Due: Thursday, September 6

1. **Sets**

- If set A has a elements and set B has b elements, how many elements are in the set $A \times B$? Explain your answer.

  The set $A \times B$ contains all ordered pairs that can be made by choosing (x, y) where $x \in A$ and $y \in B$. So, each element $a \in A$ can be paired with all of the elements of B. Therefore the size of the set $A \times B$ will be the product $a \cdot b$.

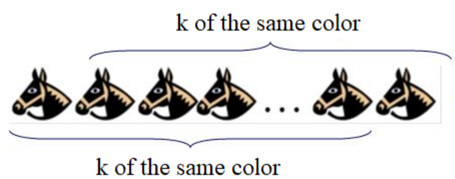
2. **Proofs**

(a) Find the error in the following proof.

Claim: All horses are the same color.

Proof: We prove that any collection of horses is monochromatic by induction on the number of horses in the collection.

Base Case: Obviously, a set of one horse is a set of horses all with the same color.
Induction Hypothesis: Assume that any set of k horses are all the same color.
Inductive Step: Consider a set of $k+1$ horses, and stand them all in a line.



The first k horses in the line form a set of k horses, and so by the Inductive Hypothesis, are all the same color. The same is true for the last k horses in the line. Therefore the entire set consists of $k+1$ horses of the same color.

The error in this proof becomes evident when we consider a set of 2 horses. The inductive step of the proof relies on the fact that the first k horses and the last k horses have an overlapping middle horse. However, with a set of 2 horses, there isn't a middle horse and the proof breaks down. The base case was incorrect and would instead have needed to show that a set of two horses has the same color.

(b) Let $S(n) = 1 + 2 + \cdots + n$ be the sum of the first n natural numbers and let $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following through induction on n.

- $S(n) = \frac{1}{2}n(n+1)$.

  Proof: We prove this through induction on n.

  Base Case: Consider the case where $n = 1$. Then $S(1) = 1$ as it should.

  Inductive Hypothesis: Assume that $S(k) = \frac{1}{2}k(k+1)$ for some $k \geq 1$.

  Inductive Step: Consider $k + 1$. We can see that

  $$S(k+1) = 1 + 2 + \cdots + k + (k+1) = S(k) + k + 1.$$

  By the inductive hypothesis, this becomes

  $$S(k) + k + 1 = \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2),$$

  when simplified. Thus

  $$S(k+1) = \frac{1}{2}(k+1)(k+2),$$

  which completes our proof.

· $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2 = S^2(n)$.

Proof: We prove this through induction on n.

Base Case: Consider the case where $n = 1$. Then $C(1) = \frac{1}{4}(1 + 2 + 1) = 1 = 1^3$ as it should.

Inductive Hypothesis: Assume that $C(k) = \frac{1}{4}k^2(k+1)^2$ for some $k \geq 1$.

Inductive Step: Consider $k + 1$. We can see that

$$C(k+1) = 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = C(k) + (k+1)^3.$$

By the inductive hypothesis, this becomes

$$C(k) + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left(k^2 + 4k + 4\right) = \frac{1}{4}(k+1)^2(k+2)^2,$$

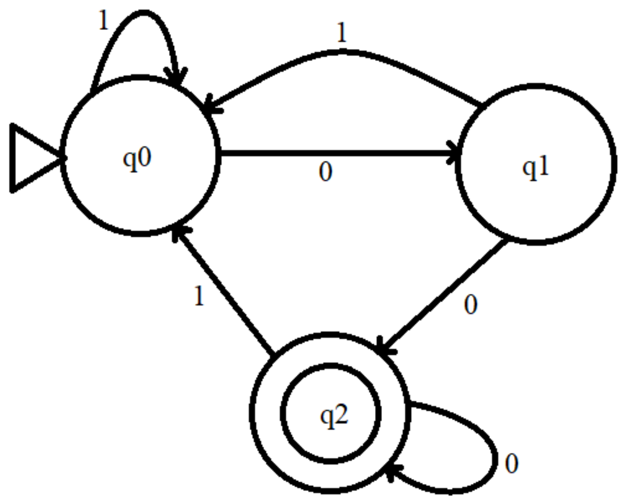
when simplified. Thus

$$C(k+1) = \frac{1}{4}(k+1)^2(k+2)^2,$$

which completes our proof.

