# THE PROTOTYPE CO-PRIME ARRAY WITH A ROBUST DIFFERENCE CO-ARRAY


*Ahmed M. A. Shaalan*[†] *and Jun Du*[†,*]

[†]University of Science and Technology of China, Hefei, China
ShaalanAhmed@outlook.com, ✉jundu@ustc.edu.cn



## ABSTRACT

In this paper, we present a new sparse co-prime array design that achieves a higher number of degrees-of-freedom for direction-of-arrival (DOA) estimation. The proposed array design adopts a sequence of displacements applied in a systematic procedure to the sensors of one of the two constituent sub-arrays of the prototype co-prime array. Accordingly, the applied displacements completely eliminate the redundant virtual sensors between the two sub-arrays and at the same time considerably increase their number of uniform DOFs. Moreover, with no cross-sensor redundancies, the new design is of ideal first weight functions, and hence, it resolves sources more robustly than other co-prime array designs in the presence of mutual coupling. Simulation results demonstrate the superior performance of the proposed co-prime structure.



***Index Terms***— Sparse arrays, co-prime arrays, difference co-arrays, mutual coupling, DOA estimation


## 1. INTRODUCTION

Over the past decade, different types of sparse arrays [1]-[15] have generated a tremendous interest in the array processing field, among which is the type of co-prime array. This array in its prototype [8] - as the sparsest structure - utilizes a pair of co-prime integers $M$ and $N_c$ as the numbers of sensors in its two constituent sub-arrays and concurrently as the uniform spacings among their sensors. However, irrespective of the number of unique degrees of freedom (DOFs) this structure can have, the number of consecutive sensors (or "uniform DOFs") is linear to the number of physical sensors $N$. Therefore, to increase the number of consecutive DOFs, augmented co-prime arrays [9] - which have been enhanced afterwards by thinned and optimized co-prime arrays [10], [11], to name a few - have emerged by doubling the number of sensors $M$, or rather, by choosing the prototype co-prime array (ProCA) parameter $M$ to be a smaller prime integer.

The generalized co-prime array in its CACIS and CADiS structures [12], which have been enhanced lately by [13], [14], in a similar vein, adopts further smaller values of $M$ for the sake of increasing the number of unique and uniform DOFs alike. But due to those compressed inter-sensor spacings one of their constituent sub-arrays is employing, especially the CACIS configuration as it has no holes in the co-array center, much more severe mutual coupling effects are brought to adversely affect the estimation performance.

In [15], authors propose enlarging the two sub-arrays spacings $M$ and $N_c$ of the ProCA to be $2M$ and $2N_c$ instead. With such a strategy, the redundant cross virtual sensors between the ProCA two sub-arrays are completely eliminated, but the number of uniform DOFs is still linear to $N$ and therefore compressive sensing (CS)-based DOA finding methods, which require extra fine-tuned parameters and have individual drawbacks [16]-[18], are a must.

In this paper, we modify the ProCA in a way that we do not compress its sub-arrays inter-sensor spacings nor do we enlarge them. We let the ProCA modified configuration proposed here retain its original spacings $M$ and $N_c$ through applying a sequence of displacements to one of the constituent sub-arrays. Those sequential displacements are proposed in order to completely eliminate the redundant cross DOFs between the two sub-arrays and at the same time to significantly increase their number of uniform DOFs, aside from having ideal first weight functions for sure. Thus, with such co-array properties, the proposed ProCA is applicable to both CS- and subspace-based DOA finding methods as it provides a decent balance of consecutive and unique DOFs. Nevertheless, here MUSIC-like algorithms will suffice to demonstrate the superiority of the proposed co-prime array design.

