Astronomy & Astrophysics manuscript no. proj2
December 20, 2019

Polytropic Stellar Model

Project II - Computational Astronomy

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December 20, 2019

ABSTRACT

Aims. Stellar structure is studied by constructing a polytropic model for main sequence stars. A computational method for determining polytropic indexes from data is developed and implemented. Regime of application and possible improvements to the model are discussed.
Methods. Numerical solutions to the Lane-Emden equation are computed from given data (mass, radius and chemical properties) and luminosity boundary conditions are satisfied employing a shooting method. Uncertainty is modeled by repeated sampling (Monte Carlo).
Results. An efficient numerical method for simulating stellar structure in polytropic conditions is developed and applied to the study of the α Centauri AB binary.

1. Introduction

As a major branch of astrophysics, the study of stars and their structure takes an important role in the understanding of the origin and dynamics of the Universe, at both local and cosmological scale, simultaneously constituting a powerful tool in the context of nuclear and plasma physics.

For the most comprehensive description of stellar interiors, observational data of multiple sources may be combined with theoretical models and numerical simulations, motivating the following study.

Modeling these objects starts by considering hydrostatic equilibrium (equation 1) and mass conservation (equation 2), where P is pressure, ρ density, $\boldsymbol{v}$ velocity, and ϕ gravitational potential, following the Poisson equation, as well as spherical symmetry.

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla P - \rho \nabla \phi \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{2}$$

For main sequence stars, one may assume a polytropic relation between the temperature T and pressure of the kind displayed in equation 3, for a real constant K.

$$P = K\rho^{1+\frac{1}{n}} \tag{3}$$

Combining these equation in terms of adimensional variables $\theta = \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}}$ and $\xi = ar$ we get the Lane-Emden equation (4), derived in detail in Monteiro & Gameiro (2019), which fully describes the thermodynamics of the stellar interior

$$\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right) = -\theta^n \tag{4}$$

by setting $a^2 = K(n+1)/4\pi G \rho_c^{1-\frac{1}{n}}$, with initial conditions $\theta(\xi = 0) = 1$ and $\frac{d\theta}{d\xi}\Big|_0 = \dot{\theta}(0) = 0$.[1]

Equation 4 has no analytical solution for almost all values of n, hence the utilization of computational methods.

From a given solution $\left(\xi, \theta, \dot{\theta}\right)$ one may describe density (therefore mass), pressure and temperature as ξ parameterized, using relations 5 to 8, where G is the gravitational constant, $\mathscr{R}$ the gas constant and μ the average molecular weight in the star center. Additionally, the luminosity is given by equation 9. (Monteiro & Gameiro (2019))

$$\rho = \frac{3M}{4\pi R^3} \frac{\xi_s}{3\left(-\theta'_s\right)} \theta^n \tag{5}$$

$$\frac{m}{M} = \left(\frac{\xi}{\xi_s}\right)^2 \frac{\theta'}{\theta'_s} \tag{6}$$

$$P = \frac{GM^2}{R^4} \frac{1}{4\pi(n+1)\left(\theta'_S\right)^2} \theta^{n+1} \tag{7}$$

$$T = \mu \frac{GM}{\mathscr{R}R} \frac{1}{(n+1)\xi_s\left(-\theta'_s\right)} \theta \tag{8}$$

$$L_r = \frac{1}{\xi_s^2\left(-\theta'_s\right)} \int_0^{\xi} \xi^2 \theta^n \varepsilon M \mathrm{d}\xi \tag{9}$$

Luminosity depends on the integral over ξ of the emissivity. With a solution of equation 4 and the aforementioned parameters, the emissivity is straightforward to calculate, as given in Monteiro & Gameiro (2019).

[1] The dot notation here and henceforth adopted is to be interpreted as differentiation with respect to parameter ξ.

