Contextuality in logical form: duality for transitive partial CABAs

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Abstract. The classical duality between sets and complete atomic Boolean algebras (CABAs) is extended to the setting of transitive partial Boolean algebras. These are partial-algebraic structures that capture the logic of contextuality, as present e.g. in quantum systems, differing from traditional, orthomodular quantum logic in that operations are only defined where physically meaningful. Specifically, we establish a dual equivalence between the category of transitive partial CABAs and a category of complete exclusivity graphs. Such graphs are interpreted as spaces of *possible worlds of maximal information*, with edges representing logical exclusivity. The result implies that any transitive partial CABA can be recovered from its graph of atoms as an algebra of cliques (i.e. sets of pairwise exclusive worlds) modulo identifying cliques that jointly exclude the same set of worlds. We also give an explicit construction of the free transitive partial CABA on a set of propositions with a commeasurability relation; this goes via an adjunction between compatibility graphs and exclusivity graphs that generalises the classical powerset self-adjunction. The duality reveals a connection between Kochen and Specker's algebraic-logical setting of partial Boolean algebras and modern approaches to contextuality.

Keywords: partial Boolean algebras, Tarski duality, Kochen–Specker contextuality, exclusivity graphs

1. Introduction

Extended summary

Partial Boolean algebras were introduced by Kochen and Specker [1, 2] in their seminal work on contextuality in quantum mechanics, as a natural (algebraic-)logical setting for contextual systems, corresponding to a calculus of partial propositional functions. They provide an alternative to traditional Birkhoff–von Neumann quantum logic in which operations such as conjunction and disjunction are partial, being only defined in the domain where they are physically meaningful. In the key example of the projectors on a Hilbert space, these operations are only defined for commuting projectors, which correspond to properties of the quantum system that can be tested *jointly.*

We investigate syntax–semantics dualities à la Stone in this partial setting. As a first step in this direction, we extend the classical Tarski dualities between finite sets and finite Boolean algebras, and more generally between sets and complete atomic Boolean algebras (CABAs), to the setting of transitive partial Boolean algebras. Specifically, we establish a dual equivalence between the category of transitive partial CABAs and a category of exclusivity graphs with an appropriate notion of morphism.

The vertices of an exclusivity graph may be interpreted as *possible worlds* of *maximal information*, with edges representing logical incompatibility or mutual exclusivity between two worlds. The classical case corresponds to complete graphs, as all possible worlds are mutually exclusive. Correspondingly, morphisms are relaxed from functions to certain kinds of relations. Given an exclusivity graph, a transitive partial CABA is constructed whose elements are cliques of the graph, i.e. sets of pairwise exclusive worlds, modulo an equivalence relation. This equivalence identifies cliques that have the same neighbourhood, i.e. that jointly exclude the same set of worlds.

The main result shows, in particular, that any transitive partial CABA can be recovered in this fashion from its graph of atoms with the logical exclusivity relation.

We also give an explicit construction of the free transitive partial CABA on a set of propositions with a compatibility relation. The construction goes via an adjunction between compatibility graphs and exclusivity graphs that generalises the powerset self-adjunction from the classical case.

The duality reveals a connection between the algebraic-logical setting of partial Boolean algebra and the modern graph- [3], hypergraph- [4], and sheaf-theoretic [5] approaches to contextuality. Under it, a transitive partial CABA witnessing contextuality, in the Kochen–Specker sense that it has no homomorphism to the two-element Boolean algebra, corresponds to a graph with no 'points', i.e. with no maps from the singleton graph. Such points are equivalently described as stable, maximum clique transversal sets.

1.1. Propositions about quantum systems

Quantum systems are described mathematically in terms of complex Hilbert spaces [6]. Measurements are represented by bounded self-adjoint operators, whose (real) eigenvalues correspond to the possible measurement outcomes. Adopting an operational perspective, the **testable properties** of the state of a system are identified with dichotomic measurements, i.e. those with two possible outcomes, say 1 and 0, thought of as *true* and *false*. Mathematically, such measurements are represented by **projectors**, i.e. bounded self-adjoint operators $p\colon \mathcal{H} \longrightarrow \mathcal{H}$ satisfying $p^2 = p$.[1]

The set $\mathsf{P}(\mathcal{H})$ of projectors on a Hilbert space $\mathcal{H}$ forms an orthomodular lattice under the inclusion ordering on the corresponding subspaces. In the tradition of algebraic logic, Birkhoff and von Neumman [7] put forward the idea that the operations *meet*, *join*, and *complement* could be interpreted as logical connectives, suggesting a calculus of propositions for quantum mechanics analogous to the classical propositional calculus with respect to *and*, *or*, and *not*. A marked difference *vis-à-vis* classical logic is the failure of the distributive law, $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. This constitutes an arguably much more radical departure, logically speaking, than for example the failure of excluded middle in intuitionistic logic.

1.2. Incompatibility as partiality

A more fundamental shortcoming of traditional quantum logic based on orthomodular lattices is that the propositional connectives lack a physically meaningful operational interpretation. To appreciate the issue, it is first important to point out that a crucial rôle in the formulation of quantum mechanics is played by the notion of **commeasurability**. In contrast to classical physics, not all pairs of measurement procedures may be performed jointly on a quantum system. Such incompatibility of measurements is embodied by the noncommutativity of the corresponding self-adjoint operators.

Now, suppose we know how to perform a test (i.e. a dichotomic measurement) for each of two properties represented by projectors p and q. If p and q commute, then these tests can be jointly performed. Hence, we can design a new dichotomic measurement procedure in which both p and q are measured, and the outcome is given by the conjunction of the individual outcomes. The measurement thus implemented corresponds to the projector $p \wedge q$.[2] But how would one implement a test for $p \wedge q$ when p and q do not commute? Similar considerations apply to the join $p \vee q$.

An alternative approach captures measurement incompatibility by **partiality**, in that the logical connectives are only defined in the domain where they are physically meaningful, i.e. for commeasurable propositions. In the key example of the projections on a Hilbert space the operations are defined for commuting projectors, which correspond to properties of a quantum system that can be tested *simultaneously*. This point of view was adopted by Kochen and Specker [8, 1] and used in

1 Idempotence implies that the eigenvalues of p must be in the set $\{0, 1\}$. Projectors on $\mathcal{H}$ are in bijection with the closed linear subspaces of $\mathcal{H}$, whereby a projector is mapped to its 1-eigenspace. This is the subspace of pure quantum states for which the property being tested can be said to hold with *certainty*. In discussing examples, we refer to the projector or the closed subspace interchangeably.

2 Mathematically, also, meet and join admit simple descriptions in terms of the operations of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators when applied to **commuting** projectors p and q, to wit: $p \wedge q = pq$ and $p \vee q = p + q - pq$. Note that if p and q do not commute, the expressions on the right-hand side of these equations do not yield projectors.

their seminal work [2] on the problem of supplementing quantum mechanics with *hidden variables* that would explain away some of its counterintuitive features. They introduced the notion of **partial Boolean algebra**, providing an alternative to traditional quantum logic à la Birkhoff–von Neumann in which disjunction and conjunction become partial operations that behave classically, i.e. according to the laws of Boolean algebra, on sets of pairwise-commeasurable elements; see section 3 for details. The language of partial Boolean algebras was used to formulate their famous no-go theorem establishing (state-independent) **contextuality** as a necessary feature of any theory matching the predictions of quantum mechanics. This was expressed as the fact that the partial Boolean algebras of projectors on a Hilbert space of dimension ≥ 3 admits no homomorphism to the two-element Boolean algebra.

1.3. Tarski duality

Duality pervades mathematics. Algebra and topology/geometry are linked by a whole landscape of dual equivalences between categories of algebraic structures and categories of spaces. An example is Gel'fand–Naĭmark duality between commutative C^*-algebras and locally compact Hausdorff spaces. In the mathematical formulation of classical physics, this is interpreted as a duality between (algebras of) observables and (spaces of) states.

In logic, such dualities relate syntax and semantics: on the one hand the Lindenbaum–Tarski algebra of sentences modulo provable equivalence in a theory T, on the other the space of models of T. For classical propositional logic, this is embodied by Stone's duality between Boolean algebras and Stone spaces [9, 10, 11]. In this article, we focus on the simpler setting that corresponds to propositional theories on a *finite* language, namely the restriction of Stone duality to a dual equivalence between the category **FinBA** of finite Boolean algebras and the category **FinSet** of finite sets. In this special case the topological aspects of Stone duality trivialise as the spaces become discrete, i.e. simply sets.

A finite Boolean algebra can be represented as the powerset of its finite set of atoms, equipped with the usual set-theoretic operations of *intersection*, *union*, and *complement*. Moreover, homomorphisms between two Boolean algebras are in one-to-one correspondence with functions between their sets of atoms in the opposite direction. In categorical terms, we have functors

$$\mathsf{At}\colon \mathbf{FinBA}^{\mathrm{op}} \longrightarrow \mathbf{FinSet}$$

mapping a finite Boolean algebra to its set of atoms, and

$$\mathcal{P}\colon \mathbf{FinSet}^{\mathrm{op}} \longrightarrow \mathbf{FinBA}$$

mapping a finite set to its powerset Boolean algebra, which yield a (contravariant) equivalence of categories $\mathbf{FinSet}^{\mathrm{op}} \simeq \mathbf{FinBA}$.

If an algebra A encodes a propositional theory T, then one may interpret its set of atoms as (mutually exclusive) **possible worlds**, or models of that theory. Through the isomorphism $A \cong \mathcal{P}(\mathsf{At}(A))$, a propositional sentence is then identified with the set of worlds in which it holds true.

The free Boolean algebra on a finite set of propositions is the Lindenbaum–Tarski algebra of the empty theory, which admits all possible worlds. That is, its models are all the truth valuations on the generating propositions, equivalently represented by all the subsets of such propositions. Thus, the free-forgetful adjunction for finite Boolean algebras corresponds under the dual equivalence to the self-adjunction of the contravariant powerset functor $\mathcal{P}\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. The situation is summarised in the following diagram:

$$\begin{tikzcd}
\mathbf{FinBA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] \arrow[dr, phantom, "\top"] & & \mathbf{FinSet}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{P}"] \arrow[dl, shift left, "\mathcal{P}"] \\
& \mathbf{FinSet} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "\mathcal{P}"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd} \tag{1}$$

This duality extends to a more general setting. Tarski characterised the Boolean algebras that arise as the *whole* powerset of a (not necessarily finite) set as being the **complete atomic Boolean algebras**, CABAs for short [12]. The above-mentioned duality generalises to this non-finite setting as a dual equivalence between the category **Set** of sets and functions and the category **CABA** of complete atomic Boolean algebras and complete Boolean algebra homomorphisms.[3] The same goes for the remarks about the free-forgetful ajunction and the powerset self-adjunction. The situation is summarised in the diagram below:

$$\begin{tikzcd}
\mathbf{CABA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{P}"] \arrow[dl, shift left, "\mathcal{P}"] \\
& \mathbf{Set} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{P}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd} \tag{2}$$

This extends the earlier triangle diagram in that the categories appearing in the vertices of (1) are full subcategories of those in the corresponding vertices of (2), and the functors in the edges of (1) are the restrictions of those in the edges of (2).[4]

1.4. Contributions

The main contribution of this article is a generalisation of the Tarski dualities just described to the setting of partial Boolean algebras. In fact, we restrict our attention to **transitive** partial Boolean algebras [15, 16], or equivalently, those that satisfy the **logical exclusivity principle** (LEP) [17]; see section 3.4 ahead for details.

The results are summarised by the following two diagrams of functors:

$$\begin{tikzcd}
\mathbf{epFinBA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{FinXGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{FinRGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd} \tag{3}$$

$$\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd} \tag{4}$$

3 This duality between **Set** and **CABA** is sometimes referred to as Tarski duality in light of the result in [12].
4 This more general form of Tarski duality does not arise as a restriction of Stone duality to complete atomic Boolean algebras, for in particular, only complete Boolean algebra homomorphisms are considered.
Yet curiously, both Stone duality $\mathbf{BA} \cong \mathbf{Stone}^{\mathrm{op}}$ between Boolean algebras and Stone spaces and the Tarski duality $\mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}$ from (2) can be seen to arise from the finite duality $\mathbf{FinBA} \cong \mathbf{FinSet}^{\mathrm{op}}$ in (1) through two related category-theoretic constructions: the former arises by taking the ind-completion on the algebraic side (obtaining Boolean algebras from finite Boolean algebras), and thus dually the pro-completion on the topological side (Stone spaces arising from finite sets),

$$\mathbf{BA} \cong \mathbf{Ind}(\mathbf{FinBA}) \cong \mathbf{Ind}(\mathbf{FinSet}^{\mathrm{op}}) \cong \mathbf{Pro}(\mathbf{FinSet})^{\mathrm{op}} \cong \mathbf{Stone}\ ;$$

the latter arises as the pro-completion on the algebraic side (obtaining CABAs from finite Boolean algebras), and dually the ind-completion on the topological side (obtaining sets from finite sets),

$$\mathbf{CABA} \cong \mathbf{Pro}(\mathbf{FinBA}) \cong \mathbf{Pro}(\mathbf{FinSet}^{\mathrm{op}}) \cong \mathbf{Ind}(\mathbf{FinSet})^{\mathrm{op}} \cong \mathbf{Set}.$$

See e.g. [13, Chapter VI] or [14, Chapter 1] for more details.
CABA also arises as a category of topological algebras, namely that of Boolean algebras on Stone spaces.

These generalise the diagrams in (1) and (2) via the following (co)reflective inclusions:

$$
\begin{tikzcd}
\mathbf{epFinBA} \arrow[r, shift left, "\mathsf{At}"] \arrow[r, phantom, "\cong^{\mathrm{op}}"] \arrow[d, bend right=40] \arrow[d, phantom, "\dashv"] & \mathbf{FinXGph} \arrow[l, shift left, "\mathcal{K}"] \arrow[d, bend left=40] \arrow[d, phantom, "\dashv"] \\
\mathbf{FinBA} \arrow[u, "\iota"'] \arrow[r, shift left, "\mathsf{At}"] \arrow[r, phantom, "\cong^{\mathrm{op}}"] & \mathbf{FinSet} \arrow[u, "\iota"] \arrow[l, shift left, "\mathcal{P}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{epCABA} \arrow[r, shift left, "\mathsf{At}"] \arrow[r, phantom, "\cong^{\mathrm{op}}"] \arrow[d, bend right=40] \arrow[d, phantom, "\dashv"] & \mathbf{XGph} \arrow[l, shift left, "\mathcal{K}"] \arrow[d, bend left=40] \arrow[d, phantom, "\dashv"] \\
\mathbf{CABA} \arrow[u, "\iota"'] \arrow[r, shift left, "\mathsf{At}"] \arrow[r, phantom, "\cong^{\mathrm{op}}"] & \mathbf{Set} \arrow[u, "\iota"] \arrow[l, shift left, "\mathcal{P}"]
\end{tikzcd}
$$

To unpack their content, we consider in turn each of the vertices and edges forming the triangles in (3) and (4).

- In the upper left vertex lies the category of transitive partial Boolean algebras that generalise finite (resp. complete atomic) Boolean algebras. These are the transitive pBAs (resp. transitive complete pBAs) whose Boolean subalgebras are finite (resp. whose complete Boolean subalgebras are CABAs).
  The paradigmatic example is that of the partial Boolean algebra $\mathsf{P}(\mathcal{H})$ of projectors on a Hilbert space $\mathcal{H}$ of finite (resp. arbitrary) dimension.
  From a logical perspective, these partial Boolean algebras can be regarded as 'algebras of propositions' where propositions might not be commeasurable, and thus propositional connectives are only partially defined.

- The universe of such a partial algebra is a set that additionally carries a compatibility or commeasurability relation, i.e. a reflexive graph. In the opposite direction, recall that a free finite Boolean algebra (resp. free CABA) is built from a finite set (resp. arbitrary set) of basic propositions: its elements correspond to the sentences built syntactically from these basic propositions, up to provable equivalence in the propositional calculus (the Lindenbaum–Tarski algebra of the empty theory). In the case of transitive partial Boolean algebras, extra data must be specified to determine which sentences are considered well formed. The extra ingredient is a compatibility relation between basic propositions. This explains the choice of reflexive (or compatibility) graphs in the bottom vertex of the triangle as the natural category from which to build a free-forgetful adjunction with **epFinBA** (resp. **epCABA**), as depicted in the left edge of the triangle.

