SCATTERING STATISTICS OF GENERALIZED SPATIAL POISSON POINT PROCESSES

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ABSTRACT

We present a machine learning model for the analysis of randomly generated discrete signals, modeled as the points of an inhomogeneous, compound Poisson point process. Like the wavelet scattering transform introduced by Mallat, our construction is naturally invariant to translations and reflections, but it decouples the roles of scale and frequency, replacing wavelets with Gabor-type measurements. We show that, with suitable nonlinearities, our measurements distinguish Poisson point processes from common self-similar processes, and separate different types of Poisson point processes.

Index Terms— Scattering transform, Poisson point process, convolutional neural network

1. INTRODUCTION

Convolutional neural networks (CNNs) have obtained impressive results for a number of learning tasks in which the underlying signal data can be modelled as a stochastic process, including texture discrimination [1], texture synthesis [2, 3], time-series analysis [4], and wireless networks [5]. In many scenarios, it is natural to model the signal data as the points of a (potentially complex) spatial point process. Furthermore, there are numerous other fields, including stochastic geometry [6], forestry [7], geoscience [8] and genetics [9], in which spatial point processes are used to model the underlying generating process of certain phenomena (e.g., earthquakes). This motivates us to consider the capacity of CNNs to capture the statistical properties of such processes.

The Wavelet scattering transform [10] is a model for CNNs, which consists of an alternating cascade of linear wavelet transforms and complex modulus nonlinearities. It has provable stability and invariance properties and has been used to achieve near state of the art results in fields such as audio signal processing [11], computer vision [12], and quantum chemistry [13]. In this paper, we examine a generalized scattering transform that utilizes a broader class of filters (which includes wavelets). We primarily focus on filters with small support, which is similar to those used in most CNNs.

Expected wavelet scattering moments for stochastic processes with stationary increments were introduced in [14], where it is shown that such moments capture important statistical information of one-dimensional Poisson processes, fractional Brownian motion, α-stable Lévy processes, and a number of other stochastic processes. In this paper, we extend the notion of scattering moments to our generalized architecture, and generalize many of the results from [14]. However, the main contributions contained here consist of new results for more general spatial point processes, including inhomogeneous Poisson point processes, which are not stationary and do not have stationary increments. The collection of expected scattering moments is a non-parametric model for these processes, which we show captures important summary statistics.

In Section 2 we will define our expected scattering moments. Then, in Sections 3 and 4 we will analyze these moments for certain generalized Poisson point processes and self-similar processes. We will present numerical examples in Section 5, and provide a short conclusion in section 6.

2. EXPECTED SCATTERING MOMENTS

Let $\psi \in \mathbf{L}^2(\mathbb{R})$ be a compactly supported mother wavelet with dilations $\psi_j(t) = 2^{-j}\psi(2^{-j}t)$ for $j \in \mathbb{Z}$, and let $X(t), t \in \mathbb{R}$, be a stochastic process with stationary increments. The first-order wavelet scattering moments are defined in [14] as $SX(j) = \mathbb{E}[|\psi_j * X|]$, where the expectation does not depend on t since $X(t)$ has stationary increments and ψ_j is a wavelet which implies $X * \psi_j(t)$ is stationary. Much of the analysis of in [14] relies on the fact that these moments can be rewritten as $SX(j) = \mathbb{E}[|\overline{\psi}_j * dX|]$, where $d\overline{\psi}_j = \psi_j$. This motivates us to define scattering moments as the integration of a filter, against a random signed measure $Y(dt)$.

To that end, let $w \in \mathbf{L}^2(\mathbb{R}^d)$ be a continuous window function with support contained in $[0,1]^d$. Denote by $w_s(t) = w\left(\frac{t}{s}\right)$ the dilation of w, and set $g_\gamma(t)$ to be the Gabor-type filter with scale $s > 0$ and central frequency $\xi \in \mathbb{R}^d$,

$$g_\gamma(t) = w_s(t)e^{i\xi\cdot t}, \quad \gamma = (s,\xi), \ \ t \in \mathbb{R}^d\,. \tag{1}$$

M.H. acknowledges support from NSF DMS #1845856 (partially supporting J.H.), NIH NIGMS #R01GM135929, and DOE #DE-SC0021152.

