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Brute Force Gravitational N Body Simulation - On Stability of Lagrange Point Orbits

Project III - Computational Astronomy

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ABSTRACT

Aims. An algorithm for studying N body astronomical systems is developed and applied to the study of the stability of orbits around Lagrange points in the solar system.
Methods. A brute force gravitational N body code is discussed. We employ a high level language implementation of brute force N body, taking full advantage of broadcasting and vectorization to easily optimize the code for a small number of bodies. Other possibilities are discussed.
Results. We analyze the stability of perturbations on Lagrange point orbits. We use numerical results to give example of stable orbits and discuss their applications for satellite orbit.

1. Introduction

The many body gravitational problem is a remarkable example of the limitations of even the most advanced methods in classical mechanics. While the two body system has an analytical solution, it would take no more than three bodies to present historical difficulties to the likes of d'Alembert and Poincarré, who struggled greatly to prove closed form analytical solutions were impossible. Only in 1912 would Sundman discover a solution in series form, albeit of little use, for its slow convergence.

Astronomical systems, however, are the perfect example of the importance of the study of many body gravitational systems. Numerical methods are for this reason crucial to the study of these.

2. Computational Methods

2.1. Code Philosophy

Out algorithm aims to optimize:

- Readability and ease of use
- Optimization for low body count
- Modular and adaptable structure

We achieve this taking advantage of scientific libraries for high level languages (*numpy* and *scipy*) which facilitate the use of vectorization and broadcasting with minimal effort.

2.2. Implementation

Brute force approach equates to solving the system of ODE (equations 1 and 2)

$$\frac{dr^k}{dt} = 2v^k \tag{1}$$

$$\frac{dv^k}{dt} = -4\sum_{i\neq j}^{N} m_j \frac{r_j^k - r_i^k}{|r_j^k - r_i^k|^3} = -4\sum_{i\neq j}^{N} m_j \Lambda_{ij}^k \tag{2}$$

for dimensionless quantities as defined in Monteiro & Gameiro (2019) (conversion factors listed in equations 3 to 7) and Λ is the k-th component of the acceleration exerted by the i-th particle on the j-th particle.

Typical scales are defined to be the sum of masses and R the largest radius of the system.

$$m_r = M \tag{3}$$

$$r_r = R \tag{4}$$

$$t_r = \left(\frac{8R^3}{GM}\right)^{1/2} \tag{5}$$

$$v_r = \left(\frac{GM}{2R}\right)^{1/2} \tag{6}$$

$$a_r = \frac{GM}{R^2} \tag{7}$$

Constructing Λ is expensive in terms of memory, but completely avoids slow high level cycles and expensive non static structures. This step constitutes the main limitation on the size of the system the algorithm is able to handle.

Using an adaptative Runge-Kutta 4 integrator with a fittingly small maximum integration step[1] we get the time evolution of the system.

[1] Determining the maximum integration step is generally done in an iterative manner. Considering the variation in energy with time one may choose a parameter such that energy is conserved.

2.3. Collisions

We avoid the numerical problems related to collisions by substituting equation 2 for equation 8 for a small ϵ when the two particles are very close.

$$\frac{dv^k}{dt} = -4 \sum_{i \neq j}^{N} m_j \frac{r_j^k - r_i^k}{|r_j^k - r_i^k|^3 + \epsilon} \tag{8}$$

This is suiting for our small number of particles. For a larger number of particles another options would be to ignore the interaction with particles of a given proximity, effectively turning off the interaction and allowing particles to go through each other.

2.4. Alternative approaches to N body simulations and scaling

For simulations of more particles one could employ better suited approaches for such endeavor.

Firstly, one would run into scaling problem in the calculation of Λ, due to memory limitations.[2]. Now one would do well to take advantage of a lower level language, and cycle over particles, still employing vectorization but only doing calculations of dimension $3N$. These algorithms, along with the ones used in our work, scale with $O(n^2)$.

For larger scale problems, a more sophisticated approach would be to use branching algorithms such as Barnes Hut method, which divides space into a grid, using a tree-searching algorithm to calculate only close interactions and estimating further ones. With this construction one would be able to bring the scaling down to a $O(\log n)$ method. Details about this implementation can be found in Iwasawa et al. (2019).

