Acyclicity and Vorob'ev's theorem

deriving monogamy of non-locality and local macroscopic averages

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THE UNIVERSITY *of* EDINBURGH
informatics

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Contextuality

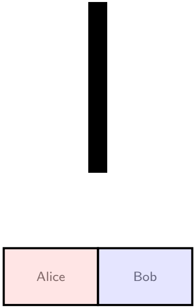
Overview

- Quantum mechanics predicts phenomena that run counter to our (classical) intuitions.
- This talk will focus on **non-locality**, and more generally, **contextuality**.
- Contextuality is a feature of **empirical data** that is a key signature of non-classicality.
- As such, we can study it in a **theory-independent** framework.

Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

They may share prior information,



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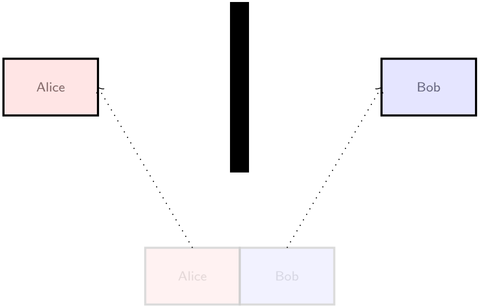

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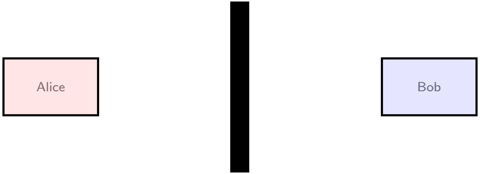

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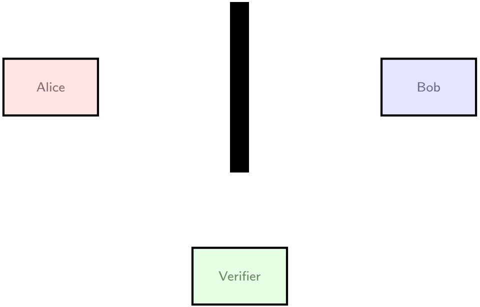

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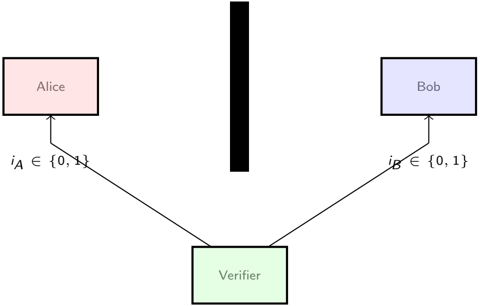

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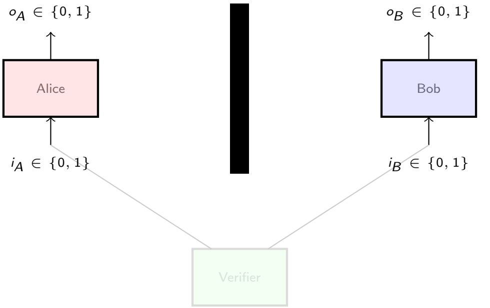

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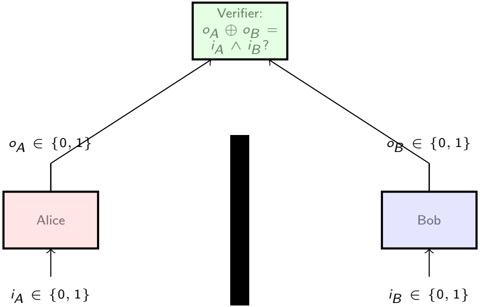

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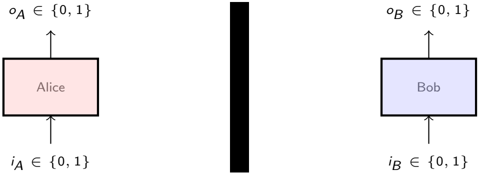

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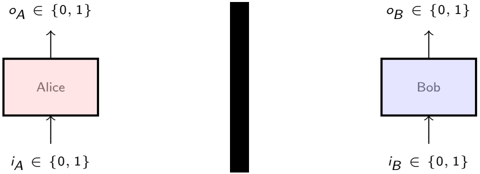

They win a play if $o_A \oplus o_B = i_A \wedge i_B$.

