DATA-DRIVEN APPROACH FOR THE FLOQUET PROPAGATOR INVERSE PROBLEM SOLUTION

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ABSTRACT

Floquet theory is a classical tool for the analysis of periodic structures' acoustics. However, it may be challenging to analyze material properties for complex cases, whereas the composite material's integral characteristics are obtained relatively straightforward. The paper shows the data-driven approach to inverse displacement data to the theoretical propagation constant approximation and consequently the material parameters using machine learning methods, namely a data-driven symbolic regression. Such an approach may be used to determine the material acoustic parameters in complex cases. Another application is a wave finite element analysis speedup. Two waveguide models are considered: axial rod and circular membrane.

Index Terms— Floquet propagator, data-driven polynomial, sparse regression, periodic structures, phononic crystal

1. INTRODUCTION

Periodic structures are well-known acoustical (also electric, optic) band-filters. Floquet theory is a classical instrument to analyze acoustical stop- and pass-bands frequencies in a periodic structure [1]. Traditionally, it is used in most cases pure theoretically for relatively simple structures like rods and beams [2] even in the most complex cases such as elastic layer [3] it is hard to find a correspondence between a stop-bands found theoretically and practically obtained measures. The Floquet theory as the theoretical ground deeply bound with wave finite element method (WFEM) analysis [4] that also have approximate techniques that are used to estimate waveguide parameters.

To connect observations data and theory, machine learning data-driven methods such as neural networks are used [5]. Modern data-driven methods do not restrict by popular neural networks. Instead, they allow building a wide range of models from complex composite pipelines [6] to algebraic expressions [7]. The latter may connect the Floquet polynomials that determine the theoretical propagator's form and the practical observations. Since the coefficients of the Floquet polynomial have a physical meaning, this can be used to determine the acoustical properties of the composite periodic structure.

The paper aims to show that the generative design of the algebraic expression has the property to mimic the complicated theory obtained by the analytical means, using simple empiric numerical experiments. Development of such methods may allow supporting theoretical derivation with approximations. In the paper, the approach that allows obtaining approximate polynomial that defines propagator and, consequently, the material parameters from the observations is proposed. The idea is not to show the inverse problem solution precision, but to show an alternative approach that is not popular in the area.

The paper is organized as follows: Sec. 2 contains basic concepts of the Floquet theory. Sec. 3 contains formulation of sparse regression problem for polynomial discovery; Sec. 4 provides the experimental application of the proposed approach. Sec. 5 outlines the main result and proposes the direction of the future work.

2. ANALYTICAL FLOQUET THEORY FOR AN AXIAL ROD

The classical periodic structure spectral problem formulation assumes that we have a periodic structure with governing operation with transitional symmetry. We consider the classical axial rod vibration problem schematically shown in Fig. 1.

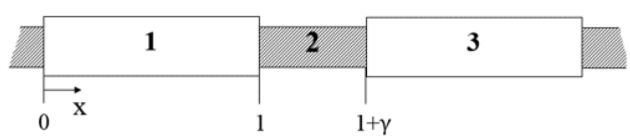

Fig. 1. Periodic structure, different colors denote different acoustical properties, γ - length of the second part.

Each segment has the governing equation (we employ the steady state in form $u(x,t) = u(x)\exp(-i\omega t)$ and wavenumber $k = \frac{\omega}{t}$). Axial rod vibraion and circular membrane vibration are essentially described by the Helmholtz equation shown in Eq. 1.

$$(\Delta + k^2)u_i = 0 \tag{1}$$

Each segment in Fig. 1 has its own dilatation wave speed c_i and length l_i. We introduce following dimensionless parameters $\sigma = \frac{c_2}{c_1}$ $\gamma = \frac{l_2}{l_1}$ and $\Omega = \frac{\omega l_1}{c_1}$. If it is not stated otherwise, parameter set Eq. 2 is used.

$$\gamma = 1;\ \sigma = \frac{1}{5} \tag{2}$$

Due to the acoustical impedance difference in such structure, destructive interference appears, and thus the wave propagation is entirely blocked. From the mathematical point of view, we could formulate the spectral problem for the periodic structure. The spectra, in this case, is the alternating series of frequency bands. The zones where wave propagation is blocked are called stop-bands and pass-bands vice versa.

