Reflection, Transmission and Absorption of Light across Conducting Surfaces

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Abstract

In these notes, I calculate the transmittance, reflectance and absorbance of a conducting surface between two dielectrics, as a function of the media's constitutive relations. We start by deriving expressions for the fields and energy flux of an electromagnetic wave moving across a three-dimensional linear dielectric medium. Then we move on to study the appropriate boundary conditions to be applied in the interface between two such media. We finish by applying these expressions to calculate the angle-resolved reflectance and transmittance of an interface were a two-dimensional conductor is placed. This problem had been considered before by Stauber et al [2]. Then we introduce the transfer-matrix method allowing us to calculate the same quantities for stratified structures.

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1.1 Electromagnetic Plane-Waves in 3D

We start by deriving an expression for the energy transported by an electromagnetic plane wave, moving in an infinite homogeneous dielectric medium with a dielectric constant ε_m (we will take the medium to be non-magnetic, i.e. $\mu_m = \mu_0$). The corresponding electric and magnetic fields of the (progressive) plane-waves can be written as follows:

$$\overrightarrow{E}_{\overrightarrow{k}}\left(\overrightarrow{r}, t\right) = \overrightarrow{E}_{0,\overrightarrow{k}} e^{i\left(\overrightarrow{k}\cdot\overrightarrow{r} - \omega t\right)} \tag{1}$$

$$\overrightarrow{B}_{\overrightarrow{k}}\left(\overrightarrow{r}, t\right) = \overrightarrow{B}_{0,\overrightarrow{k}} e^{i\left(\overrightarrow{k}\cdot\overrightarrow{r} - \omega t\right)}. \tag{2}$$

These expressions are not independent, as they are connected by the electromagnetic Maxwell's Equations, which read

$$\begin{array}{rc} \nabla\cdot\overrightarrow{\mathbf{D}}\left(\overrightarrow{r}, t\right) = & 0 \\ \nabla\cdot\overrightarrow{\mathbf{B}}\left(\overrightarrow{r}, t\right) = & 0 \\ \nabla\times\overrightarrow{\mathbf{E}}\left(\overrightarrow{r}, t\right) = & -\frac{\partial}{\partial t}\overrightarrow{\mathbf{B}}\left(\overrightarrow{r}, t\right) \\ \nabla\times\overrightarrow{\mathbf{H}}\left(\overrightarrow{r}, t\right) = & \frac{\partial}{\partial t}\overrightarrow{\mathbf{D}}\left(\overrightarrow{r}, t\right) \end{array}, \tag{3}$$

in the absence of free charges or currents. These must to be complemented by dielectric constitutive relations associated to the dielectric medium when they are propagating, i.e.

$$\begin{array}{rc} \overrightarrow{\mathbf{D}}\left(\overrightarrow{r}, t\right) = & \varepsilon_m\overrightarrow{\mathbf{E}}\left(\overrightarrow{r}, t\right) \\ \overrightarrow{\mathbf{H}}\left(\overrightarrow{r}, t\right) = & \frac{1}{\mu_0}\overrightarrow{\mathbf{B}}\left(\overrightarrow{r}, t\right) \end{array}, \tag{4}$$

where we assumed the propagation medium to be a linear one. If we plug Eqs. (4) into Eqs. (3), we get the following version of the Maxwell's Equations

$$\begin{array}{rc} \nabla\cdot\overrightarrow{\mathbf{E}}\left(\overrightarrow{r}, t\right) = & 0 \\ \nabla\cdot\overrightarrow{\mathbf{B}}\left(\overrightarrow{r}, t\right) = & 0 \\ \nabla\times\overrightarrow{\mathbf{E}}\left(\overrightarrow{r}, t\right) = & -\frac{\partial}{\partial t}\overrightarrow{\mathbf{B}}\left(\overrightarrow{r}, t\right) \\ \nabla\times\overrightarrow{\mathbf{B}}\left(\overrightarrow{r}, t\right) = & \mu_0\varepsilon_m\frac{\partial}{\partial t}\overrightarrow{\mathbf{E}}\left(\overrightarrow{r}, t\right) \end{array}, \tag{5}$$

and using the third of these equations, together with the expressions for the plane-wave fields, Eqs. (1) and (2), we arrive at the following relation between the amplitudes:

$$\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{E}}_{0,\overrightarrow{\mathbf{k}}} = 0 \tag{6}$$

$$\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}}_{0,\overrightarrow{\mathbf{k}}} = 0 \tag{7}$$

$$\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{E}}_{0,\overrightarrow{\mathbf{k}}} = \omega\overrightarrow{\mathbf{B}}_{0,\overrightarrow{\mathbf{k}}} \tag{8}$$

$$\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}}_{0,\overrightarrow{\mathbf{k}}} = -\mu_0\varepsilon_m\omega\overrightarrow{\mathbf{E}}_{0,\overrightarrow{\mathbf{k}}}. \tag{9}$$

On the one hand, Eqs. (6) and (7) simply imply that the fields are perpendicular to the propagation wave-vector, i.e. the electromagnetic waves are transverse. On the other hand, we have Eqs. (8) and (9) to be obeyed as well. First of all, these imply that $\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{E}}_{0,\overrightarrow{\mathbf{k}}}$ and $\overrightarrow{\mathbf{B}}_{0,\overrightarrow{\mathbf{k}}}$ *are a set of mutually perpendicular vectors, whose norms have to respect the following system of equations:*

$$\begin{aligned} k\left|\overrightarrow{\mathbf{E}}_{0,\overrightarrow{\mathbf{k}}}\right| - \omega\left|\overrightarrow{\mathbf{B}}_{0,\overrightarrow{\mathbf{k}}}\right| &= 0 \\ -\mu_0\varepsilon_m\omega\left|\overrightarrow{\mathbf{E}}_{0,\overrightarrow{\mathbf{k}}}\right| + k\left|\overrightarrow{\mathbf{B}}_{0,\overrightarrow{\mathbf{k}}}\right| &= 0, \end{aligned}$$

which only has a non-zero solution provided the secular determinant is null, implying

$$-\mu_0\varepsilon_m\omega^2 + k^2 = 0 \Rightarrow \omega = \pm\frac{1}{\sqrt{\mu_0\varepsilon_m}}k \tag{10}$$

and, hence, $v_m = 1/\sqrt{\mu_0\varepsilon_m}$ is the phase velocity of the plane-wave inside the propagation medium. In those conditions, the system above yields

$$\left|\overrightarrow{\mathbf{B}}_{0,\overrightarrow{\mathbf{k}}}\right| = \frac{1}{v_m}\left|\overrightarrow{\mathbf{E}}_{0,\overrightarrow{\mathbf{k}}}\right|. \tag{11}$$

The situation is represented by the "right-hand rule" depicted in Fig. 1 a).

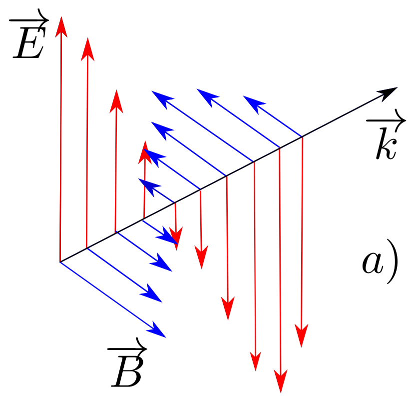

Figure 1: a) Depiction of the vector character of the electric and magnetic field components of a progressing electromagnetic wave with a wave-vector $\overrightarrow{\mathbf{k}}$ in three-dimensional space.

1.2 Energy-Density of the Electromagnetic Field

Intuitively, one understands that there must be energy stored in a set up electromagnetic field, since there was work done by conservative forces in order to place all the electric charges and currents that generate those fields. The question is then: How do we calculate that potential energy, by knowing the field configurations?

1.2.1 Electric Component

In order to answer this questions, we start by remembering that we may define electric potential $\mathcal{V}\left(\overrightarrow{\mathbf{r}},t\right)$ such that $\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) = -\nabla\mathcal{V}\left(\overrightarrow{\mathbf{r}},t\right)$. The electrostatic potential energy associated to an arbitrary distribution of free charges, $\rho_f\left(\overrightarrow{\mathbf{r}},t\right)$, which is responsible by the existence of the field, is given by:

$$\mathcal{U}_e\left(\overrightarrow{\mathbf{r}},t\right) = \frac{1}{2}\int_{\text{All Space}} d\overrightarrow{\mathbf{r}}\left\{\rho_f\left(\overrightarrow{\mathbf{r}},t\right)\mathcal{V}\left(\overrightarrow{\mathbf{r}},t\right)\right\} = \frac{1}{2}\int_{\text{All Space}} d\overrightarrow{\mathbf{r}}\left\{\nabla\cdot\overrightarrow{\mathbf{D}}\left(\overrightarrow{\mathbf{r}},t\right)\mathcal{V}\left(\overrightarrow{\mathbf{r}},t\right)\right\}, \tag{12}$$

where we used Gauss's law to write the charge density in the last equality. The factor $1/2$ comes from the double counting of the pairs of charges (see [2]). This being said, we remind ourselves that

$$\nabla\cdot\left(\overrightarrow{\mathbf{V}}\left(\overrightarrow{\mathbf{r}}\right)f\left(\overrightarrow{\mathbf{r}}\right)\right) = \nabla\cdot\left(\overrightarrow{\mathbf{V}}\left(\overrightarrow{\mathbf{r}}\right)\right)f\left(\overrightarrow{\mathbf{r}}\right) + \overrightarrow{\mathbf{V}}\left(\overrightarrow{\mathbf{r}}\right)\nabla\left(f\left(\overrightarrow{\mathbf{r}}\right)\right), \tag{13}$$

which together with Eq. (12), yields the following result:

$$\begin{aligned}\mathcal{U}_e\left(\overrightarrow{\mathbf{r}},t\right) &= \frac{1}{2}\int_{\text{All Space}} d\overrightarrow{\mathbf{r}}\left\{\nabla\cdot\left(\overrightarrow{\mathbf{D}}\left(\overrightarrow{\mathbf{r}},t\right)\mathcal{V}\left(\overrightarrow{\mathbf{r}},t\right)\right)\right\} + \frac{1}{2}\int_{\text{All Space}} d\overrightarrow{\mathbf{r}}\,\overrightarrow{\mathbf{D}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right)\\ &= \frac{1}{2}\oiint_{\mathrm{S}_\infty} d\overrightarrow{\mathbf{S}}\cdot\left\{\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right)\mathcal{V}\left(\overrightarrow{\mathbf{r}},t\right)\right\} + \frac{1}{2}\int_{\text{All Space}} d\overrightarrow{\mathbf{r}}\,\overrightarrow{\mathbf{D}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right).\end{aligned}$$

If one assumes, as usual, that all the fields die out at infinity, we arrive at the usual expression for the electrostatic energy of a field configuration:

$$\mathcal{U}_e\left(\overrightarrow{\mathbf{r}},t\right)=\frac{1}{2}\int_{\text{All Space}}d\overrightarrow{\mathbf{r}}\,\overrightarrow{\mathbf{D}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right)=\frac{\varepsilon_0\varepsilon_m}{2}\int_{\text{All Space}}d\overrightarrow{\mathbf{r}}\left|\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right)\right|^2 \tag{14}$$

thus giving an energy density of

$$u_e\left(\overrightarrow{\mathbf{r}},t\right)=\frac{\varepsilon_0\varepsilon_m}{2}\left|\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right)\right|^2. \tag{15}$$

The same expression can be obtained by the usual argument of the work done during an infinitesimal variation of the the local electric charge density.

1.2.2 Magnetic Component

It is a well-known fact that a static magnetic field produces a (Lorentz) force on any moving electric charge, that is perpendicular to its velocity. Hence, such forces will never do any work on the moving charges. However, there is a way for a magnetic field to contribute to the total energy of the system: by causing electric fields due to magnetic induction. In other words, to onset a magnetic field in space, one needs to perform work in building-up the current as a pay-off to the electromotive force generated by self-induction.

In order to calculate that energy, we may start from the usual expression for the power (rate of energy transfer) needed to create a current I going around an arbitrary circuit, $\mathcal{C}$, as drawn in Fig. 1 b). This goes as follows

$$\frac{dW\left(t\right)}{dt}=I(t)\varepsilon_{\text{EMF}}\left(t\right)=-I\left(t\right)\frac{d}{dt}\left\{\iint_{S_{\mathcal{C}}}\overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot\overrightarrow{\mathbf{dS}}\right\}. \tag{16}$$

Now, we may proceed by recognizing that $\overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right)=\nabla\times\overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}},t\right)$ and, using Stokes' Theorem, we arrive at the following expression:

$$\frac{dW\left(t\right)}{dt}=I\left(t\right)\frac{d}{dt}\left\{\oint_{\mathcal{C}}\overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot\overrightarrow{\mathbf{dl}}\right\}, \tag{17}$$

or, using the Biot-Savart's Law for the vector potential, we arrive at the following expression:

$$\begin{aligned}\frac{dW\left(t\right)}{dt}&=-\frac{4\pi}{\mu_0}I\left(t\right)\frac{d}{dt}\left\{I\left(t\right)\oint_{\mathcal{C}}\oint_{\mathcal{C}}\frac{1}{\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|}\overrightarrow{\mathbf{dl}}'\cdot\overrightarrow{\mathbf{dl}}\right\}\\&=-\frac{d}{dt}\left\{\frac{2\pi}{\mu_0}I^2\left(t\right)\oint_{\mathcal{C}}\oint_{\mathcal{C}}\frac{1}{\left|\overrightarrow{\mathbf{r}}-\overrightarrow{\mathbf{r}}'\right|}\overrightarrow{\mathbf{dl}}'\cdot\overrightarrow{\mathbf{dl}}\right\}\\&=-\frac{d}{dt}\left\{\frac{1}{2}\oint_{\mathcal{C}}\overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot I\left(t\right)\overrightarrow{\mathbf{dl}}\right\}=-\frac{d}{dt}\left\{\frac{1}{2}\oint_{\mathcal{C}}\overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot\overrightarrow{\mathbf{I}}\left(t\right)\right\}.\end{aligned} \tag{18}$$

Apart from an irrelevant current, Eq. (18) implies that the magnetic energy stored by a circuit having a current $\overrightarrow{\mathbf{I}}\left(t\right)$ is given as

$$\mathcal{U}_{\text{mag}}^{\mathcal{C}}=-W\left(t\right)=\frac{1}{2}\oint_{\mathcal{C}}\overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot\overrightarrow{\mathbf{I}}\left(t\right). \tag{19}$$

Note that Eq. (19), despite being written in terms of the vector potential, it is a gauge-invariant quantity as summing the gradient of an arbitrary differentiable function to $\overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}},t\right)$ would yield a zero contribution to the integral.

