Skip Lists

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(edits by M. Hudachek-Buswell)

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- However, searching for a particular item will still be $O(n)$.
- This could be reduced by having some items (where these items are randomly chosen) be in another list, and have these items act like a "marker". That way, you can reduce the search time by searching fewer items to see where in the list it might be.

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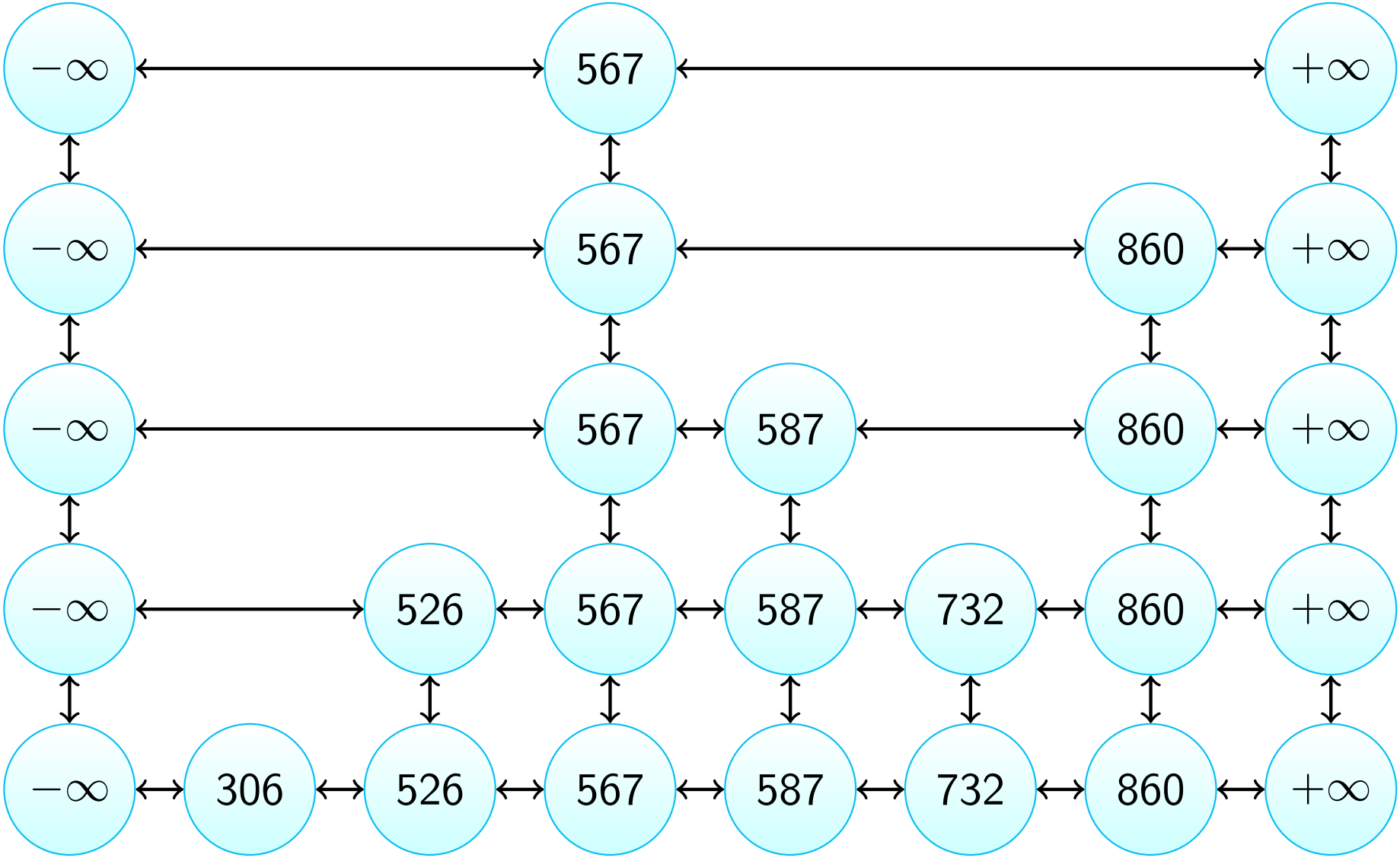
- When adding a new item, a **coin flipper** is used to determine if the item gets promoted to the next level and, if so, how many times.

Example Skip List



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- If the coin comes up heads, then the item is promoted to the next level, and another flip is done. If the coin comes up tails, then the item is not promoted, and no more flips are done.
- Because of this randomness, the resulting skip list might not be perfectly arranged (with only half of the items on a level being present on the level above it).

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- Finally, each node also contains the level the node is on; all nodes on the bottom level typically have a level of 1.
- Because each node has four references (up, down, left, right), this type of node might also be known as a **quad-node**.

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- The negative infinity phantom nodes are required; the positive infinity phantom nodes are not.
- Because negative infinity and positive infinity might not be representable in all programming languages, some other special value can be used to represent negative infinity and positive infinity.

Searching

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- Check the data in the node to the right.
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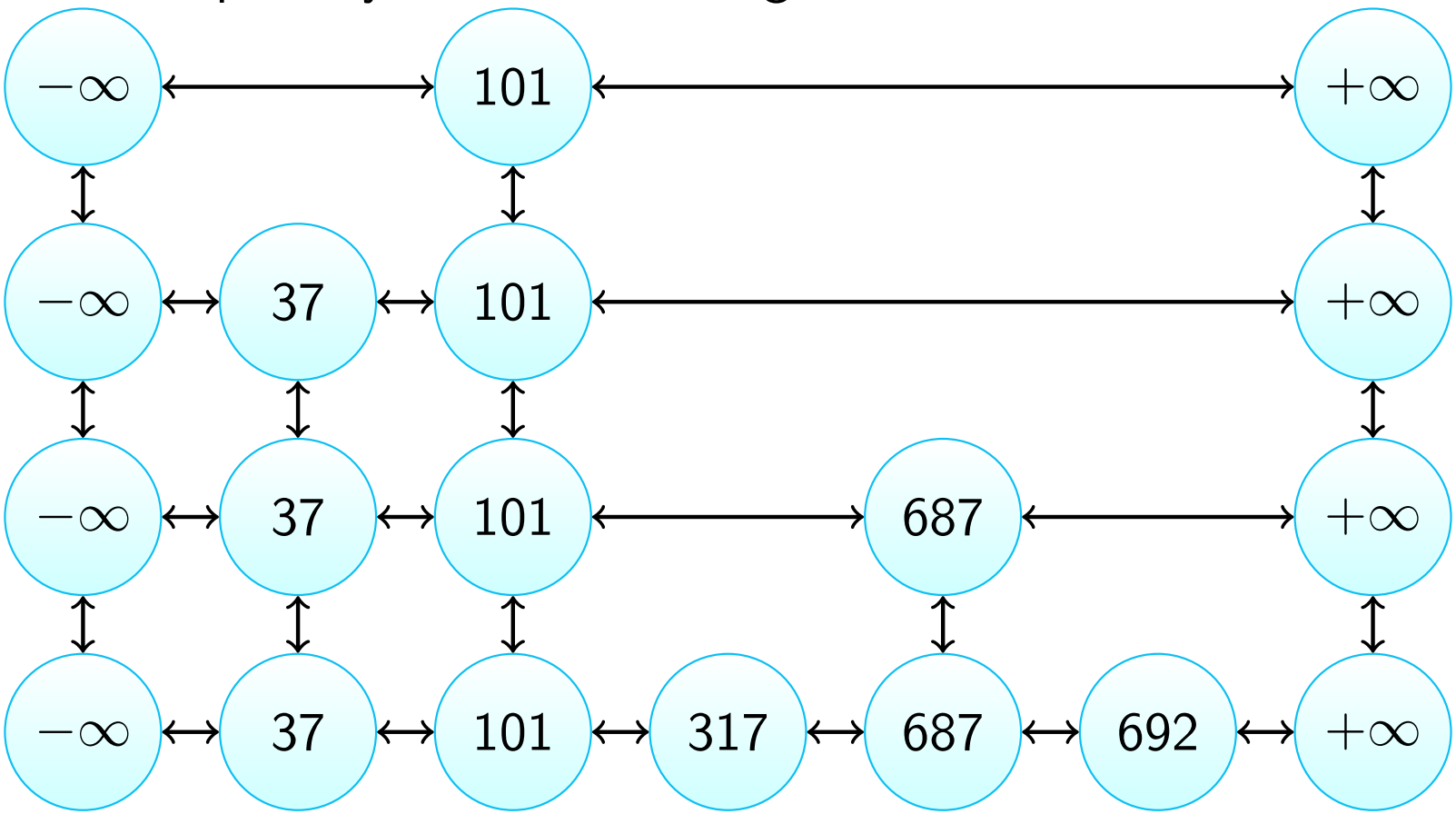
  - If the data you're looking for is greater than the data in the node, go to that node.
  - If the data you're looking for is equal to the data in the node, then you've found the data.
- Repeat the previous step, until you find the node or go off of the list, in which case the data you're looking for isn't in the skip list.

Searching

For example, if you were searching for 687:



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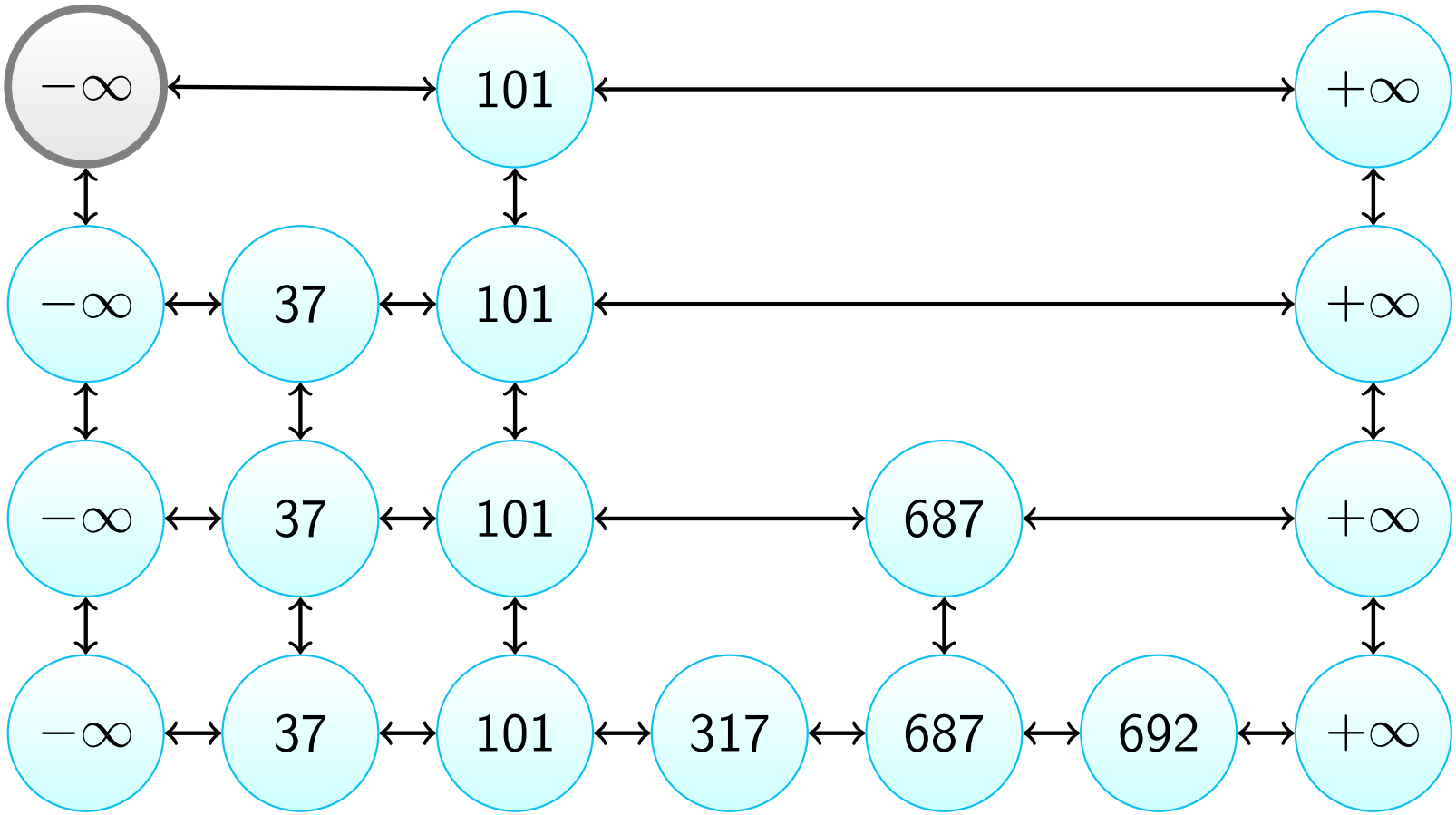

