WHAM to POMP Model Translation: Theoretical Formulation

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Contents

1 Model Overview

1.1 WHAM Model Structure

WHAM is a state-space stock assessment model that incorporates:

- Age-structured population dynamics
- Beverton-Holt stock-recruitment relationship
- Fishing and natural mortality
- Multiple data sources (catch, survey indices, age compositions)
- Environmental covariates (optional)

1.2 POMP Framework

POMP represents the system as:

$$\mathbf{X}_t = f(\mathbf{X}_{t-1}, \boldsymbol{\theta}, \boldsymbol{\epsilon}_t)$$
$$\mathbf{Y}_t = g(\mathbf{X}_t, \boldsymbol{\theta}, \boldsymbol{\eta}_t)$$

where:

- $\mathbf{X}_t$ = state variables (unobserved)
- $\mathbf{Y}_t$ = observations
- $\boldsymbol{\theta}$ = parameters
- $\boldsymbol{\epsilon}_t, \boldsymbol{\eta}_t$ = process and observation errors

2 State Variables

2.1 Notation

Symbol	Description	Dimension
$N_{a,t}$	Numbers at age a in year t	Ages $\times$ Years
SSB_t	Spawning stock biomass in year t	Years
A	Maximum age (plus group)	Scalar
T	Total number of years	Scalar

2.2 State Vector

The complete state at time t is:

$$\mathbf{X}_t = \begin{bmatrix} N_{1,t} \\ N_{2,t} \\ \vdots \\ N_{A,t} \\ \text{SSB}_t \end{bmatrix}$$

3 Parameters

3.1 Parameter Vector

$$\boldsymbol{\theta} = \begin{bmatrix} \alpha \\ \beta \\ \sigma_R \\ F \\ q \\ N_{1,1} \end{bmatrix}$$

where:

Parameter	Description	Transform	Domain
α	Beverton-Holt recruitment capacity	$\log(\alpha)$	$(0, \infty)$
β	Beverton-Holt density dependence	$\log(\beta)$	$(0, \infty)$
σ_R	Recruitment standard deviation	Identity	$(0, \infty)$
F	Fishing mortality	$\log(F)$	$(0, \infty)$
q	Catchability coefficient	$\log(q)$	$(0, 1)$

Parameter	Description	Transform	Domain
$N_{1,1}$	Initial abundance at age 1	$\log(N_{1,1})$	$(0, \infty)$

3.2 Biological Parameters (Fixed)

Parameter	Description	Typical Values
m_a	Maturity at age a	$[0, 1]$
w_a	Weight at age a (kg)	$[0.1, 2.5]$
s_a	Selectivity at age a	$[0, 1]$
M	Natural mortality	0.2

4 Process Model (rprocess)

4.1 Population Dynamics

The process model describes how the state evolves from time $t-1$ to t.

4.1.1 1. Recruitment (Age 1)

Beverton-Holt Stock-Recruitment:

$$N_{1,t} = \frac{\alpha \cdot \mathrm{SSB}_{t-1}}{1 + \beta \cdot \mathrm{SSB}_{t-1}} \cdot \exp(\epsilon_{R,t})$$

where:

$$\epsilon_{R,t} \sim \mathcal{N}(0, \sigma_R^2)$$

Process noise: Recruitment has log-normal variability:

$$N_{1,t} \sim \mathrm{LogNormal}\left(\log\left[\frac{\alpha \cdot \mathrm{SSB}_{t-1}}{1 + \beta \cdot \mathrm{SSB}_{t-1}}\right], \sigma_R^2\right)$$

4.1.2 2. Survival and Aging (Ages 2 to $A-1$)

For intermediate ages ($2 \leq a < A$):

$$N_{a,t} = N_{a-1,t-1} \cdot S_{a-1,t-1}$$

where survival is:

$$S_{a,t} = \exp(-Z_{a,t})$$

and total mortality is:

$$Z_{a,t} = M + F_t \cdot s_a$$

Components:

- M = natural mortality (constant)
- $F_t \cdot s_a$ = fishing mortality at age a
- s_a = selectivity at age a

4.1.3 3. Plus Group (Age A)

The plus group accumulates all individuals at and above age A:

$$N_{A,t} = N_{A-1,t-1} \cdot S_{A-1,t-1} + N_{A,t-1} \cdot S_{A,t-1}$$

This captures both:

1. Survivors aging into the plus group
2. Survivors remaining in the plus group

4.1.4 4. Spawning Stock Biomass

SSB is calculated as:

$$\text{SSB}_t = \sum_{a=1}^{A} N_{a,t} \cdot w_a \cdot m_a$$

where:

- w_a = weight at age a
- m_a = proportion mature at age a

4.2 Complete Process Model

$$\boxed{\begin{aligned} N_{1,t} &= \frac{\alpha \cdot \text{SSB}_{t-1}}{1 + \beta \cdot \text{SSB}_{t-1}} \cdot \exp(\epsilon_{R,t}), \quad \epsilon_{R,t} \sim \mathcal{N}(0, \sigma_R^2) \\ N_{a,t} &= N_{a-1,t-1} \cdot \exp(-Z_{a-1,t-1}), \quad 2 \leq a < A \\ N_{A,t} &= N_{A-1,t-1} \cdot \exp(-Z_{A-1,t-1}) + N_{A,t-1} \cdot \exp(-Z_{A,t-1}) \\ Z_{a,t} &= M + F_t \cdot s_a \\ \text{SSB}_t &= \sum_{a=1}^{A} N_{a,t} \cdot w_a \cdot m_a \end{aligned}}$$

Key observation: $\mathbf{X}_t$ depends only on $\mathbf{X}_{t-1}$ and parameters $\boldsymbol{\theta}$, not on $\mathbf{X}_{t-2}, \mathbf{X}_{t-3}, ...$

Thus: $p(\mathbf{X}_t|\mathbf{X}_{t-1}, \mathbf{X}_{t-2}, ..., \mathbf{X}_0) = p(\mathbf{X}_t|\mathbf{X}_{t-1})$

5 Measurement Model (dmeasure/rmeasure)

5.1 Observations

The model has two types of observations:

$$\mathbf{Y}_t = \begin{bmatrix} C_t \\ I_t \end{bmatrix}$$

where:

- C_t = aggregate catch (biomass)
- I_t = survey index (abundance or biomass)

5.2 1. Catch Observation

5.2.1 Expected Catch (Baranov Equation)

The predicted catch in year t is:

$$\hat{C}_t = \sum_{a=1}^{A} N_{a,t} \cdot \frac{F_t \cdot s_a}{Z_{a,t}} \cdot \left(1 - e^{-Z_{a,t}}\right) \cdot w_a$$

Components:

- $N_{a,t}$ = numbers at age
- $\frac{F_t \cdot s_a}{Z_{a,t}}$ = proportion of mortality due to fishing
- $1 - e^{-Z_{a,t}}$ = proportion dying during the year
- w_a = weight at age (converts numbers to biomass)

5.2.2 Observation Model

$$C_t = \hat{C}_t \cdot \exp(\eta_{C,t}), \quad \eta_{C,t} \sim \mathcal{N}(0, \sigma_C^2)$$

Equivalently (log-normal):

$$\log(C_t) \sim \mathcal{N}(\log(\hat{C}_t), \sigma_C^2)$$

where $\sigma_C = 0.1$ (10% CV).

5.3 2. Survey Index Observation

5.3.1 Expected Index

The predicted survey index is:

$$\hat{I}_t = q \sum_{a=1}^{A} s_a \cdot N_{a,t}$$

where:

- q = catchability coefficient
- s_a = survey selectivity (assumed same as fishery)
- $N_{a,t}$ = numbers at age

5.3.2 Observation Model

$$I_t = \hat{I}_t \cdot \exp(\eta_{I,t}), \quad \eta_{I,t} \sim \mathcal{N}(0, \sigma_I^2)$$

Equivalently:

$$\log(I_t) \sim \mathcal{N}(\log(\hat{I}_t), \sigma_I^2)$$

where $\sigma_I = 0.15$ (15% CV).

5.4 Complete Measurement Model

$$\boxed{\begin{aligned} \hat{C}_t &= \sum_{a=1}^{A} N_{a,t} \cdot \frac{F_t \cdot s_a}{M + F_t \cdot s_a} \cdot (1 - \exp[-(M + F_t \cdot s_a)]) \cdot w_a \\ \hat{I}_t &= q \sum_{a=1}^{A} s_a \cdot N_{a,t} \\ \log(C_t) &\sim \mathcal{N}(\log(\hat{C}_t), \sigma_C^2) \\ \log(I_t) &\sim \mathcal{N}(\log(\hat{I}_t), \sigma_I^2) \end{aligned}}$$

Key observation: Observations depend only on current state $\mathbf{X}_t$, not on past states.

