

Lógica Quântica

Lecture notes and exercise sheet 3

Functors

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Definition 1. Let \mathbf{C} and \mathbf{D} be categories. A functor F from \mathbf{C} to \mathbf{D} , written $F: \mathbf{C} \longrightarrow \mathbf{D}$, is given by;

- an object map, associating to each object A of \mathbf{C} an object $F A$ of \mathbf{D} , and
- an arrow map, associating to each arrow $f: A \longrightarrow B$ of \mathbf{C} an arrow $F f: F A \longrightarrow F B$ of \mathbf{D}

such that identities and composition are preserved (functoriality conditions):

- $F(id_A) = id_{F A}$ for all objects A of \mathbf{C} ,
- $F(g \circ f) = F g \circ F f$ for all arrows $f: A \longrightarrow B$ and $g: B \longrightarrow C$ in \mathbf{C} .

Examples

Exercise 1. Let P and Q be posets, and regard them as categories (as in exercise 1.4). Show that a functor $P \longrightarrow Q$ is the same as a monotone function. Do you need to check the functoriality conditions? Why?

Exercise 2. Let M and N be monoids, and regard them as (one-object) categories (as in exercise 1.6). Show that a functor $M \longrightarrow N$ is the same as a monoid homomorphism.

Exercise 3. Given a set X , its power set $\mathcal{P}(X)$ is the set of subsets of X , i.e. $\mathcal{P}(X) = \{S \mid S \subseteq X\}$. A function $f: X \longrightarrow Y$ determines the following two functions between the power sets of X and Y (notice the reversal in the second one!):

- the direct image function $f^{\rightarrow}: \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ is given by for any $S \subseteq X$,

$$f^{\rightarrow}(S) = \{f(x) \mid x \in S\} = \{y \in Y \mid \exists x \in S. f(x) = y\}.$$

- the inverse image function $f^{\leftarrow}: \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ is given by: for any $T \subseteq Y$,

$$f^{\leftarrow}(T) = \{x \in X \mid \exists y \in T. f(x) = y\}.$$

Show the following:

- (a) The mapping $X \mapsto \mathcal{P}(X)$ on objects and $f \mapsto f^{\rightarrow}$ on arrows determines a functor $\mathcal{P}^{\rightarrow}: \mathbf{Set} \longrightarrow \mathbf{Set}$ (known as the *covariant powerset functor*).
- (b) The mapping $X \mapsto \mathcal{P}(X)$ on objects and $f \mapsto f^{\leftarrow}$ on arrows determines a functor $\mathcal{P}^{\leftarrow}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Set}$ (known as the *contravariant powerset functor*).

Exercise 4. Show that $U: \mathbf{Mon} \longrightarrow \mathbf{Set}$ mapping a monad $\langle M, \cdot, e \rangle$ to its underlying set M (and a monoid homomorphism to itself seen as a bare function) is a functor. This is known as a *forgetful* functor because it ‘forgets’ the structure. Similar forgetful functors exist for other categories of algebraic structures, e.g. groups or vector spaces.

Exercise 5. Given a set X , write $\text{List}X$ (sometimes the notation X^* is used) for the set of lists of elements from X . We mentioned in the lectures that the assignment $X \mapsto \text{List}X$ extends to a functor $\text{List}: \mathbf{Set} \rightarrow \mathbf{Set}$, with the action on arrows (which are functions in this case) given by the ‘map’ function, i.e. for each function $f: X \rightarrow Y$, $\text{List}f: \text{List}X \rightarrow \text{List}Y$ applies f to each member of a list.

- (a) Show that this indeed determines a functor, i.e. check that it satisfies the functoriality conditions
- (b) $\text{List}X$ comes equipped with a monoid structure given by concatenation. Show that the above map can actually be extended to define functor $M: \mathbf{Set} \rightarrow \mathbf{Mon}$. What do you need to show?

Exercise 6. Recall that if V is a vector space over a field \mathbb{K} (typically, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) then its *dual vector space* V^* , whose elements are the linear functionals on V , i.e. the linear maps $V \rightarrow \mathbb{K}$.