3. Code Validation

3.0.1. Sun, Earth and Mars

Aiming to validate our code we simulate a triple system with circular orbits with parameters of the Sun, Earth and Mars.

To aid visualization we represent the system in the center of mass reference. We can see the orbits are stable, as expected, in figure 1.

In figure 2 one may see if our integration is adequately precise for sufficiently small integration step.[3]

The energy fluctuations happen as energy is exchanged between potential and kinetic energy, but no net loss is observed. This behavior is aggravated for systems where close encounters are common.

3.0.2. Sun, Earth and Moon

Studying the system Sun-Earth-Moon we validate the capability of our method to simulate more intricate orbits of hierarchical mass.

Once again employing a circular orbit for the Earth, we see Moon's orbit crossing Earth's (see figure 3) 12 times in one period around the Sun, in confirmation of our expectations.

From figure 4 we see no dissipation of energy with time, a reassuring sign that our integration step is sufficiently small.

[2] Note that $\dim\Lambda = 3N^2$.
[3] We see that for a large integration step energy is not conserved.

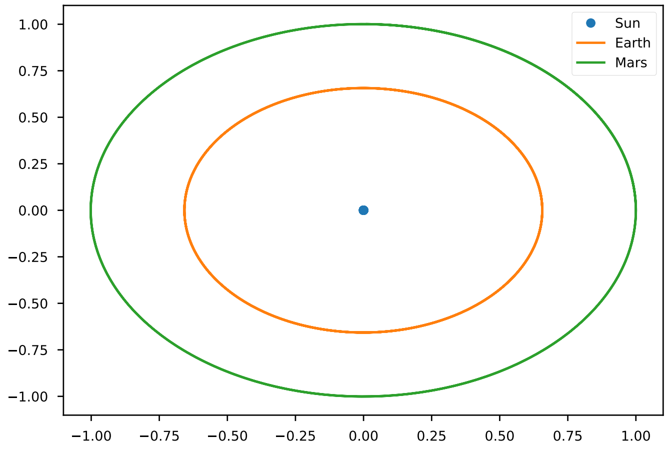

Fig. 1. Orbits of circular Sun-Earth-Mars system

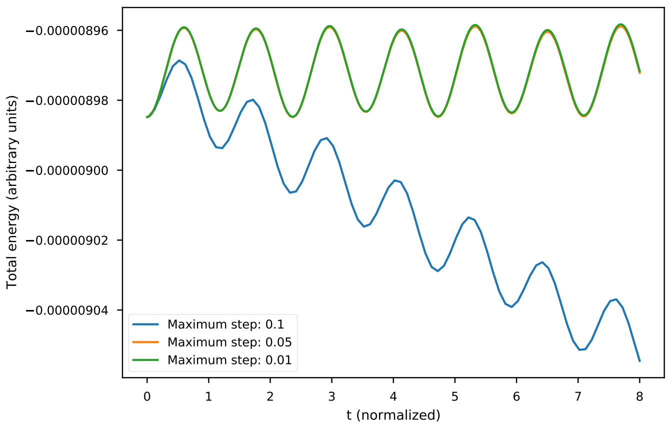

Fig. 2. Energy dissipation in Sun-Earth-Mars system with step size

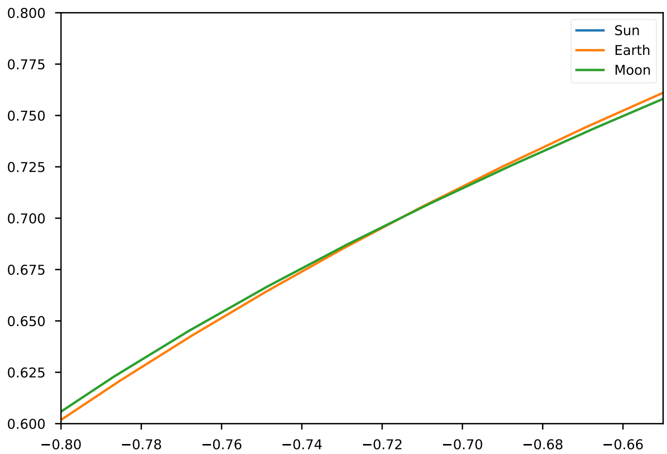

Fig. 3. Orbital motion of Earth and Moon around Sun