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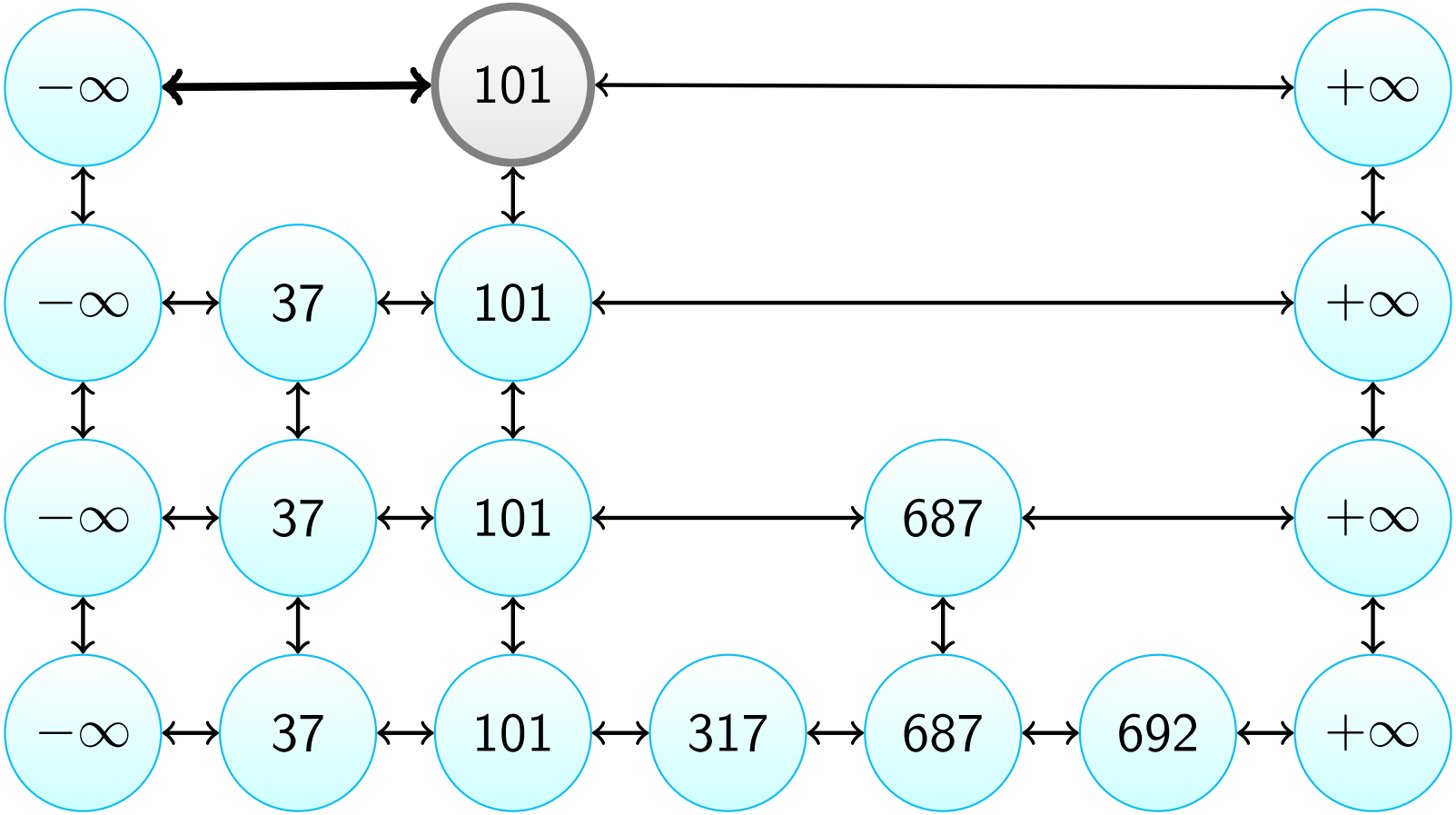

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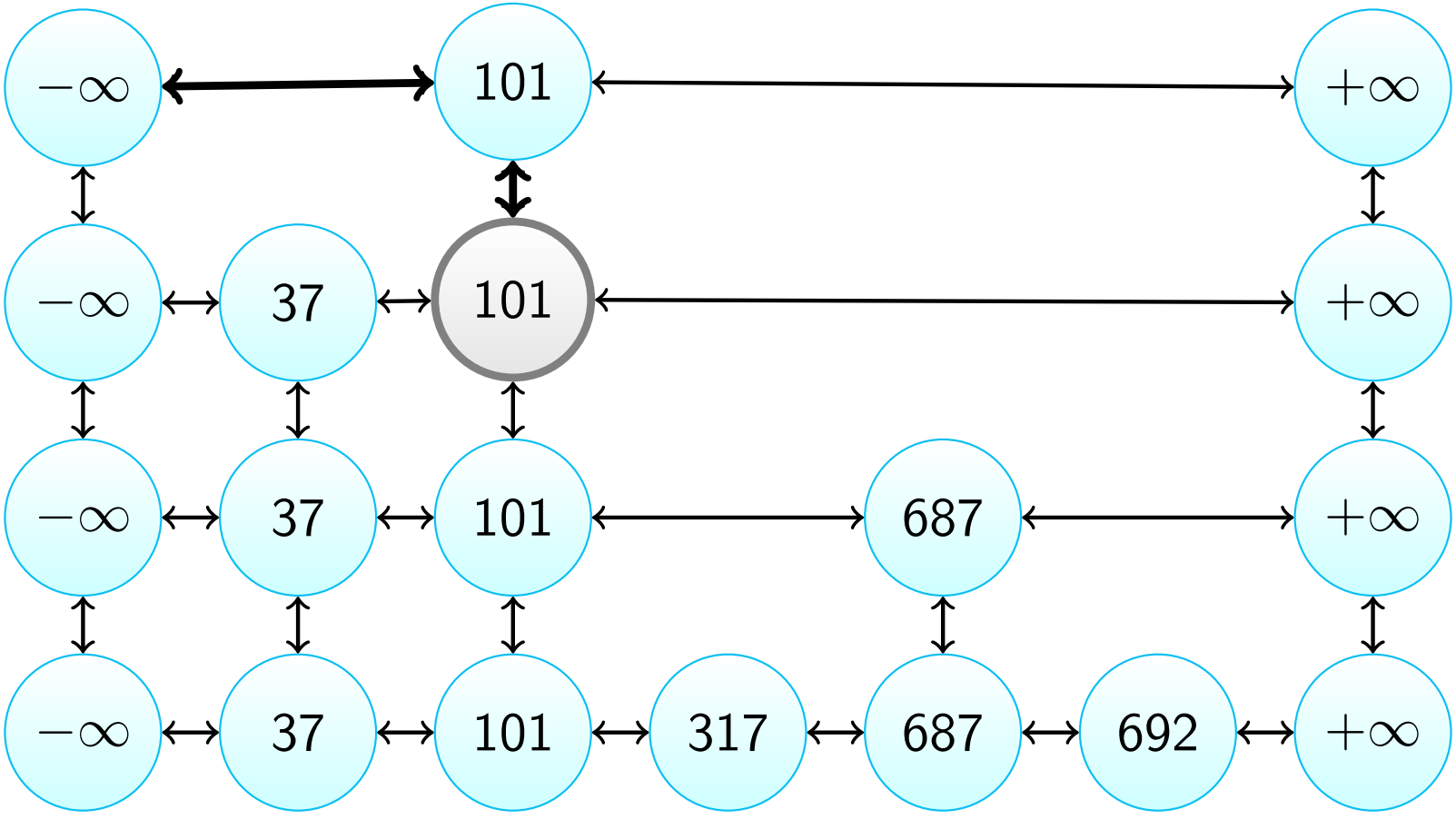

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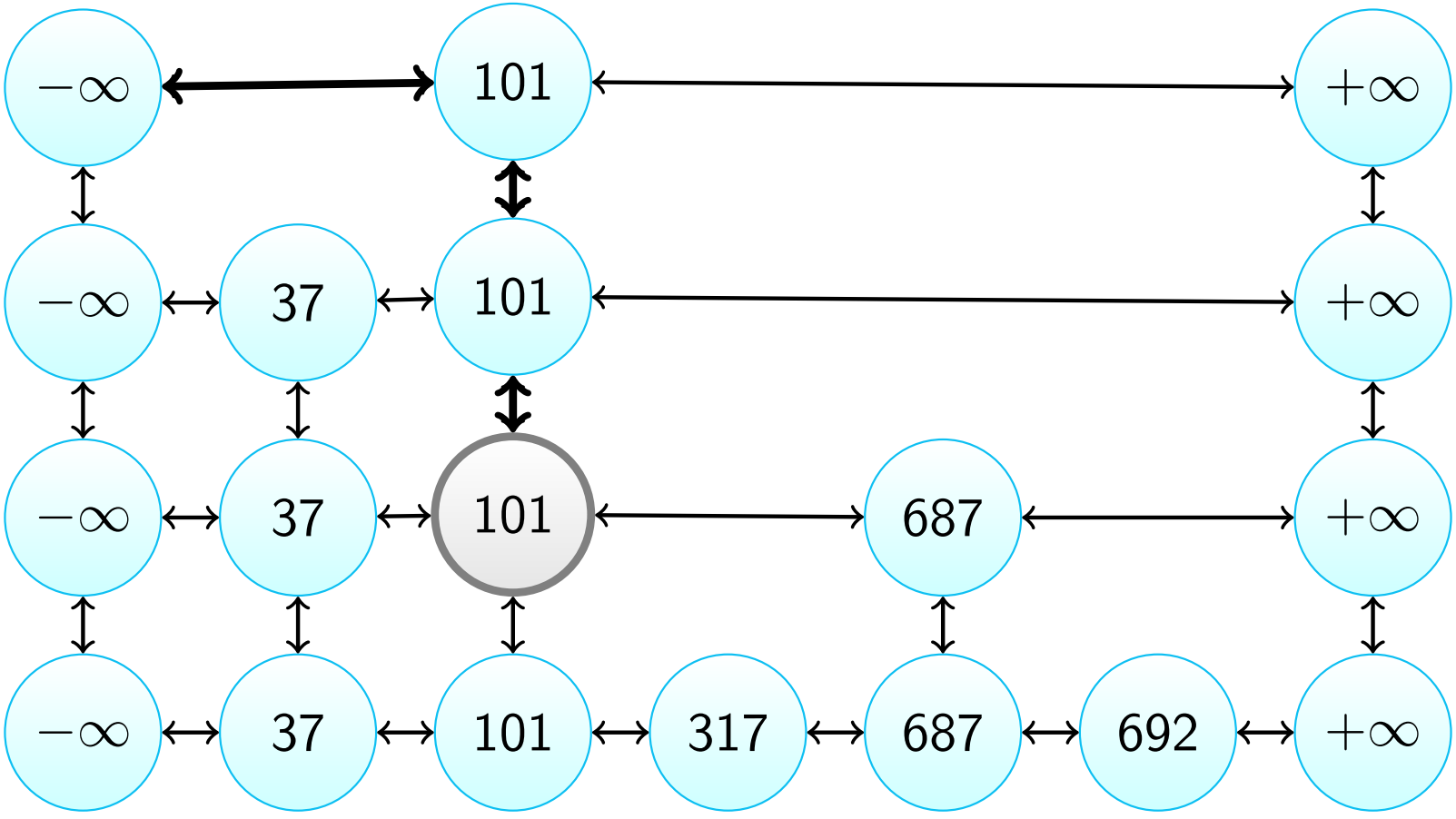

