Lógica Quântica
Lecture notes and exercise sheet 1
Categories

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Definition 1. A *category* $\mathbf{C}$ consists of

- a collection $\mathsf{Ob}(\mathbf{C})$ of *objects*,
- for each pair of objects A, B, a collection $\mathbf{C}(A, B)$ of *arrows (a.k.a. morphisms) with domain A and codomain B*, where we write $f \colon A \longrightarrow B$ to mean $f \in \mathbf{C}(A, B)$,

equipped with

- for each object A, an arrow $\mathsf{id}_A \colon A \longrightarrow A$ called the *identity* on A,
- an operation $\mathbf{C}(A, B) \times \mathbf{C}(B, C) \longrightarrow \mathbf{C}(A, C)$ called *composition*, i.e. given a pair of arrows $A \colon B \longrightarrow$ and $g \colon B \longrightarrow C$ (with the codomain of f coinciding with the domain of g) there is a *composite* arrow $g \circ f \colon A \longrightarrow C$,

satisfying the following properties: for all objects A, B, C, D and arrows $f \colon A \longrightarrow B$, $g \colon B \longrightarrow C$, $h \colon C \longrightarrow D$

- associativity: $h \circ (g \circ f) = (h \circ g) \circ f$,
- identity: $f \circ \mathsf{id}_A = f = \mathsf{id}_B \circ f$.

The associativity and identity axioms can be expressed as saying that the following diagrams commute:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathsf{id}_A"'] & B \arrow[d, "\mathsf{id}_B"] \\
A \arrow[r, "f"'] \arrow[ur, "f"] & B
\end{tikzcd}
$$

Examples of categories

Exercise 1. We write **Set** for the category whose objects are sets and arrows in $\mathbf{Set}(A, B)$ are functions from A to B, together with identity functions and function composition. Check that this satisfies the axioms of a category.

Definition 2. A *monoid* is an algebraic structure $\langle M, \cdot, e\rangle$ consisting of a set M equipped with

- a binary operation $\cdot \colon M \times M \longrightarrow M$
- an element $e \in M$

such that for all $a, b, c \in M$,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{and} \qquad a \cdot e = a = e \cdot a\ .$$

Given monoids M and N, a monoid homomorphism from M to N is a function $h \colon M \longrightarrow N$ such that

- $h(a \cdot_M b) = h(a) \cdot_N h(b)$, for all $a, b \in M$;
- $h(e_M) = e_N$.

Example 3. Some examples of monoids are:

- $(\mathbb{N}, +, 0)$, the natural numbers with addition,
- $(\mathbb{N}, \cdot, 1)$, the natural numbers with multiplication,
- idem for integers $\mathbb{Z}$, rationals $\mathbb{Q}$, reals $\mathbb{R}$, or complex numbers $\mathbb{C}$,
- strings with composition

Definition 4. A *preorder* or *preordered set* $\langle P, \leq \rangle$ consists of a set P equipped with a binary relation $\leq$ satisfying:

- reflexivity: $a \leq a$ for all $a \in P$; and
- transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in P$.

It is called a *partial order* or *partially ordered set (aka poset)* if it additionally satisfies:

- antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$.

Given preordered sets P and Q a *monotone function* from $\langle P, \leq_P \rangle$ to $\langle Q, \leq_Q \rangle$ is a function $f\colon P \longrightarrow Q$ satisfying: for all $a, b \in P$,

$$a \leq_P b \quad \text{implies} \quad f(a) \leq_Q f(b).$$

Example 5. Example of posets include:

- $(\mathbb{N}, \leq)$, natural numbers with the usual ordering;
- idem for integers, rationals, reals;
- $(P(x), \subseteq)$, the subsets of a set X ordered by inclusion;
- $(\mathbb{N}, |)$, the natural numbers with the divisibility relation, where $n \mid m$ if there is a $x \in \mathbb{N}$ such that $nx = m$;
- strings with the substring relation

Examples of preorders include:

- all of the above
- $(\mathbb{Z}, |)$, the integers with the divisibility relation where $a \mid b$ if there is a $x \in \mathbb{Z}$ such that $ax = b$;
- a directed graph with the reachability relation, which relates nodes x and y if there is a path from x to y

Exercise 2. Verify some of the listed examples are indeed monoids/preorders/posets.

