INFORMATION THEORETIC LIMITS FOR STANDARD AND ONE-BIT COMPRESSED SENSING WITH GRAPH-STRUCTURED SPARSITY

Adarsh Barik, Jean Honorio

Purdue University
Department of Computer Science
West Lafayette, Indiana, USA

ABSTRACT

In this paper, we analyze the information theoretic lower bound on the necessary number of samples needed for recovering a sparse signal under different compressed sensing settings. We focus on the weighted graph model, a model-based framework proposed by [1], for standard compressed sensing as well as for one-bit compressed sensing. We study both the noisy and noiseless regimes. Our analysis is general in the sense that it applies to any algorithm used to recover the signal. We carefully construct restricted ensembles for different settings and then apply Fano's inequality to establish the lower bound on the necessary number of samples. Furthermore, we show that our bound is tight for one-bit compressed sensing, while for standard compressed sensing, our bound is tight up to a logarithmic factor of the number of non-zero entries in the signal. *A full version of this paper is accessible at:* `https://www.cs.purdue.edu/homes/abarik/abarik_cs_icassp_full.pdf`

***Index Terms*—** Compressive sensing, weighted graph model, information theoretic bounds

1. INTRODUCTION

Sparsity has been a useful tool to tackle high dimensional problems in many fields such as compressed sensing, machine learning and statistics. Several naturally occurring and artificially created signals manifest sparsity in their original or transformed domain. For instance, sparse signals play an important role in applications such as medical imaging, geophysical and astronomical data analysis, computational biology, remote sensing as well as communications.

In compressed sensing, sparsity of a high dimensional signal allows for the efficient inference of such a signal from a small number of observations. The true high dimensional sparse signal $\boldsymbol{\beta}^* \in \mathbb{R}^d$ is not observed directly. Instead, a low dimensional linear transformation $\mathbf{y} \in \mathbb{R}^n$ is observed along with a design matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ such that:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}^* + \mathbf{e} \ , \tag{1}$$

where $\mathbf{e} \in \mathbb{R}^d$ is a zero mean independent additive noise. The true high dimensional sparse signal $\boldsymbol{\beta}^*$ is then inferred from observations $(\mathbf{X}, \mathbf{y})$. We refer to the setup in equation 1 as "standard compressed sensing". Many signal acquisition settings such as magnetic resonance imaging [9] use standard compressed sensing as their underlying model. One-bit compressed sensing [10, 6] is a generalization of the setup presented in (1), in the sense that it considers quantizing the measurements to one bit, i.e.,

$$\mathbf{y} = \operatorname{sign}(\mathbf{X}\boldsymbol{\beta}^* + \mathbf{e}) \ , \tag{2}$$

where the function sign acts on individual entries of $\mathbf{X}\boldsymbol{\beta}^* + \mathbf{e}$ and returns their sign. This kind of quantization is particularly appealing for hardware implementations.

The learning problem in compressed sensing is to recover a signal which is a *good* approximation of the true signal. The goodness of approximation can be measured by either a pre-specified distance between the inferred and the true signal, e.g., ℓ_0-norm, ℓ_1-norm., or by the similarity of their support (i.e., the indices of their non-zero entries). The algorithms for compressed sensing try to provide performance guarantees for either one or both of these measures. For instance, [11], [12], [13], [14] and [15] provide performance guarantees in terms of distance, while [16] and [17] provide performance guarantees in terms of support recovery for standard compressed sensing. The authors in [8] provide guarantees in terms of both distance and support for one-bit compressed sensing. All the aforementioned works use deterministic sparse signals. There have been some empirical studies [18, 19, 20] where Bayesian priors have been used to model stochastic sparse signals but in this paper, we do not focus on the Bayesian framework.