## 2. SPARSE ARRAY SIGNAL PROCESSING MODEL

Consider $K$ far-field uncorrelated and narrowbanded sources are intercepted by a sparse linear array of $N$ sensors from directions $\theta_k$ with powers $\sigma_k^2$, where $k = 1, 2, ..., K$. The sensor locations are given as $\mathbb{P} = \{\rho_1, \rho_2, ..., \rho_N\}d$, where $d = \lambda/2$ is normalized for convenience, and by considering the presence of mutual coupling (MC) represented by its MC matrix, the received signal vector can be modeled as

$$\mathbf{z}_{\mathbb{P}}(t) = \mathbf{CAs}(t) + \mathbf{n}_{\mathbb{P}}(t), \tag{1}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1),\ ..., \mathbf{a}(\theta_K)]$ is the array steering matrix, $\mathbf{a}(\theta_k) = [e^{j2\pi\rho_i\bar{\theta}_k},\ \rho_i \in \mathbb{P},\ i = 1, .., N]^\top$ is the array steering vector corresponding to the $k$-th signal direction, $\theta_k$,


[*] corresponding author

$T$ refers to the total number of time snapshots, and $\mathbf{s}(t) = [s_1(t),\ ..., s_K(t)]$. $\mathbf{n}_{\mathbb{P}}(t)$ indicates the white Gaussian noise vector with a covariance matrix $\sigma_n^2 \mathbf{I}_N$, where $\sigma_n^2$ is the noise power. $\mathbf{C}$ can be approximated by a $B$-banded model as

$$\mathbf{C}_{ij} = \begin{cases} \mathbf{c}_{|\rho_i-\rho_j|}, & \text{if } |\rho_i - \rho_j| \leq B, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where its entry $\mathbf{c}_{|\rho_i-\rho_j|}$ is considered as a function of only the sensor separation $\rho_i - \rho_j$ and is empirically found to be inversely proportional to this separation in the way that $1 = \mathbf{c}_0 > |\mathbf{c}_{\rho_i-\rho_j=1}| > |\mathbf{c}_2| > \cdots > |\mathbf{c}_{\rho_i-\rho_j=B}| > |\mathbf{c}_{B+1}| = 0$ [19]-[21]. Then, the coupling leakage can be evaluated by

$$\mathcal{L} = ||\mathbf{C} - diag(\mathbf{C})||_F / ||\mathbf{C}||_F, \tag{3}$$

where $||\cdot||_F$ is the Frobenius norm of the MC matrix $\mathbf{C}$ [3].

In practice, the covariance matrix is approximated using the $T$ samples as

$$\mathbf{R_{zz}} = E[\mathbf{z}_{\mathbb{P}}\mathbf{z}_{\mathbb{P}}^H] = \mathbf{C}\mathbf{A}\mathbf{R_{ss}}\mathbf{A}^H\mathbf{C}^H + \sigma_n^2\mathbf{I}_N = \approx \frac{1}{T}\sum_{t=1}^{T}\mathbf{z}_{\mathbb{P}}(t)\mathbf{z}_{\mathbb{P}}^H(t), \tag{4}$$

where $\mathbf{R_{ss}} = E[\mathbf{s}(t)\mathbf{s}^H(t)] = diag[\sigma_1^2, ..., \sigma_K^2]$ is the source covariance matrix. Vectorizing $\mathbf{R_{zz}}$ results in

$$\mathbf{r_{zz}} = vec(\mathbf{R_{zz}}) = (\mathbf{CA})^* \odot (\mathbf{AC})\tilde{\mathbf{s}} + \sigma_n^2 vec(\mathbf{I}_N) = = \tilde{\mathbf{C}}(\mathbf{A}^* \odot \mathbf{A})\tilde{\mathbf{s}} + \sigma_n^2 vec(\mathbf{I}_N), \tag{5}$$

where $\tilde{\mathbf{C}} = \mathbf{C}^* \otimes \mathbf{C}$. From Eq.(5), the sensor locations in the virtual steering matrix $\mathbf{A}^* \odot \mathbf{A}$ are giving as the following difference co-array

$$\mathbb{D} = \{\bar{\rho} = \rho_i - \rho_j, \rho_i, \rho_j \in \mathbb{P}\}. \tag{6}$$