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Note that with an appropriately chosen window function w, (1) includes dyadic wavelet families in the case that $s = 2^j$ and $|\xi| = C/s$. However, it also includes many other filters, such as Gabor filters used in the windowed Fourier transform.

Let $Y(dt)$ be a random signed measure and assume that Y is T-periodic for some $T > 0$ in the sense that for any Borel set B we have $Y(B) = Y(B + Te_i)$, for all $1 \leqslant i \leqslant d$ (where $\{e_i\}_{i \leqslant d}$ is the standard orthonormal basis for $\mathbb{R}^d$). For $f \in \mathbf{L}^2(\mathbb{R}^d)$, set $f * Y(t) := \int_{\mathbb{R}^d} f(t-u)Y(du)$. We define the first-order and second-order expected scattering moments, $1 \leqslant p, p' < \infty$, at location t as

$$S_{\gamma,p}Y(t) := \mathbb{E}[|g_\gamma * Y(t)|^p] \quad \text{and} \tag{2}$$

$$S_{\gamma,p,\gamma',p'}Y(t) := \mathbb{E}\left[||g_\gamma * Y|^p * g_{\gamma'}(t)|^{p'}\right]. \tag{3}$$

Note $Y(dt)$ is not assumed to be stationary, which is why these moments depend on t. Since $Y(dt)$ is periodic, we may also define time-invariant scattering coefficients by

$$SY(\gamma, p) := \frac{1}{T^d}\int_{[0,T]^d} S_{\gamma,p}Y(t)dt, \quad \text{and}$$

$$SY(\gamma, p, \gamma', p') := \frac{1}{T^d}\int_{[0,T]^d} S_{\gamma,p,\gamma',p'}Y(t)dt$$

In the following sections, we analyze these moments for arbitrary frequencies ξ and small scales s, thus allowing the filters g_γ to serve as a model for the learned filters in CNNs. In particular, we will analyze the asymptotic behavior of the scattering moments as s decreases to zero.

3. SCATTERING MOMENTS OF GENERALIZED POISSON PROCESSES

In this section, we let $Y(dt)$ be an inhomogeneous, compound spatial Poisson point process. Such processes generalize ordinary Poisson point processes by incorporating variable charges (heights) at the points of the process and a non-uniform intensity for the locations of the points. They thus provide a flexible family of point processes that can be used to model many different phenomena. In this section, we provide a review of such processes and analyze their first and second-order scattering moments.

Let $\lambda(t)$ be a continuous, periodic function on $\mathbb{R}^d$ with

$$0 < \lambda_{\min} := \inf_t \lambda(t) \leqslant \|\lambda\|_\infty < \infty, \tag{4}$$

and define its first and second order moments by

$$m_p(\lambda) := \frac{1}{T^d}\int_{[0,T]^d} \lambda(t)^2\, dt, \quad p = 1, 2.$$

A random measure $N(dt) := \sum_{j=1}^\infty \delta_{t_j}(dt)$ is called an inhomogeneous Poisson point process with intensity function $\lambda(t)$ if for any Borel set $B \subset \mathbb{R}^d$,

$$P(N(B) = n) = e^{-\Lambda(B)}\frac{\left(\Lambda(B)\right)^n}{n!}, \;\; \Lambda(B) = \int_B \lambda(t)\, dt,$$

and, in addition, $N(B)$ is independent of $N(B')$ for all B' that do not intersect B. Now let $(A_j)_{j=1}^\infty$ be a sequence of i.i.d. random variables independent of N. An inhomogeneous, compound Poisson point process $Y(dt)$ is given by

$$Y(dt) = \sum_{j=1}^\infty A_j \delta_{t_j}(dt). \tag{5}$$

For a further overview of these processes, we refer the reader to Section 6.4 of [15].