If, instead of a line circuit, we had a volume current density in the system, our expression for the magnetic energy could be obtained by a simple generalization of Eq. (19). I.e., we notice that

$$\mathcal{U}_{\text{mag}}^{\mathcal{C}}=\frac{1}{2}\oint_{\mathcal{C}}\overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{r}},t\right)\cdot\overrightarrow{\mathbf{I}}\left(t\right)=\frac{1}{2}\iint_{S_T}d^2\overrightarrow{\mathbf{R}}\oint_{\mathcal{C}}\delta^2\left(\overrightarrow{\mathbf{R}}-\overrightarrow{\mathbf{R}}_{\mathcal{C}}\right)\overrightarrow{\mathbf{A}}\left(\overrightarrow{\mathbf{R}},t\right)\cdot\overrightarrow{\mathbf{J}}\left(\overrightarrow{\mathbf{R}},t\right), \tag{20}$$

and, if we relax the constraint of the circuit given by the Dirac-δ, we arrive at the following volume expression

$$\mathcal{U}_{\mathrm{mag}}(t) = \frac{1}{2} \iiint_{\text{All Space}} d\vec{\mathbf{r}}\, \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}}, t).$$

This equation can be further simplified by using Ampère's Law (without the displacement current[1]) and write $\vec{\mathbf{J}}(\vec{\mathbf{r}}, t) = \mu_0^{-1} \nabla \times \vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$. This gives

$$\mathcal{U}_{\mathrm{mag}}(t) = \frac{1}{2\mu_0} \iiint_{\text{All Space}} d\vec{\mathbf{r}}\, \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) \cdot \nabla \times \vec{\mathbf{B}}(\vec{\mathbf{r}}, t), \tag{21}$$

and together with the result,

$$\nabla \cdot \left(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2\right) = \vec{\mathbf{V}}_2 \cdot \nabla \times \left(\vec{\mathbf{V}}_1\right) - \vec{\mathbf{V}}_1 \cdot \nabla \times \left(\vec{\mathbf{V}}_2\right), \tag{22}$$

we arrive at

$$\mathcal{U}_{\mathrm{mag}}(t) = \frac{1}{2\mu_0} \iiint_{\text{All Space}} d\vec{\mathbf{r}}\, \nabla \cdot \left\{\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) \times \vec{\mathbf{B}}(\vec{\mathbf{r}}, t)\right\} + \frac{1}{2\mu_0} \iiint_{\text{All Space}} d\vec{\mathbf{r}}\, \vec{\mathbf{B}}(\vec{\mathbf{r}}, t) \cdot \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{r}}, t). \tag{23}$$

Using the Divergence's Theorem, together with the assumption that the fields decay to zero very far away from all the currents, we arrive at the following expression:

$$\mathcal{U}_{\mathrm{mag}}(t) = \frac{1}{2\mu_0} \iiint_{\text{All Space}} d\vec{\mathbf{r}} \left|\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)\right|^2, \tag{24}$$

thus yielding a magnetic energy density of

$$u_{\mathrm{mag}}(\vec{\mathbf{r}}, t) = \frac{1}{2\mu_0} \left|\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)\right|^2. \tag{25}$$

1.2.3 Poynting's Theorem and Electromagnetic Energy Propagation

Both Eqs. (15) and (25) may be obtained in a more formal manner, using the very definition of the work done by a force. More precisely, the work done on an element of charge, $\rho(\vec{\mathbf{r}}, t)\, d\vec{\mathbf{r}}$, by the full electromagnetic force while it moves an infinitesimal distance $d\vec{\mathbf{l}}$ is given by:

$$\begin{aligned} \vec{\mathbf{F}}_{\mathrm{EM}} \cdot d\vec{\mathbf{l}} &= \rho(\vec{\mathbf{r}}, t)\, d\vec{\mathbf{r}} \left[\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) + \vec{\mathbf{B}}(\vec{\mathbf{r}}, t) \times \vec{\mathbf{v}}(\vec{\mathbf{r}}, t)\right] \cdot d\vec{\mathbf{l}} \\ &= \rho(\vec{\mathbf{r}}, t)\, d\vec{\mathbf{r}} \left[\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) + \vec{\mathbf{B}}(\vec{\mathbf{r}}, t) \times \vec{\mathbf{v}}(\vec{\mathbf{r}}, t)\right] \cdot \vec{\mathbf{v}}(\vec{\mathbf{r}}, t)\, dt \end{aligned} \tag{26}$$

where $\vec{\mathbf{v}}(\vec{\mathbf{r}}, t)$ is the local instantaneous velocity of the charge element. From Eq. (26) it is trivial to obtain that

$$\vec{\mathbf{F}}_{\mathrm{EM}} \cdot d\vec{\mathbf{l}} = \rho(\vec{\mathbf{r}}, t)\, \vec{\mathbf{E}}(\vec{\mathbf{r}}, t) \cdot \vec{\mathbf{v}}(\vec{\mathbf{r}}, t)\, d\vec{\mathbf{r}} dt = \vec{\mathbf{E}} \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}}, t)\, d\vec{\mathbf{r}} dt \tag{27}$$

and, if one is interested in the work done on a finite volume of three-dimensional space, $\mathcal{V}$, we arrive at the following expression:

$$W_{t \to t+dt} = \iiint_{\mathcal{V}} \left[\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}}, t)\, d\vec{\mathbf{r}}\right] dt. \tag{28}$$

Now, we wish to express Eq. (28) only in terms of the electromagnetic fields in $\mathcal{V}$, which can be done by using Ampère's law with the displacement current, i.e.

$$\nabla \times \vec{\mathbf{B}}(\vec{\mathbf{r}}, t) = \mu_0 \vec{\mathbf{J}}(\vec{\mathbf{r}}, t) + \mu_0 \varepsilon_m \frac{\partial}{\partial t} \vec{\mathbf{E}}(\vec{\mathbf{r}}, t). \tag{29}$$

[1] Lacks argument??

Hence, we get the following two terms

$$W_{t\to t+dt} = \left\{ \frac{1}{\mu_0} \iiint_{\mathcal{V}} \left[\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \cdot \nabla \times \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right) d\overrightarrow{\mathbf{r}} \right] - \varepsilon_m \iiint_{\mathcal{V}} \left[\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \cdot \frac{\partial}{\partial t} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) d\overrightarrow{\mathbf{r}} \right] \right\} dt \tag{30}$$

or, using also the identity of Eq. (22) and turning the last term into a full time partial derivative, we get

$$\begin{aligned} W_{t\to t+dt} = &\left\{ \frac{1}{\mu_0} \iiint_{\mathcal{V}} \nabla \cdot \left[\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \times \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right) d\overrightarrow{\mathbf{r}} \right] + \frac{1}{\mu_0} \iiint_{\mathcal{V}} \left[\overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right) \cdot \nabla \times \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) d\overrightarrow{\mathbf{r}} \right] \right. \\ &\left. - \frac{\varepsilon_m}{2} \frac{\partial}{\partial t} \iiint_{\mathcal{V}} \left| \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \right|^2 d\overrightarrow{\mathbf{r}} \right\} dt \\ = &\left\{ \frac{1}{\mu_0} \oiint_{S_{\mathcal{V}}} \left[\overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \times \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right) \right] \cdot d\overrightarrow{\mathbf{S}} \right. \\ &\left. - \frac{\partial}{\partial t} \left\{ \frac{\varepsilon_m}{2} \iiint_{\mathcal{V}} \left| \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \right|^2 d\overrightarrow{\mathbf{r}} + \frac{1}{2\mu_0} \iiint_{\mathcal{V}} \left| \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right) \right|^2 d\overrightarrow{\mathbf{r}} \right\} \right\} dt, \end{aligned} \tag{31}$$

where we made use of both Faraday's Law ($\nabla \times \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) = -\partial/\partial t \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right)$) and the Divergence's Theorem in the first integral. This last equation encapsulates what is named Poynting's Theorem and it basically tells us that the rate of change of the energy inside a given volume can be due to either the intrinsic time variation of both fields (second term) or to the flux of the vector $\mu_0^{-1} \overrightarrow{\mathbf{E}} \times \overrightarrow{\mathbf{B}}$ through the surface. If we consider the volume $\mathcal{V}$ as an arbitrary volume containing all charges and currents in the system, then $dW = 0$ and we arrive at the following continuity equation:

$$\frac{\partial}{\partial t} \left\{ \frac{\varepsilon_m}{2} \left| \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \right|^2 + \frac{1}{2\mu_0} \left| \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right) \right|^2 \right\} = \nabla \cdot \left(\frac{1}{\mu_0} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \times \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right) \right). \tag{32}$$

This last equation simply restates that there is an energy density (per unit volume) associated to an electromagnetic field, given as before by

$$u_{\mathrm{EM}}\left(\overrightarrow{\mathbf{r}},t\right) = \frac{\varepsilon_m}{2} \left| \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \right|^2 + \frac{1}{2\mu_0} \left| \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right) \right|^2 \tag{33}$$

and there is also a vector field

$$\overrightarrow{\mathbf{S}}\left(\overrightarrow{\mathbf{r}},t\right) = \frac{1}{\mu_0} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \times \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right), \tag{34}$$

which stands for the directed energy density transferred by the electromagnetic field across space. The vector in Eq. (34) is known as the Poynting vector.

1.3 Energy Transported by a Plane Electromagnetic Wave

Finally, we are in possession of all the tools to calculate the energy transported by an electromagnetic plane-wave, when traveling through a dielectric medium. On the one hand, we have the fields of the wave

$$\overrightarrow{\mathbf{E}}_{\overrightarrow{k}}\left(\overrightarrow{r},t\right) = \overrightarrow{\mathbf{E}}_{0,\overrightarrow{k}} e^{i\left(\overrightarrow{k}\cdot\overrightarrow{r}-\omega t\right)} \tag{35}$$

$$\overrightarrow{\mathbf{B}}_{\overrightarrow{k}}\left(\overrightarrow{r},t\right) = \overrightarrow{\mathbf{B}}_{0,\overrightarrow{k}} e^{i\left(\overrightarrow{k}\cdot\overrightarrow{r}-\omega t\right)}, \tag{36}$$

with the relation $\overrightarrow{\mathbf{B}}_{0,\overrightarrow{\mathbf{k}}} = \frac{1}{\sqrt{\mu_0 \varepsilon_m}} \hat{\mathbf{k}} \times \overrightarrow{\mathbf{E}}_{0,\overrightarrow{\mathbf{k}}}$, and also the general expression for the Poynting vector, which allows us to calculate the energy transport across space due to electromagnetic fields, i.e.

$$\overrightarrow{\mathbf{S}}\left(\overrightarrow{\mathbf{r}},t\right) = \frac{1}{\mu_0} \overrightarrow{\mathbf{E}}\left(\overrightarrow{\mathbf{r}},t\right) \times \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{r}},t\right). \tag{37}$$

Despite having these expression, one must be careful when calculating the Poynting vector associated to the fields of Eqs. (35) and (36), for they are complex representations of the real fields. Aware of this fact and using Eqs. (35)-(37), we arrive at the following pointing vector for the electromagnetic plane-wave:

$$\vec{\mathbf{S}}\left(\vec{\mathbf{r}},t\right)=\frac{1}{\mu_0}\vec{\mathbf{E}}_{0,\vec{k}}\times\vec{\mathbf{B}}_{0,\vec{k}}\cos^2\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t\right)=\frac{1}{\mu_0 v_m}\left|\vec{\mathbf{E}}_{0,\vec{k}}\right|^2\cos^2\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t\right)\hat{\mathbf{k}}. \tag{38}$$

The intensity of the wave is usually defined as the average of $\left|\vec{\mathbf{S}}\left(\vec{\mathbf{r}},t\right)\right|$ over a time period, as the detection devices usually do not have enough resolution to observe the space and time variations implied in Eq. (38). Hence, we have

$$\mathcal{I}_\perp=\frac{\omega}{2\pi}\frac{1}{\mu_0 v_m}\left|\vec{\mathbf{E}}_{0,\vec{k}}\right|^2\int_t^{t+\frac{2\pi}{\omega}}d\tau\cos^2\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega\tau\right)=\frac{1}{2\mu_0 v_m}\left|\vec{\mathbf{E}}_{0,\vec{k}}\right|^2. \tag{39}$$

If the incidence is at a certain angle α with respect to the surface's normal, we will have an effective intensity which is simply,

$$\mathcal{I}=\frac{1}{2\mu_0 v_m}\left|\vec{\mathbf{E}}_{0,\vec{k}}\right|^2\cos\alpha. \tag{40}$$

2 Boundary Conditions of Electromagnetic Fields across Conducting Interfaces

In this section, we will derive the boundary conditions associated to electric and magnetic fields, when they cross a boundary between two dielectrics with different refractive indexes, n_1 and n_2, and are separated by a conducting surface with a complex conductivity, $\sigma_S=\sigma'+i\sigma''$. As usual, these conditions can be derived directly from Maxwell's Equations in their integral form, which we restate here in non-magnetic dielectric media:

$$\begin{aligned}
\oiint_{S_\mathcal{V}}\vec{\mathbf{D}}\left(\vec{\mathbf{r}},t\right)\cdot d\vec{\mathbf{S}} &= \iiint_\mathcal{V}\rho\left(\vec{\mathbf{r}},t\right)d\vec{\mathbf{r}}\\
\oiint_{S_\mathcal{V}}\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)\cdot d\vec{\mathbf{S}} &= 0\\
\oint_\mathcal{C}\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)\cdot d\vec{\mathbf{S}} &= -\iiint d\vec{\mathbf{r}}\,\tfrac{\partial}{\partial t}\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)\\
\oint_\mathcal{C}\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)\cdot d\vec{\mathbf{S}} &= \mu_0\iiint d\vec{\mathbf{r}}\,\vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right)-\mu_0\iiint d\vec{\mathbf{r}}\,\tfrac{\partial}{\partial t}\vec{\mathbf{D}}\left(\vec{\mathbf{r}},t\right)
\end{aligned} \tag{41}$$