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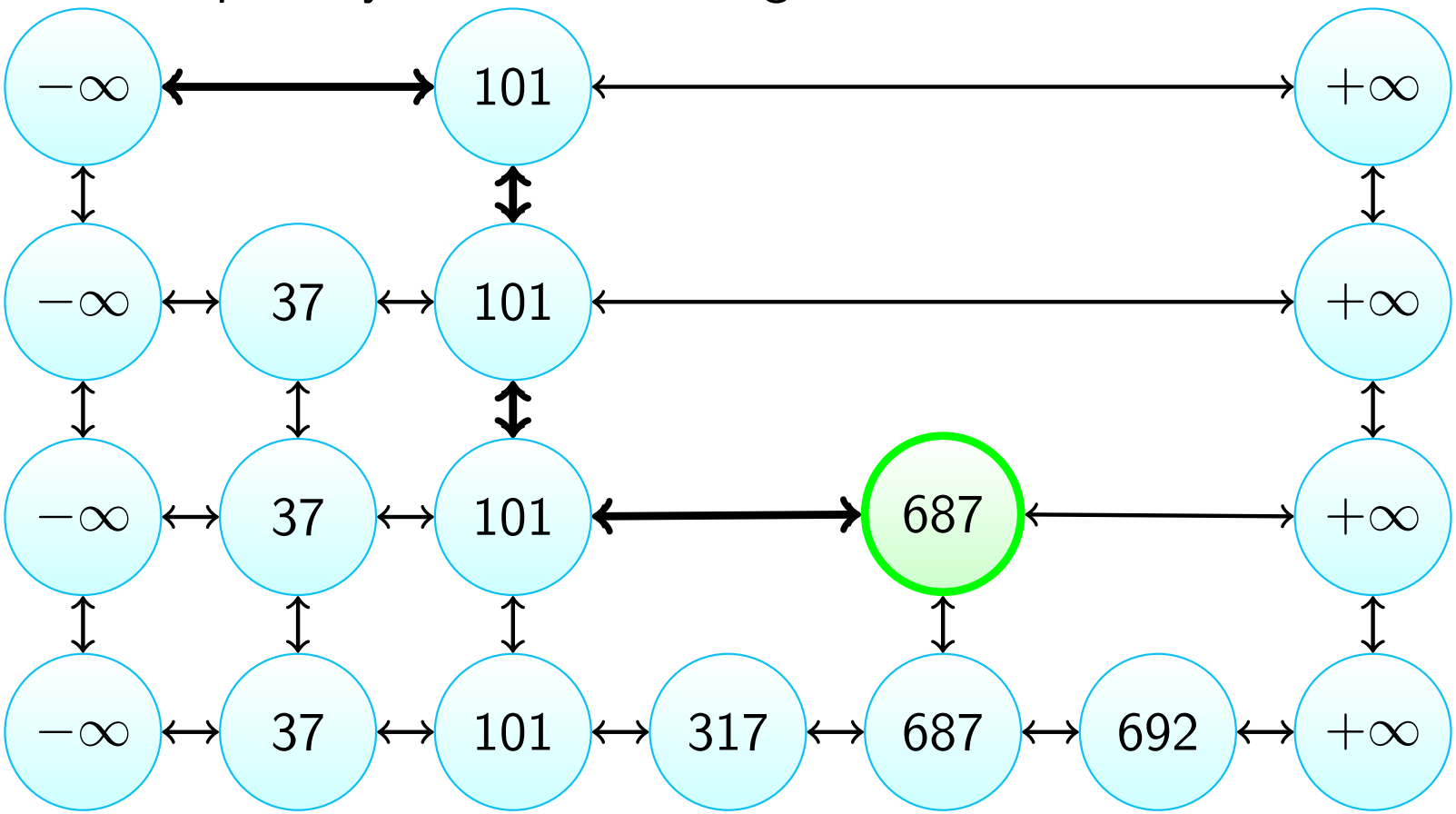
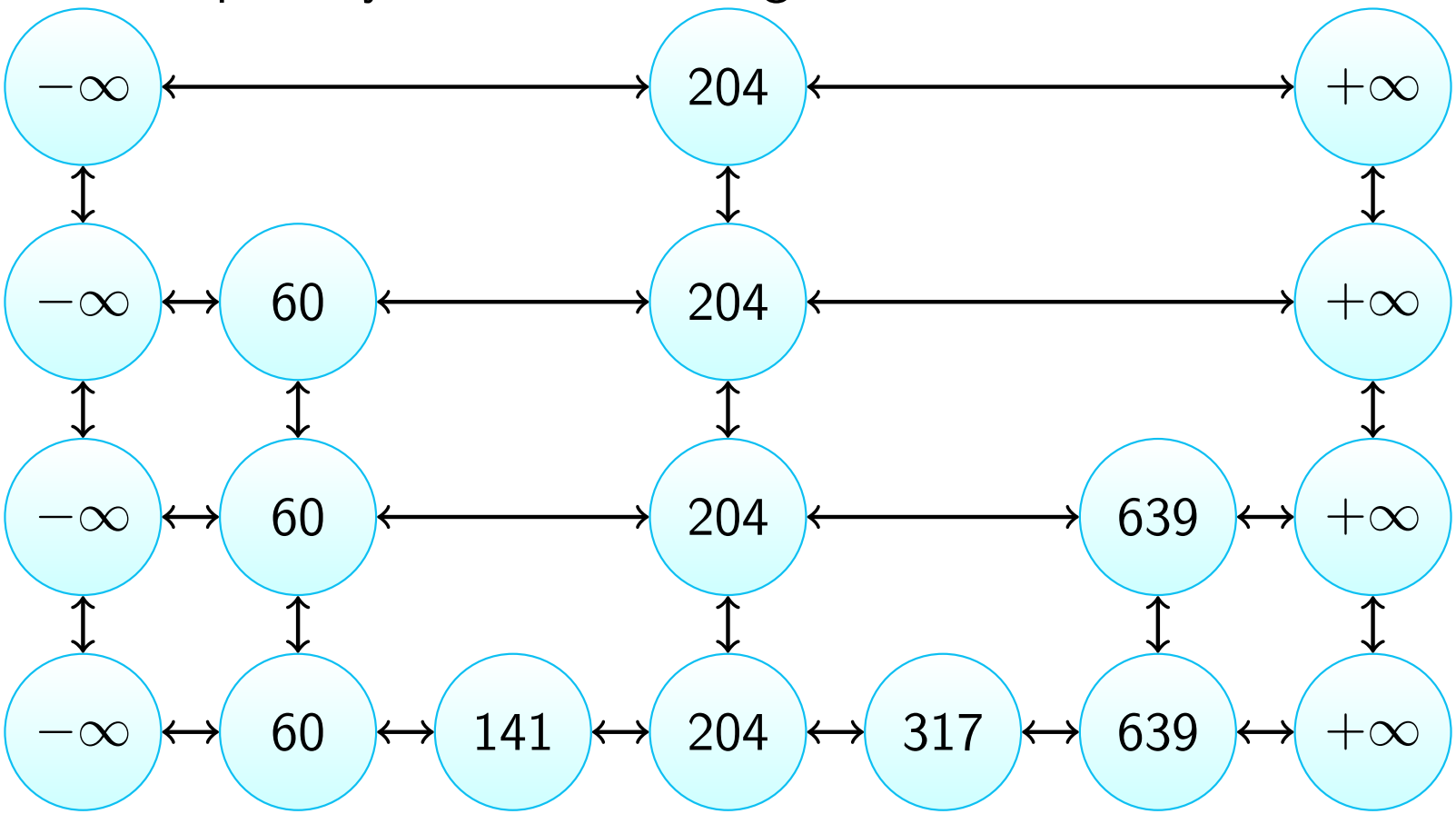


Searching

For example, if you were searching for 141:



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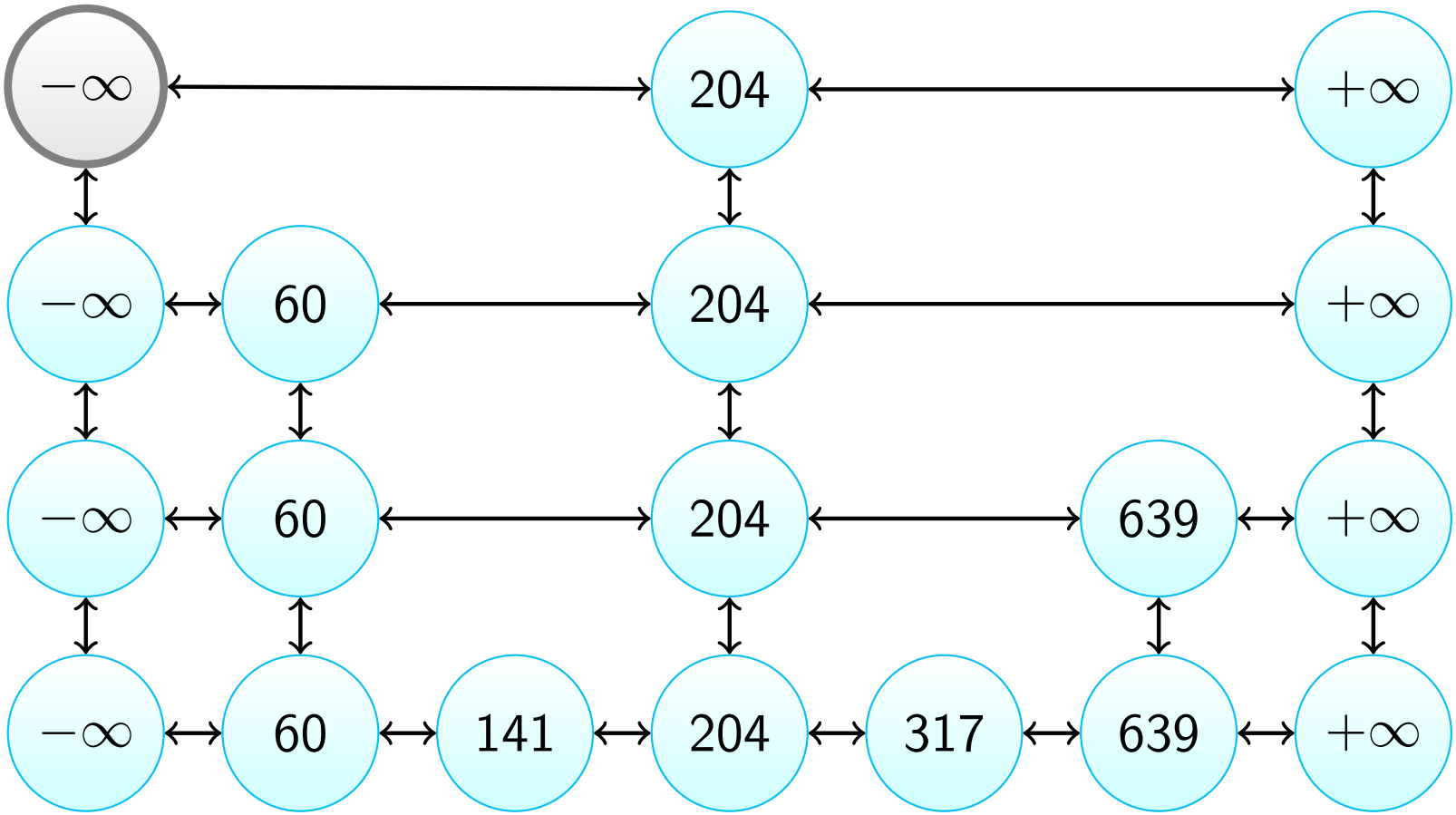

