Subspace Embedding and Linear Regression with Orlicz Norm

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General Linear Regression Formulation

- General Linear Regression as an optimization problem:
  $x^* = \text{argmin}_{x \in \mathbb{R}^d} L(Ax - b)$, where:
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- ℓ_1 Linear Regression (a.k.a Least Absolute Value Regression)
  - $L(r) = ||r||_1 = \sum_i |r_i|$
  - median estimator; robust against outliers.
  - large MSE; relatively unstable

Robust Statistics: Huber Loss

- Huber Loss: $L(r) = \sum_{i=1}^{n} f(r_i)$, where

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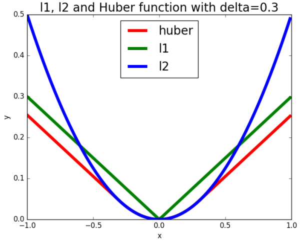

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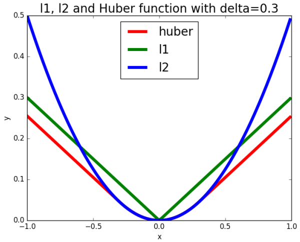

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- σ is unknown.
- f is not scale-invariant.

Our Approach: via Orlicz Norm

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- $\exists \delta_G$ s.t. G is twice-differentiable on interval $(0, \delta_G)$

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- $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix, each entry an i.i.d. random variable drawn from CDF $1 - e^{-G(t)}$.

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  $(x_1, x_2, ..., x_n) \to (x_1/u_1, x_2/u_2, ..., x_n/u_n)$. We want to bound the probability of $\|f(x)/\gamma\alpha\|_G \geq 1$, and thus $\mathbb{E}[\sum_{i=1}^n G(x_i\gamma)/(\alpha u_i)]$.
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Comparison with ℓ_1 and ℓ_2 regression

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	Gaussian	Sparse	Mixed
best performing	ℓ_2	ℓ_1	$G_{\delta=0.75}$ (Orlicz)
worst performing	ℓ_1	ℓ_2	ℓ_1

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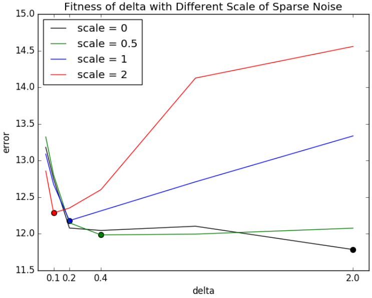

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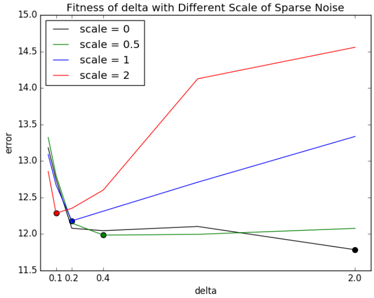
- 

scale s	best δ
0	2
0.5	1
1	0.5
2	0

A General Framework: Picking "whichever G" you want

- Consider the Orlicz norm G induced by the following function:

$$G_{\ell_{1.5}}(x) = \begin{cases} x^{1.5}/1.5 & |x| \le \delta \\ \delta^{0.5}(|x| - \delta/3) & \text{otherwise} \end{cases}$$

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- $1 + \epsilon$ approximation with non-oblivious embedding.

Thanks for watching! Questions?
Poster: # 18