- The top edge of the triangle embodies the main result of the paper, the generalisation of the Tarski duality $\mathbf{FinBA} \cong \mathbf{FinSet}^{\mathrm{op}}$ (resp. $\mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}$). As in the classical case, the rightward functor At maps a partial Boolean algebra to its atoms. But rather than regarding these as simply forming a set, they form an (exclusivity) graph: one must take into account whether two distinct atoms are compatible, and therefore mutually exclusive.
  Informally, we can think of the atoms of a partial Boolean algebra as the *possible worlds of maximal information*. Two such worlds will be linked by an edge whenever they are directly contradictory; in other words, if there is a testable proposition that distinguishes between them. The classical case corresponds to complete graphs, as all maximally specified worlds are mutually exclusive.
  In our main motivating example, the atoms of $\mathsf{P}(\mathcal{H})$ correspond to the rays of $\mathcal{H}$, or equivalently, to the elements of the projective Hilbert space, or to normalised vectors up to phase. In quantum mechanics, these are the pure states of the system. The exclusivity relation on the atoms of $\mathsf{P}(\mathcal{H})$ is given by *orthogonality*.
  Correspondingly, a homomorphism of partial Boolean algebras will not simply yield a function between their sets of atoms, but rather a (specific type of) relation between their graphs of atoms.

- The leftward functor can be thought of as a reconstruction of a transitive partial finite (resp. complete atomic) Boolean algebra from its graph of atoms. As in the classical case, elements of the algebra – which we may think of as properties – are identified with the set of atoms

below them, i.e. with the worlds where the property holds. The catch, however, is that the decomposition of an element as a join of atoms is no longer unique. While each set of pairwise exclusive worlds (a clique of the graph of atoms) can be thought of as determining a property, there may be different such cliques that correspond to the same property.
As an example, consider the partial Boolean algebra $\mathsf{P}(\mathcal{H})$ of projectors on a Hilbert space of dimension at least 2. If $\mathbf{x}$ and $\mathbf{y}$ are two orthogonal vectors in $\mathcal{H}$, then e.g.

$$\{\, \mathbb{C}\mathbf{x},\ \mathbb{C}\mathbf{y} \,\} \qquad \text{and} \qquad \{\, \mathbb{C}\,(\mathbf{x}+\mathbf{y}),\ \mathbb{C}\,(\mathbf{x}-\mathbf{y}) \,\}$$

are two different cliques in the graph of atoms but have the same join, namely the (projection onto the) 2-dimensional subspace $\mathbb{C}\mathbf{x} + \mathbb{C}\mathbf{y}$ spanned by $\mathbf{x}$ and $\mathbf{y}$.
It turns out that for transitive partial Boolean algebras this equivalence relation between cliques induced by 'having the same join' can be characterised at the level of atoms, as 'having the same common neighbourhood', or in other words, 'jointly excluding the same set of worlds'. Therefore, a finite (resp. complete atomic) partial Boolean algebra can be fully reconstructed from just the information contained in its graph of atoms, as an algebra of equivalence classes of cliques.
Of course, in case of (total) Boolean algebras, whose graph of atoms is complete, the cliques are precisely the subsets of atoms. We therefore recover the classical reconstruction of finite (resp. complete atomic) Boolean algebras from their set of atoms.

- In the upper right vertex of the triangle, the category **FinXGph** (resp. **XGph**) identifies the image of the functor At, i.e. which graphs – and which relations between them – arise as graphs of atoms from partial algebras – and homomorphisms between them – in **epFinBA** (resp. **epCABA**).
The notion of exclusivity graph thus obtained can be thought of as a generalisation of the *inequality relation* on a set. Indeed, that is precisely how **FinSet** (resp. **Set**) can be regarded as a full subcategory of **FinXGph** (resp. **XGph**).
One can break the characterisation of inequality into four independent axioms: a relation $\#$ is the inequality relation if and only if it is a coequivalence relation – i.e. it is irreflexive, symmetric, and cotransitive – and it satisfies an *identity of indiscernibles* axiom,

$$(\forall z.\ x \# z \iff y \# z) \implies x = y\ . \tag{5}$$

The notion of exclusivity graph keeps irreflexivity and symmmetry, as implicit in the terminology 'graph', as well as axiom (5). The weak link is cotransitivity,

$$x \# y \implies x \# z \ \text{ or } \ y \# z\ ,$$

the property that if two worlds are mutually exclusive, then any possible world must exclude at least one of them. Back to our running example, note that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors in $\mathcal{H}$, then $\mathbf{x} + \mathbf{y}$ is orthogonal to neither $\mathbf{x}$ nor $\mathbf{y}$.
In the definition of exclusivity graphs, cotransitivity is weakened to the following property: for $K \cup L$ a maximal clique,

$$x \# K \ \text{ and } \ y \# L \implies x \# y\ ,$$

where $x \# K$ stands for $x \# k$ for all $k \in K$. Figure 1 illustrates this condition.

- The notion of morphism between exclusivity graphs can also be understood from this point of view. Recall that a relation is a function if and only if it is functional and left-total. Morphisms of exclusivity graphs are defined to be relations satisfying two conditions. In the presence of cotransitivity in both graphs, i.e. in the case of complete (or inequality) graphs, these conditions reduce precisely to functionality and left-totality, albeit phrased unorthodoxly in terms of inequality rather than equality.
More generally, the first condition, generalising functionality, says that two worlds that are not mutually exclusive cannot be mapped to mutually exclusive worlds.

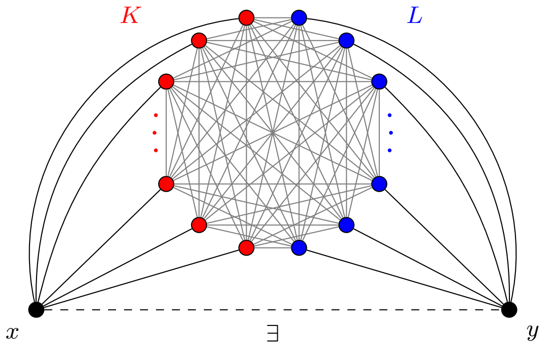

Figure 1. Depiction of condition (E1) in Definition 5.1 of exclusivity graphs: if K (in red) and L (in blue) are sets of vertices such that $K \cup L$ is a maximal clique, and x and y are two vertices respectively connected to all of K and L, then there is an edge (dashed line) between x and y.

- Finally, the free-forgetful adjunction can be translated along the duality to yield an adjunction between the categories **RGph** of compatibility graphs and **XGph** of exclusivity graphs. This generalises the power-set self-adjunction from the classical case, but the symmetry of that double-construction is somewhat broken.
  In particular, the upwards functor $\mathcal{V}$ builds an exclusivity graph whose vertices (possible worlds) are assignments of truth values to maximal subsets of compatible propositions.

1.5. Discussion

'Commutative algebra is like topology, only backwards', according to John Baez [18]. *Commutative* is a key word here. Typically, such dualities between algebra and topology work by representing an algebraic structure as some space of scalar-valued functions on a topological space, with algebraic operations being defined pointwise, and thus inheriting commutativity from the scalars.

The question of extending this 'mirror of mathematics' [19] to the *noncommutative* – or *quantum* – setting has received considerable attention. The field of noncommutative geometry [20] springs from the idea of treating noncommutative algebras *as if* they were algebras of functions on a topological space, borrowing geometric intuition and tools. This turns out to be remarkably effective, even though predicated on noncommutative spaces that are entirely fictional.

Such implausible effectiveness – or at any rate the allure – of geometric intuition motivates the quest for more concrete realisations of spectra for various kinds of 'noncommutative' algebraic structures, in a way that extends *classical* dualities such as those of Stone or Gel'fand–Naĭmark. Here we understand noncommutativity in a broad sense. We mean it to include not only C^*-algebras or von Neumann algebras, but also other algebraic structures arising in (various forms of) quantum logic. So it comes in many guises: as failure of associativity in Jordan algebras, of distributivity in orthomodular lattices, ... or – like in this article – as partiality.

This endeavour is not straightforward. Reyes [21] showed that any extension of the Zariski spectrum functor[5] to a contravariant functor from the category of (not necessarily commutative) rings to the category of topological spaces trivialises on matrix algebras $\mathbb{M}_n(\mathbf{C})$ with $n \geq 3$. Reyes also proved an analogous statement regarding extending Gel'fand–Naĭmark duality to noncommutative C^*-algebras. Van den Berg and Heunen [22, 23] extended these results to rule out locales, ringed toposes, schemes, and quantales as categories of spectra, as well as to encompass Stone and Pierce spectra. The proofs of these no-go theorems go via partial structures of 'commutative' contexts, such as partial Boolean algebras and their C^*-algebraic analogues. The obstructions arise in the partial C^*-algebra (where multiplication and addition are only defined for commuting matrices) of 3-by-3 matrices $\mathbb{M}_3(\mathbb{C})$ or in the partial Boolean algebra of projectors on a 3-dimensional Hilbert space, $\mathsf{P}(\mathbb{C}^3)$. Ultimately, such obstructions come down to the presence of contextuality. That is, they are based on the Kochen–Specker theorem [24, 2], which states that if

5 This (contravariant) functor assigns to each unital commutative ring a topological space, its Zariski spectrum, and to a ring homomorphism a continuous function in the opposite direction between the corresponding spectra.

there is a homomorphism of partial Boolean algebras from $\mathsf{P}(\mathbb{C}^3)$ to a Boolean algebra B then B is the trivial Boolean algebra where $0 = 1$, which corresponds to an inconsistent logical theory.[6]

Even though the present paper focuses on the simplest of the Stone-like dualities, it is a setting that already falls under the scope of these no-go theorems. Most directly, Theorem 8 in [22] forbidding locale-valued extensions of Stone duality to partial Boolean algebras, or at least its restriction to the full subcategory **epFinBA**, applies directly to the finitary situation we consider, and by trivial extension to the case of **epCABA**, too. The analogous results for **Top**-valued or **Set**-valued functors also hold. This is essentially due to the fact that the partial Boolean algebras $\mathsf{P}(\mathcal{H})$ of projectors on finite-dimensional Hilbert spaces – and thus $\mathsf{P}(\mathbb{C}^3)$ in particular – are objects of **epFinBA**.

Necessarily, then, our results circumvent this obstruction by adopting a notion of spectrum – more precisely, a category of spectra – that departs more radically from topological spaces. Indeeed, the authors of [23] seemed to some extent to agree with John S. Bell that 'what is proved by impossibility proofs is lack of imagination' [25]:

> 'It might be tempting to conclude from all the above impossibility results that it is hopeless to look for a good notion of spectrum for noncommutative structures. But we strongly believe that this is the wrong conclusion to draw.'

They hit the nail on the head in identifying the crux of the blockade, and therefore the key to sidestepping it (our emphasis):

> 'What our results show is merely that a category of noncommutative spectra must have *different limit behaviour* from the known categories of commutative spectra.'

And with their suggestion on how to achieve this:

> 'One of the central messages of category theory is that objects should be regarded as determined by their behaviour rather than by any internal structure. In other words, it is not the internal structure of objects that dictates what morphisms should preserve. It is the other way around: it is the morphisms connecting an object to others that determine that object's characteristics. [. . .] Historically, noncommutative spectra have almost always been pursued by generalizing the internal structure of commutative spaces (as objects). We believe the right, and optimistic, message to distill from our results is that one should let the search for noncommutative spectra be guided by morphisms instead.'

The relaxation of the functionality conditions on relations constituting the morphisms of **XGph** is precisely an example of this.

Outline of the article

Section 2 reviews the classical duality between **Set** and **CABA**. Section 3 sets the stage for our main result by recalling the definition of (transitive or LEP) partial Boolean algebra, and introducing the analogues of CABAs in this setting, i.e. the algebraic side of our duality. Section 4 shows how a transitive partial CABA can be reconstructed from its graph of atoms, while Section 5 characterises the graphs that arise as graphs of atoms of such partial algebras, dubbed complete exclusivity graphs. Section 6 focuses on morphisms, identifying a notion of morphism between exclusivity graphs that is dual to partial complete Boolean algebra homomorphisms. This completes the main duality result, which is summarised in Section 7. Section 8 looks at the free-forgetful adjunction for partial CABAs in light of this duality, yielding an explicit construction of free partial CABAs from a compatibility graph, via an adjunction between compatibility graphs and complete exclusivity graphs that generalises the classical powerset self-adjunction. Finally, Section 9 concludes by formulating open questions and directions for further work.

6 The equivalent statement in terms of partial C^*-algebra is that if there is a homomorphism of partial C^*-algebras from $\mathbb{M}_3(\mathbb{C})$ to a commutative C^*-algebra A, then A is the trivial C^*-algebra $C(\varnothing)$ of continuous functions on an empty topological space, which has a single element.

2. Background: The classical Tarski duality

We start by reviewing the classical duality between sets and CABAs [12]. This is standard material. It can be found in its modern, category-theoretic presentation in multiple references, e.g. [26, Section 2.3]. Our presentation is self-contained and fairly detailed. The purpose of laying it out is to facilitate comparison with the partial Boolean algebra case in subsequent sections.

2.1. Boolean algebras

Definition 2.1. A **Boolean algebra** is a complemented distributive lattice. Explicitly, it is an algebraic structure $\langle A; \wedge, \vee, \neg, 1, 0\rangle$ consisting of a set A equipped with binary operations $\wedge$ ('meet') and $\vee$ ('join'), a unary operation $\neg$ ('complement'), and constants 1 ('top') and 0 ('bottom'), satisfying the following equational axioms: $\wedge$ and $\vee$ are commutative operations with identity elements 1 and 0 respectively, they distribute over each other, and $\neg$ is a complement operation in the sense that $a \wedge \neg a = 0$ and $a \vee \neg a = 1$ for all $a \in A$.[7]

A **Boolean algebra homomorphism** is a function between the underlying sets preserving the operations. We write **BA** for the category of Boolean algebras.

We list some standard examples of Boolean algebras.

Example 2.2. Given a set X, its powerset $\mathcal{P}(X)$ forms a Boolean algebra with the set-theoretic operations of *intersection*, *union*, and *complement*, and the constants X itself and the *empty set*.

Example 2.3. The set of (self-adjoint) idempotent elements of a unital, commutative ($*$-)ring forms a Boolean algebra, with operations given by $e \wedge f := ef$, $e \vee f := e+f-ef$, and $\neg e := 1-e$, and Boolean-algebraic constants 1 and 0 given by the homonymous idempotent elements of the ring. In particular, this applies to the set of projectors of any commutative C^*-algebra. For noncommutative rings, the central idempotents, i.e. those commuting with all elements of the ring, form a Boolean algebra under these same operations.

Example 2.4. Let V be a set of propositional variables, and write $\Phi(V)$ for the set of propositional formulae with variables in V, given by the grammar

$$\varphi,\ \psi \ ::= \ v \in V \ \mid\ 0 \ \mid\ 1 \ \mid\ \neg\varphi \ \mid\ \varphi \wedge \psi \ \mid\ \varphi \vee \psi \ .$$

A propositional theory over V is a subset $T \subseteq \Phi_V$. It determines an equivalence relation $\equiv_T$ on Φ_V whereby

$$\varphi \equiv_T \psi \qquad \text{if and only if} \qquad T \vdash \varphi \leftrightarrow \psi \ ,$$

i.e. φ and ψ are in the same equivalence class whenever they are provably equivalent from the theory T.[8] Since this equivalence relation is in fact a congruence, the set of equivalence classes $\Phi_V/{\equiv_T}$ inherits the algebraic operations from their syntactical counterparts, e.g. $[\varphi] \wedge [\psi] := [\varphi \wedge \psi]$. Thus it forms a Boolean algebra $\mathcal{L}(T)$ called the Lindenbaum–Tarski algebra of T.
If T consists entirely of propositional tautologies (in particular if $T = \varnothing$), then the corresponding algebra is the free Boolean algebra on the set V, corresponding to equivalence classes of formulae with variables in V under provable equivalence in the classical propositional calculus.

A Boolean algebra carries a **partial order** structure on its elements whereby $a \leq b$ whenever any – and therefore all – of the following equivalent conditions hold:

$$a \wedge b = a \ , \qquad a \vee b = b \ , \qquad a \wedge \neg b = 0 \ , \qquad \neg a \vee b = 1 \ .$$

This partial order determines A as a Boolean algebra. For example, the operations $\wedge$ and $\vee$ correspond respectively to the binary infimum (greatest lower bound) and supremum (least upper bound) operations relative to the partial order.

As in the study of lattices more generally, it is often convenient to switch back and forth between the algebraic and order-theoretic perspectives.

