed approach, extensions of the MDL [19] and AIC [8] techniques can also be used for the coupled model order estimation.

3. COUPLED MODEL ORDER ESTIMATION

Next, let us consider the joint CP decomposition of L coupled N-dimensional tensors $\boldsymbol{\mathcal{X}}_0^{(l)}$, $l \in \{1, \ldots, L\}$, with the first mode in common

$$\boldsymbol{\mathcal{X}}_0^{(l)} = \boldsymbol{\mathcal{I}}_{3,R} \times_1 \boldsymbol{F}_1 \times_2 \boldsymbol{F}_2^{(l)} \times_3 \ldots \times_N \boldsymbol{F}_N^{(l)} \in \mathbb{C}^{M_1 \times M_2^{(l)} \times \ldots \times M_N^{(l)}}, \tag{3}$$

where $\boldsymbol{F}_n^{(l)} \in \mathbb{C}^{M_n^{(l)} \times R}$, $n \in \{1, \ldots, N\}$, M_1 denotes the size of the common dimension for all tensors, and R is a CPD rank satisfying $R \leq \min\{M_1, M_2^{(l)}, \ldots, M_N^{(l)}\}$. The noise corrupted versions of $\boldsymbol{\mathcal{X}}_0^{(l)}$ can be written as

$$\boldsymbol{\mathcal{X}}^{(l)} = \boldsymbol{\mathcal{X}}_0^{(l)} + \boldsymbol{\mathcal{N}}^{(l)}. \tag{4}$$

In contrast to the original algorithms for a single tensor, the coupled rank estimation is based on the coupled HOSVD of L tensors, which is expressed as follows

$$\boldsymbol{\mathcal{X}}^{(l)} = \boldsymbol{\mathcal{S}}^{(l)} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2^{(l)} \times_3 \ldots \times_N \boldsymbol{U}_N^{(l)}, \tag{5}$$

where the common factor $\boldsymbol{U}_1 \in \mathbb{C}^{M_1 \times M_1}$ is calculated from the SVD of the concatenated 1-mode unfoldings of $\boldsymbol{\mathcal{X}}^{(l)}$ [7]. Consequently, the lth global eigenvalue is defined as follows

$$\tilde{\lambda}_{l,i}^{[G]} = \prod_{n=1}^{N} \left(\sigma_{l,i}^{(n)}\right)^2, i = 1, \ldots, M, \tag{6}$$

where $M = \min\{M_n^{(l)}\}, n \in \{1, \ldots, N\}$ is the smallest dimension in all $\boldsymbol{\mathcal{X}}^{(l)}$s, $\sigma_{l,i}^{(n)}$ is the n-mode singular value, and $\sigma_{l,i}^{(1)}$ is common for all tensors $\boldsymbol{\mathcal{X}}^{(l)}$. In order to perform the coupled model order selection, we calculate the *geometric mean* of the global eigenvalues of L tensors as follows

$$\tilde{\lambda}_i^{[G]} = \sqrt[L]{\tilde{\lambda}_{1,i}^{[G]} \cdot \ldots \cdot \tilde{\lambda}_{l,i}^{[G]} \cdot \ldots \cdot \tilde{\lambda}_{L,i}^{[G]}}. \tag{7}$$

This choice of $\tilde{\lambda}_{l,i}^{[G]}$, in conjunction with the n-mode singular values obtained from the coupled HOSVD, allows combining the information from multiple tensors to ensure a more reliable rank estimation. To further detect the signal global eigenvalues, i.e., the coupled model order, the linear approximation $\hat{\lambda}_i^{[G]}$ of the C-GEVs in (7) on the logarithmic scale $\lambda_i^{(G)} = \ln \tilde{\lambda}_i^{(G)}$ can be performed starting from

Algorithm 1 Coupled LaRGE (C-LaRGE)

