Quantum Field Theory for Mathematicians

Notes for a learning seminar on the BV quantization of Yang–Mills theory
following Kevin Costello.

`http://math.tecnico.ulisboa.pt/~jhuerta/qft2024/`

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Notes revised on
15th June 2024

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— 1 —

Lecture 1

1.1   History

Date	People	What	Why	Techniques
1969	Faddeev and Popov	Gauge fixing (adding ghosts)	Quantize Yang-Mills	Berezinian integration
1973	't Hooft and Veltman	Quantized Yang-Mills	Quantize Yang-Mills	Feynman diagrams
1975	Becchi, Rouet, Stora, Tyutin (BRST)	Cohomological theory to quantize Yang-Mills	Understanding 't Hooft and Veltman	Derived invariants (Lie algebra cohomology)
1981	Batallin and Vilkovisky (BV)	Quantize systems with complicated gauge symmetries	Supergravity	Derived intersections (Koszul complexes)
1992	Henneaux	Quantize Yang-Mills using BV	Analyze Yang-Mills using BV	Derived intersections
2007	Costello	Combine BV with effective field theory	Make BV quantization rigorous	Derived everything, analysis, and homotopy theory

1.2   References

The main references for this seminar will be:

- Costello - Renormalization and Effective Field Theory [Cos11];
- Elliot, Williams, Yoo - Asymptotic Freedom in the BV Formalism [EWY18];
- Gwilliam - Factorization algebras and free field theories [Gwi].

1.3   Roadmap to BV Quantization

The space of fields $\mathcal{E}^\bullet$ is a cochain complex

$$\ldots \longrightarrow \mathcal{E}^{-1} \xrightarrow{\ Q\ } \mathcal{E}^0 \xrightarrow{\ Q\ } \mathcal{E}^1 \xrightarrow{\ Q\ } \mathcal{E}^2 \longrightarrow \ldots$$
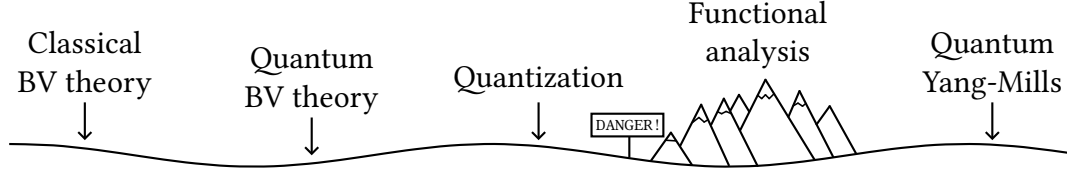
Figure 1.1: Roadmap to BV quantization.

equipped with a differential Q such that $Q^2 = 0$. Moreover, $\mathcal{E}$ admits a -1-shifted symplectic structure, that is, there exists a non degenerate pairing of degree -1

$$\langle \cdot, \cdot \rangle \colon \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathbb{R}[-1]$$

such that $\langle x, y \rangle = -(-1)^{(|x|+1)(|y|+1)} \langle y, x \rangle$. This structure defines a $+1$-shifted Poisson bracket

$$\{\cdot, \cdot\} \colon \mathcal{O}(\mathcal{E}) \otimes \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$$

where $\mathcal{O}(\mathcal{E}) \cong \mathrm{Sym}^\bullet(\mathcal{E}^\vee)$ is the (graded) commutative algebra of polynomial functions on the dual complex $\mathcal{E}^\vee$. Pick $S \in \mathcal{O}(\mathcal{E})$ obeying the **classical master equation** (CME)

$$\{S, S\} = 0.$$

The data $(\mathcal{E}, \langle \cdot, \cdot \rangle, S)$ defines a **classical BV theory**. The CME says $\{S, \cdot\}$ is a differential which makes $(\mathcal{O}(\mathcal{E}), \{S, \cdot\})$ into a cochain complex such that

$$\mathrm{H}^0 \mathcal{O}(\mathcal{E}) \cong \mathcal{O}(\mathrm{Crit}(S)),$$

where $\mathrm{Crit}(S)$ denotes the critical locus of S. We will restrict to S of the form

$$S(e) = \underbrace{\langle e, Qe \rangle}_{\substack{\text{free part} \\ \text{(kinetic +} \\ \text{mass terms)}}} + \underbrace{I(e)}_{\substack{\text{interaction} \\ \text{part (cubic} \\ \text{or higher)}}} .$$

Example 1. Why are the cubic and higher order terms called interaction terms? For electromagnetism on a manifold M we have a space of fields $\mathcal{F} = \Omega^1(M) \oplus \Omega^0(M, S)$ in degree 0. Let $F = \mathrm{d}A$ and define

$$S(A, \psi) = \int_M \underbrace{F \wedge \star F + \langle \psi, \slashed{d}\psi \rangle \, \mathrm{dvol}}_{\text{quadratic terms}} + \underbrace{\langle \psi, \slashed{A}\psi \rangle \, \mathrm{dvol}}_{\text{interaction terms}} .$$

Computing the Euler-Lagrange equations we obtain the system of differential equations

$$\star \mathrm{d} \star F = \bar{\psi} \gamma^\mu \psi \, \mathrm{d}x_\mu$$
$$\slashed{d}_A \psi = 0$$

which is coupled because of the interaction term.

1.4 Quantization in the BV formalism

The slogan of quantization in the BV formalism is to *deform the differential.* In the perturbative context we work in formal power series in $\hbar$, for example, over the ring $\mathbb{R}[[\hbar]]$. Quantization results in a cochain complex $(\mathcal{O}(\mathcal{E})[[\hbar]], \{S^q, \cdot\} + \hbar\Delta$, where Δ is called the BV Laplacian, and $S^q \in \mathcal{O}(\mathcal{E})[[\hbar]]$ satisfies the **quantum master equation** (QME)

$$(\{S^q, \cdot\} + \hbar\Delta)^2 = 0$$

Example 2. In finite dimensions, i.e. $\mathcal{F} \cong \mathbb{R}^n$, the BV fields are

$$\mathcal{E} = \mathbb{R} \longrightarrow \mathbb{R}$$

therefore

$$\mathcal{O}(\mathcal{E}) \cong \mathbb{R}\left[x^1, \ldots, x^n, \xi^1, \ldots, \xi^n\right]$$

and the BV Laplacian takes the form

$$\Delta = \sum_{\mu=1}^{n} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial x^\mu}.$$

In this form, it becomes clear that Δ is a differential operator of degree 1 such that $\Delta^2 = 0$.

The quantum action is a function of the form

$$S^q(e) = \langle e, Qe \rangle + I^q(e)$$

where $I^q \in \mathcal{O}(\mathcal{E})[[\hbar]]$ is cubic mod $\hbar$ and satisfies the QME

$$QI^q + \frac{1}{2}\{I^q, I^q\} + \hbar\Delta I^q = 0$$

which resembles the **Maurer-Cartan** (MC) **equation**. In infinite dimensions, some problems arise:

i) there may be no solution to this equation. In this case we say that quantization is obstructed (there is an anomaly);

ii) the QME in infinite dimensions is ill-defined. Some functional analysis is needed to make sense of this problem.

— 2 —

Lecture 2

In this lecture we consider a naive example that aims to exemplify how the Euler-Lagrange equations lead us to classical BV theories.

Example 3. Let $\mathcal{F}$ be a finite-dimensional vector space encoding the naive space of fields and consider an action

$$S \colon \mathcal{F} \longrightarrow \mathbb{R}.$$

We say that S is a **naive action** because it might be necessary to add additional terms to S to guarantee that it satisfies the CME. The solutions to the Euler-Lagrange equation are fields $f \in \mathcal{F}$ such that $\mathrm{d}S_f = 0$. Restricting to the case $\mathcal{F} = M$ for some finite-dimensional manifold M, we say that critical points of the action form the **critical locus** of S

$$\mathrm{Crit}(S) = \left\{ p \in M \,\middle|\, \mathrm{d}S_p = 0 \right\}.$$

Alternatively, we can characterize the critical locus of S as an intersection in T^*M

$$\mathrm{Crit}(S) = \mathrm{Graph}(\mathrm{d}S) \cap \mathrm{Graph}(M)$$

where we identify M with the zero section. It follows that

$$\mathcal{O}(\mathrm{Crit}(S)) = \mathcal{O}(\mathrm{Graph}(\mathrm{d}S)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$

We are going to consider a derived version of this construction, where the tensor product $\otimes$ is replaced by a derived tensor product $\otimes^{\mathbb{L}}$. This raises the obvious questions:
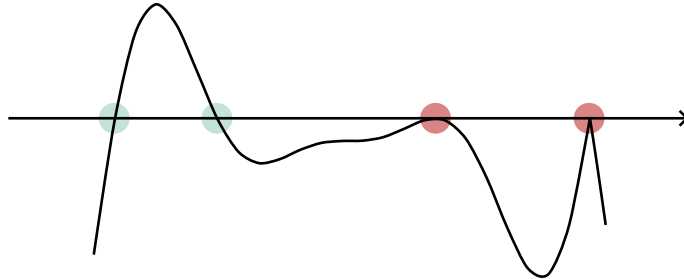

Figure 2.1: Well-behaved (in green) and badly-behaved (in red) points of an intersection.

- **Why?** This intersection might not be well-behaved, in the sense that dS and the zero section might not intersect transversally, or even smoothly, at every point, as illustrated in figure 2.1. The derived approach allows us to study these badly-behaved points using Serre's intersection formula.

- **How?** We replace $\mathcal{O}(\mathrm{Graph}(dS)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M)$ with a dg commutative algebra A such that

$$H^0 A = \mathcal{O}(\mathrm{Graph}(dS)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$

To compute the derived tensor product $\otimes^{\mathbb{L}}$ we need to resolve either $\mathcal{O}(M)$ or $\mathcal{O}(\mathrm{Graph}(dS))$ in $\mathcal{O}(T^*M)$-modules. Let us make use of Darboux coordinates to resolve

$$\mathcal{O}(\mathrm{Graph}(dS)) = \mathcal{O}(T^*M) \Big/ \left(f|_{\mathrm{Graph}(dS)} = 0 \right)$$
$$= \mathcal{O}(T^*M) \Big/ \left(p_\mu - \partial_\mu S \right).$$

Consider the resolution

$$\ldots \longrightarrow \mathcal{O}(T^*M)(\xi_1, \ldots, \xi_n) \xrightarrow{\;\xi_\mu \mapsto p_\mu - \partial_\mu S\;} \mathcal{O}(T^*M)$$
$$\Big\downarrow$$
$$\mathcal{O}(\mathrm{Graph}(dS))$$

which we extend to the left as a Koszul complex $K^{-p} = \bigwedge^p_{\mathcal{O}(T^*M)}(\xi_1, \ldots, \xi_n)$ with differential

$$d = \sum_\mu (p_\mu - \partial_\mu S) \frac{\partial}{\partial \xi_\mu}.$$

This complex freely resolves $\mathcal{O}(\mathrm{Graph}(dS))$. Alternatively, $(K^\bullet, d)$ admits a coordinate free description where

$$K^{-p} = \mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} \mathfrak{X}^p(M).$$

A model for $\mathcal{O}(\mathrm{Graph}(dS)) \otimes^{\mathbb{L}}_{\mathcal{O}(T^*M)} \mathcal{O}(M)$ is given by

$$\mathcal{O}(\mathrm{Crit}^h(S)) = K^{-\bullet} \otimes_{T^*M} \mathcal{O}(M)$$

which we call the **derived critical locus**. But notice that

$$\mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} PV^\bullet(M) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M) \cong PV^\bullet(M)$$

where $PV^\bullet(M)$ denotes the complex of polyvector fields on M. The differential is given by contracting with dS, so we write

$$\mathcal{O}(\mathrm{Crit}^h(S)) = (PV^\bullet(M), -\iota_{dS}).$$

— 3 —

Lecture 3

We want to sketch how to go from the Yang-Mills action

$$S^{\text{naive}}(A) = \int_{M^n} \text{tr}(F_A \wedge \star F_A)$$

to the Yang-Mills classical BV theory

$$\underbrace{\overset{-1}{\Omega^0(M,\mathfrak{g})}}_{\text{ghosts}} \xrightarrow{\ \text{d}\ } \underbrace{\overset{0}{\Omega^1(M,\mathfrak{g})}}_{\text{fields}} \xrightarrow{\ \text{d}\star\text{d}\ } \underbrace{\overset{1}{\Omega^{n-1}(M,\mathfrak{g})}}_{\text{antifields}} \xrightarrow{\ \text{d}\ } \underbrace{\overset{2}{\Omega^n(M,\mathfrak{g})}}_{\text{antighosts}}$$

with BV action

$$S^{\text{BV}}(e) = \langle e, Qe \rangle + I(e)$$

where

$$\langle e, f \rangle = \int_{M^n} \text{tr}(e \wedge f)$$

is the -1-shifted symplectic structure. There are some points to motivate:

i) **fields $\rightsquigarrow$ fields and antifields:** coming from the derived critical locus $\text{dCrit}(S)$;

ii) **ghosts:** coming from taking the derived coinvariants of $\mathfrak{g} \curvearrowright V$.