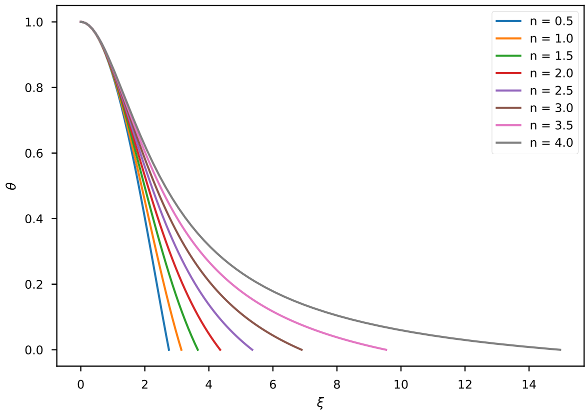

Fig. 1. Polytropic curves for varying index n

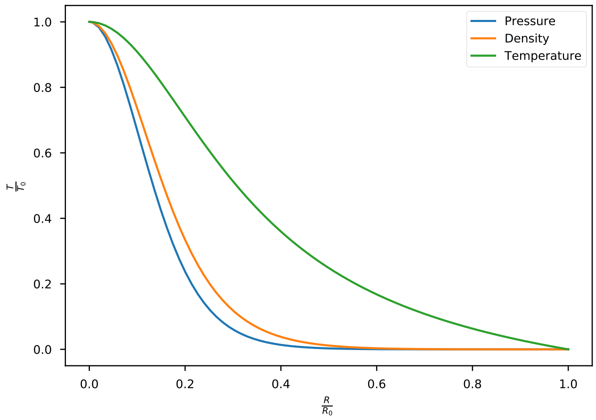

Fig. 2. Sun - Normalized parameters

2. Numerical Implementation

2.1. ODE integration and boundary conditions

We use an adaptative fourth order Runge-Kutta method to solve the initial value problem for a given n, for a data set (M, R, X, Z). Integration stops when $\theta = \theta_s$ for a boundary condition given by the density at the surface θ_s. In the simplest case of a system with no atmosphere, one may set $\theta_s = 0$. The value of ξ for which we meet our boundary condition defines de radius of the system. For this reason it is important to determine the surface values ξ_s and $\dot{\theta}$ as precisely as possible.

We define this as a terminal event of integration, allowing the integrator to iterate over variations of the last step (by successive bisections) until a set precision is attained.

For a given n we obtain a curve (ξ, θ) as seen in figure 1.

2.2. Initial conditions and integration of first step

Special attention is given to the first integration step, as we wish to avoid the division by zero on $\xi = 0$.[2] We then start on an arbitrarily small step $\delta\xi$ where we have relations

$$\theta(\delta\xi) = 1 - \frac{(\delta\xi)^2}{2} + O\left((\delta\xi)^4\right) \tag{10}$$

$$\dot{\theta}(\delta\xi) = -\delta\xi + O\left((\delta\xi)^3\right) \tag{11}$$

2.3. Shooting method for polytropic index determination

If an observed value for luminosity is given one may use a shooting method to determine a value of n for which the luminosity at ξ_s fits observed data. Once again using a successive bisection method, finding the root of the scalar function in equation 12 determined the polytropic index.

$$y(n) = 1 - \frac{L_r(\xi_s)}{L} \tag{12}$$

In the above prescription we did not need to explicitly know the value of K (from equation 3). The implications of this, along with alternative cases, will be discussed in section 5.

[2] We do this to avoid computational trouble. The singularity is otherwise removable.

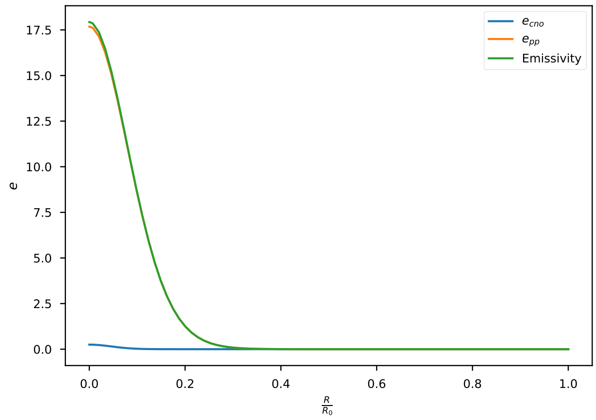

Fig. 3. Sun - Emissivity

3. Sun - Example and code verification

Firstly we take advantage of well know data for the Sun to verify the fidelity of the aforementioned method.

