Problem 1

In [1]:
```
from matplotlib import pyplot as plt
import numpy as np
import sympy as sy
from sympy import lambdify
import math
```

In [2]:
```
# define f(x)
def f(x):
    return (np.sin(x)+1)**(np.sin(np.cos(x)))
# take symbolic derivative with respect to x and lambdify the resulting function
def f_prime(X):
    def symbols():
        x = sy.symbols('x')
        f = (sy.sin(x)+1) ** (sy.sin(sy.cos(x)))
        f_prime = sy.diff(f, x)
        return sy.lambdify(x, f_prime)
    return symbols()(X)
```

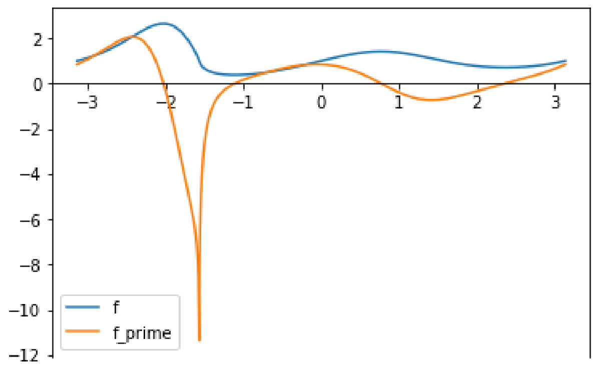
In [3]:
```
# plot f and its derivative f' over the domain [−π,π]
x_domain = np.linspace(-np.pi,np.pi,1000)

ax = plt.gca()
ax.spines["bottom"].set_position("zero")

ax.plot(x_domain, f(x_domain), label='f')
ax.plot(x_domain, f_prime(x_domain), label='f_prime')

ax.legend()
```

Out[3]:
```
<matplotlib.legend.Legend at 0x12696a3c8>
```



Problem 2

In [4]:
```
# define a function for each of the finite difference quotients
def forward_1(x, h=0.01):
    return (f(x + h) - f(x)) / h
def forward_2(x, h=0.01):
    return (-3*f(x) + 4*f(x + h) - f(x + 2*h)) / (2*h)
def backward_1(x, h=0.01):
    return (f(x) - f(x - h)) / h
def backward_2(x, h=0.01):
    return (3*f(x) - 4*f(x - h) + f(x - 2*h)) / (2*h)
def centered_2(x, h=0.01):
    return (f(x + h) - f(x - h)) / (2*h)
def centered_4(x, h=0.01):
    return (f(x - 2*h) - 8*f(x - h) + 8*f(x + h) - f(x + 2*h)) / (12*h)
```

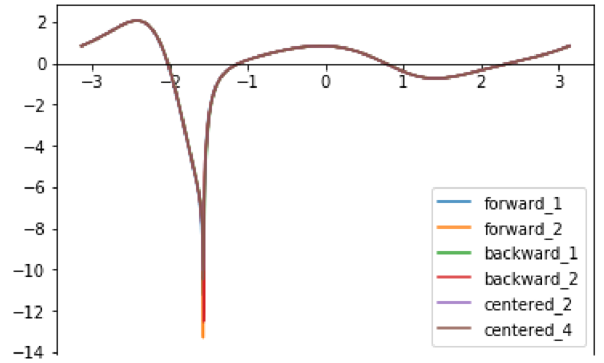
In [5]:
```
# plot the finite difference quotients for approximating f'(x0)
x_domain = np.linspace(-math.pi, math.pi, 1000)

ax = plt.gca()
ax.spines["bottom"].set_position("zero")

ax.plot(x_domain, forward_1(x_domain), label='forward_1')
ax.plot(x_domain, forward_2(x_domain), label='forward_2')
ax.plot(x_domain, backward_1(x_domain), label='backward_1')
ax.plot(x_domain, backward_2(x_domain), label='backward_2')
ax.plot(x_domain, centered_2(x_domain), label='centered_2')
ax.plot(x_domain, centered_4(x_domain), label='centered_4')

ax.legend()
```

