A NEW COPRIME-ARRAY-BASED CONFIGURATION WITH AUGMENTED DEGREES OF FREEDOM AND REDUCED MUTUAL COUPLING

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ABSTRACT

In this paper, a new type of coprime-array-based structure, named AtCADiS, is proposed to achieve increased degrees of freedom (DoFs) and reduced mutual coupling. The closed-form expressions for the sensor positions and the number of uniform DoFs (uDoFs) of AtCADiS are provided. Specifically, AtCADiS is constructed via two steps. First, we shift the leftmost sensor of tailored coprime array with displaced subarrays (tCADiS) to the right by N. Second, we increase the number of sensors of tCADiS by $\lfloor \frac{M}{2} \rfloor$ that are appropriately placed to connect the positive and negative lags of difference coarray of tCADiS and further improved its uDoFs remarkably. Finally, simulations show that AtCADiS can achieve higher number of uDoFs than the existing coprime-array-based structures by using the same number of physical sensors, which leads to stronger resolution capability and higher direction of arrival (DoA) estimation accuracy. In the presence of mutual coupling, AtCADiS can achieve comparable mutual coupling leakage compared with the existing array structures.

***Index Terms*—** sparse array, coprime array, nested array, difference coarray, direction-of-arrival estimation, mutual coupling.

1. INTRODUCTION

Recent years, coprime array (CA) [1] attracts considerable attentions in direction of arrival (DoA) estimation, because it can resolve $O(MN)$ sources using $O(M+N)$ sensors. Compared to minimum redundancy arrays (MRAs) [2], the CA has analytical expression for its geometries and difference coarrays. However, the coarray produced by the CA has some holes, which limits the size of the virtual uniform linear array (ULA), thus impacts the DoA estimation performance. For obtaining a longer virtual ULA, different coprime-based configurations have been proposed [3]$-$[14]. For instance, [3] proposes an augmented coprime array (ACA) to resolve $MN+M-1$ sources by increasing the sensor number of CA from $N+M-1$ to $N+2M-1$.

[4] proposes a coprime array with compressed inter-element spacing (CACIS). CACIS uses an integer compression factor p to adjust the inter-element separation of one subarray, while the other subarray is kept fixed. In addition, [4] proposes a coprime array with displaced subarrays (CADiS), which achieves a larger minimum inter-element spacing, a bigger number of unique lags, and a larger virtual array aperture. However, the number of consecutive lags of CADiS is reduced because the positive and negative lags are no longer connected. In [5], the authors propose the relocating extended coprime array (RECA). By relocating the right most $\lfloor \frac{M}{2} \rfloor$ elements in CADiS, RECA can increase the number of consecutive lags of CADiS by filling the holes in its central difference coarray. Three padded coprime arrays (PCAs) called PCA-I, PCA-II and extended PCA (ePCA) are proposed in [6] to augment the consecutive coarray of tailored CADiS (tCADiS). Similar works can be found in [7] and [8], which increase the number of uniform degrees of freedom (uDoFs) of CA and ACA respectively, by shifting one of its subarrays to the right while keeping the other subarray unchanged. A co-prime array with reduced sensors (CARS) [9] is proposed by using the displacement strategy, which can obtain more uDoFs with fewer sensors than ACA. Thinned coprime array (TCA) is proposed in [10], its subarray 1 is exactly the same as that of ACA while its subarray 2 has reduced sensors by removing the sensors located between $(\lfloor \frac{M}{2} \rfloor + 1)N$ and MN. Although the arrays based on coprime principle have reduced mutual coupling effect than the nested arrays [15]$-$[20], there are still some works focusing on reducing the impact of mutual coupling and increasing the number of uDoFs as well. Recently, k-times complementary coprime array (CCPA-k) [11] is proposed to enhance the uDoFs with reduced mutual coupling, it owns a hole-free difference coarray with high uDoFs. [12] proposes an unfolded augmented coprime array (UACA) to fill the holes in the difference coarray of unfolded coprime array (UCA) in [13], [14], at the same time, UACA can reduce the number of sensor pairs with significant mutual coupling. In this paper, a new type of array structure is referred to as the augmented tCADiS (AtCADiS). The proposed array structure aims to lengthen the consecutive lags of tCADiS by incorporating