4. Application: Stability of Lagrange Points

We aim to study the stability of orbits around Lagrange points. From Todoran & Roman (1992) one concludes that there is an equivalence between Lagrange points in elliptical orbits and in circular orbits. We will therefore focus our study in the simpler 2D circular case.

The Lagrange points, represented in 5, are configurations in a hierarchical three body system where the total force on the third

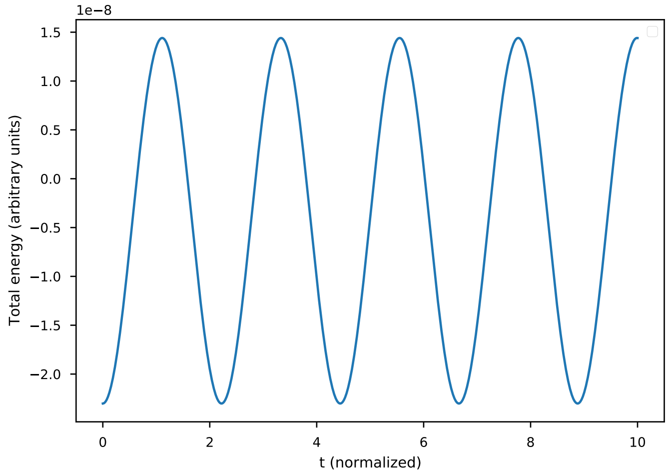

Fig. 4. Total energy in the Sun-Earth-Moon system

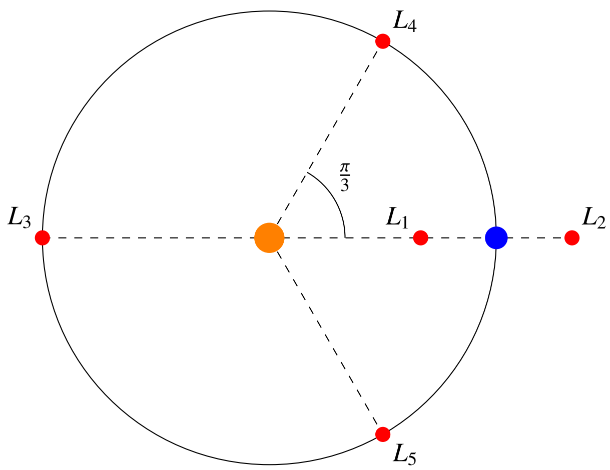

Fig. 5. Lagrange points

Parameter	Sun	Earth	Satellite
Mass (kg)	1.989×10^{30}	5.972×10^{24}	$(1 \times 10^3 / 1 \times 10^{14})$ [4]

Table 1. Sun-Earth-Satellite parameters

object, smaller than the first two, is so that it will maintain its position relative to the other bodies.

Lagrange points are extremely useful in astronomy, as they allow astronomers to place observation and measurement instruments in stable relative position, minimizing difficulties such as fuel cost and target exposure time.

There are five points in an orbit that follow this conditions, but not all of them are stable. Stable orbits are possible around L_4 and L_5 points, where the Coriolis effect acts as a restoring force. L_1, L_2 and L_3, however, do not allow for stable orbits. To verify these results we study Lagrange points L_1 and L_5 in a circular Sun-Earth like system for a satellite-like object (parameters in table 1).