3.1. First-order Scattering Asymptotics

Computing the convolution of g_γ with $Y(dt)$ gives

$$(g_\gamma * Y)(t) = \int_{\mathbb{R}^d} g_\gamma(t-u)\, Y(du) = \sum_{j=1}^\infty A_j g_\gamma(t - t_j),$$

which can be interpreted as a waveform g_γ emitting from each location t_j. Invariant scattering moments aggregate the random interference patterns in $|g_\gamma * Y|$. The results below show that the expectation of these interference patterns encode important statistical information related to the point process.

For notational convenience, we let

$$\Lambda_s(t) := \Lambda\left([t-s, t]^d\right) = \int_{[t-s,t]^d} \lambda(u)\, du$$

denote the expected number of points of N in the support of $g_\gamma(t - \cdot)$. By conditioning on $N\left([t-s,t]^d\right)$, the number of points in the support of g_γ, and using the fact that

$$\mathbb{P}\left[N\left([t-s,t]^d\right) > m\right] = \mathcal{O}\left(\left(s^d\|\lambda\|_\infty\right)^{m+1}\right)$$

one may obtain the following theorem.[1]

Theorem 1. *Let $\mathbb{E}[|A_1|^p] < \infty$, and $\lambda(t)$ be a periodic continuous intensity function satisfying* (4)*. Then for every $t \in \mathbb{R}^d$, every $\gamma = (s, \xi)$ such that $s^d\|\lambda\|_\infty < 1$, and every $m \geqslant 1$,*

$$S_{\gamma,p}Y(t) \approx \sum_{k=1}^m e^{-\Lambda_s(t)}\frac{(\Lambda_s(t))^k}{k!}\mathbb{E}\left[\left|\sum_{j=1}^k A_j w(V_j) e^{is\xi\cdot V_j}\right|^p\right], \tag{6}$$

where the error term $\varepsilon(m, s, \xi, t)$ satisfies

$$|\varepsilon(m, s, \xi, t)| \leqslant C_{m,p}\frac{\|\lambda\|_\infty}{\lambda_{\min}}\|w\|_p^p\mathbb{E}[|A_1|^p]\|\lambda\|_\infty^{m+1}s^{d(m+1)} \tag{7}$$

and $V_1, V_2, \ldots$ is an i.i.d. sequence of random variables, independent of the A_j, taking values in the unit cube $[0,1]^d$ and with density $p_V(v) = \frac{s^d}{\Lambda_s(t)}\lambda(t - vs)$ for $v \in [0,1]^d$.

[1] A proof of Theorem 1, as well as the proofs of other theorems stated in this paper, is available at https://arxiv.org/abs/1902.03537.

If we set $m = 1$, and let $s \to 0$, then one may use the fact that a small cube $[t-s, t]^d$ has at most one point of N with overwhelming probability to obtain the following result.

Theorem 2. *Let $Y(dt)$ satisfy the same assumptions as in Theorem 1. Let $\gamma_k = (s_k, \xi_k)$ be a sequence of scale and frequency pairs such that $\lim_{k\to\infty} s_k = 0$. Then*

$$\lim_{k\to\infty} \frac{S_{\gamma_k,p}Y(t)}{s_k^d} = \lambda(t)\mathbb{E}[|A_1|^p]\|w\|_p^p\,, \tag{8}$$

for all t, and consequently

$$\lim_{k\to\infty} \frac{SY(\gamma_k, p)}{s_k^d} = m_1(\lambda)\mathbb{E}[|A_1|^p]\|w\|_p^p. \tag{9}$$

This theorem shows that for small scales the scattering moments $S_{\gamma,p}Y(t)$ encode the intensity function $\lambda(t)$, up to factors depending upon the summary statistics of the charges $(A_j)_{j=1}^\infty$ and the window w. Thus even a one-layer location-dependent scattering network yields considerable information regarding the underlying data generation process.

In the case of ordinary (non-compound) homogeneous Poisson processes, Theorem 2 recovers the constant intensity. For general $\lambda(t)$ and invariant scattering moments, the role of higher-order moments of $\lambda(t)$ is highlighted by considering higher-order expansions (e.g., $m > 1$) in (6). The next theorem considers second-order expansions and illustrates their dependence on the second moment of $\lambda(t)$.