A strategy is described by the probabilities $P(o_A, o_B \mid i_A, i_B)$.

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This gives a winning probability of $3.25/4 \approx 0.81$!

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- So, what is the unwarranted assumption?
- That all variables could *in principle* be observed simultaneously.

The essence of contextuality

- Not all properties may be observed at once.
- Jointly observable properties provide **partial snapshots**.

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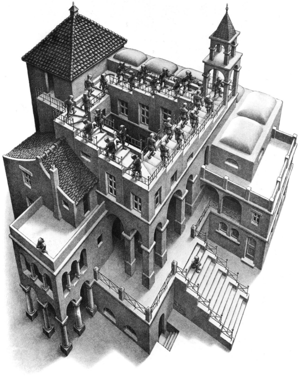

M. C. Escher, *Ascending and Descending*

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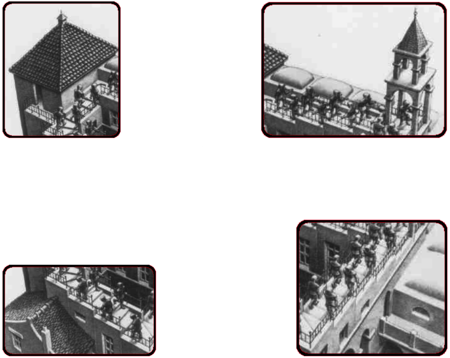

Local consistency

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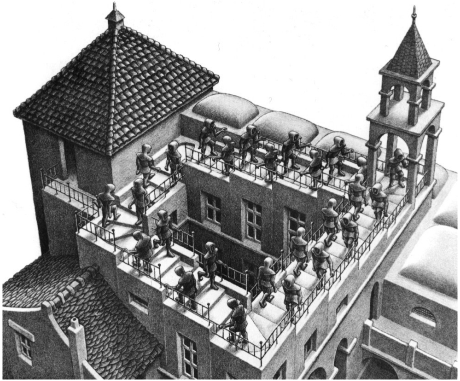

Local consistency *but* **Global inconsistency**

A recurring theme

- Non-locality and contextuality
- Relational databases
- Contraint satisfaction
- ...

Formalising empirical data

A **measurement scenario** $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- X – a finite set of measurements
- $O = (O_x)_{x \in X}$ – for each $x \in X$ a non-empty set of possible outcomes O_x
- Σ – an abstract simplicial complex on X faces are called the **measurement contexts**

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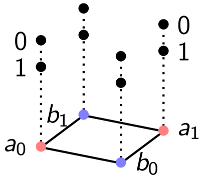

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family of finite subsets of X such that:

- it contains all the singletons: $\forall x \in X.\ \{x\} \in \Sigma$.
- it is downwards closed: $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ implies $\tau \in \Sigma$.

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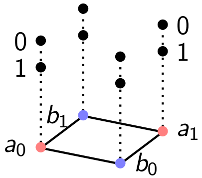

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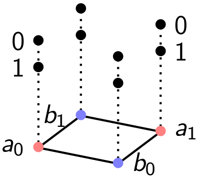

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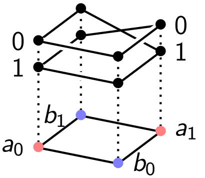
(i.e. marginals are well-defined)



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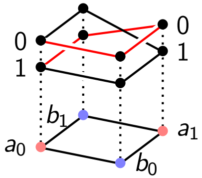

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That is, we can **glue** all the local information together into a global consistent description from which the local information can be recovered.

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The import of Bell's and Kochen–Spekker's theorems is that there are behaviours arising from quantum mechanics that are contextual.

Vorobʹev's theorem

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Vorobʹev (1962)

'Consistent families of measures and their extensions'

- In the context of game theory.
- Consider a collection of variables
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- Necessary and sufficient condition: **regularity**!

Relational databases

Codd (1970): Relational model of data

- Information is organised into tables (relations).
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- A database consists of a set of such tables, each with different attributes
- Database schema: blueprint of a database specifying attributes of each table and type of information: $\mathcal{S} = \{A_1, \ldots, A_n\}$
- Database instance: snapshot of the contents of a database at a certain time, consisting of a relation instance (i.e. a set of entries) for each table: $\{R_A\}_{A \in \mathcal{S}}$.