For Yang-Mills spacetime is a manifold M^n and $\Omega^1(M,\mathfrak{g})$ is the space of fields. However, in what follows, let M be the space of fields. Recall that

$$\begin{aligned}
\text{Crit}(S) &= \left\{ p \in M \mid \text{d}S_p = 0 \right\} \\
&= \text{Graph}(\text{d}S) \cap M
\end{aligned}$$

in T^*M. Dually

$$\mathcal{O}\big(\text{Crit}(S)\big) = \mathcal{O}\big(\text{Graph}(\text{d}S)\big) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$

By homological yoga, taking the derived intersection means that we replace the tensor product $\otimes$ with the derived tensor product $\otimes^{\mathbb{L}}$. To find $\text{Crit}^{\text{h}}(S)$ we resolve either $\mathcal{O}\big(\text{Graph}(\text{d}S)\big)$ or $\mathcal{O}(M)$ as $\mathcal{O}(T^*M)$-modules. Last time, we wrote the Koszul complex

$$K^{-p} = \text{PV}^p(M) \otimes_{\mathcal{O}(M)} \mathcal{O}(T^*M)$$

where $\text{PV}^p = \bigwedge^p \mathfrak{X}(M)$ and differential

$$Q : v_1 \wedge \cdots \wedge v_k \otimes 1 \longmapsto \sum_{i=1}^{k} (-1)^{i+1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k \otimes \big(p(v_i) - \text{d}S(v_i)\big)$$

Exercise 1. Check that $\mathrm{H}^0(K^\bullet, Q) \cong \mathcal{O}\big(\mathrm{Graph}(\mathrm{d}S)\big)$, so

$$\mathrm{Crit}^{\mathrm{h}}(S) \cong K^\bullet \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M) \cong \mathrm{PV}^\bullet(M)$$

and thus $\mathcal{O}\big(\mathrm{Crit}^{\mathrm{h}}(S)\big) \simeq (\mathrm{PV}^\bullet, -\iota_{\mathrm{d}S})$.

Exercise 2. Show that $\mathrm{H}^0\mathcal{O}(\mathrm{Crit}^{\mathrm{h}}) \cong \mathcal{O}(\mathrm{Crit})$.

We can enhance $\mathcal{O}\big(\mathrm{Crit}^{\mathrm{h}}(S)\big)$ to a sheaf on M. Following Grothendieck

$$\mathrm{Crit}^{\mathrm{h}}(S) = \big(M, \mathrm{PV}^\bullet_M, -\iota_{\mathrm{d}S}\big)$$

is an example of a **dg manifold**.

Definition 1. A dg manifold is a smooth manifold M with a sheaf $\mathcal{O}_M$ of **dg commutative algebras** (DGCAs) locally isomorphic to $\mathcal{O}_M(U) \cong \bigwedge \mathcal{E}(U)$, where $\mathcal{E}$ are the smooth sections of $E \to M$.

Ignoring the differential, we get a sheaf $\big(M, \mathrm{PV}^\bullet_M\big)$ on M such that

$$\mathrm{PV}_M = \bigwedge \mathfrak{X}_M \cong \mathrm{Sym}\,\mathfrak{X}[1].$$

The underlying graded manifold of $\mathrm{Crit}^{\mathrm{h}}(S)$ is

$$T^*[-1]M = \big(M, \mathrm{Sym}\,\mathfrak{X}[1]\big)$$

displaying the following properties:

  i) the graded manifold $T^*[-1]M$ is a -1-shifted symplectic graded manifold just as T^*M is a 0-shifted symplectic manifold;

  ii) Induced from the -1-shifted symplectic structure we get a 1-shifted Poisson bracket on $\mathcal{O}\big(T^*[-1]M\big) = \mathrm{PV}(M)$ known as the **Schouten bracket**

$$\{f, g\} = 0,$$
$$\{v, f\} = vf,$$
$$\{v, w\} = [v, w],$$
$$\{u, v \wedge w\} = \{u, v\} \wedge w + v \wedge \{u, w\}$$

for $f, g \in \mathcal{O}(M)$ and $u, v, w \in \mathrm{PV}^{-1}(M)$.

Exercise 3. Show that $-\iota_{\mathrm{d}S} = \{S, \cdot\}$.

Definition 2. A $\mathbb{P}_0$ algebra $\big(A, \mathrm{d}, \{\cdot, \cdot\}\big)$ is a DGCA (A, d) equipped with a 1-shifted Poisson bracket $\{\cdot, \cdot\} : A \otimes A \to A$ obeying:

  i) **graded skew-symmetry:**

$$\{x, y\} = -(-1)^{(|x|+1)(|y|+1)}\{y, x\};$$

  ii) **graded Poisson identity**:

$$\{x, yz\} = \{x, y\}z + (-1)^{(|x|+1)|y|}y\{x, z\}$$

  so $\{x, \cdot\}$ is a degree $|x| + 1$ derivation;

  iii) **graded Jacobi identity**:

$$\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{(|x|+1)(|y|+1)}\{y, \{x, z\}\};$$

  iv) **compatibility with differential:**

$$\mathrm{d}\{x, y\} = \{\mathrm{d}x, y\} + (-1)^{|x|+1}\{x, \mathrm{d}y\}.$$

Exercise 4. Check that the Schouten bracket defines a $\mathbb{P}_0$-algebra on $\mathcal{O}\big(T^*[-1]M\big)$.

<center>— **4** —</center>

Lecture 4

Last time we constructed the derived critical locus of $S\colon M \to \mathbb{R}$

$$\mathrm{Crit}^{\mathrm{h}}(S) = \big(T^*[-1]M, -\iota_{\mathrm{d}S}\big)$$

where we understand $T^*[1]M = \big(M, \mathrm{PV}^\bullet\big)$ as the underlying manifold M equipped with the sheaf defined by the assignment

$$\mathrm{PV}^\bullet\colon U \longrightarrow \mathrm{Sym}^\bullet_{\mathscr{O}(U)}\big(\mathcal{T}_M(U)[1]\big).$$

Proposition 1. If V is a locally-finite -1-shifted symplectic dg vector space then

$$\mathscr{O}(V) = \mathrm{Sym}(V^\vee)$$

is a $\mathbb{P}$-algebra.

Proof. The pairing $\langle \cdot, \cdot \rangle$ induces an isomorphism $V \cong V^\vee[-1]$ which we use to define a bracket

$$\{\cdot, \cdot\}\colon \mathrm{Sym}^2(V^\vee) \longrightarrow \mathbb{R}$$

which we extend as a derivation to $\mathscr{O}(V)$. ∎

4.1 Work Perturbatively

Fix a solution to the equations of motion $p \in M$, and consider $V = T_pM$ instead of M. Then we can expand S as a polynomial (or formal power series).

For us, the space of fields $\mathcal{F}$ will always be a sheaf of vector spaces on spacetime. In our example, spacetime is a point pt and $\mathcal{F} = V$ for some finite-dimensional vector space. Going forward we rewrite $T^*[-1]M \rightsquigarrow T^*[-1]V$.

Remark. Be analogy to ungraded geometry $T^*V \cong V \oplus V^\vee$ we have that

$$T^*[-1]V \cong V \oplus V^\vee[-1].$$

In infinite dimensions we consider the sheaves

$$T^*[-1]\mathcal{F} = \underbrace{\mathcal{F} \oplus \mathcal{F}^\vee[-1]}_{\substack{\text{sheaf of BV fields} \\ \text{without gauge symmetry}}}.$$

<center>10</center>

Because we are expanding around a critical point the action has takes the form

$$S(e) = \underbrace{\langle e, Qe \rangle}_{\text{Hessian of } S} + I(e), \qquad Q \colon \mathcal{F} \longrightarrow \mathcal{F}^{\vee}[-1].$$

Now let M be a finite-dimensional manifold called spacetime, $\mathcal{F}$ the space of naive fields, as a sheaf of vector spaces on M, and $\mathcal{E}$ the sheaf of BV fields. Our ultimate goal is to make sense of expressions of the form

$$\int_{\phi \in \mathcal{F}(M)} \exp\left(-\frac{S_{\text{naive}}}{\hbar}\right) D\phi.$$

If $\mathcal{F} = V$ and S is quadratic then

$$\int_V \exp\left(-\frac{S(\phi)}{\hbar}\right) d\phi = \int_V \exp\left(-\frac{\langle \phi, Q\phi \rangle}{\hbar}\right) d\phi$$
$$= \left(\frac{\pi}{\hbar}\right)^{\frac{n}{2}} \det(Q)^{-\frac{1}{2}}$$

and if $S(\phi) = \langle \phi, Q\phi \rangle + I(\phi)$ we incorporate the interaction terms by working perturbatively.

Even at finite dimensions, the case $\det(Q) = 0$ poses a bad problem when trying to apply the previous formula. However, degenerate critical points are an unavoidable feature with gauge symmetry [1]

$$\underbrace{\mathcal{G} \subset \mathcal{F}.}_{\text{nonlinear action}}$$

Example 4. Consider Yang-Mills with gauge group G and trivial gauge bundle $M \times G \to M$. The space of fields is

$$\mathcal{F}(M) = \Omega^1(M, \mathfrak{g}), \qquad \mathfrak{g} = \text{Lie}(G)$$

and the group of gauge transformations

$$\mathcal{G} = \text{Aut}(M \times G \longrightarrow M) \cong \text{Map}(M, G)$$

where

$$\text{Lie}\, \mathcal{G} \cong \text{Map}(M, \mathfrak{g}) = \Omega^0(M, \mathfrak{g}).$$

Instead of $\mathcal{G} \subset \mathcal{F}(M)$, we focus on the action of the Lie algebra of gauge transformations

$$\text{Lie}(\mathcal{G}) \subset \mathcal{F}(M).$$

Adopting the standard notation, we write that $c \in \Omega^0(M, \mathfrak{g})$ acts on $A \in \Omega^1(M, \mathfrak{g})$ by

$$c \cdot A = dc + [c, A].$$

Exercise 5. Check that

$$S_{\text{naive}}^{\text{YM}} = \int \langle F_A, F_A \rangle \, dvol$$

is invariant under infinitesimal gauge transformations.