7 There are many equivalent axiomatisations of Boolean algebras. The above is an easy-to-state, symmetrical set of independent axioms going back to Huntington [27]. In particular, note that the usual lattice-theoretic axioms of associativity and idempotency of $\wedge$ and $\vee$ as well as the absoprtion laws, $a \wedge (a \vee b) = a$ and its dual, can of course be derived from these.

8 The symbol $\vdash$ denotes the provability relation under classical propositional calculus between the set $\mathcal{P}(\Phi_V)$ of theories and the set Φ_V of formulae. The symbol $\leftrightarrow$ is syntactic sugar: $\varphi \leftrightarrow \psi := (\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi)$.

2.2. Complete atomic Boolean algebras (CABAs)

In a Boolean algebra, any finite set of elements has a supremum: this follows from the existence of binary ($\vee$) and nullary (0) suprema. But infinite sets of elements need not have one.

A Boolean algebra A is said to be **complete** if any set of elements $S \subseteq A$ has a supremum $\bigvee S$ in A (and consequently an infimum $\bigwedge S$, too). Algebraically, it has additional infinitary operations

$$\bigwedge, \bigvee : \mathcal{P}(A) \longrightarrow A \ .$$

A complete Boolean algebra homomorphism is a Boolean algebra homomorphism that preserves arbitrary suprema (equivalently, infima). We write **CBA** for the category of complete Boolean algebras and complete Boolean algebra homomorphims.

An **atom** of a Boolean algebra is a minimal non-zero element in the partial order. That is, it is an element $x \neq 0$ such that for any element $a \neq 0$ of the algebra, $a \leq x$ implies $a = x$. A (trivially) equivalent way to phrase this condition is to say that $x \neq 0$ and x is $\vee$-indecomposable: if $x = a \vee b$ then $x = a$ or $x = b$. Another useful equivalent is that $x \neq 0$ and for all a, $x \wedge a = x$ or $x \wedge a = 0$.

A common and suggestive way to think of an atom is as a model or *possible world*. It is a maximally specified event, which assigns a truth value to each proposition (element of the Boolean algebra). Thus if x is an atom, and a an element of the Boolean algebra, we say that x satisfies a if $x \leq a$. Then we can see that x satisfies $a \wedge b$ iff x satisfies a and x satisfies b, and that x satisfies $\neg a$ iff it does not satisfy a. As we shall see, an atom corresponds to a point in the space of events.

The following two lemmas state simple but key properties of atoms in any Boolean algebra. These are central to establishing the duality. We state them independently for ease of reference in the ensuing discussion.

Lemma 2.5. *Let A be a Boolean algebra, and x and x' be atoms of A. If $x \neq x'$ then $x \wedge x' = 0$*

Proof. We have $x \wedge x' \leq x$ and $x \wedge x' \leq x'$. Because x and x' are atoms, either $x \wedge x'$ is equal to 0 or it is equal to both x and x', in which case $x = x'$. □

Lemma 2.6. *Let A be a Boolean algebra, x and atom of A, and $S \subseteq A$ a subset of elements with a supremum. If $x \leq \bigvee S$, then there is an $s \in S$ with $x \leq s$.*

Proof. By definition, $x \leq \bigvee S$ means $x = x \wedge \bigvee S$, or using distribtivity,

$$x = \bigvee_{s \in S} x \wedge s \ .$$

Since x is an atom, $x \wedge s$ is either equal to 0 or to x for each $s \in S$. The fact that the join of all these elements is x means that they cannot all be equal to 0. Hence, there is an $s \in S$ such that $x \wedge s = x$ as required. □

A Boolean algebra A is said to be **atomic** if every non-zero element sits above an atom, i.e. if for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

Our focus will be on **complete atomic Boolean algebras**, or **CABA**s for short. We write **CABA** for the corresponding full subcategory of **CBA**, i.e. for the category of complete atomic Boolean algebras and complete Boolean algebra homomorphisms.

Example 2.7. Any finite Boolean algebra is atomic, and therefore trivially a CABA. More generally, the paradigmatic – and, as we shall shortly see, the 'only' (up to isomorphism) – example of a CABA is the powerset $\mathcal{P}(X)$ of an arbitrary set X, i.e. the Boolean algebra consisting of *all* the subsets of X. Being closed under arbitrary unions ensures completeness, while its atoms are the singletons $\{x\} \in \mathcal{P}(X)$ for $x \in X$, and clearly any non-empty subset of X contains a singleton.

2.3. Reconstructing a CABA from its atoms

Any subset S of the atoms of a complete Boolean algebra determines an element $\bigvee S$ of the algebra. It turns out that, for a complete Boolean algebra, being atomic implies (and is therefore equivalent to) an apparently stronger property: that any element can be written in this fashion, as a join of a set of atoms. In other words, that the assignment $S \longmapsto \bigvee S$ from sets of atoms to elements of the algebra is surjective.

Proposition 2.8. *In a CABA A, every element is the join of the atoms below it, i.e. any $a \in A$ can be written as*

$$a = \bigvee \mathcal{U}(a) \quad \textit{where} \quad \mathcal{U}(a) := \{x \in A \mid x \in \mathsf{At}(A) \textit{ and } x \leq a\}\,.$$

Proof. Clearly, $\bigvee \mathcal{U}(a) \leq a$. For the opposite inequality, we show that $a \wedge \neg \bigvee \mathcal{U}(a) = 0$. By atomicity, it suffices to show that this element has no atoms below it. Suppose that x is an atom with $x \leq a \wedge \neg \bigvee \mathcal{U}(a)$. On the one hand, $x \leq \neg \bigvee \mathcal{U}(a)$. On the other, $x \leq a$, i.e. $x \in \mathcal{U}(a)$, hence $x \leq \bigvee \mathcal{U}(a)$. This implies $x = 0$, contradicting the fact that x is an atom. □

Moreover, this decomposition of an element as a join of atoms is unique. In other words, the assignment $S \longmapsto \bigvee S$ is also injective.

Proposition 2.9. *Let S be a set of atoms in a Boolean algebra. Then $\mathcal{U}(\bigvee S) = S$.*

Proof. The inclusion $\supseteq$ is clear. For $\subseteq$, let x be an atom with $x \leq \bigvee S$. By lemma 2.6, there is a $y \in S$ with $x \leq y$. Since y is an atom and $x \neq 0$ this implies $x = y$, hence $x \in S$. □

Corollary 2.10. *For S and T sets of atoms in a CABA, $\bigvee S \leq \bigvee T$ if and only if $S \subseteq T$. Consequently, $\bigvee S = \bigvee T$ if and only if $S = T$.*

Proof. This follows from the previous proposition and monotonicity of the map $\mathcal{U}$ taking an element of the algebra to the set of atoms below it, in turn an easy consequence of transitivity of $\leq$. □

Propositions 2.8 and 2.9 form the basis for the reconstruction of a CABA from its set of atoms. Together they establish a bijection between elements of a CABA A and subsets of atoms of A, given by the maps $\bigvee$ and $\mathcal{U}$. Moreover, as shown in corollary 2.10, the partial order on A corresponds precisely to set-theoretic inclusion under this bijection. Consequently, given the order-theoretic characterisation of Boolean algebras, the *join* $\bigvee$, *meet* $\bigwedge$, and *complement* $\neg$ operations on A correspond to set-theoretic *union*, *intersection*, and *complement*. This establishes a complete Boolean algebra isomorphism between a CABA A and the powerset of its set of atoms. It is one of the natural isomorphisms that constitutes the categorical duality between **CABA** and **Set**.

2.4. Morphisms

We now consider the morphism part of the duality. The idea is to describe complete Boolean algebra homomorphisms between CABAs in terms of their sets of atoms. It turns out that such homomorphisms correspond precisely to (set-theoretic) functions between the atoms, in the opposite direction. This key result is encapsulated in the following proposition. It ultimately rests on the two easy properties about atoms established in Lemma 2.6 – one for existence, the other for uniqueness.

Proposition 2.11. *Let $h\colon A \longrightarrow B$ be a complete Boolean algebra homomorphism between CABAs A and B. For each atom y of B, there exists a unique atom x of A such that $y \leq h(x)$.*

Proof. Proposition 2.8 yields $1_A = \bigvee \mathsf{At}(A)$. Together with the fact that h is a **CBA** morphism, we obtain

$$1_B = h(1_A) = h(\bigvee \mathsf{At}(A)) = \bigvee \{h(x) \mid x \in \mathsf{At}(A)\}$$

Since y is an atom and it trivially holds that $y \leq 1_B$, Lemma 2.6 entails that $y \leq h(x)$ for some $x \in \mathsf{At}(A)$.

For uniqueness, suppose that $y \leq h(x)$ and $y \leq h(x')$ with $x, x' \in \mathsf{At}(A)$. By Lemma 2.5 we either have $x = x'$, as desired, or $x \wedge_A x' = 0_A$, in which case

$$y \leq h(x) \wedge_B h(x') = h(x \wedge_A x') = h(0_A) = 0_B$$

would contradict the fact that y is an atom. □

Proposition 2.11 determines a function $\mathsf{At}(B) \longrightarrow \mathsf{At}(A)$ from a complete Boolean algebra homomorphism $h\colon A \longrightarrow B$. Let us pause to analyse this result from a perspective that may appear gratuitous now, but which will help understand the shift to the partial setting later. Given $h\colon A \longrightarrow B$, we can define a relation $\mathsf{At}(h)\colon \mathsf{At}(B) \longrightarrow \mathsf{At}(A)$ whereby $y\ \mathsf{At}(h)\ x$ whenever $y \leq h(x)$.

The content of Proposition 2.11 is that this relation is a function. The existence part of the proof corresponds to left-totality, the uniqueness part to functionality. These rest on Lemma 2.6 and Lemma 2.5, respectively. Both of these properties need to be revised for partial Boolean algebras, leading to the variations of left-totality and functionality to be found in Definition 6.1.

2.5. The classical Tarski duality

We now have all the necessary ingredients for the duality between the categories **CABA** and **Set**. It is given by the following contravariant functors:

$$\begin{tikzcd} \mathbf{CABA} \arrow[rr, bend right, "\mathsf{At}"'] \arrow[rr, phantom, "\cong"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{P}"'] \end{tikzcd}, \tag{6}$$

The functor

$$\mathsf{At}\colon \mathbf{CABA}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is defined as follows:

- on objects: it maps a CABA A to its set of atoms.
- on morphisms: given a complete Boolean homomorphism $h\colon A \longrightarrow B$, it yields a function

  $$\mathsf{At}(h)\colon \mathsf{At}(B) \longrightarrow \mathsf{At}(A)$$

  mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

The functor

$$\mathcal{P}\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$$

is the usual contravariant powerset functor (equipped with complete Boolean algebra structure), given as follows:

- on objects: it maps a set X to its powerset $\mathcal{P}X$, which forms a CABA under the set-theoretic operations.
- on morphisms: given a function $f\colon X \longrightarrow Y$, it yields a complete Boolean algebra homomorphism

  $$\begin{aligned} \mathcal{P}(f)\colon \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) \\ (T \subseteq Y) &\longmapsto f^{-1}(T) = \{x \in X \mid f(x) \in T\} \end{aligned}$$

The duality (6) is witnessed by two natural isomorphisms:

- Given a CABA A, the isomorphism $A \cong \mathcal{P}(\mathsf{At}(A))$ maps $a \in A$ to the set

  $$\mathcal{U}(a) = \{x \in \mathsf{At}(A) \mid x \leq a\}\,.$$

  Thus, a property is identified with the set of *possible worlds* in which it holds.
- Given a set X, the bijection $X \cong \mathsf{At}(\mathcal{P}(X))$ maps $x \in X$ to the singleton set $\{x\}$, which is an atom of $\mathcal{P}(X)$. I.e. a possible world is identified with its characteristic property (which completely determines that world).[9]

9 In the more general version of Stone duality, such characteristic properties are not available (as the algebra need not be atomic). There, a possible world (point in the state space) is identified with the set of properties which hold true/false in that world, which forms an ultrafilter/prime ideal of the algebra.

3. Background: Partial Boolean algebras

We recall the basic definitions and properties of partial Boolean algebras [1, 2], and introduce the particular classes of such structures on which this article focuses.

3.1. Compatibility relations

We start by considering compatibility graphs. These are the natural replacement for sets as the underlying universes of partial Boolean algebras.

Definition 3.1. A **compatibility graph** (a.k.a. **reflexive graph**) is a pair $\langle V, \odot_V \rangle$ consisting of a set equipped with a symmetric, reflexive binary relation. A morphism of compatibility graphs is a function between the underlying sets that preserves the compatibility relation, i.e. a function $f\colon V \longrightarrow W$ such that $p \odot_V q$ implies $f(p) \odot_W f(q)$. These form a category that we denote by **RGph**.

As customary, we abuse notation by referring to a compatibility graph $\langle V, \odot_V \rangle$ as simply V, and dropping the index from the relation symbol whenever the compatibility relation is clear from context. Given a compatibility graph V, we write $\bigodot_V$ for the set of subsets of V whose elements are pairwise commeasurable,

$$\bigodot_V \;:=\; \{S \subseteq V \mid S \times S \subseteq \odot_V\} \;=\; \{S \subseteq V \mid \forall a, b \in S\textbf{.}\ a \odot b\} \ .$$

Compatibility graphs should be understood as a generalisation of sets, whereby the latter are seen as compatibility graphs for which every pair of elements is compatible. This is more formally expressed by the following proposition.

Proposition 3.2. *The functor* $\Delta\colon \mathbf{Set} \longrightarrow \mathbf{RGph}$ *mapping a set* X *to the compatibility graph* $\Delta(X) := \langle X, \top \rangle$*, consisting of* X *equipped with the universal relation* $\top = X \times X$*, reveals* **Set** *as (isomorphic to) a reflective subcategory of* **RGph**. *Moreover, the reflector is faithful.*

Proof. The assignment is clearly functorial. Moreover, the condition that a morphism preserve the compatibility relation trivialises when the codomain has the universal compatibility relation, i.e. for any compatibility graph $\langle V, \odot_V \rangle$ and a set W,

$$\mathbf{RGph}(\langle V, \odot_V \rangle, \langle W, \top \rangle) \;\cong\; \mathbf{Set}(V, W) \ . \tag{7}$$

In particular, taking $\odot_V = \top$ shows that the functor $\Delta :: V \longmapsto \langle V, \top \rangle$ is full and faithful. At the same time, the isomorphism in eq. (7) means that Δ has a left adjoint: namely, the functor $\langle V, \odot_V \rangle \longmapsto V$. □

Proposition 3.3. *The category* **RGph** *is complete and cocomplete. The 'inclusion'* $\Delta\colon \mathbf{Set} \longrightarrow \mathbf{RGph}$ *from proposition 3.2 preserves all limits (for it is a right adjoint) and coequalisers, and therefore creates them.*

Proof. Products in **RGph** are constructed as follows. Given $\{\langle V_i \rangle\}_{i \in I}$ a family of objects of **RGph**, its product is the set-theoretic (Cartesian) product $\prod_{i \in I} V_i$ equipped with the compatibility relation whereby

$$(x_i)_{i \in I} \odot (y_i)_{i \in I} \qquad \text{if and only if} \qquad \forall i \in I\textbf{.}\ x_i \odot_{V_i} y_i \ .$$

This clearly makes the projections functions $\pi_j\colon \prod_{i \in I} V_i \longrightarrow V_j$ morphisms of compatibility graphs. It is also immediate that if $f_i\colon W \longrightarrow V_i$ of **RGph** are morphisms of **RGph** then so is the function $\langle f_i \rangle_{i \in I}\colon W \longrightarrow \prod_{i \in I} V_i w(f_i(w))_{i \in I}$, yielding the required universal property.

The *equaliser* of two morphisms $f, g\colon V \longrightarrow W$ is given by the set-theoretic one, $E(f, g) = \{x \in V \mid f(x) = g(x)\}$, equipped with the compatibility relation inherited from V. Again, the proof of the universal property is as in **Set**, since the unique connecting function is an **RGph**-morphism whenever the equalising morphism is. ´

The *coequaliser* of two morphisms $f, g\colon V \longrightarrow W$ is also given by the set-theoretic one, namely the quotient of W by the smallest equivalence relation $\sim$ such that $f(x) \sim g(x)$ for all $x \in X$, equipped with the smallest compatibility relation making $W \longrightarrow W/\sim$ an **RGph**-morphism, whereby two equivalence classes are compatible if there are representatives that are compatible in W, i.e. $[y] \odot [z]$ if there are $y' \sim y$ and $z' \sim z$ with $y' \odot_W z'$. Again, the proof of the universal property goes through as the unique connecting function is an **RGph**-morphism whenever the coequalising morphism is.