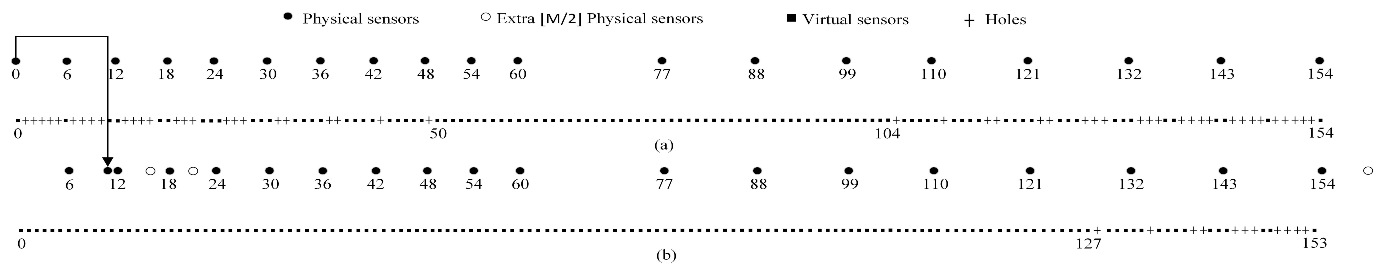

Fig. 1: (a) An example of tCADiS structure when $(M,\ N) = (6,\ 11)$ and $L = 19$. (b) An example of AtCADiS structure when $(M,\ N) = (6,\ 11)$ and $L = 22$.

$\lfloor \frac{M}{2} \rfloor$ sensors that are appropriately placed into tCADiS and shifting the leftmost sensor of tCADiS to the right by N. Throughout this paper, $(\cdot)^*$, $(\cdot)^H$, $vec(\cdot)$, $E[\cdot]$, $\otimes$, $\mathbf{I}_L$, $\|.\|_F$ and $\mathcal{N}(0, \sigma_q^2)$ denote the conjugate, conjugate transpose, vectorization, statistical expectation, Kronecker product, $L \times L$ identity matrix, Frobenius norm and zero-mean Gaussian process with covariance σ_n^2, respectively.

2. PRELIMINARIES

The tCADiS proposed in [6] contains two ULAs of N and $2M - 1 - \lfloor \frac{M}{2} \rfloor$ sensors, respectively. N and M are coprime with $N > M$. The location indices of its physical sensors are given by $\mathbb{C}_{tCADiS} = \{nMd | 0 \leq n \leq N-1\} \cup \{(mN + M(N-1) + (N+M))d | 0 \leq m \leq 2M - 2 - \lfloor \frac{M}{2} \rfloor\}$. d is the half-wavelength of incoming sources. The number of physical sensors is $L = N + 2M - 1 - \lfloor \frac{M}{2} \rfloor$. The data vector received by the sensors at positions $n_1 d,\ n_2 d,\ \cdots,\ n_L d$ can be represented as: $\mathbf{x}(t) = \sum_{q=1}^{Q} \mathbf{a}(\theta_q) s_q(t) + \mathbf{n}(t)$, with $s_q(t) \sim \mathcal{N}(0, \sigma_q^2)$, $\mathbf{n}(t) \sim \mathcal{N}(0, \sigma_n^2 \mathbf{I}_L)$. A vector is constructed as: $\tilde{\mathbf{x}} = vec(\mathbf{R_{xx}}) = \tilde{\mathbf{a}}(\theta_q)\sigma_q^2 + \sigma_n^2 \mathbf{1}_L$, where $\mathbf{R_{xx}} = E[\mathbf{x}(t)\mathbf{x}^H(t)]$, $\tilde{\mathbf{a}}(\theta_q) = \mathbf{a}^*(\theta_q) \otimes \mathbf{a}(\theta_q)$. $\mathbf{1}_L = vec(\mathbf{I}_L)$ is a deterministic noise vector. $\tilde{\mathbf{x}}$ can be considered as a received signal vector by a virtual array, with $\tilde{\mathbf{a}}(\theta_q)$ being the steering vector of the q-th source, the elements of $\tilde{\mathbf{a}}(\theta_q)$ can be described as $e^{\frac{j2\pi(n_{l_1} - n_{l_2}) d \sin(\theta_q)}{\lambda}}$. Different values of $n_{l_1} - n_{l_2}$ give rise to the difference coarray structure. An example of the difference coarray of tCADiS with $L = 19$ physical sensors are deployed as [0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 77, 88, 99, 110, 121, 132, 143, 154] is shown in Fig. 1(a). It is obvious that the difference coarray of tCADiS has a bigger number of unique lags. However, the number of its consecutive lags is reduced because the positive and negative lags are no longer connected. Accordingly, filling the holes in the centre of difference coarray of tCADiS can increase the number of uDoFs dramatically.