Thus: $p(\mathbf{Y}_t|\mathbf{X}_t, \mathbf{X}_{t-1}, ...) = p(\mathbf{Y}_t|\mathbf{X}_t)$

6 Initial Conditions (rinit)

6.1 Equilibrium Age Structure

The initial population (year 1) is based on equilibrium assumptions:

6.1.1 Age 1

$$N_{1,1} = N_0$$

where $N_0 = \exp(\log N_{1,1})$ is a parameter.

6.1.2 Ages 2 to $A-1$

Assuming equilibrium with initial fishing mortality $F_0 = 0.1$:

$$N_{a,1} = N_{a-1,1} \cdot \exp(-Z_{a-1}^{(0)})$$

where:

$$Z_a^{(0)} = M + F_0 \cdot s_a$$

6.1.3 Plus Group (Age A)

$$N_{A,1} = \frac{N_{A-1,1} \cdot \exp(-Z_{A-1}^{(0)})}{1 - \exp(-Z_A^{(0)})}$$

This represents the equilibrium plus group size.

6.1.4 Initial SSB

$$\text{SSB}_1 = \sum_{a=1}^{A} N_{a,1} \cdot w_a \cdot m_a$$

6.2 Complete Initial Conditions

$$\boxed{\begin{aligned} N_{1,1} &= \exp(\log N_{1,1}) \\ N_{a,1} &= N_{1,1} \cdot \prod_{j=1}^{a-1} \exp\left(-(M + F_0 \cdot s_j)\right), \quad 2 \le a < A \\ N_{A,1} &= \frac{N_{A-1,1} \cdot \exp(-(M + F_0 \cdot s_{A-1}))}{1 - \exp(-(M + F_0 \cdot s_A))} \\ \text{SSB}_1 &= \sum_{a=1}^{A} N_{a,1} \cdot w_a \cdot m_a \end{aligned}}$$

7 Likelihood Function

7.1 Joint Likelihood

The negative log-likelihood is:

$$-\ell(\boldsymbol{\theta}) = -\sum_{t=1}^{T} \left[\log p(C_t|\mathbf{X}_t, \boldsymbol{\theta}) + \log p(I_t|\mathbf{X}_t, \boldsymbol{\theta})\right]$$

7.2 Log-Normal Likelihood Components

7.2.1 Catch Likelihood

$$\log p(C_t|\mathbf{X}_t, \boldsymbol{\theta}) = -\frac{1}{2}\log(2\pi\sigma_C^2) - \frac{[\log(C_t) - \log(\hat{C}_t)]^2}{2\sigma_C^2}$$

7.2.2 Index Likelihood

$$\log p(I_t|\mathbf{X}_t, \boldsymbol{\theta}) = -\frac{1}{2}\log(2\pi\sigma_I^2) - \frac{[\log(I_t) - \log(\hat{I}_t)]^2}{2\sigma_I^2}$$

7.3 Complete Likelihood

$$\boxed{-\ell(\boldsymbol{\theta}) = -\sum_{t=1}^{T}\left\{-\frac{[\log(C_t) - \log(\hat{C}_t)]^2}{2\sigma_C^2} - \frac{[\log(I_t) - \log(\hat{I}_t)]^2}{2\sigma_I^2}\right\} + \text{constants}}$$

8 Selectivity Function

8.1 Logistic Selectivity

Selectivity at age follows a logistic function:

$$s_a = \frac{1}{1 + \exp(-\gamma(a - a_{50}))}$$

where:

- a_{50} = age at 50% selectivity
- γ = slope parameter (steepness)

8.2 Properties

- $s_a \in [0, 1]$ for all ages
- $s_{a_{50}} = 0.5$ (50% selected at a_{50})
- Higher $\gamma \rightarrow$ steeper selection curve

