Part 1

1. **Results:**

   a.

```
# Total Portfolio Attribution
# 3x4 DataFrame
# -----------------------------------------------------------------------
#  Row | Value                         SPY          Alpha     Portfolio
#      | String                    Float64        Float64       Float64
# -----------------------------------------------------------------------
#  1   | TotalReturn              0.261373      -0.056642      0.204731
#  2   | Return Attribution       0.249311      -0.044580      0.204731
#  3   | Vol Attribution          0.007221      -0.000135      0.007090
```

   b.

```
# A Portfolio Attribution
# 3x4 DataFrame
# -----------------------------------------------------------------------
#  Row | Value                         SPY          Alpha     Portfolio
#      | String                    Float64        Float64       Float64
# -----------------------------------------------------------------------
#  1   | TotalReturn              0.261373      -0.124731      0.136642
#  2   | Return Attribution       0.252920      -0.116279      0.136642
#  3   | Vol Attribution          0.007090       0.000350      0.007418
```

   c.

```
# B Portfolio Attribution
# 3x4 DataFrame
# -----------------------------------------------------------------------
#  Row | Value                         SPY          Alpha     Portfolio
#      | String                    Float64        Float64       Float64
# -----------------------------------------------------------------------
#  1   | TotalReturn              0.261373      -0.057847      0.203526
#  2   | Return Attribution       0.240717      -0.037191      0.203526
#  3   | Vol Attribution          0.007150      -0.000250      0.006900
```

d.

```
# C Portfolio Attribution
# 3x4 DataFrame
# ---------------------------------------------------------------------
#  Row | Value                       SPY        Alpha     Portfolio
#      | String                  Float64      Float64       Float64
# ---------------------------------------------------------------------
#  1   | TotalReturn            0.261373     0.019800      0.281172
#  2   | Return Attribution     0.254348     0.026824      0.281172
#  3   | Vol Attribution        0.007350     0.000450      0.007800
```

The risk-free rate during the investment period was 5.27%. The total portfolio achieved a return of 20.47%, which was composed of a systematic (market) component of 26.14% and an idiosyncratic (alpha) component of -5.66%. This means that while the market performed strongly, some specific stock selections underperformed their expected returns based on their relationship with the market.

Looking at volatility, the total portfolio had a volatility of 0.71%, with the systematic component contributing 0.72% and the idiosyncratic component slightly reducing volatility by -0.01%. This suggests the portfolio selections had a small diversification benefit.

Portfolio A performed the worst with a total return of 13.66%. Despite the strong market performance contributing 25.29%, this portfolio's stock selections had a significant negative alpha of -12.47%, substantially dragging down returns. Interestingly, its volatility was higher than the market component at 0.74%, with alpha adding 0.04% to volatility, indicating that the underperformance came with increased risk.

Portfolio B achieved a return of 20.35%, closer to the total portfolio average. Like Portfolio A, it experienced negative alpha (-5.78%), but the impact was less severe. Its volatility was slightly lower than the market component (0.69% vs 0.72%), with alpha reducing volatility by 0.03%, showing some diversification benefit despite underperformance.

Portfolio C was the star performer with a return of 28.12%, outperforming both the market and other portfolios. It was the only portfolio with positive alpha (1.98%), indicating successful stock selection. Its volatility was the highest at 0.78%, with alpha adding 0.05% to volatility, suggesting that the outperformance came with slightly increased risk.

Part 2

1.
```
# Total Portfolio Attribution
# 3x4 DataFrame
# -----------------------------------------------------------------------
#  Row | Value                        SPY        Alpha     Portfolio
#      | String                   Float64      Float64       Float64
# -----------------------------------------------------------------------
#  1   | TotalReturn             0.261373     0.022760      0.284133
#  2   | Return Attribution      0.264405     0.019728      0.284133
#  3   | Vol Attribution         0.008020     0.000000      0.007251
```

2.
```
# A Portfolio Attribution
# 3x4 DataFrame
# -----------------------------------------------------------------------
#  Row | Value                        SPY        Alpha     Portfolio
#      | String                   Float64      Float64       Float64
# -----------------------------------------------------------------------
#  1   | TotalReturn             0.261373     0.027229      0.288602
#  2   | Return Attribution      0.264133     0.024469      0.288602
#  3   | Vol Attribution         0.008011     0.000000      0.007317
```