3. PROPOSED ARRAY STRUCTURE

Here we present a new array structure aiming to connect between the positive and negative lags of difference coarray of tCADiS by two steps: First, we shift the leftmost sensor in the subarray of tCADiS with N sensors to the right by N. Second, we increase the number of physical sensors of tCADiS from $N + 2M - 1 - \lfloor \frac{M}{2} \rfloor$ to $N + 2M - 1$ by adding $\lfloor \frac{M}{2} \rfloor$ sensors that are appropriately placed. The proposed array, named augmented tCADiS (AtCADiS), not only achieves increased uDoFs but keeps the advantage of tCADiS in terms of countering mutual coupling. The location indices of physical sensors of the proposed AtCADiS are given by the following set:

$$\mathbb{P}_{AtCADiS} = \begin{cases} \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{A}_1 & \text{if} \quad M \leq 4, \\ \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{A}_2 & \text{if} \quad M > 4 \text{ and } M \text{ is even}, \\ \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{A}_3 & \text{if} \quad M > 4 \text{ and } M \text{ is odd}, \end{cases} \tag{1}$$

where

$$\begin{aligned} \mathbb{P}_1 &= \{n'M | 1 \leq n' \leq N-1\}. \\ \mathbb{P}_2 &= \{mN + M(N-1) + (N+M) | 0 \leq m \leq 2M - 2 - \lfloor \frac{M}{2} \rfloor\}. \\ \mathbb{A}_1 &= \{\{(a+1)N - aM | 0 \leq a \leq \lfloor \frac{M}{2} \rfloor - 1\} \cup \{a_1\}\}. \\ \mathbb{A}_2 &= \{\{aN | 1 \leq a \leq \lfloor \frac{M}{2} \rfloor - 2\} \cup \{a_1,\ a_2\}\}. \\ \mathbb{A}_3 &= \{aN | 1 \leq a \leq \lfloor \frac{M}{2} \rfloor - 1\} \cup \{a_1,\ a_3\}\}. \end{aligned} \tag{2}$$

Here $a_1 = (2M - 2 - \lfloor (M/2) \rfloor)N + M(N-1) + 2N$, $a_2 = (\lfloor \frac{M}{2} \rfloor - 2)N + k(N - M)$ with $k = 1,\ 2$ and $a_3 = (\lfloor \frac{M}{2} \rfloor - 1)N + (N - M)$. $\mathbb{P}_1$ is obtained by excluding the first sensor of tCADiS while $\mathbb{P}_2$ keep the subarray of tCADiS with $2M - 1 - \lfloor \frac{M}{2} \rfloor$ unchanged. $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}_3$ sets including the first sensor of tCADiS that is shifted from '0' position to 'N' position besides $\lfloor \frac{M}{2} \rfloor$ additional sensors that are appropriately placed.

Lemma 1 : For the structure defined by $\mathbb{P}_{AtCADiS}$, The following properties hold:
(a) Its maximum number of uDoFs when $M = 2$ is $4MN - 1$.
(b) Its maximum number of uDoFs when $M = 3$ is $4MN + 2M - 1$.
(c) Its maximum number of uDoFs when $M = 4$ is $4MN + M + 1 - 2(N - M - 1)$.
(d) Its maximum number of uDoFs when M is even and $M > 4$ is $3N(M+1) + 6M - (N+1)$.
(e) Its maximum number of uDoFs when M is odd and $M > 4$ is $3N(M+1) + 4M - 1$.
Proof:

Table 1: Summary of uDoFs and L_c of different array configurations for concrete values of $(M,\ N)$.