The first two equations yield the discontinuity (continuity) of the electric (magnetic) field across the boundary. In order to see that, let us consider a pillbox surface, as depicted in Fig 2 a). Since there are no free charges on the interface, and the pillbox has an arbitrary shape, we can see immediately that the following conditions must hold at the interface:

$$\begin{array}{ll}\vec{\mathbf{D}}_1^\perp & =\vec{\mathbf{D}}_2^\perp+\rho_S\left(\vec{\mathbf{r}},t\right)\\ \vec{\mathbf{B}}_1^\perp & =\vec{\mathbf{B}}_2^\perp\end{array}\implies\begin{array}{ll}\varepsilon_1\vec{\mathbf{E}}_1^\perp & =\varepsilon_2\vec{\mathbf{E}}_2^\perp+\rho_S\left(\vec{\mathbf{r}},t\right)\\ \vec{\mathbf{B}}_1^\perp & =\vec{\mathbf{B}}_2^\perp\end{array}, \tag{42}$$

where ρ_S is the non-homogeneous free charge density that is induced in the surface by the parallel electric field. If we have an ohmic conducting interface, there will be a surface current flowing in it, which is driven by the electric field at the boundary, i.e.

$$\vec{\mathbf{K}}\left(\vec{\mathbf{r}},t\right)=\sigma_S\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=\left[\sigma'+i\sigma''\right]\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right),$$

where $\vec{\mathbf{r}}$ is restricted to be on the interface. Besides this induced current, by continuity, we also have an induced surface charge density which is related to this current. This can be calculated as

$$\frac{\partial}{\partial t}\rho_S(y,z,t)=\frac{\partial}{\partial y}K_y(y,z,t)+\frac{\partial}{\partial z}K_z(y,z,t)$$
$$\rho_S(y,z,t)=\sigma_S\left\{\frac{\partial}{\partial y}\int_0^t E_y(0,y,z,\tau)\,d\tau+\frac{\partial}{\partial z}\int_0^t E_z(0,y,z,\tau)\,d\tau\right\},$$

if the interface is placed at the ($x=0$)-plane. Assuming that our electric field is a plane wave of the kind (with an adiabatic connection from $t\to-\infty$)

$$\overrightarrow{\mathbf{E}}(x,y,z,t)=\lim_{\gamma\to0^+}\overrightarrow{\mathbf{E}}_0 e^{i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{r}}-\omega t+\phi\right)-\gamma t},$$

it will generate an induced surface current given as

$$\begin{aligned}\rho_S(y,z,t)&=\lim_{\gamma\to0^+}i\sigma_S\left\{E_{0y}k_y e^{i\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{r}}}+E_{0z}k_z e^{i\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{r}}}\right\}\int_{-\infty}^{t}e^{-i\omega t-\gamma t}\\&=-\lim_{\gamma\to0^+}\frac{\sigma_S}{\omega+i\gamma}\left\{E_{0y}k_y e^{i\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{r}}+i\phi}+E_{0z}k_z e^{i\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{r}}+i\phi}\right\}\left\{e^{-i\omega t-\gamma t}\right\}\\&=\frac{\sigma_S}{\omega}\overrightarrow{\mathbf{E}}^{\parallel}(0,y,z,t)\cdot\overrightarrow{\mathbf{k}},\end{aligned}$$

hence, our electric fields' boundary conditions are given as:

$$\varepsilon_1\overrightarrow{\mathbf{E}}_1^{\perp}=\varepsilon_2\overrightarrow{\mathbf{E}}_2^{\perp}+\frac{\sigma_S}{\omega}\overrightarrow{\mathbf{E}}^{\parallel}_{\overrightarrow{\mathbf{k}},0}\cdot\overrightarrow{\mathbf{k}} \tag{43}$$

In these conditions we may consider the two cases depicted in Fig. 2 b), where the circuits crossing the interface are parallel (blue) or perpendicular (red) to the flow of the surface current. Now, using the two last expressions of Eq. (41), we arrive at the following expressions boundary conditions in the presence of a conducting surface between the two dielectrics:

$$\overrightarrow{\mathbf{E}}_1^{\parallel}=\overrightarrow{\mathbf{E}}_2^{\parallel} \tag{44}$$

and

$$\begin{array}{cc}\overrightarrow{\mathbf{B}}_1^{\parallel}=\overrightarrow{\mathbf{B}}_2^{\parallel} & \text{when parallel to the current}\\ \overrightarrow{\mathbf{B}}_1^{\parallel}=\overrightarrow{\mathbf{B}}_2^{\parallel}+\mu_0\left[\sigma'+i\sigma''\right]\overrightarrow{\mathbf{E}}^{\parallel}\left(\overrightarrow{\mathbf{r}},t\right) & \text{when perpendicular to the current}\end{array}. \tag{45}$$

In a more condensed notation, we have the following boundary conditions for the parallel components of the magnetic field:

$$\overrightarrow{\mathbf{B}}_1^{\parallel}=\overrightarrow{\mathbf{B}}_2^{\parallel}+\mu_0\left[\sigma'+i\sigma''\right]\hat{\mathbf{n}}\times\overrightarrow{\mathbf{E}}^{\parallel}\left(\overrightarrow{\mathbf{r}},t\right). \tag{46}$$

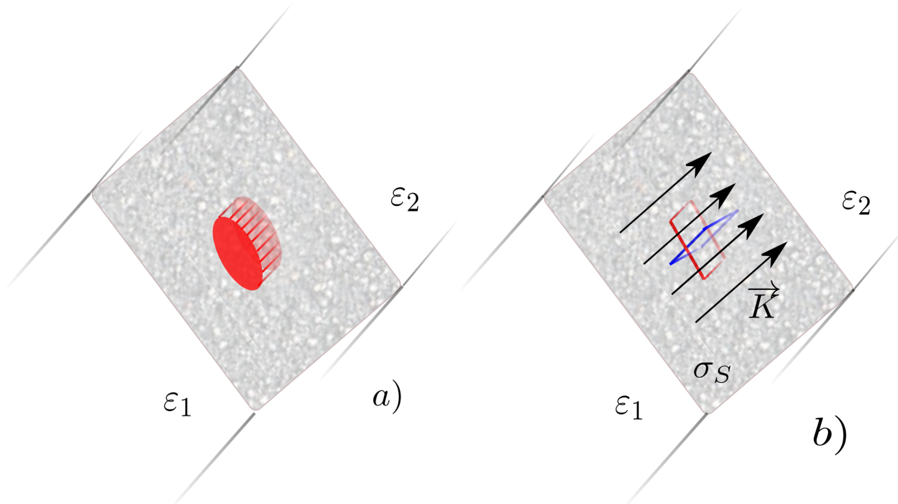

Figure 2: a) Pillbox Surface traversing the interface between the two dielectric media. b) Square circuit used to derive the boundary conditions for the parallel components.

3 Fresnel Equations for a Conducting Interface between Two Dielectric Media

In this section, we will use the toolbox developed for electromagnetic plane waves to study the reflectance, transmittance and absorbance of electromagnetic waves when they hit a conducting interface between two dielectrics at an angle θ_i. The polarization of our incident plane-wave is arbitrary but, as usual, it is helpful to decompose it into two orthogonal linearly-polarized components — *Transverse Magnetic* (TM) and *Transverse Electric* (TE). These two situations are depicted in Fig. 3.

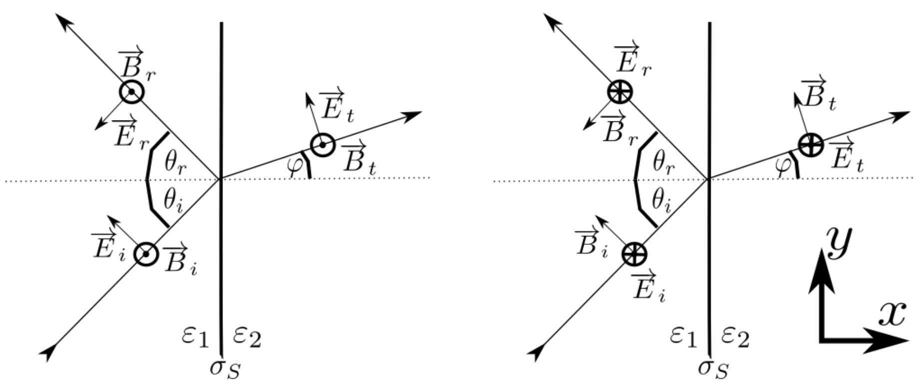

Figure 3: Scheme of the Transverse Magnetic (left panel) and Transverse Electric (right panel) setups. The cartesian system of coordinated we will be using is represented, with the z-axis being defined as out-of-plane.

In all our future calculations, our final aims will be to calculate the reflected and transmitted intensities, normalized by the incident ones. These quantities can be written in terms of the field amplitudes of each wave as follows:

$$\mathcal{T}(\theta_i) = \frac{\frac{\cos\varphi}{v_2}\left|\vec{\mathbf{E}}_t\right|^2}{\frac{\cos\theta}{v_1}\left|\vec{\mathbf{E}}_i\right|^2} = \frac{n_2\cos\varphi\left|\vec{\mathbf{E}}_t\right|^2}{n_1\cos\theta\left|\vec{\mathbf{E}}_i\right|^2} \tag{47}$$

and

$$\mathcal{R}(\theta_i) = \frac{\frac{1}{v_1}\left|\overrightarrow{\mathbf{E}}_r\right|^2}{\frac{1}{v_1}\left|\overrightarrow{\mathbf{E}}_i\right|^2} = \frac{\left|\overrightarrow{\mathbf{E}}_t\right|^2}{\left|\overrightarrow{\mathbf{E}}_i\right|^2}, \tag{48}$$

where $n_1 = \sqrt{\varepsilon_1}$ and $n_2 = \sqrt{\varepsilon_2}$ are the refractive indexes of the respective dielectric medium.

Besides reflexion and transmission, we are also expected to have dissipation in the conducting boundary. This can be described by an absorbance (per unit area) defined as follows:

$$\mathcal{A}(\theta_i) = 1 - \mathcal{R}(\theta_i) - \mathcal{T}(\theta_i). \tag{49}$$

3.1 Refraction of Transverse Magnetic (TM) Polarized Light

We will begin by dealing with the case of TM polarization. In this case, the electric field is always in the plane of incidence and so will be the surface current. Hence, the (phasor-represented) plane-waves involved can be written as follows:[2]

1. Incident Wave:

$$\overrightarrow{\mathbf{E}}_i(\overrightarrow{r}, t) = \left|\overrightarrow{\mathbf{E}}_i\right| e^{i\left(\overrightarrow{k}_i \cdot \overrightarrow{r} - \omega t\right)} \left[-\sin\theta_i \hat{\mathbf{x}} + \cos\theta_i \hat{\mathbf{y}}\right] \tag{50}$$

$$\overrightarrow{\mathbf{B}}_i(\overrightarrow{r}, t) = \left|\overrightarrow{\mathbf{B}}_i\right| e^{i\left(\overrightarrow{k}_i \cdot \overrightarrow{r} - \omega t\right)} \hat{\mathbf{z}}, \tag{51}$$

2. Reflected Wave:

$$\overrightarrow{\mathbf{E}}_r(\overrightarrow{r}, t) = -\left|\overrightarrow{\mathbf{E}}_r\right| e^{i\left(\overrightarrow{k}_r \cdot \overrightarrow{r} - \omega t + \phi_r\right)} \left[\sin\theta_r \hat{\mathbf{x}} + \cos\theta_r \hat{\mathbf{y}}\right] \tag{52}$$

$$\overrightarrow{\mathbf{B}}_r(\overrightarrow{r}, t) = \left|\overrightarrow{\mathbf{B}}_r\right| e^{i\left(\overrightarrow{k}_r \cdot \overrightarrow{r} - \omega t + \phi_r\right)} \hat{\mathbf{z}}, \tag{53}$$

3. Transmitted Wave:

$$\overrightarrow{\mathbf{E}}_t(\overrightarrow{r}, t) = \left|\overrightarrow{\mathbf{E}}_t\right| e^{i\left(\overrightarrow{k}_t \cdot \overrightarrow{r} - \omega t + \phi_t\right)} \left[-\sin\varphi \hat{\mathbf{x}} + \cos\varphi \hat{\mathbf{y}}\right] \tag{54}$$

$$\overrightarrow{\mathbf{B}}_t(\overrightarrow{r}, t) = \left|\overrightarrow{\mathbf{B}}_t\right| e^{i\left(\overrightarrow{k}_t \cdot \overrightarrow{r} - \omega t + \phi_t\right)} \hat{\mathbf{z}}, \tag{55}$$

The frequency ω is obviously the same for all three cases, as we which the boundary conditions to be satisfied at all times. The same kind of reasoning applied to the space-dependence gives rise to Snell's Laws

$$\theta_r = \theta_i \equiv \theta,$$

(from the fact that $\overrightarrow{\mathbf{k}}_i = \frac{\omega}{v_1}(\cos\theta_i \hat{\mathbf{x}} + \sin\theta_i \hat{\mathbf{y}})$ and $\overrightarrow{\mathbf{k}}_r = \frac{\omega}{v_1}(-\cos\theta_i \hat{\mathbf{x}} + \sin\theta_i \hat{\mathbf{y}})$) and also

$$n_1 \sin\theta = n_2 \sin\varphi,$$

coming from $\overrightarrow{\mathbf{k}}_t = \frac{\omega}{v_2}(\cos\varphi \hat{\mathbf{x}} + \sin\varphi \hat{\mathbf{y}})$.