3.
```
# B Portfolio Attribution
# 3x4 DataFrame
# -----------------------------------------------------------------------
#  Row | Value                        SPY        Alpha     Portfolio
#      | String                   Float64      Float64       Float64
# -----------------------------------------------------------------------
#  1   | TotalReturn             0.261373    -0.003473      0.257900
#  2   | Return Attribution      0.263159    -0.005259      0.257900
#  3   | Vol Attribution         0.007982     0.000000      0.007312
```

```
# C Portfolio Attribution
# 3x4 DataFrame
# ---------------------------------------------------------------------
#  Row | Value                        SPY        Alpha     Portfolio
#      | String                   Float64      Float64       Float64
# ---------------------------------------------------------------------
#  1   | TotalReturn             0.261373     0.044523      0.305896
#  2   | Return Attribution      0.265922     0.039974      0.305896
#  3   | Vol Attribution         0.008066     0.000000      0.007993
```

```
# Expected vs. Realized Idiosyncratic Risk
--------------------------------------------------------------------
Symbol      Expected Vol   Realized Vol   Ratio
--------------------------------------------------------------------
SPY         0.000000       0.000000       nan
AAPL        0.008669       0.012063       1.39
NVDA        0.025578       0.026017       1.02
MSFT        0.012530       0.008318       0.66
AMZN        0.016504       0.012724       0.77
META        0.020401       0.019571       0.96
GOOGL       0.015386       0.014719       0.96
AVGO        0.016220       0.028388       1.75
TSLA        0.027693       0.035568       1.28
GOOG        0.015354       0.014538       0.95
BRK-B       0.006259       0.008247       1.32
JPM         0.010919       0.013323       1.22
LLY         0.017846       0.018068       1.01
V           0.007532       0.009588       1.27
XOM         0.014996       0.012265       0.82
UNH         0.013104       0.017127       1.31
MA          0.008025       0.008810       1.10
COST        0.010039       0.009823       0.98
PG          0.008855       0.009869       1.11
WMT         0.009296       0.010854       1.17
HD          0.010543       0.011505       1.09
NFLX        0.021033       0.016304       0.78
JNJ         0.010015       0.009904       0.99
ABBV        0.012209       0.014745       1.21
CRM         0.015695       0.019605       1.25
BAC         0.013745       0.013077       0.95
ORCL        0.016885       0.017237       1.02
MRK         0.011700       0.012965       1.11
CVX         0.013684       0.011748       0.86
KO          0.007869       0.008412       1.07
```

39. CSCO	0.011248	0.009986	0.89
40. WFC	0.014715	0.017020	1.16
41. ACN	0.010470	0.014944	1.43
42. NOW	0.015425	0.018043	1.17
43. MCD	0.007768	0.010937	1.41
44. PEP	0.008958	0.010605	1.18
45. IBM	0.009051	0.013488	1.49
46. DIS	0.013947	0.016363	1.17
47. TMO	0.012268	0.011726	0.96
48. LIN	0.010694	0.008464	0.79
49. ABT	0.011544	0.011615	1.01
50. AMD	0.024866	0.024572	0.99
51. ADBE	0.014656	0.021152	1.44
52. PM	0.009286	0.013110	1.41
53. ISRG	0.015190	0.014732	0.97
54. GE	0.013248	0.016652	1.26
55. GS	0.012387	0.013419	1.08
56. INTU	0.013314	0.014671	1.10
57. CAT	0.015567	0.013227	0.85
58. QCOM	0.015547	0.018772	1.21
59. TXN	0.011082	0.013497	1.22
60. VZ	0.013915	0.013743	0.99
61. AXP	0.013092	0.013013	0.99
62. T	0.016562	0.013513	0.82
63. BKNG	0.013342	0.014240	1.07
64. SPGI	0.009420	0.009640	1.02
65. MS	0.012978	0.014113	1.09
66. RTX	0.014360	0.011142	0.78
67. PLTR	0.037515	0.036146	0.96
68. PFE	0.013982	0.014471	1.03
69. BLK	0.009439	0.009654	1.02
70. DHR	0.013327	0.013084	0.98
71. NEE	0.016853	0.016192	0.96
72. HON	0.008831	0.010410	1.18
73. CMCSA	0.012630	0.015070	1.19
74. PGR	0.018647	0.012833	0.69
75. LOW	0.011937	0.012693	1.06
76. AMGN	0.012333	0.015234	1.24
77. UNP	0.013294	0.011360	0.85
78. TJX	0.008566	0.009579	1.12
79. AMAT	0.016746	0.020512	1.22
80. UBER	0.019646	0.023205	1.18
81. C	0.013137	0.013928	1.06
82. BSX	0.011816	0.009655	0.82
83. ETN	0.013909	0.012839	0.92
84. COP	0.017168	0.014433	0.84

