Notes on Fluids and Plasmas in Astrophysics

1 Lecture 1 - 17/09/2019

A fluid

- flows (yes... indeed)
- is typically a liquid or a gas
- is composed of electrically neutral atoms or molecules.

The fluid model of a system is a macroscopic description of a system as a manifold in which macroscopic forces act.

The fluid description of a system is a valid one if the mean free path of it's constituents is much smaller than the characteristic length of the system.

$$\lambda_c << L$$

2 Lecture 2 - 19/09/2019

Most fluids present some resistance to stress and are therefore viscous. If this does not happen that fluid is called an **ideal fluid**.

2.1 Derivatives

We start by establishing some notation.

2.2 Eulerian Derivative

The Eulerian derivative describe the change of a variable with respect to time for a fixed point in space. It can be identified as a partial time derivative.

2.3 Lagrangian Derivative

The Lagrangian derivative describes the change of a variable with respect to both physical and temporal displacement. It can be identified with a total derivative.

$$\mathrm{d}Q = \frac{\partial Q}{\partial t}\mathrm{d}t + \frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z = \frac{\partial Q}{\partial t}\mathrm{d}t + (\mathrm{d}\vec{r} \cdot \nabla)Q$$

In particular, if one is describing a change in velocity, one may write

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}$$

2.4 Conservation Laws

Let V be an element of fluid limited by a surface S. The flux through an oriented element of surface $\mathrm{d}\vec{S}$ is given by

$$\frac{\partial}{\partial t}\int_V \rho \mathrm{d}V = -\oint_S \rho\vec{v}\cdot\vec{n}\mathrm{d}S$$

By the divergence theorem we have

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right]\mathrm{d}V = 0$$

and follows the continuity equation

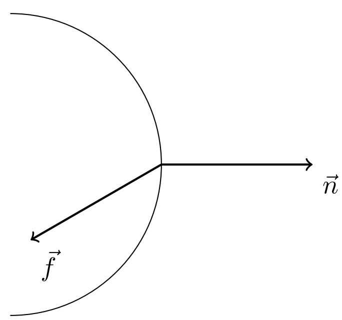
$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \tag{1}$$

2.5 Equation of Motion

Let δV be an element of fluid of mass $\rho\delta V$. From Newton's second law

$$\rho\delta V\frac{\partial\vec{v}}{\partial t} = \delta\vec{F}_{volume} + \delta\vec{F}_{\text{surface}}$$

Let $\vec{f}$ be a volumetric force density per unit mass (also called acceleration density). We have the relation $\delta\vec{F}_{\text{volume}} = \rho\delta V\vec{f}$. It is then natural to define a **total pressure** tensor P_{ij} [1] that can be interpreted as the component i of the pressure exerted in a surface of normal vector component j.



Then we have

$$\vec{f}_{\text{surface}} = \int_S \mathrm{d}\vec{f}_{\text{surface}} = \oint_S P_{ij}\mathrm{d}S_j$$

Applying the diverge theorem in the right side

$$\oint_S P_{ij}\mathrm{d}S_i = \int_V \frac{\partial P_{ij}}{\partial x_j}\mathrm{d}V_i$$

As the tensor is defined positively in the outwards direction, we add a negative sign for convenience. We then get the relation

$$\rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \rho\vec{f} - \nabla\cdot p \tag{2}$$

[1] We study fluids exclusively in euclidean geometry, that means $g_{\mu\nu} = \delta_{\mu\nu}$. To keep the notes as comprehensible as possible, we make no distinction between covariant and contravariant entities, and all indices will be written as subscripts.

3 Lecture 3 - 24/09/2019

3.1 Fluid in Static Equilibrium

For a fluid in static equilibrium ($v = 0$), the force that acts on an element of are is perpendicular to it's normal vector. That is

$$P_{ij} = p\delta_{ij} \tag{3}$$

We may now give an alternative definition of a fluid, as a substance for which movement is induced whenever a superficial force with an tangential component (also called a *shear force*) is exerted.

In this case, from equation 2 follows

$$\mathrm{d}F_{\text{sup}} = -p\,\mathrm{d}S$$

However, one may not forget that the relation 3 is no longer true for a fluid in motion. One should expect a shear force to carry momentum from the faster layers of fluid to the slower ones. These shear forces are described expressed through a **viscosity coefficient**. In other words, equation 3 is only true for non-viscous (ideal) fluids.

3.2 Ideal Fluids

For an ideal fluid the equation of motion reduces to

$$\rho\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\nabla p + \rho\vec{f}$$

which may be rewritten as

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla p + \rho\vec{f} \tag{4}$$

3.3 Equation of Energy

Remembering the **First Law of Thermodynamics**

$$\mathrm{d}Q = \mathrm{d}U + p\mathrm{d}V \tag{5}$$

we aim to apply it to a fluid in motion.