Our algorithm gives polytropic index $n = 3.197$ and normalized parameter curves (P, ρ, T) as seen in figure 2. As for emissivity, we have the expected behavior of ϵ_{cno} dominance, and the luminosity curve (with respect to relative mass) (figures 3 and 4).

One may also calculate the energy of the polytrope and compare with the expected analytical value, giving a measurement of the error of n. We use relation 15, obtained utilizing equations 13 and 14, as found in Monteiro & Gameiro (2019), getting $\delta n \approx 7 \times 10^{-7}$.

$$V = \frac{3}{5-n}\frac{GM^2}{R} \tag{13}$$

$$V = \frac{GM^2}{R}\frac{1}{\xi_s^3\theta_s'^2}\int_0^{\xi_s}\xi^3\theta^n\dot{\theta}\mathrm{d}\xi \tag{14}$$

$$\delta n = 3\xi_S^3\theta_S'^2\frac{1}{\int_0^{\xi_s}\xi^3\theta^n\dot{\theta}\mathrm{d}\xi} + 5 - n \tag{15}$$

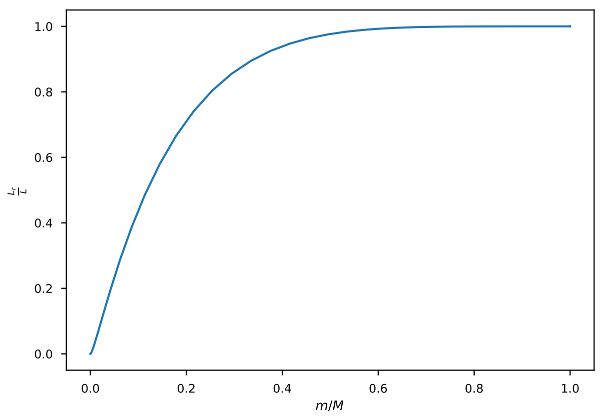

Fig. 4. Sun - Luminosity

Parameter	α Centauri A	α Centauri B
Mass ($M_\odot$)	1.105 ±0.007	0.934 ±0.006
Radius ($R_\odot$)	1.225 ±0.004	0.864 ±0.005
Luminosity ($L_\odot$)	1.47 ±0.05	0.47 ±0.02
Metallicity [3]	0.22	0.30

Table 1. Observational data

Star	Polytropic Index
Sun	3.197
α Centauri A	3.472 ±0.005
α Centauri B	2.87 ±0.02

Table 2. Polytropic indices

As will be discussed in section 4, this error is smaller than the uncertainty (dependent on the uncertainty of observed data), so we are reassured our method is appropriate, for the precision required for our application is achieved.

4. Applications - The α Centauri AB binary

In this section we use observed data from Bruntt et al. (2010) (table 1) to estimate the polytropic index of α Centauri A and B. Uncertainty is estimated by repeated sampling over normally distributed parameters of mass, radius and luminosity in accordance to experimental uncertainty.

Calculated polytropic indices of both stars can be found in table 2 and parameter curves in figures 5 to 11.

We see that to a higher polytropic index correspond curves of higher relative temperature, pressure and density (figures 5, 6 and 7).

Another relevant trend is the significant increase of CNO emissivity with higher n. Additionally, the slope of PP emissivity is proportional to the polytropic index, seeing a more rapidly decreasing (though more intensive) curve for more massive stars.

This is as expected, seeing as fusion of heavier elements (required for the CNO cycle) is more prevalent in condition of higher pressure and temperature. The higher slope in more massive stars corresponds to the proportionally smaller nuclei where radiative processes take place. None of the stars analyses are sufficiently massive to see CNO cycle dominated emissivity.