Therefore, we may now use the boundary conditions defined in the previous section and impose them at the interface between the two media. For the electric field components these conditions go as follows:

[2]Note that we are using the cartesian system of coordinated defined in Fig. 3 and the conducting surface is considered to be located at the plane defined by $x = 0$. This choice involves no loss of generality.

$$\left|\vec{\mathbf{E}}_i\right| \cos\theta_i - \left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} \cos\theta_r = \left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} \cos\varphi \tag{56}$$

$$\left|\vec{\mathbf{E}}_i\right| \sin\theta_i + \left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} \sin\theta_r = \left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} \left\{ \frac{\varepsilon_2}{\varepsilon_1} \sin\varphi + \frac{\sigma_S}{2\omega\varepsilon_1} \left|\vec{\mathbf{k}}_t\right| \sin 2\varphi \right\} \tag{57}$$

and noting that $\left|\vec{\mathbf{k}}_t\right| = \omega/v_m = \omega n_2 c^{-1}$, we get

$$\left|\vec{\mathbf{E}}_i\right| \cos\theta_i - \left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} \cos\theta_r = \left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} \cos\varphi \tag{58}$$

$$\left|\vec{\mathbf{E}}_i\right| \sin\theta_i + \left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} \sin\theta_r = \left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} \left\{ \left(\frac{n_2}{n_1}\right)^2 \sin\varphi + \frac{n_2\sigma_S}{2c\varepsilon_0 n_1^2} \sin 2\varphi \right\} \tag{59}$$

For the magnetic field we have a discontinuity due to the surface current, which now is in the y-axis' direction. I.e.

$$\left|\vec{\mathbf{B}}_i\right| + \left|\vec{\mathbf{B}}_r\right| e^{i\phi_r} = \left|\vec{\mathbf{B}}_t\right| e^{i\phi_t} + \mu_0\sigma_S \left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} \cos\varphi. \tag{60}$$

Now, we must use the expression connecting the magnetic to the electric amplitudes of each one of the plane waves to write Eqs. (56)-(60) in terms of electric field amplitudes only. In other words,

$$\frac{1}{v_1}\left|\vec{\mathbf{E}}_i\right| + \frac{1}{v_1}\left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} = \frac{1}{v_2}\left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} + \mu_0\sigma_S \left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} \cos\varphi, \tag{61}$$

or

$$\left|\vec{\mathbf{E}}_i\right| + \left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} = \left(\frac{n_2}{n_1} + \frac{\mu_0 c\sigma_S}{n_1}\cos\varphi\right) e^{i\phi_t} \left|\vec{\mathbf{E}}_t\right|, \tag{62}$$

where c is the velocity of light in the vacuum. Hence, the linear system which describes the imposition of the interface boundary conditions is written as follows:

$$\begin{cases} \left|\vec{\mathbf{E}}_i\right| \cos\theta_i - \left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} \cos\theta_r - \left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} \left\{\cos\varphi + \frac{\sigma_S}{\omega}\sin^2\varphi\right\} & = 0 \\ \left|\vec{\mathbf{E}}_i\right| \sin\theta_i + \left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} \sin\theta_r - \left|\vec{\mathbf{E}}_t\right| e^{i\phi_t} \left\{\frac{n_2}{n_1}\sin\theta + \frac{n_2\sigma_S}{2c\varepsilon_0 n_1^2}\sin 2\varphi\right\} & = 0\ , \\ \left|\vec{\mathbf{E}}_i\right| + \left|\vec{\mathbf{E}}_r\right| e^{i\phi_r} - \left(\frac{n_2}{n_1} + \frac{\mu_0 c\sigma_S}{n_1}\cos\varphi\right) e^{i\phi_t} \left|\vec{\mathbf{E}}_t\right| & = 0 \end{cases} \tag{63}$$

or

$$\begin{pmatrix} \cos\theta & -e^{i\phi_r}\cos\theta & -e^{i\phi_t}\cos\varphi \\ \sin\theta & e^{i\phi_r}\sin\theta & -\frac{n_2}{n_1}e^{i\phi_t}\left(\sin\theta + \frac{1}{2}\xi\sin 2\varphi\right) \\ 1 & e^{i\phi_r} & -\frac{n_2}{n_1}e^{i\phi_t} - \xi e^{i\phi_t}\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} \left|\vec{\mathbf{E}}_i\right| \\ \left|\vec{\mathbf{E}}_r\right| \\ \left|\vec{\mathbf{E}}_t\right| \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where we already used Snell's laws and defined $\xi = \mu_0 c\sigma_S/n_1$. This system only has a non-trivial solution if the secular determinant is zero. This means that

$$-\frac{2\xi\cos(\theta)\cos(\phi)e^{i(\phi_r+\phi_t)}\left(n_1\sin(\theta) - n_2\sin(\phi)\right)}{n_1} = 0 \tag{64}$$

which means that there is only transmission if θ and φ obey Snell's Law of transmission. In this case, our original 3×3 system is indeterminate and we can find the transmission and reflexion coefficients from just two of the equations. For instance,

$$\begin{cases} \left|\overrightarrow{\mathbf{E}}_i\right| \cos\theta - \left|\overrightarrow{\mathbf{E}}_r\right| \cos\theta e^{i\phi_r} - \left|\overrightarrow{\mathbf{E}}_t\right| \cos\varphi e^{i\phi_t} & = 0 \\ \left|\overrightarrow{\mathbf{E}}_i\right| + \left|\overrightarrow{\mathbf{E}}_r\right| e^{i\phi_r} - \left(\frac{n_2}{n_1} + \xi\cos\varphi\right) \left|\overrightarrow{\mathbf{E}}_t\right| e^{i\phi_t} & = 0 \end{cases}, \tag{65}$$

meaning that, by summing the two equations as $eq_1 + eq_2 \times \cos\theta$), we get the ratio of the transmitted and incident complex electric field amplitudes as follows:

$$2\cos\theta \left|\overrightarrow{\mathbf{E}}_i\right| - \left(\cos\varphi + \left(\frac{n_2}{n_1} + \xi\right)\cos\theta\right) \left|\overrightarrow{\mathbf{E}}_t\right| e^{i\phi_t} = 0 \Rightarrow \frac{\left|\overrightarrow{\mathbf{E}}_t\right| e^{i\phi_t}}{\left|\overrightarrow{\mathbf{E}}_i\right|} = \frac{2n_1\cos\theta}{n_1\cos\varphi + (n_2 + n_1\xi)\cos\theta}, \tag{66}$$

Additionally, by replacing this back into eq_1, we get

$$\left|\overrightarrow{\mathbf{E}}_i\right| \left(\frac{(n_2 + n_1\xi)\cos\theta - n_1\cos\varphi}{n_1\cos\varphi + (n_2 + n_1\xi)\cos\theta}\right)\cos\theta - \left|\overrightarrow{\mathbf{E}}_r\right| \cos\theta e^{i\phi_r} = 0$$

or, alternatively,

$$\left|\overrightarrow{\mathbf{E}}_r\right| = e^{-i\phi_r} \left(\frac{(n_2 + n_1\xi)\cos\theta - n_1\cos\varphi}{n_1\cos\varphi + (n_2 + n_1\xi)\cos\theta}\right) \left|\overrightarrow{\mathbf{E}}_i\right| \Rightarrow \frac{\left|\overrightarrow{\mathbf{E}}_r\right| e^{i\phi_r}}{\left|\overrightarrow{\mathbf{E}}_i\right|} = \frac{(n_2 + n_1\xi)\cos\theta - n_1\cos\varphi}{n_1\cos\varphi + (n_2 + n_1\xi)\cos\theta}. \tag{67}$$

Note that the previous equations also imply that $\phi_r = 0, \pi$ and $\phi_t = 0$ if the surface conductivity σ_S is a real quantity. In that case also, from Eqs. (66) and (67) we can get the following reflectance and transmittance:

$$\mathcal{R}(\theta) = \left(\frac{(n_2 + \mu_0 c\sigma_S)\cos\theta - n_1\cos\varphi}{n_1\cos\varphi + (n_2 + \mu_0 c\sigma_S)\cos\theta}\right)^2$$

and

$$\mathcal{T}(\theta) = \frac{4n_1 n_2 \cos\varphi \cos^2\theta}{\cos\theta\left(n_1\cos\varphi + (n_2 + \mu_0 c\sigma_S)\cos\theta\right)^2}.$$

In terms of only the incidence angle θ, and since we know that $\cos\varphi = \sqrt{1 - \left(n_1/n_2\right)^2 \sin^2\theta}$,we get:

$$\mathcal{R}(\theta) = \left(\frac{(n_2 + \mu_0 c\sigma_S)\cos\theta - n_1\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}}{n_1\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta} + (n_2 + \mu_0 c\sigma_S)\cos\theta}\right)^2 \tag{68}$$

and

$$\mathcal{T}(\theta) = \frac{4n_1 n_2\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}\cos\theta}{\left(n_1\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta} + (n_2 + \mu_0 c\sigma_S)\cos\theta\right)^2}. \tag{69}$$

The absorbance can also be calculated directly, yielding:

$$\mathcal{A}(\theta) = 1 - \mathcal{R}(\theta) - \mathcal{T}(\theta) = \frac{4\mu_0 c\sigma_S n_1\cos\theta\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}}{\left(n_1\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta} + (n_2 + \mu_0 c\sigma_S)\cos\theta\right)^2} \tag{70}$$

which rightfully converges to zero in the limit when $\sigma_S \to 0$. Note that all the calculations above were done assuming a real conductivity (i.e. purely dissipative). Plots of the quantities of Eqs. (68)-(70) are shown in Fig. **??**, as a function of the angle and for different surface conductivities. Note that the optical conductivity of pristine graphene has the following expression [2]:

$$\sigma(\omega, T) = \frac{\pi e^2}{2\hbar}\left(\frac{1}{2} - \frac{\hbar^2\omega^2}{72t^2}\right)\left(\tanh\left(\frac{\hbar\omega + 2\mu}{4k_\mathrm{B}T}\right) + \tanh\left(\frac{\hbar\omega - 2\mu}{4k_\mathrm{B}T}\right)\right), \tag{71}$$

where $t \approx 2.7$eV and μ is the chemical potential of the graphene (for undoped graphene — $\mu = 0$.).However, more important than the formula itself is to realize that at the range of wavelengths (in air) from 100nm to 1000nm, $\sigma_S \approx 10^{-4}\Omega$. Nevertheless, in the general case, the conductivity of a conducting sheet is a complex number. In that case, the previous calculations must be done with a little more care. Namely, we must have the complex reflection coefficient defined as

$$re^{i\phi_r} = \frac{(n_2 + \mu_0 c(\sigma_r + i\sigma_i))\cos\theta - n_1\cos\varphi}{n_1\cos\varphi + (n_2 + \mu_0 c(\sigma_r + i\sigma_i))\cos\theta} \tag{72}$$

and a complex transmission coefficient

$$te^{i\phi_t} = \frac{2n_1\cos\theta}{n_1\cos\varphi + (n_2 + \mu_0 c(\sigma_r + i\sigma_i))\cos\theta}. \tag{73}$$

These expressions yield the following reflectance

$$\mathcal{R}_\mathrm{TM}(\theta) = 1 - \frac{4n_1\cos\theta\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\,(c\mu_0\sigma_r + n_2)}{c^2\mu_0^2\sigma_i^2 + \left(n_1\cos\theta + (n_2 + c\mu_0\sigma_r)\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\right)^2} \tag{74}$$

and a transmittance

$$\mathcal{T}_\mathrm{TM}(\theta) = \frac{4n_1 n_2\cos\theta\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta}}{c^2\mu_0^2\sigma_i^2\left(1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta\right) + \left((c\mu_0\sigma_r + n_2)\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta} + n_1\cos\theta\right)^2}. \tag{75}$$

Finally, the energy absorbed by the surface by unit area is

$$\mathcal{A}_\mathrm{TM}(\theta) = \frac{4c\mu_0 n_1\sigma_r\cos\theta\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta}}{\cos^2\theta\left(c^2\mu_0^2\left(\sigma_i^2 + \sigma_r^2\right) + 2c\mu_0 n_2\sigma_r + n_2^2\right) + 2n_1\cos\theta\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\,(c\mu_0\sigma_r + n_2) + n_1^2\left(1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta\right)}. \tag{76}$$

Besides these intensity coefficients, the phases of introduced into the transmitted and reflected waves are also non-trivial in this case. But we do have access to them from Eqs. (72) and (73). These phases can be written as follows:

$$\tan\phi_r = \frac{2c\mu_0 n_1\sigma_i\cos\theta\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta}}{\cos^2\theta\left(c^2\mu_0^2\sigma_i^2 + (c\mu_0\sigma_r + n_2)^2\right) - n_1^2\left(1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta\right)} \text{ and } \tan\phi_t = -\frac{c\mu_0\sigma_i\cos\theta}{\cos\theta(c\mu_0\sigma_r + n_2) + n_1\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta}}. \tag{77}$$

Finally, we can assume that our sheet of graphene has a complex surface conductivity as given in [2], i.e.

$$\sigma_r(\omega, \mu, T) = \frac{\pi e^2}{2\hbar}\left(\frac{1}{2} - \frac{\hbar^2\omega^2}{72t^2}\right)\left(\tanh\left(\frac{\hbar\omega + 2\mu}{4k_\mathrm{B}T}\right) + \tanh\left(\frac{\hbar\omega - 2\mu}{4k_\mathrm{B}T}\right)\right), \tag{78}$$

$$\sigma_i\left(\omega,\mu,T\right)=\frac{2\mu e^2}{\hbar^2\omega}\left(1-\frac{2\mu^2}{9t^2}\right)-\frac{e^2}{2\hbar}\log\left(\left|\frac{\hbar\omega+2\mu}{\hbar\omega-2\mu}\right|\right)-\frac{e^2}{72\hbar}\left(\frac{\hbar\omega}{t}\right)^2\log\left(\left|\frac{\hbar\omega+2\mu}{\hbar\omega-2\mu}\right|\right). \tag{79}$$

Note that, in the limit $\mu\to 0$, the conductivity becomes purely real for a finite excitation frequency. Hence, let us assume a positive doping of the order of $\mu\approx 0.2\text{eV}$, which would be compatible with a doping with electron acceptors, like oxygen atoms. Even in that case, the effect of the imaginary part is irrelevant in when ω is an optical frequency. In Fig. 5, we depict $\mathcal{T}_{\text{TM}}\left(\theta\right)$,$\mathcal{R}_{\text{TM}}\left(\theta\right)$ and $\mathcal{A}_{\text{TM}}\left(\theta\right)$ for a polarized wave with $\lambda=500\text{nm}$, using Eqs. (78) and (79) for the optical conductivity of the graphene sheet. More interesting is to look at the limit of normal incidence and calculate what is the effect of a single graphene layer on top of a glass surface in what concerns the reflectance and transmittance of the interface. In this case, we have the following analytical expressions:

$$\mathcal{R}\left(\theta=0^{\text{o}}\right)=\frac{c^2\mu_0^2\sigma_i^2+\left(c\mu_0\sigma_r+n_2-n_1\right)^2}{c^2\mu_0^2\sigma_i^2+\left(c\mu_0\sigma_r+n_1+n_2\right)^2} \tag{80}$$

$$\mathcal{T}\left(\theta=0^{\text{o}}\right)=\frac{4n_1n_2}{c^2\mu_0^2\sigma_i^2+\left(c\mu_0\sigma_r+n_2+n_1\right)^2}. \tag{81}$$

Note that the transmittance is invariant with respect to the exchange of the media, i.e. $n_1\to n_2$, but the reflectance is not. These results are shown in Fig. 4, for frequencies in the visible range, i.e. $\lambda\in[380\text{nm},740\text{nm}]$.