## 3. PROTOTYPE CO-PRIME ARRAY (PROCA)

The prototype of the co-prime array is proposed based on utilizing two co-prime integers $N_c$ and $M$ where $M < N_c$ as the numbers of sensors in its sub-arrays $\mathbb{P}_1$ and $\mathbb{P}_2$ and concurrently as the uniform spacings among their sensors. That is, the sensor location set of the $N_c$-sensor sub-array is given as $\mathbb{P}_1 = \{nM | 0 \leq n \leq N_c - 1\}$ whereas that of the $M$-sensor sub-array is provided as $\mathbb{P}_2 = \{mN_c | 0 \leq m \leq M - 1\}$. The two sub-arrays, as such, have a shared sensor at the zeroth location and therefore the number of physical sensors is $M + N_c - 1$. The union of these two sub-arrays constitutes the ProCA sensor location set $\mathbb{P}$ and its difference co-array $\mathbb{D}$ is the union of the cross and self differences of these sub-arrays; i.e., $\mathbb{D} = \{\text{diff}(\mathbb{P}_1, \mathbb{P}_2) \cup \text{diff}(\mathbb{P}_2, \mathbb{P}_1) \cup \text{diff}(\mathbb{P}_1, \mathbb{P}_1) \cup \text{diff}(\mathbb{P}_2, \mathbb{P}_2)\}$, where

$$\begin{aligned} &\text{diff}(\mathbb{P}_1, \mathbb{P}_2) = \{nM - mN_c\}, \\ &\text{diff}(\mathbb{P}_2, \mathbb{P}_1) = \{mN_c - nM\}, \\ &\text{diff}(\mathbb{P}_1, \mathbb{P}_1) = \{(n_1 - n_2)M\} = \{\ell_1 M\}, \\ &\text{diff}(\mathbb{P}_2, \mathbb{P}_2) = \{(m_1 - m_2)N_c\} = \{\ell_2 N_c\}, \end{aligned} \tag{7}$$

where $1 \leq \{n, n_1, n_2\} \leq N_c - 1$, $0 \leq \{m, m_1, m_2\} \leq M - 1$, $-N_c + 2 \leq \ell_1 \leq N_c - 2$, and $-M + 1 \leq \ell_2 \leq M - 1$.

For ProCA, the set of self differences diff$(\mathbb{P}_1, \mathbb{P}_1)$ has $(N_c - 1)(N_c - 1)$ differences, among which only $2(N_c - 1) - 1$ differences are unique and the same is true for diff$(\mathbb{P}_2, \mathbb{P}_2)$ whose number of unique self differences is $2M - 1$. However, the $M(N_c - 1)$ cross differences (or equivalently, cross virtual sensors) diff$(\mathbb{P}_1, \mathbb{P}_2)$ - and similarly diff$(\mathbb{P}_2, \mathbb{P}_1)$ - has are unique and this is due to the merit of the co-primality of $M$ and $N_c$ as shown in the proof of proposition 1 in [12]. But with considering the union of these two cross virtual sensor sets, redundant cross virtual sensors between them do exist. That is, there are some sensors resulted by both diff$(\mathbb{P}_1, \mathbb{P}_2)$ and diff$(\mathbb{P}_2, \mathbb{P}_1)$, which means mathematically

$$n_1 M - m_1 N_c = m_2 N_c - n_2 M. \tag{8}$$

According to the proof of proposition 2 in [15], Eq.(8) holds when the two pairs $(n_1, n_2)$ and $(m_1, m_2)$ have any of these pairs of values $(1, N_c - 1), (2, N_c - 2), ..., (N_c - 1, 1)$, and $(1, M-1), (2, M-2), ..., (M-1, 1)$, respectively, and therefore, the number of those redundant sensors between the two cross-sensor sets is $(M - 1)(N_c - 1)$ or equivalently $MN_c - M - N_c + 1$. To completely eliminate these redundant sensors and increase the unique and uniform DOFs the ProCA can have, the condition in Eq.(8) must be broken. Thus, from this perspective, we propose a new co-prime array structure.

