Math 590 Homework 4 Fall 2019

Due: Tuesday, October 1

First, and observation about these proofs in general. If we are able to describe a unique and complete listing of the elements of a set, this itself is a correspondence between that set and the natural numbers $\mathbb{N}$. Since the listing is: (1) a list, and (2) unique, we know that it is one-to-one. There is only one element from the set that is associated with the 10th location in the list (by the definition of a list), and that element does not exist elsewhere on the list (unique). Since the listing is complete, we know that it is onto, because every elements from the set is in the list. Therefore, the listing is a function from the natural numbers (i.e., the location in the list) and the elements in the set, which is both one-to-one and onto.

This means that for all of these problems, to prove that a set is countable, we can simply describe a process for uniquely and completely listing the elements in the set.

1. **Integers**

   Prove that the set of integers, $\mathbb{Z}$, is a countably infinite set.

   To list the integers, simply start with 0, then proceed by increasing absolute value, alternating between positive and negative: 0, 1, -1, 2, -2, 3, -3, etc.

2. **All Rationals**

   Prove that the set of all rational numbers,

   $$\mathbb{Q} = \left\{ \frac{m}{n} \,\middle|\, m, n \in \mathbb{Z}\,,\, n \neq 0 \right\},$$

   is a countably infinite set.

   There are many ways to approach this problem. The easiest is to take the same approach as with the integers. Recall that we can count the positive rational numbers, so we can assign an ordering to them. Then our counting method here is to start with 0, then list the rationals in the same order as before, but alternating between positive and negative. So 0, 1/1, $-1/1$, 2/1, $-2/1$, 1/2, $-1/2$, etc.

3. **3-tuples**

   Prove that the set of all 3-tuples of $\mathbb{N}$ is countably infinite, i.e., prove that the set

   $$\{(i, j, k) \,|\, i, j, k \in \mathbb{N}\}$$

   is a countably infinite set. (Hint: note that positive rational numbers m/n can be rewritten as 2-tuples (m, n) of natural numbers. Can you somehow generalize the idea of that proof?)

   Given a 3-tuple (i, j, k), consider its sum $i+j+k$. Note that when we limit $i, j, k \in \mathbb{N}$, then the smallest possible sum is $i+j+k = 3$, and all possible sums are themselves natural numbers (i.e., $i+j+k \in \mathbb{N}$).

   Next we note that there are a finite number of 3-tuples that all have the same sum. For example, if we say $i+j+k = 5$, then the possible 3-tuples are:

   $$(1, 1, 3), \quad (1, 3, 1), \quad (3, 1, 1), \quad (1, 2, 2), \quad (2, 1, 2), \quad (2, 2, 1).$$

   Now, note that if we can describe a way to list our 3-tuples, then we have successfully described a way to count them (since there is a 1st thing in the list, a 2nd thing in the list, etc).

   To list the 3-tuples, consider all 3-tuples that sum to 3 and list those, then list all 3-tuples that sum to 4, then list all 3-tuples that sum to 5, etc. To make this ordering unique, we requires that each set of 3-tuples is listed in ascending order. This would generate a list like:

   $$\{(1, 1, 1)\}, \quad \{(1, 1, 2), \quad (1, 2, 1), \quad (2, 1, 1)\},$$
   $$\{(1, 1, 3), \quad (1, 2, 2), \quad (1, 3, 1), \quad (2, 1, 2), \quad (2, 2, 1), \quad (3, 1, 1)\},$$
   $$\ldots$$

4. **Computer Programs**

   (a) Given a **finite** alphabet Σ, prove that the set of all **finite** strings that can be generated from the alphabet is a countably infinite set, i.e., $\{w \,|\, w \in \Sigma^*, \; |w| \in \mathbb{N}\}$. (Hint: very similar to a proof about binary strings we saw in class.)

   Proof: First note that since our alphabet is finite, we can impose an alphabetical ordering on our strings (for example, in the same way we do with the English alphabet, where 'a' is the first and 'z' is the last letter). Note that there are a finite number of strings of length k. So, list all strings of length 0 in alphabetical order, then all strings of length 1 in alphabetical order, then all strings of length 2 in alphabetical order, then all strings of length 3 in alphabetical order, etc. We have found a unique and complete listing, so the set is countable.

(b) Assuming that computers programs are finite in length, use part (a) to prove that there are a countably infinite number of possible computer programs.

The alphabet that can be used to generate a computer program (i.e., the alphabet that is all possible symbols that can be typed into a computer program) is a finite alphabet. Furthermore, all computer programs can be (trivially) represented as strings. So, any finite computer program can be considered a finite string generated from a finite alphabet. Therefore the set of all computer programs is a subset of the set of all finite strings generated from a finite alphabet. In part (a), we proved that the set of all finite strings generated from a finite alphabet was countable, and therefore there are a countable number of computer programs.