wham to pump

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1. Latent state X_n

- In WHAM, the hidden states are the **numbers at age (NAA):**

$$X_n = (N_{1,n}, N_{2,n}, \ldots, N_{A,n})$$

- $N_{a,n}$: number of fish of age a in year n.
- These are unobserved (not directly measured), so they form the **latent Markov process** in the POMP model.

2. Observations Y_n

- The observed data in WHAM are fishery and survey data, such as:
  - Total catch
  - Survey abundance indices
  - Age composition of catch and indices
- These are modeled via the **measurement process**:

$$Y_n \sim f_{Y|X}(\cdot \mid X_n, \theta)$$

For example, expected catch is calculated with the **Baranov catch equation**, then observed with **lognormal error**.

3. State transition distribution $f_{X_n|X_{n-1}}$

- In WHAM, this is given by the **population dynamics equations**:
- **Recruitment (new age-1 fish):**

$$N_{1,n} \sim f(\mathrm{SSB}_{n-1}) \cdot e^{\varepsilon_{1,n}}$$

e.g. Beverton–Holt or Ricker stock–recruit relationship with random effects.

- **Survival / aging:**

$$N_{a,n} = N_{a-1,n-1}\, e^{-Z_{a-1,n-1}} \cdot e^{\varepsilon_{a,n}}, \quad a > 1$$

where $Z = F + M$ is total mortality.

- This defines the **Markov transition probabilities** in the POMP model.

4. Measurement Model

The predicted *aggregate catch* for fleet/index i in year y is

$$\widehat{C}_{y,i} = \sum_a \widehat{C}_{a,y,i}\, W_{a,y,i},$$

where $W_{a,y,i}$ is the empirical weight–at–age.

We assume the observed catch $C_{y,i}$ follows a **lognormal distribution**:

$$\log(C_{y,i}) \sim \mathcal{N}\Big(\log(\widehat{C}_{y,i}) - \tfrac{\sigma^2_{C,y,i}}{2},\ \sigma^2_{C,y,i}\Big)\,.$$

The observation error standard deviation is parameterized as

$$\sigma_{C,y,i} \ = \ e^{\eta_i}\, \tilde{\sigma}_{C,y,i},$$

where η_i is a fleet/index-specific scaling parameter and
$\tilde{\sigma}_{C,y,i}$ is a baseline standard deviation.

5. Parameters θ

- In WHAM, θ includes:
  - Stock–recruit parameters (α, β)
  - Selectivity parameters (e.g., logistic curve parameters)
  - Natural mortality M
  - Observation error variances (τ_C, τ_I)

6. Full joint density (POMP representation of WHAM)

The WHAM model fits directly into the POMP framework as:

$$f_{X_{0:N},Y_{1:N}}(x_{0:N}, y_{1:N}; \theta) = f_{X_0}(x_0; \theta) \prod_{n=1}^{N} f_{X_n|X_{n-1}}(x_n|x_{n-1}; \theta)\, f_{Y_n|X_n}(y_n|x_n; \theta).$$

- **State transitions** = population dynamics equations
- **Measurements** = catch and survey observation models
- **Parameters** = biological + observation parameters

7. Initial Conditions to WHAM

Concept mapping

- **POMP**: specify an initial distribution for the latent state,
$$f_{X_0}(x_0;\theta).$$
- **WHAM (NAA model)**: the first-year numbers–at–age vector
$$X_0 \equiv (N_{1,y_0},\ldots,N_{A,y_0})$$
is typically **unknown**. In practice, WHAM estimates the components $\log N_{a,y_0}$ as parameters—equivalently, f_{X_0} is a **point mass** (degenerate initial distribution).

Common choices in WHAM

1. **Fixed initial values** (when justified by external information).
2. **Stationary/equilibrium initialization** (when the transition model is time-homogeneous with a unique stationary distribution and there is a scientific rationale).
3. **Estimate** $\log N_{a,y_0}$ **as parameters** (most common; equals a point-mass f_{X_0} in POMP).

Plus group: the initial vector includes N_{A,y_0}; aging then uses the standard plus-group accumulation.

8. Covariates (Ecov) to WHAM

Concept mapping

- **POMP**: an external process $\{Z(t)\}$ may enter the **state transition** $f_{X_n|X_{n-1}}$ and/or the **measurement** model $f_{Y_n|X_n}$.
- **WHAM**: environmental covariates (e.g., **CPI**) are often modeled as a **random walk** or **AR(1)** latent series X_y^{ecov}, then linked with a **lag** ψ to population dynamics:

Recruitment impact

- $N_{1,y}$ depends on $X_{y-\psi}^{\text{ecov}}$.

Controlling (productivity/mean shift), e.g. Beverton–Holt with ecov:
$$R_y = \frac{\alpha\,\exp\big(\beta\,X_{y-\psi}\big)\,SSB_{y-1}}{1+\beta_2\,SSB_{y-1}}.$$

Limiting (capacity/ceiling effect): - modify the BH capacity term with a function of $X_{y-\psi}$, or - in Ricker, add ecov to the negative density term, e.g. $\log R_y = \log\alpha + \log SSB_{y-1} - (\beta_0+\beta_2 X_{y-\psi})\,SSB_{y-1}$.

