# Lógica Quântica
# Lecture notes and exercise sheet 3
## Functors

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**Definition 1.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A functor $F$ from $\mathbf{C}$ to $\mathbf{D}$, written $F\colon \mathbf{C} \longrightarrow \mathbf{D}$, is given by;

- an object map, associating to each object $A$ of $\mathbf{C}$ an object $F\,A$ of $\mathbf{D}$, and
- an arrow map, associating to each arrow $f\colon A \longrightarrow B$ of $\mathbf{C}$ an arrow $F\,f\colon F\,A \longrightarrow F\,B$ of $\mathbf{D}$

such that identities and composition are preserved (functoriality conditions):

- $F(id_A) = id_{F\,A}$ for all objects $A$ of $\mathbf{C}$,
- $F(g \circ f) = F\,g \circ F\,f$ for all arrows $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ in $\mathbf{C}$.

## Examples

**Exercise 1.** Let $P$ and $Q$ be posets, and regard them as categories (as in exercise 1.4). Show that a functor $P \longrightarrow Q$ is the same as a monotone function. Do you need to check the functoriality conditions? Why?

**Exercise 2.** Let $M$ and $N$ be monoids, and regard them as (one-object) categories (as in exercise 1.6). Show that a functor $M \longrightarrow N$ is the same as a monoid homomorphism.

**Exercise 3.** Given a set $X$, its power set $\mathcal{P}(X)$ is the set of subsets of $X$, i.e. $\mathcal{P}(X) = \{S \mid S \subseteq X\}$. A function $f\colon X \longrightarrow Y$ determines the following two functions between the power sets of $X$ and $Y$ (notice the reversal in the second one!):

- the direct image function $f^{\rightarrow}\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ is given by for any $S \subseteq X$,
$$f^{\rightarrow}(S) = \{f(x) \mid x \in S\} = \{y \in Y \mid \exists x \in S.\, f(x) = y\} \ .$$
- the inverse image function $f^{\leftarrow}\colon \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ is given by: for any $T \subseteq Y$,
$$f^{\leftarrow}(T) = \{x \in X \mid \exists y \in T.\, f(x) = y\} \ .$$

Show the following:

(a) The mapping $X \longmapsto \mathcal{P}(X)$ on objects and $f \longmapsto f^{\rightarrow}$ on arrows determines a functor $\mathcal{P}^{\rightarrow}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ (known as the *covariant powerset functor*).

(b) The mapping $X \longmapsto \mathcal{P}(X)$ on objects and $f \longmapsto f^{\leftarrow}$ on arrows determines a functor $\mathcal{P}^{\leftarrow}\colon \mathbf{Set}^{\mathsf{op}} \longrightarrow \mathbf{Set}$ (known as the *contravariant powerset functor*).

**Exercise 4.** Show that $U\colon \mathbf{Mon} \longrightarrow \mathbf{Set}$ mapping a monad $\langle M, \cdot, e\rangle$ to its underlying set $M$ (and a monoid homomorphism to itself seen as a bare function) is a functor. This is known as a *forgetful* functor because it 'forgets' the structure. Similar forgetful functors exist for other categories of algebraic structures, e.g. groups or vector spaces.

**Exercise 5.** Given a set $X$, write $\mathsf{List}X$ (sometimes the notation $X^*$ is used) for the set of lists of elements from $X$. We mentioned in the lectures that the assignment $X \longmapsto \mathsf{List}X$ extents to a functor $\mathsf{List}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$, with the action on arrows (which are functions in this case) given by the 'map' function, i.e. for each function $f\colon X \longrightarrow Y$, $\mathsf{List}f\colon \mathsf{List}X \longrightarrow \mathsf{List}Y$ applies $f$ to each member of a list.

(a) Show that this indeed determines a functor, i.e. check that it safisfies the functoriality conditions

(b) $\mathsf{List}X$ comes equipped with a monoid structure given by concatenation. Show that the above map can actually be extended to define functor $M\colon \mathbf{Set} \longrightarrow \mathbf{Mon}$. What do you need to show?

**Exercise 6.** Let $V$ be a vector space over a field $\mathbb{K}$ (typically, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Its *dual vector space* $V^*$ has as elements the linear functionals on $V$, i.e. the linear maps $V \longrightarrow \mathbb{K}$, with addition and scalar multiplication defined pointwise.

(a) Show that the set $V^*$ indeed has the structure of a vector space.

