Contextuality in logical form: Duality for transitive partial CABAs

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Abstract

Partial Boolean algebras were introduced by Kochen and Specker in their seminal work on contextuality in quantum mechanics, as a natural (algebraic-)logical setting for contextual systems, corresponding to a calculus of partial propositional functions. They provide an alternative to traditional Birkhoff–von Neumann quantum logic in which operations such as conjunction and disjunction are partial, being only defined in the domain where they are physically meaningful. In the key example of the projectors on a Hilbert space, the operations are only defined for commuting projectors, which correspond to properties of the quantum system that can be tested simultaneously.

We extend the classical Lindenbaum–Tarski dualities between finite sets and finite Boolean algebras, and more generally between sets and complete atomic Boolean algebras (CABAs), to the setting of (transitive) partial Boolean algebras. Specifically, we establish a dual equivalence between the category of transitive partial CABAs and a category of exclusivity graphs with an appropriate notion of morphism.

The vertices of an exclusivity graph may be interpreted as *possible worlds* of maximal information, with edges representing logical incompatibility or mutual exclusivity between two worlds. The classical case corresponds to complete graphs, as all possible worlds are mutually exclusive. Similarly, the appropriate notion of morphism is relaxed from functions to certain kinds of relations. From an exclusivity graph, a transitive partial CABA is constructed whose elements are sets of mutually exclusive worlds (cliques of the graph) modulo an equivalence relation. This equivalence identifies cliques that jointly exclude the same set of worlds, i.e. that have the same neighbourhood. The main result shows, in particular, that any transitive partial CABA can be recovered in this fashion from its graph of atoms with the logical exclusivity relation.

We also give an explicit construction of the free transitive partial CABA on a set of propositions with a compatibility relation, via an adjunction between compatibility graphs and exclusivity graphs that generalises the powerset self-adjunction from the classical case.

The duality reveals a connection between the algebraic-logical setting of partial Boolean algebra and the graph-theoretic approach to contextuality of Cabello–Severini–Winter. Under it, a transitive partial CABA witnessing contextuality, in the Kochen–Specker sense that it has no homomorphism to the two-element Boolean algebra, corresponds to a graph with no 'points', i.e. with no maps from the singleton graph, which are in bijection with stable, maximum clique transversal sets.

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1 Introduction

Propositions about quantum systems Quantum mechanical systems are described mathematically in terms of complex Hilbert spaces [33]. Measurements are represented by bounded self-adjoint operators, whose (real) eigenvalues correspond to the possible measurement outcomes. Adopting an operational perspective, the (testable) properties of the state of a system are identified with dichotomic measurements, i.e. those with two possible outcomes, say 1 and 0 (*true* and *false*). Mathematically, such measurements are represented by **projection** operators, bounded self-adjoint operators $p\colon \mathcal{H} \longrightarrow \mathcal{H}$ satisfying $p^2 = p$.[1]

The set $\mathsf{P}(\mathcal{H})$ of projections on a Hilbert space $\mathcal{H}$ forms an orthomodular lattice. In the tradition of algebraic logic, Birkhoff and von Neumman [7] put forward the idea that the operations *meet*, *join*, and *complement* could be interpreted as logical connectives, suggesting a calculus of propositions for quantum mechanics analogous to the classical propositional calculus with respect to *and*, *or*, and *not*. A marked difference *vis-à-vis* classical logic is the failure of the distributive law, $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. This constitutes an arguably much more radical departure, logically speaking, than for example intuitionistic logic.

Incompatibility as partiality A more fundamental shortcoming of traditional (orthomodular) quantum logic is that the propositional connectives lack a physically meaningful operational interpretation. To appreciate the issue, it is first important to point out that a crucial rôle in the formulation of quantum mechanics is played by the notion of **commeasurability**. In contrast to classical physics, not all pairs of measurement procedures may be performed jointly on a quantum system. Such incompatibility of measurements is embodied by noncommutativity of the corresponding self-adjoint operators.

Now, suppose that we know how to perform a test (i.e. a dichotomic measurement) for each of two properties represented by projections p and q. If p and q commute, then these tests can be jointly performed. Hence, we can design a new (dichotomic) measurement procedure wherein both p and q are measured and the outcome is the conjunction (resp. disjunction) of the individual outcomes. The measurement thus implemented corresponds to the projection $p \wedge q$ (resp. $p \vee q$).[2] But how would one implement a test for $p \wedge q$ or $p \vee q$ when p and q do not commute?

An alternative approach captures measurement incompatibility by **partiality**, so that the logical connectives are only defined in the domain where they are physically meaningful, i.e. for commeasurable propositions. In the key example of the projectors on a Hilbert space the operations are be defined for commuting projectors, which correspond to properties of a quantum system that can be tested *simultaneously*. This point of view was adopted by Kochen and Specker [25, 19] and used in their seminal work [18] on the problem of supplementing quantum mechanics with *hidden* variables that would explain away some of its counterintuitive features. They introduced the notion of **partial Boolean algebra**, providing an alternative to traditional quantum logic à la Birkhoff–von Neumann in which disjunction and conjunction become partial operations that behave classically, i.e. according to the laws of Boolean algebra, on sets of pairwise-commeasurable elements (see Section 3 for details). The language of partial Boolean algebras was used to formulate their famous no-go theorem establishing (state-independent) **contextuality** as a necessary feature of any theory matching the predictions of quantum mechanics. This was expressed as the fact that the partial Boolean algebras of projectors on a Hilbert space of dimension ≥ 3 admits no homomorphism to the two-element Boolean algebra.

There is the question of name: https://tel.archives-ouvertes.fr/tel-02459132/document calls it L–T, as does https://www.cs.cas.cz/tacl2017/slides/contributed/12.pdf

Lindenbaum–Tarski duality Duality pervades mathematics. There is a whole landscape of dual equivalences between categories of algebraic structures and categories of topological

[1] Idempotence implies that the eigenvalues of p must be in the set $\{0, 1\}$. Projections on $\mathcal{H}$ are in bijection with the closed linear subspaces of $\mathcal{H}$, whereby a projection is mapped to its 1-eigenspace. This is the subspace of pure quantum states for which the property being tested can be said to hold with *certainty*.

[2] Mathematically, also, meet and join admit simple descriptions (in terms of the operations of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators) for commuting projections p and q, to wit: $p \wedge q = pq$ and $p \vee q = p + q - pq$.

or geometric spaces.[3] In the context of logic, such dualities relate syntax and semantics: on the one hand the (Lindenbaum–Tarski) algebra of sentences modulo provable equivalence in a theory T, on the other the space of models of T. For classical propositional logic, this is embodied by Stone's duality between Boolean algebras and Stone spaces [26, 27, 24]. In this article, we focus on the simpler setting that corresponds to propositional theories on a finite language, namely the restriction of Stone duality to a dual equivalence between the category **FinBA** of finite Boolean algebras and the category **FinSet** of finite sets. In this special case the topological aspects of Stone duality trivialise as the spaces become discrete, i.e. just sets.

Indeed, a finite Boolean algebra can be represented as the powerset of its finite set of atoms, equipped with the usual set-theoretic operations of *intersection*, *union*, and *complement*. Moreover, homomorphisms between two Boolean algebras are in one-to-one correspondence with functions between their sets of atoms in the reverse direction. In categorical terms, we have two inverse (contravariant) functors: $\mathsf{At}\colon \mathbf{FinBA}^{\mathrm{op}} \longrightarrow \mathbf{FinSet}$ mapping a finite Boolean algebra to its set of atoms, and $\mathcal{P}\colon \mathbf{FinSet}^{\mathrm{op}} \longrightarrow \mathbf{FinBA}$ mapping a finite set to its powerset Boolean algebra.

If an algebra A encodes a propositional theory T, then one may interpret its set of atoms as the (mutually exclusive) **possible worlds**, or models of that theory. Through the (natural) isomorphism $A \cong \mathcal{P}(\mathsf{At}(A))$, a propositional sentence is then identified with the worlds in which it is true.

The free Boolean algebra on a finite set of propositions is the Lindenbaum–Tarski algebra of the empty theory, and as such it admits all possible worlds, i.e. its models are all the truth valuations on the generating propositions, equivalently represented by all subsets of such propositions. Thus, the free-forgetful adjunction for finite Boolean algebras corresponds under the dual equivalence to the self-adjunction of the (contravariant) powerset functor $\mathcal{P}\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. The situation is summarised in the diagram below:

$$\begin{tikzcd}
\mathbf{FinBA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] \arrow[dr, phantom, "\dashv"] & & \mathbf{FinSet}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{P}"] \arrow[dl, shift left, "\mathcal{P}"] \arrow[dl, phantom, "\dashv"] \\
& \mathbf{FinSet} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "\mathcal{P}"] &
\end{tikzcd} \tag{1}$$

In fact, this duality extends to a more general setting. Lindenbaum and Tarski characterised the Boolean algebras that arise as the *whole* powerset of a (not necessarily finite) set as being the **complete atomic Boolean algebras**, CABAs for short [28]. Indeed, the above-mentioned duality generalises to this non-finite setting as a dual equivalence between the category **Set** of sets and functions and the category **CABA** of complete atomic Boolean algebras and complete Boolean algebra homomorphisms.[4] The same goes for the remarks about the free-forgetful (and powerset self-) adjunctions. The situation is summarised in the

[3] E.g. Gel'fand–Naĭmark duality between commutative C^*-algebras and locally compact Hausdorff spaces.

diagram below:

$$\begin{tikzcd}[column sep=small]
\mathbf{CABA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] \arrow[dr, phantom, "\dashv" sloped] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{P}"] \arrow[dl, shift left, "\mathcal{P}"] \arrow[dl, phantom, "\dashv" sloped] \\
& \mathbf{Set} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "\mathcal{P}"] &
\end{tikzcd} \tag{2}$$

This extends the earlier triangle diagram in that the categories appearing in the vertices of (1) are full subcategories of those in the corresponding vertices of (2), and the functors in the edges of the former triangle are the restrictions of those in the edges of the latter.[5]

Contributions The main contribution of this article is a generalisation of the Lindenbaum–Tarski dualities just described to the setting of partial Boolean algebras. In fact, we restrict our attention to **transitive** partial Boolean algebras [15, 20], or equivalently, those that satisfy the **logical exclusivity principle** (LEP) [1]; see Section 3.4 for details.

The results are summarised by the following two diagrams of functors:

$$\begin{tikzcd}[column sep=small]
\mathbf{epFinBA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] \arrow[dr, phantom, "\dashv" sloped] & & \mathbf{FinXGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \arrow[dl, phantom, "\dashv" sloped] \\
& \mathbf{FinRGph} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "\mathcal{V}"] &
\end{tikzcd} \tag{3}$$

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[4] This duality between **Set** and **CABA** is sometimes referred to as Lindenbaum–Tarski duality in light of their 1935 result, e.g. in [22].