## 4. THE PROPOSED CO-PRIME ARRAY DESIGN

### 4.1. The Array Design and Co-array Properties

*Definition 1* (Proposed Co-prime Array Configuration): Assume that $N_c - 1$ and $M$ are the numbers of sensors in the first and second sub-arrays $\mathbb{P}_1$ and $\mathbb{P}_{2p}$, respectively, and that they are co-prime integers, where $M < N_c$ is no longer considered. $\mathbb{P}_1$ sensors are located at $\{nM | 1 \leq n \leq N_c - 1\}$, and $m_e = \left\lfloor \frac{M+1}{2} \right\rfloor$ and $m_o = \left\lfloor \frac{m_e}{2} \right\rfloor$ are two indexes. $L = \left\lfloor \frac{M-1}{2} \right\rfloor N_c$ is an initial displacement, and the proposed prototype co-prime array sensor location set then is $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_{2p}$, where $\mathbb{P}_{2p}$ is defined as follows

$$\begin{aligned} \mathbb{P}_{2p} = &\{(m_1 N_c - L - N_c - M) \cup (m_2 N_c - L - N_c) \\ &\cup (m_3 N_c - L) \cup (m_4 N_c - L + N_c)\}, \end{aligned}$$

where $0 \leq m_1 \leq m_o - 1$, $m_o \leq m_2 \leq m_e - 1$, $m_e \leq m_3 \leq M - 2$, and $m_4 = M - 1$. A concrete example of the proposed co-prime array design in accordance with Definition 1 is shown in the top of Fig. 1, where $M = 9$ and $N_c = 10$.

*Proposition 1*: For $N \geq 18$ actual sensors, the modified co-prime array according to Definition 1 has a difference co-array of (a) $2(M+1)(N_c-1)+3M+2$ unique virtual sensors, among which (b) there are $MN_c + 4M + 3N_c - 1$ contiguous virtual sensors and (c) its first $(M - 1)$ weight functions for odd $M$, where $M$ is smaller than $N_c$, are of values equal to

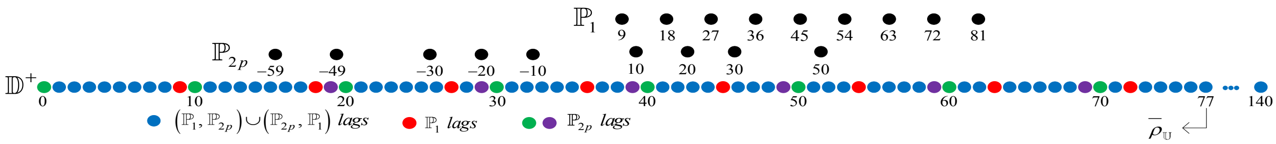



**Fig. 1**: The proposed structure configuration along with the positive contiguous part of its difference co-array.

1, while for odd $M$ that is greater than $N_c$, its first $N_c - 1$ weight functions are ones.

*Proof*: (a) This property is proved from the perspective of the contribution of the self and cross differences of the two sub-arrays $\mathbb{P}_1$ and $\mathbb{P}_{2p}$ defined in Definition 1 to the total number of unique DOFs.

Since diff$(\mathbb{P}_1, \mathbb{P}_{2p})$ and diff$(\mathbb{P}_{2p}, \mathbb{P}_1)$ are symmetric, we only provide the proof of the former. It can be observed that

$$\begin{aligned}&\{n_{1i}M\} - \{(m_{1i}N_c - L - N_c - M) \cup (m_{2i}N_c - L - N_c)\\ &\cup (m_{3i}N_c - L) \cup (m_{4i}N_c - L + N_c)\} \neq \{n_{1j}M\}-\\ &\{(m_{1j}N_c - L - N_c - M) \cup (m_{2j}N_c - L - N_c)\cup\\ &(m_{3j}N_c - L) \cup (m_{4j}N_c - L + N_c)\}\end{aligned} \tag{9}$$

if $1 \leq n_{1i}, n_{1j} \leq N_c - 1$, $0 \leq m_{1i}, m_{1j} \leq m_o - 1$, $m_o \leq m_{2i}, m_{2j} \leq m_e - 1$, $m_e \leq m_{3i}, m_{3j} \leq M - 2$, $m_{4i} = m_{4j} = M - 1$, $m_e$ and $m_o$ are defined as in Definition 1, and $M$ and $N_c$ are co-prime. That is, (9) implies that the cross difference of a sensor in $\mathbb{P}_1$ with a sensor in $\mathbb{P}_{2p}$ cannot be redundant, and therefore, this cross-difference set (and, similarly, diff$(\mathbb{P}_{2p}, \mathbb{P}_1)$) has $M(N_c - 1)$ unique cross sensors. Moreover, it can be observed from the following equation that