Relational databases

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$$R|_A := \{t|_B \mid t \in R\}$$

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- It is **totally consistent** if it has a universal relation instance: T on attributes $\bigcup \mathcal{S}$ with $\forall A \in \mathcal{S}. \;\; T|_A = R_A$

Dictionary

Databases	**Empirical models**
attributes	measurements
domain of attribute	outcome value of measurement
relation schema	set of compatible measurements
database schema	measurement scenario
tuple / entry	joint outcome

Dictionary

relation instance	distribution on joint outcomes
database instance	empirical model
projection	marginalisation
projection consistency	no-signalling condition
universal instance	global distribution
total consistency	locality / non-contextuality

An analogous question

For which database schemata does pairwise projection consistency imply total consistency?

- Necessary and sufficient condition: **acyclicity**.
- Acyclic database schemes extensively studied in late 70s / early 80s
- Many equivalent characterisations . . .

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For which database schemata does pairwise projection consistency imply total consistency?

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- Many equivalent characterisations ...
- **Turns out to be equivalent to Vorob'ev's condition!**

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- Graham reduction step: delete a vertex that belongs to only one maximal face.

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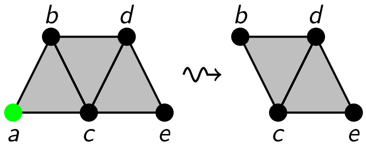

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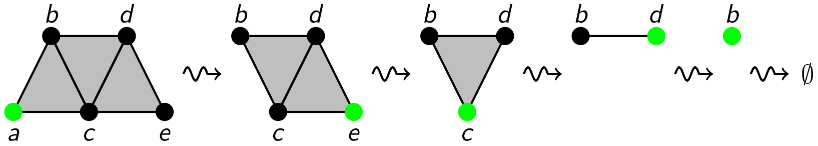

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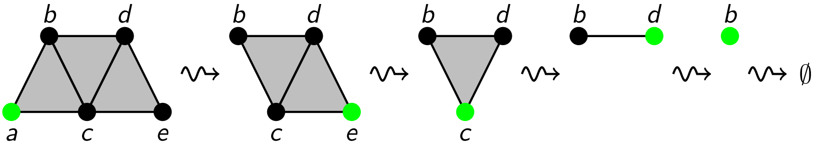

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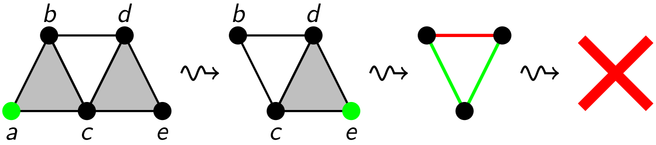


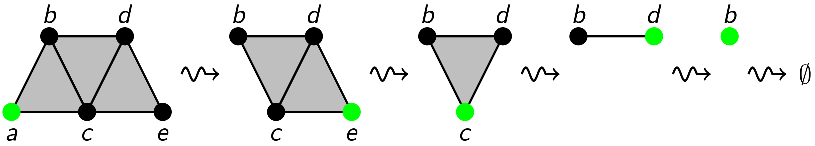
Vorob'ev's theorem

Theorem (Vorob'ev 1962, adapted)

All empirical models on Σ are extendable iff Σ is acyclic

Sketch of proof of Vorob'ev's theorem

- If Σ is acyclic,



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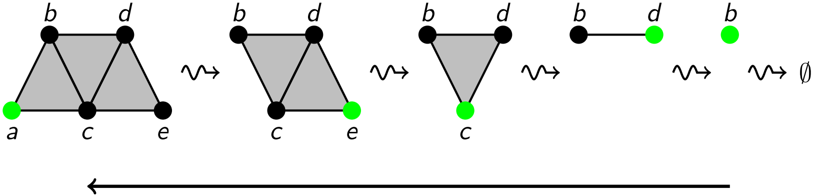

then construct a global distribution by glueing

Given distributions P_{ab} over $\{a, b\}$ and P_{bc} over $\{b, c\}$ compatible on b,