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[1] Gauge transformations preserve the action S^{naive} and the equations of motion.

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— **5** —

Lecture 5

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(John Huerta)

5.1 Ultimate Goal

Define and use the Feynman "path" integral

$$\int_{\phi \in \mathcal{F}} e^{-\frac{S(\phi)}{\hbar}} \, D\phi.$$

(Euclidean field theory)

In the constructive track: see Gonçalo on how to do this. In the BV track: we will produce a formal power series in $\hbar$.

5.2 Recall

- From now on: We work *perturbatively*, i.e., *formally* (in Algebraic Geometry speak), i.e., in *formal power series*, i.e., *infinitesimally*.

- Now M is going to be a finite dimensional manifold, denoting space-time. E.g.,

$$M = \mathbb{R}^d, \text{ or } M = \text{pt}.$$

- $\mathcal{F}$ always denotes the naive fields, a sheaf of vector spaces on M; specifically, sections of some vector bundle $F \longrightarrow M$.

  Example: Yang-Mills fields for a trivial G-bundle $M \times G \longrightarrow M$, then $\mathcal{F}(M) = \Omega^1(M, \mathfrak{g})$, where $\mathfrak{g} = \text{Lie}(G)$.

- $\mathcal{E}$ ("extended"), the space of BV-fields, a sheaf of *graded* vector spaces over M, sections of a graded vector bundle $E \longrightarrow M$. $\mathcal{E}^0(M) = \mathcal{F}(M)$.

  In the Yang-Mills example, where $d = \dim M$

$$\mathcal{E}(M) = \underbrace{\overset{-1}{\Omega^0(M, \mathfrak{g})}}_{\text{``ghosts''}} \oplus \underbrace{\overset{0}{\Omega^1(M, \mathfrak{g})}}_{\text{``fields''}} \oplus \underbrace{\overset{1}{\Omega^{d-1}(M, \mathfrak{g})}}_{\text{``anti-fields''}} \oplus \underbrace{\overset{2}{\Omega^d(M, \mathfrak{g})}}_{\text{``anti-ghosts''}}$$

<center>12</center>

5.3 BV Formulation of Gauge Theory

Input: Naive gauge theory

$$\underbrace{\mathcal{L}}_{\substack{\text{Lie algebra of} \\ \text{infinitesimal gauge} \\ \text{transformations}}} \curvearrowright \underbrace{\mathcal{F}}_{\text{space of naive fields}} \; .$$

The action may be non-linear. In the Young-Mills example it is an affine action

$$\Omega^0(M\mathfrak{g}) \curvearrowright \Omega^1(M\mathfrak{g}) \, .$$

There is a two-step process to writing down the gauge theory:

1. Take the *"stacky quotient"*

$$\mathcal{F} \rightsquigarrow \mathcal{F} /\!\!/ \mathcal{L} \qquad \text{(this lecture)}$$

2. Take the derived critical locus of S_{gauge}:

$$T^*[-1]\,(\mathcal{F}\,/\,\mathcal{L}) \, . \qquad \text{(already done)}$$

5.4 Lightning Fast Introduction to Derived Invariants

$\mathfrak{g}$ a finite dimensional Lie algebra, R a finite dimensional representation of $\mathfrak{g}$

$$\mathfrak{g} \to \mathfrak{gl}(R)$$

over some field $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$. Observe that

$$\begin{aligned} R^{\mathfrak{g}} &= \{v \in \mathbb{R} \mid Xv = 0 \text{ for all } X \in \mathfrak{g}\} \\ &= \operatorname{Hom}_{\mathfrak{g}}(\Bbbk, R) \end{aligned}$$

Derived version Hom $\rightsquigarrow \mathbb{R}\mathrm{Hom}$.

Try $R^{h\mathfrak{g}} = \mathbb{R}\mathrm{Hom}_{U\mathfrak{g}}(\Bbbk, R)$, where U is the enveloping algebra. I.e.,

$$U\mathfrak{g} = \frac{T\mathfrak{g}}{x \otimes y - y \otimes x - [x, y]}$$

where $T\mathfrak{g}$ is the tensor algebra.

Fact 1. $\mathrm{Rep}_{\mathfrak{g}} \simeq U\mathfrak{g}-\mathrm{mod}$.

To compute $\mathbb{R}\mathrm{Hom}_{U\mathfrak{g}}(\Bbbk, R)$ we need to resolve $\Bbbk$ or R as $U\mathfrak{g}$ modules.

Similar to the Koszul complex

$$\cdots \longrightarrow \overset{-k}{\Lambda^k \mathfrak{g} \otimes U\mathfrak{g}} \longrightarrow \cdots \longrightarrow \overset{-1}{\mathfrak{g} \otimes U\mathfrak{g}} \longrightarrow \overset{0}{U\mathfrak{g}}$$

with differential

$$\Lambda^{k+1}\mathfrak{g}\otimes \longrightarrow \Lambda^k \otimes U\mathfrak{g}$$

$$x_0 \wedge \cdots \wedge x_k \otimes y \longmapsto \sum_{i=0}^{k}(-1)^i x_0 \wedge \cdots \widehat{x_i} \cdots \wedge x_k \otimes x_i y$$

$$+ \sum_{i<j}(-1)^{i+j}[x_i, x_j] \wedge x_0 \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge x_k \otimes y.$$

With this differential

$$H^0\left(\Lambda^{\bullet}\mathfrak{g} \otimes U\mathfrak{g}\right) \simeq \Bbbk$$
$$H^k\left(\Lambda^{\bullet}\mathfrak{g} \otimes U\mathfrak{g}\right) = 0 \qquad \text{for } k < 0$$

Hence

$$R^{\mathrm{h}\mathfrak{g}} = \mathbb{R}\mathrm{Hom}_{U\mathfrak{g}}(\Bbbk, R)$$
$$= \mathrm{Hom}_{U\mathfrak{g}}(\Lambda^{\bullet}\mathfrak{g} \otimes U\mathfrak{g}, R)$$
$$\simeq \mathrm{Hom}_{\Bbbk}(\Lambda^{\bullet}\mathfrak{g}, R)$$

because $\Lambda^{\bullet}\mathfrak{g} \otimes U\mathfrak{g}$ is free.

Definition 3. For $\mathfrak{g}$ a Lie algebra, R a representation of $\mathfrak{g}$, the *Chevalley-Eilenberg complex* $C^{\bullet}(\mathfrak{g}, R)$ is defined as

$$C^k(\mathfrak{g}, R) = \mathrm{Hom}(\Lambda^k\mathfrak{g}, R)$$

with

$$\mathrm{d}\omega(x_0, \ldots, x_k) = \sum_{i=0}^{k}(-1)^i x_i \cdot \omega(x_0, \ldots, \widehat{x_i}, \ldots, x_k)$$

$$+ \sum_{i<j}(-1)^{i+j}\omega([x_i, x_j], x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_k)$$

Conclusion. Back to $R = O(V)$, then

$$R^{\mathrm{h}\mathfrak{g}} = O(V)^{\mathrm{h}\mathfrak{g}}$$
$$= \mathrm{Hom}_{\Bbbk}(\Lambda^{\bullet}\mathfrak{g}, O(V))$$
$$\simeq \Lambda^{\bullet}\mathfrak{g}^* \otimes O(V)$$
$$\simeq \mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes \mathrm{Sym}(V^*)$$
$$\simeq \mathrm{Sym}(\overset{0}{V^*} \oplus \overset{1}{\mathfrak{g}^*[-1]})$$
$$\simeq O(\overset{-1}{\mathfrak{g}[1]} \oplus \overset{0}{V})$$
$$=: O(V/\mathfrak{g}).$$

Hence

Definition 4. $V/\mathfrak{g} := \mathfrak{g}[1] \oplus V$

Puzzle: what happened to the differential d. It becomes a vector field on $\mathfrak{g} \oplus V$!

5.5   Back to Yang-Mills

$$V \rightsquigarrow \Omega^1(M, \mathfrak{g})$$
$$\mathfrak{g} \rightsquigarrow \Omega^0(M, \mathfrak{g})$$

Step 1: $\Omega^1(M, \mathfrak{g}) / \Omega^0(M, \mathfrak{g}) := \Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g})$

Step 2: $\mathcal{E}$ for Yang-Mills

$$T^*[-1](\overset{-1}{\Omega^0(M, \mathfrak{g})}[1] \oplus \overset{0}{\Omega^1(M, \mathfrak{g})}) \simeq \Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g})$$
$$\oplus (\Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g}))^*[-1]$$
$$\simeq \Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g})$$
$$\oplus \Omega^{d-1}(M, \mathfrak{g})[-1] \oplus \Omega^d(M, \mathfrak{g})[-2]$$

— 6 —

Lecture 6

Last time we discussed **perturbative classical BV gauge theory**. We had

$$\mathcal{L} \subset \mathcal{F}$$

with $S_{\text{naive}} \in \mathcal{O}_{\text{loc}}(\mathcal{F})$. The recipe is:

  i) take the **stacky quotient**

$$\mathcal{F} /\!/_{\mathcal{L}} = \mathcal{L}[1] \oplus \mathcal{F}$$

with a vector field Q_{CE}. The condition that S_{naive} is gauge-invariant is equivalent to

$$Q_{\text{CE}} S_{\text{naive}} = 0;$$

 ii) take the **derived critical locus**

$$T^*[-1]\left(\mathcal{F} /\!/_{\mathcal{L}}\right)$$

with differential $\{S_{\text{naive}}, \cdot\}$. The underlying space is

$$\mathcal{L}[1] \oplus \mathcal{F} \oplus \left(\mathcal{L}[1] \oplus \mathcal{F}\right)^{\vee}[-1] = \mathcal{L}[1] \oplus \mathcal{F} \oplus \mathcal{F}^{\vee}[-1] \oplus \mathcal{L}^{\vee}[-2].$$

iii) obtain the **BV action** S satisfying the CME

$$\{S, S\} = 0$$

and incorporate (somehow) S_{naive} and Q_{CE}. This means that

$$S_{\text{naive}} = S\big|_{\mathcal{F}}, \qquad \text{and} \qquad Q_{\text{CE}} = \{S, \cdot\}\big|_{\mathcal{L}[1] \oplus \mathcal{F}}.$$

Fact 2. The vector field Q_{CE} is Hamiltonian, i.e.

$$Q_{\text{CE}} = \{S_{\text{CE}}, \cdot\}$$

with respect to the -1-shifted symplectic structure, for some S_{CE}. As such, we can define

$$S = S_{\text{naive}} + S_{\text{CE}}.$$

For Yang-Mills on an oriented Riemannian n-manifold, with trivial bundle $M \times G \to M$ and $\mathfrak{g} = \text{Lie}(G)$, $\mathcal{E}$ looks like

$$\Omega^0(M, \mathfrak{g}) \longrightarrow \Omega^1(M, \mathfrak{g}) \longrightarrow \Omega^{n-1}(M, \mathfrak{g}) \longrightarrow \Omega^n(M, \mathfrak{g})$$

and the we write the action

$$S_{\text{naive}}(A) = \int_M \frac{1}{4} \langle F_A, F_A \rangle.$$

We compute that

$$S_{\text{CE}}(c, A, A^*, c^*) = \int_M \langle \text{d}_A c, A^* \rangle + \frac{1}{2} \langle [c, c], c^* \rangle.$$

Choose bases $\{T_a\}$ for $\mathfrak{g}$ and $\{e_i\}$ for V. An element of $gfrak[1] \oplus V$ can be written

$$X^a T_a[1] + v^i e_i, \qquad X^a, v^i \in \mathbb{R}.$$

Let ξ^a be the linear coordinate corresponding to $T_a[1]$ and x^i corresponding to e_i. Note that ξ^a has degree +1.