Require: N-way tensors $\boldsymbol{\mathcal{T}}^{(l)} \in \mathbb{C}^{M_1 \cdots \times M_n^{(l)} \times \cdots \times M_N^{(l)}}$
$\sigma_{l,i}^{(n)} \longleftarrow$ coupled HOSVD of $\boldsymbol{\mathcal{T}}^{(l)}$s (mode-1 is coupled)
$M = \min\{M_1, \ldots, M_N^{(l)}\} \ \forall l$
for $l = 1, \ldots, L$ **do**
  for $i = 1, \ldots, M$ **do**
    $\tilde{\lambda}_{l,i}^{[G]} = \prod_{n=1}^{N} \left(\sigma_{l,i}^{(n)}\right)^2$
  end for
end for
for $i = 1, \ldots, M$ **do**
  $\tilde{\lambda}_i^{[G]} = \sqrt[L]{\tilde{\lambda}_{1,i}^{[G]} \cdot \ldots \cdot \tilde{\lambda}_{l,i}^{[G]} \cdot \ldots \cdot \tilde{\lambda}_{L,i}^{[G]}}$
  $\lambda_i^{(G)} = \ln \tilde{\lambda}_i^{(G)}$
end for
$\hat{R}_{\text{LaRGE}} = 1$
for $k = 1, \ldots, M-1$ **do**
  linear approximation $\hat{\lambda}_{M-k}^{[G]}$
  $\delta_{M-k} = \frac{\Delta_{M-k}^{[G]}}{\left|\hat{\lambda}_{M-k}^{[G]}\right|} = \frac{\lambda_{M-k}^{[G]} - \hat{\lambda}_{M-k}^{[G]}}{\left|\hat{\lambda}_{M-k}^{[G]}\right|}$
  $\sigma_{M-k} = \sqrt{\frac{1}{k} \sum_{i=M}^{M-k} \left(\Delta_i^{[G]} - \frac{1}{k} \sum_{i=M}^{M-k} \Delta_i^{[G]}\right)^2}$
  $\text{PESDR}_k = \frac{\delta_{M-k}}{\sigma_{M-k-1}}$
  if $(\text{PESDR}_{k-1} < \rho) \wedge (\text{PESDR}_k \geq \rho)$ **then**
    $\hat{R}_{\text{LaRGE}} = M - k$
  end if
end for

the smallest C-GEV. The point where the ratio between the relative prediction error δ_{M-k} and standard deviation of the approximation error σ_{M-k-1} exceeds the predefined threshold ρ for the first time indicates the detection of the rank. This ratio is denoted as PESDR (Prediction Error to Standard Deviation Ratio). We refer to the proposed above algorithm as Coupled LaRGE (C-LaRGE), whose main steps are summarized in Algorithm 1. Moreover, to avoid estimation errors appearing due to a relatively small difference between the rank and the smallest dimension of the tensor, LaRGE with a penalty function (LaRGE PF) can be applied [20]. The penalty function ensures that the value of σ_{M-k} exceeds a certain threshold ε, which allows decreasing the outliers that may lead to wrong estimates.

Furthermore, the proposed way to define the coupled global eigenvalues can be utilized for the extension of the well known MDL [19] and AIC [8] techniques to the coupled model order estimation by replacing the eigenvalues of the covariance matrices by the C-GEVs described in (7). However, we have observed in practice that the coupled extension of LaRGE provides significantly more accurate estimates than the extensions of AIC or MDL.

4. NUMERICAL RESULTS

To find the first signal C-GEV automatically, the value of the threshold ρ should be specified. To this end, numerical experiments have been performed based on Monte Carlo simulations. Three cases can be considered when the model order is under - , over - , and correctly estimated. Therefore, the probability of false negative or missing $P_{\text{fn}}(\rho)$, the probability of false positive $P_{\text{fp}}(\rho)$, and the probability of correct detection $P_{\text{cd}}(\rho)$ are defined. A large number of Monte Carlo trials provides the functional dependency of these probabilities as a function of the value of the threshold ρ.