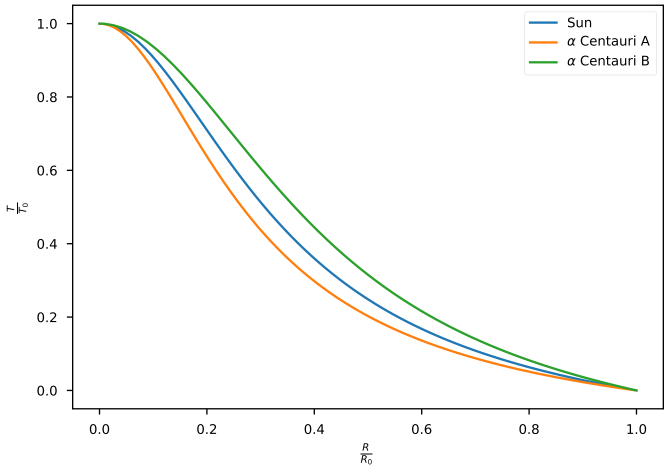

Fig. 5. Temperatures (α Centauri A, B and Sun)

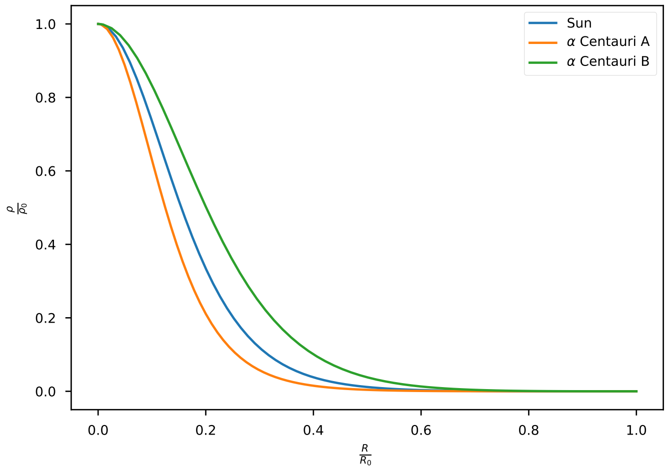

Fig. 6. Densities (α Centauri A, B and Sun)

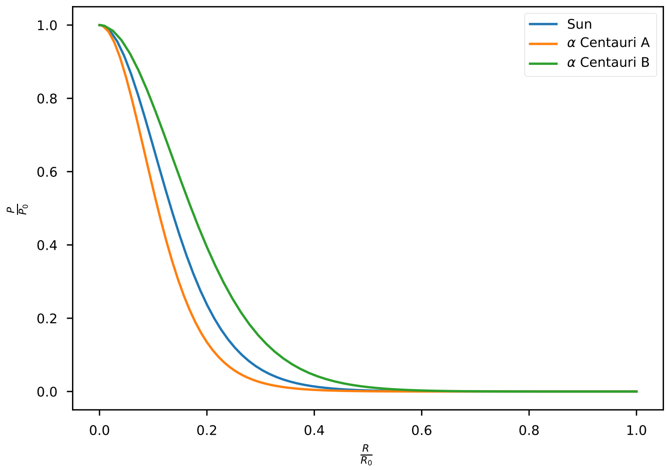

Fig. 7. Pressures (α Centauri A, B and Sun)

When it comes to luminosity curves, a similar can be seen behavior from all parts, even though absolute luminosity suffers significant variance.

Monte Carlo methods used for estimating uncertainty sampling $N = 10000$ iterations of simulated parameters within the

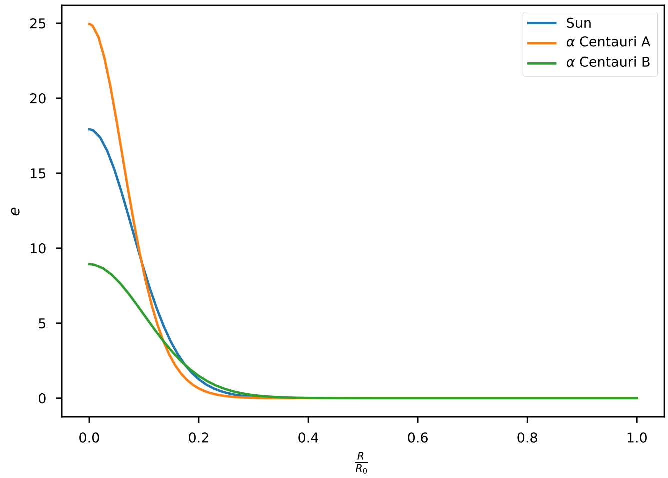

Fig. 8. Total emissivities (α Centauri A, B and Sun)