Theorem 3. *Let Y satisfy the same assumptions as in Theorem 1. If $(\gamma_k)_{k\geqslant 1} = (s_k, \xi_k)_{k\geqslant 1}$, is a sequence such that $\lim_{k\to\infty} s_k = 0$ and $\lim_{k\to\infty} s_k\xi_k = L \in \mathbb{R}^d$, then*

$$\lim_{k\to\infty}\left(\frac{SY(\gamma_k,p)}{s_k^{2d}\mathbb{E}[|A_1|^p]\mathbb{E}[|V_k|^p]} - \frac{1}{T^d}\int_{[0,T]^d}\frac{\Lambda_{s_k}(t)}{s_k^{2d}}\,dt\right)$$
$$= m_2(\lambda)\left(\frac{\mathbb{E}\left[\left|A_1 w(U_1)e^{iL\cdot U_1} + A_2 w(U_2)e^{iL\cdot U_2}\right|^p\right]}{2\|w\|_p^p\mathbb{E}[|A_1|^p]}\right), \tag{10}$$

where U_1, U_2 are independent uniform random variables on $[0,1]^d$; and $(V_k)_{k\geqslant 1}$ is a sequence of random variables independent of the A_j taking values in the unit cube with respective densities, $p_{V_k}(v) = \frac{s_k^d}{\Lambda_{s_k}(t)}\lambda(t - vs_k)$ for $v \in [0,1]^d$.

We note that the scale normalization on the left hand side of (10) is s^{-2d}, compared to a normalization of s^{-d} in Theorem 2. Thus, intuitively, (10) is capturing information at moderately small scales that are larger than the scales considered in Theorem 2. Unlike Theorem 2, which gives a way to compute $m_1(\lambda)$, Theorem 3 does not allow one to compute $m_2(\lambda)$ since it would require knowledge of $\Lambda_{s_k}(t)$ in addition to the distribution from which the charges $(A_j)_{j=1}^\infty$ are drawn. However, Theorem 3 does show that at moderately small scales the invariant scattering coefficients depend non-trivially on the second moment of $\lambda(t)$. Therefore, they can be used to distinguish between, for example, an inhomogeneous Poisson point process with intensity function $\lambda(t)$ and a homogeneous Poisson point process with constant intensity.

3.2. Second-Order Scattering Moments of Generalized Poisson Processes

Our next result shows that second-order scattering moments encode higher-order moment information about the $(A_j)_{j=1}^\infty$.

Theorem 4. *Let $Y(dt)$ satisfy the same assumptions as in Theorem 1. Let $\gamma_k = (s_k, \xi_k)$ and $\gamma'_k = (s'_k, \xi'_k)$ be sequences of scale-frequency pairs with $s'_k = cs_k$ for some $c > 0$ and $\lim_{k\to\infty} s_k\xi_k = L \in \mathbb{R}^d$. Let $1 \leqslant p, p' < \infty$ and $q = pp'$. Assume $\mathbb{E}|A_1|^q < \infty$, and let $K := \left\|g_{c,L/c} * |g_{1,0}|^p\right\|_{p'}^{p'}$. Then,*

$$\lim_{k\to\infty}\frac{S_{\gamma_k,p,\gamma'_k,p'}Y(t)}{s_k^{d(p'+1)}} = K\lambda(t)\mathbb{E}[|A_1|^q], \quad \textit{and} \tag{11}$$

$$\lim_{k\to\infty}\frac{SY(\gamma_k,p,\gamma'_k,p')}{s_k^{d(p'+1)}} = Km_1(\lambda)\mathbb{E}[|A_1|^q]\,. \tag{12}$$

Theorem 2 shows first-order scattering moments with $p = 1$ are not able to distinguish between different types of Poisson point processes at very small scales if the charges have the same first moment. However, Theorem 4 shows second-order scattering moments encode higher-moment information about the charges, and thus are better able to distinguish them (when used in combination with the first-order coefficients). In Sec. 4, we will see first-order invariant scattering moments can distinguish Poisson point processes from self-similar processes if $p = 1$, but may fail to do so for larger values of p.