To this end, we have generated four random three-dimensional tensors $\boldsymbol{\mathcal{X}}^{(1)}, \boldsymbol{\mathcal{X}}^{(2)}, \boldsymbol{\mathcal{X}}^{(3)}, \boldsymbol{\mathcal{X}}^{(4)}$ with known rank $R = 5$, dimensions $103 \times 350 \times 30$, and the second factor matrix in common for all tensors according to the coupled CPD model. The columns in the factor matrices are correlated according to the vector $r = [0.2, 0.3, 0.4]^{\mathrm{T}}$ as in [21]. Gaussian noise is added to each tensor with SNR = 0 dB. The SNR is defined as

$$\mathrm{SNR} = 10 \cdot \log_{10} \frac{\mathbb{E}\left\{\|\boldsymbol{\mathcal{X}}\|_{\mathrm{F}}^{2}\right\}}{\mathbb{E}\left\{\|\boldsymbol{\mathcal{N}}\|_{\mathrm{F}}^{2}\right\}}, \mathrm{dB}, \tag{8}$$

where the $\mathbb{E}\left\{\cdot\right\}$ denotes the expected value. In Figure 1 the results of 1000 Monte Carlo simulations are shown. The threshold $\rho \le 0.595$ leads to a probability of false negative less than the 0.01. On the other hand, the probability of false positive becomes less than 0.01 with a threshold $\rho \ge 0.585$. Therefore, for the automatic detection of the first signal C-GEV, the threshold can be set in the range between 0.585 and 0.595. The probability of correct detection $P_{\text{cd}}(\rho)$ is equal to one in this range.

Figure 2 shows the probability of detection vs. the SNR performance of C-LaRGE PF, LaRGE PF as well as 1-D and 3-D versions of AIC and MDL. Four tensors coupled in the first mode, $\boldsymbol{\mathcal{X}}^{(1)}, \boldsymbol{\mathcal{X}}^{(2)}, \boldsymbol{\mathcal{X}}^{(3)}, \boldsymbol{\mathcal{X}}^{(4)}$ having dimensions $16 \times 16 \times 16$ each, with given rank $R = 7$ have been drawn randomly. The SNR of $\boldsymbol{\mathcal{X}}^{(1)}$ is fixed at 7 dB, while for the rest of the tensors it changes from -30 to 20 dB. The coupling of global eigenvalues from four tensors has enhanced the rank detection capability of MDL, AIC, and LaRGE as compared to those cases where we estimate the rank individually. The left-most curves indicate the significant performance enhancement of C-LaRGE and C-LaRGE PF over the rest of the methods.

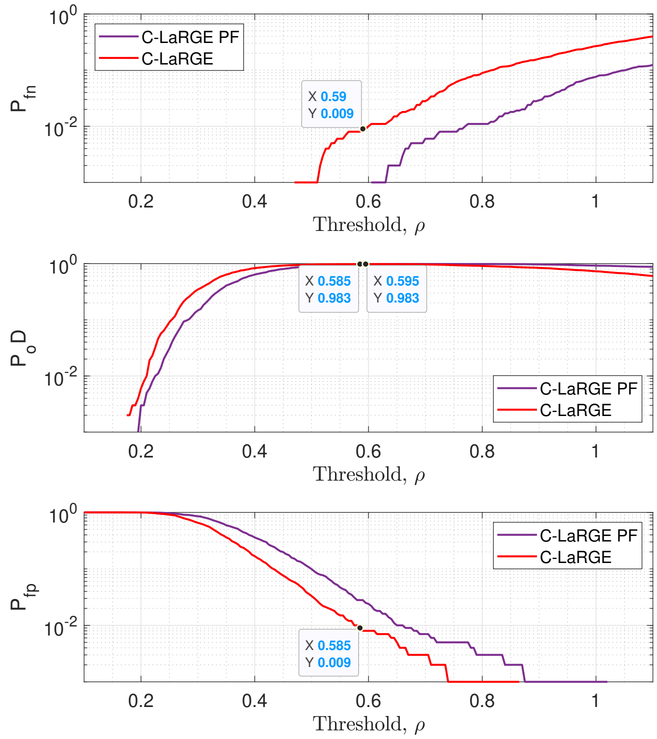

Fig. 1: Probability of false negative $P_{\text{fn}}(\rho)$ (top), probability of correct detection $P_{\text{cd}}(\rho)$ (middle), and probability of false positive $P_{\text{fp}}(\rho)$ (bottom) vs. Threshold. SNR=0 dB, $R = 5$, 1000 trials.