The spectral problem requires interfacial and periodicity conditions. We write them as Eq. 3 and Eq. 4 respectively.

$$u_1(1) = u_2(1);\ u_1'(1) = u_2'(1) \tag{3}$$

$$u_1(0) = \Lambda u_2(1+\gamma);\ u_1'(0) = \Lambda u_2'(1+\gamma) \tag{4}$$

With general solution for the axial rod vibration equation in a form $u(x) = b_{i,1}\exp(ikx) + b_{i,2}\exp(-ikx)$ we obtain the system of algebraic equations with respect to the unknown displacements $b_{i,j}$ that has the determinant Eq. 5

$$D(\Lambda, \Omega) = \Lambda^2 + \Lambda\left(\frac{1}{\sigma} + \sigma\right)\sin(\Omega)\sin\left(\frac{\gamma\Omega}{\sigma}\right) - \\ - 2\Lambda\cos(\Omega)\cos\left(\frac{\gamma\Omega}{\sigma}\right) + 1 \tag{5}$$

The parameter $\Lambda = \exp(iK_B)$ could be interpreted as Bloch's parameter in the cases where transitional symmetry is preserved. Zoned where $\mathrm{abs}(\Lambda) = 1$ are thus interpreted as the pass-bands and vice versa zones where $\mathrm{abs}(\Lambda) \neq 1$ are stop-bands.

We emphasize the properties of the determinant Eq.5. It has equal coefficients for terms Λ^2 and Λ^0. This property will be used below for functional formulation.

2.1. Forcing problem

Data-driven models require observations data and a parametric form of the model that could fit the data. In the paper, we consider an approximation for two different problems. First is axial rod vibrations [2], which has the analytical solution shown above. Second is the case for which analytical solution is not readily available, radially periodic membrane vibrations in the polar coordinates [8].

Since experimental data is hard to share, we use a numerical solution of two forcing problems to create the observations-like data. The first one is the displacement field of the rod with a finite number of periodic insertions.

Apart from the interfacial conditions Eq. 3, forcing conditions are required. The forcing conditions usually contain force applied to the structure, and Sommerfeld, or radiation, conditions at the infinity in the force direction. For the axial rod vibration problem with n periodic insertions we write forcing conditions as Eq. 6.

$$u_1'(0) = -f;\ b_{n,2} = 0 \tag{6}$$

Moving to the radially periodic membrane, we introduce the coordinate dependency of the propagator, i.e., $\Lambda = \Lambda(r)$. We assume that the center of the membrane is left out of the computation domain with its vicinity r. Thus, we reformulate the forcing problem correspondingly.

Interfacial conditions are left the same, but now the coordinate r is taken into account. In the Eq. 7-Eq. 8 we write explicitly the radially periodic membrane problem.

$$u_1(r) = \Lambda u_2(1+\gamma+r);\ u_1'(r) = \Lambda u_2'(1+\gamma+r) \tag{7}$$

$$u_1'(r) = -f;\ b_{n,2} = 0 \tag{8}$$

Numerical solutions of both forcing problems are used to obtain the data for further data-driven polynomial discovery.

3. PROBLEM FORMULATION

To obtain data for the data-driven models first we solve forcing problems with n periodic inserts Eq.3-Eq.6 and Eq.7-Eq.8 for unknown constants.

We use constants found to form the Floquet periodicity coefficient Λ approximation in form Eq. 9.

$$F_n(x, \Omega) = \frac{D_n(x, \Omega)}{D_n(x + (1+\gamma), \Omega)} \tag{9}$$

In Eq. 9 with $D_n(x, \Omega)$ displacement of the structure with displacements found from forcing problem is designated.

On the other hand, we search for Floquet polynomial in form Eq. 10.

$$P_k(\vec{A}, \Omega) = a_2^{(k)}(\Omega)\Lambda^2 + a_1^{(k)}(\Omega)\Lambda + a_0^{(k)}(\Omega) \tag{10}$$

In Eq. 10 in the paper we use unknown coefficients in form $a_2^{(k)}(\Omega) = \sum\limits_{j=1}^{j=k} A_j^{(2)} * \cos(B_j^{(2)}\Omega)$, where k is the order of approximation and upper index (2) in $B^{(2)}$ is the power of Λ term, i.e. we consider Λ^2. Vector $\vec{A} = \{A_1^{(2)}, B_1^{(2)}, ..., A_k^{(0)}, B_k^{(0)}\}$ is the vector of unknown amplitudes and frequencies of cosines that form coefficients.

To find the form of the polynomial Eq. 10 that fits the data we tabulate first the displacement Eq. 9 as shown in Eq.11.

$$\{\vec{D}1, \vec{D}2\} = \{F_n(x, \Omega_i), \frac{1}{F_n(x, \Omega_i)}\},\ i = 1, N \tag{11}$$

The coordinate x could be taken arbitrarily for the Cartesian coordinate models. However, for the polar coordinate system, the distance from the origin r affects the resulting form of $\Lambda(r)$. With N we designate the number of frequency data points taken (which is also arbitrary).