Array configurations	ACA $(M,\ N)$	TCA $(M,\ N)$	RECA $(M,\ N)$	ePCA $(M,\ N)$	AtCADiS $(M,\ N)$
$L=8$	$(2,5)$	–	–	–	$(2,5)$
uDoFs, $L_c(0.3)$	$23,\ 0.2883$	–	–	–	$39,\ 0.2661$
$L=12$	$(4,5)$	$(5,6)$	$(3,7)$	$(4,5)$	$(3,7)$
uDoFs, $L_c(0.3)$	$47,\ 0.2282$	$69,\ 0.1687$	$81,\ 0.1864$	$83,\ 0.1823$	$89,\ 0.1828$
$L=30$	$(8,15)$	$(11,15)$	$(7,17)$	$(9,13)$	$(8,15)$
uDoFs, $L_c(0.3)$	$255,\ 0.1461$	$351,\ 0.1053$	$401,\ 0.1123$	$417,\ 0.1072$	$437,\ 0.1119$
$L=50$	$(13,25)$	$(19,23)$	$(13,25)$	$(13,25)$	$(13,25)$
uDoFs, $L_c(0.3)$	$675,\ 0.1118$	$911,\ 0.0799$	$1051,\ 0.0820$	$1077,\ 0.0820$	$1101,\ 0.0821$

According to [6], the hole locations in the difference coarray of tCADiS can be represented by the following set $\mathbb{H}$ as: $\mathbb{H}=\pm\mathbb{H}_1\cup\pm\mathbb{H}_2$, where $\mathbb{H}_1=\{h_1|h_1=MN-bM-cN\}\cap\{0,\ 1,\ 2,\cdots,(N-1)(M-1)\}$, $\mathbb{H}_2=\{h_2|h_2=I-h_1,h_1\in\mathbb{H}_1\}$, $1\leq b\leq N-2$, $1\leq c\leq M-1$ and $I=M(N-1)+(M+N)+(2M-2-\lfloor M/2\rfloor)N$. Excluding the first sensor of tCADiS will produce new holes located at $\pm\mathbb{H}'$ as: $\mathbb{H}'=\{\{h'|h'=mN+M(N-1)+(N+M)\}\cup\{M(N-1)\}\}$ where $0\leq m\leq 2M-2-\lfloor\frac{M}{2}\rfloor$. For AtCADiS, the locations of the additional $\lfloor\frac{M}{2}\rfloor$ sensors are appropriately placed such that all the holes in $\pm\mathbb{H}_1$ and $\pm\mathbb{H}'$ can be filled by the newly created virtual sensors produced by the cross difference between $\mathbb{P}_1,\mathbb{P}_2$, and $\mathbb{A}_1,\mathbb{A}_2$ or $\mathbb{A}_3$ except the rightmost hole and leftmost of $\pm\mathbb{H}'$ located at $\pm(m_oN+M(N-1)+(N+M))$ where $m_o=2M-2-\lfloor\frac{M}{2}\rfloor$. Moreover, these newly created virtual sensors can also lengthen the consecutive lags of tCADiS by filling the holes in $\mathbb{H}_2$ (or $-\mathbb{H}_2$) whose positions smaller than l_1,l_2,l_3,l_4 and l_5 (or greater than $-l_1$, $-l_2$, $-l_3$, $-l_4$ and $-l_5$) if $M=2,\ M=3,\ M=4,\ M>4$ is even and $M>4$ is odd respectively, where $l_1=(4MN-1)/2$, $l_2=(4MN+2M-1)/2$, $l_3=(4MN+M+1-2(N-M-1))/2$, $l_4=(3N(M+1)+6M-(N+1))/2$ and $l_5=(3N(M+1)+4M-1)/2$. $\Box$

To illustrate **Lemma 1**, A concrete example of the AtCADiS configuration with $M=6,\ N=11,\ L=22$ and its corresponding difference coarray is shown in Fig. 1(b). It is obvious that the AtCADiS has 3 extra sensors compared with tCADiS in Fig. 1(a) located at $[a_1,\ a_2]=[159,\ 16,\ 21]$. Besides, the sensor of tCADiS located at 'mN' position with $m=0$ is shifted to 'N' position with $N=11$. The extra sensors not only can make the positive and negative lags of difference coarray of tCADiS are connected but also can lengthen its consecutive lags remarkably. As a result, the one-side consecutive lags of AtCADiS in Fig. 1(b) is 128 located at $[0,\ \frac{3N(M+1)+6M-(N+1)-1}{2}]=[0,\ 127]$ compared with tCADiS in Fig. 1(a) which has 55 one-side consecutive lags located at $[50,\ 104]$.