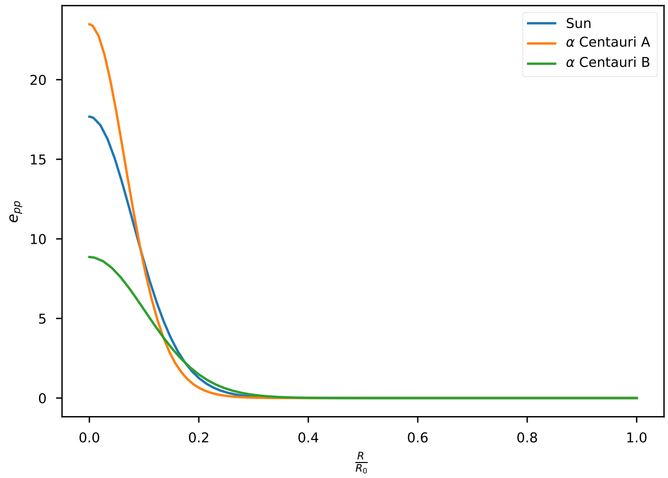

Fig. 9. PP Emissivities (α Centauri A, B and Sun)

experimental uncertainty. The resulting index distribution can be inspected in figures 12 and 13 for α Centauri A and B, respectively.

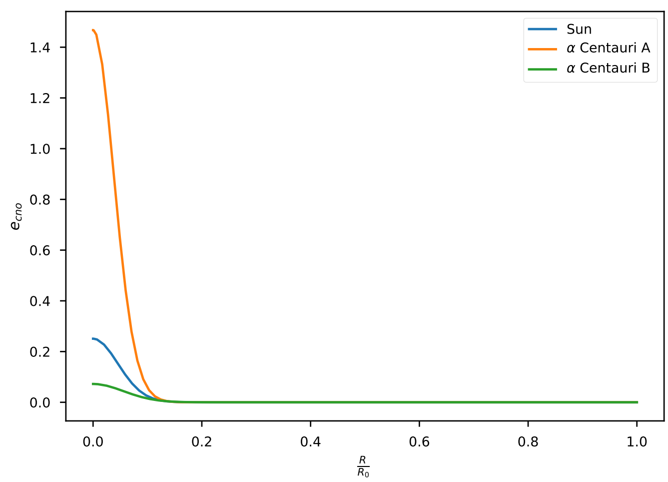

Fig. 10. CNO emissivities (α Centauri A, B and Sun)

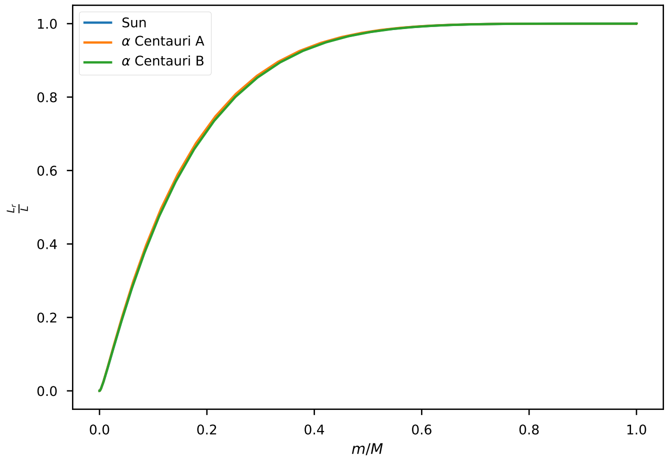

Fig. 11. Luminosities (α Centauri A, B and Sun)

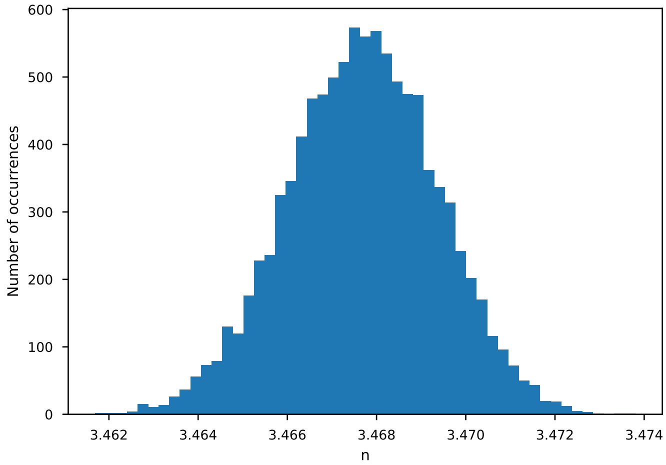

Fig. 12. α Centauri A - Monte Carlo index distribution

One may notice that these distributions are not symmetrical. This means only that the expressions through which our error is propagated have non-linear effects on said values, resulting in possibly asymmetrical distributions.