$$\sum_{x \in O} P(a, b = x, y) = \sum_{z \in O} P(b, c = y, z) \ ,$$

we can define an extension:

$$P(a, b, c = x, y, z) = \frac{P(a, b = x, y) P(b, c = y, z)}{P(b = y)} \ .$$

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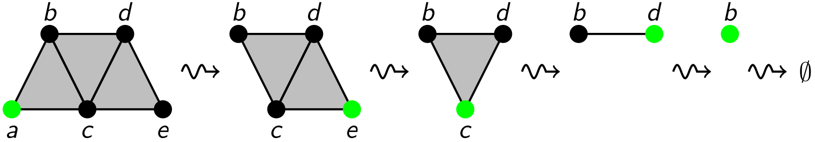

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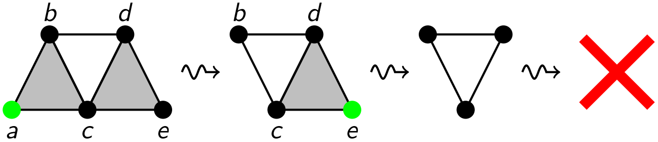

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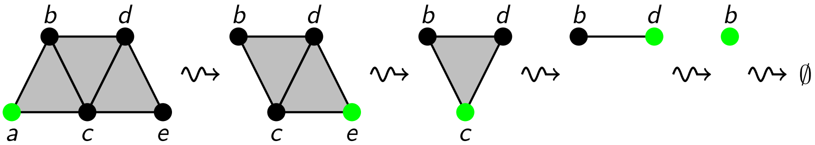

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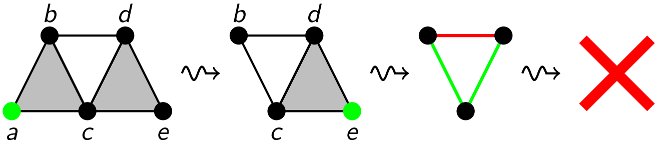

There is a "cycle"!

Examples of glueing

- Relational databases:
  - R_1 on attributes A_1, R_2 on attributes A_2
  - Define the natural join $R_1 \bowtie R_2$ on $A_1 \cup A_2$:

$$R_1 \bowtie R_2 \ := \left\{ t \in D^{A \cup B} \mid t|_{A_1} \in R_1, t|_{A_2} \in R_2 \right\}$$

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  - Both of these are examples of distributions
  - $\langle \mathbb{R}_{\geq 0}, +, \cdot, 0, 1 \rangle$: probability
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- Flori–Fritz: metric spaces

- Logic: Robinson Joint Consistency Theorem
  - Let T_i be a theory over the language L_i, with $i \in \{1, 2\}$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

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Σ is acyclic if and only if for all $\sigma \in \Sigma$ $\mathsf{lk}_\Sigma(\sigma)$ is contractible to a disjoint union of points.

An application: monogamy of non-locality and average macroscopic locality

Monogamy and average macroscopic locality

- Average macro correlations from micro models are local
(Ramanathan, Paterek, Kay, Kurzyński & Kaszlikowski 2011: multipartite quantum models)

- Monogamy of violation of Bell inequalities
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- connect and generalise the results above

- a structural explanation related to Vorob'ev's theorem

- Let us look at a simple illustrative example.

Macroscopic measurements

- (Micro) dichotomic measurement: a single particle is subject to an interaction a and collides with one of two detectors: outcomes 0 and 1.
- The interaction is probabilistic: $p(a = x)$, $x = 0, 1$.

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- Consider beam (or region) of N particles, differently prepared.
- Subject each particle to the interaction a: the beam gets divided into 2 smaller beams hitting each of the detectors.
- Outcome represented by the intensity of resulting beams:
  $I_a \in [0, 1]$ proportion of particles hitting the detector 1.
- We are concerned with the mean, or expected, value of such intensities.

Macroscopic average behaviour

- This mean intensity can be interpreted as the average behaviour among the particles in the beam or region:
  if we would randomly select one of the N particles and subject it to the microscopic measurement a, we would get outcome 1 with probability I_a:

$$I_a = \sum_{i=1}^{N} p_i(a = 1) \ .$$

- The situation is analogous to statistical mechanics, where a macrostate arises as an averaging over an extremely large number of microstates, and hence several different microstates can correspond to the same macrostate.