Natural mortality impact

Allow ecov to affect M:

$$\log M_{a,y} = \mu_{M,a} + \gamma X_{y-\psi}.$$

Multiple covariates are allowed; each can have its **own process** and **link**.

In short

Recasting WHAM as a POMP means:

- **NAA** = latent states (X_n)
- **Fishery/survey data** = observations (Y_n)
- **Population dynamics** = transition model
- **Catch equations + survey error** = measurement model

Classification of WHAM quantities in POMP terms

Quantity	Type	Notes
Numbers-at-age (NAA), $N_{a,n}$	**Latent state**	Hidden Markov process; survival + recruitment
Recruitment, $N_{1,n}$	Latent state	First element of NAA each year
Fishing mortality, F_n, s_a	Parameter (often w/ RE)	F_n fully selected F (time-varying); s_a selectivity-at-age
Natural mortality, M_a	Parameter	Age-specific, constant, or ecov-linked
Weights-at-age, w_a	Data / fixed covariate	Biological input, not estimated

Quantity	Type	Notes
Maturity-at-age, mat_a	Data / fixed covariate	Biological input, not estimated
Spawning stock biomass (SSB)	**Derived variable**	$\text{SSB}_n = \sum_a N_{a,n} w_a \, \text{mat}_a$

WHAM: Nonlinear Mixed-Effects State-Space Model

1. Parameter Layer

Fixed effects (β): - $\alpha, \beta \rightarrow$ Beverton–Holt or Ricker stock–recruit parameters
- $\mu_{M_a} \rightarrow$ mean natural mortality
- $\nu_1, \nu_2 \rightarrow$ mean selectivity parameters (a_{50}, k)
- $q \rightarrow$ catchability
- $\phi_X, \mu_X \rightarrow$ mean and autocorrelation for environmental covariate

Random effects ($\eta \sim \mathcal{N}(0, G)$): - $\varepsilon_{a,y} \rightarrow$ NAA process deviations (recruitment / abundance)
- $\delta_{a,y} \rightarrow$ natural mortality deviations (log M)
- $\zeta_{p,y} \rightarrow$ selectivity parameter deviations (logit-scale)
- $\omega_y \rightarrow$ environmental innovations (Ecov process)

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2. State Process Layer

- $\log N_{a,y} \;=\; f\big(N_{a-1,y-1},\, F,\, M,\, \text{SSB}\big) \;+\; \varepsilon_{a,y}$
- $\log M_{a,y} \;=\; \mu_{M_a} \;+\; \delta_{a,y}$
- $\text{logit}(a_{50,y}) \;=\; \nu_1 \;+\; \zeta_{1,y}$
- $\text{logit}(k_y) \;=\; \nu_2 \;+\; \zeta_{2,y}$
- $X_y \;=\; \mu_X\,(1-\phi_X) \;+\; \phi_X\, X_{y-1} \;+\; \omega_y$

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3. Observation Layer

Catch, survey, and age–composition data are modeled as:

- $\log C_{y,i} \;\sim\; \mathcal{N}\big(\log \widehat{C}_{y,i},\, \tau_C^2\big)$
- $\log I_{y,i} \;\sim\; \mathcal{N}\big(\log \widehat{I}_{y,i},\, \tau_I^2\big)$

Age compositions:

Logistic–normal (overdispersed comps) - Latent logits $\boldsymbol{\ell}_{y,i} \sim \mathcal{N}\big(\boldsymbol{\mu}_{y,i},\, \Sigma\big)$
- Proportions from softmax:
$$p_{a,y,i} \;=\; \frac{\exp(\ell_{a,y,i})}{\sum_{a'=1}^{A} \exp(\ell_{a',y,i})}$$

4. **Estimation / Fitting (TMB Framework)**

- Optimize fixed effects (β)
- Integrate random effects (η) by Laplace approximation
- Obtain marginal likelihood L(β, σ^2, ρ)

Mathematical Representation

WHAM can be written as a nonlinear mixed-effects model on link scales:

$$\text{Linear predictor} = X\beta + Z\eta + \varepsilon$$

where

$$\eta = \begin{bmatrix} \varepsilon_{a,y} \\ \delta_{a,y} \\ \zeta_{p,y} \\ \omega_y \end{bmatrix} \sim \mathcal{N}(0, G)$$

- Z: incidence matrix, determining which random effects enter each equation
- η: vector of Gaussian random deviations
- G: covariance structure describing correlation across age, year, or parameters

Random-Effect Components and Covariance Structures

Component	Symbol	Meaning	Covariance G_i Structure
NAA process deviations	$\varepsilon_{a,y}$	recruitment or abundance year-to-year variation	IID / AR(1) / 2D-AR(1)
Natural mortality deviations	$\delta_{a,y}$	smooth variation of log M over age & year	IID / 2D-AR(1)
Selectivity parameter deviations	$\zeta_{p,y}$	temporal changes in selectivity curve parameters	IID / parameter×year 2D-AR(1)
Environmental innovations	ω_y	stochastic evolution of environmental covariate	random walk / AR(1)

All components combine as a block-diagonal covariance matrix:

$$G = \mathrm{blockdiag}(G_{\mathrm{NAA}},\, G_M,\, G_{\mathrm{Sel}},\, G_{\mathrm{Ecov}}).$$