Quantum Mechanics II - Solutions

Rui Peixoto

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1 Time Independent Perturbation Theory

1. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + \varepsilon_0 q x$$

with eigenvalues

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$$

and eigenvectors

$$\psi_n = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & x \in [0,a] \\ 0 & x \notin [0,a] \end{cases}.$$

(a) The perturbation on the first energy level, is first order, is

$$\Delta E = \langle \phi_1 | W | \phi_1 \rangle = \frac{2\varepsilon_0 q}{a} \int_0^a x \sin^2\left(\frac{\pi x}{a}\right) dx = \frac{\varepsilon_0 q}{a} \int_0^a x \left(1 - \cos\left(\frac{2\pi x}{a}\right)\right) dx = \frac{\varepsilon_0 q a}{2}$$

(b) The perturbation on the wavefunction gives

$$\Delta\phi_1 = \sum_{n\neq 1}^{\infty} \frac{\langle \phi_n | W | \phi_1 \rangle}{E_1 - E_n} = \sum_{n=2}^{\infty} \frac{4\varepsilon_0 q m a}{\hbar^2 \pi^2 (1-n^2)} \int_0^a x \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx$$

as every even term is null, it simplifies to

$$= \sum_{n=1}^{\infty} \frac{4\varepsilon_0 q m a}{\hbar^2 \pi^2 n (n+1)} \int_0^a x \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{(2n+1)\pi x}{a}\right) dx$$

2. The Hamiltonian of the unperturbed system is

$$H_0 = \hbar\omega\left(N + \frac{1}{2}\mathbb{I}\right)$$

with eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

Let us introduce a perturbation $V = \varepsilon q x$. Recalling ladder operators defined as

$$\begin{cases} a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right) \\ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right) \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) \\ p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right) \end{cases}$$

we may write

$$V = \varepsilon q\sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right).$$

(a) The energy perturbation on first order is

$$\Delta E = \langle n\,|V|\,n\rangle = \varepsilon q\sqrt{\frac{\hbar}{2m\omega}}\langle n\left|a^\dagger + a\right|n\rangle = 0$$

and on second order

$$\Delta E_n = \sum_{m\neq n}^{\infty}\frac{|\langle m\,|V|\,n\rangle|^2}{E_n - E_m} = \frac{q^2\varepsilon^2\hbar}{2m\omega}\sum_{m\neq n}^{\infty}\frac{\left|\langle m\left|a^\dagger + a\right|n\rangle\right|^2}{\hbar\omega\,(n-m)}$$

$$= \frac{q^2\varepsilon^2\hbar}{2m\omega^2}\sum_{m\neq n}^{\infty}\frac{n\langle m|n-1\rangle^2 + (n+1)\,\langle m|n+1\rangle^2}{(n-m)} = -\frac{q^2\varepsilon^2\hbar}{2m\omega^2}$$

(b) The problem admits an analytical solution by noticing

$$H = H_0 + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \varepsilon q x = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x + \frac{\varepsilon q}{m\omega^2}\right)^2 - \frac{\varepsilon^2q^2}{2m\omega^2}$$

by setting $X = x + \frac{\varepsilon q}{m\omega^2}$ we get

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2X^2 - \frac{\varepsilon^2q^2}{2m\omega^2} = \hbar\omega\,(N+1) - \frac{\varepsilon^2q^2}{2m\omega^2}$$

with eigenvalues

$$E_n\text{'} = E_n - \frac{\varepsilon^2q^2}{2m\omega^2}$$

so the perturbed solution energy on second order is exact.

(c) The first order correction to the wave function is

$$|\psi_n\rangle = |n\rangle + \sum_{m\neq n}^{\infty} \frac{\langle m\,|V|\,n\rangle}{E_n - E_m}\,|m\rangle = \frac{q^2\varepsilon^2\hbar}{2m\omega} \sum_{m\neq n}^{\infty} \frac{\langle m\,\left|a^\dagger + a\right|\,n\rangle}{\hbar\omega\,(n-m)}\,|m\rangle =$$

$$= |n\rangle + \frac{q\varepsilon}{\omega}\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right).$$

3. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \alpha x^4 == \hbar\omega\left(N + \frac{1}{2}\mathbb{I}\right) + \alpha\frac{\hbar}{2m\omega}\left(a^{\dagger 2} + a^2 + 2N + \mathbb{I}\right)$$

(a) The perturbation to the first energy level is given by

$$\Delta E_n = \frac{\hbar}{2m\omega}\left\langle n\left|a^{\dagger 2} + a^2 + 2N + \mathbb{I}\right|n\right\rangle = \frac{\hbar}{2m\omega}\left(2\hbar n + 1\right).$$

(b) The perturbed wave function is

$$|\psi_n\rangle = |n\rangle + \frac{\hbar}{2m\omega}\sum_{m\neq n}^{\infty}\frac{\langle m\left|a^{\dagger 2} + a^2 + 2N + \mathbb{I}\right|n\rangle}{E_n - E_m}\,|m\rangle$$

$$= |n\rangle + \frac{\hbar^2}{4m\omega^2}\left[\sqrt{(n+1)(n+2)}\,|n+2\rangle - \sqrt{n(n-1)}\,|n-2\rangle\right].$$