Finally, *coproducts* in **RGph** are constructed as follows. Given $\{\langle V_i \rangle\}_{i \in I}$ a family of objects of **RGph**, their coproduct is the set-theoretic coproduct (disjoint union) $\coprod_{i \in I} V_i$ equipped with a compatibility relation whereby for $x \in V_i$ and $y \in V_j$,

$$\iota_i(x) \odot \iota_j(y) \qquad \text{if and only if} \qquad i = j \text{ and } x \odot_{V_i} y$$

where $\iota_j \colon V_j \longrightarrow \coprod_{i \in I} V_i$ stand for the injection functions. This definition makes each of these injections a morphism in **RGph**, and the same is true of the set-theoretic $\coprod_{i \in I} f_i \colon \coprod_{i \in I} V_i \longrightarrow W$ whenever $f_i \colon V_i \longrightarrow W$ are **RGph**-morphisms.

The first three constructions above reduce to the corresponding (co)limits in **Set** whenever the objects involved are of the form $\langle V, \top \rangle$, i.e. in the image of the inclusion Δ. Hence, Δ preserves them (note that for limits this was already a consequence of the fact that Δ is a right adjoint). Moreover, Proposition 3.2 means that Δ is monadic. Therefore, it creates all existing limits as well as the colimits it preserves.

Note that this is not true for coproducts, as the coproduct of (at least two) objects in the image of Δ is not itself in it: the compatibility relation only relates elements coming from the same summand and is therefore not the universal relation. $\square$

3.2. Partial Boolean algebras

We now recall the basic definition of partial Boolean algebra. This can be seen as a relaxation of the notion of Boolean algebra where the binary operations $\wedge, \vee$ become partial, defined only on commeasurable elements. Commeasurability is specified by a compatibility graph as in the previous section.

Definition 3.4. A **partial Boolean algebra** is a structure

$$\langle A, \odot_A, 0_A, 1_A, \neg_A, \vee_A, \wedge_A \rangle$$

consisting of:

- a compatibility graph $\langle A, \odot_A \rangle$, whose relation is read as 'commeasurability',
- constants $0_A, 1_A \in A$,
- a (total) unary operation $\neg_A \colon A \longrightarrow A$,
- (partial) binary operations $\vee_A, \wedge_A \colon \odot_A \longrightarrow A$ defined on commeasurable elements,

satisfying the following property: for any set $S \in \bigodot_A$ of pairwise-commeasurable elements, there is a set $T \in \bigodot_A$ of pairwise-commeasurable elements with $S \subseteq T$ such that T forms a Boolean algebra under the restrictions of the operations.[10]

A homomorphism of partial Boolean algebras is a morphism of compatibility graphs which preserves the constants and operations wherever defined. We write **pBA** for the resulting category of partial Boolean algebras.

Again, we usually omit subscripts whenever the algebra is clear from context. There is an evident forgetful functor $U \colon \mathbf{pBA} \longrightarrow \mathbf{RGph}$.

Example 3.5. Any Boolean algebra is a partial Boolean algebra, where the commeasurability relation is the universal relation. In fact, Boolean algebras form a reflective subcategory of **pBA**

10An equivalent way of formulating this condition, closer to a direct first-order (in fact, Horn theory) axiomatisation, is to require: (i) that the operations preserve commeasurability in the sense that for each operation f of arity n, and elements $a_1, \ldots, a_n, b$,

$$a_1 \odot b, \ \ldots, \ a_n \odot b \implies f(a_1, \ldots, a_n) \odot b$$

(note that this implies that the constants are commeasurable with all other elements), and (ii) that for any triple a, b, c of pairwise-commeasurable elements, the axioms of Boolean algebra are satisfied.

(see e.g. [17]). The forgetful functor $U\colon \mathbf{pBA} \longrightarrow \mathbf{RGph}$ generalises the usual forgetful functor $U\colon \mathbf{BA} \longrightarrow \mathbf{Set}$ in the sense depicted in the commutative diagram:

$$\begin{tikzcd} \mathbf{pBA} \arrow[r, "U"] & \mathbf{RGph} \\ \mathbf{BA} \arrow[u, hook, "\iota"] \arrow[r, "U"] & \mathbf{Set} \arrow[u, hook, "\Delta=\langle -,\top\rangle"'] \end{tikzcd}$$

Example 3.6. The paradigmatic – and motivating – example of a partial Boolean algebra is the collection of projectors on a Hilbert space $\mathcal{H}$, or equivalently, of closed subspaces of a Hilbert space $\mathcal{H}$. Commeasurability and the operations are given by the following definitions (here stated in terms of projections): given projections p, q,

- $p \odot q$ if and only if $pq = qp$;
- the constants are the 0 and 1 projectors;
- $\neg p := 1 - p$;
- for $p \odot q$, $p \wedge q := pq$;
- for $p \odot q$, $p \vee q := p + q - pq$;

Example 3.7. It is clear from the description in terms of projectors, which are the self-adjoint, idempotent elements in the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, that this construction can be generalised to the set of projectors on any C^*-algebra, and even beyond. Given a not-necessarily-commutative unital $(*$-$)$ring R, the set

$$E_{(\mathsf{sa})}(R) := \left\{ e \in R \mid e^2 = e \text{ (and } e = e^*) \right\}$$

of its (self-adjoint) idempotent elements forms a partial Boolean algebra, with the commeasurability relation and operations defined as for projectors in example 3.6.

As for Boolean algebras, there is a useful order-theoretic perspective.

Definition 3.8. Let A be a partial Boolean algebra. Given $a, b \in A$, we write $a \leq b$ to mean that $a \odot b$ and $a \wedge b = a$.[11]

The restriction of this relation to any Boolean subalgebra of A coincides with the partial order of that Boolean algebra. But in spite of the notation, for a general partial Boolean algebra A, the relation $\leq$ need not be (globally) a partial order. This will be the case, however, for the partial Boolean algebras that we consider in this article, a point to which we return in section 3.4.

3.3. Completeness and atomicity for partial Boolean algebras

The most general form of the result presented in this article requires the partial version of complete Boolean algebras, previously studied in [28, Section 2.1].

Definition 3.9. A **partial complete Boolean algebra** is a partial Boolean algebra with an additional (partial) operation defined on sets of commeasurable elements

$$\bigvee \colon \bigodot \longrightarrow A\ ,$$

satisfying the following property: any set $S \in \bigodot$ is contained in a set $T \in \bigodot$ which forms a complete Boolean algebra under the restriction of the operations.

A homomorphism of partial complete Boolean algebras is a homomorphism of partial Boolean algebras which preserves arbitrary joins (equivalently, meets) of commeasurable elements. We write **pCBA** for the resulting category of partial complete Boolean algebras.

We now turn to the analogue of atomicity in the partial setting. The following definitions are straightforward generalisations from the total case.

11Condition $a \wedge b = a$ can equivalently be written as any of the following: $a \vee b = b$, $\neg a \vee b = 1$, $a \wedge \neg b = 0$.

Definition 3.10. An **atom** of a partial Boolean algebra A is an element $x \in A$ with $x \neq 0$ such that for any other element $a \in A$ with $a \neq 0$, if $a \leq x$ then $a = x$.

Given an element $a \in A$, we write $\mathcal{U}(a)$ for the set of atoms below a,

$$\mathcal{U}(a) := \{x \in A \mid x \text{ is an atom and } x \leq a\} \ .$$

Definition 3.11. A partial Boolean algebra A is said to be **atomic** if every non-zero element sits above an atom, i.e. if $\mathcal{U}(a) \neq \varnothing$ for all $a \in A \setminus \{0\}$.

Our focus will be on the partial analogues of finite Boolean algebras and of CABAs.

Definition 3.12. A partial CABA is a partial Boolean algebra that is complete and atomic. We write **pCABA** for the full subcategory of **pCBA** whose objects are partial CABAs.

Definition 3.13. A partial Boolean algebra A is said to be **locally finite** if any set of commeasurable elements, i.e. any member of $\bigodot_A$, is finite. We write **pFinBA** for the full subcategory of **pBA** (equivalently, of **pCBA** or **pCABA**) consisting of locally finite partial Boolean algebras.

In other words, a partial Boolean algebra is locally finite when all its Boolean subalgebras are finite Boolean algebras. From this it immediately follows that locally finite partial Boolean algebras are, in particular, partial CABAs.

Example 3.14. The partial Boolean algebra $\mathsf{P}(\mathcal{H})$ of projectors on a Hilbert space $\mathcal{H}$ is a partial CABA. More generally, so is the partial Boolean algebra of projections in a von Neumann algebra with only type I factors. If $\mathcal{H}$ is a finite-dimensional Hilbert space then the partial Boolean algebra $\mathsf{P}(\mathcal{H})$ is locally finite.

3.4. Transitivity or the logical exclusivity principle

In this article, we focus on a special class of partial Boolean algebras, namely those that satisfy the logical exclusivity principle, or equivalently, transitivity. These include the paradigmatic example of projectors on a Hilbert space, and indeed the partial Boolean algebra of projections in any von Neumann algebra.

Definition 3.15. Two elements a and b of a partial Boolean algebra are said to be **exclusive**, written $a \perp b$, if there is a element c such that $a \leq c$ and $b \leq \neg c$.

In a Boolean algebra, this condition could be equivalently phrased as $a \wedge b = 0$, or as $a \leq \neg b$. In a partial Boolean algebra, too, these conditions are equivalent whenever $a \odot b$. However, exclusivity is in general a weaker requirement than demanding $a \odot b$ and $a \wedge b = 0$. The reason is that there may exist exclusive elements that are not commeasurable.

Definition 3.16. A partial Boolean algebra is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are exclusive are also commeasurable.

We write **epBA** (resp. **epFinBA**, **epCBA**, **epCABA**) for the full subcategory of **pBA** (resp. **pFinBA**, **pCBA**, **pCABA**) whose objects are partial Boolean algebras satisfying LEP.

In partial Boolean algebras satisfying LEP, we indeed have that $a \perp b$ is equivalent to $a \odot b$ and $a \wedge b = 0$.

The LEP condition turns out to coincide with another well-studied condition on partial Boolean algebras, namely transitivity of the $\leq$ relation [15, 16, 29].

Definition 3.17. A partial Boolean algebra is said to be **transitive** if $\leq$ is a transitive relation, i.e. for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

Transitivity can fail in general for a partial Boolean algebra, since one need not have $a \odot c$ under the stated hypotheses. In any partial Boolean algebra, $\leq$ is reflexive and symmetric, so transitivity is equivalent to requiring that it be a partial order (globally) on A.

Proposition 3.18 ([17, Proposition X]). *A partial Boolean algebra satisfies LEP if and only if it is transitive.*

An important property of this class of pBAs, which we will use often, is that the exclusivity relation is downwards-closed.

Proposition 3.19. *Let A be a transitive partial Boolean algebra and $a, a', b, b' \in A$ with $a' \leq a$ and $b' \leq b$. Then $a \perp b$ implies $a' \perp b'$.*

Proof. Immediate from Definition 3.15, of $\perp$, and transitivity of $\leq$. □

4. Partial CABAs from their graphs of atoms

In this section we show how a partial CABA can be reconstructed from its (graph of) atoms.

4.1. Graphs

We first fix some notational conventions on graphs, which will come in handy in later sections.

Definition 4.1. A (simple, undirected) **graph** $\langle X, \#_X \rangle$ is a set equipped with a symmetric irreflexive relation. Elements of the set X are called vertices, while unordered pairs $\{x, y\}$ with $x \# y$ are called edges (and the vertices x and y are said to be adjacent).

We postpone the discussion of morphisms to section 5, as we shall be interested in a notion that differs from the usual graph homomorphisms.

We make use of the following notational conventions. Let X be a graph. Given a vertex $x \in X$ and sets of vertices $S, T \subseteq X$, we write:

- $x \# S$ to mean that $x \# y$ for all $y \in S$;
- $S \# T$ to mean that $x \# y$ for all $x \in S$ and $y \in T$;
- $x^{\#} := \{y \in X \mid y \# x\}$ to denote the **neighbourhood** of the vertex x;
- $S^{\#} := \bigcap_{x \in S} x^{\#} = \{y \in X \mid y \# S\}$ to denote the **common neighbourhood** of the set S.

An important property of the common neighbourhood function $(\text{–})^{\#} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is that it is a self-dual (antitone) Galois connection: for any $S, T \subseteq X$,

$$S \subseteq T^{\#} \qquad \text{iff} \qquad S \# T \qquad \text{iff} \qquad T \subseteq S^{\#} \ . \tag{8}$$

This has a number of important consequences, which we list for later reference.

Lemma 4.2.

(i) *$(\text{–})^{\#}$ is an antitone map, i.e. $S \subseteq T$ implies $T^{\#} \subseteq S^{\#}$.*
(ii) *$(S \cup T)^{\#} = S^{\#} \cap T^{\#}$.*
(iii) *$(S \cap T)^{\#} \supseteq S^{\#} \cup T^{\#}$.*
(iv) *The double neighbourhood function $(\text{–})^{\#\#}$ is a closure operator.*
(v) *If S is closed, then so is $S^{\#}$, i.e. $S^{\#\#\#} = S^{\#}$.*
(vi) *If $S \subseteq T$ then $S \cap T^{\#} = \varnothing$.*

Proof. These are straightforward consequences of eq. (8). Note that item iii and the $\subseteq$ part of item ii follow directly – and dually – from item i and the universal properties of union and intersection. The $\supseteq$ part of item ii, which breaks the symmetry between the two items, is then proven using item iii. Together with the inflationary property $S \subseteq S^{\#\#}$, part of item iv, this implies $S \cup T \subseteq S^{\#\#} \cup T^{\#\#} \subseteq (S^{\#} \cap T^{\#})^{\#}$, which by eq. (8) is equivalent to $S^{\#} \cap T^{\#} \subseteq (S \cup T)^{\#}$. □

The standard concept of clique will play a central rôle in our reconstruction.

Definition 4.3. A **clique** of a graph X is a set of pairwise adjacent vertices, i.e. a set $K \subseteq X$ such that $x \# y$ for all $x, y \in K$ with $x \neq y$ (equivalently, such that $x \# (K \setminus \{x\})$ for all $x \in K$). The **clique number** of a graph is the cardinality of the largest clique in the graph.

Note that we do not assume graphs to be finite. We will regard graphs as generalising sets, and the appropriate generalisation of finite sets will be (possibly infinite) graphs with finite clique number.

4.2. Graph of atoms

In the total case, the set of atoms was enough to determine a CABA. For partial algebras, additional structure is needed.

Definition 4.4. The **graph of atoms** of a partial Boolean algebra A, denoted $\mathsf{At}(A)$, has as vertices the atoms of A and has an edge between atoms x and y, written $x \# y$, if and only if $x \perp y$.

So, the graph $\mathsf{At}(A)$ is the set of atomic events together with their exclusivity relation. These atoms can be interpreted as worlds of maximal information and the relation corresponds to incompatibility between such worlds, in the sense that there is a proposition that distinguishes between them. In the classical case where A is a total algebra, $\mathsf{At}(A)$ is the complete graph on the set of atoms since any pair of atomic events is exclusive.

Note that in a transitive partial Boolean algebra, the exclusivity relation $\perp$ on atoms coincides with the commeasurability relation $\odot$. Indeed, in any partial Boolean algebra, commeasurable atoms must be exclusive (for otherwise their meet would be a non-zero element smaller than the atoms). Conversely, transitivity implies that exclusive elements be commeasurable. However, we choose to emphasise exclusivity rather than commeasurability since this perspective is more useful in guiding our intuition. This will become even more apparent when we move on to consider morphisms.

A notational remark: We prefer the notation $\#$ instead of $\perp$ when referring to exclusivity between atoms, in order to stress that we are referring to properties of the graph $\mathsf{At}(A)$. While $\perp$ is a relation on (all) elements of A, $\#$ is a relation on atoms only.

4.3. Transitive partial CABA from its graph of atoms

Recall that in a CABA any element is uniquely written as a join of atoms, concretely as the join of all the atoms sitting below it (Proposition 2.8). That is,

$$a = \bigvee \mathcal{U}(a) \qquad \text{with} \qquad \mathcal{U}(a) = \{x \in \mathsf{At}(A) \mid x \leq a\} \ .$$

The following proposition provides an analogous result in the (transitive) partial setting. In contrast with the classical case, the decomposition of an element as a join of atoms is not necessarily unique. In particular, the atoms in $\mathcal{U}(a)$ are typically not pairwise commeasurable, hence their join need not even be defined.