Exercise 3. Examples of categories are given by mathematical structures and structure-preserving functions:

(a) Check that monoids and monoid homomorphism form a category **Mon** with the usual identities and composition on functions (i.e. defined as in **Set**). What do you need to verify?

(b) Show that preorder (resp. posets) and monotone functions form a category **PreOrd** (resp. **Pos**).

(c) Check that vector spaces over a field $\mathbb{K}$ (e.g. the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$) and linear maps form a category, $\mathbf{Vect}_{\mathbb{K}}$.

Exercise 4. A preorder $\langle P, \leq \rangle$ can be regarded as a category whose objects are the elements of P and there is a single arrow $a \longrightarrow b$ iff $a \leq b$. Show that this is well defined. In particular, explain why reflexivity and transitivity of $\leq$ imply the existence of identity and composites, respectively. Do you need to verify the associativity and identity equations? Conversely, show that a category $\mathbf{C}$ with at most one arrow between any two objects determines a preorder on $\mathsf{Ob}(\mathbf{C})$. Conclude that there is a correspondence between preorders and categories with at most one arrow between any two objects.

Exercise 5. Look up the definition of isomorphism in definition 6 below. A category is *skeletal* if any two isomorphic objects are equal. Explain why a poset, but not a general preorder, is skeletal when regarded as a category as per exercise 4.

Exercise 6. Show that one-object categories are the same as monoids.

Exercise 7. Given sets A and B, a *relation* R from A to B, written $R \colon A \longrightarrow B$, is a subset $R \subseteq A \times B$. It is usual to write aRb to mean that $(a, b) \in R$. Show that sets (as objects) and relations (as arrows) form a category **Rel**, with identity on A given by the relation

$$\mathsf{id}_A := \{(a, a) \mid a \in A\} = \{(a, a') \in A \times A \mid a = a'\},$$

and composition of relations $R \colon A \longrightarrow B$ and $S \colon B \longrightarrow C$ given by $S \circ R \colon A \longrightarrow C$

$$S \circ R := \{(a, c) \in A \times C \mid \exists b \in B. aRb \text{ and } bSc\}.$$

Exercise 8. Show that you can define a category $\mathbf{Mat}_{\mathbb{K}}$ whose objects are the natural numbers and where an arrow $a \longrightarrow b$ is an $b \times a$ matrix with entries on $\mathbb{K}$.

Exercise 9. Note that the definition of relational composition (composition in **Rel**) from exercise 7 can be written as

$$a(S \circ R)b \qquad \text{iff} \qquad \bigvee_{b \in B} (aRb \wedge bSc).$$

where $\vee$ represents disjunction (or) and $\wedge$ conjunction (and). Note the similarity with matrix composition

$$(M \cdot N)_{ik} = \sum_j M_{ij} N_{jk}$$

Explain that relations can be regarded as Boolean-valued matrices.
(NB: the Booleans are not a field, but note that the construction of $\mathbf{Mat}_{\mathbb{K}}$ from exercise 8 requires only that $\mathbb{K}$ be a semiring; see `https://en.wikipedia.org/wiki/Semiring` for the definition)

New categories from old

Exercise 10. (Opposite category) Given a category $\mathbf{C}$, one can define its *dual category*, $\mathbf{C}^{\mathsf{op}}$, having the same objects as $\mathbf{C}$ and all the arrows reversed. Give this construction explicitly, defining identities and composition in $\mathbf{C}^{\mathsf{op}}$ in terms of the corresponding operations in $\mathbf{C}$, and showing that it does indeed form a category.

Exercise 11. (Product category) Given two categories $\mathbf{C}_1$ and $\mathbf{C}_2$, their *product* is a category $\mathbf{C}_1 \times \mathbf{C}_2$ whose objects are pairs (A, B) of objects $A \in \mathsf{Ob}\mathbf{C}_1$ and $B \in \mathsf{Ob}(\mathbf{C}_2)$ and arrows in $(A, B) \longrightarrow (C, D)$ are pairs (f, g) consisting of arrows $f \colon A \longrightarrow C$ in $\mathbf{C}$ and $g \colon B \longrightarrow D$ in $\mathbf{D}$. Give the definition of $\mathbf{C}_1 \times \mathbf{C}_2$ explicitly, defining identities and compositions, and showing that it forms a category.