Macroscopic average behaviour: multipartite

- Multipartite macroscopic measurements:
  - several 'macroscopic' sites consisting of a large number of microscopic sites/particles;
  - several (macro) measurement settings at each site.
- Average macroscopic Bell experiment: the (mean) values of the macroscopic intensities indicate the behaviour of a randomly chosen tuple of particles: one from each of the beams, or sites.

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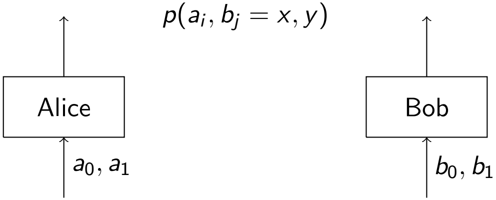
- We shall show that, as long as there are enough particles (microscopic sites) in each macroscopic site, such average macroscopic behaviour is always local no matter which no-signalling model accounts for the underlying microscopic correlations.

Non-locality



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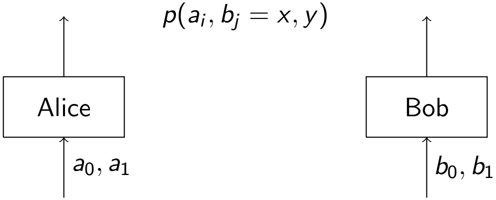

	00	01	10	11
$a_0 b_0$	$1/2$	0	0	$1/2$
$a_0 b_1$	$3/8$	$1/8$	$1/8$	$3/8$
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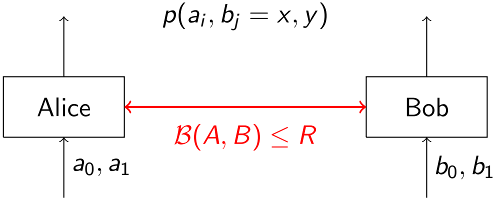

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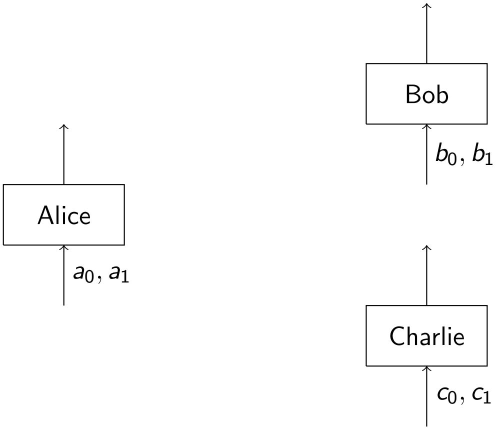
Monogamy of non-locality



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  where x, y, z are possible outcomes.

- Consider the subsystem composed of A and B only, given by marginalisation (in QM, partial trace):

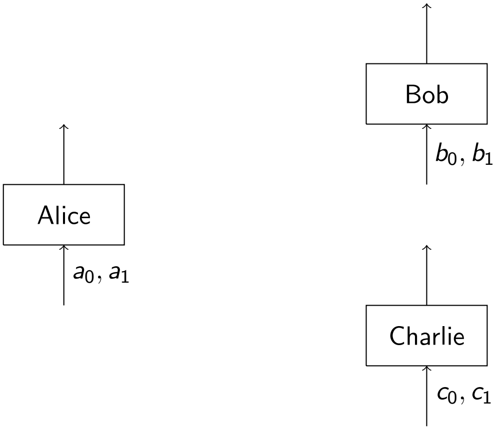
$$p(a_i, b_j = x, y) = \sum_z p(a_i, b_j, c_k = x, y, z)$$

  (this is independent of c_k due to no-signalling).