As the second step we tabulate Floquet polynomial (Eq. 10) solutions at the same frequency points as shown in Eq. 12.

$$E(\vec{A}) = \{P_k^{(1)}(\vec{A}, \Omega_i), P_k^{(2)}(\vec{A}, \Omega_i)\},\ i = 1, N \tag{12}$$

We use all vectors defined above to formulate functional for the sparse regression. This method is also called LASSO [9]. To find the best fitting polynomial Eq. 10 we solve minimization problem shown in Eq. 13.

$$\min_{\vec{A}} \|E(\vec{A}) - \vec{D}\|_2 + \lambda\|\vec{A}\|_1 + \mu\|a_2^{(k)} - a_0^{(k)}\|_1 \tag{13}$$

With $\|\cdot\|_p$ we designate L_p -norm. In Eq. 13 first term determine how close is polynomial to the data. The second term is the normalizing term that allows to fit as short a polynomial as possible and avoid overfitting. The last term is, as shown above, is defined by the physical nature of the problem. It shows that the waves propagating in both directions should be equal. Multipliers λ and μ are the normalization weights. The normalization weights are used for the optimization problem fine-tuning.

In the following section, we consider how the solution of the problem Eq.13 is applied to find the form of the Floquet propagator.

4. EXPERIMENTAL RESULTS

4.1. Axial rod vibration case

The axial rod vibration case has the analytical solution that may be used to assess the convergence and stability of the algorithm. To compare results we use analytical solution for the parameter set Eq. 2 which has the form Eq. 14.

$$\Lambda^2 + (\frac{8}{5}\cos(4\Omega) - \frac{18}{5}\cos(6\Omega))\Lambda + 1 \tag{14}$$

After the polynomial is found, we solve the minimization problem to reverse the material properties. Namely, we assume that after the minimization process, we obtain the polynomial Eq. 15.

$$D(\Lambda, \Omega) = a_2(\Omega)\Lambda^2 + a_1(\Omega)\Lambda + a_0(\Omega) \tag{15}$$

Since we know the nature of the problem and the meaning of the polynomial Eq. 15 coefficients, we can normalize it further to obtain more interpretable results. For normalization we use averaged value of the coefficient $a_2(\Omega)$ to obtain following polynomial in form Eq. 16.

$$D_{norm}(\Lambda, \Omega) = \Lambda^2 + b_1(\Omega)\Lambda + 1 \tag{16}$$

In Eq. 16 coefficient $b_1 = \frac{a_1(\Omega)}{a_{norm}}$ with normalizing coefficient in form $a_{norm} = \frac{1}{\Omega_{term}} \int\limits_0^{\Omega_{term}} a_2(\Omega)d\Omega$. We note that normalization of the coefficient $a_0(\Omega)$ gives value of 1 only when coefficients are exact, in this case we neglect the difference between $a_2(\Omega)$ and $a_0(\Omega)$. The difference between Eq. 15 and Eq. 16 is shown in Fig. 3 for the 50 uniformly taken with respect to the frequency $\Omega \in [0, 2]$ data points obtained from Eq. 9.

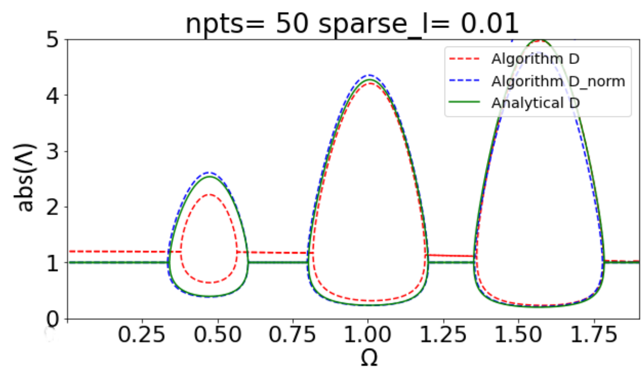

Fig. 2. Difference between theoretical and data-driven polynomials: analytical Eq. 5 (green), data-driven Eq. 15 (red, dashed) and normalized data-driven Eq. 16 (blue, dashed).

From normalized equation we can restore the material parameters σ and γ. To restore parameters we take theoretical coefficient from Eq. 5 and the obtained from the normalized polynomial Eq. 16 coefficient $b_1(\Omega)$ and solve the minimization problem in form Eq. 17.

$$\arg\min_{\gamma,\sigma} \int\limits_0^{\Omega_{term}} \Big[\left(\frac{1}{\sigma} + \sigma\right)\sin(\Omega)\sin\left(\frac{\gamma\Omega}{\sigma}\right) - \\ - 2\cos(\Omega)\cos\left(\frac{\gamma\Omega}{\sigma}\right) - b_1(\Omega)\Big] d\Omega \tag{17}$$

The resulting restored parameters σ and γ boxplots for 20 runs for a different amount of uniformly taken data points are shown in Fig. 3.