$$\begin{aligned}&\{n_{11}M\} - \{(m_{11}N_c - L - N_c - M) \cup (m_{21}N_c - L - N_c)\\ &\cup (m_{31}N_c - L) \cup (m_{41}N_c - L + N_c)\} \neq \{(m_{12}N_c - L-\\ &N_c - M) \cup (m_{22}N_c - L - N_c) \cup (m_{32}N_c - L) \cup (m_{42}N_c\\ &- L + N_c)\} - \{n_{12}M\}\end{aligned} \tag{10}$$

if $1 \leq n_{11}, n_{12} \leq N_c - 1$, $0 \leq m_{11}, m_{12} \leq m_o - 1$, $m_o \leq m_{21}, m_{22} \leq m_e - 1$, $m_e \leq m_{31}, m_{32} \leq M - 2$, $m_{41} = m_{42} = M - 1$, and $M$ and $N_c$ are co-prime. That is, different from (8), the sequence of the three displacements $L$, $N_c$ and $M$ applied in a systematic procedure to the sensors of $\mathbb{P}_2$ as shown in $\mathbb{P}_{2p}$ completely eliminates the $M(N_c - 1)$ redundant cross differences in $\{\text{diff}(\mathbb{P}_1, \mathbb{P}_2) \cup \text{diff}(\mathbb{P}_2, \mathbb{P}_1)\}$, and thus, the number of cross sensors the union set $\{\text{diff}(\mathbb{P}_1, \mathbb{P}_{2p}) \cup \text{diff}(\mathbb{P}_{2p}, \mathbb{P}_1)\}$ can have is $2(M(N_c - 1))$. Moreover, with the adopted displacements, $\mathbb{P}_{2p}$ becomes non-uniform and its unique self differences are included in $\{\ell N_c, -(\lfloor (M + 1)/2\rfloor + \lceil (M + 1)/4\rceil) \leq \ell \leq \lfloor (M + 1)/2\rfloor + \lceil (M + 1)/4\rceil\}$ and $\{N_c + M + \ell N_c, -M \leq \ell \leq M\}$. However, $2\lceil (M + 1)/4\rceil$ of $\mathbb{P}_{2p}$ self sensors co-locate with certain sensors in the set $\{\text{diff}(\mathbb{P}_1, \mathbb{P}_{2p}) \cup \text{diff}(\mathbb{P}_{2p}, \mathbb{P}_1)\}$, while the unique self sensors of $\mathbb{P}_1$, which are as in (7), do not anymore. Therefore, considering the cross sensors in (10) and the unique self sensors of $\mathbb{P}_1$ and $\mathbb{P}_{2p}$, the total number of unique DOFs the modified ProCA can have is $2M(N_c - 1) + 2(N_c - 1) - 2 + 3M + 4$.