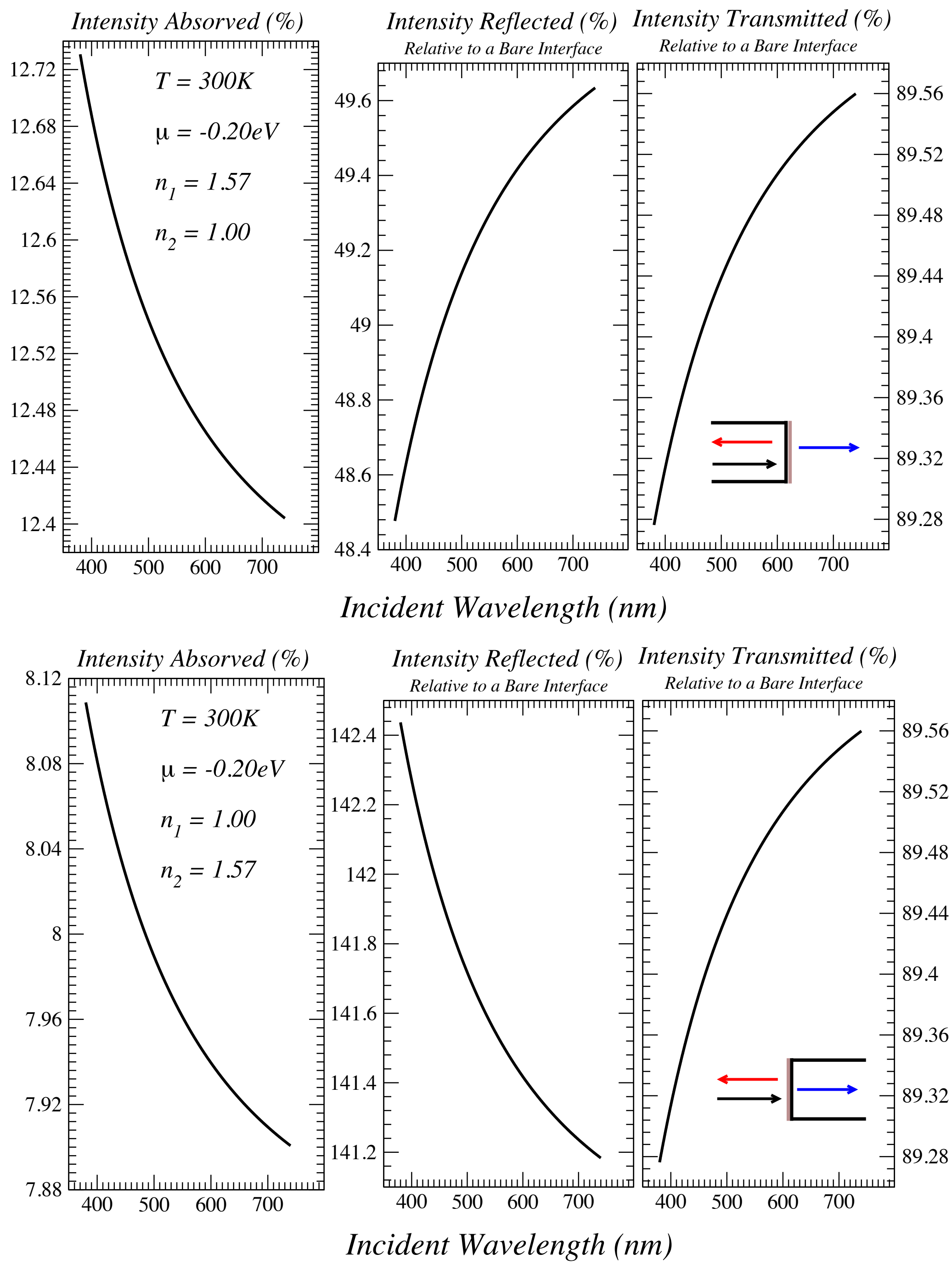

Figure 4: Normalized reflected, transmitted and absorbed radiation for a conducting sheet of graphene placed in an interface between a glass ($n = 1.0$) and air ($n = 1.57$). The situations corresponding to incidence from the glass and air side are considered. The temperature used was again $T = 300K$ and $\mu = -0.2$eV.

3.2 Total Reflection of Transverse Magnetic Waves

Before moving onto the next polarization, we need remark that all the calculations above were done assuming that transmission onto medium 2 is possible. Such is true for any angle if $n_2 \geq n_1$, but otherwise, it only happens below the critical angle for total internal reflexion, i.e.

$$\sin\theta_{\mathrm{C}} = \frac{n_2}{n_1}\sin\left(\frac{\pi}{2}\right) = \frac{n_2}{n_1}.$$

Above θ_{C}, the wave in medium 2 is evanescent in the perpendicular direction, i.e.

$$\overrightarrow{\mathbf{E}}_t\left(\overrightarrow{r},t\right) = \overrightarrow{\mathbf{E}}_{t,0}e^{i(k_{t,y}y+k_{t,z}z-\omega t+\phi_t)}e^{-\kappa x} \tag{82}$$

$$\overrightarrow{\mathbf{B}}_t\left(\overrightarrow{r},t\right) = \overrightarrow{\mathbf{B}}_{t,0}e^{i(k_{t,y}y+k_{t,z}z-\omega t+\phi_t)}e^{-\kappa x}. \tag{83}$$

In this case, we have different relations between the two fields associated to the wave. In order to obtain these, we use again Faraday's Law — $\nabla\times\overrightarrow{\mathbf{E}}_t\left(\overrightarrow{r},t\right) = -\partial/\partial t\overrightarrow{\mathbf{B}}_t\left(\overrightarrow{r},t\right)$ — yielding

$$(i\kappa, k_{t,y}, k_{t,z})\times\overrightarrow{\mathbf{E}}_t\left(\overrightarrow{r},t\right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ i\kappa & k_{t,y} & k_{t,z} \\ E_{t,x} & E_{t,y} & E_{t,z} \end{vmatrix} = \omega\overrightarrow{\mathbf{B}}_t\left(\overrightarrow{r},t\right),$$

hence, we must have

$$\begin{cases} k_{t,y}E_{t,z} - k_{t,z}E_{t,y} = \omega B_{t,x} \\ k_{t,z}E_{t,x} - i\kappa E_{t,z} = \omega B_{t,y} \\ i\kappa E_{t,y} - k_{t,y}E_{t,x} = \omega B_{t,z} \end{cases}.$$

Since our magnetic field components along $\hat{y}$ and $\hat{x}$ are continuous, then $B_{t,x} = B_{t,y} = 0$, for a TM polarized incident wave.

[TO BE WRITTEN...]

3.3 Refraction of Transverse Electric (TE) Polarized Light

In this case, we have take the conventions defined in the right panel of Fig 3. Hence, the fields associated to each wave will be:

1. Incident Wave:

$$\overrightarrow{\mathbf{E}}_i\left(\overrightarrow{r},t\right) = -\left|\overrightarrow{\mathbf{E}}_i\right|e^{i\left(\overrightarrow{k}_i\cdot\overrightarrow{r}-\omega t\right)}\hat{\mathbf{z}} \tag{84}$$

$$\overrightarrow{\mathbf{B}}_i\left(\overrightarrow{r},t\right) = \left|\overrightarrow{\mathbf{B}}_i\right|e^{i\left(\overrightarrow{k}_i\cdot\overrightarrow{r}-\omega t\right)}\left[-\sin\theta_i\hat{\mathbf{x}}+\cos\theta_i\hat{\mathbf{y}}\right], \tag{85}$$

2. Reflected Wave:

$$\overrightarrow{\mathbf{E}}_r\left(\overrightarrow{r},t\right) = -\left|\overrightarrow{\mathbf{E}}_r\right|e^{i\left(\overrightarrow{k}_r\cdot\overrightarrow{r}-\omega t+\phi_r\right)}\hat{\mathbf{z}} \tag{86}$$

$$\overrightarrow{\mathbf{B}}_r\left(\overrightarrow{r},t\right) = -\left|\overrightarrow{\mathbf{B}}_r\right|e^{i\left(\overrightarrow{k}_r\cdot\overrightarrow{r}-\omega t+\phi_r\right)}\left[\sin\theta_r\hat{\mathbf{x}}+\cos\theta_r\hat{\mathbf{y}}\right], \tag{87}$$

3. Transmitted Wave:

$$\overrightarrow{\mathbf{E}}_t\left(\overrightarrow{r},t\right) = -\left|\overrightarrow{\mathbf{E}}_t\right|e^{i\left(\overrightarrow{k}_t\cdot\overrightarrow{r}-\omega t+\phi_t\right)}\hat{\mathbf{z}} \tag{88}$$

$$\overrightarrow{\mathbf{B}}_t\left(\overrightarrow{r},t\right) = \left|\overrightarrow{\mathbf{B}}_t\right|e^{i\left(\overrightarrow{k}_t\cdot\overrightarrow{r}-\omega t+\phi_t\right)}\left[-\sin\varphi\hat{\mathbf{x}}+\cos\varphi\hat{\mathbf{y}}\right], \tag{89}$$

Again, the frequency is the same and Snell's geometrical laws are valid. But note, however, that in this case the effect of discontinuity in the perpendicular component of the electric field is not important. That comes about from the fact that, by definition, one has $\overrightarrow{\mathbf{E}}\cdot\overrightarrow{\mathbf{k}}=0$. This means that we can write our version of the fields' boundary conditions for this polarization as follows:

$$\begin{cases} \left|\overrightarrow{\mathbf{E}}_i\right| + e^{i\phi_r}\left|\overrightarrow{\mathbf{E}}_r\right| - e^{i\phi_t}\left|\overrightarrow{\mathbf{E}}_t\right| & = 0 \\ \sin\theta\left|\overrightarrow{\mathbf{B}}_i\right| + \sin\theta e^{i\phi_r}\left|\overrightarrow{\mathbf{B}}_r\right| - \sin\varphi e^{i\phi_r}\left|\overrightarrow{\mathbf{B}}_r\right| & = 0 \\ \cos\theta\left|\overrightarrow{\mathbf{B}}_i\right| - \cos\theta e^{i\phi_r}\left|\overrightarrow{\mathbf{B}}_r\right| - \cos\varphi e^{i\phi_r}\left|\overrightarrow{\mathbf{B}}_t\right| + \mu_0 c\sigma_S e^{i\phi_t}\left|\overrightarrow{\mathbf{E}}_t\right| & = 0 \end{cases}$$

or, using the fact that $\left|\overrightarrow{\mathbf{B}}\right| = v_m^{-1}\left|\overrightarrow{\mathbf{E}}\right|$, we get to the following system:

$$\begin{cases} \left|\overrightarrow{\mathbf{E}}_i\right| + e^{i\phi_r}\left|\overrightarrow{\mathbf{E}}_r\right| - e^{i\phi_t}\left|\overrightarrow{\mathbf{E}}_t\right| & = 0 \\ \sin\theta\left|\overrightarrow{\mathbf{E}}_i\right| + \sin\theta e^{i\phi_r}\left|\overrightarrow{\mathbf{E}}_r\right| - \frac{n_2}{n_1}\sin\varphi e^{i\phi_r}\left|\overrightarrow{\mathbf{E}}_r\right| & = 0 \\ \cos\theta\left|\overrightarrow{\mathbf{E}}_i\right| - \cos\theta e^{i\phi_r}\left|\overrightarrow{\mathbf{E}}_r\right| - \left(\frac{n_2}{n_1}\cos\varphi e^{i\phi_r} + \frac{\mu_0 c\sigma_S}{n_1}e^{i\phi_t}\right)\left|\overrightarrow{\mathbf{E}}_t\right| & = 0 \end{cases}.$$

As a matrix system, we have the following result:

$$\begin{pmatrix} 1 & e^{i\phi_r} & -e^{i\phi_t} \\ \sin\theta & \sin\theta e^{i\phi_r} & -\frac{n_2}{n_1}\sin\varphi e^{i\phi_t} \\ \cos\theta & -\cos\theta e^{i\phi_r} & -\left(\frac{n_2}{n_1}\cos\varphi + \frac{\mu_0 c\sigma_S}{n_1}\right)e^{i\phi_t} \end{pmatrix} \cdot \begin{pmatrix} \left|\overrightarrow{\mathbf{E}}_i\right| \\ \left|\overrightarrow{\mathbf{E}}_r\right| \\ \left|\overrightarrow{\mathbf{E}}_t\right| \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{90}$$

which only yields a non-zero solution if the secular determinant is non-zero, i.e. if

$$\frac{e^{i\phi_t}\left(n_1\sin\theta - n_2\sin\varphi\right)\left(\xi n_1\left(-1+e^{i\phi_r}\right)e^{i\phi_t} - 2n_2\cos\varphi e^{i\phi_r}\right)}{n_1^2} = 0.$$

This equation is automatically satisfied if Snell's Law of transmission is obeyed — $n_1\sin\theta = n_2\sin\varphi$. So, in principle, we have transmission at all angles.