In the case of two terms in every coefficient $a_i(\Omega)$ in Eq. 10, we may obtain the closed-form expression that allows obtaining parameters from amplitudes and frequencies of two cosine functions. From boxplots, in Fig. 3 we see that the obtained parameter distribution covers theoretical value of parameters $\sigma = 1/5$ and $\gamma = 1$. Also, we note that with an advanced optimization algorithm for the polynomial Eq. 10 discovery, we may obtain better results.

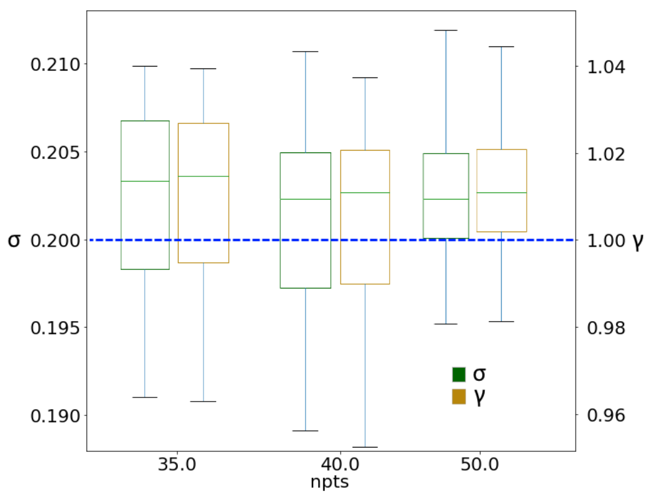

Fig. 3. Restored parameters σ and γ distributions, blue dashed line - theoretical value.

The results obtained in this section show that the polynomial of a required form may be obtained by simple machine learning tools such as LASSO regression. More advanced tools such as a mathematical expression discovery [10, 11] may be used to find the polynomial structure without any preliminary assumptions like pre-determined polynomial form Eq. 10.

4.2. Radially periodic membrane

In the previous section, we test the algorithm with the known solution. In this section, we recreate approximation from the displacement data for the radially periodic membrane problem. As shown in [8], the polynomial in form Eq. 15 may be considered as the approximation for a differential equation that defines propagator (like polynomial Eq. 5).

Following [8], we consider propagator $\Lambda(r)$ for the circular periodic membrane problem that shows stop- and pass-bands depending on a coordinate r. Thus, to obtain the data used within the minimization procedure Eq. 13, we need to consider several forcing problems with different coordinates r.

In this case we also can normalize polynomials to obtain the form Eq. 16, which allows to use the standard Floquet stop-band definition abs$(\Lambda) \neq 1$ and consider periodic circular membrane as the Floquet propagator. The result of normalization is shown in Fig. 4.

Since Floquet theory cannot be applied to circular membrane problem directly, we may only use indirect methods to validate results. The results may be validated, for example, with symmetrical cell eigenfrequencies [2]. Moreover, the obtained results agree with the results obtained theoretically in [8].

Thus, the process described in the Sec. 4.1 may also be

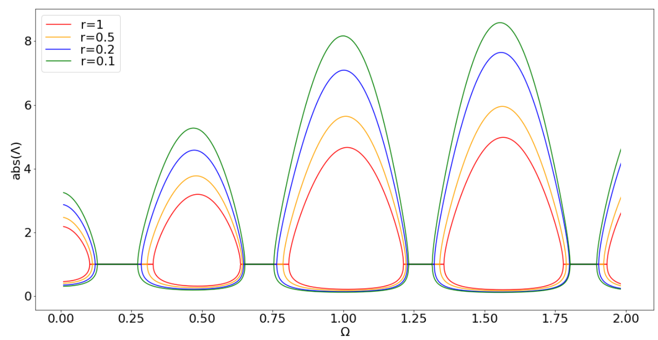

Fig. 4. Normalized data-driven polynomials for different values of displacements data coordinate r.

used in problems where the classical Floquet theory is not applicable to formulate the approximation that allows using Floquet propagator terminology.

5. CONSLUSION

In the paper, we formulate the machine learning data-driven approach that allows obtaining a propagator approximation from the displacement data. It is of interest in a different direction of phononic crystals properties analysis. The main advantages of the proposed approach are:

- Such an approach may be used in acoustical problems where the Floquet theory is not applicable, for example, in problems where translational symmetry is not preserved (like the circular membrane problem).
- The algorithm may be used to formulate approximations for the complex Floquet propagator like the elastic layer problem and allows analyzing the general acoustical properties with straightforward tools.
- It may be used to formulate approximation of the WFEM for complex problems and possibly solve the numerical problems faster.

6. DATA AND CODE AVAILABILITY

The scripts and pre-computed data may be obtained from repository `https://github.com/ITMO-NSS-team/data-driven-Floquet`.

7. ACKNOWLEDGEMENTS

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