- (a) Show that the set V^* indeed has the structure of a vector space.
- (b) Show that there is a functor $(-)^*: \mathbf{Vect}_{\mathbb{K}}^{\text{op}} \rightarrow \mathbf{Vect}_{\mathbb{K}}$ mapping (on objects) each vector space V to its dual V^* and (on arrows) a linear map $f: V \rightarrow W$ to the linear map $f^*: W^* \rightarrow V^*$ defined by $f(\phi) = \phi \circ f$.

Exercise 7. Given a linear map $f: H \rightarrow K$ between Hilbert spaces H and K , its adjoint is the unique linear map $f^\dagger: K \rightarrow H$ such that, for all $v \in H$ and $w \in K$,

$$\langle f(v) | w \rangle = \langle v | f^\dagger(w) \rangle.$$

Show that this construction is functorial.

Exercise 8. Given a set X we can construct a vector space with basis X . This is called the *free vector space* on X (over a field \mathbb{K}). The elements of this vector space are the formal \mathbb{K} -linear combinations, the expressions

$$\sum_{x \in X} k_x x$$

with $k_x \in \mathbb{K}$ and $k_x = 0$ for all but finitely many x (i.e. the set $\{x \in X \mid k_x \neq 0\}$ is finite).

- (a) Verify that this indeed forms a vector space.¹
- (b) Extend this object map to a functor $F: \mathbf{Set} \rightarrow \mathbf{Vect}_{\mathbb{K}}$. That is, define the arrow map and verify the functoriality axioms.

Exercise 9. Let G be a group (regarded as a category in the sense of exercise 1.6²). Show that:

- (a) a functor $G \rightarrow \mathbf{Set}$ is the same as a G -set, a set with a group action of G on it; see https://en.wikipedia.org/wiki/Group_action.
- (b) a functor $G \rightarrow \mathbf{Vect}_{\mathbb{K}}$ is the same as a group representation of G ; see https://en.wikipedia.org/wiki/Group_representation.

Functors as arrows

Exercise 10 (The category of categories). Show how one can form a category whose objects are small³ categories and whose arrows are functors.

¹Note that it is built from the set X without imposing any constraints except for the equations imposed by the definition of vector space. Hence the terminology *free*.

²Note that a group is, in particular, a monoid (of a special kind). As a category, it is therefore a one-object category. Among these, it is characterised by the property that every arrow is an isomorphism. For this reason, a category (with any number of objects) where every arrow is an iso is known as a *groupoid*.

³Small means that the class of objects is a set (not a general class). This restriction is necessary to avoid a paradox, for the same reason that there is no ‘set of all sets’ (check We could similarly take all locally small categories, those for which $\mathbf{C}(A, B)$ is a set for any pair of objects A, B).

Bifunctors

Exercise 11. Recall the definition of product category from exercise 1.11. Show that the product category construction gives (category-theoretic) products in **Cat**.

A functor whose domain is a product category, i.e. a functor $f: \mathbf{C}_1 \times \mathbf{C}_2 \longrightarrow \mathbf{D}$ is called a *bifunctor*.

Exercise 12. Define a functor $\text{SWAP}: \mathbf{C} \times \mathbf{D} \longrightarrow \mathbf{D} \times \mathbf{C}$, which swaps the order of the components (as its type suggests). Verify that it does indeed satisfy functoriality.

Exercise 13. Let \mathbf{C} be a category with all binary products (i.e. where any two objects have a product). Show that $- \times -: \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$, which maps a pair of objects to their product and a pair of arrows f and g to $f \times g$ from exercise 2.12, is a functor. What do you need to check?

What is the dual fact that holds for a category with all binary coproducts?

Exercise 14. Show that the tensor product of vector spaces (which is neither a product nor a coproduct in $\mathbf{Vect}_{\mathbb{K}}$) gives a bifunctor $- \otimes -: \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$.

Exercise 15. Recall the contravariant power set functor from exercise 3 and the dual vector space functor from exercise 6. Note that both are contravariant endofunctors, i.e. functors $\mathbf{C}^{\text{op}} \longrightarrow \mathbf{C}$ for some category \mathbf{C} ($\mathbf{C} = \mathbf{Set}$ in one case and $\mathbf{C} = \mathbf{Vect}_{\mathbb{K}}$ in the other).