9 Complete Mathematical Formulation

9.1 State-Space Representation

Process Model:

$$N_{1,t} = \frac{\alpha \cdot \mathrm{SSB}_{t-1}}{1 + \beta \cdot \mathrm{SSB}_{t-1}} \cdot \exp(\epsilon_{R,t}), \quad \epsilon_{R,t} \sim \mathcal{N}(0, \sigma_R^2)$$

$$N_{a,t} = N_{a-1,t-1} \cdot \exp(-(M + F_t \cdot s_a)), \quad 2 \leq a < A$$

$$N_{A,t} = N_{A-1,t-1} \cdot \exp(-(M + F_t \cdot s_{A-1})) + N_{A,t-1} \cdot \exp(-(M + F_t \cdot s_A))$$

$$\mathrm{SSB}_t = \sum_{a=1}^{A} N_{a,t} \cdot w_a \cdot m_a$$

Measurement Model:

$$\hat{C}_t = \sum_{a=1}^{A} N_{a,t} \cdot \frac{F_t \cdot s_a}{M + F_t \cdot s_a} \cdot (1 - e^{-(M+F_t \cdot s_a)}) \cdot w_a$$

$$\hat{I}_t = q \sum_{a=1}^{A} s_a \cdot N_{a,t}$$

$$\log(C_t) \sim \mathcal{N}(\log(\hat{C}_t), \sigma_C^2)$$

$$\log(I_t) \sim \mathcal{N}(\log(\hat{I}_t), \sigma_I^2)$$

Key observation: $\mathbf{X}_t$ depends only on $\mathbf{X}_{t-1}$ and parameters $\boldsymbol{\theta}$, not on $\mathbf{X}_{t-2}, \mathbf{X}_{t-3}, ...$

Thus: $p(\mathbf{X}_t | \mathbf{X}_{t-1}, \mathbf{X}_{t-2}, ..., \mathbf{X}_0) = p(\mathbf{X}_t | \mathbf{X}_{t-1})$

Key observation: Observations depend only on current state $\mathbf{X}_t$, not on past states.

Thus: $p(\mathbf{Y}_t | \mathbf{X}_t, \mathbf{X}_{t-1}, ...) = p(\mathbf{Y}_t | \mathbf{X}_t)$

WHAM includes:

- **Process noise**: $\epsilon_t \sim \mathcal{N}(0, \sigma_R^2)$ in recruitment
- **Observation noise**: $\eta_{C,t}, \eta_{I,t} \sim \mathcal{N}(0, \sigma^2)$

Thus the model is stochastic.

10 Comparison: WHAM vs POMP

10.1 Similarities

Feature	WHAM	POMP
Population dynamics	Exponential survival	Exponential survival
Recruitment	Beverton-Holt	Beverton-Holt
Catch equation	Baranov	Baranov

Feature	WHAM	POMP
Observation errors	Log-normal	Log-normal
Age structure	Yes	Yes

10.2 Differences

Feature	WHAM	POMP
Estimation	TMB (Laplace approx)	Particle filter
Time steps	Flexible (sub-annual)	Discrete (annual)
Random effects	Extensive	Process noise only
Environmental covariates	Built-in	Manual implementation
Computational speed	Very fast	Moderate

11 Appendix: Parameter Table

11.1 Default Parameter Values

Parameter	Symbol	Default Value	Units
BH capacity	α	10^6	recruits
BH density dep.	β	10^5	1/kg
Recruit SD	σ_R	0.4	-
Fishing mortality	F	0.2	year^{-1}
Catchability	q	0.3	-
Initial abundance	$N_{1,1}$	10^6	fish
Natural mortality	M	0.2	year^{-1}
Catch CV	σ_C	0.1	-
Index CV	σ_I	0.15	-

12 Appendix: Notation Summary

12.1 Indices

- t : time (year), $t = 1, ..., T$
- a : age, $a = 1, ..., A$

12.2 State Variables

- $N_{a,t}$: numbers at age a in year t
- SSB_t : spawning stock biomass in year t

12.3 Parameters

- α : Beverton-Holt recruitment capacity
- β : Beverton-Holt density dependence

- σ_R : recruitment standard deviation
- F_t : fishing mortality in year t
- q : catchability coefficient
- M : natural mortality rate

12.4 Biological Data

- m_a : maturity at age a
- w_a : weight at age a
- s_a : selectivity at age a

12.5 Observations

- C_t : catch in year t
- I_t : survey index in year t

12.6 Derived Quantities

- $Z_{a,t}$: total mortality at age a in year t
- $S_{a,t}$: survival at age a in year t
- $\hat{C}_t$: predicted catch in year t
- $\hat{I}_t$: predicted index in year t