(b) Besides increasing the unique DOFs, the sequence of displacements in Definition 1, in the first place, is also intended to increase the number of uniform DOFs the modified ProCA can have. For instance, the first displacement by $-L$ increases the one-sided number of uniform DOFs of the ProCA, $\bar{\rho}_{\mathbb{U}}$, from $M + N_c - 1$ to be $M + N_c - 1 + L$, while the second and third displacements $-N_c$ and $-M$, respectively, in their turn, augment $\bar{\rho}_{\mathbb{U}}$ to be $M + N_c - 1 + L + N_c + M$. Hence, to prove that this sequentially displaced sub-array, $\mathbb{P}_{2p}$, along with $\mathbb{P}_1$ do generate this considerable number of sensors contiguously from 0 to $\bar{\rho}_{\mathbb{U}} = 2M + 2N_c + \lfloor((M - 1)/2)\rfloor N_c - 1$, we need to examine the union of their self and cross differences, which constitute $\mathbb{D}$ (as we mentioned earlier). And interestingly enough, the cross-difference sets of $\mathbb{P}_1$ and $\mathbb{P}_{2p}$ represent all the consecutive virtual sensors from 1 to $\bar{\rho}_{\mathbb{U}}$, except for three sets of holes which are happened to be filled by the self differences of $\mathbb{P}_1$ and $\mathbb{P}_{2p}$. Specifically, the first set has holes located at $\ell M$ where $1 \leq \ell \leq N_c - \lfloor (M + 1)/4\rfloor$, while the second and third sets have holes located at $\{\ell N_c | 0 \leq \ell \leq M - \lfloor (M + 1)/4\rfloor\}$ and at $\{M + N_c + N_c\ell | 0 \leq \ell \leq \lfloor (M + 1)/2\rfloor\}$, respectively. Illustratively, for $M = 9$ and $N_c = 10$, it can be observed from the bottom of Fig.1 that the blue circles representing the virtual sensors resulted from $\{\text{diff}(\mathbb{P}_1, \mathbb{P}_{2p}) \cup \text{diff}(\mathbb{P}_{2p}, \mathbb{P}_1)\}$ non-contiguously extend from 1 to $\bar{\rho}_{\mathbb{U}} = 77$. However, red circles which are for the self sensors of $\mathbb{P}_1$, and green and purple circles that are for the self sensors of $\mathbb{P}_{2p}$ - due to its non-uniformity - come to be aligned with the empty locations (or "holes") in the two cross-lag sets segment and this completes the proof.

(c) The modified design introduced in this paper has no redundant cross lags, suggesting that only the virtual sensors representing the inter-sensor spacings of $\mathbb{P}_1$ and $\mathbb{P}_{2p}$ and their multiples are redundant and hence their weight functions are functions of the numbers of sensors in $\mathbb{P}_1$ and $\mathbb{P}_{2p}$. Therefore, for this structure, the weight functions of the first critical separation 1 to the smaller inter-sensor spacing whether it is $M - 1$ or $N_c - 1$ depending on the optimal distribution of $N$, as it will be introduced next, are of values equal to 1.

### 4.2. The Optimal Distribution of $N$

For a given number of sensors $N$, $M$ and $N_c$ which maximize the number of consecutive lags are obtained by solving the following optimization problem:

$$\begin{aligned} \max_{M,\, N_c} \quad & MN_c + 4M + 3N_c - 1 \\ \text{s.t.} \quad & M + N_c = N + 1, \\ & M \text{ and } N_c \text{ are co-prime integers,} \end{aligned} \tag{11}$$

and the solutions are presented as follows: If $N$ is an odd number, $M$ is $\lceil \frac{N+2}{2} \rceil$ or is $\lceil \frac{N+2}{2} \rceil + 1$ in case the ceiled $M$ is an even number and $N_c$ is $N - M + 1$. But if $N$ is even, $M = \frac{N}{2}$ and is odd. Otherwise, $N_c = \frac{N}{2}$ and $M$ is $N - N_c + 1$.

### 4.3. Discussion

In [15], it is concluded that the redundant self lags cannot be avoided, and the reduction of the cross lags is the aim within reach. Thus, the maximum number of unique lags can be achieved in the context of co-prime array when cross-lag redundancies are completely eliminated. In the proposed co-prime array design therein, the aim achieved through doubling the spacing for one of the sub-arrays at least. Specifically, when $\tilde{N}_c = 2N_c - 1$ and $\tilde{M} = 2M - 1$, the total number of unique lags is $2M(N_c - 1) + 2M - 1$ and the uniform capacity is linear to $N$. But when $\tilde{N}_c = 2N_c - 1$ and $\tilde{M} = M$ - as this setting is adopted by authors therein, the total number of unique lags is $2M(N_c - 1) + 2M - 1$ and the uniform capacity is of a modest increase.