4. COMPARISON TO SELF-SIMILAR PROCESSES

We show first-order scattering moments can distinguish between Poisson point processes and certain self-similar processes, such as α-stable processes, or fractional Brownian motion (fBM). These results generalize those in [14] both by considering p^{th} scattering moments and more general filters.

For a stochastic process $X(t)$, $t \in \mathbb{R}$, we consider the convolution of the filter g_γ with the noise dX defined by $g_\gamma * dX(t) := \int_{\mathbb{R}} g_\gamma(t-u)\,dX(u)$, and define (in a slight abuse of notation) the first-order scattering moments at time t by $S_{\gamma,p}X(t) := \mathbb{E}[|g_\gamma * dX(t)|^p]$. In the case where $X(t)$ is a compound, inhomogeneous Poisson (counting) process, $Y = dX$ will be a compound Poisson random measure and these scattering moments will coincide with those defined in (2).

The following theorem analyzes the small-scale first-order scattering moments when X is either an α-stable process, or an fBM. It shows the small-scale asymptotics of the corresponding scattering moments are guaranteed to differ from those of a Poisson point process when $p = 1$. We also

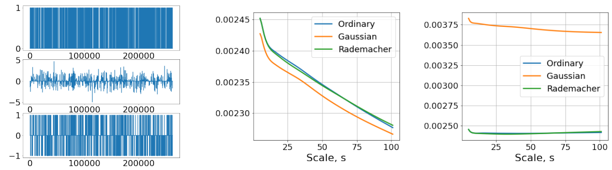

Fig. 1. First-order invariant scattering moments of homogeneous compound Poisson point processes with the same intensity λ_0 and different A_i. **Left:** Realizations of the process with arrival rates given by *Top:* $A_i = 1$ *Middle:* A_i are normal random variables *Bottom:* A_i are Rademacher random variables. **Middle:** Plots of normalized first-order scattering $\frac{SY(s,\xi,1)}{s\|w\|_1}$ moments with $p = 1$.**Right:** Plots of normalized first-order scattering $\frac{SY(s,\xi,2)}{s\|w\|_2^2}$ moments with $p = 2$.

note that both α-stable processes and fBM have stationary increments and thus $S_{\gamma,p}X(t) = SX(\gamma, p)$ for all t.

Theorem 5. *Let* $1 \leqslant p < \infty$*, and let* $\gamma_k = (s_k, \xi_k)$ *be a sequence of scale-frequency pairs with* $\lim_{k\to\infty} s_k = 0$ *and* $\lim_{k\to\infty} s_k\xi_k = L \in \mathbb{R}$*. Then, if* $X(t)$ *is a symmetric* α*-stable process,* $p < \alpha \leqslant 2$*, we have*

$$\lim_{k\to\infty} \frac{SX(\gamma_k, p)}{s_k^{p/\alpha}} = \mathbb{E}\left[\left|\int_0^1 w(u)e^{iLu}\, dX(u)\right|^p\right].$$

Similarly, if $X(t)$ *is an fBM with Hurst parameter* $H \in (0, 1)$ *and* w *has bounded variation on* $[0, 1]$*, then*

$$\lim_{k\to\infty} \frac{SX(\gamma_k, p)}{s_k^{pH}} = \mathbb{E}\left[\left|\int_0^1 w(u)e^{iLu}\, dX(u)\right|^p\right].$$

This theorem shows that first-order invariant scattering moments distinguish inhomogeneous, compound Poisson processes from both α-stable processes and fractional Brownian motion except in the cases where $p = \alpha$ or $p = 1/H$. In particular, these measurements distinguish Brownian motion, from a Poisson point process except in the case where $p = 2$.

5. NUMERICAL ILLUSTRATIONS

We carry out several experiments to numerically validate the previously stated results. In all of our experiments, we hold the frequency ξ constant while letting s decrease to zero.