Proposition 2 (Berezin, Leites). Consider

$$\text{Der}\left(\mathcal{O}(\mathbb{R}^n) \otimes \bigwedge W^\vee \right)$$

where $\{\theta_a\}$ is a basis of W, and θ^a denotes the respective dual basis elements. Then

$$\text{Der}\left(\mathcal{O}(\mathbb{R}^n) \otimes \bigwedge W^\vee \right) = \left(\mathcal{O}(\mathbb{R}^n) \otimes \bigwedge W^\vee \right) \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^i} \right\}$$

where $\frac{\partial}{\partial \theta^i}$ is the degree -1 derivation such that

$$\frac{\partial}{\partial \theta^i} \theta^1 \dots \theta^n = (-1)^{i+1} \theta^1 \dots \hat{\theta}^k \dots \theta^n \delta_{i,k}.$$

The idea of the computation is to determine the coefficients of the derivation Q_{CE} by computing $Q_{\text{CE}} x^i$ and $Q_{\text{CE}} \xi^a$. We obtain

$$Q_{\text{CE}} = \xi^a \rho_a^i \frac{\partial}{\partial x^i} \underbrace{\left(-\frac{1}{2} f_{bc}^a \xi^b \xi^c \frac{\partial}{\partial \xi^a} \right)}_{\frac{1}{2}[c,c]}$$

where $\rho : \mathfrak{g} \to \mathfrak{X}(V)$ and $[T_a, T_b] = f_{a,b}^c T_c$.

Definition 5. A **perturbative classical BV theory** consists of the data:
  i) a graded vector bundle $E \to M$;
 ii) a -1-shifted symplectic structure

$$E \boxtimes E \longrightarrow \text{Dens } M;$$

iii) a local action functional $S \in \mathcal{O}_{\text{loc}}(\mathcal{E})$ that is at least quadratic and satisfies the CME.

Let $\mathcal{E}$ be the topological vector space (TVS) of global smooth sections, and $\mathcal{E}_c$ the TVS of compactly supported sections. Then $\mathcal{O}(\mathcal{E})$ is the completion of the symmetric algebra on $\mathcal{E}_c^\vee$. We define

$$\mathcal{O}(\mathcal{E})_{\mathrm{loc}}(\mathcal{E}) \subseteq \mathcal{O}(\mathcal{E})$$

where $F \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ is a sum of terms of the form

$$e \in \mathcal{E}_c \longmapsto \int_M \mathrm{D}_1 e \ldots \mathrm{D}_n e \, \Omega$$

for $\mathrm{D}_i \colon \mathcal{E} \to C^\infty(M)$ and Ω a density on M.

Proposition 3. For a classical BF theory $\mathcal{E}$ we can write

$$S = \int_M \langle e, Qe \rangle + I(e)$$

where $Q \colon \mathcal{E} \to \mathcal{E}$ is a differential operator of degree $+1$, squares to zero, and $I \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ is at least cubic and satisfies the QME

$$QI + \frac{1}{2}\{I, I\} = 0.$$

— 7 —
Lecture 7 - Rui Peixoto

7.1 Quantum BV in Finite Dimensions

In this section we start out journey towards the quantization of BV theories. Recall that out goal is to provide a **homological approach** to compute integrals of the form

$$\int_{\phi \in \mathcal{F}} e^{-\frac{S}{\hbar}} D\phi.$$

We will introduce the quantum BV complex as a generalized **divergence complex**. In finite dimensions this is an obscured version of the de Rham complex, where we have the usual homological approach to integration.

For this chapter we return to the finite-dimensional case, which corresponds to the case where spacetime is a point $M = \text{pt}$. Consider:

i) a n-dimensional **graded** vector space V of **fields**;

ii) a -1-**shifted symplectic pairing** $\langle \cdot, \cdot \rangle \colon V \otimes V \to \mathbb{R}$;

iii) an **action** $S \in \mathcal{O}(V)$ such that

$$\{S, S\} = 0, \qquad \text{\small classical master equation (CME)}$$

where $\{\cdot, \cdot\} \colon \mathcal{O}(V) \otimes \mathcal{O}(V) \to \mathcal{O}(V)$ is the $+1$-**shifted Poisson bracket** induced by the symplectic pairing.

Such a setup makes $\big(\mathcal{O}(V), \{\cdot, \cdot\}\big)$ into a cochain complex such that

$$H^0 \mathcal{O}(V) \cong \mathcal{O}(\text{Crit}(S)).$$

Passing to the derived critical locus, we have seen that functions on $\text{Crit}^{\text{h}}(S)$ form a commutative dg algebra

$$\mathcal{O}\big(\text{Crit}^{\text{h}}(S)\big) = \big(\text{PV}^{\bullet}(V), -\iota_{dS}\big)$$

which we call the **classical BV complex**. Passing to the quantum version amounts to deforming this complex by changing the differential.

7.2 Integration in Finite Dimensions

Fixing a nonvanishing top form $\mu \in \Omega^n(V)$ defines a map

$$\int_V \colon \mathcal{O}(V) \longrightarrow \mathbb{R}$$

$$f \longmapsto \int_V f\mu$$

where we make the necessary assumptions on $\mathcal{O}$ such that f is integrable with respect to μ. This map depends only on **cohomological data**. Explicitly, we have that

$$\int_M f\mu = \frac{[f\mu]}{[\mu]}$$

where $[\cdot]$ denotes the cohomology class in $H_{dR}(V)$. Note that the denominator is just a normalizing factor. If we pick $\mu \in \Omega^n(V)$ such that $\int_V \mu = 1$, computing the integral of f with respect to μ boils down to computing a class in cohomology

$$\int_V f\mu = [f\mu].$$

The pairing by integration with μ is nondegenerate so it defines an isomorphism $\mathcal{O}(V) \to \Omega^n(V)$ which we can extends to an isomorphism of complexes

$$m_\mu \colon \operatorname{PV}^{-k}(V) \longrightarrow \Omega^{n-k}(V)$$
$$X \longmapsto \iota_X \mu.$$

Example 5. In coordinates, let $\mu = dx^1 \wedge \cdots \wedge dx^n$ and $X = f\frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_k}}$. Then

$$m_\mu(X) = f m_\mu\left(\frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_k}}\right)$$
$$= \sigma f dx^1 \ldots d\hat{x}^{j_i} \ldots d\hat{x}^{j_k} \ldots dx^n$$

where the terms $d\hat{x}^{j_i}$ are omitted, and $\sigma = \pm 1$ is such that

$$\sigma dx^1 \ldots dx^n = dx^{j_1} \ldots dx^{j_k} \wedge dx^1 \ldots dx^n.$$

The **divergence operator** on $\operatorname{PV}^\bullet(V)$

$$\operatorname{div}_\mu = m_\mu^{-1} d m_\mu$$

is obtained by pulling back the de Rham differential on $\Omega^\bullet$ using μ.

$$
\begin{array}{ccccccc}
\ldots & \longrightarrow & \operatorname{PV}^{-2}(V) & \xrightarrow{\operatorname{div}_\mu} & \operatorname{PV}^{-1}(V) & \xrightarrow{\operatorname{div}_\mu} & \mathcal{O}(V) \\
& & \downarrow{\scriptstyle m_\mu} & & \downarrow{\scriptstyle m_\mu} & & \downarrow{\scriptstyle m_\mu} \\
\ldots & \longrightarrow & \Omega^{n-2}(V) & \xrightarrow{d} & \Omega^{n-1}(V) & \xrightarrow{d} & \Omega^n(V)
\end{array}
$$

Because it admits this definition in the finite-dimensional case, we say that the divergence complex is an *obscured* version of the usual de Rham complex. At this point, one might wonder: why do we not just use the de Rham complex to begin with? The point is that, unlike the de Rham complex, the divergence complex generalizes to the infinite-dimensional case.

Recall how before we recovered the space of functions on the critical locus $\operatorname{Crit}(S)$ from $\operatorname{PV}^\bullet$ by passing to cohomology in degree 0. This crucial information is encoded in the de Rham complex in degree n, where top forms live. In infinite dimensions, this data escapes as the de Rham complex ceases to be bounded above. However, it still resides in degree 0 in the divergence complex. In this sense, the natural approach in quantum field theory is to generalize the divergence operator to the infinite-dimensional case.

Example 6. Let $V = \mathbb{R}$ and μ_{Leb} be the Lebesgue measure. A simple computation recovers the usual divergence operator

$$\text{div}_{\text{Leb}}\left(f\frac{\partial}{\partial x}\right) = m_{\mu}^{-1}\text{d}m_{\mu}\left(f\frac{\partial}{\partial x}\right)$$

$$= m_{\mu}^{-1}(\text{d}f)$$

$$= m_{\mu}^{-1}\left(\frac{\partial f}{\partial x}\mu\right)$$

$$= \frac{\partial f}{\partial x}.$$

If we write the generator of the vector fields as $\xi = \frac{\partial}{\partial x}$ then

$$\text{div}_{\text{Leb}} = \frac{\partial}{\partial x}\frac{\partial}{\partial \xi} = \Delta_{\text{BV}}$$

takes the form of the usual BV Laplacian. It is straightforward to generalize to the n-dimensional case

$$\text{div}_{\text{Leb}} = \Delta_{\text{BV}} = \sum_{i=1}^{n}\frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi^i}.$$

Example 7. Consider again $V = \mathbb{R}$ but let μ_S be a Gaussian measure of the form

$$\mu_S = e^{-\frac{S}{\hbar}}\mu_{\text{Leb}}.$$

In this case, we see that

$$\text{div}_S = m_{\mu_S}^{-1}\text{d}m_{\mu_S}\left(f\frac{\partial}{\partial x}\right)$$

$$= m_{\mu_S}^{-1}\text{d}\left(fe^{-\frac{S}{\hbar}}\right)$$

$$= m_{\mu_S}^{-1}\left(\frac{\partial f}{\partial x}e^{-\frac{S}{\hbar}}\mu_{\text{Leb}} - \frac{1}{\hbar}\frac{\partial S}{\partial x}fe^{-\frac{S}{\hbar}}\mu_{\text{Leb}}\right)$$

$$= \frac{\partial f}{\partial x} - \frac{1}{\hbar}\frac{\partial S}{\partial x}f.$$

from which we conclude that

$$\text{div}_S = -\frac{1}{\hbar}\iota_{\text{d}S} + \Delta_{\text{BV}}.$$

If $\hbar \neq 0$ (and is not formal) we can multiply by $\hbar$ to obtain a differential

$$\hbar\,\text{div}_S = -\iota_{\text{d}S} + \hbar\Delta_{\text{BV}}$$

on $\text{PV}^{\bullet}(V)$ that we recognize as a deformation of the classical BV differential given by contracting with $\text{d}S$. Alternatively, we can also write the divergence operator in terms of the Schouten bracket

$$\text{div}_S = \{S, \cdot\} + \hbar\Delta_{\text{BV}}.$$

Proposition 4 (Wick's Lemma). We can recover (a baby version of) Wick's Lemma employing just cohomological methods. This lemma gives a formula to compute the moments of the Gaussian measure associated to a free field theory. Let

$$S = \frac{x^2}{2}, \qquad \hbar \operatorname{div}_S = -x\frac{\partial}{\partial \xi} + \hbar\frac{\partial}{\partial x}\frac{\partial}{\partial \xi}$$

with respect to which we compute

$$\operatorname{div}_S\left(x^{n-1}\xi\right) = -x^n + \hbar(n-1)x^{n-2}.$$

Passing to cohomology we get

$$\left[x^n\right] = \hbar(n-1)\left[x^{n-2}\right]$$

which we apply recursively to conclude that

$$\left[x^n\right] = \int_V x^n \mu = \begin{cases} \hbar^{\frac{n}{2}}(n-1)!! & \text{if } n \in 2\mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases}.$$

