

# Lógica Quântica

## Assessment 2

2022–2023

Throughout this text, we work in a dagger symmetric monoidal category  $\mathbf{C}$ , that is, a symmetric monoidal category equipped with an involutive contravariant endofunctor  $\dagger$ . We can therefore use the graphical language of dagger symmetric monoidal categories, where the action of the dagger is represented by vertical reflection of the diagrams.

**Monoids** Recall the following definition of monoid in a monoidal category.

**Definition 1.** A *monoid* in  $\mathbf{C}$  is a triple  $(A, m, u)$  consisting of an object  $A$  of  $\mathbf{C}$  and two arrows

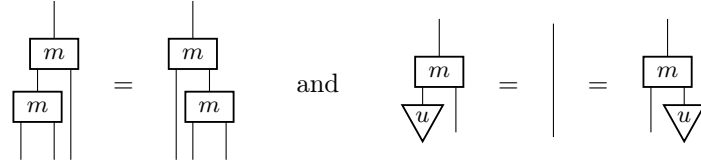
$$m: A \otimes A \longrightarrow A \quad \text{and} \quad u: I \longrightarrow A,$$

known as *multiplication* and *unit*, satisfying the following equational properties

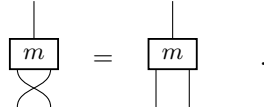
- *associativity*:  $m \circ (m \otimes \text{id}_A) = m \circ (\text{id}_A \otimes m)$ ;
- *unitality*:  $m \circ (\text{id}_A \otimes u) = \text{id}_A = m \circ (u \otimes \text{id}_A)$ .

The monoid is said to be *commutative* if  $m \circ \sigma_{A,A} = m$  where  $\sigma_{A,A}: A \otimes A \longrightarrow A \otimes A$  is the swap map.

We can write these equations using the graphical language: associativity and unitality are rendered as



while commutativity is



In fact, we can choose a more intuitive notation to represent  $m$  and  $u$ :



This leads to the following alternative way to present the definition of a monoid. We will use this style from now on.

**Definition 2.** A *monoid* in  $\mathbf{C}$  is a triple  $(A, m, u)$  consisting of an object  $A$  of  $\mathbf{C}$  and two arrows

$$\multimap: A \otimes A \longrightarrow A \quad \text{and} \quad \bullet: I \longrightarrow A \quad ,$$

known as *multiplication* and *unit*, satisfying

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad (\text{associativity})$$

and

$$\begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array} = \begin{array}{c} \text{Diagram 5} \end{array} \quad (\text{unitality})$$

The monoid is said to be *commutative* if

$$\begin{array}{c} \text{Diagram 6} \end{array} = \begin{array}{c} \text{Diagram 7} \end{array} \quad (\text{comutativity})$$

**Dagger monoids and comonoids** As we are in a dagger category, to each monoid  $(A, \multimap, \bullet)$  corresponds a comonoid  $(A, \smile, \spadesuit)$  given by *comultiplication* and *counit* maps

$$\smile = (\multimap)^\dagger: A \longrightarrow A \otimes A \quad \text{and} \quad \spadesuit = (\bullet)^\dagger: A \longrightarrow I$$

satisfying

$$\begin{array}{c} \text{Diagram 8} \end{array} = \begin{array}{c} \text{Diagram 9} \end{array} \quad (\text{coassociativity})$$

and

$$\begin{array}{c} \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \end{array} = \begin{array}{c} \text{Diagram 12} \end{array} \quad (\text{counitality})$$

The monoid  $(A, \multimap, \bullet)$  is commutative if and only if the comonoid  $(A, \smile, \spadesuit)$  is *cocommutative*:

$$\begin{array}{c} \text{Diagram 13} \end{array} = \begin{array}{c} \text{Diagram 14} \end{array} \quad (\text{cocommutativity})$$

In summary, in a dagger category, monoids and comonoids always come in pairs.

**Classical structures** We now consider conditions on the interaction of such monoid/comonoid pairs. These are relevant to define the concept of classical structure, which in the case of  $\mathbf{FHilb}$  captures precisely the notion of orthonormal basis.

**Definition 3.** A *dagger Frobenius structure* in  $\mathbf{C}$  is a monoid/comonoid pair satisfying the following equality known as the *Frobenius law*:

$$\begin{array}{c} \text{Diagram 15} \end{array} = \begin{array}{c} \text{Diagram 16} \end{array} = \begin{array}{c} \text{Diagram 17} \end{array} \quad (\text{Frobenius})$$

The dagger Frobenius structure is said to be *special* if it satisfies

$$\begin{array}{c} \text{Diagram 18} \end{array} = \begin{array}{c} \text{Diagram 19} \end{array} \quad (\text{speciality})$$

As discussed in the lectures, classical structures in **FHilb** are in one-to-one correspondence with orthonormal bases of a Hilbert space. For example, take the Hilbert space of a qubit,  $\mathbb{C}^2$ . Picking a specific orthonormal basis, e.g. the computational basis

determines the comultiplication map

which maps

thus *copying* the states in the computational basis. The corresponding multiplication is its adjoint

Moreover, the unit and counit are

More generally, the  $m$ -to- $n$  *spider* for this classical structure (i.e. the unique arrow  $A^{\otimes m} \rightarrow A^{\otimes n}$  representable as a (any) connected diagram built out of the (co)multiplication, (co)unit, identities, and swap by sequential and parallel composition) is

**Phases** We now introduce the new notion of phase. This is the central concept explored in this assessment questions.

$$\begin{array}{c} \triangle a \\ \downarrow \\ \bullet \\ \downarrow \\ \nabla a \end{array} = \begin{array}{c} | \\ \bullet \end{array} . \quad (\text{phase})$$


 $\quad$  (phase shift)

**Exercise 1** (Phase group). Let  $(A, \downarrow, \bullet)$  be a classical structure. The goal of this first question is to show that phases form a group under  $\downarrow$ . Step by step:<sup>1</sup>

(a) Show that

$$\downarrow_0 := \bullet$$

is a phase.