The authors in [2] initially proposed a model-based sparse recovery framework. Under this framework, they showed that the sufficient number of samples for correct recovery is logarithmic with respect to the cardinality of the sparsity model (the sparsity model is defined only by the support of the sparse signal $\boldsymbol{\beta}^*$), i.e., the number of supports in the sparsity model. The model of [2] considered signals with common sparsity structure and small cardinality. Later, [1] proposed a weighted graph model for graph-structured sparsity and accompanied

Table 1: Sample Complexity Results for Structured Sparsity Models (d is the dimension of the true signal, s is the signal sparsity, i.e., the number of non-zero entries, g is the number of connected components, $\rho(G)$ is the maximum weight degree of graph G, B is the weight budget in the weighted graph model, K is the block sparsity, J is the number of entries in a block and N is the total number of blocks in the block structured sparsity model

	Standard Compressed Sensing		One-bit Compressed Sensing	
Sparsity Structure	Our Lower Bound	Upper Bound	Our Lower Bound	Upper Bound
Weighted Graph Model	$\tilde{\Omega}(s(\log \rho(G)\frac{B}{s}) + g\log\frac{d}{g})$	$O(s(\log \rho(G)\frac{B}{s})+g\log\frac{d}{g})$ [1]	$\Omega(s(\log \rho(G)\frac{B}{s})+g\log\frac{d}{g})$	NA
Tree Structured	$\tilde{\Omega}(s)$	$O(s)$ [2]	$\Omega(s)$	NA
Block Structured	$\tilde{\Omega}(KJ + K\log\frac{N}{K})$	$O(KJ + K\log\frac{N}{K})$ [2]	$\Omega(KJ + K\log\frac{N}{K})$	NA
Regular s-sparsity	$\tilde{\Omega}(s\log\frac{d}{s})$	$O(s\log\frac{d}{s})$ [3, 4, 5]	$\Omega(s\log\frac{d}{s})$	$O(s\log\frac{d}{s})$[6, 7, 8]

it with a nearly linear time recovery algorithm. They also analyzed the sufficient number of samples for efficient recovery. Notice that regular sparse signals, i.e., signals without any additional sparsity structure can be easily modeled using the weighted graph model. Independently, there has been a long line of work ([5, 4, 3]) which deals with algorithms with support recovery guarantees for regular sparse signals.

In this paper, we analyze the necessary condition on the sample complexity for exact sparse recovery. While our proof techniques can also be applied to any model-based sparse recovery framework, we apply our method to get the necessary number of samples to perform efficient recovery on a weighted graph model. We provide results for both the noisy and noiseless regimes of compressed sensing. We also extend our results to one-bit compressed sensing. We note that a lower bound on sample complexity was previously provided in [21] when the observer has access to only the measurements $\mathbf{y}$. Here, we analyze the more relevant setting in which the observer has access to the measurements $\mathbf{y}$ along with the design matrix $\mathbf{X}$. Compared to [21], we additionally analyze one-bit compressed sensing in detail. Table 1 shows a comparison of our information theoretic lower bounds on sample complexity under different settings with the existing upper bounds available in the literature. Note that our bounds for one-bit compressed sensing match upper bounds by [6]. The author in [22] provided information-theoretic lower bounds for regular sparsity in standard compressed sensing. For this case, along with other instances of standard compressed sensing our bounds are tight up to a factor of $\log s$, where s is the number of non-zero entries in $\boldsymbol{\beta}^*$.

2. PROBLEM DESCRIPTION

In this section, we introduce the observation model and later specialize it for specific problems such as standard compressed sensing and one-bit compressed sensing.