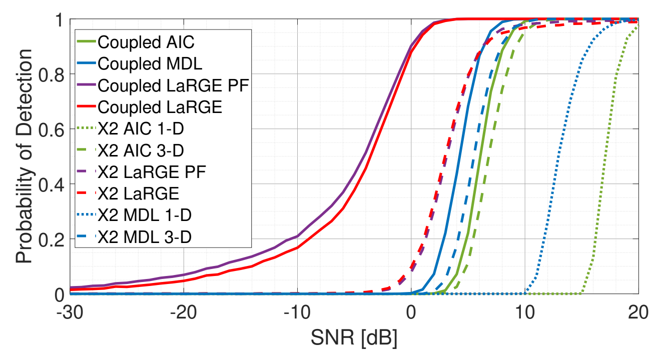

Fig. 2: Probability of detection vs. SNR. Simulation parameters: $\boldsymbol{\mathcal{X}}_1, \boldsymbol{\mathcal{X}}_2, \boldsymbol{\mathcal{X}}_3, \boldsymbol{\mathcal{X}}_4 \in \mathbb{C}^{16\times16\times16}$, $R = 7$, LaRGE threshold = 0.57, C-LaRGE threshold = 0.59, LaRGE PF/C-LaRGE PF threshold = 0.7, common correlation $r = [0.2, 0.3, 0.4]$, 10 000 trials.

5. JOINT DECOMPOSITION OF EEG, MEG MAGNETOMETER, AND GRADIOMETER TENSORS

The developed algorithm has been applied to simultaneously recorded EEG and MEG Magnetometer and Gradiometer measurements via 128 EEG electrodes and 306 MEG channels (MAG + GRAD-1 + GRAD-2) during an Intermittent Photic Stimulation (IPS). The experiment was performed on 12 healthy participants, aged between 21 and 42 years [17].

In the first step, the individual α-rhythm was measured during 60 seconds of MEG at rest. Then, by means of the Discrete Fourier