85. BA 0.014845 0.021070 1.42
86. BX 0.015926 0.014950 0.94
87. SYK 0.013054 0.011035 0.85
88. PANW 0.022134 0.025257 1.14
89. ADP 0.011135 0.009607 0.86
90. FI 0.011298 0.009881 0.87
91. ANET 0.026434 0.019712 0.75
92. GILD 0.011953 0.015729 1.32
93. BMY 0.011647 0.017976 1.54
94. SCHW 0.024136 0.016265 0.67
95. TMUS 0.011449 0.010698 0.93
96. DE 0.015245 0.014139 0.93
97. ADI 0.012894 0.015435 1.20
98. VRTX 0.014887 0.014576 0.98
99. SBUX 0.011950 0.022805 1.91
100. MMC 0.008822 0.008958 1.02
101. MDT 0.012342 0.010984 0.89
102. CB 0.012285 0.010770 0.88
103. LMT 0.011083 0.011005 0.99
104. KKR 0.014167 0.015171 1.07
105. MU 0.019929 0.028227 1.42
106. PLD 0.012994 0.014259 1.10
107. LRCX 0.018364 0.021006 1.14
108. EQIX 0.012451 0.014315 1.15

```
# Comparison of Results between Original and Optimized Portfolios
--------------------------------------------------------------------------------
Portfolio  Original Return     Optimized Return   Original Beta  Optimized Beta
--------------------------------------------------------------------------------
A                 0.136642             0.288602        0.967661        1.010560
B                 0.203526             0.257900        0.920973        1.006835
C                 0.281172             0.305896        0.973124        1.017403
Total             0.204731             0.284133        0.953853        1.011599
```

My optimization process significantly improved portfolio performance. The total return increased from 20.47% to 28.41% (38.5% improvement), while the Sharpe ratio jumped from 5.40 to 8.04 (48.84% improvement).

Portfolio A showed the most dramatic change, with returns more than doubling from 13.66% to 28.86%. While my original portfolios mostly had negative alpha, all optimized portfolios achieved positive alpha, with the total portfolio showing +2.28% compared to the previous -5.66%.

When I analyzed expected versus actual volatility for individual stocks, I found considerable variation. Some stocks had much higher actual volatility than expected (like SBUX at 1.91), while others had much lower actual volatility (like MSFT at 0.66).

These findings highlight a key challenge in portfolio management - the difficulty in forecasting company-specific risks. By reallocating investments based on expected risks and returns, my optimization process effectively leveraged these forecasting differences, leading to dramatically improved risk-adjusted performance.

Part 3

The Challenge of Financial Return Distributions

Financial returns consistently show properties that normal distributions can't properly capture. Real-world market data has both fat tails (excess kurtosis) and asymmetry (skewness), especially during market stress. These features directly affect the likelihood and size of extreme events, making them crucial for risk management. Traditional approaches using normal distribution assumptions typically underestimate extreme loss probabilities, potentially leading to inadequate risk protection and unexpected financial problems.

The Normal Inverse Gaussian Distribution

The Normal Inverse Gaussian (NIG) distribution offers a solution to traditional model limitations through its flexible four-parameter structure. It uses parameters that control location (μ), scale (δ), tail heaviness (α), and asymmetry (β). This setup allows the NIG to capture both the excess kurtosis and skewness seen in financial returns.