4. DISCUSSION

4.1. Degrees of freedom

Lemma 1 demonstrates that the achievable uDoFs of AtCADiS is determined by $(M,\ N)$, where M and N are coprime integers and $N>M$. Accordingly, for a fixed number of sensors L, the number of uDoFs of AtCADiS can be maximized by choosing an optimal $(M,\ N)$ combination.

Lemma 2. For the geometry $\mathbb{P}_{AtCADiS}$ with $L=2M+N-1$ sensors, the optimal M and N are given by

$$M=\begin{cases}\frac{3L+5}{12} & \text{if} \quad \text{M is even,}\\ \frac{3L+1}{12} & \text{if} \quad \text{M is odd,}\end{cases} \tag{3}$$

and $N=L-2M+1$. It is important to say that the values of M and N have given by **Lemma 2** may not be coprime or integers. Thus, the adjacent integers closest to the expressions given by **Lemma 2** should be taken as long as $(M,\ N)$ are coprime and $M<N$.

TABLE 1 provides a uDoFs comparison between AtCADiS and the existing coprime-based-structures by using different $(M,\ N)$ combinations. In the comparison, both the classic coprime-based-structures and the newly appeared ones are considered, including ACA, TCA, RECA and ePCA. For a given L, thus the M and N values of each array configuration are its optimal solution. We can observe that AtCADiS has the biggest number of uDoFs than the above configurations. Moreover, the number of physical sensors for TCA, RECA and ePCA should be at least $12,\ 9$ and 12 respectively, unlike AtCADIS which can work for $L\geq 6$.

4.2. Mutual coupling

It is said in [19] that two factors can influence the performance of coarray in DoA estimation. The first one is the uDoF, namely the size of virtual ULA. As the number of uDoF increases, the estimation error decreases. The second factor is the mutual coupling which have negative impact on DoA estimation performance. By incorporating mutual coupling effect, the data vector received by the sensors at positions $n_1d,\ n_2d,\ \cdots,\ n_Ld$ can be rewritten as follows: $\mathbf{x}(t)=\mathbf{C}\sum_{q=1}^{Q}\mathbf{a}(\theta_q)s_q(t)+\mathbf{n}(t)$, where $\mathbf{C}$ is a mutual coupling matrix. Less mutual coupling means that $\mathbf{C}$ is closer to the identity matrix, which decreases the estimation error. The coupling leakage of an array is defined as [19]: $L_c=\frac{\|\mathbf{C}-\mathrm{diag}(\mathbf{C})\|_{\mathrm{F}}}{\|\mathbf{C}\|_F}$. A comparison of mutual coupling leak-

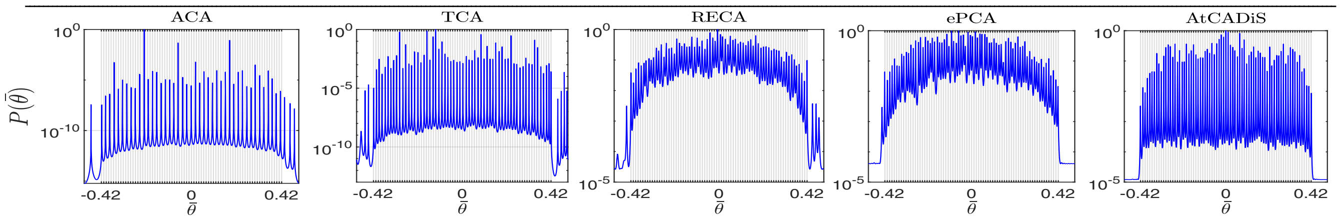

Fig. 2: Comparison among ACA, TCA, RECA, ePCA, AtCADiS and their MUSIC spectra $P(\bar{\theta})$ when $L = 18$ physical sensors, $Q = 75$ signals are distributed uniformly over $\bar{\theta} = [-0.42,\ 0.42]$, SNR = 0 dB and $T = 1000$ snapshots.