[5] This more general form of Lindenbaum–Tarski duality does not arise as a restriction of Stone duality to complete atomic Boolean algebras, for in particular, only complete Boolean algebra homomorphisms are considered.
Yet curiously, both Stone duality $\mathbf{BA} \cong \mathbf{Stone}^{\mathrm{op}}$ between Boolean algebras and Stone spaces and the Lindenbaum–Tarski duality $\mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}$ from (2) can be seen to arise from the finite duality $\mathbf{FinBA} \cong \mathbf{FinSet}^{\mathrm{op}}$ in (1) through two related category-theoretic constructions: the former arises by taking the ind-completion on the algebraic side (obtaining Boolean algebras from finite Boolean algebras), and thus dually the pro-completion on the topological side (Stone spaces arising from finite sets),

$$\mathbf{BA} \ \cong\ \mathbf{Ind}(\mathbf{FinBA}) \ \cong\ \mathbf{Ind}(\mathbf{FinSet}^{\mathrm{op}}) \ \cong\ \mathbf{Pro}(\mathbf{FinSet})^{\mathrm{op}} \ \cong\ \mathbf{Stone}\ ;$$

the latter arises as the pro-completion on the algebraic side (obtaining CABAs from finite Boolean algebras), and dually the ind-completion on the topological side (obtaining sets from finite sets),

$$\mathbf{CABA} \ \cong\ \mathbf{Pro}(\mathbf{FinBA}) \ \cong\ \mathbf{Pro}(\mathbf{FinSet}^{\mathrm{op}}) \ \cong\ \mathbf{Ind}(\mathbf{FinSet})^{\mathrm{op}} \ \cong\ \mathbf{Set}.$$

See e.g. [17, Chapter VI] or [22, Chapter 1] for more details.
CABA also arises as a category of topological algebras, namely that of Boolean algebras on Stone spaces.

$$
\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\top"] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
\tag{4}
$$

These generalise the diagrams in (1) and (2) via the following (co)reflective inclusions:

$$
\begin{tikzcd}
\mathbf{epFinBA} \arrow[r, shift left, "\mathsf{At}"] \arrow[r, phantom, "\cong^{\mathrm{op}}"] \arrow[d, bend right, ""'] \arrow[d, phantom, "\dashv"] & \mathbf{FinXGph} \arrow[l, shift left, "\mathcal{K}"] \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\mathbf{FinBA} \arrow[u, "\iota"'] \arrow[r, shift left, "\mathsf{At}"] \arrow[r, phantom, "\cong^{\mathrm{op}}"] & \mathbf{FinSet} \arrow[u, "\iota"] \arrow[l, shift left, "\mathcal{P}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{epCABA} \arrow[r, shift left, "\mathsf{At}"] \arrow[r, phantom, "\cong^{\mathrm{op}}"] \arrow[d, bend right] \arrow[d, phantom, "\dashv"] & \mathbf{XGph} \arrow[l, shift left, "\mathcal{K}"] \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\mathbf{CABA} \arrow[u, "\iota"'] \arrow[r, shift left, "\mathsf{At}"] \arrow[r, phantom, "\cong^{\mathrm{op}}"] & \mathbf{Set} \arrow[u, "\iota"] \arrow[l, shift left, "\mathcal{P}"]
\end{tikzcd}
$$

Let us consider in turn each of the vertices and edges forming the triangles in (3) and (4).

To Do: add references to sections where each of these things is discussed

- The upper left vertex has the category of transitive partial Boolean algebras that generalise finite (resp. complete atomic) Boolean algebras. These are the transitive pBAs (resp. complete pBAs) whose (total) Boolean subalgebras are finite (resp. whose complete Boolean subalgebras are CABAs)
  The paradigmatic example is that of the partial Boolean algebra $\mathsf{P}(\mathcal{H})$ of projectors – or closed subspaces – of a Hilbert space $\mathcal{H}$ of finite (resp. arbitrary) dimension.
  From a logical perspective, these partial Boolean algebras can be regarded as 'algebras of propositions' where propositions might not be commeasurable, and thus propositional connectives are only partially defined.

- The universe of such a partial algebra is a set that additionally carries a compatibility or commeasurability relation. In the opposite direction, recall that a free finite Boolean algebra (resp. free CABA) is built from a finite set (resp. arbitrary set) of basic propositions: its elements correspond to the sentences built syntactically from these basic propositions, up to provable equivalence in the propositional calculus (the Lindenbaum–Tarski algebra of the empty theory). In the case of transitive partial Boolean algebras, an extra parameter can be specified that affects which sentences are considered to be well-formed, namely a compatibility relation between the basic propositions. This explains the choice of reflexive (or compatibility) graphs, in the bottom vertex of the triangles, as the natural category from which to build a free-forgetful adjunction (the left edge of the triangle) with **epFinBA** (resp. **epCABA**).

  give example?

- The top edge of the triangle embodies the main result of the paper, the generalisation of the Lindenbaum–Tarski duality $\mathbf{FinBA} \cong \mathbf{FinSet}^{\mathrm{op}}$ (resp. $\mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}$). As in the classical case, the rightward functor At maps a partial Boolean algebra to its atoms. But rather than regarding these as simply forming a set, they form an (exclusivity) graph: one must take into account whether two distinct atoms are compatible, and therefore mutually exclusive.

As an example, the atoms of $\mathsf{P}(\mathcal{H})$ correspond to the rays of $\mathcal{H}$, or equivalently, to the elements of the projective Hilbert space, or to normalised vectors up to phase. In quantum mechanics, these are the pure states of the system. The exclusivity relation on the atoms of $\mathsf{P}(\mathcal{H})$ is given by *orthogonality*.

Informally, we can think of the atoms of a partial Boolean algebra as the *possible worlds of maximal information*. Two such worlds will be linked by an edge whenever they are directly contradictory; in other words, if there is a testable proposition that distinguishes between them. The classical case corresponds to complete graphs, as all maximally specified worlds are mutually exclusive.

Similarly, a homomorphism of partial Boolean algebras will not simply yield a function between their sets of atoms, but rather a (specific type of) relation between their graphs of atoms.

add more on the morphisms?

- The leftward functor can be thought of as a reconstruction of a transitive partial finite (resp. complete atomic) Boolean algebra from its graph of atoms. As in the classical case, elements of the algebra – which we may think of as properties – are identified with the set of atoms below them, i.e. with the worlds where the property holds. The catch, however, is that the decomposition of an element as a join of atoms is no longer unique. While each set of pairwise exclusive worlds (a clique of the graph of atoms) can be thought of as a property, there may be different such cliques that correspond to the same property.
As an example, consider the partial Boolean algebra $\mathsf{P}(\mathcal{H})$ of projectors on a Hilbert space of dimension ≥ 2. If $|0\rangle$ / $\mathbf{x}$ and $|1\rangle$ / $\mathbf{y}$ are two orthogonal vectors in $\mathcal{H}$, then e.g.

$$\{\, |0\rangle \,,\, |1\rangle \,\} \qquad \text{and} \qquad \{\, \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \,\}$$

$$\{\, \mathbf{x}, \, \mathbf{y} \,\} \qquad \text{and} \qquad \{\, \frac{\mathbf{x}+\mathbf{y}}{\sqrt{2}}, \, \frac{\mathbf{x}-\mathbf{y}}{\sqrt{2}} \,\}$$

are two different cliques in the graph of atoms which have the same join, namely the (projector onto the) 2-dimensional subspace spanned by $|0\rangle$ and $|1\rangle$ / $\mathbf{x}$ and $\mathbf{y}$.
It turns out that for transitive partial Boolean algebras this equivalence relation between cliques induced by 'having the same join' can be characterised at the level of atoms, as 'having the same common neighbourhood', or in other words, 'jointly excluding the same set of worlds'. Therefore, a finite (resp. complete atomic) partial Boolean algebra can be fully reconstructed from just the information contained in its graph of atoms, as an algebra of equivalence classes of cliques.
Of course, in case of (total) Boolean algebras, whose graph of atoms is complete, the cliques are precisely the subsets of atoms, and we thus recover the classical reconstruction of finite (resp. complete atomic) Boolean algebras from their set of atoms.

- In the upper right vertex of the triangle, the category **FinXGph** (resp. **XGph**) identifies precisely the image of the functor At, i.e. which graphs – and which relations between them – arise as graphs of atoms from partial algebras – and homomorphims between them – in **epFinBA** (resp. **epCABA**).
The obtained notion of exclusivity graph can be thought of as a generalisation of the *inequality relation* on a set. This is precisely how **FinSet** (resp. **Set**) can be seen as a full subcategory of **FinXGph** (resp. **XGph**).
One can break the characterisation of $\neq$ into four (independent) axioms: a relation $\#$ is the inequality relation if and only if it is a coequivalence relation – i.e. it is irreflexive,

name?

symmetric, and cotransitive – and it satisfies an *identity of indiscernibles* axiom,

$$(\forall z.\ x \# z \iff y \# z) \implies x = y\ . \tag{5}$$

The notion of exclusivity graph keeps irreflexivity and symmmetry (as implicit in the terminology 'graph') as well as axiom (5). The weak link is cotransitivity,

$$x \# y \implies x \# z \text{ or } y \# z\ ,$$

the property that if two worlds are mutually exclusive, then any possible world must exclude at least one of them. Back to our running example, note that if $|0\rangle$ / $\mathbf{x}$ and $|1\rangle$ / $\mathbf{y}$ are orthogonal vectors in $\mathcal{H}$, then $|0\rangle + |1\rangle$ / $\mathbf{x}+\mathbf{y}$ is orthogonal to neither $|0\rangle$ / $\mathbf{x}$ nor $|1\rangle$ / $\mathbf{y}$.

In the definition of exclusivity graphs, cotransitivity is weakened to the following property: for $K \cup L$ a maximal clique,

is this the best formulation?

$$x \# K \text{ and } y \# L \implies x \# y\ .$$

Blah

comments here on interpretation

The notion of morphism between exclusivity graphs can also be understood from this point of view. Recall that a relation is a function if and only if it is functional and left-total. Morphisms of exclusivity graphs are defined to be relations satisfying two conditions. In the presence of cotransitivity on both graphs, i.e. in the case of complete (or inequality) graphs, these conditions reduce precisely to functionality and left-totality, albeit phrased slightly unorthodoxly in terms of inequality rather than equality.

- Finally, the free-forgetful adjunction can also be translated along the duality to yield an adjunction between the categories **RGph** of compatibility graphs and **XGph** of exclusivity graphs. This generalises the power-set self-adjunction from the classical case, but the symmetry of this double-construction is somewhat broken.
In particular, the upwards functor $\mathcal{V}$ builds an exclusivity graph whose vertices (possible worlds) are assignments of truth values to maximal subsets of compatible propositions.

One thing that is missing here which perhaps might be useful to clarify is the status of P

Discussion 'Commutative algebra is like topology, only backwards', John Baez put it pithily [4]. *Commutative* is a key word here. Typically, such dualities between algebra and topology work by representing an algebraic structure as some space of (scalar) functions on a topological space, with algebraic operations being defined pointwise, and thus inheriting commutativity from said scalars.

The question of extending this 'mirror of mathematics' [21] to the *noncommutative* – or *quantum* – setting has received considerable attention. The field of noncommutative geometry [8] springs from the idea of treating noncommutative algebras *as if* they were algebras of functions on a topological space, borrowing geometric intuition and tools. This turns out to be remarkably effective even if predicated on noncommutative spaces that are entirely fictional / fictitious.

Such implausible effectiveness, or at any rate the allure, of geometric intuition motivates the quest for more concrete realisations of spectra for various kinds of 'noncommutative' algebraic structures, in a way that extends *classical* dualities such as Stone's of Gel'fand–Naĭmark's. Here we understand noncommutativity in a broad sense. We mean it to include not only C^*-algebras or von Neumann algebras, but also other algebraic structures arising in (various forms of) quantum logic. So it comes in many guises: as failure of associativity in

Jordan algebras, of distributivity in orthomodular lattices, ... or – like in this article – as partiality.

