WHAM to POMP Model: Essential Formulas

STATE VARIABLES

State vector: $\mathbf{X}_t = [N_{1,t}, N_{2,t}, ..., N_{A,t}, \text{SSB}_t]^T$

- $N_{a,t}$ = Numbers at age a in year t
- SSB_t = Spawning stock biomass
- A = Maximum age (plus group)

PROCESS MODEL (Population Dynamics)

Recruitment (Age 1)

$$N_{1,t} = \frac{\alpha \cdot \text{SSB}_{t-1}}{1 + \beta \cdot \text{SSB}_{t-1}} \cdot \exp(\epsilon_t), \quad \epsilon_t \sim \mathcal{N}(0, \sigma_R^2)$$

Survival & Aging (Ages 2 to $A-1$)

$$N_{a,t} = N_{a-1,t-1} \cdot e^{-Z_{a-1,t-1}}$$

Plus Group (Age A)

$$N_{A,t} = N_{A-1,t-1} \cdot e^{-Z_{A-1,t-1}} + N_{A,t-1} \cdot e^{-Z_{A,t-1}}$$

Total Mortality

$$Z_{a,t} = M + F_t \cdot s_a$$

Spawning Stock Biomass

$$\text{SSB}_t = \sum_{a=1}^{A} N_{a,t} \cdot w_a \cdot m_a$$

MEASUREMENT MODEL (Observations)

Catch (Baranov Equation)

$$\hat{C}_t = \sum_{a=1}^{A} N_{a,t} \cdot \frac{F_t \cdot s_a}{Z_{a,t}} \cdot (1 - e^{-Z_{a,t}}) \cdot w_a$$

$$\log(C_t) \sim \mathcal{N}(\log(\hat{C}_t), \sigma_C^2)$$

Survey Index

$$\hat{I}_t = q \sum_{a=1}^{A} s_a \cdot N_{a,t}$$

$$\log(I_t) \sim \mathcal{N}(\log(\hat{I}_t), \sigma_I^2)$$

PARAMETERS

Parameter	Description	Transform
α	BH recruitment capacity	$\log(\alpha)$
β	BH density dependence	$\log(\beta)$
σ_R	Recruitment SD	identity
F	Fishing mortality	$\log(F)$
q	Catchability	$\log(q)$
$N_{1,1}$	Initial abundance	$\log(N_{1,1})$

Fixed parameters: M (natural mortality), w_a (weight), m_a (maturity), s_a (selectivity)

SELECTIVITY

Logistic:

$$s_a = \frac{1}{1 + e^{-\gamma(a - a_{50})}}$$

INITIAL CONDITIONS

Equilibrium with $F_0 = 0.1$:

$$N_{1,1} = \exp(\log N_{1,1})$$

$$N_{a,1} = N_{a-1,1} \cdot e^{-(M + F_0 \cdot s_{a-1})}, \quad 2 \leq a < A$$

$$N_{A,1} = \frac{N_{A-1,1} \cdot e^{-(M + F_0 \cdot s_{A-1})}}{1 - e^{-(M + F_0 \cdot s_A)}}$$

LIKELIHOOD

$$\log \mathcal{L}(\boldsymbol{\theta}) = \sum_{t=1}^{T} \left[-\frac{[\log C_t - \log \hat{C}_t]^2}{2\sigma_C^2} - \frac{[\log I_t - \log \hat{I}_t]^2}{2\sigma_I^2} \right]$$