(b) Show that there is a functor $(-)^*\colon \mathbf{Vect}_{\mathbb{K}}^{\mathsf{op}} \longrightarrow \mathbf{Vect}_{\mathbb{K}}$ mapping (on objects) each vector space $V$ to its dual $V^*$ and (on arrows) a linear map $f\colon V \longrightarrow W$ to the linear map $f^*\colon W^* \longrightarrow V^*$ defined by $f(\phi) = \phi \circ f$.

**Exercise 7.** Given a linear map $f\colon H \longrightarrow K$ between Hilbert spaces $H$ and $K$, its adjoint is the unique linear map $f^\dagger\colon K \longrightarrow H$ such that, for all $v \in H$ and $w \in K$,

$$\langle f(v), w\rangle = \langle v, f^\dagger(w)\rangle \,.$$

Show that this construction is functorial, i.e. it defines a functor $(-)^\dagger\colon \mathbf{Hilb} \longrightarrow \mathbf{Hilb}$.

**Exercise 8.** Given a set $X$ we can construct a vector space with basis $X$. This is called the *free vector space* on $X$ (over a field $\mathbb{K}$). The elements of this vector space are the formal $\mathbb{K}$-linear combinations, the expressions

$$\sum_{x\in X} k_x x$$

with $k_x \in \mathbb{K}$ and $k_x = 0$ for all but finitely many $x$ (i.e. the set $\{x \in X \mid k_x \neq 0\}$ is finite).

(a) Verify that this indeed forms a vector space.[1]

(b) Extend this object map to a functor $F\colon \mathbf{Set} \longrightarrow \mathbf{Vect}_{\mathbb{K}}$. That is, define the arrow map and verify the functoriality axioms.

**Exercise 9.** Let $G$ be a group (regarded as a category in the sense of exercise 1.6[2]). Show that:

(a) a functor $G \longrightarrow \mathbf{Set}$ is the same as a $G$-set, a set with a group action of $G$ on it; see `https://en.wikipedia.org/wiki/Group_action`.

(b) a functor $G \longrightarrow \mathbf{Vect}_{\mathbb{K}}$ is the same as a group representation of $G$; see `https://en.wikipedia.org/wiki/Group_representation`.

## Functors as arrows

**Exercise 10** (The category of categories)**.** Show how one can form a category whose objects are small[3]categories and whose arrows are functors.

[1]Note that it is built from the set $X$ without imposing any constraints except for the equations imposed by the definition of vector space. Hence the terminology *free*.

[2]Note that a group is, in particular, a monoid (of a special kind). As a category, it is therefore a one-object category. Among these, it is characterised by the property that every arrow is an isomorphism. For this reason, a category (with any number of objects) where every arrow is an iso is known as a *grupoid*.

[3]Small means that the class of objects is a set (not a general class). This restriction is necessary to avoid a paradox, for the same reason that there is no 'set of all sets' (check Cantor's beautiful diagonal argument ). We could similarly take all locally small categories, those for which $\mathbf{C}(A, B)$ is a set for any pair of objects $A, B$.

## Bifunctors

**Exercise 11.** Recall the definition of product category from exercise 1.11. Show that the product category construction gives (category-theoretic) products in $\mathbf{Cat}$.

A functor whose domain is a product category, i.e. a functor $f\colon \mathbf{C}_1 \times \mathbf{C}_2 \longrightarrow \mathbf{D}$ is called a *bifunctor*.

**Exercise 12.** Define a functor $\mathsf{SWAP}\colon \mathbf{C} \times \mathbf{D} \longrightarrow \mathbf{D} \times \mathbf{C}$, which swaps the order of the components (as its type suggests). Verify that it does indeed satisfy functoriality.

**Exercise 13.** Let $\mathbf{C}$ be a category with all binary products (i.e. where any two objects have a product). Show that $- \times -\colon \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$, which maps a pair of objects to their product and a pair of arrows $f$ and $g$ to $f \times g$ from exercise 2.12, is a functor. What do you need to check?

What is the dual fact that holds for a category with all binary coproducts?

**Exercise 14.** Show that the tensor product of vector spaces (which is neither a product nor a coproduct in $\mathbf{Vect}_{\mathbb{K}}$) gives a bifunctor $- \otimes -\colon \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$.

**Exercise 15.** Recall the contravariant power set functor from exercise 3 and the dual vector space functor from exercise 6. Note that both are contravariant endofunctors, i.e. functors $\mathbf{C}^{\mathsf{op}} \longrightarrow \mathbf{C}$ for some category $\mathbf{C}$ ($\mathbf{C} = \mathbf{Set}$ in one case and $\mathbf{C} = \mathbf{Vect}_{\mathbb{K}}$ in the other).