This endeavour is not straightforward. Reyes [23] showed that any extension of the Zariski spectrum functor[6] to a contravariant functor from noncommutative rings to the category of topological spaces trivialises on matrix algebras $\mathbb{M}_n(\mathbf{C})$ with $n \geq 3$, as well as an analogous statement regarding extending Gel'fand–Naĭmark duality to noncommutative C^*-algebras. Van den Berg and Heunen [29, 31] extended this result to rule out locales, ringed toposes, schemes, and quantales as categories of spectra, as well as to encompass Stone and Pierce spectra. The proof of these results goes via partial structures of 'commutative' contexts, such as partial Boolean algebras and their C^*-algebraic analogues. The obstructions arise in the partial C^*-algebra of matrices $\mathbb{M}_3(\mathbb{C})$ or the partial Boolean algebra $\mathsf{P}(\mathbb{C}^3)$, and ultimately come down to the presence of contextuality. That is, they are predicated on the Kochen–Specker theorem [5, 18], the statement that if there is a homomorphims of partial Boolean algebras from $\mathsf{P}(\mathbb{C}^3)$ to a Boolean algebra B then B is the trivial Boolean algebra, where $0 = 1$, which corresponds to an inconsistent logical theory.[7]

Even though the present manuscript focuses on the most trivial of the Stone-like dualities, it is a setting that already falls under the scope of these no-go theorems. Most directly, Theorem 8 in [29] forbidding locale-valued extensions of Stone duality to partial Boolean algebras, or at least its restriction to the full subcategory **epFinBA**, applies directly to the finitary situation we consider, and by trivial extension to the case of **epCABA**, too. The analogous results for **Top**-valued or **Set**-valued functors also hold. This is essentially due to the fact that the partial Boolean algebras $\mathsf{P}(\mathcal{H})$ of projectors on finite-dimensional Hilbert spaces – and thus $\mathsf{P}(\mathbb{C}^3)$ in particular – are objects of **epFinBA**.

Necessarily, then, our results circumvent this obstruction by adopting a notion of spectrum – more precisely, a category of spectra – that departs more radically from topological spaces. Indeeed, the authors of [31] would seem to some extent to agree with John S. Bell that 'what is proved by impossibility proofs is lack of imagination' [6]:

> 'It might be tempting to conclude from all the above impossibility results that it is hopeless to look for a good notion of spectrum for noncommutative structures. But we strongly believe that this is the wrong conclusion to draw.'

They hit the nail on the head in identifying the crux of the blockade, and therefore the key to sidestepping it (our emphasis):

> 'What our results show is merely that a category of noncommutative spectra must have *different limit behaviour* from the known categories of commutative spectra.'

And with their suggestion on how to achieve this:

> 'One of the central messages of category theory is that objects should be regarded as determined by their behaviour rather than by any internal structure. In other words, it is not the internal structure of objects that dictates what morphisms should preserve. It is the other way around: it is the morphisms connecting an object to

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[6] This (contravariant) functor assigns to each unital commutative ring a topological space, its Zariski spectrum, and to a ring homomorphism a continuous function in the opposite direction between the corresponding spectra.

[7] The equivalent statement in terms of partial C^*-algebra is that if there is a homomorphism of partial C^*-algebras from $\mathbb{M}_3(\mathbb{C})$ to a commutative C^*-algebra, then this is the trivial C^*-algebra $C(\varnothing)$ of continuous functions on an empty topological space, which has a single element.

> others that determine that object's characteristics. [. . .] Historically, noncommutative spectra have almost always been pursued by generalizing the internal structure of commutative spaces (as objects). We believe the right, and optimistic, message to distill from our results is that one should let the search for noncommutative spectra be guided by morphisms instead.'

The relaxation of the function conditions on relations constituting the morphisms of **XGph** is precisely an example of this.[8]

2 Background: The classical Lindenbaum–Tarski duality

We start by briefly reviewing the classical Lindenbaum–Tarski duality between sets and CABAs [28]. This is standard material. It can be found in its modern, category-theoretic presentation in multiple references, e.g. [32, Section 2.3].

2.1 Boolean algebras

A **Boolean algebra** is an algebraic structure

$$\langle A; \wedge, \vee, \neg, 1, 0\rangle$$

consisting of a set A equipped with binary operations $\wedge$ and $\vee$, a unary operation $\neg$, and constants 1 and 0, satisfying the usual Boolean algebra axioms.[9] A Boolean algebra homomorphism is a function between the underlying sets preserving the operations. We write **BA** for the category of Boolean algebras.

Let us list some standard examples of Boolean algebras.

▶ **Example 1.** Given a set X, its powerset $\mathcal{P}(X)$ forms a Boolean algebra with the set-theoretic operations of *intersection*, *union*, and *complement*, and constants X itself and the *empty set*.

▶ **Example 2.** The set of (self-adjoint) idempotent elements of a unital, commutative ($*$-)ring R forms a Boolean algebra, with operations given by $e \wedge f := ef$, $e \vee f := e + f - ef$, and $\neg e := 1 - e$, and Boolean-algebraic constants 1 and 0 given by the homonymous idempotent elements of the ring R. In particular, this applies to the set of projectors of any commutative C^*-algebra.

▶ **Example 3.** Let V be a set of propositional variables, and write $\Phi(V)$ for the set of propositional formulae with variables in V, given by the grammar

$$\varphi,\ \psi \ ::= \ v \in V \ \mid \ 0 \ \mid \ 1 \ \mid \ \neg\varphi \ \mid \ \varphi \wedge \psi \ \mid \ \varphi \vee \psi \ .$$

A propositional theory over V is a subset $T \subset \Phi_V$. It determines an equivalence relation $\equiv_T$ on Φ_V whereby

$$\varphi \equiv_T \psi \qquad \text{if and only if} \qquad T \vdash \varphi \leftrightarrow \psi \ ,$$

[8] More pointedly still [but possible for later], in much the same way as one can regard the construction of **XGph** (dually, of **epCABA**) out of **Set** (resp **CABA**) as factoring via a variant of the **Pro** (resp. **Ind**) completion where more formal limits (colimits) beyond cofiltered (filtered) ones are constructed.

[9] There are many equivalent sets of axioms for Boolean algebras. A minimal, convenient example are Huntington's axioms: $\wedge$ and $\vee$ are commutative operations with identity elements 1 and 0 respectively, they distribute over each other, and $\neg$ is a complement operation in the sense that $a \wedge \neg a = 0$ and $a \vee \neg a = 1$ for all $a \in A$.

i.e. whenever φ and ψ are provably equivalent from T.[10] The set of equivalence classes $\Phi_V/{\equiv_T}$, with operations inherited from their syntactical counterparts,[11] forms a Boolean algebra $\mathcal{L}(T)$ called the Lindenbaum–Tarski algebra of T.
If T consists entirely of propositional tautologies (in particular if $T = \varnothing$), then the corresponding algebra is the free Boolean algebra on the set V, corresponding to equivalence classes of formulae with variables in V under provable equivalence in the propositional calculus.

A Boolean algebra carries a **partial order** structure on its elements whereby $a \leq b$ whenever any – and thus all – of the following equivalent conditions hold:

$$a \wedge b = a \qquad\qquad a \wedge \neg b = 0$$
$$a \vee b = b \qquad\qquad \neg a \vee b = 1$$

This partial order determines A as a Boolean algebra. For example, the operations $\wedge$ and $\vee$ correspond respectively to the binary infimum (greatest lower bound) and supremum (least upper bound) operations relative to the partial order.

As in the study of lattices more generally, it is often convenient to switch back and forth between the algebraic and order-theoretic perspectives.

2.2 Complete atomic Boolean algebras

In a Boolean algebra, any finite set of elements has a supremum – this follows from the existence of binary ($\vee$) and nullary (0) suprema. But infinite sets of elements need not have one.

A Boolean algebra A is said to be **complete** if any set of elements $S \subseteq A$ has a supremum $\bigvee S$ in A (and consequently an infimum $\bigwedge S$, too). Algebraically, it has additional infinitary operations

$$\bigwedge, \bigvee \colon \mathcal{P}(A) \longrightarrow A \ .$$

A complete Boolean algebra homomorphism is a Boolean algebra homomorphism that preserves arbitrary suprema (equivalently, infima). We write **CBA** for the category of complete Boolean algebras and complete Boolean algebra homomorphims.

An **atom** of a Boolean algebra is a minimal non-zero element in the partial order. [Option 1] That is, it is an element $x \neq 0$ such that for any element a of the algebra, $a \leq x$ implies $a = 0$ or $a = x$. / [Option 2] That is, it is an element $x \neq 0$ such that for any element a of the algebra, $0 \neq a \leq x$ implies $a = x$. A (trivially) equivalent way to phrase this condition is to say that $x \neq 0$ and x is $\vee$-indecomposable: if $x = a \vee b$ then $x = a$ or $x = b$.

A common and suggestive way to think of an atom is as a *possible world*, a maximally specified event, … as we shall see, a point in the event space (or event set, rather).

insert possible worlds maybe this should go to the next subsection, as a motivation for the duality.

A Boolean algebra A is said to be **atomic** if every non-zero element sits above an atom, i.e. if for all $a \in A \setminus \{0\}$ there is an atom x with $x \leq a$.

Our focus will be on **complete atomic Boolean algebras**, or **CABA**s for short. We write **CABA** for the corresponding full subcategory of **CBA**, i.e. for the category of complete atomic Boolean algebras and complete Boolean algebra homomorphisms.

[10] The symbol $\vdash$ denotes the provability relation between the set $\mathcal{P}(\Phi_V)$ of theories and the set Φ_V of formulae. The symbol $\leftrightarrow$ is syntactic sugar: $\varphi \leftrightarrow \psi := (\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi)$.

[11] E.g. $[\varphi] \wedge [\psi] := [\varphi \wedge \psi]$. This is well-defined because $\equiv_T$ is a congruence.

▶ **Example 4.** Any finite Boolean algebra is atomic, and therefore trivially a CABA. More generally, the paradigmatic – and, as we shall shortly see, the 'only' (up to isomorphism) – example of a CABA is the powerset $\mathcal{P}(X)$ of an arbitrary set X, i.e. the Boolean algebra consisting of *all* the subsets of X. Being closed under arbitrary unions ensures completeness, while its atoms are the singletons $\{x\} \in \mathcal{P}(X)$ for $x \in X$, and clearly any non-empty subset of X contains a singleton.

2.3 Reconstructing a CABA from its atoms

Any subset S of the atoms of a complete Boolean algebra determines an element $\bigvee S$ of the algebra. It turns out that, for a complete Boolean algebra, being atomic implies (and is therefore equivalent to) an apparently stronger property: that any element can be written in this fashion, as a join of a set of atoms. In other words, that the assignment $S \longmapsto \bigvee S$ from sets of atoms to elements of the algebra is surjective.

▶ **Proposition 5.** *In a CABA A, every element is the join of the atoms below it, i.e. any $a \in A$ can be written as*

$$a = \bigvee \mathcal{U}(a) \quad \textit{where} \quad \mathcal{U}(a) := \{x \in A \mid x \textit{ is an atom and } x \leq a\}\,.$$

Proof. Clearly, $\bigvee \mathcal{U}(a) \leq a$. For the opposite inequality, we show that $a \wedge \neg \bigvee \mathcal{U}(a) = 0$. By atomicity, it suffices to show that this element has no atoms below it. Suppose that x is an atom with $x \leq a \wedge \neg \bigvee \mathcal{U}(a)$. On the one hand, $x \leq \neg \bigvee \mathcal{U}(a)$. On the other, $x \leq a$, i.e. $x \in \mathcal{U}(a)$, hence $x \leq \bigvee \mathcal{U}(a)$. This implies $x = 0$, contradicting the fact that x is an atom. ◀

Moreover, this is the unique such decomposition of an element as a join of atoms.