Proposition 4.5. *Let A be a transitive partial CABA and let $a \in A$. Then $a = \bigvee K$ for any clique K of $\mathsf{At}(A)$ which is maximal in $\mathcal{U}(a)$.*

Proof. Let K be a clique of $\mathsf{At}(A)$ that is maximal in $\mathcal{U}(a)$. Being a clique in $\mathsf{At}(A)$, K is a set of pairwise-commeasurable elements of A, i.e. $K \in \bigodot$, and thus $\bigvee K$ is defined.

Since $K \subseteq \mathcal{U}(a)$, all $k \in K$ satisfy $k \leq a$ and thus in particular $k \odot a$. Hence, $K \cup \{a\} \in \bigodot$, i.e. it is a set of pairwise-commeasurable elements of A, implying that it can be extended to a complete Boolean subalgebra of A. From $k \leq a$ for all $k \in K$, it then follows that $\bigvee K \leq a$.

It remains to show that $a \leq \bigvee K$, or equivalently, that $a \wedge \neg \bigvee K = 0$. Suppose for a contradiction that this is not the case. Then atomicity implies there is an atom $x \leq a \wedge \neg \bigvee K$. By transitivity, $x \leq a$ and $x \leq \neg k$ (hence $x \# k$) for all $k \in K$. This makes $K \cup \{x\}$ a clique of atoms contained in $\mathcal{U}(a)$, contradicting the maximality of K. □

The proposition above states that an element a is the join of *any* clique that is maximal in $\mathcal{U}(a)$. So, if K and L are two such maximal cliques, it yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not all commeasurable.

As we have just established that all elements of the partial algebra can be written as a join of (a clique of) atoms, then the key to reconstructing a partial CABA from its atoms lies in characterising the equalities that arise between joins of cliques of atoms. To be sure, such equalities only allow us to reconstruct the partial CABA as a set, i.e. to recover its elements. More generally, we ought to characterise inequalities, i.e. the $\leq$ relation, between such joins – which then suffices to reconstruct the algebraic operations. Proposition 4.8 below provides such a characterisation in terms of the graph-theoretic structure of the atoms.

Lemma 4.6. *Let A be a transitive atomic partial Boolean algebra. For any elements $a, b \in A$, one has $a \leq b$ if and only if $\mathcal{U}(a) \subseteq \mathcal{U}(b)$.*

Proof. The forward implication follows from transitivity: $a \leq b$ and $x \leq a$ implies $x \leq b$. The converse implication uses atomicity: $\mathcal{U}(a) \subseteq \mathcal{U}(b)$ implies

$$\mathcal{U}(a \wedge \neg b) = \mathcal{U}(a) \cap \mathcal{U}(\neg b) \subseteq \mathcal{U}(b) \cap \mathcal{U}(\neg b) = \mathcal{U}(b \wedge \neg b) = \varnothing \ ,$$

which by atomicity implies that $a \wedge \neg b = 0$, i.e. $a \leq b$. □

Lemma 4.7. *Let A be a transitive partial complete Boolean algebra. For any cliques K and L in $\mathsf{At}(A)$,*

$$\bigvee K \perp \bigvee L \qquad \text{if and only if} \qquad K \# L \ .$$

Proof. First, unravelling definitions, note that $K \# L$ means that $k \odot l$ and $k \wedge l = 0$ for all $k \in K$ and $l \in L$. This implies that $K \cup L$ is a set of pairwise commeasurable elements, hence it can be extended to a complete Boolean subalgebra. Consequently, $\bigvee K \odot \bigvee L$, and using distributivity,

$$\bigvee K \wedge \bigvee L = \bigvee \{k \wedge l \mid k \in K, l \in L\} = 0 \ .$$

Conversely, by Proposition 3.19 and since A is transitive, $\bigvee K \perp \bigvee L$ implies $k \perp l$ (i.e. $k \# l$) for each $k \in K$ and $l \in L$. □

Proposition 4.8. *Let A be a transitive partial CABA. Given cliques K and L in $\mathsf{At}(A)$,*

$$\bigvee K \leq \bigvee L \qquad \textit{iff} \qquad L^{\#} \subseteq K^{\#} \qquad \textit{iff} \qquad K \subseteq L^{\#\#} \ .$$

Proof. The previous lemmas yield the following sequence of equivalences:

$$\begin{array}{ll}
& \bigvee K \leq \bigvee L \\
\Leftrightarrow & \\
& \neg \bigvee L \leq \neg \bigvee K \\
\Leftrightarrow & \qquad \{ \text{ Lemma 4.6 } \} \\
& \forall x \in \mathsf{At}(A).\ x \leq \neg \bigvee L \Rightarrow x \leq \neg \bigvee K \\
\Leftrightarrow & \qquad \{ \ a \leq \neg b \text{ iff } a \perp b \ \} \\
& \forall x \in \mathsf{At}(A).\ x \perp \bigvee L \Rightarrow x \perp \bigvee K \\
\Leftrightarrow & \qquad \{ \text{ Lemma 4.7 } \} \\
& \forall x \in \mathsf{At}(A).\ x \# L \Rightarrow x \# K \\
\Leftrightarrow & \\
& L^{\#} \subseteq K^{\#} \\
\Leftrightarrow & \qquad \{ \ (-)^{\#} \text{ is Galois connection, eq. (8) } \} \\
& K \subseteq L^{\#\#}
\end{array}$$

□

In particular, this yields a description of the atoms below an element $\bigvee L$, namely

$$\mathcal{U}(\bigvee L) = L^{\#\#} \ .$$

Corollary 4.9. *Let A be a transitive partial CABA. Given cliques K and L in $\mathsf{At}(A)$,*

$$\bigvee K = \bigvee L \qquad \textit{iff} \qquad K^{\#} = L^{\#} \qquad \textit{iff} \qquad K^{\#\#} = L^{\#\#} \ .$$

4.4. Reconstructing a transitive partial CABA from its graph of atoms

As we have already hinted, Corollary 4.9 yields a characterisation of the elements of a transitive partial CABA A from its atoms. Namely, considering the equivalence relation on cliques of $\mathsf{At}(A)$ given by

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

the elements of A are in one-to-one correspondence with $\equiv$-equivalence classes.

Moreover, Proposition 4.8 reconstructs the partial order structure, i.e. the relation $\leq$, of A. This is sufficient to describe the algebraic structure of a transitive partial CABA from its graph of atoms. Explicitly, this is given as follows:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \cup K$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
- $[K] \vee [L] = [K' \cup L']$, for K', L' as in the previous item.
- $[K] \wedge [L] = [K' \cap L']$, for K', L' as in the previous item.

5. Complete exclusivity graphs

We showed in the previous section that a transitive partial CABA is determined by its graph of atoms. This means that the mapping $A \longmapsto \mathsf{At}(A)$ is injective (on objects). A natural question is then to characterise its image, that is, which graphs arise as $\mathsf{At}(A)$ for some transitive partial CABA A? In other words, what conditions are required on a graph for the construction of the previous section to yield a transitive partial CABA.

5.1. Definition of exclusivity graphs

Definition 5.1. A **complete exclusivity graph** is a graph X satisfying the following two conditions:

(E1) if $K, L \subseteq X$ are disjoint such that $K \cup L$ is a maximal clique, then $K^{\#} \# L^{\#}$, i.e. for any vertices $x, y \in X$, if $x \# K$ and $y \# L$ then $x \# y$;

(E2) for any vertices $x, y \in X$, $x^{\#} \subseteq y^{\#}$ implies $x = y$.

The first of these conditions is depicted in Figure 1. It can be equivalently rephrased as follows: for any cliques K and L, if $K \# L$ and $K^{\#} \cap L^{\#} = \varnothing$ then $K^{\#} \# L^{\#}$.

One can think of $x \# y$ in a complete exclusivity graph as signifying that the possible worlds x and y can be distinguished unambiguously: there is a testable property that is *false* in one world and *true* in the other. The 'classical' case would then correspond to a complete graph, as any two possible worlds can be distinguished. Indeed, a helpful intuition for Definition 5.1 is to regard exclusivity graphs as generalising sets with the inequality relation $\neq$, i.e. complete graphs.[12] Note that a graph is already a symmetric and irreflexive relation. In order for $\#$ to be an inequivalence relation (the complement of an equivalence relation), one additionally needs cotransitivity:

$$x \# z \qquad \text{implies} \qquad x \# y \ \text{or} \ y \# z \ .$$

Condition (E2), which eliminates redundant vertices, would then imply that $\#$ is the inequality relation, bridging this last gap from inequivalence to inequality.

The weak link in this characterisation of inequality, which is loosened in relaxing sets to complete exclusivity graphs, is therefore cotransitivity: condition (E1) is a weaker version of it.

Proposition 5.2. *Let X be a graph. If the relation $\#_X$ is cotransitive, then it satisfies condition (E1).*

12Indeed, a consequence of our results is that the category **Set** of sets and functions, when sets are seen as complete graphs in this fashion, forms a coreflective subcategory of that of exclusivity graphs.

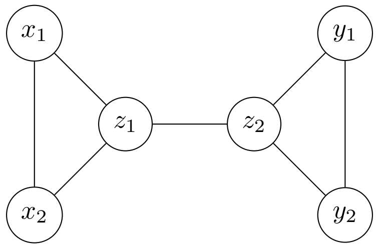

Figure 2. Example of a graph satisfying condition (E2) but not condition (E1). The former can be seen by inspection of the vertex neighbourhoods:

$$x_1^\# = \{z_1, x_2\} \qquad y_1^\# = \{z_2, y_2\} \qquad z_1^\# = \{z_2, x_1, x_2\}$$
$$x_2^\# = \{z_1, x_1\} \qquad y_2^\# = \{z_1, y_1\} \qquad z_2^\# = \{z_1, y_1, y_2\}$$

For the latter, take $K = \{z_1\}$ and $L = \{z_2\}$. These are disjoint and their union $K \cup L = \{z_1, z_2\}$ is a maximal clique. Moreover, $x_1 \# K$ and $y_1 \# L$ but we do not have that $x_1 \# y_1$, falsifying the condition.

Proof. Let K and L be two disjoint cliques such that $K \cup L$ is maximal, and let x and y be two vertices satisfying $x \# K$ and $y \# L$.

For each $k \in K$, since $x \# k$, cotransitivity implies $x \# y$ or $y \# k$. The fact that $K \cup L$ is a maximal clique precludes $y \# K \cup L$. Since $y \# L$ by assumption, there must therefore be a $k \in K$ with no edge to y. This forces the remaining option, namely $x \# y$. □

In contrast, Figure 2 shows an example of a graph for which condition (E1) fails.

5.2. Graph of atoms is complete exclusivity graph

The first task is to check that $\mathsf{At}(A)$ is indeed a complete exclusivity graph for any transitive partial CABA A. The following lemma will be useful, and it helps to illuminate the origin of condition $(E1)$.

Lemma 5.3. *Let A be a transitive partial CABA. If $K, L \subseteq \mathsf{At}(A)$ with $K \cap L = \varnothing$ and $K \cup L$ a maximal clique, then $\bigvee K = \neg \bigvee L$.*

Proof. K and L are disjoint subsets of a clique, hence $K \# L$. So, by Lemma 4.7, $\bigvee K \wedge \bigvee L = 0$, i.e. $\bigvee K \leq \neg \bigvee L$.

The other direction follows from maximality. The fact that $K \cup L$ is a maximal clique implies by proposition 4.5 that its join, $\bigvee (K \cup L) = \bigvee K \vee \bigvee L$, is equal to 1. Hence, $\neg \bigvee L \leq \bigvee K$. □

Proposition 5.4. *Let A be a transitive partial CABA. Then $\mathsf{At}(A)$ is a complete exclusivity graph.*

Proof. For condition (E1), let $K, L \subseteq \mathsf{At}(A)$ such that $K \cap L = \varnothing$ and $K \cup L$ is a maximal clique. By the previous lemma, $\bigvee K = \neg \bigvee L$. Given atoms x and y, by Lemma 4.7, $x \# K$ is equivalent to $x \leq \neg \bigvee K$ while $y \# L$ is equivalent to $y \leq \neg \bigvee L = \bigvee K$. Together, they imply $x \perp y$ using the LEP condition.

For condition (E2), given any atoms $x, y \in \mathsf{At}(A)$, the inclusion $x^\# \subseteq y^\#$ is equivalent to $y \leq x$ by Proposition 4.8. Since x and y are both atoms (minimal non-zero elements) we conclude that $x = y$. □

5.3. The 'powerset' of a complete exclusivity graph

Given a complete exclusivity graph X, we construct a partial CABA $\mathcal{K}(X)$ as follows. Consider the set of cliques of X, and define an equivalence relation on it whereby

$$K \equiv L \;:\Leftrightarrow\; K^\# = L^\#$$

for cliques K and L. The elements of the algebra $\mathcal{K}(X)$ are the $\equiv$-equivalence classes. Set

- $0 := [\varnothing]$;

- $1 := [M]$ for any maximal clique M;
- $\neg[K] := [L]$ for any L maximal in $K^{\#}$, i.e. for any clique L such that $L \# K$ and $L \cup K$ is a maximal clique;
- $[K] \odot [L]$ if and only if there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique;
- $[K] \wedge [L] := [K' \cap L']$ where K' and L' as in the previous item;
- $\bigvee_i [K_i] := [L]$ for any clique L with $L^{\#} = (\bigcup_i K_i)^{\#}$.[13]

We must show first that this is well-defined as a partial algebraic structure on $\equiv$-equivalence classes. A clique K is maximal if and only if $K^{\#} = \varnothing$, hence all maximal cliques are in the same equivalence class, making 1 well-defined. For negation, we use the following useful result.

Proposition 5.5. *Let K, L be cliques in a complete exclusivity graph. The following are equivalent:*

- *L is a maximal clique in $K^{\#}$;*
- *K and L are disjoint and $K \cup L$ is a maximal clique, i.e. $K \# L$ and $(K \cup L)^{\#} = \varnothing$;*
- *$K^{\#\#} = L^{\#}$*

Proof. Equivalence between the first two items is immediate. For the other two, first note the following equivalence

$$K \# L \qquad \text{iff} \qquad L \subseteq K^{\#} \qquad \text{iff} \qquad K^{\#\#} \subseteq L^{\#} \ ,$$

where the second equivalence comes from $(-)^{\#}$ being antitone in the forward direction and from $K \subseteq K^{\#\#}$ in the converse.

Now, given the other side of the inequality, $L^{\#} \subseteq K^{\#\#}$, one obtains

$$(K \cup L)^{\#} = K^{\#} \cap L^{\#} \subseteq K^{\#} \cap K^{\#\#} = \varnothing \ .$$

Conversely, assuming that $(K \cup L)^{\#} = \varnothing$ and K, L disjoint, the exclusivity condition (E1) implies $K^{\#} \# L^{\#}$, i.e. $L^{\#} \subseteq K^{\#\#}$. □

Corollary 5.6. *Let K, L be cliques in a complete exclusivity graph. Then L is maximal in $K^{\#\#}$ if and only if $K^{\#} = L^{\#}$.*

This result shows that negation $\neg[K]$ is well-defined: the property of L being maximal in $K^{\#}$ depends only on $L^{\#}$, hence it is an invariant under the equivalence, as stated explicitly below.

Corollary 5.7. *Let K, L, L' be cliques in a complete exclusivity graph such that L is maximal in $K^{\#}$. Then L' is maximal in $K^{\#}$ if and only if $L \equiv L'$*

We now turn our attention to the relation $\odot$. By the way it is defined, it is clearly invariant under the equivalence $\equiv$, and moreover, it is a reflexive and symmetric relation. We now consider a more explicit characterisation.

Proposition 5.8. *Let K, L be cliques in a complete exclusivity graph. Then the following are equivalent:*

- *$[K] \odot [L]$, i.e. there exist K', L' with $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.*
- *The four sets*
$$K^{\#\#} \cap L^{\#\#}, \quad K^{\#\#} \cap L^{\#}, \quad K^{\#} \cap L^{\#\#}, \quad K^{\#} \cap L^{\#},$$
have empty common neighbourhood[14].

