Lógica Quântica
Lecture notes and exercise sheet 4
Natural transformations

Rui Soares Barbosa

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Definition 1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and let $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ and $G\colon \mathbf{C} \longrightarrow \mathbf{D}$ be functors. A *natural transformation* η from F to G, written $\eta\colon F \Longrightarrow G$, is given by an arrow $\eta_A\colon FA \longrightarrow GA$ of $\mathbf{D}$ for each object $A \in \mathsf{Ob}(\mathbf{C})$ satisfying the following property, called *naturality*: for every arrow $h\colon A \longrightarrow B$ in $\mathbf{C}$,

$$\eta_B \circ Fh = Gh \circ \eta_A \ ,$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\eta_B"'] & GB
\end{tikzcd}$$

The natural transformation η is called a *natural isomorphism* if each η_A is an iso.

Exercise 1. Recall the functor $\mathsf{List}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ from exercise 3.5. Show that the following are natural transformations:

(a) $\mathsf{reverse}\colon \mathsf{List} \Longrightarrow \mathsf{List}$ where

$$\mathsf{reverse}_X : \mathsf{List}X \longrightarrow \mathsf{List}X :: [x_1, \ldots, x_n] \longmapsto [x_n, \ldots, x_1] \ ;$$

(b) $\mathsf{unit}\colon \mathsf{Id} \Longrightarrow \mathsf{List}$ where

$$\mathsf{unix}_X : X \longrightarrow \mathsf{List}X :: x \longmapsto [x] \ ;$$

(c) $\mathsf{flatten}\colon \mathsf{List} \circ \mathsf{List} \Longrightarrow \mathsf{List}$ where

$$\mathsf{flatten}_X : \mathsf{List}(\mathsf{List}X) \longrightarrow \mathsf{List}X :: [[x_{1,1}, \ldots, x_{1,k_1}], \ldots, [x_{n,1}, \ldots, x_{n,k_n}]] \longmapsto [x_{1,1}, \ldots, x_{1,k_1}, \ldots, x_{n,1}, \ldots, x_{n,k_n}] \ ;$$

(d) $\mathsf{concat}\colon \mathsf{List} \times \mathsf{List} \Longrightarrow \mathsf{List}$ where

$$\mathsf{concat}_X : \mathsf{List}X \times \mathsf{List}X \longrightarrow \mathsf{List}X :: ([x_1, \ldots, x_n], [y_1, \ldots, y_m],) \longmapsto [x_1, \ldots, x_n, y_1, \ldots, y_m] \ .$$

Exercise 2. Let $D\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be the functor $D = \times \circ \langle \mathsf{Id}, \mathsf{Id} \rangle$ mapping a set X to $X \times X$ and a function f to $f \times f$, and let $P_1\colon \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$ be the functor mapping (X, Y) to X and (f, g) to f.

(a) Show that $\Delta_X : X \longrightarrow X \times X :: x \longmapsto (x, x)$ defines a natural transformation $\mathsf{Id} \Longrightarrow D$.

(b) Show that $p_{X,Y} : X \times Y \longrightarrow X :: (x, y) \longmapsto x$ defines a natural transformation $\times \Longrightarrow P_1$.

(c) Show that the Δ and p are the only natural transformations between these functors.

Exercise 3. Define analogous transformations Δ and p for any category $\mathbf{C}$ with binary products.

Exercise 4. Let $\mathbf{C}$ be a category with a terminal object T. Let $K_T \colon \mathbf{C} \longrightarrow \mathbf{C}$ be the constant functor with value T, i.e. K_T maps any object to T and any arrow to id_T. Show that the canonical arrows to the terminal object, $\tau_A \colon A \longrightarrow T$ define a natural transformation $\mathsf{Id}_{\mathbf{C}} \Longrightarrow K_T$.

Exercise 5 (Uniform deleting)**.** Let $\mathbf{C}$ be a category and T be any object of $\mathbf{C}$. A category $\mathbf{C}$ has *uniform deleting* to T if there is a natural transformation $e \colon \mathsf{Id}_{\mathbf{C}} \Longrightarrow K_T$ with $e_T = \mathsf{id}_T$. Show that $\mathbf{C}$ has uniform deleting to T if and only if T is terminal.

Exercise 6. Recall the definition of dual of a vector space from exercise 3.6. Let V be a finite-dimensional vector space.

(a) Show that V is isomorphic to its dual V^* and second dual V^{**}.

(b) Show that the isomorphism $V \cong V^{**}$ is natural, while there is no natural isomorphism $V \cong V^*$. Note how this is related to *basis independence*.

Exercise 7. Let $\mathbf{C}$ be a category with binary products and a terminal object $\mathbf{1}$. Show that there are natural isomorphisms:

(a) $a_{A,B,C} \colon A \times (B \times C) \xrightarrow{\cong} (A \times B) \times C$ (Hint: $\langle\langle \pi_1, \pi_1 \circ \pi_2\rangle, \pi_2 \circ \pi_2\rangle$.

(b) $s_{A,B} \colon A \times B \xrightarrow{\cong} B \times A$

(c) $l_A \colon \mathbf{1} \times A \xrightarrow{\cong} A$

(d) $r_A \colon A \times \mathbf{1} \xrightarrow{\cong} A$

Exercise 8. Let P, Q be posets (seen as categories) and $f, g \colon P \longrightarrow Q$ be functors, i.e. monotone functions. When is there a natural transformation $f \Longrightarrow g$?

Yoneda lemma