In rigor, one would give generally different uncertainty values corresponding for each side of the distribution. On our case, however, we choose to give a more conservative value, for reasons discussed in section 6.

5. Valid regime and alternative methods

5.1. Alternative systems

As referenced in section 2, the way one formulates the problem may have consequences on the choice of adequate boundary conditions, if any are needed.

A solution of equation 4 sets a value for ξ_s. A choice of M and R provides a conversion factor to physical quantities (pressure, density, temperature). This factors determine the value of K. For some systems, however, like neutron stars, the polytropic relation is given by physical considerations. This automatically sets a relation between the radius and mass of the object, allowing us to construct the whole model from scarcer data.

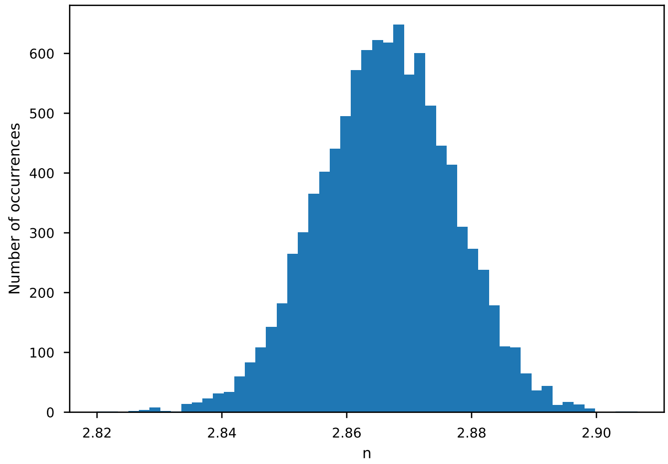

Fig. 13. α Centauri B - Monte Carlo index distribution

5.2. Valid regime

Even though equation 4 is general, given the assumptions employed to construct our model, it may only be applied only in a given regime. This enforces condition such as:

- electromagnetic forces can be ignored
- system is fully characterized by a single polytropic index
- system is in equilibrium

If using the shooting method, the star need be from the main sequence, due to the assumptions made of the radiative processes leading to the luminosity calculation.

6. Model pathologies and possible improvements

The main pathology of our method lies in the assumption that the system is described by a single polytropic condition. This assumption ignores, for example, the fact that there might be an atmosphere affecting our calculation.

Multilayered systems could easily be included in our model (and even code, given the modular nature of it). One would need only iteratively set boundary conditions for each layer based on previous integration results.

This still does now allow for a full simulation of stellar structure, as a fitting model of the atmosphere would need to be included, automatically setting a boundary condition for the limiting density of our polytrope, θ_s, until which our algorithm would proceed in just the same way.

One could also include electromagnetic interactions, by using equations such as found in Takisa & Maharaj (2013).

A more significant improvement would be to include time dependence, permitting the study of evolution. This, however, would require the time integration of all four state equations, now out of equilibrium. This approach is much more computationally intensive, but employed in super computers, using methods for hydrodynamic equations such as found in Kifonidis & Müller (2012).

7. Conclusion

- The polytropic model of stellar interiors was developed and the condition of its applications discussed.
- An efficient numerical method for solving the Lane-Emden equation was implemented.
- An algorithm for determining polytropic indices of main sequence stars from observed data (mass, radius, luminosity and chemical abundances) was presented.
- This algorithm was applied to the study of the α Centauri AB binary. Stellar indices were determined and their influence on stellar structure analysed from constructed models.
- Model limitations were studied and possible improvements discussed.

Acknowledgements. This work was done under the orientation of professor Mário João Monteiro in the context of the curricular unit of Computational Astronomy.

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