Proof. Suppose that $K \cup L$ is a clique. Then $(K \setminus L) \# L$ and $(L \setminus K) \# K$, hence

$$K \cup L = (K \cap L) \cup (K \cap L^{\#}) \cup (K^{\#} \cap L) \ .$$

Then, we can calculte the common neighborhood of

$$K \cap L, \quad K \cap L^{\#}, \quad K^{\#} \cap L^{\#}, \quad K^{\#} \cap L^{\#},$$

13In particular, binary joins are given by $[K] \vee [L] := [K' \cup L']$ for representatives K' and L' with $K' \cup L'$ a clique. The same is true for arbitrary joins, but it requires showing that binary compatibility implies arbitrary compatibility.

to conclude that it is empty:

$$
\begin{aligned}
& ((K \cap L) \cup (K \cap L^{\#}) \cup (K^{\#} \cap L) \cup (K^{\#} \cap L^{\#}))^{\#} \\
= \\
& ((K \cap L) \cup (K \cap L^{\#}) \cup (K^{\#} \cap L))^{\#} \cap (K^{\#} \cap L^{\#})^{\#} \\
= \\
& (K \cup L)^{\#} \cap (K \cup L)^{\#\#} \\
= \\
& \varnothing
\end{aligned}
$$

Since $K \subseteq K^{\#\#}$ and $L \subseteq L^{\#\#}$, the four sets in the statement are larger than the ones just considered, hence their common neighbourhood smaller, and therefore also empty.

This establishes the property in case $K \cup L$ is a clique. To extend it to any K, L with $[K] \odot [L]$ it is enough to notice that it only depends on $K^{\#}$ and $L^{\#}$, hence it is invariant under the equivalence relation.

For the converse, suppose that the sets in the statement have empty common neighbourhood and choose

$$M_{11} \subseteq K^{\#\#} \cap L^{\#\#}, \quad M_{10} \subseteq K^{\#\#} \cap L^{\#}, \quad M_{01} \subseteq K^{\#} \cap L^{\#\#}, \quad M_{00} \subseteq K^{\#} \cap L^{\#},$$

a maximal clique in each of them. By construction, and bearing in mind the remark in the footnote, the sets $M_{00}, M_{01}, M_{10}, M_{11}$ are disjoint and their union is a maximal clique in the graph. We show that the condition of the first item will be fulfilled by setting

$$K' = M_{11} \cup M_{10}, \qquad L' = M_{11} \cup M_{01} \ .$$

Proposition 5.5 gives, in particular, that $(M_{11} \cup M_{10})^{\#} = (M_{01} \cup M_{00})^{\#\#}$, and consequently,

$$(M_{11} \cup M_{10})^{\#} = (M_{10} \cup M_{11})^{\#\#} \subseteq (K^{\#} \cap (L^{\#\#} \cup L^{\#}))^{\#\#} \subseteq (K^{\#})^{\#\#} = K^{\#} \ .$$

Using this we conclude that $K' = M_{11} \cup M_{10}$, which is clearly contained in $K^{\#\#}$, is in fact a maximal clique in $K^{\#\#}$, since

$$(K')^{\#} \cap K^{\#\#} = (M_{11} \cup M_{10})^{\#} \cap K^{\#\#} \subseteq K^{\#} \cap K^{\#\#} = \varnothing \ .$$

Proposition 5.5 or Corollary 5.6 then yield $K' \equiv K$. A similar argument establishes $L' \equiv L$. □

The four sets in the proposition above are defined in terms of $K^{\#}$ and $L^{\#}$ alone and therefore do not depend on the choice of representative. Moreover, they can be thought of as representing the equivalence classes of (any) maximal cliques $M_{\kappa\lambda}$ within them. In that language, the partial binary operations are given as

$$[K] \wedge [L] = [M_{11}] \qquad \text{and} \qquad [K] \vee [L] = [M_{11} \cup M_{10} \cup M_{11}] \ .$$

This is enough to show that the finitary operations in $\mathcal{K}(X)$ are well defined.

We still need to show the crucial property of partial Boolean algebra, that any set of commeasurable elements extends to a total Boolean algebra, besides dealing with arbitrary joins and meets. In fact, the result above can be regarded as a special case – and a first step in that direction. It is instructive to see how. One way to state the extension property for a partial CABA A is as follows: for any set S of commeasurable elements in A, the inclusion of S in A can be extended to a Boolean algebra homomorphism $F(S) \longrightarrow B \subseteq A$ from the free CABA on the set S to (some Boolean subalgebra B of) A. Proposition 5.8 establishes this for a set S with two elements, $S = \{[K], [L]\}$. We now unpack this more explicitly.

The free CABA on a set S is given concretely by the double powerset construction. The atoms of $F(S)$ correspond to valuations $v \colon S \longrightarrow \{0, 1\}$, and the elements of $F(S)$ are sets of

14Note that, for any K, L, these sets are in each other's neighbourhood, i.e. they are pairwise related by the relation $\#$ lifted to subsets. Intuitively, the point of the additional condition is that they 'cover' the whole space.

such valuations. One can also think of a set V of valuations on S as encoding a formula with propositional variables in S, in disjunctive normal form:

$$\bigvee_{v\in V}\bigwedge_{a\in S}\neg^{v(a)}a \qquad \text{with} \quad \neg^{\alpha}a = \begin{cases} a & \text{if } \alpha = 1\ , \\ \neg a & \text{if } \alpha = 0\ . \end{cases}$$

In the two-element case, the atoms of $F(\{[K],[L]\})$ are assignments $\langle [K] \longmapsto \kappa, [L] \longmapsto \lambda\rangle$ given by pairs $(\kappa,\lambda) \in \{0,1\}^2$. Choose cliques $M_{\kappa\lambda}$ as in the proof of Proposition 5.8 and put $M := \bigcup_{\kappa\lambda} M_{\kappa\lambda}$. Mapping each atom to the corresponding $M_{\kappa\lambda}$ determines a homomorphism from the free algebra $F(S)$ to the powerset of M

$$\begin{aligned} F(S) &\longrightarrow \mathcal{P}(M) \\ V &\longmapsto \bigcup_{\langle\kappa,\lambda\rangle\in V} M_{\kappa\lambda}\ . \end{aligned}$$

It maps a set of assignments

$$\{\, \langle\kappa_1,\lambda_1\rangle\, ,\ \ldots\ ,\ \langle\kappa_n,\lambda_n\rangle \,\}\ ,$$

which one can think of as the formula

$$(\neg^{\kappa_1}[K] \wedge \neg^{\lambda_1}[L]) \ \vee\ \cdots\ \vee\ (\neg^{\kappa_n}[K] \wedge \neg^{\lambda_n}[L])\ ,$$

to the subset of M

$$M_{\kappa_1\lambda_1} \ \cup\ \cdots\ \cup\ M_{\kappa_n\lambda_n}\ .$$

The consequence of Proposition 5.8 is that the quotient map of the equivalence relation $\equiv$ is a Boolean algebra isomorphism between the powerset $\mathcal{P}(M)$ and a subalgebra of $\mathcal{K}(X)$.

The goal is then to extend this result to an analogous statement for any set S of commeasurable elements. Before proceeding, we consider yet another characterisation of commeasurability in $[K] \odot [L]$, which provides a handy test.

Lemma 5.9. *Let K, L be cliques in a complete exclusivity graph, and write $A := K^\#$ and $B := L^\#$ for simplicity. Then $[K] \odot [L]$ is equivalent to each of the following inequalities:*

(i) $(A \cap (B \cup B^\#))^\# \subseteq A^\#$,

(ii) $(A^\# \cap (B \cup B^\#))^\# \subseteq A$,

(iii) $(B \cap (A \cup A^\#))^\# \subseteq B^\#$,

(iv) $(B^\# \cap (A \cup A^\#))^\# \subseteq B$.

Proof. The reverse inclusions are all trivial, so one really has equality in all four conditions. Taking for example item i, observe that

$$(A \cap (B \cup B^\#))^\# \ =\ ((A\cap B) \cup (A \cap B^\#))^\# \ =\ (A\cap B)^\# \cap (A \cap B^\#)^\#\ . \tag{9}$$

From the characterisation of $[K] \odot [L]$ given in Proposition 5.8 in terms of the four sets, and from Proposition 5.5 using the exclusivity property, one obtains

$$((A\cap B) \cup (A\cap B^\#))^\# = ((A^\# \cap B) \cup (A^\# \cap B^\#))^{\#\#}\ ,$$

and therefore,

$$(A \cap (B \cup B^\#))^\# = (A^\# \cap (B \cup B^\#))^{\#\#} \subseteq (A^\#)^{\#\#} = A^\#\ .$$

Each of the other conditions in items ii to iv is similarly derived from $[K] \odot [L]$.

In the opposite direction, in light of eq. (9), it is clear that items i and ii together (or similarly items iii and iv together) imply $[K] \odot [L]$ by forcing the common neighbourhood of the four sets to be empty,

$$(A\cap B)^\# \cap (A \cap B^\#)^\# \cap (A^\# \cap B)^\# \cap (A^\# \cap B^\#)^\# \subseteq A^\# \cap A = \varnothing\ .$$

It remains to show that items i and ii actually imply each other – and each of the four conditions, in fact. We establish that 'i implies iii'. Since the condition in item iii is invariant under exchanging A and $A^{\#}$, all the equivalences follow.

We establish the following sequence of inclusions

$$\begin{aligned}
& (A \cap B)^{\#} \cap (A^{\#} \cap B)^{\#} \cap B \\
= \quad & \{ \text{ by } A \cap B^{\#} \subseteq B^{\#} \text{, and thus equivalently, } B \subseteq (A \cap B^{\#})^{\#} \ \} \\
& (A \cap B)^{\#} \cap (A^{\#} \cap B)^{\#} \cap (A \cap B^{\#})^{\#} \cap B \\
\subseteq \quad & \{ \text{ by item i, } (A \cap B)^{\#} \cap (A \cap B^{\#})^{\#} \subseteq A^{\#} \ \} \\
& (A^{\#} \cap B)^{\#} \cap A^{\#} \cap B \\
= \quad & \{ \text{ since } S^{\#} \cap S = \varnothing \text{ for any } S \ \} \\
& \varnothing
\end{aligned}$$

Using $B = B^{\#\#}$, the conclusion can be rewritten as

$$(A \cap B) \cup (A^{\#} \cap B)^{\#} \cap (B^{\#})^{\#} = \varnothing \ .$$

The exclusivity condition (E1) then yields item iii as desired. □

We now show an analogue of proposition 5.8 for sets of three commeasurable elements. In light of the previous remarks, the upshot is that any three-element set of commeasurable elements in $\mathcal{K}(X)$ extends to a homomorphic image of the free Boolean algebra $F(3)$. This, in fact, is enough to establish that $\mathcal{K}(X)$ is a partial Boolean algebra, because the equational axioms of Boolean algebra need only involve up to three variables.

Proposition 5.10. *Let K, L, M be cliques in a complete exclusivity graph whose equivalence classes are pairwise commeasurable, i.e. $\{[K], [L], [M]\} \in \odot$. The eight sets*

$$K^{\square_1} \cap L^{\square_2} \cap M^{\square_3} \ , \qquad \square_i \in \{\#, \#\#\}$$

are pairwise non-intersecting and have empty common neighbourhood.

Proof. Pairwise non-intersection is easy: for any two distinct sets

$$K^{\square_1} \cap L^{\square_2} \cap M^{\square_3} \qquad \text{and} \qquad K^{\blacksquare_1} \cap L^{\blacksquare_2} \cap M^{\blacksquare_3}$$

there is an $i \in \{1, 2, 3\}$ such that $\square_i \neq \blacksquare_i$ and the result follows from $K \cap K^{\#} = \varnothing$ if $i = 1$ or the analogous property for L or M if $i = 2, 3$.

Intuitively, the key step is to show that the fact that $[K]$ and $[L]$ are commeasurable with $[M]$ implies that each of the propositions built out of $[K]$ and $[L]$ in proposition 5.8 is also commeasurable with $[M]$.

For simplicity, set $A := K^{\#}, B := L^{\#}, C := M^{\#}$ and define $\tilde{A} := A \cup A^{\#}$ and similarly for $\tilde{B}$ and $\tilde{C}$. The overall aim is to show

$$(\tilde{A} \cap \tilde{B} \cap \tilde{C})^{\#} = \varnothing$$

We first establish that (the equivalence class represented by) $A \cap B$ is commeasurable with (that represented by) C, in the form of lemma 5.9–ii:

$$\begin{array}{cl} & ((A\cap B)^{\#}\cap\tilde{C})^{\#} \\ = & \quad\{\text{ by lemma 4.2–ii }\} \\ & ((A^{\#}\cup B^{\#})\cap\tilde{C})^{\#} \\ = & \quad\{\text{ by distributivity }\} \\ & ((A^{\#}\cap\tilde{C})\cup(B^{\#}\cap\tilde{C}))^{\#} \\ = & \quad\{\text{ by lemma 4.2–ii }\} \\ & ((A^{\#}\cap\tilde{C})^{\#}\cap(B^{\#}\cap\tilde{C}))^{\#} \\ = & \quad\{\text{ by } [K],[L]\odot[M] \text{ in the form lemma 5.9–ii }\} \\ & A\cap B \end{array}$$

Using the implication 'ii implies i' from lemma 5.9, this turns into

$$(A\cap B\cap\tilde{C})^{\#}=(A\cap B)^{\#}\ .$$

The same argument establishes

$$(A^{\square_1}\cap B^{\square_2}\cap\tilde{C})^{\#}=(A^{\square_1}\cap B^{\square_2})^{\#}$$

for any $\square_1,\square_2$. Then, $[K]\odot[L]$ in the form of 5.8 obtains

$$(\tilde{A}\cap\tilde{B}\cap\tilde{C})^{\#}=(\tilde{A}\cap\tilde{B})^{\#}=\varnothing\ .$$

□

As discussed, the proposition above suffices to establish the finitary version of our result, for locally finite partial Boolean algebras. For the full partial CABA result, we need an infinitary version. Our proof requires the axiom of choice.

Proposition 5.11. *Let $\{K_i\}_{i\in I}$ be a set of cliques in a complete exclusivity graph whose equivalence classes are pairwise commeasurable, i.e. $\{K_i\}_{i\in I}\in\bigodot$. The sets*

$$\bigcap_{i\in I}K_i^{\square_i}\ ,\qquad \square_i\in\{\#,\#\#\}$$

are pairwise non-intersecting and have empty common neighbourhood.

Proof. The sets in question are indexed by functions $v\colon I\longrightarrow 2$ (where $2=\{0,1\}$), with each $v(i)$ determining the choice of $\square_i$. Pairwise non-intersection follows as in proposition 5.8: given two different sets represented by functions v and v', there is an $i\in I$ where $v(i)\neq v'(i)$ and the result follows from $K_i\cap K_i^{\#}=\varnothing$.

For the main argument, the key step is also already contained in the proof of proposition 5.8. The argument given there can be trivially adapted to yield the following: if $\{L_j\}_{j\in J}$ is a set of cliques and M is a clique satisfying $[L_j]\odot[M]$ for all $j\in J$, then

$$\left(\bigcap_{j\in J}L_j^{\square_j}\cap(M^{\#}\cup M^{\#\#})\right)^{\#}=\left(\bigcap_{j\in J}L_j^{\square_j}\right)^{\#}\ .$$

So, if the set $\{L_j\}_{j\in J}$ has the desired property that the right-hand side in the equation above is empty, then so does the set $\{L_j\}_{j\in J}\cup\{M\}$.

We show the main statement by transfinite induction. Fix a well-ordering of the given set of cliques, e.g. assuming that the index set I is a (von Neumann) ordinal. We want to show that for any ordinal α less than (i.e. in) I, the sets

$$A_v:=\bigcap_{\gamma<\alpha}K_\gamma^{\square_{v(\gamma)}}\qquad\text{with}\qquad \square_b=\begin{cases}\# & \text{if } b=0\ ,\\ \#\# & \text{if } b=1\ ,\end{cases}$$

indexed by functions $v\colon \alpha \longrightarrow 2$, or $v \in 2^\alpha$, have empty common neighbourhood, i.e. that

$$\left(\bigcap_{\gamma<\alpha}(K_\gamma^{\#} \cup K_\gamma^{\#\#})\right)^{\#} = \left(\bigcup_{v\in 2^\alpha} A_v\right)^{\#} = \bigcap_{v\in 2^\alpha} A_v^{\#} = \varnothing \ .$$

The base case is trivial. For the successor case ($\alpha = \beta + 1$), assume that the property holds for β. Each $v \in 2^\alpha$ can be decomposed into its restriction to β and its value at β, so we can write $v = \langle w, b\rangle$ with $b \in \{0, 1\}$. Then,

$$A_{\langle w,b\rangle} = A_w \cap K_\beta^{\square_b} \ ,$$

and the above-mentioned result states that

$$(A_{\langle w,0\rangle} \cup A_{\langle w,1\rangle})^{\#} = (A_w \cap (K_\beta^{\#} \cup K_\beta^{\#\#}))^{\#} = A_w^{\#} \ .$$

The result then follows from the induction hypothesis.