7.3   Quantum Master Equation

The **quantum master equation** (QME) can be formulated as the requirement that the divergence operator squares to zero

$$\operatorname{div}_S^2 = 0$$

such that $\left(\mathrm{PV}^\bullet, \operatorname{div}_S\right)$ is a cochain complex. We have seen that

$$\hbar \operatorname{div}_S(X) = \{S, X\} + \hbar\Delta_{\mathrm{BV}}(X), \qquad \forall X \in \mathrm{PV}^{-2}(V)$$

therefore

$$\left(\hbar \operatorname{div}_S(X)\right)^2 = \{S, \{S, X\}\} + \hbar\{S, \Delta_{\mathrm{BV}}X\}$$
$$+ \hbar\Delta_{\mathrm{BV}}\{S, X\} + \hbar^2\Delta_{\mathrm{BV}}^2 X$$
$$= \left\{\frac{1}{2}\{S, S\} + \hbar\Delta_{\mathrm{BV}}S, X\right\}$$

where we employed the graded Jacobi identity for the Schouten bracket

$$\{\{S, S\}, X\} = 2\{S, \{S, X\}\}$$

and compatibility with the differential Δ_{BV}

$$\Delta_{\mathrm{BV}}\{S, X\} = \{\Delta_{\mathrm{BV}}S, X\} - \{S, \Delta_{\mathrm{BV}}X\}.$$

Because X is arbitrary, we conclude that $\operatorname{div}_S^2 = 0$ if

$$\frac{1}{2}\{S, S\} + \hbar\Delta_{\mathrm{BV}}S = 0.$$

which we recognize as the **Maurer-Cartan** equation.

We conclude with one more reformulation of the QME. A straightforward but lengthy computation shows that for $X, Y \in \mathrm{PV}^\bullet$

$$\operatorname{div}_S(X \wedge Y) = \operatorname{div}_S X \wedge Y + (-1)^{|X|}X \operatorname{div}_S Y + \{X, Y\}.$$

This is a consequence of the fact that, even though m_μ defines a chain map, it is not a Lie algebra morphisms with respect to the wedge product. The failure of div_S to be a derivation is measured by the bracket. Using this identity, we can formulate the QME as the condition

$$\mathrm{div}_S\left(e^{-\frac{S}{\hbar}}\right) = 0.$$

To see this, note that

$$\mathrm{div}_S(e^{-\frac{S}{\hbar}}) = \left\{S, e^{-\frac{S}{\hbar}}\right\} + \hbar\Delta_{\mathrm{BV}}e^{-\frac{S}{\hbar}} = -\frac{1}{\hbar^2}\left(\frac{1}{2}\{S,S\} + \hbar\Delta_{\mathrm{BV}}S\right)e^{-\frac{S}{\hbar}}$$

vanishes if

$$\frac{1}{2}\{S,S\} + \Delta_{\mathrm{BV}}S = 0.$$

— **8** —

Lecture 8 - Manuel Araújo

8.1 Feynman Integrals in Finite Dimensions

Our study will involve integrals of the form

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, \mathrm{e}^{-Q(x,x)} = \frac{\pi^{\frac{n}{2}}}{\sqrt{\det Q}}$$

for a nondegenerate quadratic form Q.

Expectation value of monomials

We are interested in expectation values

$$\langle\!\langle x_{i_1} \dots x_{i_{2m}} \rangle\!\rangle = \frac{\int_{\mathbb{R}^n} \mathrm{d}^n x \, \mathrm{e}^{-\frac{1}{2}Q(x,x)} x_{i_1} \dots x_{i_{2m}}}{\int_{\mathbb{R}^n} \mathrm{d}^n x \, \mathrm{e}^{-\frac{1}{2}Q(x,x)}}.$$

To compute these consider

$$W(J) = \int_{\mathbb{R}^n} \mathrm{d}^n x \, \mathrm{e}^{-\frac{1}{2}Q(x,x) + \langle J, x \rangle}$$

which is such that

$$\langle\!\langle x_{i_1} \dots x_{i_{2m}} \rangle\!\rangle = \left. \frac{\partial}{\partial J_{i_1}} \dots \frac{\partial}{\partial J_{i_{2m}}} \right|_{J=0} \frac{W(J)}{W(0)}$$

Completing the square we get

$$
\begin{aligned}
W(J) &= \mathrm{e}^{\frac{1}{2}\langle J, Q^{-1} J \rangle} \int_{\mathbb{R}^n} \mathrm{d}^n x \, \mathrm{e}^{-\frac{1}{2}Q(x,x) + \langle J, x \rangle - \frac{1}{2}\langle J, Q^{-1} x \rangle} \\
&= \mathrm{e}^{\frac{1}{2}\langle J, Q^{-1} J \rangle} \int_{\mathbb{R}^n} \mathrm{d}^n x \, \mathrm{e}^{-\frac{1}{2}Q(x - Q^{-1}J, x - Q^{-1}J)} \\
&= \mathrm{e}^{\frac{1}{2}\langle J, Q^{-1} J \rangle} \underbrace{\int_{\mathbb{R}^n} \mathrm{d}^n y \, \mathrm{e}^{-\frac{1}{2}Q(y,y)}}_{W(0)}
\end{aligned}
$$

therefore

$$\langle\!\langle x_{i_1} \dots x_{i_{2m}} \rangle\!\rangle = \left. \frac{\partial}{\partial J_{i_1}} \dots \frac{\partial}{\partial J_{i_{2m}}} \right|_{J=0} \mathrm{e}^{-\frac{1}{2}\langle J, Q^{-1} J \rangle}$$

and Taylor expanding the exponential we obtain

$$
\langle\!\langle x_{i_1} \dots x_{i_{2m}} \rangle\!\rangle = \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_{2m}}}\Big|_{J=0} \sum_{p=0}^{\infty} \frac{1}{2^p p!} \left(\sum_{j,k} J_j J_k Q_{jk}^{-1} \right)^p
$$

$$
= \frac{1}{2^m m!} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_{2m}}}\Big|_{J=0} \left(\sum_{j,k} J_j J_k Q_{jk}^{-1} \right)^m
$$

$$
= \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} Q_{i_{\sigma_1} i_{\sigma_2}}^{-1} \cdots Q_{i_{\sigma_{2m-1}} i_{\sigma_{2m}}}^{-1}.
$$

We can exploit the symmetry of this formula by noticing that there exists a free action of the group

$$
S_m \ltimes \mathbb{Z}_2^m
$$

corresponding to permutations of the elements in each pair, as well as the order of the pairings. We call the equivalence classes of permutations corresponding to picking (unordered) pairs of elements by pairings, and write the corresponding quotient group

$$
\text{Matchings}_{2m} = S_{2m} \Big/ S_m \ltimes \mathbb{Z}_2^m.
$$

which we will often abbreviate to just M_{2m}. The size of any orbit is $|O(\sigma)| = 2^m m!$ so we sum on matchings to obtain

$$
\langle\!\langle x_{i_1} \dots x_{i_{2m}} \rangle\!\rangle = \sum_{\sigma \in M_{2m}} Q_{i_{\sigma_1} i_{\sigma_2}}^{-1} \cdots Q_{i_{\sigma_{2m-1}} i_{\sigma_{2m}}}^{-1}
$$

$$
= \sum_{\sigma \in M_{2m}} (Q^{-1})^{\otimes m} \circ \sigma \circ \left(x_{i_1} \otimes \cdots \otimes x_{i_{2m}} \right).
$$

Example 8. Consider a monomial $\psi = x_1 x_2 x_3 x_4$, then

$$
\langle\!\langle x_1 x_2 x_3 x_4 \rangle\!\rangle = \text{(diagram)} + \text{(diagram)} + \text{(diagram)}
$$

$$
= Q_{12}^{-1} Q_{34}^{-1} + Q_{14}^{-1} Q_{23}^{-1} + Q_{13}^{-1} Q_{24}^{-1}.
$$

On the other hand, for the monomial $\psi = x^4$ there is further symmetry, so we get

$$
\langle\!\langle x^4 \rangle\!\rangle = 3 Q_{11}^{-1} Q_{11}^{-1}.
$$

For a general finite-dimensional vector space V the expectation value corresponds to a map

$$
\langle\!\langle \cdot \rangle\!\rangle \colon TV^{\vee} \longrightarrow \mathbb{R}
$$

where $TV^{\vee}$ denotes the tensor algebra of V. Picking a top form μ on V we can write

$$
\langle\!\langle \phi_{i_1} \otimes \cdots \otimes \phi_{i_{2m}} \rangle\!\rangle = \frac{\int_V \mu e^{-\frac{1}{2} Q(x,x)} \phi_{i_1}(x) \dots \phi_{i_{2m}}(x)}{\int_V \mu e^{-\frac{1}{2} Q(x,x)}}
$$

$$
= \sum_{[\sigma]} (Q^{-1})^{\otimes m} \circ \sigma \circ (\phi_{i_1} \otimes \cdots \otimes \phi_{i_{2m}})
$$

The graphical representation systematize the computations. In this context, a **graph** consists of:

i) a finite set V of **vertices**;

ii) a finite set HE of **half-edges** (an even number of them);

iii) an **incidence** map $i : HE \to V$;

iv) a **matching** σ on HE.

A graph automorphism permutes vertices and half-edges, respecting i and σ.

Example 9. The following graph has a nontrivial automorphism
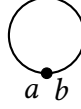
corresponding to permuting half-edges $a \leftrightarrow b$.

Expectation value of symmetric tensors

Consider homogeneous elements $\Psi_1, \ldots, \Psi_r \in \operatorname{Sym} V^\vee$ such that

$$\Psi_a = \sum (\psi_a)_{i_1, \ldots, i_{d_a}} x_{i_1} \ldots x_{i_{d_a}}$$

where $|\Psi_a| = d_a$ and $2m = \sum_{a=1}^r d_a$. We compute the expectation value

$$\langle\!\langle \Psi_1 \otimes \cdots \otimes \Psi_r \rangle\!\rangle = \sum_{[\sigma]} (Q^{-1})^{\otimes m} \circ \sigma \circ (\Psi_1 \otimes \cdots \otimes \Psi_r).$$

Example 10. For $\Psi \in \operatorname{Sym}^4 V^\vee$ we have

In general, on $\langle\!\langle \Psi_1 \otimes \cdots \otimes \Psi_r \rangle\!\rangle$ there exists an action of

$$S_{d_1} \times \cdots \times S_{d_r}$$

therefore we can write

$$\left\langle\!\!\left\langle \frac{1}{d_1!} \Psi_1 \otimes \cdots \otimes \frac{1}{d_r!} \Psi_r \right\rangle\!\!\right\rangle = \sum_{[\sigma]} \frac{1}{|\operatorname{Stab}_\sigma|} (Q^{-1})^{\otimes m} \circ \sigma \circ (\Psi_1 \otimes \cdots \otimes \Psi_r)$$

where

$$[\sigma] \in \left(\textstyle\prod_{a=1}^r S_{d_a} \right) \Big\backslash \operatorname{Matchings}_{2m}.$$

Example 11. Employing the previous formula we see that

$$\left\langle\!\!\left\langle \frac{1}{4!} \Psi \right\rangle\!\!\right\rangle = \frac{1}{|\operatorname{Stab}_\sigma|} \raisebox{-1em}{\includegraphics{dummy}} = \frac{1}{8} \sum_{i,j,k,l} \Psi_{ijkl} (Q^{-1})_{ik} (Q^{-1})_{jl}.$$

Example 12. Let $\Psi_1, \Psi_2 \in \mathrm{Sym}^3 V^\vee$. We can verify the previous formula by explicitly computing

Example 13. The example $\Psi \in \mathrm{Sym}^3 V^\vee$ exhibits more symmetry. As before, we have

In this case, there exists an action of $(S_3 \times S_3) \ltimes S_2$ therefore

where we identify the denominators with number of automorphisms of the respective graph.