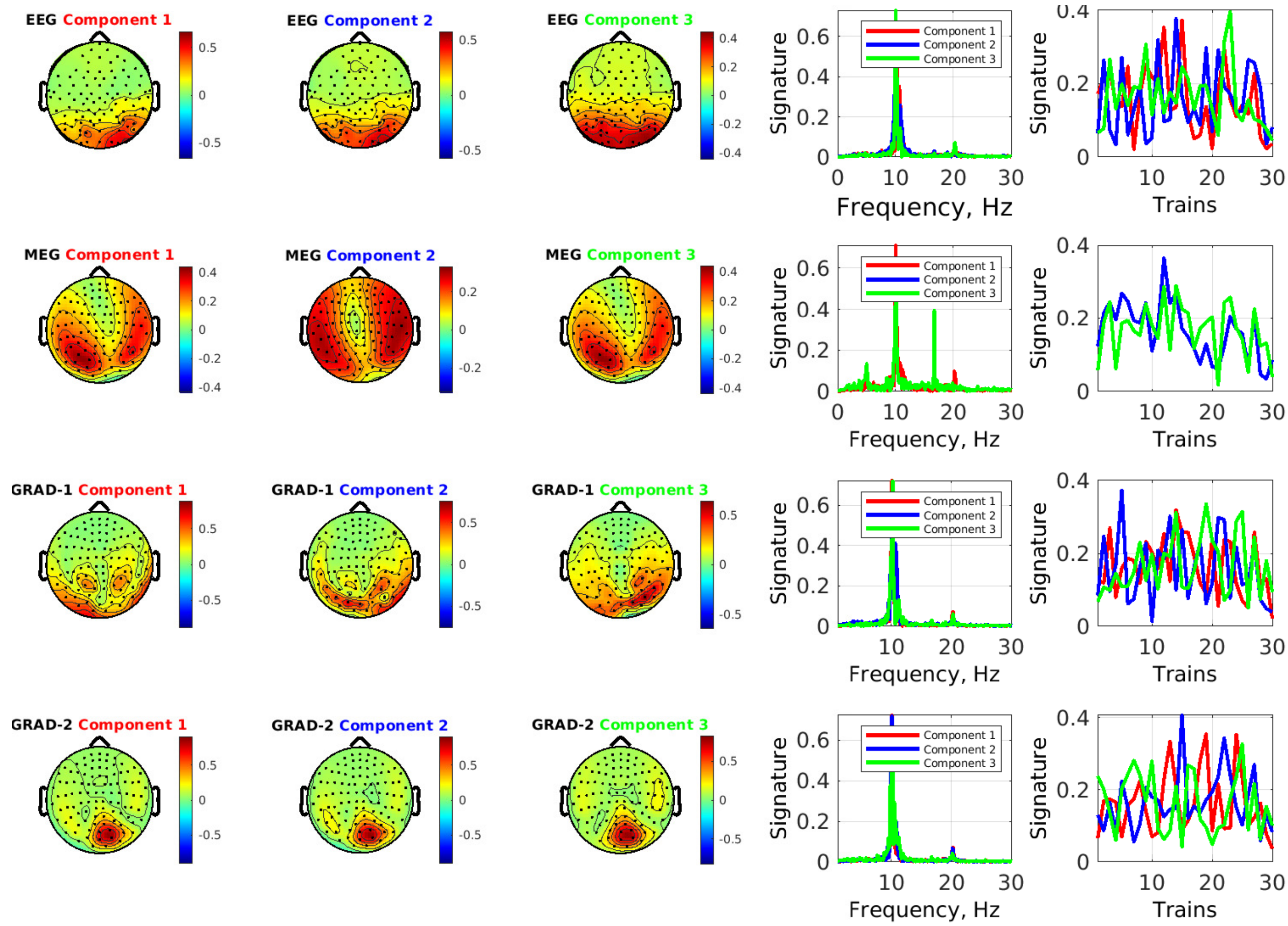

Fig. 3: The estimated factor matrices for volunteer No. 5 (f_α = 10.7 Hz) at stimulation frequency 0.95 f_α and estimated rank $\hat{R}$ = 3. The results of C-SECSI show that the MEG frequency mode estimate is not coupled.

Transform (DFT) from the occipital MEG channels, the individual α-frequencies f_α were calculated. The flickering light stimulation was conducted at different frequencies with closed eyes. For each stimulation frequency, 30 stimulus trains, i.e., repetitions, are presented. Each train consists of 40 periods of on/off light stimulation. Further details on the experiment are provided in [17].

In the preprocessing steps, the jointly measured signals were filtered with a bandpass filter in the frequency range from 3 to 40 Hz. Then, to obtain the distribution of the signals in the frequency domain, the DFT was computed and four tensors – EEG, MEG-MAG, MEG-GRAD-1, MEG-GRAD-2 – were constructed with dimensions *Channels* × *Frequency* × *Trains*. In this configuration, we assume that the factor matrix in the frequency domain is common for all tensors.

The proposed C-LaRGE algorithm was used on the four tensors with the preprocessed data. Then the coupled CP decomposition was calculated via the coupled SECSI framework for an arbitrary number of tensors [7] using the jointly estimated tensor rank $\hat{R}$ = 3. The estimated factor matrices for volunteer No. 5 and stimulation frequency 0.95 f_α are shown in Fig. 3. It can be observed that with the jointly estimated rank, the coupled CP decomposition results in extracting clear dominant components from the recorded signals in all tensor dimensions. As a comparison, the results of a separate model order estimation of these tensors have resulted in different estimated ranks for each tensor, i.e., $\hat{R}$ = 3 for EEG and MEG-MAG tensors, and $\hat{R}$ = 1 for the two MEG-Gradiometer tensors.

6. CONCLUSIONS

In this paper, we have introduced an extension of the LaRGE algorithm for coupled rank estimation of multiple noise-corrupted tensors. The coupling of the global eigenvalues from different but coupled tensors is performed by calculating their geometric mean. These coupled global eigenvalues are then used to estimate the coupled rank with the C-LaRGE algorithm.

Monte Carlo simulations have provided a suitable threshold to be used for C-LaRGE. Moreover, extensive simulations have shown that the proposed C-LaRGE method outperforms its uncoupled counterpart as well as other classical methods, especially in low SNR scenarios. Such a scenario is observed in measured biomedical data tensors, where extremely sensitive EEG and MEG sensors record the brain activity. Finally, a practical scenario for the coupled model order estimation as well as for the coupled CPD has been presented, where four coupled tensors from a biomedical experiment have been used, and the dominant components have been extracted successfully.

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