▶ **Proposition 6.** *x is an atom and $x \leq \bigvee S$, there is $a \in S$ with $x \leq a$.*
$\mathcal{U}(\vee S) = \bigcup_{s \in S} \mathcal{U}(a)$

Proof. Let $x \in S$ ◀

▶ **Proposition 7.** $\mathcal{U}(\vee S) = S$

Proof. The $\supset$ is clear. For $\subset$, let x be an atom with $x \leq \bigvee S$. By **??**, there is a $y \in S$ with $x \leq y$. Since y is an atom and $x \neq 0$ this implies $x = y$, hence $x \in S$. ◀

▶ **Proposition 8.** *For S and T subsets of atoms in a CABA, $\bigvee S \leq \bigvee T$ if and only if $S \subseteq T$. Consequently, $\bigvee S = \bigvee T$ if and only if $S = T$.*

Proof. The 'if' direction is immediate. For the 'only if' direction, complete this proof

◀

[lost stuff]Moreover, if S is a set of atoms and $x \notin S$, then $x \notin \mathcal{U}(\bigvee S)$

The two propositions above form the basis for the reconstruction of a CABA from its set of atoms. Together they establish a bijection between elements of a CABA A and subsets of atoms. Moreover, as shown in Proposition 8, the partial order $\leq$ on A corresponds precisely to set-theoretic inclusion. Consequently, given the order-theoretic characterisation of Boolean algebras, the *join* $\vee$, *meet* $\wedge$ and *negation* $\neg$ operations on A correspond to set-theoretic *union*, *intersection*, and *complement*. This establishes a complete Boolean algebra isomorphism between a CABA A and the powerset of its set of atoms. It is one of the natural isomorphisms that constitutes the categorical duality between **CABA** and **Set**.

2.4 Morphisms

Blah blah

▶ **Lemma 9.** *Let $h\colon A \longrightarrow B$ be a complete Boolean algebra homomorphism. For each atom y of B, there exists a unique atom x of A such that $y \leq h(x)$.*

Proof. Facts about atoms in any BA:

- If $x \neq x'$ are atoms, then $x \wedge_A x' = 0$.
- If x is an atom and $x \leq \bigvee S$, there is $a \in S$ with $x \leq a$.

Existence
A complete atomic implies $1_A = \bigvee \mathsf{At}(A)$. Hence,

$$1_B = h(1_A) = h(\bigvee \mathsf{At}(A)) = \bigvee \{h(x) \mid x \in \mathsf{At}(A)\}$$

Since $y \leq 1_B$, we conclude $y \leq h(x)$ for some $x \in \mathsf{At}(A)$.
Uniqueness
If $y \leq h(x)$ and $y \leq h(x')$, then $y \leq h(x) \wedge_B h(x') = h(x \wedge x')$, hence $x = x'$. ◀

If x and x' are distinct atoms of A, then $x \wedge_A x' = 0$. Moreover, the supremum of all the atoms is the top element, $\bigvee \mathsf{At}(A) = 1_A$. Consequently, given any complete Boolean homomorphism $h\colon A \longrightarrow B$, we have $h(x) \wedge_B h(x') = 0$ for all $x, x' \in \mathsf{At}(A)$ with $x \neq x'$ and $\bigvee \{h(x) \mid x \in \mathsf{At}(A)\} = 1_B$. This induces a partition on the atoms of B: (1) for any $y \in \mathsf{At}(B)$, $b \leq h(x)$ and $b \leq h(x')$ implies $b \leq h(x) \wedge_B h(x')$ and thus $x = x'$; (2) for

2.5 The classical Lindenbaum–Tarski duality

The duality between the categories **CABA** and **Set** is witnessed by the following contravariant functors:

$$\begin{tikzcd} \mathbf{CABA} \arrow[rr, bend right, "\mathsf{At}"'] \arrow[rr, phantom, "\cong"] && \mathbf{Set}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{P}"'] \end{tikzcd}, \tag{6}$$

The functor

$\mathsf{At}\colon \mathbf{CABA}^{\mathrm{op}} \longrightarrow \mathbf{Set}$

is defined as follows:

- on objects: it maps a CABA A to its set of atoms.
- on morphisms: given a complete Boolean homomorphism $h\colon A \longrightarrow B$, it yields a function

  $\mathsf{At}(h)\colon \mathsf{At}(B) \longrightarrow \mathsf{At}(A)$

  mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

The functor

$\mathcal{P}\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$

is the usual contravariant powerset functor (equipped with complete Boolean algebra structure), given as follows:

- on objects: it maps a set X to its powerset $\mathcal{P}X$, which forms a CABA under the usual set-theoretic operations.
- on morphisms: given a function $f\colon X \longrightarrow Y$, it yields a complete Boolean algebra homomorphism

$$\begin{aligned} \mathcal{P}(f)\colon \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) \\ (T \subseteq Y) &\longrightarrow f^{-1}(T) = \{x \in X \mid f(x) \in T\} \end{aligned}$$

The duality (6) is witnessed by two natural isomorphisms:

- Given a CABA A, the isomorphism $A \cong \mathcal{P}(\mathsf{At}(A))$ maps $a \in A$ to the set

$$\mathcal{U}(a) = \{x \in \mathsf{At}(A) \mid x \leq a\}\,.$$

  Thus, a property is identified with the set of *possible worlds* in which it holds.
- Given a set X, the bijection $X \cong \mathsf{At}(\mathcal{P}(X))$ maps $x \in X$ to the singleton set $\{x\}$, which is an atom of $\mathcal{P}(X)$. I.e. a possible world is identified with its characteristic property (which completely determines that world).[12]

3 Background: Partial Boolean algebras

We recall the basic definitions and properties of partial Boolean algebras, and introduce the particular classes of such structures on which this article focuses.

3.1 Compatibility relations

We start by considering compatibility relations. These are the natural replacement for sets as the underlying universe of a partial Boolean algebra.

▶ **Definition 10.** *A* ***compatibility graph*** *(a.k.a.* ***reflexive graph****) is a pair $\langle V, \odot_V \rangle$ consisting of a set equipped with a symmetric, reflexive binary relation. A morphism of compatibility graphs is a function between the underlying sets that preserves the compatibility relation, i.e. a function $f\colon V \longrightarrow W$ such that $p \odot_V q$ implies $f(p) \odot_W f(q)$. These form a category which we denote by* **RGph**.

As usual, we abuse notation by referring to a compatibility graph $\langle V, \odot_V \rangle$ as simply V, and dropping the index from the relation symbol whenever the compatibility graph is clear from context.

Given a compatibility graph V, we write $\bigodot_V$ for the set of subsets of V whose elements are pairwise commeasurable,

$$\bigodot_V \;\coloneqq\; \{S \subseteq V \mid S \times S \subseteq \odot_V\} \;=\; \{S \subseteq V \mid \forall a, b \in S.\, a \odot b\}\;.$$

Compatibility graphs should be understood as a generalisation of sets, in that the latter are seen as compatibility graphs where every pair of elements is compatible. This is more formally expressed by the proposition below.

[12] In the more general version of Stone duality, such characteristic properties are not available (as the algebra need not be atomic). There, a possible world (point in the state space) is identified with the set of properties which hold true/false in that world, which forms an ultrafilter/prime ideal of the algebra.

▶ **Proposition 11.** *The functor $\Delta\colon \mathbf{Set} \longrightarrow \mathbf{RGph}$ mapping a set X to the compatibility graph $\Delta(X) := \langle X, \top\rangle$, consisting of X equipped with the universal relation $\top = X \times X$, reveals* **Set** *as (equivalent to) a reflective subcategory of* **RGph**.

Proof. The assignment is clearly functorial. Moreover, the condition that a morphism preserve the compatibility relation trivialises in the case of the universal compatibility relation in the codomain, i.e.

$$\mathbf{RGph}(\langle V, \odot_V\rangle, \langle W, \top\rangle) \;\cong\; \mathbf{Set}(V, W)\ . \tag{7}$$

In particular, taking $\odot_V = \top$ shows that the functor $\Delta :: V \longmapsto \langle V, \top\rangle$ is full and faithful. Finally, the isomorphism in Equation (7) means that Δ has a left adjoint, namely the functor given by $\langle V, \odot_V\rangle \mapsto V$. ◀

▶ **Proposition 12.** *The category* **RGph** *is complete and cocomplete. The 'inclusion'* $\mathbf{Set} \longrightarrow \mathbf{RGph}$ *from Proposition 11 preserves all limits (for it is a right adjoint) and coequalisers.*

missing roof / details here, limits and colimits are given the obvious definitions.

3.2 Partial Boolean algebras

We now recall the basic definition of partial Boolean algebra. These can be seen as a relaxation of the notion of Boolean algebra where the binary operations $\wedge, \vee$ become partial, defined only on commeasurable elements. This commeasurability is given by a compatibility graph as in the previous section.

▶ **Definition 13.** *A* ***partial Boolean algebra*** *is a structure*

$$\langle A, \odot_A, 0_A, 1_A, \neg_A, \vee_A, \wedge_A\rangle$$

consisting of:

- *a compatibility graph $\langle A, \odot_A\rangle$, where the relation is read as 'commeasurability',*
- *constants $0_A, 1_A \in A$,*
- *a (total) unary operation $\neg_A\colon A \longrightarrow A$,*
- *(partial) binary operations $\vee_A, \wedge_A\colon \odot_A \longrightarrow A$ defined on commeasurable elements,*

satisfying the following property: for any set $S \in \bigodot_A$ of pairwise-commeasurable elements, there is a set $T \in \bigodot_A$ of pairwise-commeasurable elements with $S \subseteq T$ such that T forms a Boolean algebra under the restrictions of the operations.[13]

A homomomorphism of partial Boolean algebras is a morphism of compatibility graphs which preserves the constants and the operations whenever defined. We write **pBA** *for the resulting category of partial Boolean algebras.*

Again, we usually omit subscripts whenever the algebra is clear from context. There is an evident forgetful functor $U\colon \mathbf{pBA} \longrightarrow \mathbf{RGph}$.

[13] An equivalent way of formulating this condition, closer to a direct equational axiomatisation, is to require: (i) that the operations preserve commeasurability in the sense that for each operation f of arity n, and elements $a_1, , \ldots, a_n, b$,

$$a_1 \odot b,\ \ldots,\ a_n \odot b \implies f(a_1, \ldots, a_n) \odot b$$

(note that this implies that the constants are commeasurable with all other elements), and (ii) that for any triple a, b, c of pairwise-commeasurable elements, the axioms of Boolean algebra are satisfied.

▶ **Example 14.** Any Boolean algebra is a partial Boolean algebra, where the commeasurability relation is the universal relation. In fact, Boolean algebras are a reflective subcategory of **pBA** (e.g. [1]). The forgetful functor $U\colon \mathbf{pBA} \longrightarrow \mathbf{RGph}$ generalises the usual forgetful functor $U\colon \mathbf{BA} \longrightarrow \mathbf{Set}$ in the sense that:

$$\begin{tikzcd}
\mathbf{pBA} \arrow[r, "U"] & \mathbf{RGph} \\
\mathbf{BA} \arrow[u, hook, "\iota"] \arrow[r, "U"] & \mathbf{Set} \arrow[u, hook', "\Delta=\langle -,\top\rangle"']
\end{tikzcd}$$

▶ **Example 15.** The paradigmatic – and motivating – example of a partial Boolean algebra is the collection of projector operators on a Hilbert space $\mathcal{H}$, or equivalently, of closed subspaces of a Hilbert space $\mathcal{H}$. Commeasurability and the operations are given by the following definitions (here stated in terms of projector operators): given projectors p, q,

- $p \odot q$ if and only if $pq = qp$;
- the constants are the 0 and 1 projectors;
- $\neg p := 1 - p$;
- for $p \odot q$, $p \wedge q := pq$;
- for $p \odot q$, $p \vee q := p + q - pq$;

▶ **Example 16.** It is clear from the description in terms of projectors, which are the self-adjoint, idempotent elements in the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, that this construction can be generalised to the set of projectors on any C^*-algebra, and even beyond. Given a not necessarily commutative unital ($*$-)ring R, the set

$$E_{(\mathsf{sa})}(R) := \left\{ e \in R \mid e^2 = e \text{ (and } e = e^*) \right\}$$

of its (self-adjoint) idempotent elements forms a partial Boolean algebra, with the commeasurability relation and operations defined as for projectors in Example 15.