For the limit case (α is a limit ordinal), assume that the property holds for all smaller ordinals, that is $\bigcap_{w\in 2^\beta} A_w^{\#} = \varnothing$ for any $\beta < \alpha$. For any $v \in 2^\alpha$, we show that

$$A_v = \bigcap_{\beta<\alpha} A_{v|_\beta} \ .$$

The inclusion $\subseteq$ follows from $A_v \subseteq A_{v|_\beta}$ for each $\beta < \alpha$. The reverse inclusion uses the fact that α is a limit ordinal: for each $\gamma < \alpha$ there is an ordinal β with $\gamma < \beta < \alpha$, hence $A_{v|_\beta} \subseteq K_\gamma^{\square_{v|_\beta(\gamma)}} = K_\gamma^{\square_{v(\gamma)}}$. From this, it follows that

$$\bigcup_{v\in 2^\alpha} A_v = \bigcup_{v\in 2^\alpha}\bigcap_{\beta<\alpha} A_{v|_\beta} = \bigcap_{\beta<\alpha}\bigcup_{w\in 2^\beta} A_w$$

Using commeasurability of A_w ($w \in 2^\beta$) with each $K_{\beta'}$ ($\beta < \beta' < \alpha$) obtains a dual formula using neighbourhoods, from which the result follows:

$$\bigcap_{v\in 2^\alpha} A_v^{\#} = \left(\bigcap_{\beta<\alpha}\bigcup_{w\in 2^\beta} A_w\right)^{\#} = \bigcup_{\beta<\alpha}\bigcap_{w\in 2^\beta} A_w^{\#} \ .$$

□

Combining the results of this section, we obtain the following.

Theorem 5.12. *For any complete exclusivity graph X, $\mathcal{K}(X)$ is a transitive partial CABA. Moreover, its graph of atoms is (isomorphic to) X.*

Proof. We have seen that the operations of $\mathcal{K}(X)$ are well defined and that it forms a complete partial Boolean algebra. Transitivity, or the logical exclusivity principle, follows easily from its description: $[K] = \neg[L]$ means $K^{\#\#} = L^{\#}$, $[K'] \leq [K]$ and $[L'] \leq [L]$ mean $K^{\#} \subseteq K'^{\#}$ and $L^{\#} \subseteq L'^{\#}$, from which it follows that $K' \subseteq K'^{\#\#} \subseteq K^{\#\#} = L^{\#\#} \subseteq L'^{\#}$, and thus $K' \# L'$ as desired. The atoms of $\mathcal{K}(X)$ are equivalence classes of single-element cliques $[\{x\}]$, which condition (E2) implies have a single representative. □

Together with the results from the last section, we establish a bijection between complete exclusivity graphs and transitive partial CABAs, given by At and $\mathcal{K}$.

6. Morphisms

We now turn to the morphism part of duality. Generalising Section 2.4, the goal is to characterise homomorphisms in **pCABA** in terms of the graphs of atoms of the domain and codomain. As discussed at the end of that section, the key idea is to move from functions to certain relations.

6.1. Morphisms of complete exclusivity graphs

Definition 6.1. Let X and Y be complete exclusivity graphs. A **morphism of complete exclusivity graphs** $X \longrightarrow Y$ is a relation $R\colon X \longrightarrow Y$ satisfying:

(M1) $R^{\circ} \circ \#_Y \circ R \subseteq \#_X$, i.e. for all $x, x' \in X$ and $y, y' \in Y$, if xRy, $x'Ry'$, and $y \#_Y y'$ then $x \#_X x'$;

(M2) if $K \subseteq Y$ is a maximal clique of Y, its preimage $R^{-1}(K) := \{x \in X \mid \exists y \in K .\, xRy\}$ contains a maximal clique of X.

(M3) for each vertex $y \in Y$, its preimage is a double neighbourhood closed set, $R^{-1}(\{y\})^{\#\#} = R^{-1}(\{y\})$.

We write **XGph** for the resulting category of complete exclusivity graphs.

In order to gain some intuition about this definition, it is worth calculating what it restricts to in the setting of complete graphs (i.e. sets). Condition (M1) reduces to

$$xRy,\ x'Ry',\ \text{and } y \neq y' \qquad \text{implies} \qquad x \neq x' \ ,$$

or contrapositively,

$$xRy,\ x'Ry',\ \text{and}\, x = x' \qquad \text{implies} \qquad y = y' \ ,$$

which is to say that the relation R is functional. On the other hand, for any graphs X, Y, condition (M2) applied to $K = \varnothing$ yields the requirement that every vertex of x has an image by R, i.e. left-totality. In the case of complete graphs (i.e. sets), this is all the condition reduces to. Condition (M3) is empty in this case. Together, these two conditions mean that a complete exclusivity graph morphism R between complete graphs is precisely a function. In other words, **Set** is (equivalent to) a *full* subcategory of **XGph**

6.2. Homomorphisms of transitive partial CABAs and morphisms of exclusivity graphs

Proposition 6.2. *Let A and B be transitive partial CABAs and $h\colon A \longrightarrow B$ a partial complete Boolean algebra homomorphism. The relation $R_h\colon \mathsf{At}(B) \longrightarrow \mathsf{At}(A)$ given by*

$$xR_h y \qquad \textit{iff} \qquad x \leq h(y)$$

for all $x \in \mathsf{At}(B)$ and $y \in \mathsf{At}(A)$ is a morphism of complete exclusivity graphs. Moreover, the assignment $h \longmapsto R_h$ is functorial.

Proof. For condition (M1), suppose $xR_h y$ and $x'R_h y'$, i.e. $x \leq h(y)$ and $x' \leq h(y')$. Then $y \# y'$, meaning $y \perp y'$, implies $h(y) \perp h(y')$ because h is a partial Boolean algebra homomorphism. Since B is transitive, Proposition 3.19 entails $x \perp x'$.

For condition (M3), given a $y \in \mathsf{At}(A)$, its pre-image under R_h is

$$R_h^{-1}(y) = \{x \in \mathsf{At}B \mid xR_h y\} = \{x \in \mathsf{At}B \mid x \leq h(y)\} = \mathcal{U}(h(y)) \ ,$$

which is a double-neighbourhood closed set.

For condition (M2), suppose K is a clique in $\mathsf{At}(A)$. Then $\bigvee K$ exists in A, and by virtue of h being a complete pBA homomorphism, $\{h(y) \mid y \in K\} \in \bigodot_B$ and $h(\bigvee K) = \bigvee \{h(y) \mid y \in K\}$. The clique K's pre-image is

$$R_h^{-1}(K) = \bigcup_{y \in K} R^{-1}(y) = \bigcup_{y \in K} \mathcal{U}(h(y)) \ ,$$

which is a unions of pairwise exclusive sets, i.e. $\mathcal{U}(h(y)) \# \mathcal{U}(h(y'))$ for $y \neq y'$. A clique M maximal $R_h^{-1}(K)$ will therefore be a union of cliques M_y each maximal in \$$Uh(y)$. Consequently, its join is

$$\bigvee M = \bigvee \left\{ \bigvee M_y \mid y \in K \right\} = \bigvee \{h(y) \mid y \in K\} = h(\bigvee K) \ .$$

If moreover K is a maximal clique in $\mathsf{At}(A)$, then $\bigvee K = 1$ and therefore $h(\bigvee K) = h(1_A) = 1_B$, and so M is a maximal clique in $\mathsf{At}(B)$.

Functoriality is a consequence of transitivity of the partial algebras:

$$\begin{aligned}
& x(R_g \circ R_h)z \\
\iff \quad & \{ \text{ relational composition } \} \\
& \exists y.\ xR_h y \text{ and } yR_g z \\
\iff \quad & \{ \text{ definition of } R_h \text{ and } R_g \} \\
& x \leq h(y) \text{ and } y \leq g(z) \\
\implies \quad & \{ \text{ } g \text{ is a homomorphism } \} \\
& x \leq h(y) \text{ and } h(y) \leq h(g(z)) \\
\implies \quad & \{ \text{ transitivity } \} \\
& x \leq (h \circ g)(z) \\
\iff \quad & \{ \text{ definition of } R_{h\circ g} \} \\
& xR_{h\circ g} z
\end{aligned}$$

□

Proposition 6.3. *Let X and Y be complete exclusivity graphs and $R\colon X \longrightarrow Y$ a morphism of complete exclusivity graphs. The function $h_R\colon \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)$ given for each clique K of $\mathcal{K}(Y)$ by $h_R([K]) := [L]$ where L is any clique maximal in $R^{-1}(K)$ is well defined and is a complete Boolean algebra homomorphism.*

Proof. This requires only conditions (M1) and (M2). The third condition is only necessary to make this injective. Condition (M1) implies that the function h_R maps exclusive atoms to exclusive elements of B: it states that $y \# y'$ implies $R^{-1}(y) \# R^{-1}(y')$, and therefore $[L] \perp [L']$ for any cliques L, L' maximal in $R^{-1}(y)$, $R^{-1}(y')$, i.e. $h_R(y) \perp h_R(y')$. The definition of h_R extends to arbitrary elements in the only possible way for a homomorphism:

$$h_R([K]) = h_R(\bigvee_{y\in K} [\{y\}]) = \bigvee_{y\in K} h_R([\{y\}]) \ .$$

Condition (M2) states that $h_R(1_{\mathcal{K}(Y)}) = 1_{\mathcal{K}(X)}$.

Suppose that K and K' are complementary cliques in Y, i.e. $K \# K'$ and $K \cup K'$ is a clique maximal in Y, so that $[K] = \neg[K']$. Then $R^{-1}(K) \# R^{-1}(K')$ from (M1), while (M2) implies that $R^{-1}(K) \cup R^{-1}(K') = R^{-1}(K \cup K')$ contains a clique maximal in X. Taking any two cliques L, L' maximal in $R^{-1}(K)$, $R^{-1}(K')$ respectively, we conclude that they are complementary cliques in X and therefore that $[L] = \neg[L']$. This shows that h_R preserves negation (as long as it is well defined): $[K] = \neg[K']$ implies $h_R(K) = h_R(K')$.

We now use the property just proved to show that h_R is well defined. Suppose K_1, K_2 are cliques in Y with $K_1 \equiv K_2$, and pick K' to be any clique maximal in $K_1^\# = K_2^\#$, i.e. any clique complementary to K_1 or K_2, and therefore to both. Pick L_1, L_2, L' maximal cliques in $R^{-1}([K_1])$, $R^{-1}([K_2])$, $R^{-1}([K'])$, respectively. Applying the previous property twice, we conclude that $[L_1] = \neg[L] = [L_2]$, establishing that h is a well-defined function.

For commeasurability, let K_1, K_2 be cliques in Y such that $[K_1] \odot [K_2]$, and pick $K_i' \equiv K_i$ $(i = 1, 2)$ such that $K_1 \cup K_2$ is a clique. Then

$$K_1 \cup K_2 = (K_1 \cap K_2) \cup (K_1 \cap K_2^\#) \cup (K_1^\# \cap K_2)$$

where the union is disjoint. Picking any three cliques M_{11}, M_{10}, M_{01} each maximal in the preimage of one of these components, condition (M1) guarantees that they form a clique. Moreover, $L_1 := M_{11} \cup M_{10}$ is maximal in

$$R^{-1}(K_1) = R^{-1}((K_1 \cap K_2) \cup (K_1 \cap K_2^\#)) = R^{-1}(K_1 \cap K_2) \cup R^{-1}(K_1 \cap K_2^\#) \ ,$$

and similarly $L_2 := M_{11} \cup M_{01}$ is maximal in $R^{-1}(K_2)$, showing $h_R([K_1]) \odot h_R([K_2])$.

Finally, we show that it preserves joins. Let $\{K_i\}_{i\in I}$ be a set of cliques in Y such that the equivalence classes $\{[K_i]\}_{i\in I}$ are pairwise commeasurable in $\mathcal{K}(Y)$. Find representatives $K'_i \equiv K_i$ $(i \in I)$ such that $\bigcup K_i$ is a clique. The proof follows by an argument similar to that for commeasurability: breaking K into disjoint cliques indexed by functions $I \longrightarrow 2$, and picking cliques in X maximal in each of their pre-images. $\square$

Proposition 6.4. *Let A and B be transitive partial CABAs. Then* $\mathbf{epCABA}(A,B) \cong \mathbf{XGph}(\mathsf{At}(B), \mathsf{At}(A))$.

Proof. We show that the morphism assignments given by Propositions 6.2 and 6.3 are essentially inverses of each other. In summary, the mapping $h \longrightarrow R_h$ is injective since homomorphisms of partial CABAs are determined by their actions on atoms. The fact that it is surjective, or that $R \longrightarrow h_R$ is injective, reduces to condition (M3) – which was not necessary to construct a homomorphism h_R.

Starting from a homomorphism h, $h_{R_h}([K]) = [L]$ for L a clique maximal in $R_h^{-1}(K) = \bigcup_{y\in K} \mathcal{U}(h(y))$, whose join is $\bigvee L = \bigvee_{y\in K} h(y) = h(\bigvee K)$.

Starting from a morphism of complete exclusivity graphs R,

$$
\begin{aligned}
& [x] R_{h_R} [y] \\
\iff\; & \\
& [x] \leq h_R([y]) \\
\iff\; & \\
& [x] \leq [L] \text{ for some } L \text{ maximal in } R^{-1}(y) \\
\iff\; & \\
& L^{\#} \subseteq x^{\#} \text{ for some } L \text{ maximal in } R^{-1}(y) \\
\iff\; & \\
& R^{-1}(y)^{\#} \subseteq x^{\#} \\
\iff\; & \\
& x \in R^{-1}(y)^{\#\#} \\
\iff\; & \quad \{ \text{ condition item (M3) } \} \\
& x \in R^{-1}(y) \\
\iff\; & \\
& xRy
\end{aligned}
$$

$\square$

6.3. Contextuality

Kochen and Specker characterised contextuality in partial Boolean algebraic terms [2]. The strongest nonclassicality condition they considered was the lack of a (partial Boolean algebra) homomorphism to the two-element Boolean algebra, also known as a valuation. They showed that this property holds for the partial Boolean algebra $\mathsf{P}(\mathcal{H})$ of projectors on a Hilbert space of dimension at least 3. In the infinite-dimensional case, it would be natural to require such valuations to also preserve arbitrary joins, i.e. to be complete partial Boolean algebra homomorphisms.

Under the duality, such homomorphisms to the two-element Boolean algebra correspond to morphisms of exclusivity graphs $K_1 \longrightarrow \mathsf{At}(A)$ where K_1 is the graph with a single vertex. Such a relation is then determined by the image of this unique vertex, which is a subset S of the atoms of A. Conditions (M1) and (M2) from the definition of morphism of complete exclusivity graphs reduce to the following properties of this subset:

- it is an independent (or stable) set, i.e. no two elements of S are connected by an edge;
- it is a maximal clique transversal, i.e. it has a vertex in each maximal clique.

Therefore, a partial CABA satisfies the Kochen–Specker contextuality condition (of having no homomorphism to the two-element Boolean algebra) if and only if its graph of atoms has no independent set of vertices that is a maximal clique transversal. In other words, there is no way to pick a set containing one and only one element from each maximal clique.

7. Duality for transitive partial CABAs

We combine the results from previous sections as the main theorem of the paper, a categorical duality generalising the classical Tarski duality summarised in Section 2.5.

Theorem 7.1. *The category* **epCABA** *of transitive partial CABAs and complete partial Boolean algebra homomorphisms (Definitions 3.12 and 3.16) is contravariantly equivalent to the category* **XGph** *of complete exclusivity graphs and their morphisms (Definitions 5.1 and 6.1).*

Theorem 7.2. *The category* **epFinBA** *of locally finite transitive partial Boolean algebras and partial Boolean algebra homomorphisms (Definitions 3.13 and 3.16) is contravariantly equivalent to the category* **FinXGph** *of complete exclusivity graphs with finite clique number and their morphisms (Definitions 4.3, 5.1 and 6.1)*

The duality between the categories **epCABA** and **XGph** is witnessed by the following contravariant functors:

$$\begin{tikzcd} \mathbf{epCABA} \arrow[rr, bend right, "\mathsf{At}"'] \arrow[rr, phantom, "\cong"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{K}"'] \end{tikzcd} , \tag{10}$$

The functor

$$\mathsf{At}\colon \mathbf{epCABA}^{\mathrm{op}} \longrightarrow \mathbf{XGph}$$

is defined as follows:

- on objects: it maps a transitive partial CABA A to its graph of atoms, whose vertices are atoms and edges denote orthogonality. This forms an exclusivity graph.
- on morphisms: given a partial complete Boolean homomorphism $h\colon A \longrightarrow B$, it yields a relation

$$\mathsf{At}(h)\colon \mathsf{At}(B) \longrightarrow \mathsf{At}(A)$$

  whereby an atom x of B is related to an atom y of A if and only if $x \leq h(y)$.This is a morphism of exclusivity graphs.

