Lógica Quântica
Lecture notes and exercise sheet 2
Universal properties

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Intial and terminal objects

Definition 1. An *initial object* in a category $\mathbf{C}$ is an object I of $\mathbf{C}$ such that for every object A of $\mathbf{C}$ there exists a unique arrow from I to A.

A *terminal object* in a category $\mathbf{C}$ is an object T of $\mathbf{C}$ such that for every object A of $\mathbf{C}$ there exists a unique arrow from A to T.

A *zero object* in a category $\mathbf{C}$ is an object that is both initial and terminal in $\mathbf{C}$.

Exercise 1. Show that:

(a) in **Set**, $\emptyset$ is initial and a singleton set is terminal;

(b) **Rel** has a zero object, $\emptyset$;

(c) **Vect** has a zero object, the zero-dimensional space.

Exercise 2. In a poset regarded as a category (exercise 1.4), what is a terminal (respectively, initial) object?

Exercise 3. Show that any arrow from a terminal to an initial object is an iso.

The following exercise aims to show that terminal (and initial) objects are essentially unique: they are unique up to unique isomorphism.

Exercise 4. In a category $\mathbf{C}$ show that

(a) If T and T' are terminal then they are isomorphic, and there is a unique isomorphism between them.

(b) If T is terminal and A is isomorphic to T then A is terminal.

(c) If T is terminal and there is an epic arrow $f\colon T \longrightarrow A$ then A is terminal.

Products and coproducts

Definition 2. Let A and B be objects in a category $\mathbf{C}$. A product of A and B is an object $A \times B$ together with arrows

$$\pi_A\colon A \times B \longrightarrow A \qquad \text{and} \qquad \pi_B\colon A \times B \longrightarrow B$$

such that for every object C and arrows

$$f\colon C \longrightarrow A \qquad \text{and} \qquad g\colon C \longrightarrow B$$

there exists a unique arrow $\langle f, g\rangle\colon C \longrightarrow A \times B$ such that

$$\pi_A \circ \langle f, g\rangle = f \qquad \text{and} \qquad \pi_B \circ \langle f, g\rangle = g,$$

i.e. such that the following diagram commutes:

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\
& C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Definition 3. Let A and B be objects in a category $\mathbf{C}$. A coproduct of A and B is an object $A+B$ together with arrows

$$\iota_A \colon A \longrightarrow A \times B \qquad \text{and} \qquad \iota_B \colon B \longrightarrow A \times B$$

such that for every object C and arrows

$$f \colon A \longrightarrow C \qquad \text{and} \qquad g \colon B \longrightarrow C$$

there exists a unique arrow $[f, g] \colon A + B \longrightarrow C$ such that

$$[f, g] \circ \iota_A = f \qquad \text{and} \qquad [f, g] \circ \iota_B = g,$$

i.e. such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\iota_A"'] \arrow[dr, "f"] & A + B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "\iota_B"] \arrow[dl, "g"'] \\
& C &
\end{tikzcd}$$

That is, the product $A \times B$ is the 'most general' object that admits an arrow to each of A and B; the coproduct $A + B$ is the 'most general' object that admits an arrow from each of A and B.

Exercise 5. Show that in **Set** products and coproducts are given by the cartesian product $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ and disjoint union $A + B = \{\iota_A(a) \mid a \in A\} \cup \{\iota_B(b) \mid b \in A\}$.

Exercise 6. Show that in a poset regarded as a category, products are greatest lower bounds and coproducts are least upper bounds.

Exercise 7. Characterise the products and coproducts in the category **Pos** of posets and monotone functions.

Exercise 8. A *discrete category* is a category whose only arrows are the identity arrows. When do such categories have initial and terminal objects? What pairs of objects have products or coproducts?

Exercise 9. Show that (co)products of two objects A and B are unique up to unique isomorphism.