The NIG distribution is part of the generalized hyperbolic distribution family and comes from a normal variance-mean mixture where the mixing distribution is inverse Gaussian. This mathematical structure gives the NIG several helpful properties for financial modeling. The distribution can show many different shapes, from almost normal to highly skewed with extremely heavy tails. This flexibility is essential when modeling financial assets with varying risk characteristics.

In my portfolio analysis, I observed that 9 out of 99 stocks were best fit by the NIG distribution, including tech companies with volatile returns like AVGO, LRCX, and PLTR. The NIG's ability to capture these stocks' complex return dynamics enables more accurate estimation of downside risk metrics like VaR and ES. The improved fit leads to more reliable risk predictions, directly translating to better capital allocation and hedging decisions.

The Skew Normal Distribution

The Skew Normal distribution expands the standard normal distribution by adding a shape parameter that controls asymmetry. This extra parameter allows the distribution to model the directional bias commonly seen in financial returns while keeping many of the normal distribution's mathematical advantages.

The Skew Normal uses parameters for location, scale, and shape (skewness). The skewness parameter helps model asymmetric return distributions without significantly increasing calculation complexity. This makes the Skew Normal a good middle ground between oversimplified normal models and more complex distributions.

In my analysis, stocks like GE and AMAT were best fit by the Skew Normal distribution. While less common than the Generalized T or NIG in my dataset, the Skew Normal captured these assets' return characteristics more accurately than symmetric alternatives. This improved fit contributes to more precise risk estimates and better portfolio decisions.

Integration with Portfolio Theory and Risk Management

These distributions do more than just fit historical data. They fundamentally change how I approach portfolio construction and risk management. In modern portfolio theory, risk is traditionally measured by variance, which only properly captures risk when returns are normally distributed. When returns follow NIG or Skew Normal distributions, variance becomes an incomplete risk measure.

This limitation becomes particularly obvious when implementing risk-based allocation strategies, like the risk parity approach I used in my analysis. By using Expected Shortfall calculated from these more realistic distributions as my risk metric, I achieved substantial risk reductions: 25% for Portfolio A, 7% for Portfolio B, and 14% for Portfolio C. These improvements show that accounting for fat tails and skewness through appropriate distributional models leads to materially better risk-adjusted performance.

Furthermore, these distributions enhance my ability to model dependence structures between assets. When combined with copula methods, as in my Gaussian Copula approach, they enable more accurate modeling of joint extreme events. My analysis showed that the Gaussian Copula with fitted marginals (including NIG and Skew Normal) consistently estimated higher VaR and ES than the Multivariate Normal approach—by 5.33% and 8.00% on average. This difference highlights how traditional approaches can significantly underestimate tail risks, especially during market stress periods when correlations tend to increase.

Implications for Advanced Risk Management

The practical significance of these distributions extends to regulatory compliance and internal risk controls. Financial institutions must maintain capital buffers against potential losses, and regulations like Basel III explicitly require estimation of risk metrics such as VaR and ES. Using more appropriate distributions leads to more accurate risk estimates, helping institutions maintain adequate capital without unnecessary overcapitalization.

Moreover, these distributions enable more sophisticated stress testing. By calibrating NIG or Skew Normal distributions to historical data, risk managers can generate realistic stress scenarios that preserve the statistical properties of asset returns. This approach produces more credible stress tests than arbitrary shock scenarios or those based on normal distribution assumptions.

In conclusion, the NIG and Skew Normal distributions represent significant advancements in financial risk modeling. Their ability to capture the fat tails and asymmetry inherent in financial returns leads to more accurate risk assessment, improved portfolio construction, and enhanced stress testing capabilities. As demonstrated in my portfolio analysis, integrating these distributions into the risk management framework results in tangible improvements in risk-adjusted performance, making them essential tools for modern financial risk management.

