

Lógica Quântica

Assessment 1

2022–2023

Part 1

Definition 1. Given a linear map $f: H \rightarrow K$ between Hilbert spaces H and K , its adjoint is the unique linear map $f^\dagger: K \rightarrow H$ such that, for all $v \in H$ and $w \in K$,

$$\langle f(v), w \rangle = \langle v, f^\dagger(w) \rangle.$$

Exercise 1. Show that this construction is functorial, i.e. it defines a functor $(-)^\dagger: \mathbf{Hilb}^{\text{op}} \rightarrow \mathbf{Hilb}$. Moreover, show that this functor is involutive, i.e. for any $f: H \rightarrow K$, $(f^\dagger)^\dagger = f$. ◀

Part 2

Definition 2. A *monoidal category* is a category \mathbf{C} equipped with:

- a functor $\otimes: \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ (called *tensor*);
- an object I of \mathbf{C} (called the *unit*);
- natural isomorphisms α, λ, ρ with components

$$\begin{aligned} \alpha_{A,B,C}: A \otimes (B \otimes C) &\xrightarrow{\cong} (A \otimes B) \otimes C \\ \lambda_A: I \otimes A &\xrightarrow{\cong} A \quad \rho_A: A \otimes I \xrightarrow{\cong} A \end{aligned}$$

(called the *associator*, the *left unitor*, and the *right unitor*, respectively);

satisfying the equations

$$(\text{id}_A \otimes \lambda_B) \circ \alpha_{A,I,B} = \rho_A \otimes \text{id}_B \quad \text{and} \quad (\text{id}_A \otimes \alpha_{B,C,D}) \circ \alpha_{A,B \otimes C,D} \circ (\alpha_{A,B,C} \otimes \text{id}_D) = \alpha_{A,B,C \otimes D} \circ \alpha_{A \otimes B,C,D}$$

which are expressed by the commutativity of the following *triangle* a *pentagon* diagrams:

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ & \searrow \rho_A \otimes \text{id}_B \quad \swarrow \text{id}_A \otimes \lambda_B & \\ & (A \otimes B) & \end{array}$$

$$\begin{array}{ccccc} & & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\ & \nearrow \alpha_{A,B,C} \otimes \text{id}_D & & & \searrow \text{id}_A \otimes \alpha_{B,C,D} \\ ((A \otimes B) \otimes C) \otimes D & & & & A \otimes (B \otimes (C \otimes D)) \\ & \searrow \alpha_{A \otimes B,C,D} & & \nearrow \alpha_{A,B,C \otimes D} & \\ & (A \otimes B) \otimes (C \otimes D) & & & \end{array}$$

These equations guarantee what is called *coherence*: that all diagrams involving only α , λ , and ρ commute.

In any monoidal category, one can reason using string diagrams (and you may use those in this exercise if you prefer).

We have seen that any category \mathbf{C} with binary products \times and a terminal object $\mathbf{1}$ is a monoidal category: exercise 3.13, exercise 4.7. Moreover, we have seen in exercises 4.2 and 4.3 that, in the case of products there are natural transformations Δ , p , q with components

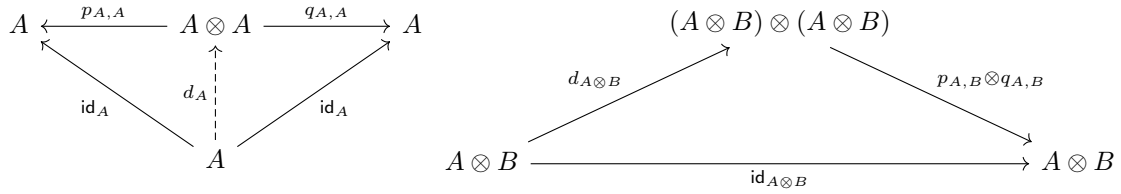
$$\delta_A: A \longrightarrow A \times A, \quad p_{X,Y}: X \times Y \longrightarrow X, \quad q_{X,Y}: X \times Y \longrightarrow Y,$$

which one can interpret as *copying* and *projections*. The goal of this second part is to show a converse to this, therefore characterising products as precisely monoidal structures (tensors) that admit copying and projections in some sense.

Exercise 2. Let \mathbf{C} be a monoidal category and suppose there are natural transformations with components of type

$$d_A: A \longrightarrow A \otimes A, \quad p_{X,Y}: X \otimes Y \longrightarrow X, \quad q_{X,Y}: X \otimes Y \longrightarrow Y,$$

such that the following diagrams commute:



Show that \otimes gives a product structure, i.e. that $A \otimes B$ is the categorical product of A and B . ◀