Out[5]: <matplotlib.legend.Legend at 0x126ea3d30>

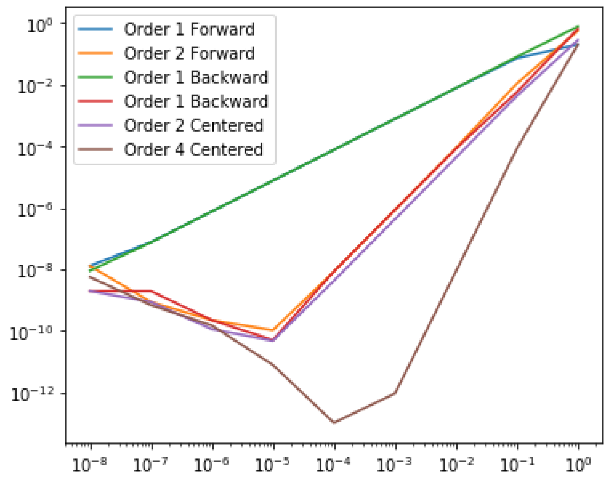


In [6]:
```
xr=np.random.uniform(-math.pi,math.pi)
xapp = centered_4(xr)
```

Problem 3

In [7]:
```
def absolute_error(x):
    h = np.logspace(-8, 0, 9)
    ax = plt.figure(figsize=(6,5)).gca()
    ax.loglog(h, abs(forward_1(x,h) - f_prime(x)),label='Order 1 Forward')
    ax.loglog(h, abs(forward_2(x, h) - f_prime(x)),label='Order 2 Forward')
    ax.loglog(h, abs(backward_1(x, h) - f_prime(x)),label='Order 1 Backward')
    ax.loglog(h, abs(backward_2(x, h) - f_prime(x)),label='Order 1 Backward')
    ax.loglog(h, abs(centered_2(x, h) - f_prime(x)),label='Order 2 Centered')
    ax.loglog(h, abs(centered_4(x, h) - f_prime(x)),label='Order 4 Centered')
    ax.legend()

absolute_error(1)
```



Problem 4

In [8]:

```
import warnings
warnings.filterwarnings("ignore")

# load the data
import pandas as pd
df = pd.DataFrame(np.load('plane.npy'), columns=['t','alpha','beta'])
df.head()
```

Out[8]:

	t	alpha	beta
0	7.0	56.25	67.54
1	8.0	55.53	66.57
2	9.0	54.80	65.59
3	10.0	54.06	64.59
4	11.0	53.34	63.62

In [9]:

```
# convert to radians
df['alpha'] = np.deg2rad(df['alpha'])
df['beta'] = np.deg2rad(df['beta'])
df.head()
```

Out[9]:

	t	alpha	beta
0	7.0	0.981748	1.178795
1	8.0	0.969181	1.161866
2	9.0	0.956440	1.144761
3	10.0	0.943525	1.127308
4	11.0	0.930959	1.110378

In [10]:

```
# compute the coordinates
df['xt'] = 500 * np.tan(df['beta']) / (np.tan(df['beta']) - np.tan(df['alpha']))
df['yt'] = 500 * np.tan(df['beta']) * np.tan(df['alpha']) / (np.tan(df['beta']) - np.tan(df['a
df.head()
```

Out[10]:

	t	alpha	beta	xt	yt
0	7.0	0.981748	1.178795	1311.271337	1962.456239
1	8.0	0.969181	1.161866	1355.936476	1975.114505
2	9.0	0.956440	1.144761	1401.918398	1987.346016
3	10.0	0.943525	1.127308	1450.497006	2000.840713
4	11.0	0.930959	1.110378	1498.640350	2013.512411

In [11]:
```
df['xt_prime'] = 0
df['yt_prime'] = 0

# using a forward difference quotient for t = 7
df['xt_prime'][0] =  df['xt'][1] - df['xt'][0]
df['yt_prime'][0] =  df['yt'][1] - df['yt'][0]

# using a backward difference quotient for t = 14
df['xt_prime'][7] =  df['xt'][7] - df['xt'][6]
df['yt_prime'][7] =  df['yt'][7] - df['yt'][6]

# using a centered difference quotient for t = 8, 9, . . . , 13
for t in range(1, 7):
    df['xt_prime'][t] =  (df['xt'][t+1] - df['xt'][t-1]) / 2
    df['yt_prime'][t] =  (df['yt'][t+1] - df['yt'][t-1]) / 2
df.head()
```