Now, we can we the two independent equations,

$$\begin{cases} \left|\overrightarrow{\mathbf{E}}_i\right| + e^{i\phi_r}\left|\overrightarrow{\mathbf{E}}_r\right| - e^{i\phi_t}\left|\overrightarrow{\mathbf{E}}_t\right| & = 0 \\ \cos\theta\left|\overrightarrow{\mathbf{E}}_i\right| - \cos\theta e^{i\phi_r}\left|\overrightarrow{\mathbf{E}}_r\right| - \left(\frac{n_2}{n_1}\cos\varphi e^{i\phi_r} + \frac{\mu_0 c\sigma_S}{n_1}e^{i\phi_t}\right)\left|\overrightarrow{\mathbf{E}}_t\right| & = 0 \end{cases}, \tag{91}$$

in order to find the conditions we are after, i.e. the transmittance, absorbance and reflectance. If we start by doing $eq_2 + eq_1\times\cos\theta$, we get

$$\cos\theta\left|\overrightarrow{\mathbf{E}}_i\right| - \frac{1}{2}\left(\cos\theta + \frac{n_2}{n_1}\cos\varphi + \frac{\mu_0 c\sigma_S}{n_1}\right)e^{i\phi_t}\left|\overrightarrow{\mathbf{E}}_t\right| = 0$$

thus yielding the following ratio of complex amplitudes

$$\frac{\left|\overrightarrow{\mathbf{E}}_t\right|e^{i\phi_t}}{\left|\overrightarrow{\mathbf{E}}_i\right|} = \frac{2n_1\cos\theta}{n_1\cos\theta + n_2\cos\varphi + \mu_0 c\sigma_S}. \tag{92}$$

Replacing Eq. (92) in the first line of the system in Eq. 91 we get to the following result:

$$\left(\frac{n_2\cos\varphi + \mu_0 c\sigma_S - n_1\cos\theta}{n_1\cos\theta + n_2\cos\varphi + \mu_0 c\sigma_S}\right)\left|\overrightarrow{\mathbf{E}}_i\right| + e^{i\phi_r}\left|\overrightarrow{\mathbf{E}}_r\right| = 0,$$

yielding the following ratio

$$\frac{\left|\overrightarrow{\mathbf{E}}_r\right| e^{i\phi_r}}{\left|\overrightarrow{\mathbf{E}}_i\right|} = \frac{n_2 \cos\varphi - n_1 \cos\theta + \mu_0 c\sigma_S}{n_1 \cos\theta + n_2 \cos\varphi + \mu_0 c\sigma_S}. \tag{93}$$

Once again, we can write the corresponding transmittance, reflectance and absorbance across the surface, as a function of the incidence angle, for a single layer with a complex conductivity, $\sigma_S = \sigma_r + i\sigma_i$. This yields the following results:

$$\mathcal{T}_{\mathrm{TE}}(\theta) = \frac{4n_1 n_2 \cos\theta \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta}}{c^2\mu_0^2\sigma_i^2 + \left(c\mu_0\sigma_r + n_2\cos\theta + n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\right)^2} \tag{94}$$

and

$$\mathcal{R}_{\mathrm{TE}}(\theta) = \frac{c^2\mu_0^2\sigma_i^2 + \left(c\mu_0\sigma_r + n_2\cos\theta - n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\right)^2}{c^2\mu_0^2\sigma_i^2 + \left(c\mu_0\sigma_r + n_2\cos\theta + n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\right)^2}, \tag{95}$$

which yield an absorbed energy per unit area in the graphene sheet of

$$\begin{aligned}\mathcal{A}(\theta) = 4c\mu_0 n_1\sigma_r\cos\theta & \left(c^2\mu_0^2\left(\sigma_i^2+\sigma_r^2\right) + 2n_1\cos\theta\left(c\mu_0\sigma_r + n_2\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\right)\right. \\ & \left. +2c\mu_0 n_2\sigma_r\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta} + n_1^2\cos^2\theta + n_2^2\left(1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta\right)\right)^{-1}\end{aligned} \tag{96}$$

which obviously goes to zero in the limit when $\sigma_r \to 0$. In Fig. 5, we depict $\mathcal{T}_{\mathrm{TE}}(\theta)$,$\mathcal{R}_{\mathrm{TE}}(\theta)$ and $\mathcal{A}_{\mathrm{TE}}(\theta)$ for a polarized wave with $\lambda = 500$nm incident on a glass-to-air and air-to-glass interface. Additionally, one can also check that they reduce to Eqs. (80)-(81) in the limit of normal incidence, as they should. At normal incidence, there is no distinction between a TE and a TM polarized wave.

3.4 Total Reflection of Transverse Electric Waves

[TO BE WRITTEN...]

3.5 Transmission of Non-polarized Light

In some experiments, the incident light beam is not polarized. Instead, the polarization of the wave changes randomly in time, meaning that one will actually observe an average of the transmittances/reflectances for each of the polarizations analyzed above. This means that the effective values are as follows:

$$\mathcal{R}_{\text{eff}}(\theta) = \frac{1}{2}\left(\mathcal{R}_{\text{TM}}(\theta) + \mathcal{R}_{\text{TE}}(\theta)\right) =$$

$$= \frac{1}{2}\left(1 + \frac{c^2\mu_0^2\sigma_i^2 + \left(c\mu_0\sigma_r + n_2\cos\theta - n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\right)^2}{c^2\mu_0^2\sigma_i^2 + \left(c\mu_0\sigma_r + n_2\cos\theta + n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\right)^2}\right.$$
$$\left. - \frac{4n_1\cos\theta\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\,(c\mu_0\sigma_r + n_2)}{\left(1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta\right)\left(c^2\mu_0^2\sigma_i^2 + (c\mu_0\sigma_r + n_2)^2\right) + 2n_1\cos\theta\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\,(c\mu_0\sigma_r + n_2) + n_1^2\cos^2\theta}\right) \tag{97}$$

and,

$$\mathcal{T}_{\text{eff}}(\theta) = \frac{1}{2}\left(\mathcal{T}_{\text{TM}}(\theta) + \mathcal{T}_{\text{TE}}(\theta)\right) =$$

$$= \frac{2n_1n_2\left(1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta\right)}{c^2\mu_0^2\sigma_i^2\left(1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta\right) + \left(\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\,(c\mu_0\sigma_r + n_2) + n_1\cos\theta\right)^2}$$
$$+ \frac{2n_1n_2\left(1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta\right)}{c^2\mu_0^2\sigma_i^2 + \left(c\mu_0\sigma_r + n_2\cos\theta + n_1\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta}\right)^2}. \tag{98}$$

Plots of these two quantities are shown in Fig. 5, as a function of the angle, for a glass-to-air and air-to-glass interfaces. These are then compared to the corresponding curves in the case of a bare interface, with no graphene deposited.

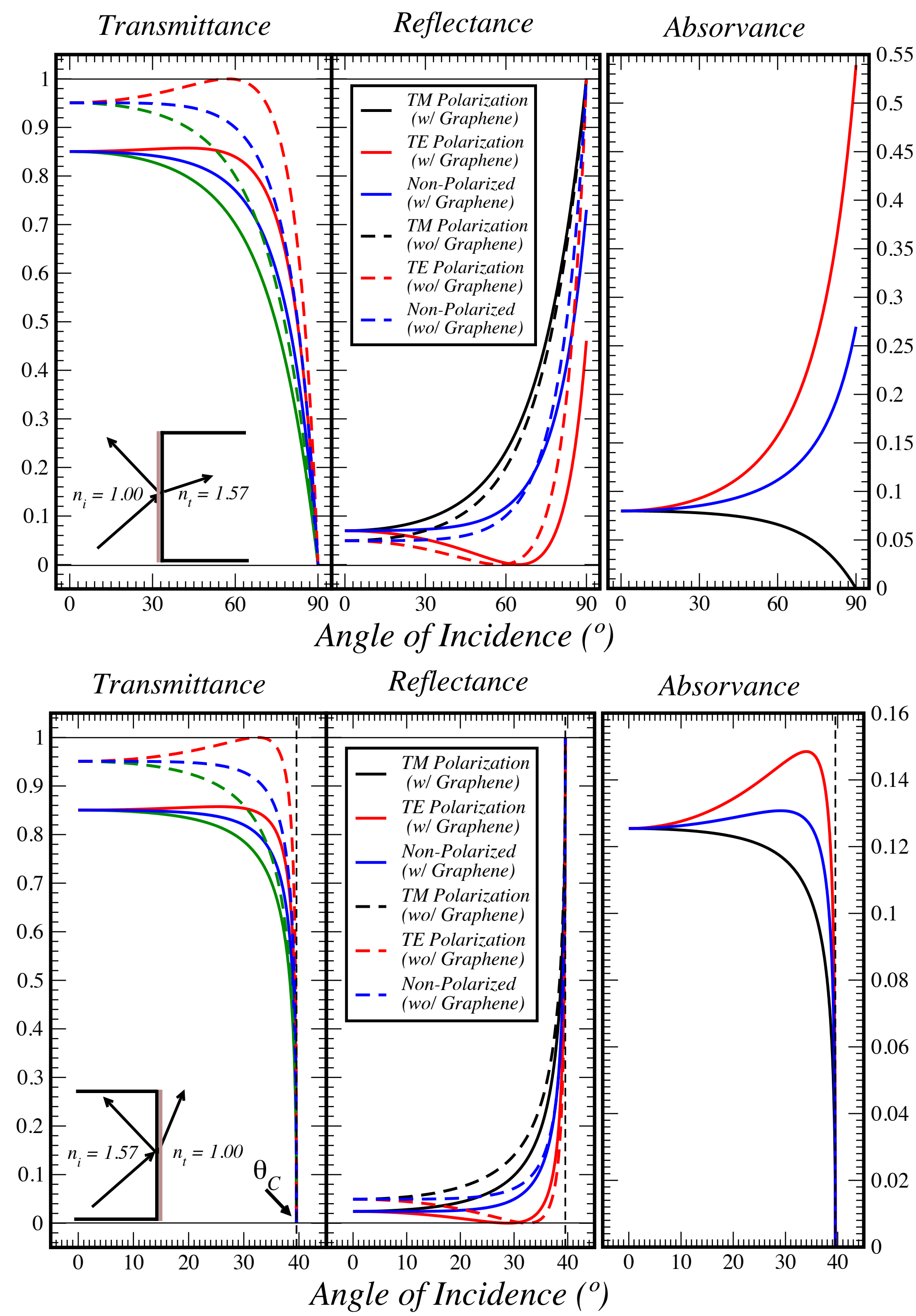

Figure 5: Plots of the Transmittance, Reflectance and Fraction of absorbed intensity for a wave incident on an interface between air and a glass ($n = 1.57$). We compare the curves obtained with and without a pristine graphene sheet deposed on the surface of the glass. We considered a negative chemical potential of $\mu = -0.2$eV and a temperature of $T = 300K$. The top panels correspond to an air-to-glass interface, while the bottom ones show the opposite incidence (see schemes in the corners).

4 Transfer-Matrix Method for Normal Incidence with Conducting Interfaces

Till now, we considered only the transmission of electromagnetic waves across a single graphene layer placed on top of a glass surface. However, in many experiments, one is interested in the case where several layers of graphene, separated by dielectric media, scatter the incident wave. Treating this problem may be done easily by a transfer-matrix method. In a first approach, we describe the case when all the layers of graphene are parallel to each other and the incidence of EM radiation is normal to those surfaces. This situation does not distinguish between TM and TE polarizations and is depicted in Fig. 6.

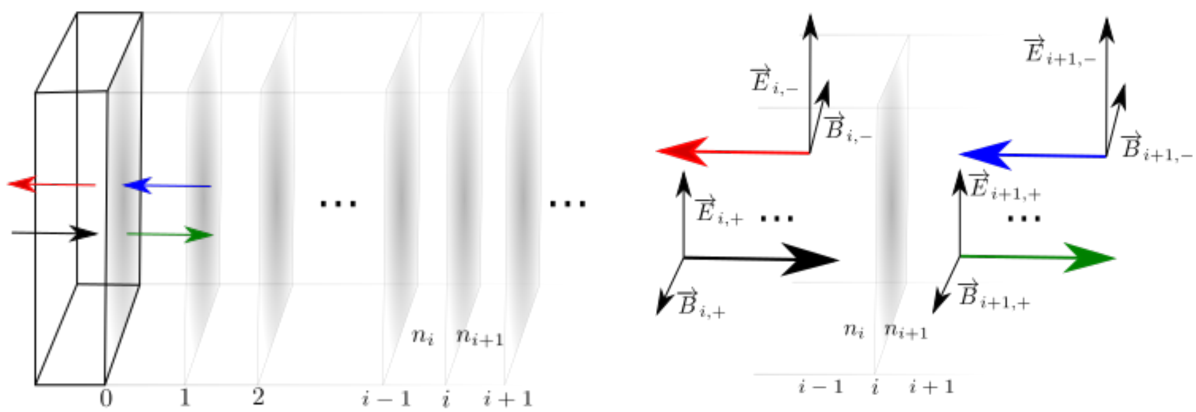

Figure 6: Left Panel: Scheme of the Transfer Matrix Method (TMM) for a layered structure of parallel graphene single-layers separated by dielectric media. Right Panel: Conventions used to define the transfer-matrix for the electric and magnetic fields at a single graphene interface. Only normal incidence is considered.