3. **Describing DFAs**

For the following deterministic finite automaton M_1:



(a) Write out the full mathematical description of M_1.

i. $Q = \{q_0, q_1, q_2\}$,

ii. $\Sigma = \{0, 1\}$,

iii. δ is defined by:

	0	1
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_2	q_0

iv. q_0 is given as the start state,

v. $F = \{q_2\}$.

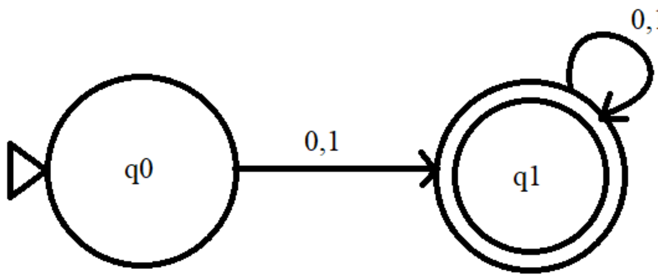
(b) Determine what language M_1 recognizes.

$$L(M_1) = \{w \mid w \text{ ends in } 00\}.$$

4. **Creating DFAs**

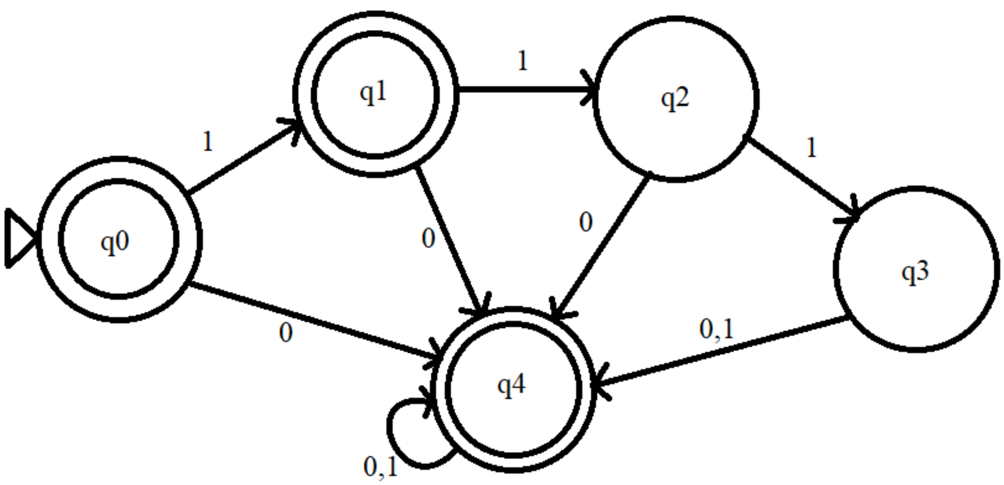
Draw DFAs for the following languages (you may assume the alphabet is always $\{0, 1\}$):

(a) $\{w \mid w \neq \varepsilon\}$



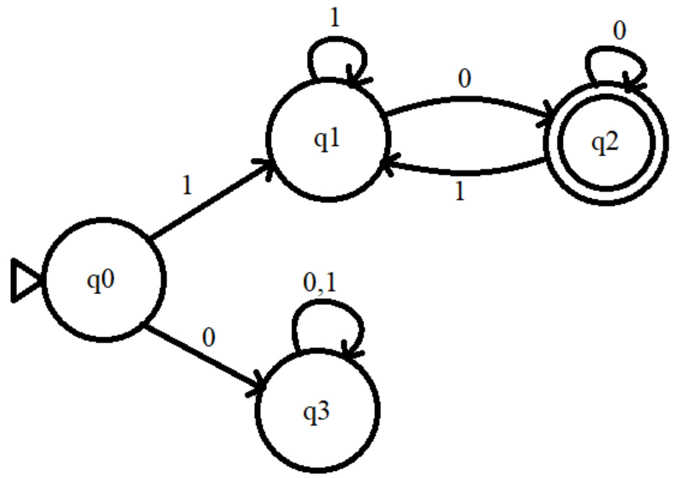
- q_0 is 'empty string'
- q_1 is 'not empty string'

