homework3

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1 Problem 1

Recurrence equation is: $S_n = S_{n-1} + 2S_{n-2} + 2S_{n-3}$ for $n \geq 3$
Initial case: $S_0 = 1 \quad S_1 = 3 \quad S_2 = 7$
Justification: If we pick one color as the end tile, there are three possibilities: Blue, Red, and Green. For Blue and Green, each color can have another two possibilities: RB, BB, RG, and GG. For BB and GG, each can have only one another possibility: RBB and RGG.

2 Problem 2

a). $f_n = f_{n-1} + 4f_{n-2} + 2f_{n-3} + 2n$

to find homogeneous part: $x^3 - x^2 - 4x - 2 = 0 \rightarrow$
$\mathrm{x}_1 = 1 + \sqrt{3} \quad x_2 = 1 - \sqrt{3} \quad x_3 = -1$
therefore, the homogeneous part is: $f_n = a_1(1+\sqrt{3})^n + a_2(1-\sqrt{3})^n + a_3(-1)^n$

to find a particular solution $g(n) = p_1 n + p_0$:
$p_1 n + p_0 = p_1(n-1) + p_0 + 4(p_1(n-2) + p_0) + 2(p_1(n-3) + p_0) + 2n \Rightarrow$
$(6\mathrm{p}_1 + 2)n + (6p_0 - 15p_1) = 0 \Rightarrow \quad \mathrm{p}_1 = -\frac{1}{3} \quad p_0 = -\frac{5}{6}$

therefore, the solution is: $f_n = a_1(1+\sqrt{3})^n + a_2(1-\sqrt{3})^n + a_3(-1)^n - \frac{1}{3}n - \frac{5}{6}$

to solve a_1, a_2, a_3, we have:
$f_0 = a_1 + a_2 + a_3 - \frac{5}{6} = 0$
$f_1 = a_1(1+\sqrt{3}) + a_2(1-\sqrt{3}) - a_3 - \frac{7}{6} = 1$
$f_2 = a_1(1+\sqrt{3})^2 + a_2(1-\sqrt{3})^2 + a_3 - \frac{9}{6} = 2$
$\Rightarrow \mathrm{a}_1 = \frac{30-11\sqrt{3}}{18} a_2 = \frac{30+11\sqrt{3}}{18} a_3 = -\frac{5}{2}$
the final answer is:
$f_n = \frac{30-11\sqrt{3}}{18}(1+\sqrt{3})^n + \frac{30+11\sqrt{3}}{18}(1-\sqrt{3})^n - \frac{5}{2}(-1)^n - \frac{1}{3}n - \frac{5}{6}$

b). $t_n = 2t_{n-1} + t_{n-2} + 2^n \Rightarrow \quad \mathrm{g(n)} = \mathrm{p}_0 2^n$

to find homogeneous part: $x^2 - 2x - 1 = 0 \Rightarrow$ x = 1-$\sqrt{2}$ and $x = 1 + \sqrt{2}$
therefore, the homogeneous part is: $t_n = a_1(1-\sqrt{2})^n + a_2(1+\sqrt{2})^n$

to find a particular solution $g(n) = p_0 2^n$:
$p_0 2^n = 2p_0 2^{n-1} + p_0 2^{n-2} + 2^n \Rightarrow$ $p_0 = -4$
therefore, the solution is: $t_n = a_1(1-\sqrt{2})^n + a_2(1+\sqrt{2})^n - 2^{n+2}$
to solve a_1, a_2, we have:
$t_0 = a_1 + a_2 - 4 = 0$
$t_1 = a_1(1-\sqrt{2}) + a_2(1+\sqrt{2}) - 8 = 2$
$\Rightarrow$ $a_1 = \frac{2\sqrt{2}-3}{\sqrt{2}}$ $a_2 = \frac{2\sqrt{2}+3}{\sqrt{2}}$
the final answer is: $t_n = \frac{2\sqrt{2}-3}{\sqrt{2}}(1-\sqrt{2})^n + \frac{2\sqrt{2}+3}{\sqrt{2}}(1+\sqrt{2})^n - 2^{n+2}$

3 Problem 3

The recursive relation for the sequence T_n is: $T_n = 2T_{n-1} + 1$ for $n \geq 2$
Initial case: $T_1 = 1$
to find homogeneous part: $x^n = 2x^{n-1} \Rightarrow$ x = 2
$\Rightarrow$ $T_n = a_1 2^n$
to find $g(n)$:
$g(n) = p_0 \Rightarrow$ $p_0 = 2p_0 + 1 \rightarrow$ $p_0 = -1$
therefore: $T_n = a_1 2^n - 1$
to find a_1: $T_1 = 2a_1 - 1 = 1 \Rightarrow$ $a_1 = 1 \Rightarrow$
$T_n = 2^n - 1$
Proof: $T_n = 2^n - 1$ (using math induction):
Base case: $n = 1$: $T_1 = 2^1 - 1 = 1$
Assumption: $T_n = 2^n - 1$ is correct for all $n \geq 2$
Induction: $T_{n+1} = 2T_n + 1$ according to the recursive definition
$T_{n+1} = 2 * (2^n - 1) + 1$
$T_{n+1} = 2^{n+1} - 1$
Proof is done