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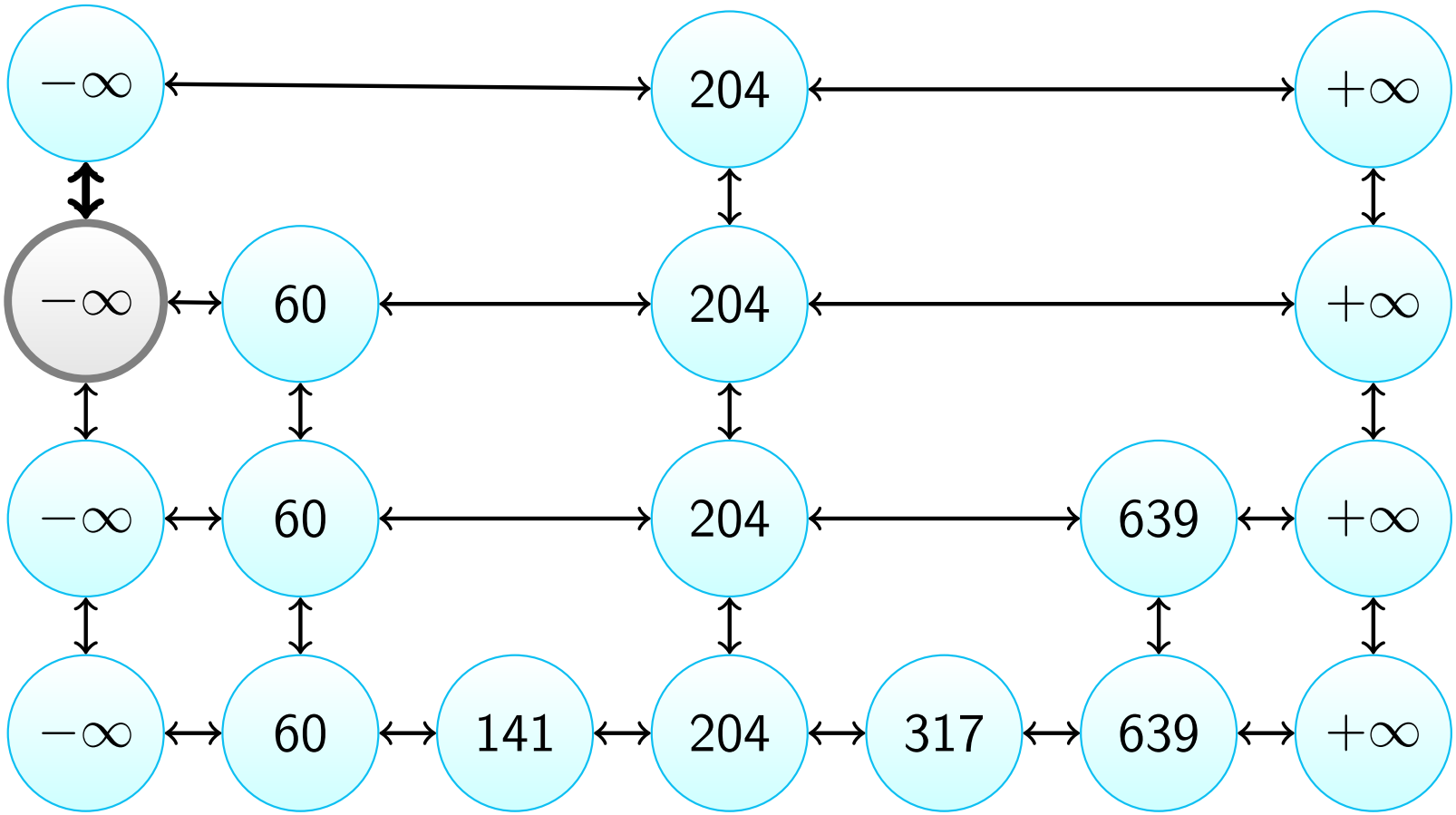

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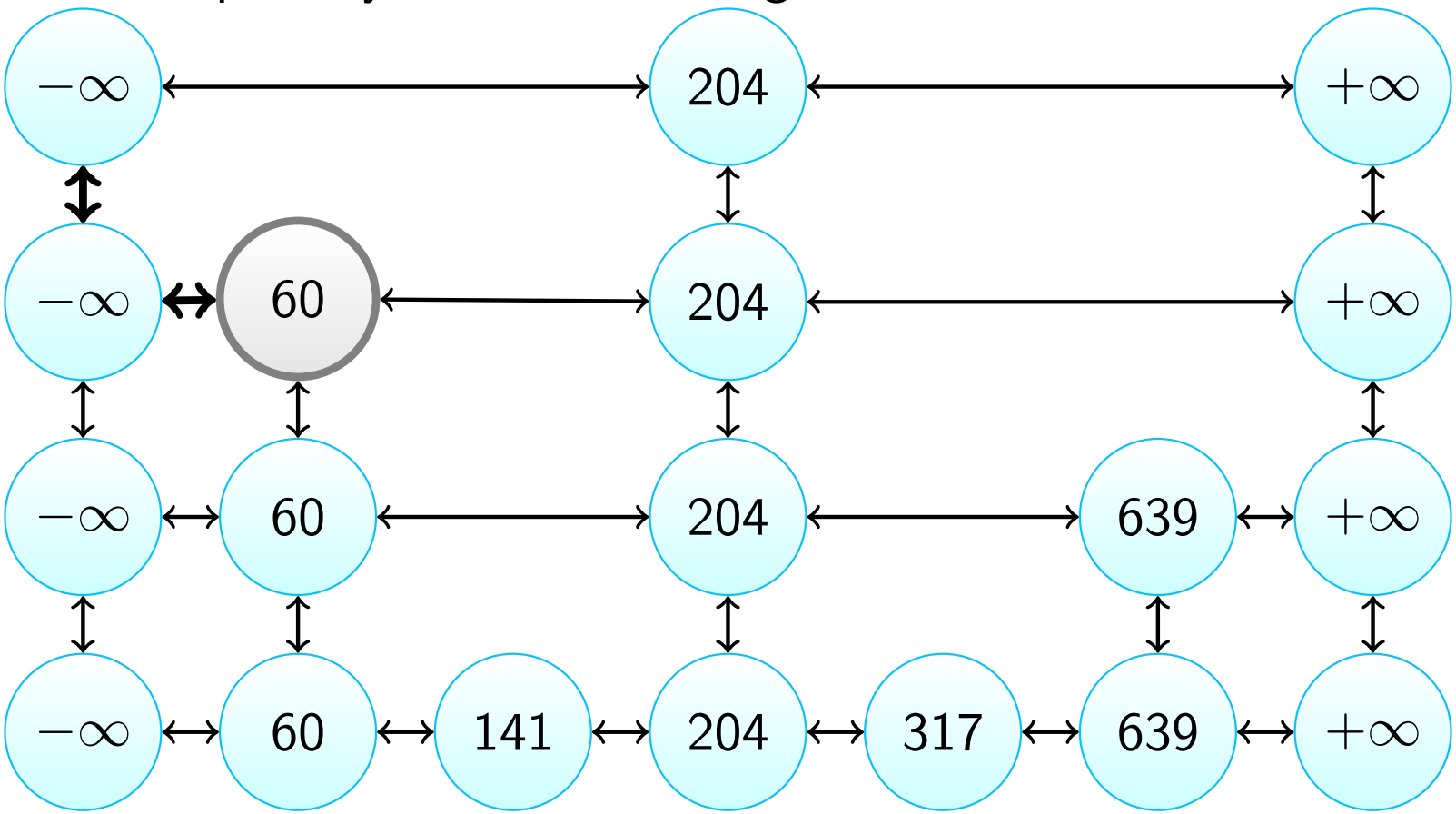

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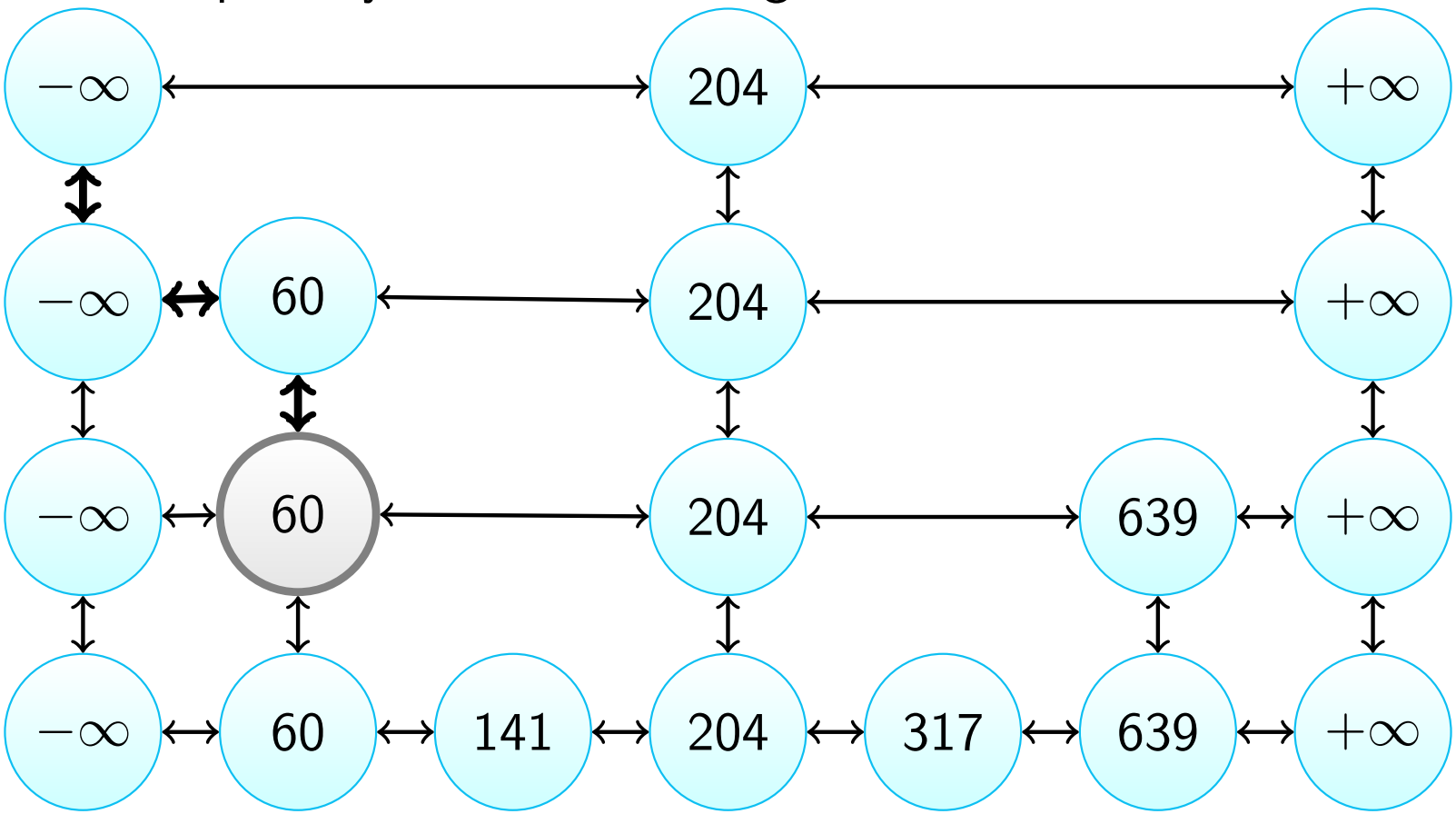