4.1 General Theory

Making use of the boundary conditions defined earlier for the electric and magnetic fields at an interface between two dielectric media of refractive indexes n_i and n_{i+1}, with a graphene sheet deposed in between (as shown in the right panel of Fig. 6), we obtain the following system of equations for the (complex) amplitudes at the boundary:

$$\begin{cases} E^0_{i,+} + E^0_{i,-} = E^0_{i+1,+} + E^0_{i+1,-} \\ B^0_{i,+} - B^0_{i,-} = B^0_{i+1,+} - B^0_{i+1,-} + \mu_0 \sigma_S \left(E^0_{i,+} + E^0_{i,-}\right) \end{cases} . \tag{99}$$

Additionally, we know that there are relations between the amplitudes of the electric and magnetic field components of a linearly polarized plane-wave propagating inside a dielectric medium. More precisely, with the conventions define in the right panel of Fig. 6:

$$\begin{cases} B_{i,+} = \frac{n_i}{c} E_{i,+} \\ B_{i,-} = \frac{n_i}{c} E_{i,-} \end{cases} ,$$

which allows for the following generalization of the system in Eq. (99):

$$\begin{cases} E^0_{i,+} + E^0_{i,-} = E^0_{i+1,+} + E^0_{i+1,-} \\ E^0_{i,+} - E^0_{i,-} = \frac{n_{i+1}}{n_i} \left(E^0_{i+1,+} - E^0_{i+1,-}\right) + \frac{\mu_0 c \sigma_S}{n_i} \left(E^0_{i,+} + E^0_{i,-}\right) \\ B^0_{i,+} + B^0_{i,-} = \frac{n_i}{n_{i+1}} \left(B^0_{i+1,+} + B^0_{i+1,-}\right) \\ B^0_{i,+} - B^0_{i,-} = B^0_{i+1,+} - B^0_{i+1,-} + \frac{\mu_0 c \sigma_S}{n_i} \left(B^0_{i,+} + B^0_{i,-}\right) \end{cases} , \tag{100}$$

thus yielding, after some trivial algebra, the following system of equations

$$\begin{cases} E^0_{i,+} + E^0_{i,-} = E^0_{i+1,+} + E^0_{i+1,-} \\ \left(1 - \frac{\mu_0 c \sigma_S}{n_i}\right) E^0_{i,+} - \left(1 + \frac{\mu_0 c \sigma_S}{n_i}\right) E^0_{i,-} = \frac{n_{i+1}}{n_i} E^0_{i+1,+} - \frac{n_{i+1}}{n_i} E^0_{i+1,-} \\ B^0_{i,+} + B^0_{i,-} = \frac{n_i}{n_{i+1}} B^0_{i+1,+} + \frac{n_i}{n_{i+1}} B^0_{i+1,-} \\ \left(1 - \frac{\mu_0 c \sigma_S}{n_i}\right) B^0_{i,+} - \left(1 + \frac{\mu_0 c \sigma_S}{n_i}\right) B^0_{i,-} = B^0_{i+1,+} - B^0_{i+1,-} \end{cases}, \tag{101}$$

or, in matrix form,

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ \left(1 - \frac{\xi}{n_i}\right) & -\left(1 + \frac{\xi}{n_i}\right) & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \left(1 - \frac{\xi}{n_i}\right) & -\left(1 + \frac{\xi}{n_i}\right) \end{pmatrix}}_{\mathcal{M}_1} \cdot \begin{pmatrix} E^0_{i,+} \\ E^0_{i,-} \\ B^0_{i,+} \\ B^0_{i,-} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ \frac{n_{i+1}}{n_i} & -\frac{n_{i+1}}{n_i} & 0 & 0 \\ 0 & 0 & \frac{n_i}{n_{i+1}} & \frac{n_i}{n_{i+1}} \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{\mathcal{M}_2} \cdot \begin{pmatrix} E^0_{i+1,+} \\ E^0_{i+1,-} \\ B^0_{i+1,+} \\ B^0_{i+1,-} \end{pmatrix}, \tag{102}$$

where $\xi = \mu_0 c \sigma_S$. The matrix $\mathcal{M}_2$ can be explicitly inverted, since $\det(\mathcal{M}_2) = 4n_i/n_{i+1}$. This yields,

$$\mathcal{M}_2^{-1} = \frac{1}{2} \begin{pmatrix} 1 & \frac{n_{i+1}}{n_i} & 0 & 0 \\ 1 & -\frac{n_{i+1}}{n_i} & 0 & 0 \\ 0 & 0 & \frac{n_i}{n_{i+1}} & 1 \\ 0 & 0 & \frac{n_i}{n_{i+1}} & -1 \end{pmatrix}$$

and, multiplying both sides of Eq. (102) by $\mathcal{M}_2^{-1}$, we arrive at the following recursion relation for the complex field amplitudes:

$$\begin{pmatrix} E^0_{i+1,+} \\ E^0_{i+1,-} \\ B^0_{i+1,+} \\ B^0_{i+1,-} \end{pmatrix} = \frac{1}{2} \underbrace{\begin{pmatrix} 1 + \frac{n_i}{n_{i+1}}\left(1 - \frac{\xi}{n_i}\right) & 1 - \frac{n_i}{n_{i+1}}\left(1 + \frac{\xi}{n_i}\right) & 0 & 0 \\ 1 - \frac{n_i}{n_{i+1}}\left(1 - \frac{\xi}{n_i}\right) & 1 + \frac{n_i}{n_{i+1}}\left(1 + \frac{\xi}{n_i}\right) & 0 & 0 \\ 0 & 0 & \frac{n_{i+1}}{n_i} + \left(\frac{\xi}{n_i} + 1\right) & \frac{n_{i+1}}{n_i} - \left(1 + \frac{\xi}{n_i}\right) \\ 0 & 0 & \frac{n_{i+1}}{n_i} - \left(1 + \frac{\xi}{n_i}\right) & \frac{n_{i+1}}{n_i} + \left(\frac{\xi}{n_i} + 1\right) \end{pmatrix}}_{\mathcal{T}_G(n_i,,n_{i+1},\xi)} \cdot \begin{pmatrix} E^0_{i,+} \\ E^0_{i,-} \\ B^0_{i,+} \\ B^0_{i,-} \end{pmatrix}. \tag{103}$$

The matrix $\mathcal{T}_G$ is almost the transfer matrix for a "chunk" of material. However, this "chunk" is not composed only by the infinitesimal width of graphene which established the boundary conditions. Before that there is a finite width D_i of dielectric material which leads to the accumulation of a complex phase on the fields of the two counter-propagating EM plane-waves. The phase accumulated during this optical path length can be described by the following matrix:

$$\mathcal{T}_\phi(n_i, D_i, \omega) = \begin{pmatrix} e^{i n_i D_i \omega c^{-1}} & 0 & 0 & 0 \\ 0 & e^{-i n_i D_i \omega c^{-1}} & 0 & 0 \\ 0 & 0 & e^{i n_i D_i \omega c^{-1}} & 0 \\ 0 & 0 & 0 & e^{-i n_i D_i \omega c^{-1}} \end{pmatrix},$$

which allows us to write the following transfer-matrix for the chunk i:

$$\mathcal{T}_i(n_i,, n_{i+1}, \xi, D_i, \omega) = \mathcal{T}_G(n_i,, n_{i+1}, \xi) \cdot \mathcal{T}_\phi(n_i, D_i, \omega),$$

thus yielding our final recursion relation:

$$\begin{pmatrix} E^0_{i+1,+} \\ E^0_{i+1,-} \\ B^0_{i+1,+} \\ B^0_{i+1,-} \end{pmatrix} = \underbrace{\frac{e^{\frac{i\omega D_i n_i}{c}}}{2} \begin{pmatrix} \left(1+\left(1-\frac{\xi}{n_i}\right)\frac{n_i}{n_{i+1}}\right) & \left(1-(1+\frac{\xi}{n_i})\frac{n_i}{n_{i+1}}\right)e^{-\frac{2i\omega D_i n_i}{c}} & 0 & 0 \\ \left(1-\left(1-\frac{\xi}{n_i}\right)\frac{n_i}{n_{i+1}}\right) & \left(1+\left(1+\frac{\xi}{n_i}\right)\frac{n_i}{n_{i+1}}\right)e^{-\frac{2i\omega D_i n_i}{c}} & 0 & 0 \\ 0 & 0 & \left(\frac{n_{i+1}}{n_i}+(1+\xi)\right) & \left(\frac{n_{i+1}}{n_i}-(1+\xi)\right)e^{-\frac{2i\omega D_i n_i}{c}} \\ 0 & 0 & \left(\frac{n_{i+1}}{n_i}-(1+\xi)\right) & \left(\frac{n_{i+1}}{n_i}+(1+\xi)\right)e^{-\frac{2i\omega D_i n_i}{c}} \end{pmatrix}}_{\mathcal{T}_i\left(n_i,,n_{i+1},\xi,D_i,\omega\right)} \cdot \begin{pmatrix} E^0_{i,+} \\ E^0_{i,-} \\ B^0_{i,+} \\ B^0_{i,-} \end{pmatrix}.$$

Note that $\mathcal{T} \to \mathcal{T}_G$ in the limit when $D_i \to 0$, as it should.

If we have a layered structure like the one in the left panel of Fig. 6, we can relate the amplitudes of the fields in the two ends by successively applying $\mathcal{T}_i = \mathcal{T}\left(n_i,,n_{i+1},\xi,D_i,\omega\right)$. More precisely, we have the following relation:

$$\begin{pmatrix} E^0_{out,+} \\ 0 \\ B^0_{out,+} \\ 0 \end{pmatrix} = \underbrace{\mathcal{T}\left(n_{out},,n_{N-1},\xi,D_{N-1},\omega\right) \cdot \mathcal{T}_{N-2} \cdots \mathcal{T}_1 \cdot \mathcal{T}_G\left(n_{in},n_{i+1},\xi\right)}_{\mathcal{T}_T} \cdot \begin{pmatrix} E^0_{in,+} \\ E^0_{in,-} \\ B^0_{in,+} \\ B^0_{in,-} \end{pmatrix},$$

where N is the total number of Graphene layers deposed and $\mathcal{T}_T$ is the total transfer-matrix of the layered system. This is a 4×4 matrix, whose elements can be related to the (complex) transmission and reflection coefficients of the whole system. In general, we have

$$\begin{pmatrix} E^0_{out,+} \\ 0 \\ B^0_{out,+} \\ 0 \end{pmatrix} = \begin{pmatrix} t^T_{11} & t^T_{12} & t^T_{13} & t^T_{14} \\ t^T_{21} & t^T_{22} & t^T_{23} & t^T_{24} \\ t^T_{31} & t^T_{32} & t^T_{33} & t^T_{34} \\ t^T_{41} & t^T_{42} & t^T_{43} & t^T_{44} \end{pmatrix} \cdot \begin{pmatrix} E^0_{in,+} \\ E^0_{in,-} \\ B^0_{in,+} \\ B^0_{in,-} \end{pmatrix}, \tag{104}$$

where we assumed, as a boundary condition at the last layer, that there is only a forward propagating wave. Despite its messy structure, we see that Eq. (104), due to the block structure of the matrices involved, has the simpler structure of Eq. (105):

$$\begin{pmatrix} E^0_{out,+} \\ 0 \\ B^0_{out,+} \\ 0 \end{pmatrix} = \begin{pmatrix} t^T_{11} & t^T_{12} & 0 & 0 \\ t^T_{21} & t^T_{22} & 0 & 0 \\ 0 & 0 & t^T_{33} & t^T_{34} \\ 0 & 0 & t^T_{43} & t^T_{44} \end{pmatrix} \cdot \begin{pmatrix} E^0_{in,+} \\ E^0_{in,-} \\ B^0_{in,+} \\ B^0_{in,-} \end{pmatrix}, \tag{105}$$

and hence the following useful relations emerge:

$$\begin{cases} E^0_{out,+} = t^T_{11} E^0_{in,+} + t^T_{12} E^0_{in,-} \\ t^T_{21} E^0_{in,+} + t^T_{22} E^0_{in,-} = 0 \end{cases}. \tag{106}$$

These two equations imply that the reflection and transmission coefficients of the whole system are as follows:

$$\begin{cases} r = \frac{E^0_{in,-}}{E^0_{in,+}} = -\frac{t^T_{21}}{t^T_{22}} \\ t = \frac{E^0_{out,+}}{E^0_{in,+}} = t^T_{11} - \frac{t^T_{12} t^T_{21}}{t^T_{22}} \end{cases}$$

and, correspondingly, the reflectance, transmittance and absorbance of the whole system are

$$\begin{cases} \mathcal{R}_T = |r|^2 = \frac{\left|t^T_{21}\right|^2}{\left|t^T_{22}\right|^2} \\ \mathcal{T}_T = \frac{n_{out}}{n_{in}} |t|^2 = \frac{n_{out}}{n_{in}} \left|t^T_{11} - \frac{t^T_{12} t^T_{21}}{t^T_{22}}\right|^2 \\ \mathcal{A}_T = 1 - \mathcal{R}_T - \mathcal{T}_T \end{cases},$$

which is all we wish to know about the system in question.