  Similarly define $p(a_i, c_k = x, z)$. (A and C)

Monogamy of non-locality

Given a Bell inequality $\mathcal{B}(-,-,) \leq R$,



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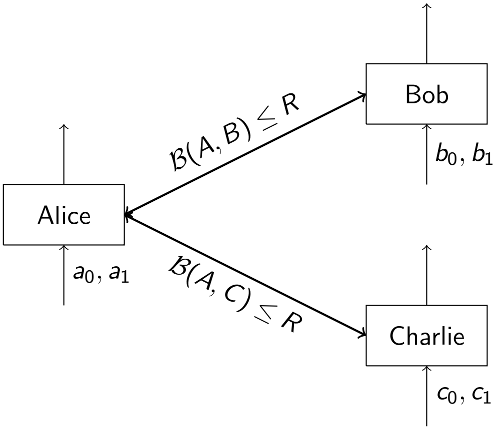

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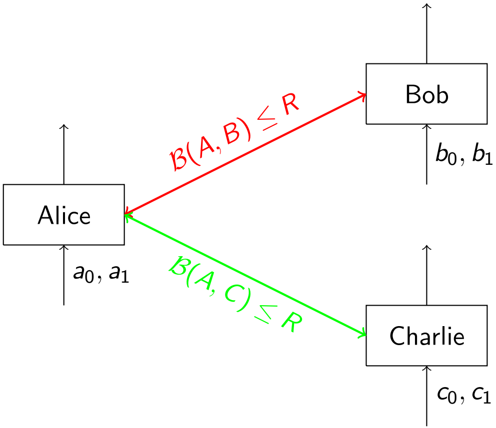

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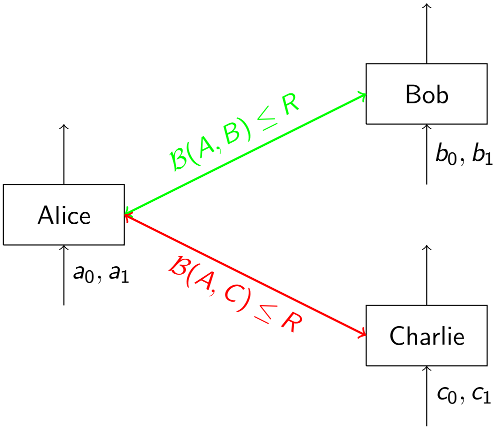

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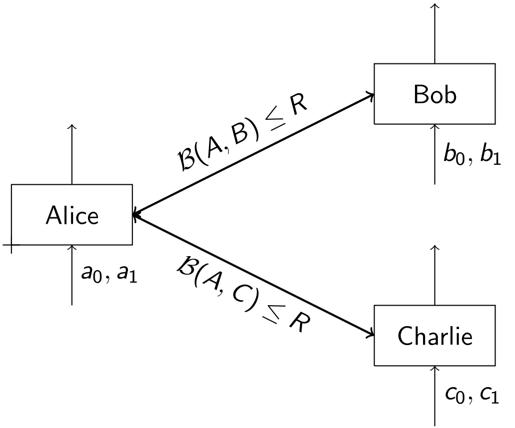

Monogamy relation: $\mathcal{B}(A, B) + \mathcal{B}(A, C) \leq 2R$

Macroscopic average behaviour: tripartite example

- We regard sites B and C as forming one 'macroscopic' site, M, and site A as forming another.
- In order to be 'lumped together', B and C must be symmetric/of the same type: the symmetry identifies the measurements $b_0 \sim c_0$ and $b_1 \sim c_1$, giving rise to 'macroscopic' measurements m_0 and m_1.

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- Given a (tripartite) empirical model $p(a_i, b_j, c_k \mapsto x, y, z)$, the 'macroscopic' average behaviour is a bipartite model (with two macro sites A and M) given by the following average of probabilities of the partial models:

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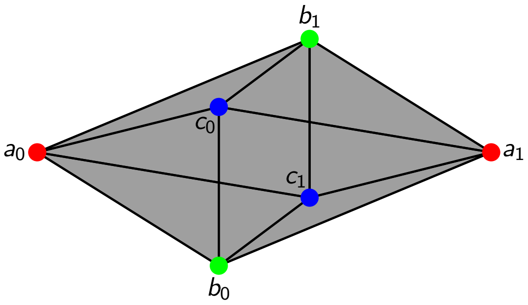
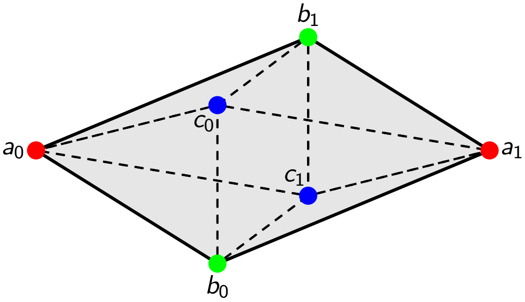
The average model p_{a_i,m_j} **satisfies a bipartite Bell inequality** if and only if in the microscopic model Alice is **monogamous** with respect to violating it with Bob and Charlie.