2.1. Notation

In what follows, we list down the notations which we use throughout the paper. The unobserved true d-dimensional signal is denoted by $\boldsymbol{\beta}^* \in \mathbb{R}^d$. The inferred signal is represented by $\hat{\boldsymbol{\beta}} \in \mathbb{R}^d$. We call a signal $\boldsymbol{\beta} \in \mathbb{R}^d, s < d$ an s-sparse signal if $\boldsymbol{\beta}$ contains only s non-zero entries. The n-dimensional observations are denoted by $\mathbf{y} \in \mathbb{R}^n, n \ll d$. We denote the design matrix by $\mathbf{X} \in \mathbb{R}^{n\times d}$. The $(i,j)^{\text{th}}$ element of the design matrix is denoted by $\mathbf{X}_{ij}, \forall\ 1 \le i \le n, 1 \le j \le d$. The i^{th} row of $\mathbf{X}$ is denoted by $\mathbf{X}_{i.}, \forall\ 1 \le i \le n$, and the j^{th} column of $\mathbf{X}$ is denoted by $\mathbf{X}_{.j}, \forall\ 1 \le j \le d$,. We assume that the true signal $\boldsymbol{\beta}^*$ belongs to a set $\mathcal{F}$, which is defined more formally later. The number of elements in a set A is denoted by $|A|$. The measurement vector $\mathbf{y} \in \mathbb{R}^n$ is a function $f(\mathbf{X}\boldsymbol{\beta}^* + \mathbf{e})$ of $\mathbf{X}, \boldsymbol{\beta}^*$ and $\mathbf{e}$ where $\mathbf{e} \in \mathbb{R}^n$ is Gaussian noise with i.i.d. entries, each with mean 0 and variance σ^2. For brevity, we use function notation f while keeping in mind that f acts on each row of $\mathbf{X}\boldsymbol{\beta}^* + \mathbf{e}$. The probability of the occurrence of an event E is denoted by $\mathbb{P}(E)$. The shorthand notation $[p]$ is used to denote the set $\{1, 2, \ldots, p\}$. Other notations specific to weighted graph models are defined later in Section 3. For brevity and comparisons, we use the order notations $\tilde{o}$ and $\tilde{\Omega}$ which ignore logarithmic factors of s.

2.2. Observation Model

We define a general observation model. The learning problem is to estimate the unobserved true s-sparse signal $\boldsymbol{\beta}^*$ from noisy observations. Since $\boldsymbol{\beta}^*$ is a high dimensional signal, we do not sample it directly. Rather, we observe a function of its inner product with the rows of a randomized matrix $\mathbf{X}$. Formally, the i^{th} measurement $\mathbf{y}_i$ comes from the model, $\mathbf{y}_i = f(\mathbf{X}_{i.}\boldsymbol{\beta}^* + \mathbf{e}_i)$, where $f : \mathbb{R} \to \mathbb{R}$ is a fixed function. We observe n such i.i.d. samples and collect them in the measurement vector $\mathbf{y} \in \mathbb{R}^n$. We can express this mathematically by,

$$\mathbf{y} = f(\mathbf{X}\boldsymbol{\beta}^* + \mathbf{e})\ . \tag{3}$$

Our task is to recover an estimate $\hat{\boldsymbol{\beta}} \in \mathbb{R}^d$ of $\boldsymbol{\beta}^*$ from the observations $(\mathbf{X}, \mathbf{y})$. By choosing an appropriate function f, we can describe specific instances of compressed sensing.

2.2.1. Standard Compressed Sensing

The standard compressed sensing is a special case of equation (3) by choosing $f(x) = x$. Then we simply have,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}^* + \mathbf{e} \,. \tag{4}$$

2.2.2. One-bit Compressed Sensing

The problem of signal recovery in one-bit compressed sensing has been introduced recently [10]. In this setup, we do not have access to linear measurements but rather observations come in the form of a single bit. This can be modeled by choosing $f(x) = \text{sign}(x)$. In other words, we have,

$$\mathbf{y} = \text{sign}(\mathbf{X}\boldsymbol{\beta}^* + \mathbf{e}) \,. \tag{5}$$

Note that we lose lot of information by limiting the observations to a single bit. It is known that for the noiseless case, unlike standard compressed sensing, one can only recover $\boldsymbol{\beta}^*$ up to scaling[6].