3.3 Completeness and atomicity for partial Boolean algebras

The most general form of the result presented in this article requires the partial version of complete Boolean algebras [30, Section 2.1].

▶ **Definition 17.** *A* ***partial complete Boolean algebra*** *is a partial Boolean algebra with an additional (partial) operation*

$$\bigvee \colon \bigodot \longrightarrow A$$

satisfying the following property: any set $S \in \bigodot$ is contained in a set $T \in \bigodot$ which forms a complete Boolean algebra under the restriction of the operations.

A homomorphism of partial complete Boolean algebras is a homomorphism of partial Boolean algebras which preserves arbitrary joins (equivalently, meets) of commeasurable elements. We write **pCBA** *for the resulting category of partial complete Boolean algebras.*

▶ **Definition 18.** *Let A be a partial Boolean algebra. Given $a, b \in A$, we write $a \leq b$ to mean that $a \odot b$ and $a \wedge b = a$.*[14]

[14] Condition $a \wedge b = a$ can equivalently be written as any of the following: $a \vee b = b$, $\neg a \vee b = 1$, $a \wedge \neg b = 0$.

The restriction of this relation to any Boolean subalgebra of A coincides with the partial order of that Boolean algebra. But in spite of the chosen notation, $\leq$ need not be (globally) a partial order on a general partial Boolean algebra A. This will be the case, however, for the partial Boolean algebras that we consider in this article, a point to which we return in Section 3.4.

The following definitions are straightforward generalisations from the total case.

▶ **Definition 19.** *An **atom** of a partial Boolean algebra A is an element $x \in A$ with $x \neq 0$ and such that for any $a \in A$, $a \leq x$ implies $a = 0$ or $a = x$.*

Given an element $a \in A$, we write $\mathcal{U}(a)$ for the set of atoms below a,

$$\mathcal{U}(a) := \{x \in A \mid x \text{ is an atom and } x \leq a\} \ .$$

▶ **Definition 20.** *A partial Boolean algebra A is said to be **atomic** if every non-zero element sits above an atom, i.e. if $\mathcal{U}(a) \neq \varnothing$ for all $a \in A \setminus \{0\}$.*

Our focus will be on the partial analogues of finite Boolean algebras and of CABAs.

▶ **Definition 21.** *A partial CABA is a partial Boolean algebra that is complete and atomic. We write* **pCABA** *for the full subcategory of* **pCBA** *whose objects are partial CABAs.*

▶ **Definition 22.** *A partial Boolean algebra A is said to be [Option 1] finite / [Option 2] finitary / [Option 3] f.d. if any set of commeasurable elements (i.e. any member of $\bigodot_A$) is finite. We write* **pFinBA** *for the full subcategory of* **pBA** *(equivalently,* **pCBA** *or* **pCABA***) consisting of [Option 1] finite / [Option 2] finitary / [Option 3] f.d.* partial Boolean algebras.

In other words, a partial Boolean algebra is [Option 1] finite / [Option 2] finitary / [Option 3] f.d. if all its Boolean subalgebras are finite Boolean algebras. From this it is immediate that [Option 1] finite / [Option 2] finitary / [Option 3] f.d. partial Boolean algebras are, in particular, partial CABAs.

▶ **Example 23.** The partial Boolean algebra $\mathsf{P}(\mathcal{H})$ for a finite-dimensional Hilbert space $\mathcal{H}$ is [Option 1] finite / [Option 2] finitary / [Option 3] f.d..

▶ **Example 24.** Examples of partial CABAs are the partial Boolean algebras $\mathsf{P}(\mathcal{H})$ for arbitrary Hilbert spaces $\mathcal{H}$, and more generally, the partial Boolean algebras of projectors in a von Neumann algebra with only type I factors.

3.4 Transitivity or the logical exclusivity principle

In this article, we focus on a special class of partial Boolean algebras, namely those that satisfy the logical exclusivity principle, or equivalently, transitivity. These include the paradigmatic example of projectors on a Hilbert space, and indeed the pBA of projections on any von Neumann algebra.

▶ **Definition 25.** *Two elements a and b of a partial Boolean algebra are said to be **exclusive**, written $a \perp b$, if there is a element c such that $a \leq c$ and $b \leq \neg c$.*

In a Boolean algebra, this condition could be equivalently phrased as $a \wedge b = 0$, or as $a \leq \neg b$. In a partial Boolean algebra, too, the conditions are equivalent whenever $a \odot b$. However, exclusivity is in general a weaker requirement than demanding $a \odot b$ and $a \wedge b = 0$. The reason is that there may exist exclusive elements that are not commeasurable.

▶ **Definition 26.** *A partial Boolean algebra is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are exclusive are also commeasurable.*

We write **epBA** *(resp.* **epFinBA**, **epCBA**, **epCABA***) for the full subcategory of* **pBA** *(resp.* **pFinBA**, **pCBA**, **pCABA***) whose objects are partial Boolean algebras satisfying LEP.*

In partial Boolean algebras satisfying LEP, we indeed have that $a \perp b$ is equivalent to $a \odot b$ and $a \wedge b = 0$

The LEP condition turns out to coincide with another well-studied condition on partial Boolean algebras, transitivity of the $\leq$ relation [15, 20, 16].

▶ **Definition 27.** *A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.*

Transitivity can fail in general for a partial Boolean algebra, since one need not have $a \odot c$ under the stated hypotheses. In any partial Boolean algebra, $\leq$ is reflexive and symmetric, so transitivity is equivalent to requiring that it be a partial order (globally) on A

▶ **Proposition 28** ([1, Proposition X])**.** *A partial Boolean algebra satisfies LEP if and only if it is transitive.*

An important property of exclusivity in this class of pBAs, which we will use often, is that it is downwards-closed.

▶ **Proposition 29.** *Let A be a transitive partial Boolean algebra and $a, a', b, b' \in A$ with $a' \leq a$ and $b' \leq b$. Then $a \perp b$ implies $a' \perp b'$.*

4 Partial CABAs and their graphs of atoms

In this secton we show how a partial CABA can be reconstructed from its (graph of) atoms.

4.1 Graphs

Before diving in, we fix some notational conventions on graphs, which will come in handy in later sections.

▶ **Definition 30.** *A (simple, undirected) **graph** $\langle X, \# \rangle$ is a set equipped with a symmetric irreflexive relation. Elements of the set X are called vertices, while unordered pairs $\{x, y\}$ with $x \# y$ are called edges (and the vertices x and y are said to be adjacent).*

We postpone the discussion of morphisms to Section 5, as we shall be interested in a notion that differs from the usual graph homomorphisms.

We make use of the following notational conventions. Let $\langle X, \# \rangle$ be a graph. Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ to mean that $x \# y$ for all $y \in S$;
- $S \# T$ to mean that $x \# y$ for all $x \in S$ and $y \in T$;
- $x^{\#} := \{y \in X \mid y \# x\}$ to denote the **neighbourhood** of the vertex x;
- $S^{\#} := \bigcap_{x \in S} x^{\#} = \{y \in X \mid y \# S\}$ to denote the **common neighbourhood** of the set S.

An important property of the common neighbourhood function $(-)^{\#}\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is that it is a self-dual (antitone) Galois connection: for any $S, T \subset X$,

$$S \subset T^{\#} \qquad \text{iff} \qquad S \# T \qquad \text{iff} \qquad T \subset S^{\#} \ .$$

Consequently, the double neighborhood function, $S \longmapsto S^{\#\#}$, is a closure operator on the powerset of X.

The standard concept of clique will play a central rôle in our reconstruction.

▶ **Definition 31.** *A* ***clique*** *of a graph $\langle X, \# \rangle$ is a set of pairwise adjacent vertices, i.e. a set $K \subset X$ such that $x \# y$ for all $x, y \in K$ with $x \neq y$ (equivalently, such that $x \# K \setminus \{x\}$ for all $x \in K$). The* ***clique number*** *of a graph is the cardinality of the largest clique in the graph.*

Note that we do not assume graphs to be finite. We will regard graphs as generalising sets, and the appropriate generalisation of finite sets will be (possibly infinite) graphs with finite clique number.

4.2 Atomicity

In the total case, the set of atoms was enough to determine a CABA. For partial algebras, additional structure is needed.

▶ **Definition 32.** *The* ***graph of atoms*** *of a partial Boolean algebra A, denoted $\mathsf{At}(A)$, has as vertices the atoms of A and has an edge between atoms x and y, written $x \# y$, if and only if $x \perp y$.*

So, $\mathsf{At}(A)$ is the set of atomic events together with their exclusivity relation. These atoms can be interpreted as worlds of maximal information and the relation corresponds to incompatibility between such worlds, in the sense that there is a proposition that distinguishes between them.[15]

A notational remark: we will prefer the notation $\#$ instead of $\perp$ when referring to exclusivity between atoms, in order to stress that we are referring to properties of the graph $\mathsf{At}(A)$.

In the classical case where A is a total algebra, $\mathsf{At}(A)$ is the complete graph on the set of atoms since any pair of atomic events is exclusive.

4.3 Transitive partial CABA from its graph of atoms

Recall that in a CABA any element is uniquely written as a join of atoms, namely as the join of all the atoms sitting below said element, i.e.

$$a = \bigvee \mathcal{U}(a) \qquad \text{with} \qquad \mathcal{U}(a) = \{x \in \mathsf{At}(A) \mid x \leq a\} \ .$$

The following proposition provides the analogous result in the (transitive) partial setting. In contrast with the classical case, the decomposition of an element as a join of atoms is not

[15] In a transitive partial Boolean algebra, the exclusivity relation on atoms coincides with the commeasurability relation. Indeed, in any partial Boolean algebra, commeasurable atoms must be exclusive (for otherwise their meet would be a non-zero element smaller than the atoms), and conversely, transitivity implies that exclusive elements be commeasurable. However, we choose to emphasise exclusivity rather than commeasurability since this perspective is more useful in guiding our intuition. This will hopefully become even more apparent when we move on to consider morphisms.

necessarily unique. In particular, the atoms in $\mathcal{U}(a)$ may not be pairwise commeasurable, hence their join need not even be defined.

▶ **Proposition 33.** *Let A be a transitive partial CABA and let $a \in A$. Then $a = \bigvee K$ for any clique K of $\mathsf{At}(A)$ which is maximal in $\mathcal{U}(a)$.*

Proof. Let $a \in A$ and let K be a clique of $\mathsf{At}(A)$ that is maximal in $\mathcal{U}(a)$. Being a clique in $\mathsf{At}(A)$, K is a set of pairwise-commeasurable elements of A, i.e. $K \in \bigodot$, and thus $\bigvee K$ is defined.