- Observe that both are functors that send each object A to the arrows from A to a fixed object D . What is this D in each of the cases?
- One can generalise this idea (at least the set-theoretic part). Given a (locally small) category \mathbf{C} and an object D of \mathbf{C} , the contravariant Hom functor at D ,

$$\mathbf{C}(-, D): \mathbf{C}^{\text{op}} \longrightarrow \mathbf{Set},$$

is defined as follows:

- on objects: for an objects A of \mathbf{C} ,

$$\mathbf{C}(-, D)(A) = \mathbf{C}(A, D);$$

- on morphisms: for an arrow $f: A \longrightarrow B$ of \mathbf{C} ,

$$\mathbf{C}(-, D)(f): \mathbf{C}(B, D) \longrightarrow \mathbf{C}(A, D) :: g \longmapsto g \circ f.$$

Show that this is indeed functorial.

- Similarly, define a covariant Hom functor at D ,

$$\mathbf{C}(D, -): \mathbf{C} \longrightarrow \mathbf{Set}.$$

- Generalise the two Hom functors to obtain a bifunctor

$$\mathbf{C}(-, -): \mathbf{C}^{\text{op}} \times \mathbf{C} \longrightarrow \mathbf{Set}.$$

Describe how it is defined on morphisms, and check functoriality.

Properties of functors

Definition 2. A functor $f: \mathbf{C} \longrightarrow \mathbf{D}$ is said to be

- faithful* if for all pair of objects A and B of \mathbf{C} , the map

$$F_{A,B}: \mathbf{C}(A, B) \longrightarrow \mathbf{D}(fA, fB)$$

sending f to Ff is injective;

- full* if for all A and B , $F_{A,B}$ is surjective;

- *essentially surjective* if for any object B of \mathbf{D} there is an object A of \mathbf{C} such that $FA \cong B$;
- *an equivalence* if it is faithful, full, and essentially surjective;
- *an isomorphism of categories* if there is a functor $G: \mathbf{D} \rightarrow \mathbf{C}$ such that $G \circ F = \text{id}_{\mathbf{C}}$ and $F \circ G = \text{id}_{\mathbf{D}}$.

Exercise 16. Recall the functors from exercise 4 and exercise 5, the *forgetful* functor $U: \mathbf{Mon} \rightarrow \mathbf{Set}$ and the *free* functor $M: \mathbf{Set} \rightarrow \mathbf{Mon}$. For each of them: is it faithful? is it full?

Exercise 17. Functions can be seen as a special class of relations. Build a functor $R: \mathbf{Set} \rightarrow \mathbf{Rel}$ that acts as the identity on objects and maps each function $f: X \rightarrow Y$ to the relation

$$Rf = \{x, f(x) \mid x \in A\} = \{(x, y) \mid x \in A, y \in B, f(x) = y\}.$$

Is it faithful? Is it full?

Exercise 18. Recall the category $\mathbf{Mat}_{\mathbb{K}}$ from exercise 1.8. Show that there is an equivalence between $\mathbf{Mat}_{\mathbb{K}}$ and $\mathbf{Vect}_{\mathbb{K}}$. Why is this not an isomorphism?

Exercise 19. What conditions on \mathbf{C} must hold to make the functor $- \times -: \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ from exercise 13 faithful (resp. full)?

Preservation and reflection

Exercise 20. Let P be any property of arrows. A functor F is said to *preserve* P when for all f , f satisfies P implies that Ff satisfies P . It is said to *reflect* P when for all f , Ff satisfies P implies f satisfies P .

- Show that any functor preserves isos.
- Show that functors do not necessarily reflect isos by providing a counterexample: a functor F and arrow f such that Ff is an iso but f is not.
- Show that full and faithful functors reflect isos.
- Show that faithful functors reflect monics and epics.
- Show (through an example) that functors need not reflect monics or epics.
- Show that equivalences preserve monics and epics.
- Show that full and faithful need not preserve monics and epics.