(a) Observe that both are functors that send each object $A$ to the arrows from $A$ to a fixed object $D$. What is this $D$ in each case?

(b) One can generalise this idea (at least the set-theoretic part). Given a (locally small) category $\mathbf{C}$ and an object $D$ of $\mathbf{C}$, the contravariant Hom functor at $D$,

$$\mathbf{C}(-, D)\colon \mathbf{C}^{\mathsf{op}} \longrightarrow \mathbf{Set},$$

is defined as follows:

- on objects: for an objects $A$ of $\mathbf{C}$,

$$\mathbf{C}(-, D)(A) = \mathbf{C}(A, D);$$

- on morphisms: for an arrow $f\colon A \longrightarrow B$ of $\mathbf{C}$,

$$\mathbf{C}(-, D)(f)\colon \mathbf{C}(B, D) \longrightarrow (A, D) :: g \longmapsto g \circ f.$$

Show that this is indeed functorial.

(c) Similarly, define a covariant Hom functor at $D$,

$$\mathbf{C}(D, -)\colon \mathbf{C} \longrightarrow \mathbf{Set}.$$

(d) Generalise the two Hom functors to obtain a bifunctor

$$\mathbf{C}(-, -)\colon \mathbf{C}^{\mathsf{op}} \times \mathbf{C} \longrightarrow \mathbf{Set}.$$

Describe how it is defined on morphisms, and check functoriality.

## Properties of functors

**Definition 2.** A functor $f\colon \mathbf{C} \longrightarrow \mathbf{D}$ is said to be

- *faithful* if for all pair of objects $A$ and $B$ of $\mathbf{C}$, the map

$$F_{A,B}\colon \mathbf{C}(A, B) \longrightarrow \mathbf{D}(FA, FB)$$

sending $f$ to $Ff$ is injective;

- *full* if for all $A$ and $B$, $F_{A,B}$ is subjective;

- *essentially surjective* if for any object $B$ of $\mathbf{D}$ there is an object $A$ of $\mathbf{C}$ such that $FA \cong B$;
- *an equivalence* if it is faithful, full, and essentially subjective;
- *an isomorphism of categories* if there is a functor $G\colon \mathbf{D} \longrightarrow \mathbf{C}$ such that $G \circ F = \mathsf{id}_{\mathbf{C}}$ and $F \circ G = \mathsf{id}_{\mathbf{D}}$.

**Exercise 16.** Recall the functors from exercise 4 and exercise 5, the *forgetful* functor $U\colon \mathbf{Mon} \longrightarrow \mathbf{Set}$ and the *free* functor $M\colon \mathbf{Set} \longrightarrow \mathbf{Mon}$. For each of them: is it faithful? is it full?

**Exercise 17.** Functions can be seen as a special class of relations. Build a functor $R\colon \mathbf{Set} \longrightarrow \mathbf{Rel}$ that acts as the identity on objects and maps each function $f\colon X \longrightarrow Y$ to the relation

$$R\,f = \{(x, f(x)) \mid x \in A\} = \{(x, y) \mid x \in A, y \in B, f(x) = y\}.$$

Is it faithful? Is it full?

**Exercise 18.** Recall the category $\mathbf{Mat}_{\mathbb{K}}$ from exercise 1.8. Show that there is an equivalence between $\mathbf{Mat}_{\mathbb{K}}$ and $\mathbf{Vect}_{\mathbb{K}}$. Is this an isomorphism? Why?

**Exercise 19.** Show that $\mathbf{Rel}$ is isomorphic to $\mathbf{Rel}^{\mathsf{op}}$.

**Exercise 20.** What conditions on $\mathbf{C}$ must hold to make the functor $- \times -\colon \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ from exercise 13 faithful (resp. full)?

## Preservation and reflection

**Exercise 21.** Let $P$ be any property of arrows. A functor $F$ is said to *preserve* $P$ when for all $f$, $f$ satisfies $P$ implies that $F\,f$ satisfies $P$. It is said to *reflect* $P$ when for all $f$, $F\,f$ satisfies $P$ implies $f$ satisfies $P$.

(a) Show that any functor preserves isos.

(b) Show that functors do not necessarily reflect isos by providing a counterexample: a functor $F$ and an arrow $f$ such that $F\,f$ is an iso but $f$ is not.

(c) Show that full and faithful functors reflect isos.

(d) Show that faithful functors reflect monics and epics.

(e) Show (through an example) that functors need not reflect monics or epics.

(f) Show that equivalences preserve monics and epics.

(g) Show that full and faithful need not preserve monics and epics.