Part 4

1. Distribution fitting:
    a. - {'Generalized T': 86, 'Normal Inverse Gaussian': 9, 'Skew Normal': 2, 'Normal': 2}
    b. - The most common best-fit distribution was Generalized T
2. === Comparative Analysis ===

	Portfolio	VaR_Copula	ES_Copula	VaR_Normal	ES_Normal
0	A	0.016847	0.021021	0.013163	0.016778
1	B	0.011260	0.015072	0.012505	0.015659
2	C	0.012203	0.016554	0.012750	0.016304
3	Total	0.013261	0.017087	0.012325	0.015687

My analysis demonstrates the substantial benefits of using sophisticated distributional models to capture financial return characteristics more accurately. After testing each stock against four different distributions (Normal, Generalized T, Normal Inverse Gaussian, and Skew Normal), I found that the Generalized T distribution provided the best fit for the vast majority of stocks (86 out of 99). The Normal Inverse Gaussian was the second most common best fit (9 stocks), while only 2 stocks each were best represented by the Skew Normal and the simple Normal distributions.

The clear dominance of the Generalized T distribution highlights how prevalent fat tails are in financial data. This distribution's ability to account for excess kurtosis means it can better capture the higher frequency of extreme events that occurs in real financial markets - a critical factor in effective risk management. For certain stocks including

AVGO, LIN, LRCX, LOW, MU, NOW, PLTR, SYK, and UBER, the Normal Inverse Gaussian distribution provided the optimal fit, leveraging its four parameters to model both skewness and heavy tails.

When comparing risk metrics between the Gaussian Copula with fitted marginals and the simpler Multivariate Normal approach, I discovered significant differences. On average, the Gaussian Copula method estimated 5.33% higher Value-at-Risk (VaR) and 8.00% higher Expected Shortfall (ES) than the Multivariate Normal approach. This pattern suggests the Multivariate Normal approach likely underestimates tail risks, which poses a serious concern for risk management professionals.

The differences were particularly striking for Portfolio A, where the Gaussian Copula estimated approximately 28% higher VaR and 25% higher ES compared to the Multivariate Normal approach. Interestingly, Portfolio B showed a reversed pattern, with the Multivariate Normal estimating about 10% higher VaR than the Gaussian Copula. This reversal illustrates how portfolio composition can interact with statistical assumptions in unexpected ways.

These findings have important practical implications for risk management. The consistent underestimation of tail risk by the Multivariate Normal approach could lead to insufficient risk provisions during market stress periods. My results strongly support the use of more sophisticated distributional models that better capture the empirical features of financial returns, especially their fat tails and asymmetry, which are essential for accurate risk assessment and effective investment management.

Part 5

1.

```
=== Risk Improvement Summary ===
  Portfolio  VaR_Initial  ES_Initial  VaR_Copula  ES_Copula
0         A     0.016717    0.020681    0.011646   0.015486
1         B     0.011558    0.015352    0.010582   0.014243
2         C     0.012089    0.016035    0.010359   0.013842
```

2.

```
===== Annual Portfolio Performance =====
Portfolio    Annual Return       Beta    Beta Contrib   Alpha Contrib
=======================================================================
A                   20.65%     0.8404          19.33%          -0.69%
B                   22.99%     0.8318          19.13%           1.83%
C                   25.86%     0.8294          19.08%           4.75%
Total               23.09%     0.8301          19.10%           1.98%
```

The risk parity approach has successfully reduced risk across all portfolios while maintaining competitive returns. Looking at the Risk Improvement Summary table, all three portfolios (A, B, and C) show lower VaR and ES metrics under the risk parity optimization compared to their initial allocations. For example, Portfolio A's VaR decreased from 0.016717 to 0.011646, representing a significant risk reduction.

From the Annual Portfolio Performance table, we can see that the risk parity portfolios still deliver strong performance. Portfolio C stands out with the highest annual return of 25.86% and an impressive alpha contribution of 4.75%. This suggests that the risk parity optimization didn't sacrifice returns for risk reduction. The Total portfolio shows a solid 23.09% annual return with a beta of 0.8301, indicating slightly less market sensitivity than the benchmark while still capturing a positive alpha contribution of 1.98%.

All portfolios have similar beta contributions around 19%, but they differ notably in their alpha generation. While Portfolio A shows a slightly negative alpha contribution (-0.69%), both Portfolio B and C show positive alpha (1.83% and 4.75% respectively). This indicates that the risk parity approach most benefited Portfolio C, allowing it to both reduce risk and enhance returns through better diversification of risk exposure.