4.2 Numerical Implementation and Results

We built the entire transfer-matrix theory having 4×4 matrices as the main ingredients. However, we note that both $\mathcal{T}_G$ and $\mathcal{T}_i$ are bock diagonal matrices with 2×2 blocks in either the electric and magnetic sectors. This fact indicates that, for normal incidence, it is redundant to care about both sectors and one can use only the electric field equations, i.e.

$$\begin{pmatrix} E^0_{i+1,+} \\ E^0_{i+1,-} \end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix} \left(1+\left(1-\frac{\xi(\omega)}{n_i}\right)\frac{n_i}{n_{i+1}}\right)e^{\frac{i\omega D_i n_i}{c}} & \left(1-\left(1+\frac{\xi(\omega)}{n_i}\right)\frac{n_i}{n_{i+1}}\right)e^{-\frac{i\omega D_i n_i}{c}} \\ \left(1-\left(1-\frac{\xi(\omega)}{n_i}\right)\frac{n_i}{n_{i+1}}\right)e^{\frac{i\omega D_i n_i}{c}} & \left(1+\left(1+\frac{\xi(\omega)}{n_i}\right)\frac{n_i}{n_{i+1}}\right)e^{-\frac{i\omega D_i n_i}{c}} \end{pmatrix}}_{\mathbb{T}_i(n_i,n_{i+1},D_i,\omega)} \cdot \begin{pmatrix} E^0_{i,+} \\ E^0_{i,-} \end{pmatrix},$$

and consequently

$$\mathbb{T}_T = \mathbb{T}_{N-1}\left(n_{out}, n_{N-1}, D_{N-1}, \omega\right) \cdot \mathbb{T}_{N-2}\left(n_{N-1}, n_{N-2}, D_{N-2}, \omega\right) \cdots \mathbb{T}_1\left(n_2, n_1, D_1, \omega\right) \cdot \mathbb{T}_G\left(n_{in}, n_1, \omega\right),$$

where

$$\mathbb{T}_G\left(n_{in}, n_1, \omega\right) = \frac{1}{2}\begin{pmatrix} 1+\frac{n_i}{n_{i+1}}\left(1-\frac{\xi}{n_i}\right) & 1-\frac{n_i}{n_{i+1}}\left(1+\frac{\xi}{n_i}\right) \\ 1-\frac{n_i}{n_{i+1}}\left(1-\frac{\xi}{n_i}\right) & 1+\frac{n_i}{n_{i+1}}\left(1+\frac{\xi}{n_i}\right) \end{pmatrix} \tag{107}$$

and, again, $\xi(\omega) = \mu_0 c\left(\sigma_r(\omega) + i\sigma_i(\omega)\right)$. Note that, in this formalism, the reflection coefficient of a single layer of graphene, placed in between two dielectrics n_1 and n_2, is simply

$$r = \frac{n_2 - n_1 - \frac{n_2}{n_1}\mu_0 c\left[\sigma_r(\omega) + i\sigma_i(\omega)\right]}{n_2 + n_1 + \frac{n_2}{n_1}\mu_0 c\left[\sigma_r(\omega) + i\sigma_i(\omega)\right]}$$

and the transmission coefficients

$$t = \frac{1}{2}\left\{1 + \frac{n_2}{n_1}\left(1 - \frac{\xi(\omega)}{n_1}\right) - \frac{\left[1-\frac{n_2}{n_1}\left(1-\frac{\xi(\omega)}{n_1}\right)\right]\left[1-\frac{n_2}{n_1}\left(1+\frac{\xi(\omega)}{n_1}\right)\right]}{1+\frac{n_2}{n_1}\left(1+\frac{\xi(\omega)}{n_1}\right)}\right\},$$

which reduce to the coefficients calculated previously in the case of a single graphene sheet and an arbitrary angle of incidence. We can now implement the numerical calculation of $\mathbb{T}_T$ for an arbitrary system. For this we only need to provide the following information to the computer:

1. Index of refraction of the incidence and transmission media — n_{in} and n_{out};
2. Complex conductivity of the graphene sheets (which is entirely determined by ω);
3. The widths (D_i) and indexes of refraction (n_i) of all the dielectric separators.

Such situation is schematically represented in Fig. , for different situations. Namely, we consider both the cases in which there is a fixed distance between the graphene sheets in the layered structure and also the

5 Transfer-Matrix Method for Oblique Incidence with Conducting Interfaces

6 Field Reconstruction

6.1 Method

Employing the afore constructed machinery, the system

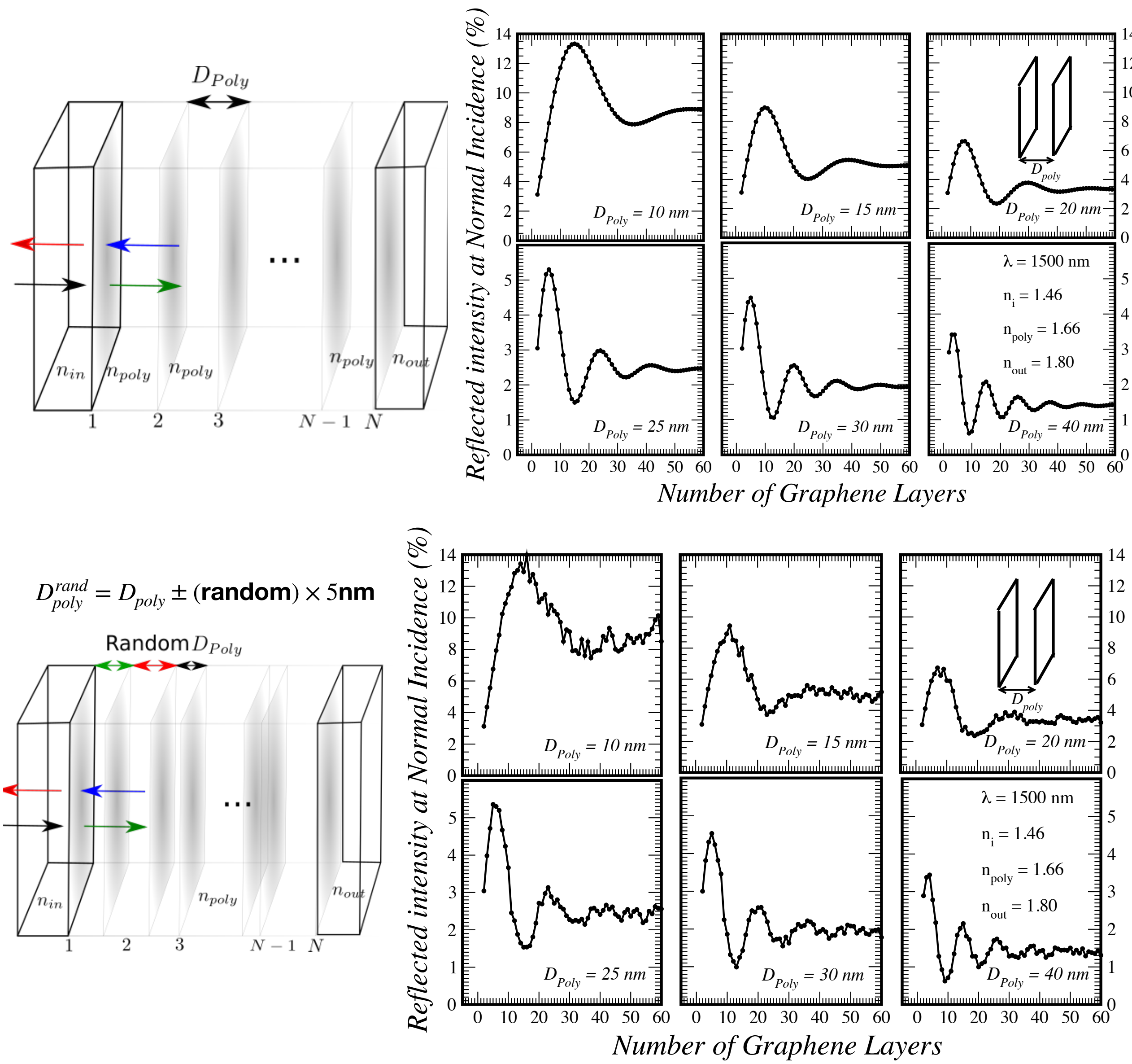

Figure 7: Plots of the percentage of reflected intensity by the whole structure, as a function of the number of deposed graphene layers. In the upper panel, we represent this quantity for different (fixed) values for the distance between graphene planes, ranging from 10nm to 40nm. In the lower panel, we represent the same quantity but with a random realization of distances between planes, with a symmetric tolerance of 10nm around the average D_{poly}.

$$\begin{pmatrix} E_{out,+} \\ 0 \end{pmatrix} = \mathcal{T}_T \begin{pmatrix} E_{in,+} \\ E_{in,-} \end{pmatrix}$$

when normalized to the incident electric field becomes

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = \mathcal{T}_T \begin{pmatrix} 1 \\ r \end{pmatrix}.$$

Knowing the r coefficient we must only successively apply matrix equilvalent to each layer to the vector $\begin{pmatrix} 1 \\ r \end{pmatrix}$ to construct the field at each layer of the system. Notice that to achieve this we must build the system twice, once to discover r, and then again to construct the field given the initial conditions.

6.2 Results

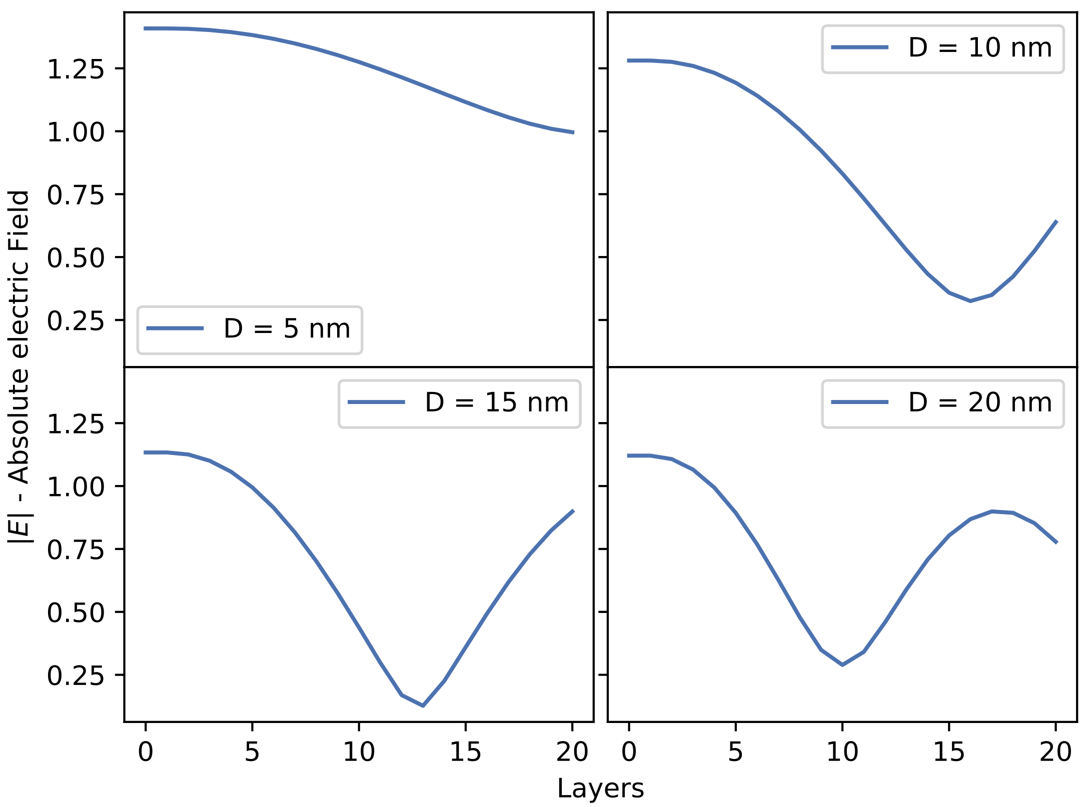

Figure 8: Absolute value of electric field at each layer of a glass-polymer-air 10 graphene layer system with refrative index $n = 1.6$ and wavelength $\lambda = 1000$ nm, for polymer widths of $d \in \{5, 10, 15, 20\}$ nm.

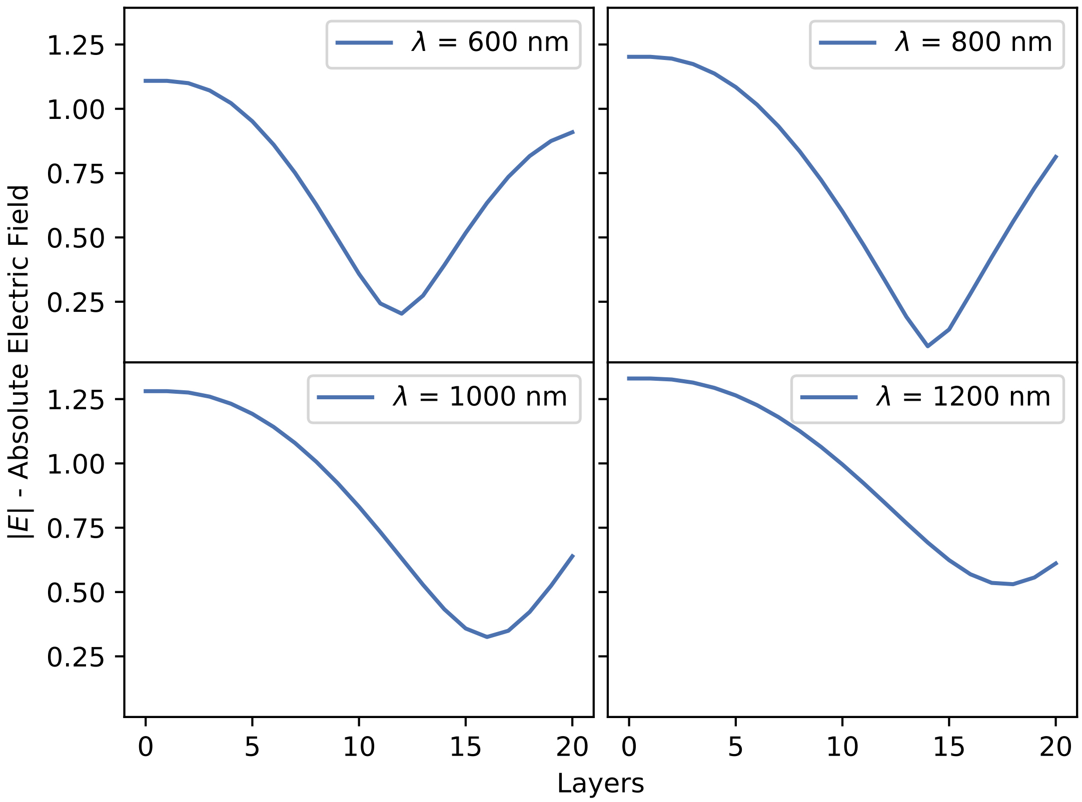

Figure 9: Absolute value of electric field at each layer of a glass-polymer-air 10 graphene layer system with refrative index $n = 1.6$ and polymer width $d = 10$ nm, for wavelengths of $\lambda \in \{600, 800, 1000, 1200\}$ nm.

7 The two layer case - analysis

In the simple case of two graphene layers, we have the expression for the reflected intensity

References

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