Structural Reason



- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.

Structural Reason



- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.

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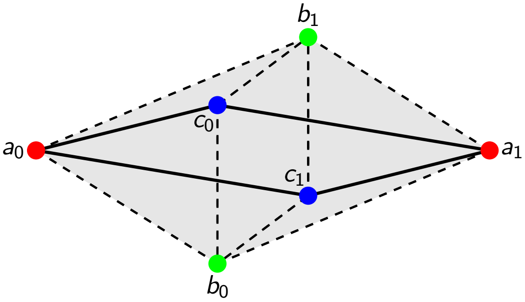

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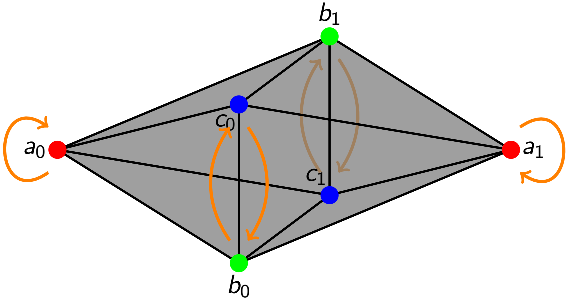

- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.
- We identify B and C: $b_0 \sim c_0$, $b_1 \sim c_1$.
- The macro scenario arises as a quotient.

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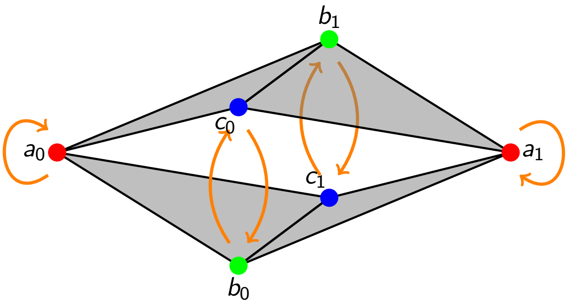

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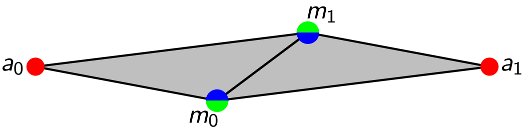

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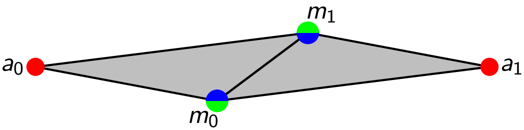

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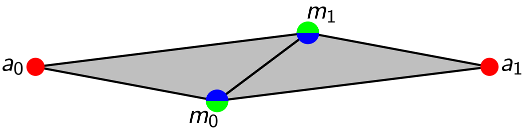
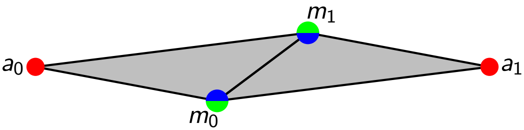


- This quotient complex is **acyclic**.

Structural Reason



- This quotient complex is **acyclic**.
- Therefore, no matter from which micro model p_{a_i,b_j,c_k} we start, the averaged macro correlations p_{a_i,m_j} are local.

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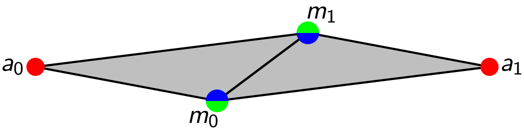

- This quotient complex is **acyclic**.
- Therefore, no matter from which micro model p_{a_i,b_j,c_k} we start, the averaged macro correlations p_{a_i,m_j} are local.
- In particular, they satisfy any Bell inequality.
- Hence, the original tripartite model also satisfies a monogamy relation for any Bell inequality.

Questions...

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