Since $K \subset \mathcal{U}(a)$, all $k \in K$ satisfy $k \leq a$ and thus in particular $k \odot a$. Hence, $K \cup \{a\} \in \bigodot$, i.e. it is a set of pairwise-commeasurable elements of A, implying that it can be extended to a complete Boolean subalgebra of A. Consequently, from $k \leq a$ for all $k \in K$, we derive that $\bigvee K \leq a$.

It remains to show that $a \leq \bigvee K$, or equivalently, that $a \wedge \neg \bigvee K = 0$. Suppose for a contradiction that this is not the case. Then atomicity implies there is an atom $x \leq a \wedge \neg \bigvee K$. By transitivity, $x \leq a$ and $x \leq \neg k$ (hence $x \# k$) for all $k \in K$. This makes $K \cup \{x\}$ a clique of atoms contained in $\mathcal{U}(a)$, contradicting the maximality of K. ◀

The proposition above states that an element a is the join of *any* clique that is maximal in $\mathcal{U}(a)$. So, if K and L are two such maximal cliques, it yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not all commeasurable.

We have just established that all elements of the partial algebra can be written as a join of atoms. Then the key to reconstructing a partial CABA from its atoms lies in characterising this kind of equalities between joins of cliques of atoms. To be sure, such equalities only allow us to reconstruct the partial CABA as a set, i.e. to recover its elements. More generally, we can characterise inequalities, i.e. the $\leq$ relation, between such joins – which is then sufficient to reconstruct the algebraic operations.

Proposition 36 below provides such a characterisation in terms of the graph-theoretic structure of the atoms.

▶ **Lemma 34.** *Let A be a transitive atomic partial Boolean algebra. For any elements $a, b \in A$, one has $a \leq b$ if and only if $\mathcal{U}(a) \subset \mathcal{U}(b)$*

Proof. The forward implication follows by transitivity, since $a \leq b$ and $x \leq a$ implies $x \leq b$.

The converse implication uses atomicity: for any atom x we have that

$$x \leq a \wedge \neg b \quad \Leftrightarrow \quad x \leq a \text{ and } x \leq \neg b \quad \Rightarrow \quad x \leq b \text{ and } x \leq \neg b \quad \Leftrightarrow \quad x = 0$$

hence $\mathcal{U}(a \wedge \neg b) = 0$, which by atomicity implies $a \wedge \neg b = 0$, i.e. $a \leq b$.

The converse implication uses atomicity: $a \not\leq b$ means that $a \wedge \neg b \neq 0$ and atomicity implies that there is an atom x such that $x \leq a \wedge \neg b$, showing that $\mathcal{U}(a) \not\subset \mathcal{U}(b)$. ◀

this seems to be re-using a trick. probably can be simplified.

▶ **Lemma 35.** *Let A be a transitive partial complete Boolean algebra. For any cliques K and L in $\mathsf{At}(A)$, we have*

$$\bigvee K \perp \bigvee L \qquad \textit{iff} \qquad K \# L\ .$$

Proof. First, unravelling definitions, note that $K \# L$ is equivalent to

$$k \odot l \qquad \text{and} \qquad k \wedge l = 0$$

for all $k \in K$ and $l \in L$. This implies that $K \cup L \in \odot$, hence $K \cup L$ can be extended to a complete Boolean subalgebra. Consequently, we have that $\bigvee K \odot \bigvee L$, and moreover, using distributivity,

$$\bigvee K \wedge \bigvee L = \bigvee \{k \wedge l \mid k \in K, l \in L\} = 0 \ .$$

Conversely, by Proposition 29 and transitivity of A, $\bigvee K \perp \bigvee L$ implies $k \perp l$ (i.e. $k \# l$) for each $k \in K$ and $l \in L$. ◀

▶ **Proposition 36.** *Let A be a transitive partial CABA. Given cliques K and L in $\mathsf{At}(A)$,*

$$\bigvee K \leq \bigvee L \qquad \textit{iff} \qquad L^{\#} \subset K^{\#} \qquad \textit{iff} \qquad K \subset L^{\#\#} \ .$$

Proof. The previous lemmas yield the following sequence of equivalences:

$$x \bigvee K \leq \bigvee L$$

$$\Leftrightarrow$$

$$\neg \bigvee L \leq \neg \bigvee K$$

$$\Leftrightarrow \qquad \{ \text{ Lemma 34 } \}$$

$$\forall x \in \mathsf{At}(A).\ x \leq \neg \bigvee L \Rightarrow x \leq \neg \bigvee K$$

$$\Leftrightarrow \qquad \{ \ a \leq \neg b \text{ iff } a \perp b \ \}$$

$$\forall x \in \mathsf{At}(A).\ x \perp \bigvee L \Rightarrow x \perp \bigvee K$$

$$\Leftrightarrow \qquad \{ \text{ Lemma 35 } \}$$

$$\forall x \in \mathsf{At}(A).\ x \# L \Rightarrow x \# K$$

$$\Leftrightarrow$$

$$L^{\#} \subset K^{\#}$$

$$\Leftrightarrow \qquad \{ \ (-)^{\#} \text{ is a Galois connection } \}$$

$$K \subset L^{\#\#}$$

◀

In particular, we obtain a descripyionis yields a get that for any clique L in $\mathsf{At}(A)$,

$$\mathcal{U}(\bigvee L) = L^{\#\#} \ .$$

▶ **Corollary 37.** *Let A be a transitive partial CABA. Given cliques K and L in $\mathsf{At}(A)$, $\bigvee K = \bigvee L$ iff $K^{\#} = L^{\#}$ iff $K^{\#\#} = L^{\#\#}$.*

4.4 Transitive partial CABA from its graph of atoms

As we have already hinted at, Corollary 37 yields a characterisation of the elements of a transitive partial CABA A from its atoms. Namely, considering the equivalence relation on cliques of $\mathsf{At}(A)$ given by

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

the elements of A are in one-to-one correspondence with $\equiv$-equivalence classes.

Moreover, Proposition 36 reconstructs the partial order structure, i.e. the relation $\leq$, of A. This is sufficient to describe the algebraic structure of a transitive partial CABA from its graph of atoms.

incomplete

▶ **Proposition 38.** *AA*

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^\#$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
- $[K] \vee [L] = [K' \cup L']$.
- $[K] \wedge [L] = [K' \cap L']$.

5 Exclusivity graphs

We saw in the previous section that a transitive partial CABA is determined by its graph of atoms. This shows that the mapping $A \longmapsto \mathsf{At}(A)$ is injective (on objects). A natural question is therefore to characterise its image, i.e. which graphs arise as $\mathsf{At}(A)$ for some transitive partial CABA A.

5.1 Definition of exclusivity graphs

▶ **Definition 39.** *A* ***complete exclusivity graph*** *is a graph $\langle X, \# \rangle$ satisfying the following conditions:*

(E1) if $K, L \subset X$ disjoint such that $K \cup L$ is a maximal clique, then $K^\# \# L^\#$, i.e. for any vertices $x, y \in X$, if $x \# K$ and $y \# L$ then $x \# y$;
(E2) for any vertices $x, y \in X$, $x^\# \subseteq y^\#$ implies $x = y$.

The first of these conditions can be equivalently rephrased as saying that for any cliques K and L, if $K \# L$ and $K^\# \cap L^\# = \varnothing$ then $K^\# \# L^\#$. The second is equivalently stated as follows: for any $x, y \in X$, $x^\# \subseteq y^\#$ implies $x = y$.

One can think of $x \# y$ in a complete exclusivity graph as signifying that the possible worlds x and y can be distinguished unambiguously: there is a testable property that is *false* in one world and *true* in the other. The 'classical' case would then correspond a complete graph, as every possible world can be distinguished. Indeed, a helpful intuition to break down Definition 39 is to regard complete exclusivity graphs as generalising sets with the inequality relation $\neq$ (i.e. complete graphs).[16] Note that a graph is already a symmetric and irreflexive relation. In order for $\#$ to be an inequivalence relation (the complement of an equivalence relation), one additionally would need cotransitivity:

$$x \# z \qquad \text{implies} \qquad x \# y \text{ or } y \# z \ .$$

Condition (E2), which eliminates redundant vertices, would then imply that $\#$ is the inequality relation. The weak link, which is loosened in relaxing sets to exclusivity graphs, is then cotransitivity: condition (E1) is a weaker version of it.

Figure 1 shows an example of a graph where this condition fails.

---

[16] A consequence of our results is that the category **Set** of sets and functions, when seen as complete graphs in this fashion, forms a coreflective subcategory of that of exclusivity graphs.

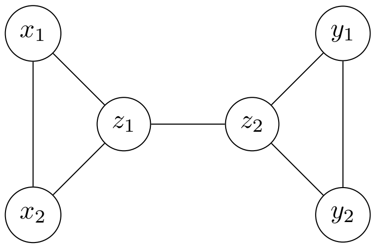

Figure 1 Example of a graph satisfying condition (E2) but not condition (E1). The former can be seen by inspection of the vertex neighbourhoods:

$$x_1^\# = \{z_1, x_2\} \qquad y_1^\# = \{z_2, y_2\} \qquad z_1^\# = \{z_2, x_1, x_2\}$$
$$x_2^\# = \{z_1, x_1\} \qquad y_2^\# = \{z_1, y_1\} \qquad z_2^\# = \{z_1, y_1, y_2\}$$

For the latter, take $K = \{z_1\}$ and $L = \{z_2\}$. These are disjoint and their union $K \cup L = \{z_1, z_2\}$ is a maximal clique. Moreover, $x_1 \# K$ and $y_1 \# L$ but we do not have that $x_1 \# y_1$, falsifying the condition.

5.2 Graph of atoms is exclusive

The first task is to check that $\mathsf{At}(A)$ is indeed an exclusivity graph for any transitive partial CABA A. The following lemma will be useful, and it helps illuminating condition $(E1)$.

▶ **Lemma 40.** *Let A be a transitive partial CABA. If $K, L \subset \mathsf{At}(A)$ with $K \cap L = \varnothing$ and $K \cup L$ a maximal clique, then $\bigvee K = \neg \bigvee L$.*

Proof. Since K and L are disjoint subsets of a clique, we have that $K \# L$. Hence, by Lemma 35, $\bigvee K \wedge \bigvee L = 0$, i.e. $\bigvee K \leq \neg \bigvee L$.

The other direction follows from maximality. The fact that $K \cup L$ is a maximal clique implies by Proposition 33 that its join, $\bigvee (K \cup L) = \bigvee K \vee \bigvee L$, is equal to 1. Hence, $\neg \bigvee L \leq \bigvee K$.

(alternatively, being maximal means that $(K \cup L)^\# = \varnothing$, hence $\mathcal{U}(\bigvee K \vee \bigvee L) = (K \cup L)^{\#\#} = \varnothing^\# = \mathsf{At}(A)$).

◀

▶ **Proposition 41.** *Let A be a transitive partial CABA. Then $\mathsf{At}(A)$ is an exclusivity graph.*

Proof. For condition (E1), let $K, L \subset \mathsf{At}(A)$ such that $K \cap L = \varnothing$ and $K \cup L$ is a maximal clique. By the previous lemma, $\bigvee K = \neg \bigvee L$. Given atoms x and y, by Lemma 35, $x \# K$ is equivalent to $x \leq \neg \bigvee K$ while $y \# L$ is equivalent to $y \leq \neg \bigvee L = \bigvee K$. Together, they imply $x \perp y$ using the LEP condition.