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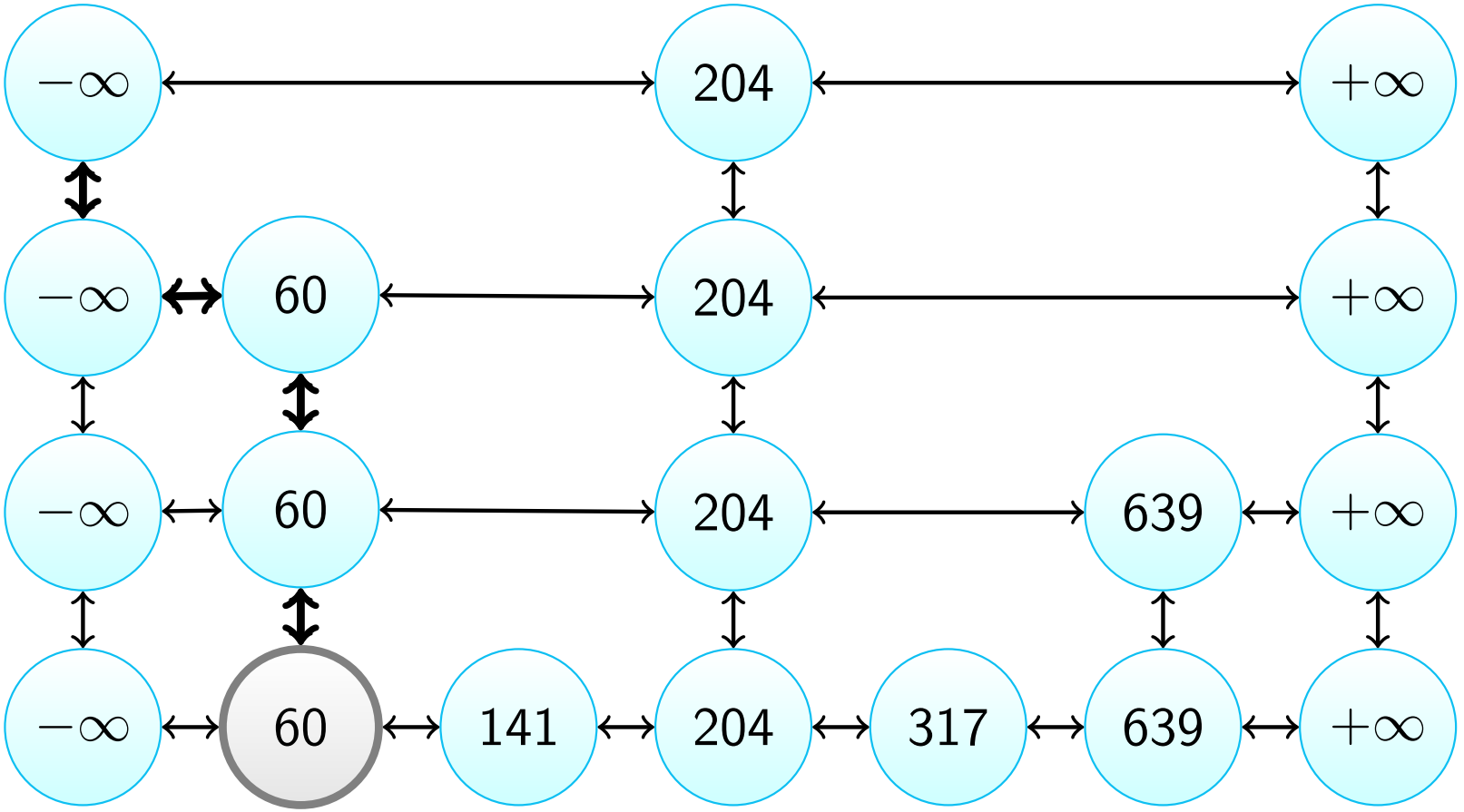

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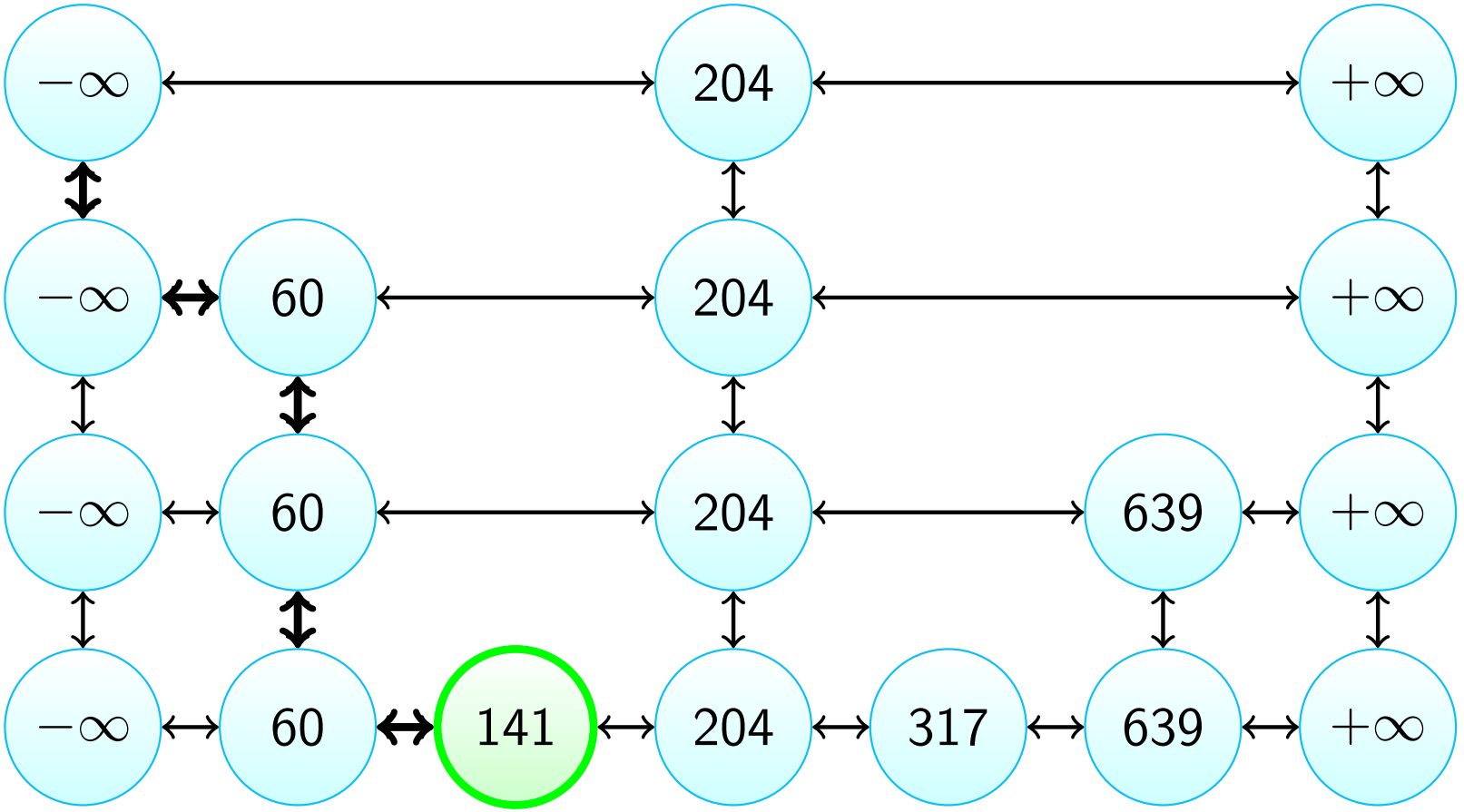

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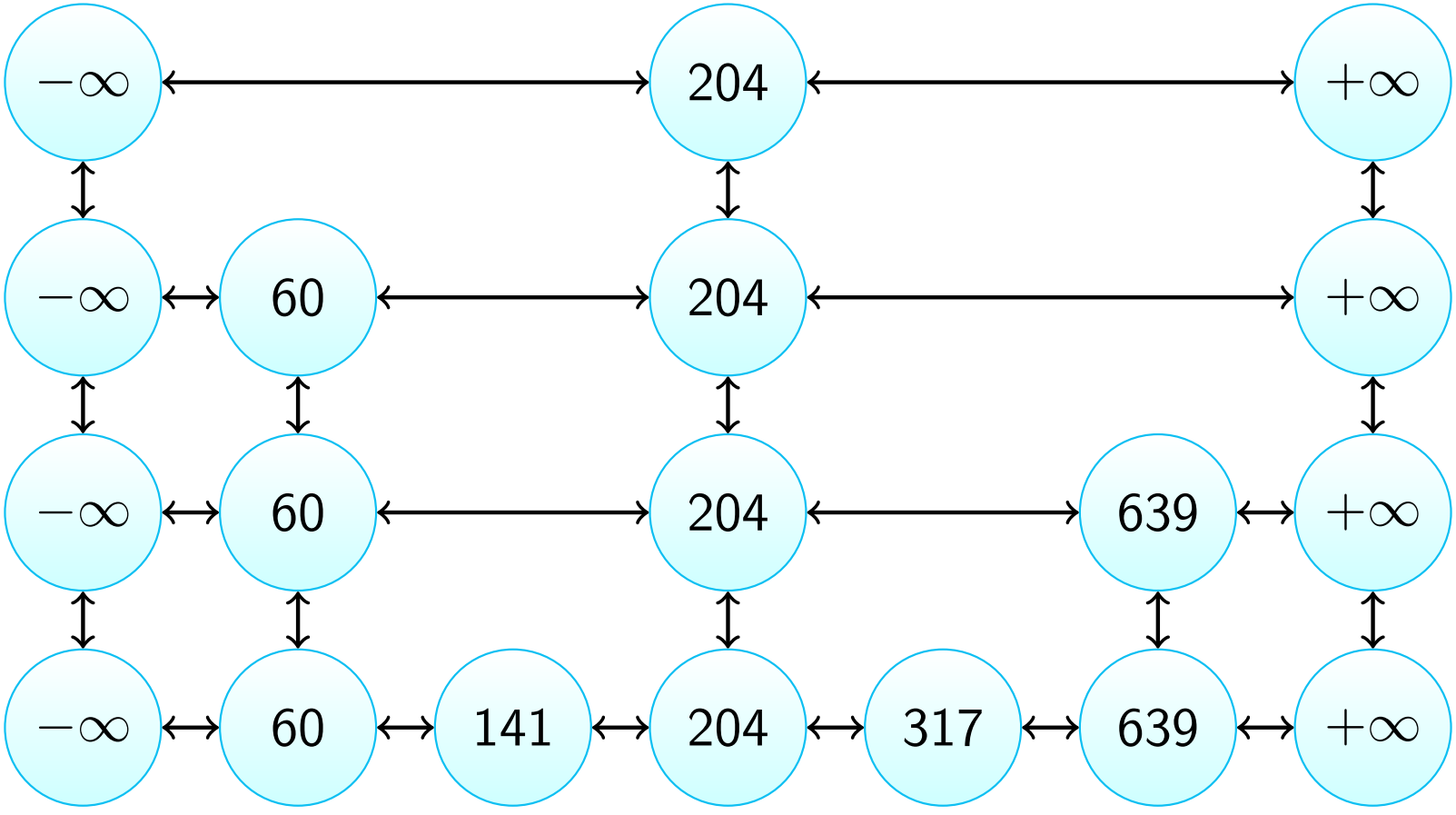