The functor

$$\mathcal{K}\colon \mathbf{XGph}^{\mathrm{op}} \longrightarrow \mathbf{epCABA}$$

is defined as follows:

- on objects: it maps an exclusivity graph X to the transitive partial complete Boolean algebra $\mathcal{K}(X)$ thus defined:
  - its elements are cliques of X modulo the equivalence relation that identifies cliques with the same neighbourhood, $K \equiv L \;:\Leftrightarrow\; K^{\#} = L^{\#}$.
  - Commeasurability is the smallest reflexive, symmetric relation such that $[K] \odot [L]$ whenever $K \cup L$ is a clique. That is, $[K] \odot [L]$ if and only if there are cliques $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
  - The constants are given by $0 := [\varnothing]$ and $1 = [K]$ for any maximal clique K (i.e. any clique K with $K^{\#} = \varnothing$).
  - Negation is given by $\neg[K] := [L]$ where L is any clique maximal in $K^{\#}$.
  - Given cliques K and L with $[K] \odot [L]$, meets are given by $[K] \wedge [L] := [K' \cap L']$ for representatives K', L' as in the item above.
  - Given pairwise compatible cliques $\{K_i\}_{i\in I}$, arbitrary joins are given by $\bigvee_i [K_i] := [\bigcup K_i']$ for $K_i' \equiv K_i$ with $\bigcup_{i\in I} K_i'$ a clique.

- on morphisms: given a relation $R\colon X \longrightarrow Y$ which is a morphism of complete exclusivity graphs, one obtains a partial complete Boolean algebra homomorphism

$$\mathcal{K}(f)\colon \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)$$

mapping, for any clique K of Y, the equivalence class $[K]$ to the equivalence class of any clique maximal in the set

$$R^{-1}(K) = \{x \in X \mid \exists y \in K.\, xRy\} \ .$$

The duality (10) is witnessed by two natural isomorphisms:

- Given a transitive partial CABA A, the isomorphism $A \cong \mathcal{K}(\mathsf{At}(A))$ maps $a \in A$ to the equivalence class of any clique maximal in the set

$$\mathcal{U}(a) = \{x \in \mathsf{At}(A) \mid x \leq a\} \ .$$

Thus, a property is identified with the set of (maximally specified) *possible worlds* in which it holds.

- Given a complete exclusivity graph X, the bijection $X \cong \mathsf{At}(\mathcal{K}(X))$ maps $x \in X$ to the singleton set $\{x\}$ – or more accurately, to its equivalence class, which happens only contain the singleton $\{x\}$ itself – which is an atom of $\mathcal{K}(X)$. As such, a possible world is identified with its characteristic property, which completely determines that world.

8. The free-forgetful adjunction

We now consider the free-forgetful adjunction between partial CABAs and compatibility graphs. The free-forgetful adjunction between **pBA** and **RGph** was described in our earlier work [17] as an example of application of a general inductive construction.

Here, we present a different explicit construction, going via the duality. The classical analogy to have in mind is the explicit description of a free CABA through a double-powerset construction, as mentioned in the introduction.

Definition 8.1. Given a compatibility graph V, its **graph of assignments** $\mathcal{A}(V)$ is the graph whose vertices are pairs $\langle C, \gamma\colon C \longrightarrow \{0,1\}\rangle$ where C is a maximal compatible set in V, and whose edge relation is given by

$$\langle C, \gamma\rangle \;\#\; \langle D, \delta\rangle \qquad \text{iff} \qquad \exists x \in C \cap D.\, \gamma(x) \neq \delta(x) \ .$$ [15]

In future versions of this work, we aim to include a direct proof (i.e. not appealing to the earlier description of the free partial Boolean algebra in [17]) that this yields the free-forgetful adjuntion.

9. Outlook

We conclude with some remarks on open questions and pathways for further research. These are presented here in order of increasing scope, from the concrete to the general.

9.1. A more concrete reconstruction: double-neighbourhood closures of cliques

An immediate concrete question regarding the duality presented in this article seeks a more concrete reconstruction of the partial CABA from an exclusivity graph. To that end, it would be helpful to characterise, for any exclusivity graph, the double-neighbourhood-closed sets that are closures of cliques, i.e. the sets of the form $K^{\#\#}$ where K is a clique.

15An alternative description is that $\mathcal{A}(V)$ is the graph whose vertices are pairs $\langle Y, N\rangle$ where $Y \cup N \subseteq V$ is a maximal compatible set and $Y \cap N = \varnothing$, and whose edge relation is given by

$$\langle Y_1, N_1\rangle \;\#\; \langle Y_2, N_2\rangle \qquad \text{iff} \qquad (Y_1 \cap N_2) \cup (N_1 \cap Y_2) \neq \varnothing \ .$$

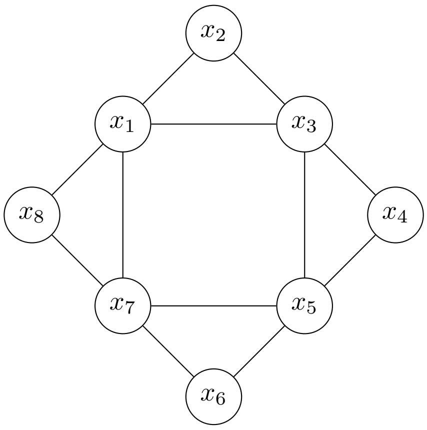

Figure 3. Example of an exclusivity graph (dual to a transitive partial CABA A) with $-^{\#\#}$-closed sets that do not correspond to elements of A. The set of vertices $S = \{x_1, x_5\}$ has neighbourhood $S^{\#} = \{x_3, x7\}$ and bi-neighbourhood $S^{\#\#} = S$, i.e. it's bi-neighbourhood closed. However, it is not the closure of any clique: it has two maximal cliques consisting of single atoms, which are distinct but not commeasurable.

The motivation behind this question is as follows. In section 4 the reconstruction of a partial CABA from its graph of atoms is achieved by taking equivalence classes of cliques under a relation whereby $K \equiv L$ if and only if $K^{\#} = L^{\#}$. Of course, one could equivalently write $K^{\#\#} = L^{\#\#}$. This double-neighbourhood operation $(-)^{\#\#}$ is a closure operation. Moreover, for any element a of the partial CABA, the set $\mathcal{U}(a)$ of atoms under it is given as the closure $K^{\#\#}$ of any clique K maximal in $\mathcal{U}(a)$.

This strongly suggests taking the sets $K^{\#\#}$ themselves, instead of equivalence classes of cliques, as the elements of the reconstructed partial algebra. There is a catch, however. Unfortunately, not all double-neighbourhood closed sets, i.e. sets S of vertices satisfying $S^{\#\#} = S$, in a complete exclusivity graph arise as the closure of a clique. A simple counterexample is presented in Figure 3.

Can we characterise which double-neighbourhood closed sets arise as the closure of cliques?

9.2. The spatial landscape of partial Boolean algebras

Three somewhat orthogonal directions for generalising the duality results presented here immediately suggest themselves. These define three axes in the diagram of categories of partial algebras depicted in Figure 4.

- Transitivity, or the logical exclusivity principle, played a seemingly crucial rôle in the proofs of our results. But can we get away without it? I.e. can this result be generalised to the non-transitive case? What extra information on the atoms is necessary in order to reconstruct a non-transitive partial (complete atomic) Boolean algebra? What is the corresponding generalisation of exclusivity graphs?

- Compatibility in quantum mechanics is given by the binary relation of commutativity. Thus a set of observables is compatible if they are pairwise compatible. This *binarity* of compatibility is built into the definition of partial Boolean algebras through that of the category **RGph** of compatibility graphs.
  General frameworks for studying contextuality, such as e.g. [5, 4], include more general kinds of compatibility structure, described by abstract simplicial complexes (or hypergraphs). Such generality is motivated both by theory-independent analysis of contextuality for quantum information and computation, and by applications in other domains.
  Simplicial complexes generalise compatibility graphs in the sense that **RGph** can be seen as a reflective subcategory the category **Simp** of simplicial complexes.[16] The corresponding appropriate generalisation of partial Boolean algebras seems to coincide with Czelakowski's notion of 'partial Boolean algebras in a broader sense' from [30]. How much of the duality can

$$
\begin{tikzcd}
& \mathbf{epDLat_{bs}} \arrow[rr, tail] & & \mathbf{pDLat_{bs}} \\
\mathbf{epDLat} \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pDLat} \arrow[ur, tail] & \\
& \mathbf{epBA_{bs}} \arrow[uu, hook] \arrow[rr, tail] & & \mathbf{pBA_{bs}} \arrow[uu, hook] \\
\mathbf{epBA} \arrow[uu, hook] \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pBA} \arrow[uu, hook] \arrow[ur, tail] & \\
& \mathbf{epCABA_{bs}} \arrow[uu, hook] \arrow[rr, tail] & & \mathbf{pCABA_{bs}} \arrow[uu, hook] \\
\mathbf{epCABA} \arrow[uu, hook] \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pCABA} \arrow[uu, hook] \arrow[ur, tail] &
\end{tikzcd}
$$

Figure 4. Diagram of inclusions of categories of partial Boolean algebras (and partial distributive lattices). All arrows are inclusions of categories, and horizontal ones are moreover reflective. The picture suggests possible avenues to generalise this paper's results along three axes: rightwards dropping transitivity, backwards generalising from binary to simplicial compatibility, upwards dropping completeness and atomicity towards Stone (and Priestley) duality. This paper describes a duality for the category **epCABA** in the foreground lower left vertex: the remainder of the picture is open to future exploration.

be generalised to this *broader* setting?

- Perhaps more challenging, another natural generalisation to consider would be to non-atomic (transitive) partial Boolean algebras. The goal would be to find a generalisation of the classical Stone duality between Boolean algebras and Stone spaces to the (transitive) partial setting. A relevant class of examples this would cover is that of partial Boolean algebras $\mathsf{P}(\mathcal{A})$ of projections on a von Neumann algebra $\mathcal{A}$ with at least one factor not of type I.
  Mirroring the classical case, one expects that it is key to follow Stone's precept to 'always topologise' [31]. But what is the appropriate generalisation of topological spaces from sets (inequality relations) to more relaxed forms of exclusivity? What is the corresponding generalisation of continuity from functions to relations between topologised complete exclusivity graphs?

- Along the same line, we could aim to push even further beyond partial Boolean algebras to a partial version of distributive lattices, aiming to generalise the notions of Priestley or spectral spaces.

16In fact, **RGph** embeds in **Simp** in two different ways, as a reflective and a coreflective subcategory. It is a coreflexive subcategory if we think of compatibility graphs as simplicial complexes of dimension ≤ 1, with the right adjoint being given by the 1-skeleton functor $\mathsf{sk}_1 \colon \mathbf{Simp} \longrightarrow \mathbf{RGph}$. However, the inclusion $\mathbf{RGph} \longrightarrow \mathbf{Simp}$ we have in mind maps a compatibility graph into the simplicial complex whose faces are its cliques. This has a left adjoint, which happens to be sk_1 as well.

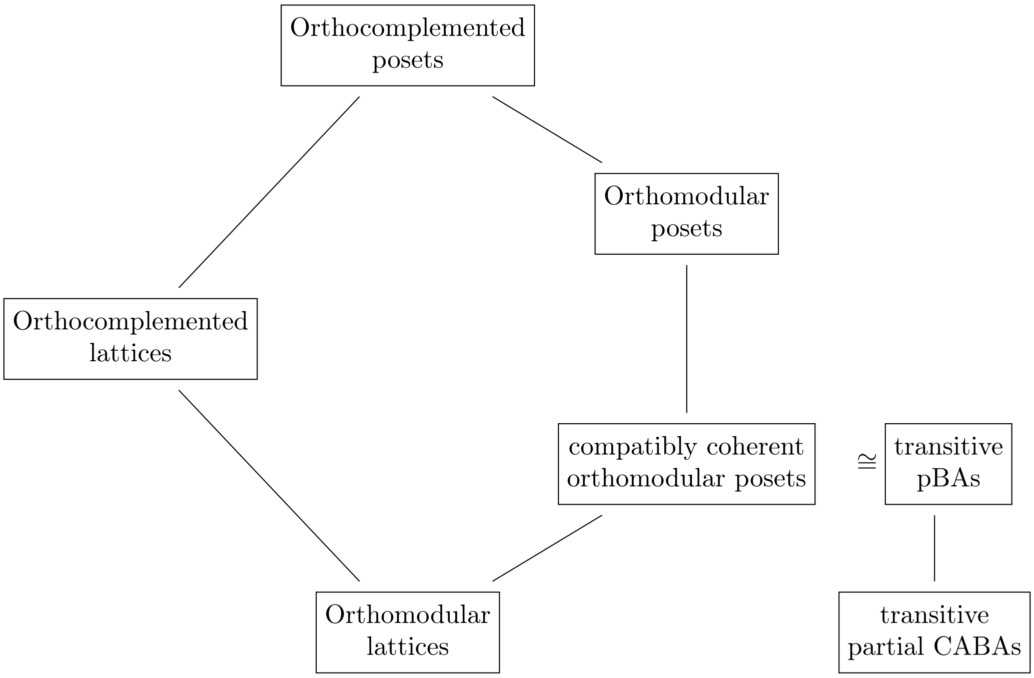

Figure 5. Hasse diagram with inclusions between various classes of orthocomplemented posets.

9.3. The wider spatial landscape of 'quantum' logics

Various classes of algebraic and/or order-theoretic structures have been considered in the wider context of algebraic quantum logic. Generalising orthomodular lattices gives rise to the hierarchy of structures depicted in Figure 5. This takes as its starting point the notion of orthocomplemented poset: a bounded poset with an involutive, antitone, unary operation $(-)'$ satisfying that the bottom element 0 is the infimum of x and x' for all elements x. This notion is then built upon through the (independent) additional requirements of lattice structure and orthomodularity.

Gudder [15] showed that transitive/LEP partial Boolean algebras sit within this hierarchy as a special class of orthomodular posets [32], those that satisfy an additional condition known as being compatibly coherent. This establishes a connection between the world of partial Boolean algebras and that of orthocomplemented posets.

In the other branch of the Hasse diagram of Figure 5 sits the class of orthocomlemented lattices, or ortholattices. This has been studied from a logical perspective as the algebraisation of so-called orthologic, also known as 'minimal quantum logic' [33, 34, 35]. Moreover, Goldblatt proved a Stone representation theorem for ortholattices [36]. Perhaps not surprisingly, the central ideas are closely related to those in the present paper – notably, ortholattices are represented as sets closed under a double neighbourhood construction on a graph (symmetric, irreflexive relation). However, the precise connection between both representation results is yet to be ironed out. In particular, Goldblatt's result constructs an ortholattice from any graph (not just complete exclusivity graphs) by taking all the double-neighbourhood closed sets (not just the closures of cliques), revealing some subtle differences with our representation result.

Two paths to establish a meaningful comparison between the two results are: to consider the restriction of the ortholattice result to the case of finite ortholattices, or to first generalise our result to transitive partial Boolean algebras (no longer necessarily complete and atomic). Answering the first question raised in this outlook section would also facilitate this comparison.

A broader aim is to extend both of these results in search of a mirror image of Figure 5 in the 'spatial' side. Importantly, a notable aspect seemingly missing from Goldblatt's representation theorem for ortholattices is what happens to morphisms. The dual representation of morphisms is a key facet to keep in sight in the comparisons and extensions suggested above. A first step in this regard would be, of course, to extend Goldblatt's representation theorem into a fully-fledged categorical duality.

9.4. The quest for noncommutative spaces

As a wider – and necessarily more speculative – point, one might hope that further investigation along these lines, extending classical dualities to various special cases of 'quantum' structures, may lead one to identify some germs of generality for a more encompassing duality theory, providing appropriate notions of spectra for a host of 'noncommutative' or 'quantum' structures.

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