4.1. Lagrange point L_1

One may compute the conditions for L_1 by solving the quintic equation 9 for the distance to the Earth-like object r, where M_1,

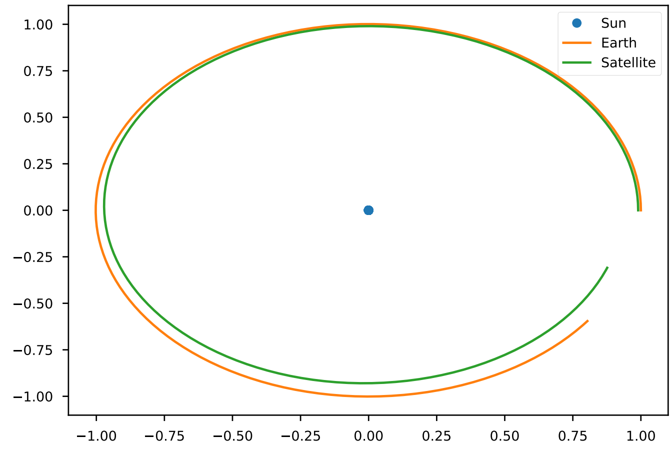

Fig. 6. Unstable orbit for approximate L_1

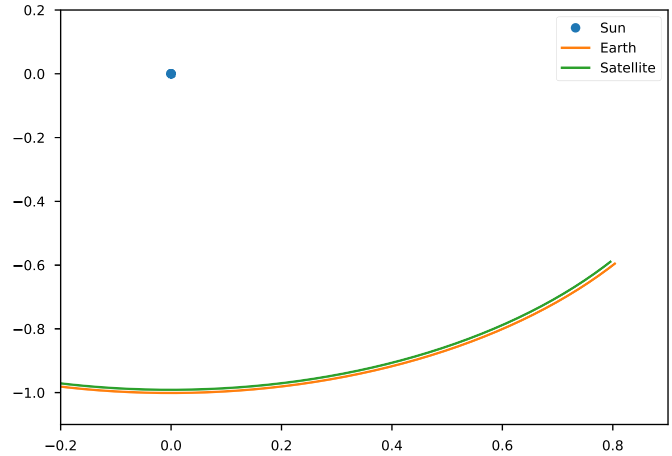

Fig. 7. Stable orbit around L_1

M_2, R are the masses of the Sun-line object, the Earth-line object and the radius between them, respectively.

$$\frac{M_1}{(R-r)^2} = \frac{M_2}{r^2} + \frac{M_1}{R^2} - \frac{r\,(M_1+M_2)}{R^3} \tag{9}$$

Because $M_1 << M_2$ one could approximate r by the Hill sphere's radius, as in equation 10.

$$r \approx R\sqrt[3]{\frac{M_2}{3M_1}} \tag{10}$$

However, when we do so, we see the approximation is not good enough for the Sun-Earth system, as can be seen in figure 6, where we do not get a stable orbit.

Using a consecutive bisection method to numerically solve equation 9 will however provide us with a stable orbit (see figure 7).

In figure 8 we can see the variation of the relative radii of Satellite-like object with time for small perturbations of the form $r' = r(1+b)$. It is clear that even for very relatively small perturbations the orbit rapidly becomes unstable, transitioning to a non radially stationary orbit.

Additionally, we verify that the movement of the Satellite-like object does not depend on its mass, as long as it preserves the system's hierarchy.