2.3. Problem Setting

We assume that the nature picks a true s-sparse signal $\boldsymbol{\beta}^*$ uniformly at random from a set of signals $\mathcal{F}$. Then observations are generated using the model described in equation (3). The function f is chosen appropriately for different settings. We also assume that the observer has access to the design matrix $\mathbf{X}$. Thus, observations are denoted by $(\mathbf{X}, \mathbf{y})$. This procedure can be interpreted as a Markov chain which is described as $\boldsymbol{\beta}^* \rightarrow \big(\mathbf{X}, \mathbf{y} = f(\mathbf{X}\boldsymbol{\beta}^* + \mathbf{e})\big) \rightarrow \hat{\boldsymbol{\beta}}$. We use this Markov chain in our proofs. We assume that the true signal $\boldsymbol{\beta}^*$ comes from a weighted graph model. We state our results for standard sparse compressed sensing and one-bit compressed sensing. We note that our arguments for establishing information theoretic lower bounds are not algorithm specific. We use Fano's inequality [23] to prove our result by carefully constructing restricted ensembles. Any algorithm which infers $\boldsymbol{\beta}^*$ from this particular ensemble would require a minimum number of samples. The use of restricted ensembles is customary for information theoretic lower bounds [24, 25].

3. WEIGHTED GRAPH MODEL (WGM)

We assume that the true s-sparse signal comes from a weighted graph model. This encompasses many commonly seen sparsity patterns in signals such as tree structured sparsity, block structured sparsity as well as the regular s-sparsity without any additional structure. Next, we introduce the Weighted Graph Model (WGM) which was proposed by [1].

The Weighted Graph Model is defined on an underlying graph $G = (V, E)$ whose vertices represent the coefficients of the unknown $s-$sparse vector $\boldsymbol{\beta}^* \in \mathbb{R}^d$ i.e. $V = [d] = \{1, 2, \ldots, d\}$. Moreover, the graph is weighted and thus we introduce a weight function $w : E \rightarrow \mathbb{N}$. Borrowing some notations from [1], $w(F)$ denotes the sum of edge weights in a forest $F \subseteq G$, i.e., $w(F) = \sum_{e \in F} w_e$. We also assume an upper bound on the total edge weight which is called the weight budget and is denoted by B. The number of non-zero coefficients of $\boldsymbol{\beta}^*$ is denoted by the sparsity parameter s. The number of connected components in a forest F is denoted by $\gamma(F)$. The weight-degree $\rho(v)$ of a node $v \in V$ is the largest number of adjacent nodes connected by edges with the same weight, i.e., $\rho(v) = \max_{b \in \mathbb{N}} |\{(v', v) \in E \mid w(v', v) = b\}|$. We define the weight-degree $\rho(G)$ of G to be the maximum weight-degree of any $v \in V$. Next, we define the Weighted Graph Model on coefficients of $\boldsymbol{\beta}^*$ as follows:

Definition 1 ([1]). *The $(G, s, g, B) - WGM$ is the set of supports defined as $\mathbb{M} = \{S \subseteq [d] \mid |S| = s$ and $\exists\, F \subseteq G$ with $V_F = S, \gamma(F) = g,\ w(F) \leq B\}$.*

4. MAIN RESULTS

In this section, we state our results for the standard compressed sensing and one-bit compressed sensing. We consider both the noisy and noiseless cases. We establish an information theoretic lower bound on the sample complexity for signal recovery on a WGM (See Appendix A for detailed proofs).

4.1. Results for Standard Compressed Sensing

For standard compressed sensing, the recovery is not exact for the noisy case but it is sufficiently close to the true signal in ℓ_2-norm with respect to the noise. Our setup, in this case, uses a Gaussian design matrix. The formal statement of our result is as follows.