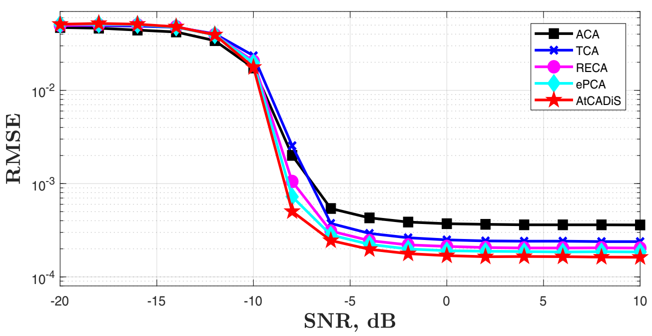

Fig. 3: RMSE versus SNR when $T = 1000$.

age L_c with $c_0 = 1, c_1 = 0.3e^{j\pi/3}$ and $c_i = c_1 e^{-j(i-1)\pi/8}/i$ for $2 \leq i \leq B$ with $B = 100$ is provided in TABLE I. From TABLE 1, ACA has the biggest values of L_c, RECA, ePCA and AtCADiS almost possess the same mutual coupling leakage while TCA shows limited superiority compared with others. Unlike the nested arrays [15]$-$[20] which their enhancement in the terms of uDoFs come at the expense of increased mutual coupling, AtCADiS has a good balance between mutual coupling and number of uDoFs.

5. NUMERICAL SIMULATIONS

In this section, extensive simulations are performed to evaluate the performance of the proposed AtCADiS configuration with the existing sparse arrays.

5.1. MUSIC spectra

To compare the MUSIC spectra of the proposed structure AtCADiS and other considered configurations listed in TABLE 1, $L = 18$ physical sensors with $(M,\ N) = (7,\ 9)$ for TCA and $(M,\ N) = (5,\ 9)$ for ACA, RECA, ePCA and AtCADiS are used. Then, the maximum number of resolvable signals for ACA, TCA, RECA, ePCA and AtCADiS is $49,\ 69,\ 81,\ 86$ and 90, respectively. The estimation results using the spatial smoothing MUSIC algorithm [21] are demonstrated in Fig. 2. It can observe that the ACA and TCA fail to detect all 75 signals because of the limitation in their number of uDoFs. The RECA and ePCA possess false peaks, while the AtCADiS can detect all 75 signals correctly.

5.2. RMSE performance

Here we investigate the root mean square error (RMSE) performance of AtCADiS. The RMSE of DoA estimation is defined as: RMSE $= \sqrt{\frac{1}{RQ}\sum_{i=1}^{R}\sum_{q=1}^{Q}(\hat{\theta}_q(i) - \bar{\theta}_q)^2}$, where $\bar{\theta}_q$ is the true DoA of the q-th signal source and $\hat{\theta}_q(i)$ indicates the estimate of $\bar{\theta}_q$ for the i-th $(i = 1, 2, \cdots, R)$ simulation. In the simulation, $L = 22$ physical sensors can be used to build the arrays listed in TABLE I. Fig. 3 shows the RMSE performance of AtCADiS. Moreover, the geometries of ACA, TCA, RECA, and ePCA are also simulated. One possible way is that we use $(M,\ N) = (8,\ 11)$ for TCA and $(M,\ N) = (6,\ 11)$ for ACA, RECA, ePCA and AtCADiS. $Q = 35$ sources are distributed uniformly over $\bar{\theta} = [-0.42,\ 0.42]$ and $T = 1000$ snapshots are used. One can observe that AtCADiS has the best RMSE performance, this is because AtCADiS has the higher number of uDoFs.

6. CONCLUSION

In this paper, a coprime-array-based structure, called AtCADiS is proposed. AtCADiS has a general closed-form expression for its sensor locations, and it also has the closed-form expressions for its achievable number of uDoFs. Our motivations not only contribute to obtain the highest uDoFs, but also achieve comparable mutual coupling. Extensive simulations show that AtCADiS can achieve more uDoFs than the existing structures like ACA, TCA, RECA and ePCA by using the same number of physical sensors, which in turn leads to stronger resolution capability and higher DoA estimation accuracy. Actually, the coarray of AtCADiS still has holes, which limits the size of its virtual ULA. So in future work, we will try to specify the hole locations of the coarray of AtCADiS and propose a new structure with less or without holes in its coarray.

7. ACKNOWLEDGMENTS

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