Compound Poisson point processes with the same intensities: We generated three homogeneous compound Poisson point processes, all with intensity $\lambda(t) \equiv \lambda_0 = 0.01$, where the charges $A_{1,j}$, $A_{2,j}$, and $A_{3,j}$ are chosen so that $A_{1,j} = 1$ uniformly, $A_{2,j} \sim \mathcal{N}(0, \sqrt{\frac{\pi}{2}})$, and $A_{3,j}$ are Rademacher random variables. The charges of the three signals have the same first moment $\mathbb{E}[|A_{i,j}|] = 1$ and different second moment with $\mathbb{E}[|A_{1,j}|^2] = \mathbb{E}[|A_{3,j}|^2] = 1$ and $\mathbb{E}[|A_{2,j}|^2] = \frac{\pi}{2}$. As predicted by Theorem 2, Figure 1 shows first-order scattering

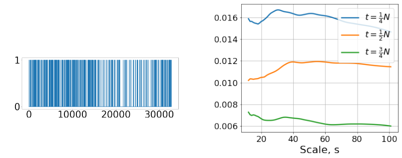

Fig. 2. First-order scattering moments for inhomogeneous Poisson point processes. **Left:** Sample realization with $\lambda(t) = 0.01(1 + 0.5\sin(\frac{2\pi t}{N}))$. **Right:** Time-dependent scattering moments $\frac{S_{\gamma,p}Y(t)}{s\|w\|_p^p}$ at $t_1 = \frac{N}{4}$, $t_2 = \frac{N}{2}$, $t_3 = \frac{3N}{4}$. Note that the scattering coefficients at times t_1, t_2, t_3 converges to $\lambda(t_1) = 0.015$, $\lambda(t_2) = 0.01$, $\lambda(t_3) = 0.005$.

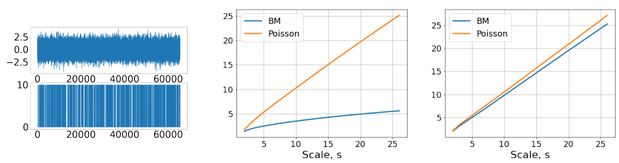

Fig. 3. First-order invariant scattering moments for standard Brownian motion and Poisson point process. **Left:** Sample realizations *Top:* Brownian motion. *Bottom:* Ordinary Poisson point process. **Middle:** Normalized scattering moments $\frac{SY_{\text{poisson}}(x,\xi,p)}{\lambda\mathbb{E}|A_1|^p\|w\|_p^p}$ and $\frac{SX_{\text{BM}}(s,\xi,p)}{\lambda\mathbb{E}|Z|^p\|w\|_p^p}$ for Poisson and BM with $p = 1$. **Right:** The same but with $p = 2$.

moments will not be able to distinguish between the three processes with $p = 1$, but will distinguish the process with Gaussian charges from the other two when $p = 2$.

Inhomogeneous, non-compound Poisson point processes: We also consider an inhomogeneous, non-compound Poisson point processes with intensity function $\lambda(t) = 0.01(1 + 0.5\sin(\frac{2\pi t}{N}))$ (where we estimate $S_{\gamma,p}Y(t)$, by averaging over 1000 realizations). Figure 2 plots the scattering moments for the inhomogeneous process at different times, and shows they align with the true intensity function.

Poisson point process and self similar process: We consider a Brownian motion compared to a Poisson point process with intensity $\lambda = 0.01$ and charges $(A)_{j=1}^{\infty} \equiv 10$. Figure 3 shows the convergence rate of the first-order scattering moments can distinguish these processes when $p = 1$ but not when $p = 2$.

6. CONCLUSION

We have constructed Gabor-filter scattering transforms for random measures on $\mathbb{R}^d$. Our work is closely related to [14] but considers more general classes of filters and point processes (although we note that [14] provides a more detailed analysis of self-similar processes). In future work, it would be interesting to explore the use of these measurements for tasks such as, e.g., synthesizing new signals.

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