Lógica Quântica
Lecture notes and exercise sheet 6
Multiplicative linear logic

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Gentzen sequent calculus system for intuitionistic logic

We now consider a variation of the natural deduction calculus for intuitionistic logic. Sequents are taken to have lists instead of sets of formulae to the left of the turnstile, and the proof system has explicit structural rules to manipulate these. Moreover, there are left and right rules for each connective instead of introduction and elimination rules.

Formulae are built out of basic propositions with binary connectives $\wedge$ (conjunction) and $\supset$ (implication). A *sequent* is a judgement of the form

$$\Gamma \vdash A$$

where A is a formula and $\Gamma = A_1, \ldots, A_n$ a finite list (not a set!) of formulae.

Proofs are constructed using the *structural* rules:

$$\frac{}{A \vdash A}\ \mathrm{Id} \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \mathrm{Exch}$$

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ \mathrm{Contr} \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ \mathrm{Weak} \quad ,$$

the following *left* and *right* rules for each connective:

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge\text{-R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge\text{-L}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset\text{-R} \qquad \frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \supset B, \Delta \vdash C}\ \supset\text{-L} \quad ,$$

and the *cut rule*:

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \mathrm{Cut}$$

Proposition 1 (Gentzen's Haupsatz or Cut elimination)**.** *The cut rule is admissible in the above sequent calculus proof system without Cut.*

Exercise 1. Give interpretations of the structural rules in any cartesian closed category.

Exercise 2. Show that this proof system is equivalent to the natural deduction proof system from exercise sheet 5. That is, show that each of the left and right rules is admissible in the natural deduction system and vice-versa.

Gentzen sequent calculus system for multiplicative linear logic

Multiplicative linear logic is obtained from the above by dropping the *contraction* and *exchange* rules. This gives rise to a *resource-sensitive* logic. To emphasize the difference, one typically renames the connectives to $\otimes$ and $\multimap$. Since the exchange rule is kept, it is more convenient in practice to treat the left-hand side of sequents as a multiset instead of the list and leave applications of the exchange rule implicit.

Formulae are built out of basic propositions with binary connectives $\otimes$ (multiplicative conjunction) and $\multimap$ (linear implication). A *sequent* is a judgement of the form

$$\Gamma \vdash A$$

where A is a formula and $\Gamma = A_1, \ldots, A_n$ a finite list of formulae.

Proofs are constructed using the following rules:

$$\frac{}{A \vdash A}\ \text{Id} \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \text{Exch}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes\text{-R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\text{-L}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap\text{-R} \qquad \frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\ \multimap\text{-L}\ ,$$

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \text{Cut}$$

Exercise 3. Consider the rule

$$\frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ \multimap\text{-E}\ .$$

The goal is to show that we could have equivalently have used it instead of $\multimap$-L.

(a) Show that $\multimap$-E is admissible (using Id, Cut, and $\multimap$-L)

(b) Show that $\multimap$-L is admissible in the system consisting of Id, Cut, and $\multimap$-E.

(c) What is a disadvantage of $\multimap$-E?

In fact, the above system satisfies Cut elimination, but if one replaces $\multimap$-L by $\multimap$-E then it no longer would.

Exercise 4. Consider the following sequents. Can you construct proofs of them in linear logic? Either give the proof or show that it is impossible to do so (making crucial use of Cut elimination).

(a) $\vdash A \multimap A$

(b) $A \vdash A \otimes A$

(c) $\vdash (A \otimes B) \multimap A$

(d) $(A \otimes B) \otimes C \vdash A \otimes (C \otimes B)$

(e) $B \multimap C, A \multimap B \vdash A \multimap C$

(f) $\vdash (A \wedge B) \multimap A$

(g) $A \multimap (B \supset C) \vdash B \supset (A \supset C)$

(h) $A \multimap (B \otimes B) \vdash A \multimap B$

(i) $\vdash A \multimap (B \multimap A)$

(j) $A \supset (B \supset C) \vdash (A \multimap B) \multimap (A \multimap C)$

(TBC)