Exercise 10. Show that the uniqueness part of the definition of product can be expressed equationally as

$$h = \langle \pi_A \circ h, \pi_B \circ h \rangle$$

for all h of the appropriate type (which?). That is, show that in the definition of product, one could replace

'there exists a unique arrow $\langle f, g \rangle \colon C \longrightarrow A \times B$'

by

'there exists an arrow $\langle f, g \rangle \colon C \longrightarrow A \times B$ and, moreover, for all arrows h, $h = \langle \pi_A \circ h, \pi_B \circ h \rangle$.'

Exercise 11. Show that the following laws hold in any category where the product $A \times B$ exists.

(a) *product reflection*: $\langle \pi_A, \pi_B \rangle = \pi_{A \times B}$

(b) *product fusion*: $\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$ for all f, g, h suitably typed (what is their type?).

Exercise 12. Given arrows $f\colon A \longrightarrow A'$ and $g\colon B \longrightarrow B'$ and supposing that the products $A \times B$ and $A' \times B'$ exist, define an arrow

$$f \times g\colon A \times B \longrightarrow A' \times B'$$

by

$$f \times g = \langle f \circ \pi_A, g \circ \pi_B \rangle.$$

Show that the following laws hold:

(a) *product absorption*: $(f \times g) \circ \langle h, k \rangle = \langle f \circ h, g \circ k \rangle$

(b) $\mathsf{id}_A \times \mathsf{id}_B = \mathsf{id}_{A \times B}$

(c) $(f \times g) \circ (h \times k) = (f \circ h) \times (g \circ k)$.

Exercise 13. What are the corresponding (dual) statements for coproducts to exercise 10, exercise 11, and exercise 12?

Exercise 14. Given a category $\mathbf{C}$ and objects A, B of $\mathbf{C}$, define a new category $\mathsf{Pair(A, B)}$ whose objects are triples (P, p_A, p_B) with P an object of $\mathbf{C}$ and $p_A\colon P \longrightarrow A$, $p_B\colon P \longrightarrow B$ arrows of $\mathbf{C}$, and an arrow of type $(P, p_A, p_B) \longrightarrow (Q, q_A, q_B)$ is an arrow $h\colon P \longrightarrow Q$ of $\mathbf{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Q \arrow[dl, "q_A"'] \arrow[dr, "q_B"] & \\
A & & B \\
 & P \arrow[uu, "h"] \arrow[ul, "p_A"] \arrow[ur, "p_B"'] &
\end{tikzcd}$$

(a) Show that $\mathsf{Pair(A, B)}$ is indeed a well-defined category.

(b) Can you phrase what it means to be a product of A and B (in $\mathbf{C}$) as a special object of the category $\mathsf{Pair(A, B)}$?

(c) Observe that exercise 9 follows from a similar result proved earlier (which?).

(d) How do these statements dualise for coproduct?

Biproduct

Sometimes products and coproducts coincide. A biproduct is an object that is both a product and a coproduct 'in a compatible way'.

Definition 4. Let A and B be objects in a category $\mathbf{C}$. A biproduct of A and B is an object $A \oplus B$ together with arrows

$$\begin{tikzcd}
A \arrow[r, shift left, "\iota_A"] & A \oplus B \arrow[l, shift left, "\pi_A"] \arrow[r, shift right, "\pi_B"'] & B \arrow[l, shift right, "\iota_B"']
\end{tikzcd}$$

such that $(A \oplus B, \pi_A, \pi_B)$ is a product of A and B, $(A \oplus B, \iota_A, \iota_B)$ is a coproduct of A and B and moreover,

$$\pi_A \circ \iota_A = \mathsf{id}_A$$
$$\pi_B \circ \iota_B = \mathsf{id}_B$$
$$\iota_A \circ \pi_A \circ \iota_B \circ \pi_B = \iota_B \circ \pi_B \circ \iota_A \circ \pi_A : w$$

Exercise 15. Show that **Rel** has biproducts for every pair of elements, given by the disjoint union of sets.

Exercise 16. Show that **Vect** has biproducts for every pair of elements, given by the direct sum of vector spaces.