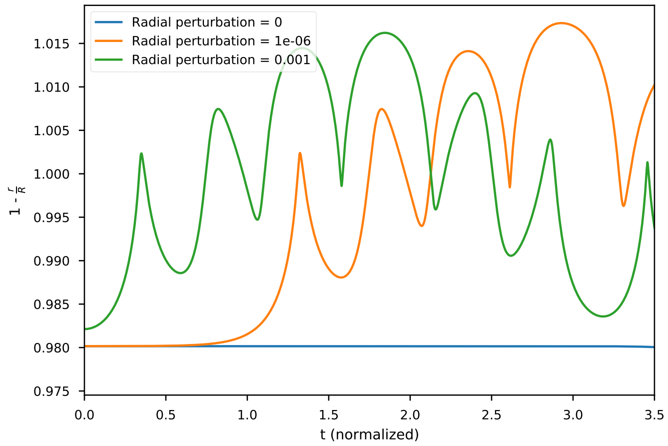

Fig. 8. Relative radii of the Satellite-like object with perturbations

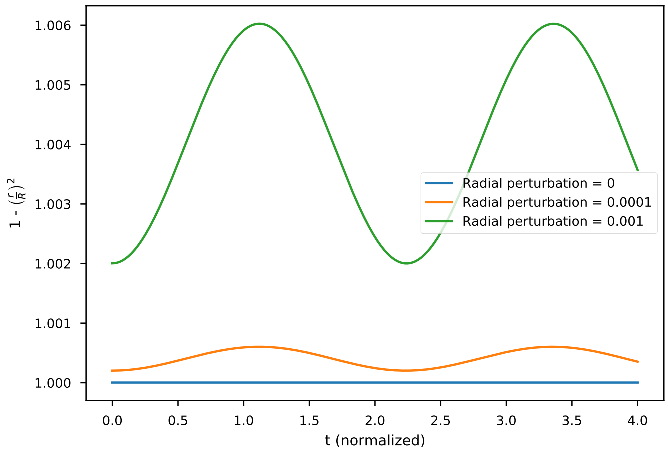

Fig. 10. Radial perturbation in L_5 orbit

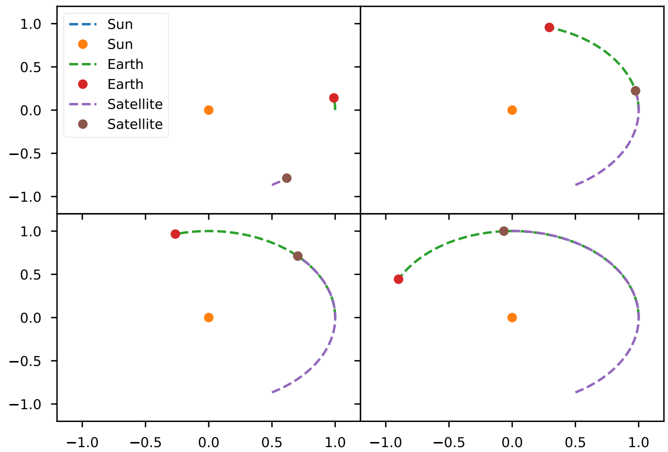

Fig. 9. Evolution of Sun-Earth-Satellite-like system in L_5 orbit (left to right)

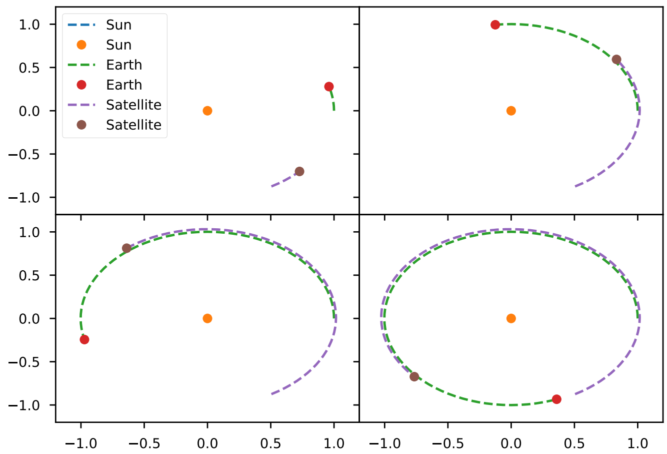

Fig. 11. Orbit of L_5 with a radial perturbation $b \approx 0.01$

4.2. Lagrange point L_5

Lagrange point L_5 lies in the trajectory of the Earth-like object, with a phase difference of $-\frac{\pi}{3}$. In this case, the initial conditions of the Satellite-like object can be determined analytically with simple vector projection.

The evolution of this system can be seen in figure 9. Additionally, if a radial perturbation is introduced (see 10), we see a very different behavior than that of the L_1 orbit. Instead of the divergent behavior we see for a perturbation in the radius like the first case, we now see an oscillatory behavior.

This is a consequence of having a stable orbit. Another difference is that a radial perturbation will not significantly alter the period (for $b < 0.2$), like it does in the case of the L_1 point. This is clear when we represent the trajectory for a perturbed radius in figure 11.