Theorem 1 (Standard Compressed Sensing, Noisy Case)**.** *There exist a (G, s, g, B)-WGM $\mathbb{M}$ and a finite set of weights ψ such that if nature picks $\boldsymbol{\beta}^*$ uniformly at random from $\mathcal{F} = \{\boldsymbol{\beta} \mid \boldsymbol{\beta}_i = 0, \text{if } i \notin S, \boldsymbol{\beta}_i \in \psi, \text{if } i \in S, S \in \mathbb{M}\}$ and draws $n \in \tilde{o}(\log|\mathcal{F}|) = \tilde{o}((s-g)(\log\rho(G) + \log\frac{B}{s-g}) + g\log\frac{d}{g} + (s-g)\log\frac{g}{s-g} + s\log 2)$ i.i.d. observations from a family of distributions $\mathcal{D}$ indexed by $\boldsymbol{\beta} \in \mathbb{R}^d$ following equation* (4)*, i.e., $(\mathbf{X}_{i.}, \mathbf{y}_i) \sim D(\boldsymbol{\beta})$ which represents $\mathbf{X}_{i.} \sim \mathcal{N}(0, \frac{1}{n})^d, \mathbf{y}_i = \mathbf{X}_{i.}\boldsymbol{\beta} + \mathbf{e}_i, \mathbf{e}_i \sim \mathcal{N}(0, \sigma^2), \forall i \in \{1 \ldots n\}$ then,*

$$\inf_{\hat{\boldsymbol{\beta}}} \sup_{D \in \mathcal{D}} \mathop{\mathbb{P}}_{\substack{\boldsymbol{\beta}^* \sim \text{Unif}(\mathcal{F}) \\ (\mathbf{X}, \mathbf{y}) \sim D(\boldsymbol{\beta}^*)^n}} (\|\boldsymbol{\beta}^* - \hat{\boldsymbol{\beta}}(\mathbf{X}, \mathbf{y})\| \geq C\|\mathbf{e}\|) \geq \frac{1}{10}$$

for $0 < C \leq C_0$, where $\hat{\boldsymbol{\beta}} : \mathbb{R}^{n \times d} \times \mathbb{R}^n \rightarrow \mathbb{R}^d$ is any procedure used to infer $\boldsymbol{\beta}^*$, and $\mathcal{D}$ is the family of all distributions indexed by $\boldsymbol{\beta} \in \mathbb{R}^d$ with support on $\mathbb{R}^{n \times d} \times \mathbb{R}^n$.

We provide a similar result for the noiseless case. In this case recovery is exact. We use a Bernoulli design matrix for our proofs. Note that a Bernoulli random variable denoted by $\text{Bernoulli}(0.5)$ takes a value -1 with probability 0.5 and $+1$ with probability 0.5. In what follows, we state our result.

Theorem 2 (Standard Compressed Sensing, Noiseless Case)**.** *There exist a (G, s, g, B)-WGM $\mathbb{M}$ and a finite set of weights ψ such that if nature picks $\boldsymbol{\beta}^*$ uniformly at random from $\mathcal{F} = \{\boldsymbol{\beta} \mid \boldsymbol{\beta}_i = 0, \text{if } i \notin S, \boldsymbol{\beta}_i \in \psi, \text{if } i \in S, S \in \mathbb{M}\}$ and draws $n \in \tilde{o}(\log|\mathcal{F}|) = \tilde{o}((s-g)(\log\rho(G) + \log\frac{B}{s-g}) + g\log\frac{d}{g} + (s-g)\log\frac{g}{s-g} + s\log 2)$ i.i.d. observations from a family of distributions $\mathcal{D}$ indexed by $\boldsymbol{\beta} \in \mathbb{R}^d$ following equation* (4) *with zero noise, i.e., $(\mathbf{X}_{i.}, \mathbf{y}_i) \sim D(\boldsymbol{\beta})$ which represents $\mathbf{X}_{i.} \sim \frac{1}{\sqrt{n}}\text{Bernoulli}(0.5)^d, \mathbf{y}_i = \mathbf{X}_{i.}\boldsymbol{\beta}, \forall i \in \{1 \dots n\}$ then,*