Out[11]:

	t	alpha	beta	xt	yt	xt_prime	yt_prime
0	7.0	0.981748	1.178795	1311.271337	1962.456239	44	12
1	8.0	0.969181	1.161866	1355.936476	1975.114505	45	12
2	9.0	0.956440	1.144761	1401.918398	1987.346016	47	12
3	10.0	0.943525	1.127308	1450.497006	2000.840713	48	13
4	11.0	0.930959	1.110378	1498.640350	2013.512411	46	12

In [12]:
```
# compute the values of the speed
df['speed'] = np.sqrt(df['xt_prime'] ** 2 + df['yt_prime'] ** 2)
df.head()
```

Out[12]:

	t	alpha	beta	xt	yt	xt_prime	yt_prime	speed
0	7.0	0.981748	1.178795	1311.271337	1962.456239	44	12	45.607017
1	8.0	0.969181	1.161866	1355.936476	1975.114505	45	12	46.572524
2	9.0	0.956440	1.144761	1401.918398	1987.346016	47	12	48.507731
3	10.0	0.943525	1.127308	1450.497006	2000.840713	48	13	49.729267
4	11.0	0.930959	1.110378	1498.640350	2013.512411	46	12	47.539457

Problem 5

In [13]:
```
def high_dim(g, p, h):
    n = len(g)
    m = len(p)
    I = np.identity(m)
    J = np.zeros((n, m))
    for i, fun in enumerate(g):
        for j, pt in enumerate(p):
            func = sy.lambdify((x, y), fun, 'numpy')
            xleft = pt - h*I[:,j]
            xright = pt + h*I[:,j]
            J[i, j] = (func(xright[0], xright[1]) - func(xleft[0], xleft[1])) / (2*h)
    return J
```

In [14]:
```
x = sy.Symbol('x')
y = sy.Symbol('y')

g = [x**2, x**3-y]
p=[1,1]
h=0.01
high_dim(g, p, h)
```

Out[14]:
```
array([[ 2.    ,  0.    ],
       [ 3.0001, -1.    ]])
```

Problem 7

In [15]:
```python
import time
from autograd import numpy as anp
from autograd import grad

def Timer(N):
    Time1=np.zeros(N,dtype='float')
    Time2=np.zeros(N,dtype='float')
    Time3=np.zeros(N,dtype='float')
    Error1=1e-18*np.ones(N,dtype='float')
    Error2=np.zeros(N,dtype='float')
    Error3=np.zeros(N,dtype='float')
    dg = grad(lambda x: (anp.sin(x)+1)**(anp.sin(anp.cos(x))) )
    for i in range(N):
        xp = np.random.uniform(low=-np.pi, high=np.pi)
        time_begin_1 = time.clock()
        result1 = f_prime(xp)
        time_end_1 = time.clock()
        Time1[i] = time_end_1-time_begin_1

        time_begin_2 = time.clock()
        result2 = centered_4(xp)
        time_end_2 = time.clock()
        Time2[i] = time_end_2-time_begin_2
        Error2[i] = abs(result2-result1)

        time_begin_3 = time.clock()
        result3 = dg(xp)
        time_end_3 = time.clock()
        Time3[i] = time_end_3-time_begin_3
        Error3[i] = abs(result3-result1)

    return Time1,Time2,Time3,Error1,Error2,Error3
```

In [16]:
```python
Time1,Time2,Time3,Error1,Error2,Error3 = Timer(200)
ax = plt.figure(figsize=(10,8)).gca()
ax.loglog(Time1,Error1,'ro',label='Sympy',color='blue')
ax.loglog(Time2,Error2,'ro',label='Difference Quotients',color='orange')
ax.loglog(Time3,Error3,'ro',label='Autograd',color='green')
ax.legend()
```

Out[16]: <matplotlib.legend.Legend at 0x12708a3c8>