7. (22 points.) HMMs

You sometimes get colds, which make you sneeze. You also get allergies, which make you sneeze. Sometimes you are well, which doesn't make you sneeze (much). You decide to model the process using the following HMM, with hidden states $X \in \{well, allergy, cold\}$ and observations $E \in \{sneeze, quiet\}$:

$P(X_1)$

well	1
allergy	0
cold	0

$P(X_t \mid X_{t-1} = well)$

well	0.7
allergy	0.2
cold	0.1

$P(X_t \mid X_{t-1} = allergy)$

well	0.6
allergy	0.3
cold	0.1

$P(X_t \mid X_{t-1} = cold)$

well	0.2
allergy	0.2
cold	0.6

Transitions

$P(E_t \mid X_t = well)$

quiet	1.0
sneeze	0.0

$P(E_t \mid X_t = allergy)$

quiet	0.0
sneeze	1.0

$P(E_t \mid X_t = cold)$

quiet	0.0
sneeze	1.0

Emissions

Note that colds are "stickier" in that you tend to have them for multiple days, while allergies come and go on a quicker time scale. However, allergies are more frequent. Assume that on the first day, you are well.

(a) **(2 pts)** Imagine you observe the sequence *quiet, sneeze, sneeze*. What is the probability that you were well all three days and observed these effects?

Consider parts (e) and (f) only!

(b) **(4 pts)** What is the posterior distribution over your state on day 2 (X_2) if $E_1 = quiet$, $E_2 = sneeze$?

(c) **(4 pts)** What is the posterior distribution over your state on day 3 (X_3) if $E_1 = quiet$, $E_2 = sneeze$, $E_3 = sneeze$?

(d) **(4 pts)** What is the Viterbi (most likely) sequence for the observation sequence *quiet, sneeze, sneeze, sneeze, quiet, quiet, sneeze, quiet, quiet*? *Hint: you should not have to do extensive calculations.*

Consider parts (e) and (f) only!

Imagine you are monitoring your state using the particle filtering algorithm, and on a given day you have 5 particles on *well*, 2 on *cold*, and 3 on *allergy* before making an observation on that day.

(e) **(4 pts)** If you observe *sneeze*, what weight will each of your particles have?

(f) **(4 pts)** After resampling, what is the expected number of particles you will have on *cold*?