$$\inf_{\hat{\boldsymbol{\beta}}} \sup_{D \in \mathcal{D}} \mathop{\mathbb{P}}_{\substack{\boldsymbol{\beta}^* \sim \text{Unif}(\mathcal{F}) \\ (\mathbf{X},\mathbf{y}) \sim D(\boldsymbol{\beta}^*)^n}} (\boldsymbol{\beta}^* \neq \hat{\boldsymbol{\beta}}(\mathbf{X}, \mathbf{y})) \geq \frac{1}{2}$$

where $\hat{\boldsymbol{\beta}} : \mathbb{R}^{n \times d} \times \mathbb{R}^n \to \mathbb{R}^d$ is any procedure used to infer $\boldsymbol{\beta}^*$, and $\mathcal{D}$ is the family of all distributions indexed by $\boldsymbol{\beta} \in \mathbb{R}^d$ with support on $\mathbb{R}^{n \times d} \times \mathbb{R}^n$.

We note that when $s \gg g$ and $B \geq s - g$ then $\tilde{\Omega}((s-g)(\log\rho(G) + \log\frac{B}{s-g}) + g\log\frac{d}{g} + (s-g)\log\frac{g}{s-g} + s\log 2)$ is roughly $\tilde{\Omega}(s(\log\rho(G) + \log\frac{B}{s}) + g\log\frac{d}{g})$.

4.2. Results for One-bit Compressed Sensing

In this setting, we provide a construction which works for both the noisy and noiseless case. We use a Bernoulli design matrix for both of these setups. Our first result in this setting shows that even in our restricted ensemble recovering the true signal exactly is difficult.

Theorem 3 (One-bit Compressed Sensing, Exact Recovery)**.** *There exist a (G, s, g, B)-WGM $\mathbb{M}$ and a finite set of weights ψ such that if nature picks $\boldsymbol{\beta}^*$ uniformly at random from $\mathcal{F} = \{\boldsymbol{\beta} \mid \boldsymbol{\beta}_i = 0, \text{if } i \notin S, \boldsymbol{\beta}_i \in \psi, \text{if } i \in S, S \in \mathbb{M}\}$ and draws $n \in o(\log|\mathcal{F}|) = o((s-g)(\log\rho(G) + \log\frac{B}{s-g}) + g\log\frac{d}{g} + (s-g)\log\frac{g}{s-g} + s\log 2)$ i.i.d. observations from a family of distributions $\mathcal{D}$ indexed by $\boldsymbol{\beta} \in \mathbb{R}^d$ following equation* (5)*, i.e., $(\mathbf{X}_{i.}, \mathbf{y}_i) \sim D(\boldsymbol{\beta})$ which represents $\mathbf{X}_{i.} \sim \frac{1}{\sqrt{n}}\text{Bernoulli}(0.5)^d, \mathbf{y}_i = \text{sign}(\mathbf{X}_{i.}\boldsymbol{\beta} + \mathbf{e}_i), \mathbf{e}_i \sim \mathcal{N}(0, \sigma^2), \forall i \in \{1 \dots n\}$ then,*

$$\inf_{\hat{\boldsymbol{\beta}}} \sup_{D \in \mathcal{D}} \mathop{\mathbb{P}}_{\substack{\boldsymbol{\beta}^* \sim \text{Unif}(\mathcal{F}) \\ (\mathbf{X},\mathbf{y}) \sim D(\boldsymbol{\beta}^*)^n}} (\boldsymbol{\beta}^* \neq \hat{\boldsymbol{\beta}}(\mathbf{X}, \mathbf{y})) \geq \frac{1}{2}$$

where $\hat{\boldsymbol{\beta}} : \mathbb{R}^{n \times d} \times \{-1, +1\}^n \to \mathbb{R}^d$ is any procedure used to infer $\boldsymbol{\beta}^*$, and $\mathcal{D}$ is the family of all distributions indexed by $\boldsymbol{\beta} \in \mathbb{R}^d$ with support on $\mathbb{R}^{n \times d} \times \{-1, +1\}^n$.