— 9 —

Lecture 9 - Manuel Araújo

9.1 Feynman Integrals - Part 2

Groupoid of graphs with prescribed vertices

Let V_d be the set of vertices with valency $d \in \mathbb{Z}_{>0}$. The groupoid $\text{Graphs}_{V_1,\dots,V_D}$ consists of the data:

i) **objects:** matchings in HE (set of half-edges);

ii) **isomorphisms:** collections of morphisms

$$\varphi_d \colon V_d \longrightarrow V_d, \qquad 0 \le d \le D$$

and $\varphi \colon HE \to HE$ respecting the incidence maps.

The action of the group

$$G = \prod_{d=0}^{D} \underbrace{(S_d)^{V_d}}_{\substack{\text{permute} \\ \text{half-edges}}} \ltimes \underbrace{S_{V_d}}_{\substack{\text{permute} \\ \text{vertices}}}$$

on Matchings_{2m} is such that

$$\text{Stab}_\sigma \cong \text{Aut}\,\Gamma_\sigma, \qquad \forall \sigma \in {}_G \backslash \text{Matchings}_{2m}$$

where Γ_σ is the graph corresponding to a matching σ, in the obvious way. Fixing $g \in G$ and $\sigma \in M_{2m}$ defines a *canonical* isomorphism

$$\Gamma_\sigma \longrightarrow \Gamma_{g \cdot \sigma}$$

and we identify $\text{Graphs}_{V_0,\dots,V_D}$ with the *action groupoid* of G acting on Matchings_{2m}.

Example 14. There is an isomorphism of graphs

$$\begin{aligned} i &\mapsto j \\ j &\mapsto i \end{aligned}$$
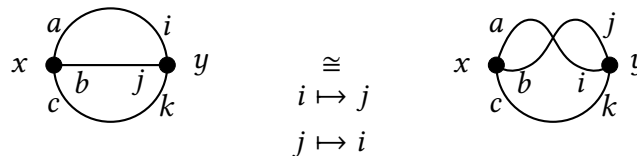
but no such graph isomorphism exists for between the following graphs.

Example 15. Consider homogeneous polynomials $P_d \in \text{Sym}^d V^\vee$. We compute

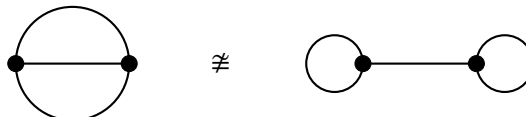$$\langle\langle P_0(x)^{V_0} \dots P_D(x)^{V_D} \rangle\rangle = \sum_{[\sigma]} \frac{|G|}{|\text{Aut}\,\Gamma_\sigma|} (Q^{-1})^{\otimes m} \circ \sigma \circ (P_0^{V_0} \otimes \dots \otimes P_D^{V_D})$$
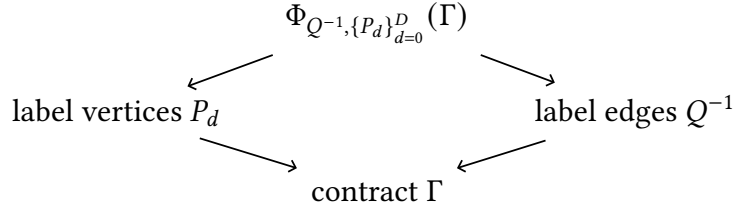
where

$$|G| = \prod_{d=0}^{D} (d!)^{V_d} V_d!.$$

We can rewrite this as

$$\langle\langle P_0(x)^{V_0} \dots P_D(x)^{V_D} \rangle\rangle = \left(\prod_{d=0}^{D} (d!)^{V_d} V_d! \right) \sum_{\Gamma} \frac{1}{|\text{Aut}\,\Gamma|} \Phi_{Q^{-1}, \{P_d\}}(\Gamma)$$

where we relabel the sum as being over graphs, to which we apply the following procedure.

$$\Phi_{Q^{-1}, \{P_d\}_{d=0}^{D}}(\Gamma)$$

label vertices P_d $\qquad\qquad$ label edges Q^{-1}

contract Γ

Example 16. For $\Psi \in \text{Sym}^2 V^\vee$ we check that

$$\langle\langle \Psi^3 \rangle\rangle = 8 \cdot 3! \left(\frac{1}{8 \cdot 3!} \,\, \text{⊖⊖⊖} \,\, + \frac{1}{8} \,\, \text{⊖◯} \,\, + \frac{1}{6} \,\, \text{◯} \right).$$

Perturbed Gaussian

We define a **perturbed** Gaussian integral

$$\int_{\mathbb{R}^n}^{\text{pert}} dx\, e^{-\frac{1}{2}Q(x,x) + p(x)} = \left(\int_{\mathbb{R}^n}^{\text{pert}} dx\, e^{\frac{1}{2}Q(x,x)} \right) \langle\langle e^{p(x)} \rangle\rangle$$

where

$$p(x) = \sum_{d=0}^{D} \frac{g_d P_d}{d!}, \qquad P_d \in \text{Sym}^d V^\vee.$$

We write

$$e^{p(x)} = \prod_{d=0}^{D} e^{\frac{g_d P_d}{d!}} = \sum_{V_0, \dots, V_D} \left(\prod_{d=0}^{D} \frac{g_d^{V_d}}{V_d!(d!)^{V_d}} \right) P_0(x)^{V_0} \dots P_D(x)^{V_D}$$

therefore

$$\langle\langle e^{p(x)} \rangle\rangle = \sum_{V_0, \dots, V_D} g_0^{V_0} \cdots g_D^{V_D} \sum_{\Gamma \in \text{Graphs}_{V_0, \dots, V_D}} \frac{1}{|\text{Aut}\,\Gamma|} \Phi_{Q^{-1}, \{P_d\}}(\Gamma)$$

$$= \underbrace{\sum_{\Gamma}}_{\substack{\text{sum over} \\ \text{all graphs}}} \frac{1}{|\text{Aut}\,\Gamma|} \underbrace{\Phi_{Q^{-1}, \{g_d P_d\}}}_{\substack{\text{change labels} \\ P_d \mapsto g_d P_d}} \cdot$$

Example 17. Let

$$Q(x,x) = x^2, \qquad p(x) = \frac{\lambda}{4!}x^4.$$

In this case

$$I^p(\lambda) = \int_{\mathbb{R}^n}^{\text{pert}} dx\, e^{-\frac{1}{2}x^2 + \frac{\lambda}{4!}x^4}$$

$$= \sqrt{2\pi}\left(1 + \frac{1}{8}\,\bigcirc\!\!\bigcirc\,\lambda \;+\; \frac{1}{8^2 2}\,\bigcirc\!\!\bigcirc\lambda\bigcirc\!\!\bigcirc\lambda \right.$$

$$\left. + \frac{1}{4!2}\,\lambda\bigcirc\!\!\!\!\bigcirc\,\lambda \;+\; \frac{1}{2^4}\,\bigcirc\!\!\bigcirc^{\lambda\;\lambda}\!\!\bigcirc\!\!\bigcirc \;+\; \mathcal{O}(\lambda^3). \right)$$

The n-th coefficient of this series expansion is given by

$$\underbrace{\sum_{\Gamma}}_{\substack{n \text{ vertices} \\ \text{of valency } 4}} \frac{1}{|\operatorname{Aut}\Gamma|} = \frac{(4n-1)!!}{n!4^n}.$$

This expression for $I^p(\lambda)$ has radius of convergence zero. Asymptotically, one can say that for all $N > 0$ there exists M_N such that

$$\left| I^p(\lambda) - \sqrt{2\pi}\sum_{n=0}^{N} \lambda^n \frac{(4n-1)!!}{n!4^n} \right| \le M_N |\lambda|^{N+1}$$

for $\lambda < 0$ provided that $|\lambda|$ is sufficiently small.

Connected graphs

It can be useful to rewrite the usual expression in term of connected graphs. To achieve this, we decompose the sum with respect to the number of connected components of the graphs

$$\sum_{\Gamma} \frac{1}{|\operatorname{Aut}\Gamma|}\Phi(\Gamma) = \overbrace{\sum_{k=0}^{\infty}}^{\substack{\text{connected} \\ \text{components}}} \sum_{\gamma_1 < \cdots < \gamma_k} \overbrace{\sum_{r_1,\ldots,r_k=1}^{\infty}}^{\text{valencies}} \left(\prod_{i=1}^{k} \frac{1}{r_i!|\operatorname{Aut}\gamma_i|}\right)\Phi(\gamma_1)^{r_1}\ldots\Phi(\gamma_k)^{r_k}$$

$$= \underbrace{\prod_{\gamma \text{ connected}}}_{\substack{\text{finitely-many} \\ \text{nonzero valencies}}} \left(\sum_{r=0}^{\infty} \frac{1}{r!|\operatorname{Aut}\gamma|^r}\Phi(\gamma)^r\right)$$

$$= \prod_{\gamma \text{ connected}} \exp\left(\frac{1}{|\operatorname{Aut}\gamma|}\Phi(\gamma)\right)$$

$$= \exp\left(\sum_{\gamma \text{ connected}} \frac{1}{|\operatorname{Aut}\gamma|}\Phi(\gamma)\right).$$

Example 18. We can repeat the same computation summing on connected diagrams

$$I^p(\lambda) = \sqrt{2\pi} \exp\left(\frac{1}{8} \; \lambda \; + \frac{1}{4!2} \; \lambda \; \lambda \; + \frac{1}{2^4} \; \lambda \; \lambda \; + \mathcal{O}(\lambda^3)\right)$$

$$= \sqrt{2\pi} \exp\left(\frac{\lambda}{8} + \frac{\lambda^2}{4!} + \frac{\lambda^2}{16} + \mathcal{O}(\lambda^3)\right)$$

$$= \sqrt{2\pi}\left(1 + \frac{\lambda}{8} + \frac{\lambda^2}{24!} + \frac{\lambda^2}{16} + \frac{\lambda^2}{8^2 2} + \mathcal{O}(\lambda^3)\right).$$