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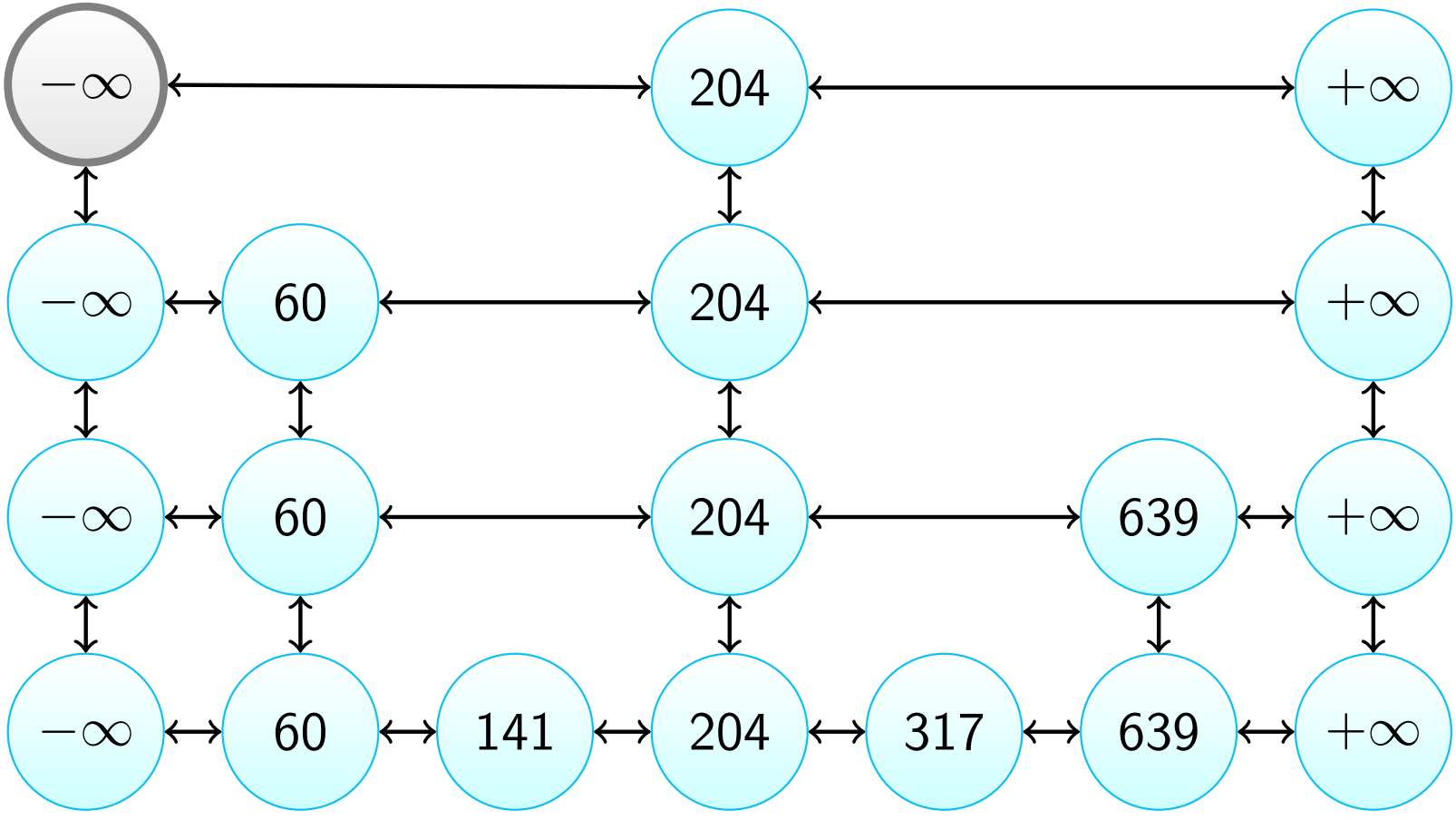

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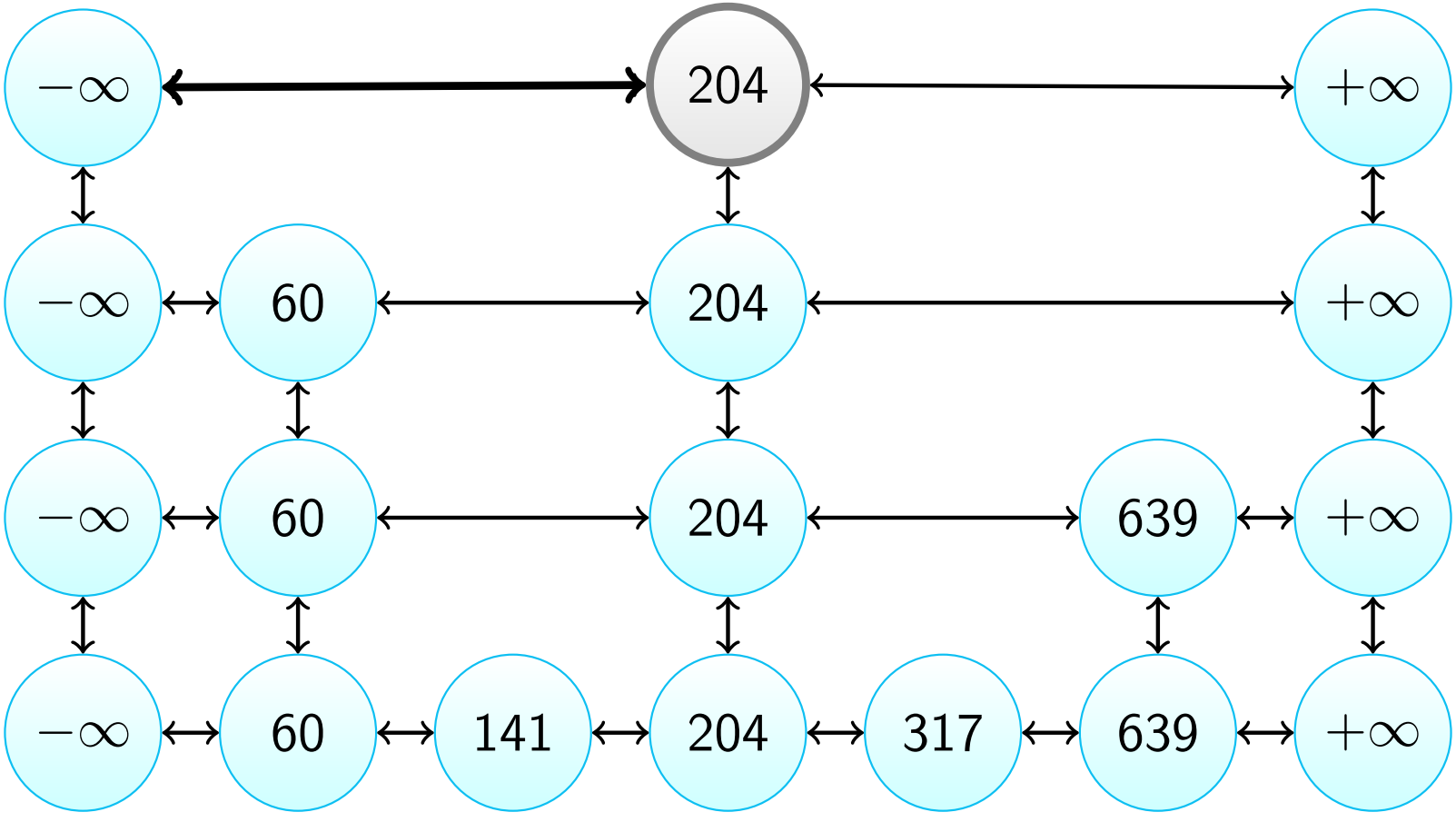

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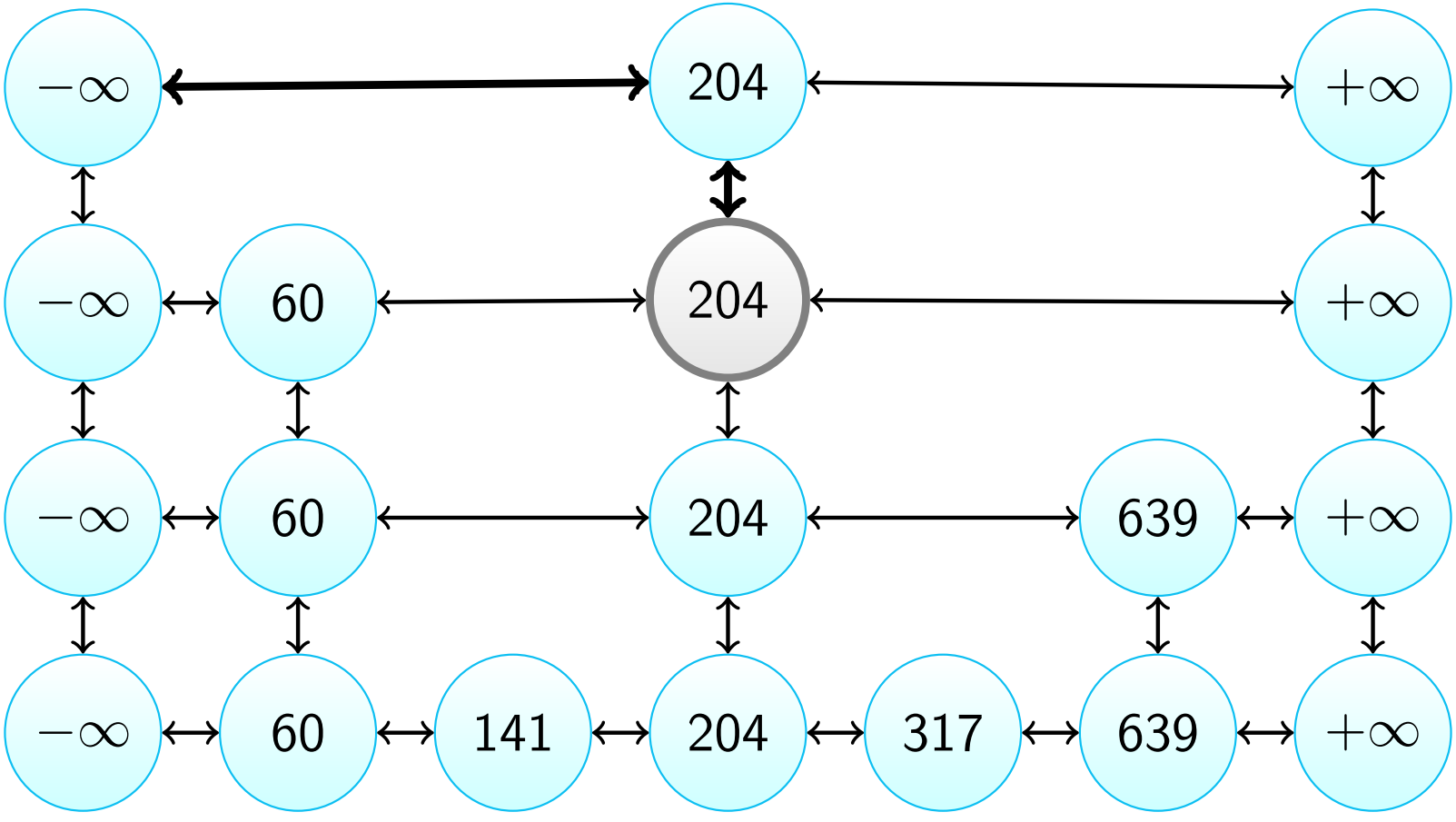

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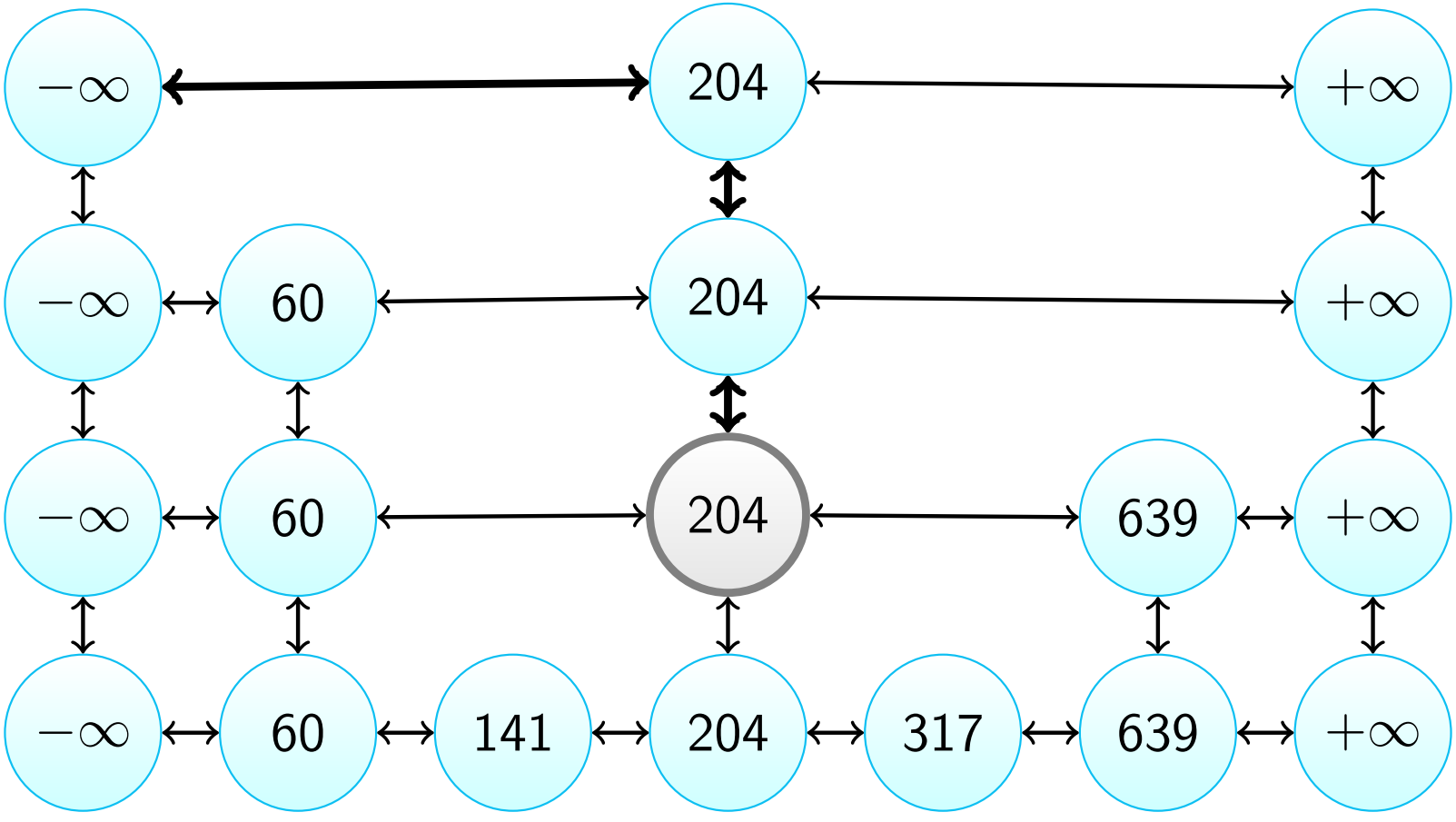