Exercise 12. (Slice category) Given category $\mathbf{C}$ and an object $X \in \mathsf{Ob}(\mathbf{C})$, the slice category $\mathbf{C}/X$ has as objects all the arrows of $\mathbf{C}$ with codomain X, and an arrow between $f \colon A \longrightarrow X$ and $g \colon B \longrightarrow X$ is an arrow $h \colon A \longrightarrow B$ in $\mathbf{C}$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Complete the definition of $\mathbf{C}/X$ by defining identities and composition and showing that the axioms of a category hold.

Exercise 13. (Slice category) Given category $\mathbf{C}$ and an object $X \in \mathsf{Ob}(\mathbf{C})$, the co-slice category $X/\mathbf{C}$ is defined by

$$X/\mathbf{C} := (\mathbf{C}^{\mathsf{op}}/X)^{\mathsf{op}}$$

Unfold the definitions to get an explicit description of this construction.

Monics, epics, isos

Definition 6. An arrow $f\colon A \longrightarrow B$ in a category $\mathbf{C}$ is said to be

- *epic* (or an epimorphism) if for all $g, h\colon B \longrightarrow C$,

$$g \circ f = h \circ f \qquad \Longrightarrow \qquad g = h\ ;$$

- *monic* (or a monomorphism) if for all $g, h\colon C \longrightarrow A$,

$$f \circ g = f \circ h \qquad \Longrightarrow \qquad g = h\ ;$$

- *split epic* if it has a right inverse (aka a section), i.e. there is an arrow $s\colon B \longrightarrow A$ such that $f \circ s = \mathsf{id}_B$;
- *split monic* if it has a left inverse (aka a retraction), i.e. there is an arrow $r\colon B \longrightarrow A$ such that $r \circ f = \mathsf{id}_A$;
- *iso* (or an isomorphism) if it has a two-sided inverse, i.e. there is an arrow $g\colon B \longrightarrow A$ such that $f \circ g = \mathsf{id}_B$ and $g \circ f = \mathsf{id}_A$

Exercise 14. Show that, in **Set**, a function $f\colon A \longrightarrow B$ is[1]

(a) injective if and only if it is monic;

(b) surjective if and only if it is epic;

(c) bijective if and only if it is an iso.

Exercise 15. In categories of mathematical structures it is not always the case that isomorphisms correspond to bijections. Demonstrate this in **Pos** by building a bijective monotone function that is not an iso.

Exercise 16. Show that an arrow is epic (resp. split epic) in $\mathbf{C}$ if and only if it is monic (resp. split monic) in $\mathbf{C}$.

Exercise 17. Show that, in any category, given arrows $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$,

(a) if f and g are monic then so is $g \circ f$

(b) if $g \circ f$ is monic then so is f

(c) if f is split monic then it is monic

(d) if f and g are split monic then so is $g \circ f$

Use exercise 16 to obtain corresponding (dual) results for epis.

Exercise 18. In any category, an iso is clearly split monic and split epic. Consequently, they are also monic and epic by the previous exercise.

(a) Show the converse of the first statement: if an arrow is split monic and split epic then it is an iso (note that this is not immediate because the left and right inverses could in principle be different).

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[1] Recall definitions: a function $f\colon A \longrightarrow B$ is

- *injective* if $\forall a_1, a_2 \in A. f(a_1) = f(a_2) \implies a_1 = a_2$;
- *surjective* if $\forall b \in B \exists a \in A. b = f(a)$.

(b) Conclude that isos have unique inverses.

(c) Conclude using results from the previous exercise that the composite of two isos is an iso.

(d) Consider the inclusion of the naturals $\mathbb{N}$ in the integers $\mathbb{Z}$ as a morphism $\langle \mathbb{N}, +, 0 \rangle \longrightarrow \langle \mathbb{Z}, +, 0 \rangle$ in the category **Mon** of monoids. Show that this provides a counterexample to the second statement, i.e. that this monoid homomorphism is monic and epic but not m iso.

Exercise 19. Given arrows $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ in any category $\mathbf{C}$, show that if $g \circ f$ and g are isos then so is f.