Quantum Mechanics II - Solutions

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1 Time Independent Perturbation Theory

1. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + \varepsilon_0 q x$$

with eigenvalues

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$$

and eigenvectors

$$\psi_n = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & x \in [0,a] \\ 0 & x \notin [0,a] \end{cases}.$$

(a) The perturbation on the first energy level, in first order, is

$$\Delta E = \langle \phi_1 | W | \phi_1 \rangle = \frac{2\varepsilon_0 q}{a} \int_0^a x \sin^2\left(\frac{\pi x}{a}\right) dx = \frac{\varepsilon_0 q}{a} \int_0^a x \left(1 - \cos\left(\frac{2\pi x}{a}\right)\right) dx = \frac{\varepsilon_0 q a}{2}$$

(b) The perturbation on the wave function gives

$$\Delta \phi_1 = \sum_{n \neq 1}^{\infty} \frac{\langle \phi_n | W | \phi_1 \rangle}{E_1 - E_n} = \sum_{n=2}^{\infty} \frac{4\varepsilon_0 q m a}{\hbar^2 \pi^2 (1 - n^2)} \int_0^a x \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx$$

as every even term is null, it simplifies to

$$= \sum_{n=1}^{\infty} \frac{4\varepsilon_0 q m a}{\hbar^2 \pi^2 n (n+1)} \int_0^a x \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{(2n+1)\pi x}{a}\right) dx$$

2. The Hamiltonian of the unperturbed system is

$$H_0 = \hbar\omega\left(N + \frac{1}{2}\mathbb{I}\right)$$

with eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

Let us introduce a perturbation $V = \varepsilon q x$. Recalling ladder operators defined as

$$\begin{cases} a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right) \\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right) \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) \\ p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right) \end{cases}$$

we may write

$$V = \varepsilon q\sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right).$$

(a) The energy perturbation on first order is

$$\Delta E = \langle n\,|V|\,n\rangle = \varepsilon q\sqrt{\frac{\hbar}{2m\omega}}\langle n\left|a^\dagger + a\right|n\rangle = 0$$

and on second order

$$\Delta E_n = \sum_{m\neq n}^{\infty}\frac{\left|\langle m\,|V|\,n\rangle\right|^2}{E_n - E_m} = \frac{q^2\varepsilon^2\hbar}{2m\omega}\sum_{m\neq n}^{\infty}\frac{\left|\langle m\left|a^\dagger + a\right|n\rangle\right|^2}{\hbar\omega\,(n-m)}$$

$$= \frac{q^2\varepsilon^2\hbar}{2m\omega^2}\sum_{m\neq n}^{\infty}\frac{n\langle m|n-1\rangle^2 + (n+1)\,\langle m|n+1\rangle^2}{(n-m)} = -\frac{q^2\varepsilon^2\hbar}{2m\omega^2}$$

(b) The problem admits an analytical solution by noticing

$$H = H_0 + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \varepsilon q x = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x + \frac{\varepsilon q}{m\omega^2}\right)^2 - \frac{\varepsilon^2 q^2}{2m\omega^2}$$

by setting $X = x + \frac{\varepsilon q}{m\omega^2}$ we get

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2X^2 - \frac{\varepsilon^2q^2}{2m\omega^2} = \hbar\omega\,(N+1) - \frac{\varepsilon^2q^2}{2m\omega^2}$$

with eigenvalues

$$E_n\text{'} = E_n - \frac{\varepsilon^2q^2}{2m\omega^2}$$

so the perturbed solution energy on second order is exact.

(c) The first order correction to the wave function is

$$|\psi_n\rangle = |n\rangle + \sum_{m\neq n}^{\infty} \frac{\langle m\,|V|\,n\rangle}{E_n - E_m}\,|m\rangle = \frac{q^2\varepsilon^2\hbar}{2m\omega}\sum_{m\neq n}^{\infty}\frac{\langle m\,|a^\dagger + a|\,n\rangle}{\hbar\omega\,(n-m)}\,|m\rangle =$$

$$= |n\rangle + \frac{q\varepsilon}{\omega}\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right).$$

3. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \alpha x^4 == \hbar\omega\left(N + \frac{1}{2}\mathbb{I}\right) + \alpha\frac{\hbar}{2m\omega}\left(a^{\dagger 2} + a^2 + 2N + \mathbb{I}\right)$$

(a) The perturbation to the first energy level is given by

$$\Delta E_n = \frac{\hbar}{2m\omega}\left\langle n\,\middle|a^{\dagger 2} + a^2 + 2N + \mathbb{I}\middle|\,n\right\rangle = \frac{\hbar}{2m\omega}\,(2\hbar n + 1)\,.$$

(b) The perturbed wave function is

$$|\psi_n\rangle = |n\rangle + \frac{\hbar}{2m\omega}\sum_{m\neq n}^{\infty}\frac{\left\langle m\,\middle|a^{\dagger 2} + a^2 + 2N + \mathbb{I}\middle|\,n\right\rangle}{E_n - E_m}\,|m\rangle$$

$$= |n\rangle + \frac{\hbar^2}{4m\omega^2}\left[\sqrt{(n+1)\,(n+2)}\,|n+2\rangle - \sqrt{n\,(n-1)}\,|n-2\rangle\right].$$

4. Let the Hamiltonian be $H = AJ^2 + BJ_z^2$ and $J = 1$.

(a) The eigenvalues are

$$E_m = \hbar^2\left(2A - BM^2\right) = \begin{cases} 2A\hbar^2 & \text{if } M = 0 \\ \hbar^2\,(2A + B) & \text{if } M \in \{-1, 1\} \end{cases}$$

corresponding to eigenvectors $|J, M\rangle \in \{|1,1\rangle\,, |1,0\rangle\,, |1,-1\rangle\}\,.$

(b) A perturbation of the form $V = \alpha J_x^2$ may be written as ladder operators

$$\begin{cases} J_+ = J_x + iJ_y \\ J_- = J_x - iJ_y \end{cases} \Leftrightarrow \begin{cases} J_x = \frac{1}{2}\,(J_- + J_+) \\ J_y = \frac{i}{2}\,(J_- - J_+) \end{cases} \Rightarrow V = \frac{\alpha}{4}\left(J_-^2 + J_+^2 + 2J^2 - 2J_z^2\right).$$

i. Now there is the degenerate case of $M \neq 0$. We treat each case seperately. If $M = 0$:

$$\Delta E_{M=0} = \frac{\alpha}{4}\left\langle 1,0\,\middle|J_-^2 + J_+^2 + 2J^2 - 2J_z^2\middle|\,1,0\right\rangle = \hbar^2\alpha$$

else, if $M \in \{-1, 1\}$, we diagonalize the perturbation in the degenerate subspace:

$$\frac{\alpha}{4}\left\langle 1,\pm 1\,\middle|J_-^2 + J_+^2 + 2J^2 - 2J_z^2\middle|\,1,\pm 1\right\rangle = \frac{\alpha}{4}\left\langle 1,\pm 1\,\middle|J_-^2 + J_+^2\middle|\,1,\pm 1\right\rangle$$

$$= \frac{\alpha}{4}\left\langle 1,\pm 1\,\middle|J_\mp^2\middle|\,1,\pm 1\right\rangle = \frac{\hbar^2\alpha}{2}$$

resulting in a perturbed energy of $\Delta E_{M=\pm 1} = \frac{\hbar^2\alpha}{2}$.

ii. The perturbed wave function for $M = 0$:

$$|\psi_{M=0}\rangle = |1,0\rangle + \frac{\alpha}{4}\sum_{M=\pm 1}\frac{\left\langle 1,\pm 1\,\middle|J_-^2 + J_+^2 + 2J^2 - 2J_z^2\middle|\,1,0\right\rangle}{E_{M=0} - E_{M=\pm 1}}\,|1,\pm 1\rangle = |1,0\rangle$$

and for $M = \pm 1$:

$$|\psi_{M=\pm 1}\rangle = |1,\pm 1\rangle + \frac{\alpha}{4}\frac{\left\langle 1,0\,\middle|J_-^2 + J_+^2 + 2J^2 - 2J_z^2\middle|\,1,\pm 1\right\rangle}{E_{M=\pm 1} - E_{M=0}}\,|1,0\rangle = |1,\pm 1\rangle$$