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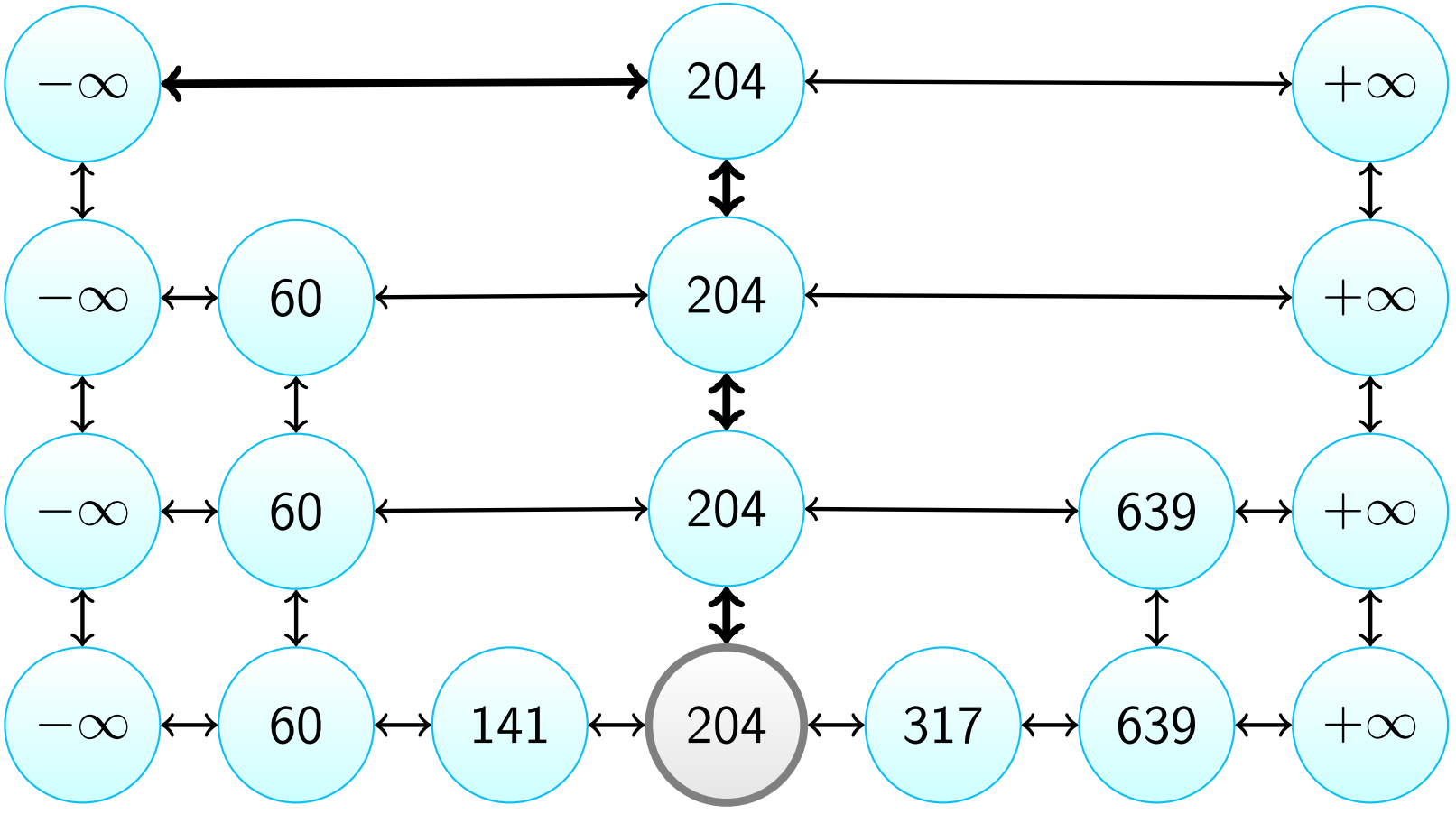

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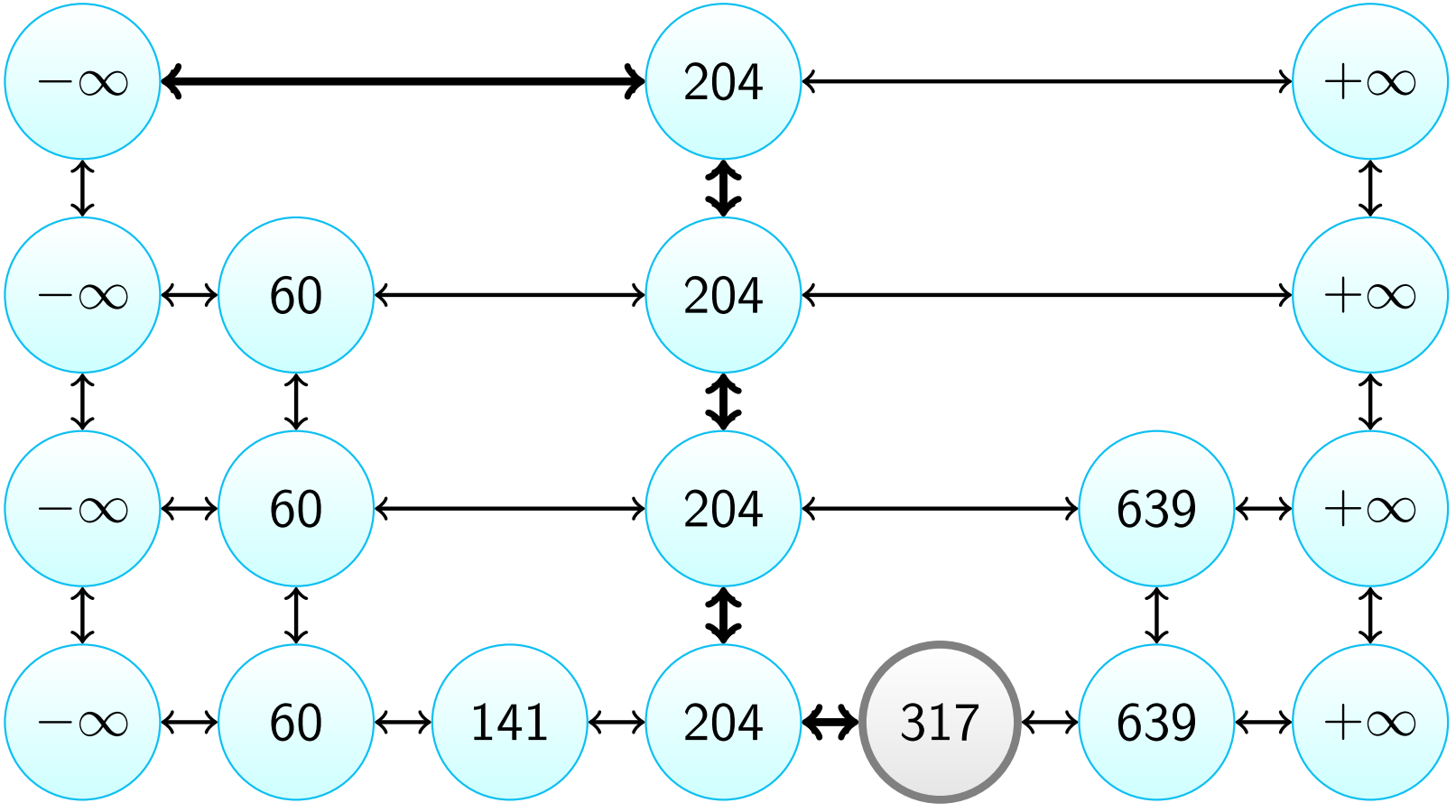

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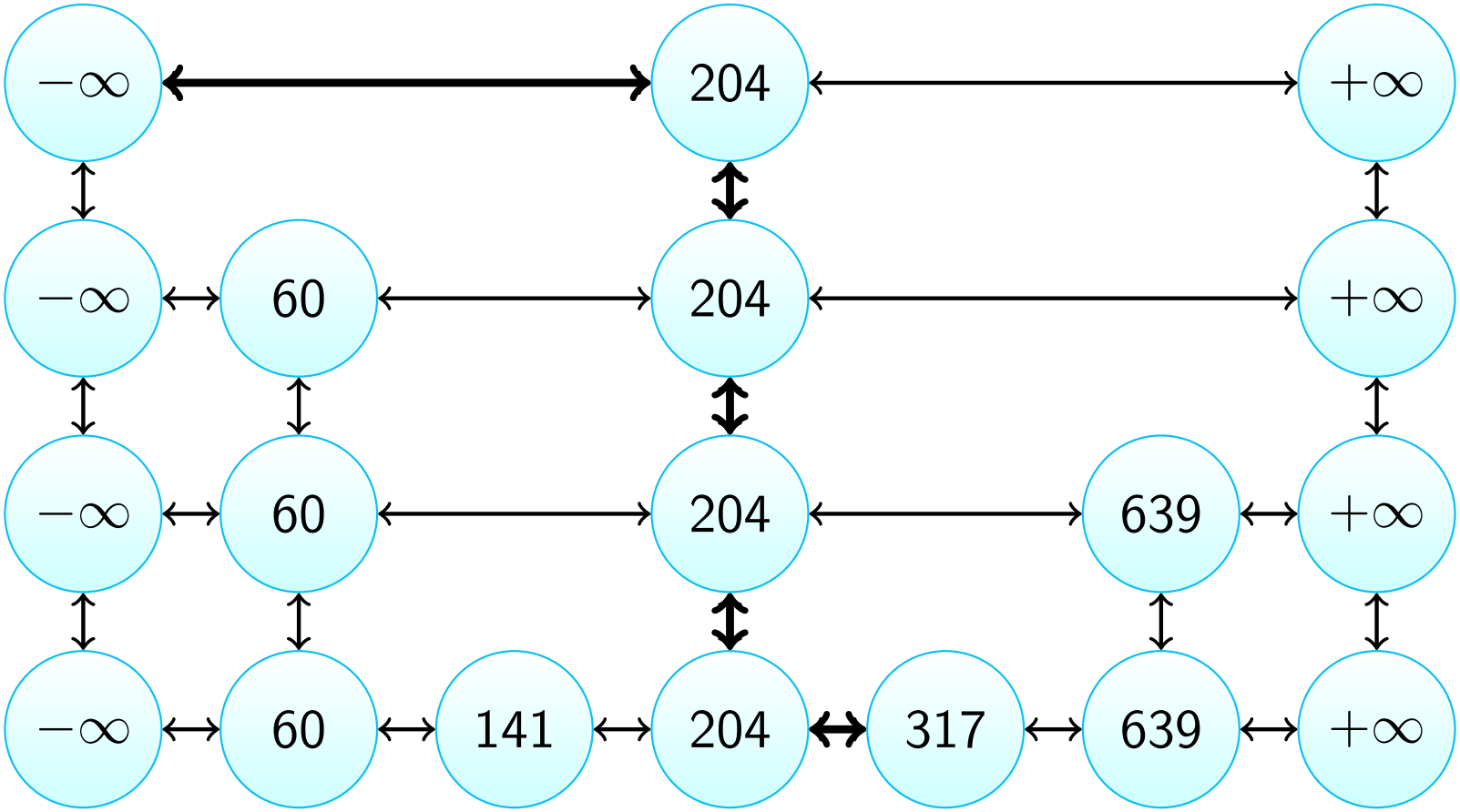

405 is not in the skip list.