Our second result provides a bound on the necessary number of samples for approximate signal recovery, which we state formally below.

Theorem 4 (One-bit Compressed Sensing, Approximate Recovery)**.** *There exist a (G, s, g, B)-WGM $\mathbb{M}$ and a finite set of weights ψ such that if nature picks $\boldsymbol{\beta}^*$ uniformly at random from $\mathcal{F} = \{\boldsymbol{\beta} \mid \boldsymbol{\beta}_i = 0, \text{if } i \notin S, \boldsymbol{\beta}_i \in \psi, \text{if } i \in S, S \in \mathbb{M}\}$ and draws $n \in o(\log|\mathcal{F}|) = o((s-g)(\log\rho(G) + \log\frac{B}{s-g}) + g\log\frac{d}{g} + (s-g)\log\frac{g}{s-g} + s\log 2)$ i.i.d. observations from family of distributions $\mathcal{D}$ indexed by $\boldsymbol{\beta} \in \mathbb{R}^d$ following equation* (5)*, i.e., $(\mathbf{X}_{i.}, \mathbf{y}_i) \sim D(\boldsymbol{\beta})$ which represents $\mathbf{X}_{i.} \sim \frac{1}{\sqrt{n}}\text{Bernoulli}(0.5)^d, \mathbf{y}_i = \text{sign}(\mathbf{X}_{i.}\boldsymbol{\beta} + \mathbf{e}_i), \mathbf{e}_i \sim \mathcal{N}(0, \sigma^2), \forall i \in \{1 \dots n\}$ then,*

$$\inf_{\hat{\boldsymbol{\beta}}} \sup_{D \in \mathcal{D}} \mathop{\mathbb{P}}_{\substack{\boldsymbol{\beta}^* \sim \text{Unif}(\mathcal{F}) \\ (\mathbf{X},\mathbf{y}) \sim D(\boldsymbol{\beta}^*)^n}} \left(\left\| \frac{\hat{\boldsymbol{\beta}}(\mathbf{X}, \mathbf{y})}{\|\hat{\boldsymbol{\beta}}(\mathbf{X}, \mathbf{y})\|} - \frac{\boldsymbol{\beta}^*}{\|\boldsymbol{\beta}^*\|} \right\| \geq \epsilon \right) \geq \frac{1}{2}$$

for some $\epsilon > 0$, where $\hat{\boldsymbol{\beta}} : \mathbb{R}^{n \times d} \times \{-1, +1\}^n \to \mathbb{R}^d$ is any procedure used to infer $\boldsymbol{\beta}^*$, and $\mathcal{D}$ is the family of all distributions indexed by $\boldsymbol{\beta} \in \mathbb{R}^d$ with support on $\mathbb{R}^{n \times d} \times \{-1, +1\}^n$.

Our proof techniques can be applied to prove lower bounds of the sample complexity for several specific sparsity structures as long as one can bound the cardinality of the model. For well-known sparsity structures such that tree-structured sparsity, block sparsity and regular s-sparsity our lower bounds for standard compressed sensing match with respective upper bounds up to a factor of $\log s$. For one-bit compressed sensing, we provide novel lower bounds for tree-structured sparsity and block sparsity. The use of the model-based framework for one-bit compressed sensing remains an open area of research and our information theoretic lower bounds on sample complexity may act as a baseline comparison for the algorithms proposed in the future. In the case of regular s-sparsity, the bound on the sample complexity for one-bit compressed sensing is tight as it matches the current upper bound [6] (See Table 1 and Appendix B for details).

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