Expectation value of perturbed Gaussian

To compute expectation values note that

$$\langle\!\langle \Psi_1 \dots \Psi_r \rangle\!\rangle_{\text{pert}} = \frac{\int_{\mathbb{R}^n}^{\text{pert}} \mathrm{d}x \mathrm{e}^{-\frac{1}{2}Q(x,x)+p(x)} \Psi_1(x) \dots \Psi_r(x)}{\int_{\mathbb{R}^n}^{\text{pert}} \mathrm{e}^{-\frac{1}{2}Q(x,x)+p(x)}}$$

$$= \frac{\langle\!\langle \mathrm{e}^{p(x)} \Psi_1 \dots \Psi_r \rangle\!\rangle}{\langle\!\langle \mathrm{e}^{p(x)} \rangle\!\rangle}$$

where each term Ψ_j is of the form

$$\Psi_j = \sum_{d \geq 0} \frac{1}{d!} \Psi_{j,d}$$

for $\Psi_{j,d}$ homogeneous polynomials of degree d, therefore

$$\langle\!\langle \mathrm{e}^{p(x)} \Psi_1 \dots \Psi_r \rangle\!\rangle = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det Q}} \sum_{\Gamma} \frac{1}{|\operatorname{Aut}\Gamma|} \Psi(\Gamma)$$

where now we are summing over graphs Γ colored by $\{0, \dots, r\}$ where each nonzero color appears *exactly once*, and where $\Phi(\Gamma)$ encodes the following procedure.
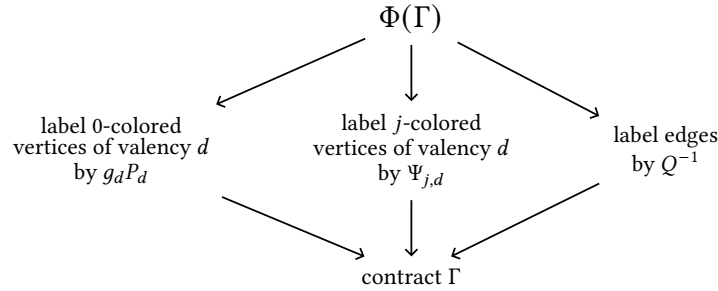
Decomposing each graph Γ into a 0-colored part components and the rest we get

$$\Gamma = \tilde{\Gamma} \amalg \underbrace{\gamma_1^{m_i} \amalg \dots \amalg \gamma_k^{m_k}}_{\text{0-colored}}$$

where the graphs γ_i are connected 0-colored and (each connected component of) $\tilde{\Gamma}$ has at least one nonzero color. Therefore

$$
\begin{aligned}
\langle\!\langle e^{p(x)} \Psi_1 \ldots \Psi_r \rangle\!\rangle &= \sum_{k=0}^{\infty} \sum_{\gamma_1 < \cdots < \gamma_k} \sum_{m_1,\ldots,m_k=1}^{\infty} \frac{1}{|\operatorname{Aut}\Gamma|} \Phi(\Gamma) \\
&= \left(\sum_{k=0}^{\infty} \sum_{\gamma_1 < \cdots < \gamma_k} \sum_{m_1,\ldots,m_k} \left(\prod_j \frac{1}{m_j! |\operatorname{Aut}\gamma_j|^{m_j}} \right) \Phi(\gamma_1)^{m_1} \ldots \Phi(\gamma_k)^{m_k} \right) \\
&\qquad \cdot \left(\sum_{\tilde{\Gamma}} \frac{1}{|\operatorname{Aut}\tilde{\Gamma}|} \Phi(\tilde{\Gamma}) \right) \\
&= \exp\left(\underbrace{\sum_{\gamma} \frac{1}{|\operatorname{Aut}\gamma|} \Phi(\gamma)}_{\substack{\text{0-colored} \\ \text{connected}}} \right) \left(\sum_{\tilde{\Gamma}} \Phi(\tilde{\Gamma}) \right)
\end{aligned}
$$

which, when normalized, wields the simpler formula

$$
\langle\!\langle \Psi_1 \ldots \Psi_r \rangle\!\rangle_{\text{pert}} = \underbrace{\sum_{\Gamma} \frac{1}{|\operatorname{Aut}\Gamma|} \Phi(\Gamma).}_{\substack{\text{connected components} \\ \text{with at least one} \\ \text{nonzero color}}}
$$

Planck's constant

In this section we study a different type of perturbed Gaussian integrals. Let us consider formal power series on $\mathbb{R}[\![\hbar]\!]$ for some paremter $\hbar$, and fix a polynomial of degree at least 3 (we justify this requirement later)

$$
p = \sum_{d=3}^{D} \frac{P_d}{d!}.
$$

Then we compute

$$
\begin{aligned}
\int_{\mathbb{R}^n}^{\text{pert}} dx \exp\left[\frac{1}{\hbar}\left(-\frac{1}{2}Q(x,x) + p(x) \right) \right] &= \frac{(2\pi\hbar)^{\frac{n}{2}}}{\sqrt{\det Q}} \sum_{\Gamma} \frac{1}{|\operatorname{Aut}\Gamma|} \Phi_{\hbar Q^{-1}, \{\hbar P_d\}}(\Gamma) \\
&= \frac{(2\pi\hbar)^{\frac{n}{2}}}{\sqrt{\det Q}} \sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{|\operatorname{Aut}\Gamma|} \Phi_{\hbar Q^{-1}, \{\hbar P_d\}}
\end{aligned}
$$

where $\chi(\Gamma)$ is the *Euler characteristic* of the graph. Note that

$$
|E| - |V| = \frac{1}{2}|HE| - |V| = \sum_{d=3}^{D} V_d \underbrace{\left(\frac{d}{2} - 1 \right)}_{>0}
$$

holds if p is at least cubic. Finally, for expectation values we get a familiar result

$$
\int_{\mathbb{R}^n}^{\text{pert}} dx \exp\left[\frac{1}{\hbar}\left(-\frac{1}{2}Q(x,x) + p(x) \right) \right] \Psi_1 \ldots \Psi_r = \frac{(2\pi\hbar)^{\frac{n}{2}}}{\sqrt{\det Q}} \sum_{\Gamma} \frac{\hbar^{r-\chi(\Gamma)}}{|\operatorname{Aut}\Gamma|} \Phi(\Gamma).
$$

where we sum over graphs with at least one nonzero color in each connected component.

— 10 —

Lecture 10

10.1 Scalar QFT in the Wilsonian Sense

The main reference for this lecture is [Cos11, Sections 1.1 to 1.5 and 2.1 to 2.7]. Our goal is to give a Wilsonian definition of scaler QFT. Consider the data:

i) **spacetime:** smooth Riemannian manifold M (we consider $M = \mathbb{R}^n$);

ii) **scalar fields:** smooth functions $\varphi \colon M \to \mathbb{R}$;

iii) **action functional:** a local functional

$$S(\varphi) = \int_M -\frac{1}{2}\varphi\big(D + m^2\big)\varphi + \underbrace{I(\varphi)}_{\substack{\text{interactions terms} \\ \text{(cubic or higher)}}}$$

where we call $m > 0$ the *mass parameter* and D denotes the Laplacian.

For the Dirichlet problem on some domain $U \subseteq M$

$$\begin{cases} D\varphi(x) + \lambda\varphi(x), & \text{if } x \in U \\ \varphi(x) = 0, & \text{if } x \in \partial U \end{cases}$$

we write the associated eigenfunctions φ_n with corresponding eigenvalues λ_n. It is known that the inverse Laplacian operator is compact and self-adjoint. From the spectral theorem follows that

$$0 < \underbrace{\lambda_1 \leq \cdots \leq \lambda_n}_{\text{energies}} \leq \cdots \longrightarrow \infty.$$

In this context, **observables** are functionals $O : C^\infty(M) \to \mathbb{C}$.

Example 19. The evaluation map is an observable

$$O_x(\varphi) = \varphi(x), \quad \forall x \in M.$$

The physical information of the theory is encoded in the correlation functions, which we compute (up to normalization) using the Feynman sum of histories approach

$$\langle O_1, \ldots, O_n \rangle = \int_{\varphi \in C^\infty(M)} e^{\frac{1}{\hbar}S(\varphi)} O_1(\varphi) \ldots O_n(\varphi) D\varphi.$$

To proceed, we restrict to *low-energy fields*

$$C_{\leq\Lambda} = C^\infty(M)_{\leq\Lambda}$$

corresponding to the space spanned by eigenfunctions with associated energy $\lambda_n \leq \Lambda$ (in principle finite-dimensional), and *low-energy observables*

$$(O\colon C_{\leq\Lambda} \longrightarrow \mathbb{C}) \in \mathrm{Obs}_{\leq\Lambda}.$$

Then

$$\langle O_1,\dots,O_n \rangle = \int_{\varphi\in C_{\leq\Lambda}} e^{\frac{1}{\hbar}S^{\mathrm{eff}}[\Lambda](\varphi)} O_1(\varphi)\dots O_n(\varphi)\mathrm{D}\varphi$$

for some low-energy **effective action** $S^{\mathrm{eff}}[\Lambda]$. For low-energy observable we have

$$O \in \mathrm{Obs}_{\leq\Lambda'} \implies O \in \mathrm{Obs}_{\leq\Lambda}, \quad 0 < \Lambda' < \Lambda$$

which motivates the decomposition of fields into *low-* and *high-energy* parts

$$\varphi = \varphi_L + \varphi_H, \quad \varphi_L \perp \varphi_H$$

where φ_L is the projection of φ on $C_{\leq\Lambda'}$, and φ_H the corresponding parallel component in $C_{\leq\Lambda}$. Then

$$\int_{\varphi_L\in C_{\leq\Lambda'}} e^{\frac{1}{\hbar}S^{\mathrm{eff}}[\Lambda'](\varphi_L)} O_1(\varphi_L)\dots O_n(\varphi_L)\mathrm{D}\varphi$$

$$= \int_{\varphi\in C_{\leq\Lambda}} e^{\frac{1}{\hbar}S^{\mathrm{eff}}[\Lambda](\varphi)} O_1(\varphi)\dots O_n(\varphi)\mathrm{D}\varphi$$

$$= \int_{\varphi_L}\left(\int_{\varphi_H} e^{\frac{1}{\hbar}S^{\mathrm{eff}}[\Lambda](\varphi_L+\varphi_H)}\right) O_1(\varphi_L)\dots O_n(\varphi_L)\mathrm{D}\varphi$$

implying that

$$e^{\frac{1}{\hbar}S^{\mathrm{eff}}[\Lambda'](\varphi_L)} = \int_{\varphi_H} e^{\frac{1}{\hbar}S^{\mathrm{eff}}[\Lambda](\varphi_L+\varphi_H)}\mathrm{D}\varphi_H.$$

Taking the logarithm we obtain the **renormalization group equation** (RGE)

$$S^{\mathrm{eff}}[\Lambda'](\varphi_L) = \hbar\log\int_{\varphi_H} e^{\frac{1}{\hbar}S^{\mathrm{eff}}[\Lambda](\varphi_L+\varphi_H)}\mathrm{D}\varphi_H.$$

Renormalization Group Equation

Assume that

$$S^{\mathrm{eff}}[\Lambda](\varphi) = \int_M -\frac{1}{2}\varphi\big(D + m^2\big)\varphi + \underbrace{I^{\mathrm{eff}}[\Lambda](\varphi)}_{\text{effective interaction}}.$$

From the linearity of the Laplacian and the RGE follows that

$$I^{\mathrm{eff}}[\Lambda'](\varphi_L) = \hbar\log\int_{\varphi_H} \exp\left(-\frac{1}{2\hbar}\varphi_H\big(D+m^2\big)\varphi_H + \frac{1}{\hbar}I^{\mathrm{eff}}[\Lambda](\varphi_L+\varphi_H)\right)\mathrm{D}\varphi_H.$$

