Notes on Quantum Mechanics II

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Part I
Time Dependent Perturbation Theory

1 Interaction Representation

We define the interactiong representation

$$|\psi(t)\rangle_S = e^{-i\frac{H_0 t}{\hbar}} |\psi(t)\rangle_I$$

where

$$|\psi(t)\rangle_I = \mathcal{U}(t, t_0) |\psi(t_0)\rangle_I .$$

From the Shrodinger equation

$$i\hbar\partial_t |\psi(t)\rangle_I = i\hbar\partial_t \left(e^{i\frac{H_0 t}{\hbar}} |\psi(t)\rangle_S\right) = -H_0 e^{-i\frac{H_0 t}{\hbar}} |\psi(t)\rangle_S + i\hbar e^{i\frac{H_0 t}{\hbar}} [H_0 + V(t)\mathbb{I}] |\psi(t)\rangle_S$$

writing $\mathbb{I} = e^{-i\frac{H_0 t}{\hbar}} e^{i\frac{H_0 t}{\hbar}}$ we get

$$i\hbar\partial_t |\psi(t)\rangle_I = V_I(t) |\psi(t)\rangle_I$$

from where follows

$$i\hbar\partial_t \mathcal{U}_I(t, t_0) = V_I(t)\mathcal{U}(t, t_0).$$

Integrating the left hand side we get

$$\int_{t_0}^{t} i\hbar\partial_t \mathcal{U}_I(t, t_0)\, dt = i\hbar [\mathcal{U}_I(t, t_0) - \mathbb{I}]$$

so

$$\mathcal{U}_{\mathcal{I}}(t, t_0) = \mathbb{I} - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\mathcal{U}_I(t', t_0) dt'$$

1.1 Iterative Solution

For the first order we get

$$\mathcal{U}_{\mathcal{I}}(t,t_0)=\mathbb{I}-\frac{i}{\hbar}\int_{t_0}^{t}dt' V_I(t')+\frac{1}{\hbar^2}\int_{t_0}^{t}dt' V_I(t')\int_{t_0}^{t'}dt'' V_I(t'')+\dots$$

$$=\sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t}dt_1 V_I(t_1)\dots\int_{t_0}^{t_{n-1}}dt_n V_I(t_n)$$

2 Transition Probability

Let $|\psi(t_0)\rangle=|\phi_i\rangle$, then

$$P_{i\to f}=|\langle\phi_f|\psi(t)\rangle|^2=|c_f(t)|^2=|b_f(t)|^2=|\langle\phi_f|\mathcal{U}_{\mathcal{I}}(t,t_0)|\phi_i\rangle|^2.$$

In first order

$$P_{i\to f}\approx\frac{1}{h^2}\left|\int_{t_0}^{t}dt'\langle\phi_f|V_I(t)|\phi_i\rangle\right|^2$$

but remembering that $V_I=e^{i\frac{H_0 t}{\hbar}}V(t)e^{-i\frac{H_0 t}{\hbar}}$, and $|\phi_j\rangle$ that are eigenstates of the Hamiltonian, it simplifies

$$P_{i\to f}\approx\frac{1}{h^2}\left|\int_{t_0}^{t}dt' e^{i\omega_{fi}t'}\langle\phi_f|V(t)|\phi_i\rangle\right|^2$$

for $\omega_{fi}=\frac{E_f-E_i}{\hbar}$.

3 Sinusoidal perturbation

Let

$$V(t)=\mathcal{V}\cos\omega t$$