We start by noticing that a sufficiently small element of fluid is trivially in thermodynamic equilibrium. Let δm be the mass of an element of fluid. Let also $\mathrm{d}q = \frac{\mathrm{d}Q}{\delta m}$ and $\mathrm{d}\epsilon = \frac{\mathrm{d}u}{\delta m}$. Then

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon}{\mathrm{d}t} + p\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right)$$

From equation 1

$$p\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right) = -\frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{p}{\rho}\nabla\cdot\vec{v}$$

Substituting

$$\rho\frac{\mathrm{d}\epsilon}{\mathrm{d}t} + p\nabla\cdot\vec{v} = \rho\frac{\mathrm{d}q}{\mathrm{d}t}$$

If one defines

$$-\mathcal{L} = \rho\frac{\mathrm{d}q}{\mathrm{d}t} \tag{6}$$

the relation simplifies to

$$\rho\frac{\mathrm{d}\epsilon}{\mathrm{d}t} + p\nabla\cdot\vec{v} = -\mathcal{L} \tag{7}$$

If the variations of heat are a case of simply **heat flux**

$$\mathcal{F} = -k\nabla T$$

with k being the thermal conductivity coefficient. $\mathcal{F}$ is related to $\mathcal{L}$ through

$$\mathcal{L} = \nabla\cdot\mathcal{F} = -\nabla\cdot(k\nabla T)$$

Substituting in equation 7

$$\rho\left(\frac{\partial\epsilon}{\partial t} + (\vec{v}\cdot\nabla)\epsilon\right) + p\nabla\cdot\vec{v} - \nabla\cdot(k\nabla T) = 0$$

Other phenomena may contribute to $\mathcal{L}$ with, for example

- gain of heat by viscous dissipation of movement
- radiation
- convection

3.4 Euler's Equation in Conservative Form

In component form equation 1 and 2 become

$$v_i\frac{\partial\rho}{\partial t} = -v_i\frac{\partial}{\partial x_j}(\rho v_j)$$

$$\rho\frac{\partial v_i}{\partial t} + \left(v_j\frac{\partial}{\partial x_j}\right)v_i = -\frac{\partial p}{\partial x_i} + \rho f_i$$

Summing the equations and simplifying

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_j} + \rho f_i$$

If $f_i = 0$ we get

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial T_{ij}}{\partial x_i} = 0 \tag{8}$$

where $T_{ij} = p\delta_{ij} + pv_iv_j$ is the **momentum flux** tensor. In fact, the total momentum of an element with volume V is

$$p = \int_V \rho v_i \mathrm{d}V$$

and it's time derivative

$$\frac{\partial}{\partial t}\int_V \rho v_i \mathrm{d}V = -\int_V \frac{\partial T_{ij}}{\partial x_i}\mathrm{d}V = -\oint_S T_{ij}\mathrm{d}S_i$$

3.5 Energy Equation in Conservative Form

The energy density is

$$\rho\epsilon + \frac{1}{2}\rho v^2$$

So one may write

$$\frac{\partial}{\partial t}\left(\rho\epsilon + \frac{1}{2}\rho v^2\right) = -\frac{\partial}{\partial x_j}\left(\epsilon\rho v_j\right) + \frac{\partial}{\partial x_j}\left(p v_j\right) + \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) - \frac{1}{2}\left(\delta_{ij} v_j v_i\right)\frac{\partial}{\partial x_j}\left(\rho v_j\right) - \rho v_i\left(v_j\frac{\partial}{\partial x_j}\right) v_i v_i$$

and simplifying

$$\frac{\partial}{\partial t}\left(\rho\epsilon + \frac{1}{2}\rho v^2\right) - \nabla\cdot\left[\rho\vec{v}\left(\frac{1}{2}v^2 + w\right) - k\nabla T\right] \tag{9}$$

where $w = \epsilon + \frac{p}{\rho}$ is the **enthalpy density**.

4 Lecture 4 - 26/09/2019

4.1 Bernoulli's Principle

For stationary stream $\left(\frac{\partial\vec{v}}{\partial t} = 0\right)$, we may define current lines that are tangent to $\vec{v}$ at every point. If the stream is stationary, the current lines trace the path of a given element of fluid. Let $\vec{f} = \vec{f}_{\text{grav}} = -\nabla\Phi$ where Φ is the gravitational potential.

Considering the identity

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$$

Taking Euler's equation for a stationary fluid (eq. 4)

$$(\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{f}$$

Using the previous identity

$$\nabla\left(\frac{v^2}{2}\right) - \vec{v}\times(\nabla\times\vec{v}) = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

Integrating along the path of the current

$$\int_l \left[\nabla\left(\frac{v}{2}\right) - \vec{v}\times(\nabla\times\vec{v}) + \frac{1}{\rho}\nabla p + \nabla\Phi\right]\mathrm{d}\vec{l} = 0$$

which gives

$$\frac{1}{2}v^2 + \int\frac{\mathrm{d}p}{\rho} + \Phi = \text{const} \tag{10}$$

In the case of an **incompressible fluid** $\int\frac{\mathrm{d}p}{\rho} = \frac{p}{\rho}$ we get

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gh = \text{const} \tag{11}$$