(b) $\{w \mid w \neq 11 \text{ and } w \neq 111\}$



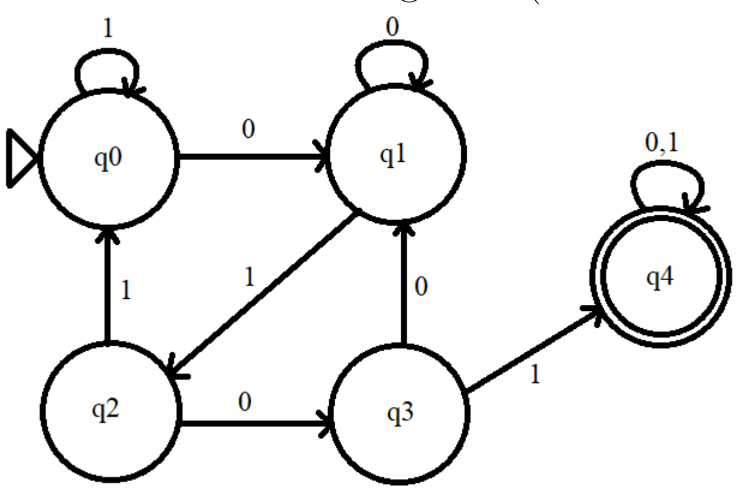
- q_0 is 'currently the empty string'
- q_1 is 'currently the string 1'
- q_2 is 'currently the string 11'
- q_3 is 'currently the string 111'
- q_4 is 'has a 0 or more than three 1s'

(c) $\{w \mid w \text{ begins with a 1 and ends with a 0}\}$



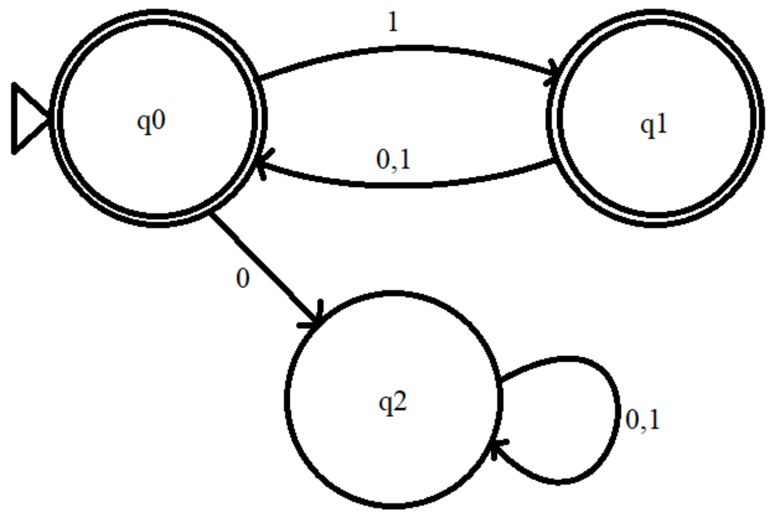
- q_0 is 'empty string'
- q_1 is 'begins with a 1 and ends with a 1'
- q_2 is 'ends with a 0, but began with a 1'
- q_3 is 'began with a 0'

(d) $\{w \mid w \text{ contains the substring } 0101\, (w = x0101y \text{ for some } x \text{ and } y)\}$



- q_0 is 'last saw a 1, not part of string 0101'
- q_1 is 'last saw a 0, possibly first 0 of 0101'
- q_2 is 'have seen 01'
- q_3 is 'have seen 010'
- q_4 is 'have seen 0101'

(e) $\{w \mid \text{every odd position of } w \text{ is a } 1\}$



- q_0 is 'just looked at an even position'
- q_1 is 'just saw a 1 in an odd position'
- q_2 is 'saw a 0 in an odd position'