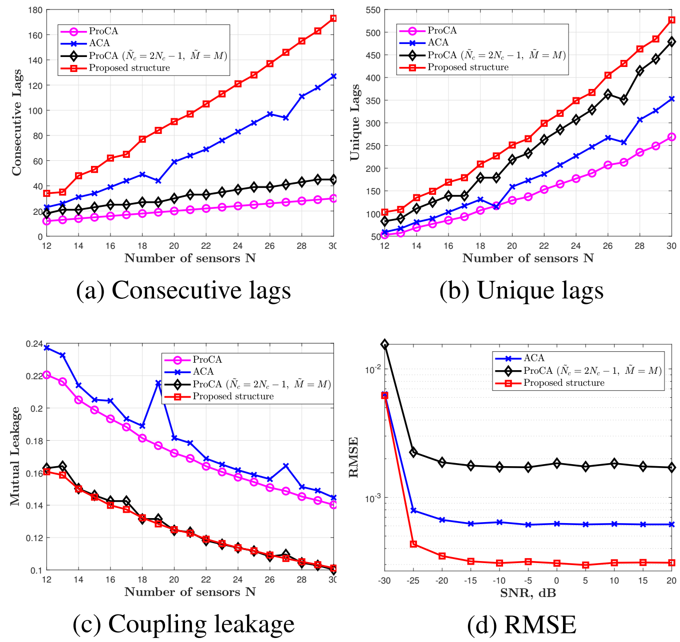
However, in the co-prime array design proposed here and in accordance with Proposition 1, due to the non-uniformity of $\mathbb{P}_{2p}$, the self lags of the $N_c$-sensor sub-array are no longer a subset of the cross-lag set and the redundant cross lags between the two constituent sub-arrays as well as certain self-lag redundancies of the $M$-sensor sub-array are eliminated. Therefore, the larger unique and uniform capacities of virtual sensors it owns in contrast.

## 5. NUMERICAL RESULTS

### 5.1. Degrees of Freedom and Mutual Coupling

In this part, we compare the unique and uniform DOFs for four sparse arrays: ProCA [8], ACA [9], ProCA with $\tilde{N}_c = 2N_c - 1$ and $\tilde{M} = M$ [15], and the modified ProCA proposed here. Uniform lags for the four arrays under consideration are plotted in Fig. 2(a), while their unique lags are plotted in Fig. 2(b) for a fixed number of $N$ in the range from 12 to 35 sensors. It is seen that, the proposed structure achieves larger capacities of consecutive and unique lags than the other compared arrays, which follows the discussions in Section 4.3.

The mutual coupling modeled with $c_1 = 0.3$ and $B = 30$ and being represented by its leakage for the same above sparse arrays versus the same range of $N$ is plotted in Fig. 2(c), and from which it can be observed that the proposed array design faces the same mutual coupling effect as the ProCA with $\tilde{N}_c = 2N_c - 1$ and $\tilde{M} = M$ does.




(a) Consecutive lags (b) Unique lags

(c) Coupling leakage (d) RMSE

**Fig. 2**: Lags, mutual coupling leakage and estimation accuracy comparison for sparse arrays.

### 5.2. Estimation Accuracy

In this part, by means of the RMSE and via 100 Monte Carlo trials, we examine the estimation performance of the last three sparse arrays under consideration in the previous part. We consider the uniform DOFs and use the spatially smoothed root-MUSIC [22] for DOA estimation. $N = 18$ sensors, $K = 22$ sources that are uniformly distributed in the normalized range $[-0.4,\ 0.4]$, $c_1 = 0.2$, $B = 50$, and $T = 1000$. Fig. 2(d) compares the RMSE as a function of the input SNR. The proposed design performs the best among all sparse arrays due to the increased consecutive lags and reduced MC.

## 6. CONCLUDING REMARKS

With the availability of a considerable number of uniform DOFs, the sparsest co-prime array structure with the highest number of unique DOFs yet has been proposed. The introduced perspective of displacing one of the ProCA constituent sub-arrays in a systematic sequence guaranteed the modified structure these properties, thereby possessing a robust difference co-array that can perform well in a real-world estimation environment, such as the one contaminated by MC effects.

## 7. ACKNOWLEDGEMENT

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