We are interested in finite-dimensional integrals of the form

$$W(P,I) = \int_U \exp\left(\frac{1}{2\hbar}\Phi(x,x) + \frac{1}{\hbar}I(x+a)\right)$$

for some nondegenerate negative-definite quadratic form Φ, understood as a Feynman diagram expansion. Here P is the integral kernel of $(D + m^2)^{-1}$ (propagator)

$$P(x, y) = \int_{\tau=0}^{\infty} e^{-\tau m^2} \, d\tau \, \underbrace{K_\tau^0(x, y)}_{\text{heat kernel}}$$

where we write

$$K_\tau^0(x, y) = \int_{f \in \Omega_{x,y}} \exp\left(\int_0^\tau \|df\|^2 \, ds\right) dW$$

for

$$\Omega_{x,y} = \{f : [0, \tau] \to M \mid f(0) = x, f(\tau) = y\}.$$

In terms of eigenfunctions

$$K_\tau^0(x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n \tau} \varphi_n(x) \varphi_n(y)$$

and $K_\tau = K_\tau^0 e^{\tau m^2}$ is such that

$$P(x, y) = \int_{\tau=0}^{\infty} K_\tau(x, y) \, d\tau.$$

Length scale instead of energy scale

The high-energy regimes correspond to small scales:

$$\text{short lengths} \leftrightsquigarrow \text{high energy}.$$

Because of this, the RGE should relate different length scales

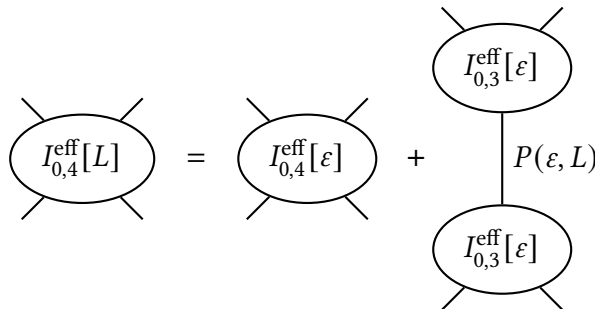$$P(\varepsilon, L)(x, y) = \int_{l=\varepsilon}^{L} K_l(x, y) \, dl.$$

Again, the RGE for relating different scales

$$I^{\text{eff}}[L] = W\big(P(\varepsilon, L), I^{\text{eff}}[\varepsilon]\big)$$

is given in terms of Feynman diagrams. Expanding in powers

$$I^{\text{eff}} = \sum_{i,j} \hbar^j \varphi^k I_{j,k}.$$

Example 20. Some examples of the diagrammatic approach are

and

$$
\left(I_{1,1}^{\mathrm{eff}}[L] \right) \quad = \quad \left(I_{1,1}^{\mathrm{eff}}[\varepsilon] \right) \quad + \quad \left(I_{0,3}^{\mathrm{eff}}[\varepsilon] \right) P(\varepsilon, L) \quad .
$$

Definition 6. A **perturbative QFT** with space of fields and action functional as prescribed earlier, is given by a set of interactions $I[L]$ such that:

i) the RGE holds for any positive scales:

$$
I[L] = W\big(p(\varepsilon, L), I[\varepsilon]\big), \qquad \forall \varepsilon, L \in (0, \infty];
$$

ii) the components $I_{j,k}$ are **local:** if

$$
S^{\mathrm{eff}}[L](\varphi) = \sum_i f_i(L) \Theta_i(\varphi)
$$

then Θ_i are local functionals.

$$—\ 11\ —$$

Lecture 11 - Leander Stecker

11.1 Elliptic Operators and Complexes

Let M be a closed, oriented Riemannian manifold, and $E, F \to M$ complex vector bundles over M. A **differential operator**

$$P \colon \Gamma(E) \longrightarrow \Gamma(F)$$

is a $\mathbb{C}$-linear map that, in local coordinates (U, x^i) of M with local trivializations of E and F, has the form

$$P = \sum_{|\alpha| \leq k} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

where α is a multi-index and

$$A^\alpha \in \mathrm{Hom}(E|_U, F|_U)$$

is a bundle map over U. The number $k \in \mathbb{Z}_{\geq 0}$ is called the **order** of P.

Definition 7. The **principal symbol** $\sigma(P)$ is a section of the pullback bundle $\pi^* \mathrm{Hom}(E, F)$ over T^*M, as in the following diagram.

$$
\begin{array}{ccc}
\pi^* \mathrm{Hom}(E, F) & \longrightarrow & \mathrm{Hom}(E, F) \\
\downarrow & \lrcorner & \downarrow \\
T^*M & \xrightarrow{\ \ \pi\ \ } & M
\end{array}
$$

At $\xi \in T^*M$, $\sigma(P)$ is given by

$$\sigma(P)_\xi = \sum_{|\alpha| \leq k} A^\alpha(x) \xi_\alpha$$

where for $\alpha = (\alpha_1, \dots, \alpha_n)$, we write $\xi = \xi_i \mathrm{d}x^i$. Since $\sigma(P)$ is the only symbol we will need, let us just call it the *symbol* of P.

Lemma 1. The symbol of P is equivalently defined by

$$\sigma(P)_\xi = \lim_{t \to \infty} (it)^{-k} e^{-itf} P e^{itf}$$

where f is any smooth function such that

$$\mathrm{d}f(x) = \xi.$$

Proof. In a local trivialization over a chart, for a local section s, we compute

$$P e^{itf} s = \sum_{|\alpha| \leq k} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \left(e^{itf} s \right)$$

$$= \sum_{|\alpha| \leq k} (it)^k \frac{\partial^{|\alpha|} f}{\partial x^\alpha} e^{itf} A^\alpha s + \underset{\text{order in } t}{\text{lower}}$$

hence

$$\lim_{t \to \infty} (it)^{-k} e^{-itf} P e^{itf} s = \sum_{|\alpha| = k} df \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} \right) A^\alpha s.$$

Evaluating at x gives $\sigma(P)_\xi$. $\blacksquare$

Example 21. The exterior derivative

$$d \colon \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

can be writen in local coordinates as

$$d = \sum_{i=1}^n \frac{\partial}{\partial x^i} dx^i \wedge$$

therefore

$$\sigma(d)_\xi = \sum_{i=1}^n \xi_i dx^i \wedge = \xi \wedge$$

so the symbol is given by the wedge product.

Example 22. The Laplacian

$$\Delta \colon C^\infty(M) \longrightarrow C^\infty(M)$$

can be writen in normal coordinates around $p \in M$

$$\Delta|_p = -\sum_{i=1}^n \left(\frac{\partial}{\partial x^i} \right)^2$$

hence $\sigma(\Delta)_\xi = -|\xi|^2$.

Example 23. The Hodge Laplacian

$$\Delta \colon \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

is given by

$$\Delta = dd^\star + d^\star d$$

$$= -\sum_{i=1}^n \left(\frac{\partial}{\partial x^i} \right)^2 \left(\iota_{\frac{\partial}{\partial x^i}} dx^i + dx^i \iota_{\frac{\partial}{\partial x^i}} \right)$$

$$= -\sum_{i=1}^n \left[\iota_{\frac{\partial}{\partial x^i}}, dx^i \right] \left(\frac{\partial}{\partial x^i} \right)^2$$

where $\iota_{\frac{\partial}{\partial x^i}}$ denotes the interior product and dx^i is the operator $dx^i \wedge$ in $\Omega^\bullet(M)$.

Remarkably, we still have $\sigma(\Delta)_\xi = -|\xi|^2$. To see this, we need the following proposition.

Proposition 5. For differential operators P and P' we have

$$\sigma(P)_\xi \sigma(P')_\xi = \sigma(PP')_\xi.$$

Proof. If P has order k and P' has order k', then PP' has order $k + k'$ and

$$\lim_{t \to \infty} \left((it)^{-k} e^{-itf} P e^{itf} \right) \left((it)^{-k'} e^{-itf} P' e^{itf} \right)$$
$$= \lim_{t \to \infty} \left((it)^{-k-k'} e^{-itf} PP' e^{itf} \right).$$

We also need that

$$\sigma(\mathrm{d}^\star)_\xi = -\iota_{\xi^\#}$$

where $\xi^\# \in T_x M$ is the unique vector such that $\xi = g(\xi^\#, \cdot)$. Then

$$
\begin{aligned}
\sigma(\Delta)_\xi &= \sigma(\mathrm{d}\mathrm{d}^\star + \mathrm{d}^\star \mathrm{d})_\xi \\
&= \sigma(\mathrm{d}\mathrm{d}^\star)_\xi + \sigma(\mathrm{d}^\star \mathrm{d})_\xi \\
&= \left[\xi, -\iota_{\xi^\#} \right] \\
&= -|\xi|^2.
\end{aligned}
$$

Definition 8. An operator $P\colon \Lambda(E) \to \Lambda(F)$ is **elliptic** if the symbol $\sigma(P)_\xi$ is invertible for all $\xi \neq 0$.

Remember that, for all $\xi \in T_x^* M$, the symbol is a linear operator $\sigma(P)_\xi \colon E_x \to F_x$.

Definition 9. An operator $P : \Lambda(E) \to \Lambda(E)$ is called a **generalized Laplacian** if

$$\sigma(P)_\xi = -|\xi|_{\tilde{g}}^2$$

for some metric $\tilde{g}$ on M, not necessarily the one we started with.

The previous definition comes from [BGV96], but we have taken the opposite sign convention.

11.2   Elliptic Complexes

Suppose $P\colon \Gamma(E) \to \Gamma(F)$ is elliptic. The symbol $\sigma(P)$ defines a bundle map over T^*M

$$0 \longrightarrow \pi^* E \xrightarrow{\ \sigma(P)\ } \pi^* F \longrightarrow 0$$

which is exact over T^*M away from the zero section. Now let $(E^\bullet, Q)$ be the $\mathbb{Z}$-graded vector bundle $E^\bullet \to M$, and

$$Q\colon \Gamma(E^\bullet) \longrightarrow \Gamma(E^\bullet)$$

a differential operator of cohomological degree 1 such that $Q^2 = 0$.

Definition 10. The complex $(E^\bullet, Q)$ is an **elliptic complex** if the symbol complex $(\pi^* E, \sigma(Q))$ is exact over $T^*M \to M$.

Example 24. To the de Rham complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^n(M)$$

corresponds the symbol complex for $\xi \in \Omega^1(M)$

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\xi} \Omega^1(M) \xrightarrow{\xi} \ldots \xrightarrow{\xi} \Omega^n(M) \ .$$

If $\xi \neq 0$ and $\xi \wedge \beta = 0$, then $\beta = \xi \wedge \alpha$ for some $\alpha \in \Omega^\bullet(M)$. We conclude that the symbol complex is exact, and thus $(\Omega^\bullet, \mathrm{d})$ is an elliptic complex.

Example 25. The Yang-Mills complex is not elliptic. We have

$$0 \longrightarrow \Omega^0(M, \mathfrak{g}) \xrightarrow{\mathrm{d}} \Omega^1(M, \mathfrak{g}) \xrightarrow{\mathrm{d}\star\mathrm{d}} \Omega^{n-1}(M, \mathfrak{g}) \xrightarrow{\mathrm{d}} \Omega^n(M, \mathfrak{g})$$

with corresponding symbol complex

$$M \times \mathfrak{g} \xrightarrow{\xi\wedge} T^*M \otimes \mathfrak{g} \xrightarrow{Q} \bigwedge^{n-1} T^*M \otimes \mathfrak{g} \xrightarrow{\xi\wedge} \bigwedge^n T^*M \otimes \mathfrak{g}$$

where

$$Q = \sigma(\mathrm{d}\star\mathrm{d})_\xi \alpha = \sum_{j,k} \xi_j \xi_k \mathrm{d}x^i \wedge \star(\mathrm{d}x^i \wedge \alpha).$$

For $n \geq 2$ this complex is not exact, hence the original complex not elliptic.

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