4.3. Possible orbits around Lagrange points

4.3.1. Oscillations around L_1

While effectively less stable, Lagrange points L_1, L_2 are extremely important because they constitute the best position to observe solar, earthly and outer solar system phenomena, simultaneously providing ease of communication back to observatories.

However, a satellite on the precise Lagrange point is directly obstructing the Sun-Earth-outer solar system line of sight. To allow for multiple instruments in conflict free positions and to avoid line of sight problems, semi-stable orbits around these points are utilized.

One simple example of such orbit, at L_1, may be obtained by giving the Satellite-like object's position a z component. The planar orbit is not significantly changed, but now we see the object oscillates in the z axis, as can be seen in figure 12.

These orbits, along with other more complex (with slight periodic corrections, which form the know Lissajous orbits), allow for a much more efficient and practical use of this astronomical real-estate.

4.3.2. Influece of Jupiter on L_5 stability

Including another large body on the simulation, in this case, a Jupiter-like body, we see (figure 13) the stability of the orbit is broken and oscillations are induced.

This effect means that even the stable orbits of L_4 and L_5 require a given amount of orbit compensation.

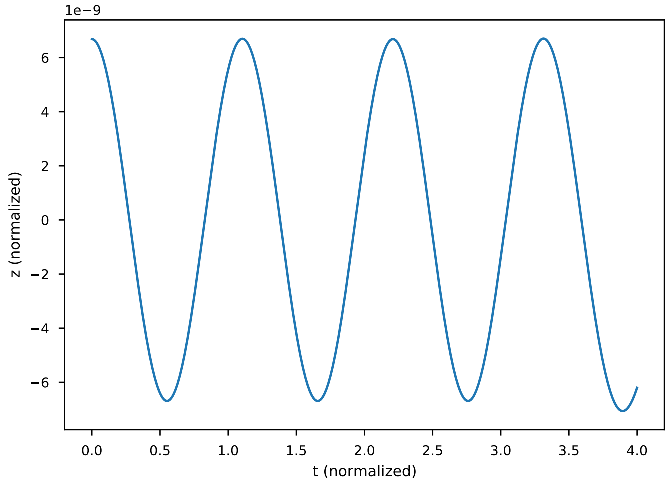

Fig. 12. z oscillations around orbit L_1 in stable regime

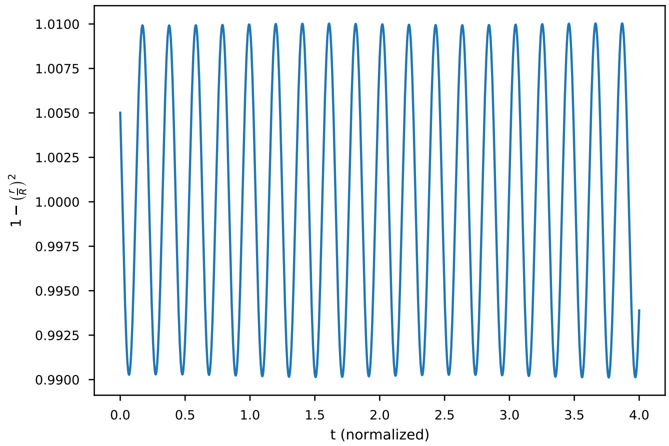

Fig. 13. Radial flutuations cause by Jupiter-like object on L_5 orbit

5. Conclusion

- An efficient, high level, understandable brute force gravitational N body code was developed for small systems.
- Algorithm was validated for the circular Sun-Earth-Mars and Sun-Earth-Moon systems.
- N body method was applied to the study of Lagrange points.
- Considerations were made about the stability of orbits around L_1 and L_5 points.
- Stability of L_5 orbits and instability of L_1 orbits were confirmed and simulated.
- The need for numerical solution for the quintic equation for L_1 radius determination was confirmed.
- Influence of Jupiter on the L_4 and L_5 orbits was studied.
- The importance of Lagrange points and practical orbits around even unstable points are possible with minimal trajectory correction was studied.

Full code may be found at `https://github.com/ruimpz/nbody`. Feedback is welcome.

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