(b) Show that if states  $a: I \rightarrow A$  and  $b: I \rightarrow B$  are phases, then the state

$$\downarrow_{a+b} := \downarrow_a \downarrow_b$$

is also a phase.

(c) Show that if  $a$  is a phase, then the state

$$\downarrow_{-a} := \uparrow_a \bullet$$

is a phase.

(d) Show that for any state  $a: I \rightarrow A$ ,  $(-a) + a = 0$ , i.e. that

$$\downarrow_{-a} \downarrow_a = \bullet$$

◀

**Exercise 2** (Phase shift group). The goal is to interpret the operations above in terms of phase shifts maps instead of phase states.

(a) Show that

$$\text{circle}(a+b) = \text{circle}(a) \text{circle}(b)$$

(b) What is the zero-phase shift

$$\text{circle}(0) = \text{cup} \downarrow_0 = \text{cup} \bullet$$

equal to?

<sup>1</sup>Note that we already know, from the fact that  $(A, \downarrow, \bullet)$  is a commutative monoid, that  $\downarrow$  determines an operation on states that is associative, commutative, and has neutral element  $\bullet$ , i.e. it is a monoid on states. We need to show that the states that are phases are a submonoid of this, and moreover that every phase has an inverse under this operation.

(c) Observe that

$$\begin{array}{c} | \\ \bullet \\ -a \\ | \end{array} = \left( \begin{array}{c} | \\ \bullet \\ a \\ | \end{array} \right)^\dagger$$

and conclude that phase shifts are unitary. (Recall that an arrow  $f: A \rightarrow A$  is unitary if  $f \circ f^\dagger = \text{id}_A = f^\dagger \circ f$ .)

◀

**Exercise 3.** In **FHilb**, consider the concrete classical structure given for  $\mathbb{C}^2$  (the Hilbert space of one qubit) by the computational basis as explained above.

(a) What states are its phases?

(b) What is the form of the corresponding phase shifts?

◀

**Complementary classical structures** The notion of complementary captures the idea of bases being mutually unbiased.

**Definition 6.** Let  $A$  be an object equipped with two classical structures  $(A, \bullet, \bullet)$  and  $(A, \phi, \phi)$ . These classical structures are said to be *complementary* if they satisfy

$$\begin{array}{c} | \\ \bullet \\ \circlearrowleft \\ \circ \\ | \end{array} = \begin{array}{c} | \\ \bullet \\ | \\ \circ \\ | \end{array} \quad (\text{complementarity})$$

In **FHilb**, an example of two complementary classical structures on  $\mathbb{C}^2$  are the classical structures defined from the computational or  $Z$  basis  $\{|0\rangle, |1\rangle\}$  and from the  $X$  basis  $\{|+\rangle, |-\rangle\}$ , where

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

I.e. the classical structures with spiders given by

$$\begin{array}{c} \overbrace{\quad \quad \quad}^n \\ \circlearrowleft \\ \underbrace{\quad \quad \quad}_m \end{array} := |0\rangle^{\otimes n} \langle 0|^{\otimes m} + |1\rangle^{\otimes n} \langle 1|^{\otimes m} \quad \text{and} \quad \begin{array}{c} \overbrace{\quad \quad \quad}^n \\ \circlearrowright \\ \underbrace{\quad \quad \quad}_m \end{array} := |+\rangle^{\otimes n} \langle +|^{\otimes m} + |-\rangle^{\otimes n} \langle -|^{\otimes m} \quad (\star)$$

**Definition 7.** Let  $(A, \phi, \phi)$  be a classical structure. With respect to the structure  $(A, \phi, \phi)$ , a state  $a: I \rightarrow A$  is said to be

- *copyable* if

$$\begin{array}{c} \cup \\ \circ \\ | \\ \triangle \\ a \end{array} = \begin{array}{c} | \\ \triangle \\ a \end{array} \quad \begin{array}{c} | \\ \triangle \\ a \end{array} \quad (\text{copyable})$$

- *deletable* if <sup>2</sup>

$$\begin{array}{c} \circ \\ | \\ \triangle \\ a \end{array} = \quad (\text{deletable})$$

<sup>2</sup>The right-hand side of the equation is blank on purpose. The empty diagram represents the arrow  $\text{id}_I: I \rightarrow I$ , the unit scalar.

- *self-conjugate* if



In particular, the states  $|0\rangle$  and  $|1\rangle$  (resp. the states  $|+\rangle$  and  $|-\rangle$ ) are copyable, deletable, and self-conjugate with respect to the corresponding classical structure, denoted by black dot (resp. white dots) in Equation  $(\star)$  above.

**Exercise 4.** Let  $(A, \bullet, \bullet)$  and  $(A, \circ, \circ)$  be two complementary classical structures, and let  $a: I \rightarrow A$  be a state. Show that if  $a$  is copyable, deletable, and self-conjugate for  $(A, \circ, \circ)$ , then it is a phase for  $(A, \bullet, \bullet)$ .  $\blacktriangleleft$