For condition (E2), given any atoms $x, y \in \mathsf{At}(A)$, the containment $x^\# \subset y^\#$ is equivalent to $y \leq x$ by Proposition 36. Since x and y are both atoms (minimal non-zero elements) we conclude that $x = y$. ◀

5.3 The 'powerset' of an exclusivity graph

Given an exclusivity graph $G = (X, \#)$, we construct a partial Boolean algebra $\mathcal{K}(G)$ as follows. Consider the set of cliques of G. Define an equivalence relation on it whereby

$$K \equiv L \;:\Leftrightarrow\; K^\# = L^\#$$

for cliques K, L. The elements of the algebra $\mathcal{K}(G)$ are the $\equiv$-equivalence classes. Set

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $LcupK$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
- $[K] \vee [L] = [K' \cup L']$.
- $[K] \wedge [L] = [K' \cap L']$.

▶ **Proposition 42.** *For any complete exclusivity graph G, $\mathcal{K}(G)$ is a transitive partial CABA.*

Proof. The facts that $\odot$ is a well-defined relation on equivalence classes and that it is symmetric and reflexive is immediate from the definition.

◀

6 Morphisms

Morphism part of duality – still very sketchy

6.1 Morphisms of exclusivity graphs

maybe new section here

▶ **Definition 43.** *A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R\colon X \longrightarrow Y$ satisfying:*

(M1) xRy, $x'Ry'$, and $y \# y'$ imply $x \# x'$;
(M2) if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.

the second condition perhaps can be weakened

We write **XGph** *for the resulting category of complete exclusivity graphs.*

In order to gain some intuition about this definition, it is worth calculating what it restricts to in the setting of complete graphs (i.e. sets). Condition (M1) reduces to

$$xRy,\ x'Ry',\ \text{and } y \neq y' \qquad \text{implies} \qquad x \neq x'\ ,$$

or contrapositively,

$$xRy,\ x'Ry',\ \text{and} x = x' \qquad \text{implies} \qquad y = x'\ ,$$

which is to say that the relation R is functional. On the other hand, since there is only one maximal clique, condition (M2) reduces to the requirement that $R^{-1}(Y) = X$ (i.e. R is left-total). Together, these two conditions mean that a exclusivity graph morphism R between complete graphs is precisely a function. In other words, **Set** is (equivalent to) a *full* subcategory of **XGph**

6.2 AAA

Given $h\colon A \longrightarrow B$ define yR_hx iff $y \leq h(x)$.

▶ **Proposition 44.** *Let A and B be transitive partial CABAs.* $\mathbf{epCABA}(A, B) \cong \mathbf{XGph}(\mathsf{At}(B), \mathsf{At}(A))$.

R_h is a morphism of exclusivity graphs:

Condition (M1): Suppose yR_hx and $y'R_hx'$, i.e. $y \leq h(x)$ and $y' \leq h(x')$. Then $x \wedge x' = 0$ implies by h being a homomorphism that $h(x) \wedge h(x') = h(x \wedge x') = h(0) = 0$, i.e. $h(x) \perp h(x')$. By transitivity, then, $y \perp y'$.

Note: this was the uniqueness part in the (classical) proof that R_h is a function.

Condition (M2): Suppose K is a clique in $\mathsf{At}(A)$. Then $\bigvee K$ exists in A, and by virtue of h being a complete pBA homomorphism, $\{h(x) \mid x \in K\} \in \bigodot_B$ and $\bigvee\{h(x) \mid x \in K\} = h(\bigvee K)$, and moreover this is a join of pairwise orthogonal elements, i.e. $h(x) \perp h(x')$ for any $x, x' \in K$ with $x \neq x'$. If moreover K is a maximal clique we have $\bigvee K = 1$ and therefore $\bigvee\{h(x) \mid x \in K\} = h(\bigvee K) = h(1_A) = 1_B$. Now, for each $x \in K$, let L_x be some maximal clique in $\mathcal{U}(h(x))$. Note that $\mathcal{U}(h(x)) = \{y \in \mathsf{At}B \mid yR_h x\} = R^{-1}(x)$, hence $L_x \subset R^{-1}(x)$. By Proposition 33, $\bigvee L_X = h(x)$. For any $x \neq x'$, since $h(x) \perp h(x')$, we have that $L_x \# L_{x'}$. Consequently $L := \bigcup_{x \in K} L_x$ is a clique and it is contained in $\bigcup_{x \in K} R^{-1}(x) = R^{-1}(K)$. Its join is

$$\bigvee L = \bigvee \left\{ \bigvee L_x \mid x \in K \right\} = \bigvee \{h(x) \mid x \in K\} = 1_B \ ,$$

hence this is a maximal clique.

6.3 Contextuality

Homomorphism $A \longrightarrow 2$ corresponds under the duality to morphisms of exclusivity graps $K_1 \longrightarrow \mathsf{At}(A)$, where K_1 is the graph with a single vertex. Such a relation is then determined by the image of this unique vertex, a subset of atoms of A satisfying:

- it is an independent (or stable) set
- it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

add some remarks here

7 The duality

We summarise the previous sections as the main result of the paper, a categorical duality generalising the classical duality summarised in Section 2.5.

The duality between the categories **epCABA** and **XGph** is witnessed by the following contravariant functors:

$$\begin{tikzcd} \mathbf{epCABA} \arrow[rr, bend right, "\mathsf{At}"'] \arrow[r, phantom, "\cong"] & {} & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{K}"'] \end{tikzcd} \ , \tag{8}$$

The functor

$$\mathsf{At}\colon \mathbf{epCABA}^{\mathrm{op}} \longrightarrow \mathbf{XGph}$$

is defined as follows:

- on objects: it maps a transitive partial CABA A to its graph of atoms, whose vertices are atoms and edges denote orthogonality. This forms an exclusivity graph.
- on morphisms: given a partial complete Boolean homomorphism $h\colon A \longrightarrow B$, it yields a relation

  $$\mathsf{At}(h)\colon \mathsf{At}(B) \longrightarrow \mathsf{At}(A)$$

  whereby an atom y of B is related to an atom x of A if and only if $y \leq h(x)$. This is a morphism of exclusivity graphs.

The functor

$$\mathcal{K}\colon \mathbf{XGph}^{\mathrm{op}} \longrightarrow \mathbf{epCABA}$$

admits two alternative descriptions. The first description is as follows:

- on objects: it maps an exclusivity graph $\langle X, \# \rangle$ to the transitive partial complete Boolean algebra $\mathcal{K}(X)$ thus defined:
  - its elements are cliques of X modulo the equivalence relation that identifies cliques with the same neighbourhood, $K \equiv L \;:\Leftrightarrow\; K^{\#} = L^{\#}$.
  - Commeasurability is the smallest co relation such that $[K] \odot [L]$ whenever $K \cup L$ is a clique. I.e. we have $[K] \odot [L]$ iff there are cliques $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
  - The constants are given by $0 := [\varnothing]$ and $1 = [K]$ for any maximal clique K (note that a clique K is maximal iff $K^{\#} = \varnothing$, thus they are all equivalent).
  - Negation is given by $\neg[K] = [K^{\#}]$.
  - Given cliques K and L with $[K] \odot [L]$, meets and joins are given by $[K] \wedge [L] := [K' \cap L']$ and $[K] \vee [L] := [K' \cup L']$ for representatives K', L' as in the item above.

fix this to show complete ojins and meets: need the binarity imply any commeasurable coming from exc. graph condition

- on morphisms: given a relation $R\colon X \longrightarrow Y$ which is a morphism of exclusivity graphs, one obtains a partial complete Boolean algebra homomorphism

  $$\mathcal{K}(f)\colon \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)$$

  mapping, for any clique K, the equivalence class $[K]$ to the equivalence class of any clique maximal in the set

  $$R^{-1}(K) = \{x \in X \mid \exists y \in K.\, xRy\} \ .$$

The second description is as follows:

- on objects: it maps an exclusivity graph $\langle X, \# \rangle$ to the transitive partial complete Boolean algebra $\mathcal{K}(X)$ thus defined:
  - its elements are neighbourhood-regular subsets of vertices, i.e. $S \subset X$ with $S^{\#\#} = S$, that happen to be the closure of a clique, i.e. $S = K^{\#\#}$ for some clique K (in fact, for any clique maximal in S).
  - Commeasurability

    complete
  - The constants are given by $0 := [\varnothing]$ and $1 = [K]$ for any maximal clique K (note that a clique K is maximal iff $K^{\#} = \varnothing$, thus they are all equivalent).
  - Negation is given by $\neg S = S^{\#}$.
  - Given elements S and T with $S \odot T$, meets and joins are given by $S \wedge T := S \cap T$ and $S \vee T := (S \cup T)^{\#\#}$.

fix this to show complete ojins and meets: need the binarity imply any commeasurable coming from exc. graph condition

- on morphisms: given a relation $R\colon X \longrightarrow Y$ which is a morphism of exclusivity graphs, one obtains a partial complete Boolean algebra homomorphism

  $$\begin{aligned}\mathcal{K}(f)\colon \mathcal{K}(Y) &\longrightarrow \mathcal{K}(X)\\ S &\longrightarrow R^{-1}(S) = \{x \in X \mid \exists y \in S.\, xRy\} \ .\end{aligned}$$

The duality (8) is witnessed by two natural isomorphisms:

- Given a transitive partial CABA A, the isomorphism $A \cong \mathcal{K}(\mathsf{At}(A))$ maps $a \in A$ to the set

$$\mathcal{U}(a) = \{x \in \mathsf{At}(A) \mid x \leq a\}$$

(or to the equivalence class of any clique maximal in $\mathcal{U}(a)$). Thus, a property is identified with the set of (maximally specified) *possible worlds* in which it holds.
- Given an exclusivity graph X, the bijection $X \cong \mathsf{At}(\mathcal{K}(X))$ maps $x \in X$ to the singleton set $\{x\}$ (or the equivalence class of the single-element clique $\{x\}$), which is an atom of $\mathcal{K}(X)$. I.e. a possible world is identified with its characteristic property (which completely determines that world).

In summary, we have establish the following results.

▶ **Theorem 45.** *The category* **epCABA** *of transitive partial CABAs and complete partial Boolean algebra homomorphisms (Definitions 21 and 26) is contravariantly equivalent to the category* **XGph** *of exclusivity graphs and their morphisms (Definition 39)*

▶ **Theorem 46.** *The category* **epFinBA** *of [Option 1] finite / [Option 2] finitary / [Option 3] f.d. transitive partial Boolean algebras and partial Boolean algebra homomorphisms (**??**) is contravariantly equivalent to the category* **FinXGph** *of exclusivity graphs with finite clique number and their morphisms (**??**)*

8 The free-forgetful adjunction

what to name this?

▶ **Definition 47.** *Given a compatibility graph V, its* ***graph of assignments*** *$\mathcal{A}(V)$ is the graph whose vertices are pairs $\langle C, \nu \colon C \longrightarrow \{0,1\}\rangle$ where C is a maximal compatible set, and the edge relation is given by*

$$\langle C, \nu\rangle \;\#\; \langle D, \xi\rangle \qquad \textit{iff} \qquad \exists x \in C \cap D.\, \nu(x) \neq \xi(x) \ .$$

▶ **Definition 48.** *Given a compatibility graph V, its* ***graph of assignments*** *$\mathcal{A}(V)$ is the graph whose vertices are pairs $\langle Y, N\rangle$ where $Y \cup N \subset V$ is a maximal compatible set and $Y \cap N = \varnothing$, and the edge relation is given by*

$$\langle Y_1, N_1\rangle \;\#\; \langle Y_2, N_2\rangle \qquad \textit{iff} \qquad (Y_1 \cap N_2) \cup (N_1 \cap Y_2) \neq \varnothing \ .$$

▶ **Proposition 49.** *For any compatibility graph V, $\mathcal{A}(V)$ is an exclusivity graph.*

Proof. Let $K = (\langle C_i, \nu_i\rangle)i \in I$ and $L = (\langle D_j, \xi_j\rangle)j \in J$ such that $K \cap L = \varnothing$ and $K \cup L$ is a maximal clique in $\mathcal{A}(G)$. Moreover, let $x = \langle C, \nu\rangle$ and $y = \langle D, \xi\rangle$ be two vertices such that $x \;\#\; K$ and $y \;\#\; L$.

◀

9 Outlook

We conclude with some remarks on open questions and pathways for further research. These are presented here in order of increasing scope, from the concrete to the general.

Immediate questions about this result

An immediate concrete question related to the result presented in this note is to find a nice characterisation, for any exclusivity graph, of neighbourhood-regular sets that are the closures of cliques, i.e. of the sets of the form $K^{\#\#}$ where K is a clique.

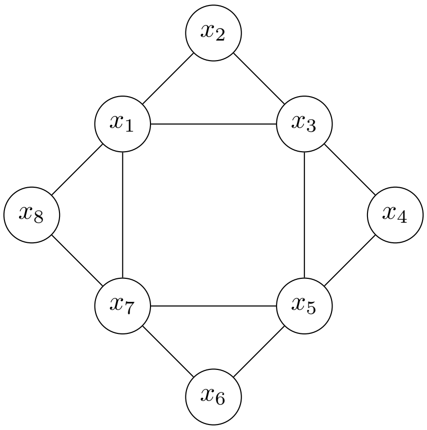

Figure 2 Example of an exclusivity graph (dual to a transitive partial CABA A) with $-^{\#\#}$-closed sets that do not correspond to elements of A. The set of vertices $S = \{x_1, x_5\}$ has neighbourhood $S^{\#} = \{x_3, x7\}$ and bi-neighbourhood $S^{\#\#} = S$, i.e. it's bi-neighbourhood closed. However, it is not the closure of any clique: it has two maximal cliques consisting of single atoms, which are distinct but not commeasurable.

The (spatial) landscape of partial Boolean algebra

Three somewhat orthogonal directions for generalising the duality results presented here immediately suggest themselves. These define three axes in the diagram of categories of partial algebras depicted in Figure 3.

- Transitivity, or the logical exclusivity principle, played a seemingly crucial rôle in the proofs of our results. But can we get away without it? I.e. can this result be generalised to the non-transitive case? What extra information on the atoms is necessary in order to reconstruct a non-transitive partial (complete atomic) Boolean algebra? What is the corresponding generalisation of exclusivity graphs?

- Compatibility in quantum mechanics is given by commutativity, a binary relation. Thus a set of observables is compatible if they are pairwise compatible. This *binarity* of compatibility is built into the definition of partial Boolean algebras through that of the category **RGph** of compatibility graphs.
  General frameworks for studying contextuality, such as e.g. [2, 3], include more general kinds of compatibility structure, described by an abstract simplicial complex (or a hypergraph). This is motivated both by theory-independent analysis of contextuality for quantum information or computation, and by applications in other domains.
  Simplicial complexes generalise compatibility graphs in the sense that **RGph** can be seen as a reflective subcategory of the category **Simp** of simplicial complexes.[17] The corresponding appropriate generalisation of partial Boolean algebras seems to coincide with the notion of 'partial Boolean algebras in a broader sense' from [9]. How much of the duality can be generalised to this *broader* setting?

[17] In fact, **RGph** embeds in **Simp** in two different ways, as a reflective and a coreflective subcategory. It is a coreflexive subcategory if we think of compatibility graphs as simplicial complexes of dimension ≤ 1, with the right adjoint being given by the 1-skeleton functor $\mathsf{sk}_1 \colon \mathbf{Simp} \longrightarrow \mathbf{RGph}$. However, the inclusion $\mathbf{RGph} \longrightarrow \mathbf{Simp}$ we have in mind maps a compatibility graph into the simplicial complex

- Perhaps more challenging, a natural generalisation to consider would be to non-atomic (transitive) partial Boolean algebras. The goal would be to find a generalisation of the classical Stone duality between Boolean algebras and Stone spaces to the (transitive) partial setting. A relevant class of examples that this would cover is that of partial Boolean algebras $\mathsf{P}(\mathcal{A})$ of projectors on a von Neumann algebras with at least one factor not of type I.
  Mirroring the classical case, one expects that it is key to follow Stone's precept to 'always topologise'. So, what is the appropriate generalisation of topological spaces from sets (inequality relations) to more relaxed forms of exclusivity? What is the corresponding generalisation of continuity from functions to relations between topologised exclusivity graphs?

- Along the same line, we could aim to push it even further to a partial version of distributive lattices, aiming to generalise the notions of spectral spaces, Prestley spaces, and/or pairwise Stone spaces.

The wider (spatial) landscape of 'quantum' logics

Various classes of algebraic and/or order-theoretic structures have been considered in the wider context of (algebraic) quantum logic. A different way of slicing down the notion of orthomodular lattice gives rise to the hierarchy of structures depicted in Figure 4.This takes as its starting point the notion of orthocomplemented poset: bounded poset with an involutive, antitone unary operation $(-)'$ such that x and x' have a greatest lower bound, $x \wedge x'$, equal to the bottom element 0. This is then built upon through the (independent) additional requirements of lattice structure or orthomodularity. .

Note: these are all property, not structure

Gudder [15] showed that transitive/LEP partial Boolean algebras sit within this hierarchy as a special class of orthomodular posets [12], those that satisfy an additional condition known as being compatibly coherent. This establishes a connection between the world of partial Boolean algebras and that of orthocomplemented posets.

In the other branch of the Hasse diagram sits the class of orthocomlemented lattices, or ortholattices. This has been studied from a logical perspective as the algebraisation of so-called orthologic, also known as 'minimal quantum logic' [11, 13, 10]. Moreover, Goldblatt proved a Stone representation theorem for ortholattices [14]. Perhaps not surprisingly, the central ideas are very closely related to those in the present paper – notably, ortholattices are represented as sets closed under a double neighbourhood construction on a graph (symmetric, irreflexive relation). However, the precise connection between both results is yet to be ironed out. In particular, Goldblatt's result constructs an ortholattice from any graph (not just exclusivity graphs) by taking all the neighbourhood-regular sets (not just the closures of cliques), revealing some subtle differences with the present result.

[Option 1] Two paths to establish a meaningful comparison/connection/bridge between the two results are: to consider the restriction of the ortholattice result to the case of finite ortholattices, or to first generalise our result to transitive partial Boolean algebras (no longer necessarily complete and atomic). Answering the first question raised in this outlook section would also facilitate this comparison.

[Option 2] In order to establish a meaningful comparison/connection/bridge between the

whose faces are its cliques. This has a left adjoint, which happens to be sk_1 as well.

$$
\begin{tikzcd}
& \mathbf{epDLat_{bs}} \arrow[rr, tail] & & \mathbf{pDLat_{bs}} \\
\mathbf{epDLat} \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pDLat} \arrow[ur, tail] & \\
& \mathbf{epBA_{bs}} \arrow[uu, tail] \arrow[rr, tail] & & \mathbf{pBA_{bs}} \arrow[uu, tail] \\
\mathbf{epBA} \arrow[uu, tail] \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pBA} \arrow[uu, tail] \arrow[ur, tail] & \\
& \mathbf{epCABA_{bs}} \arrow[uu, tail] \arrow[rr, tail] & & \mathbf{pCABA_{bs}} \arrow[uu, tail] \\
\mathbf{epCABA} \arrow[uu, tail] \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pCABA} \arrow[uu, tail] \arrow[ur, tail] &
\end{tikzcd}
$$

Figure 3 Diagram of inclusions of categories of partial Boolean algebras (and partial distributive lattices). All arrows are inclusions of categories, and horizontal ones are moreover reflective. The picture suggests possible avenues to generalise this paper's results along three axes: rightwards dropping transitivity, backwards generalising from binary to simplicial compatibility, upwards dropping completeness and atomicity towards Stone (and Priestley) duality. This paper describes a duality for the category **epCABA** in the foreground lower left vertex: the remainder of the picture is open to future exploration.

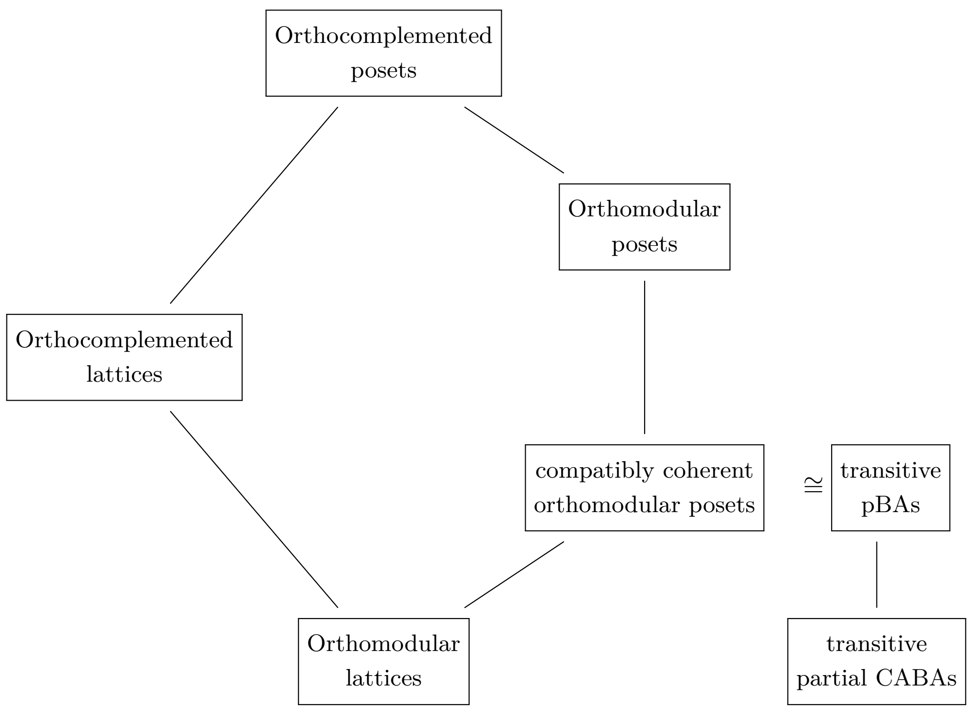

Figure 4 Hasse diagram with inclusions between various classes of orthocomplemented posets.

two results two possible paths are: to consider the restriction of the ortholattice result to the case of finite ortholattices, or to first generalise our result to transitive partial Boolean algebras (no longer necessarily complete and atomic). Answering the first question raised in this outlook section would also facilitate this comparison.

A broader aim is to extend both of these results in search of a mirror image of Figure 4 in the 'spatial' side. Importantly, a notable aspect seemingly missing from Goldblatt's representation theorem for ortholattices is what happens to morphisms. The (dual) representation of morphisms is a key facet to keep in sight in the comparisons and extensions suggested above. A first step in this regard would be, of course, to extend Goldblatt's representation theorem into a fully-fledged categorical duality.

The quest for noncommutative spaces / Towards noncommutative dualities

As a wider – and necessarily less specified – point, one might hope that further investigation along these lines, extending classical dualities to various special cases of 'quantum' structures, may lead one to identify germs of generality for a more encompassing duality theory providing appropriate notions of spectra for a host of 'noncommutative' or 'quantum' structures.

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