Searching

```
procedure SEARCH(data, node)
    if node is not valid then
        return FALSE
    else
        while data > node.next.data do
            node ← node.next
        end while
        if data = node.next.data then
            return TRUE
        else
            return SEARCH(data, node.down)
        end if
    end if
end procedure
```

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- Repeat the previous step until you reach the bottom level.

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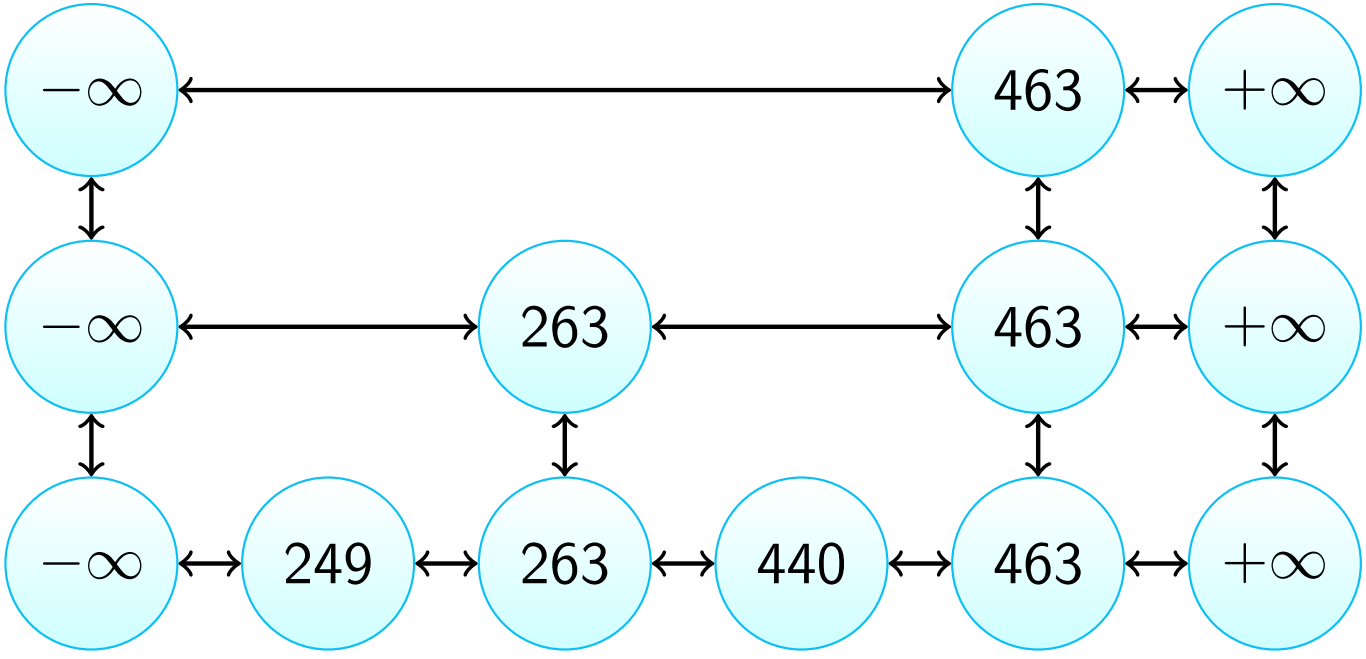
- Repeat the previous step until you reach the bottom level.
- Handling of duplicate items is implementation-defined.

Adding

For example, if you were adding 310, and the coin flipper gave HHHTTTHTTHHTTTT:



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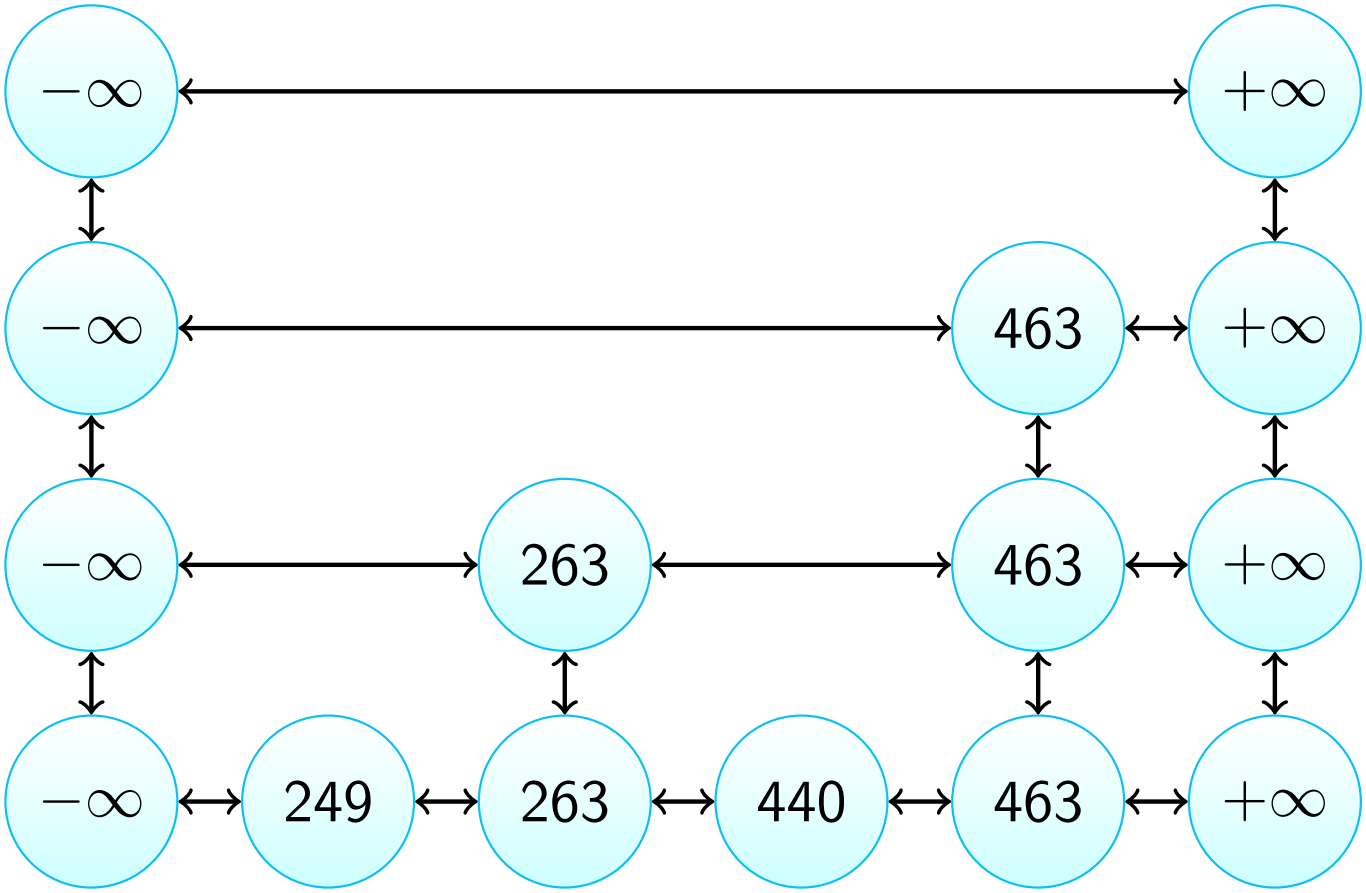

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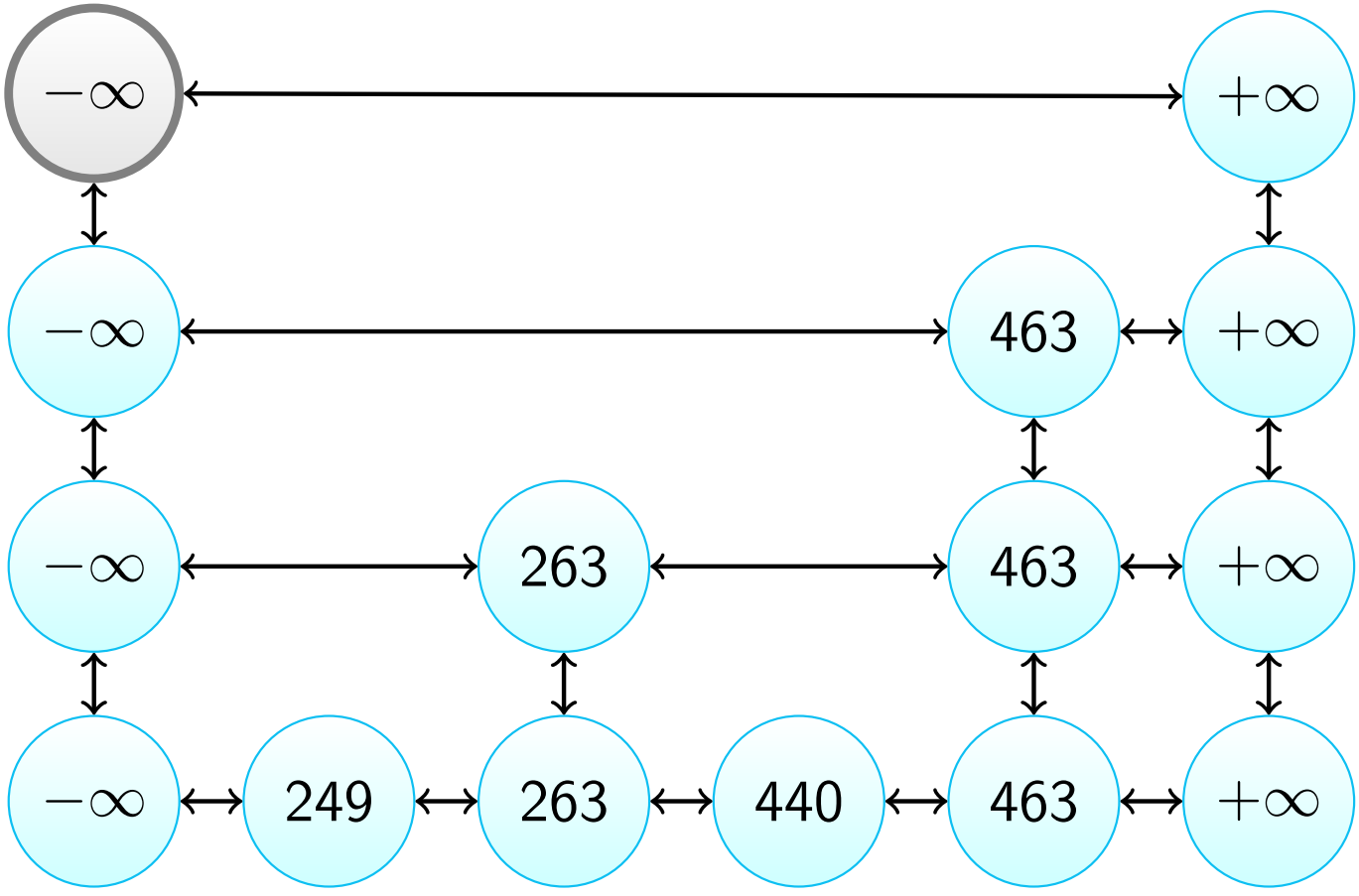

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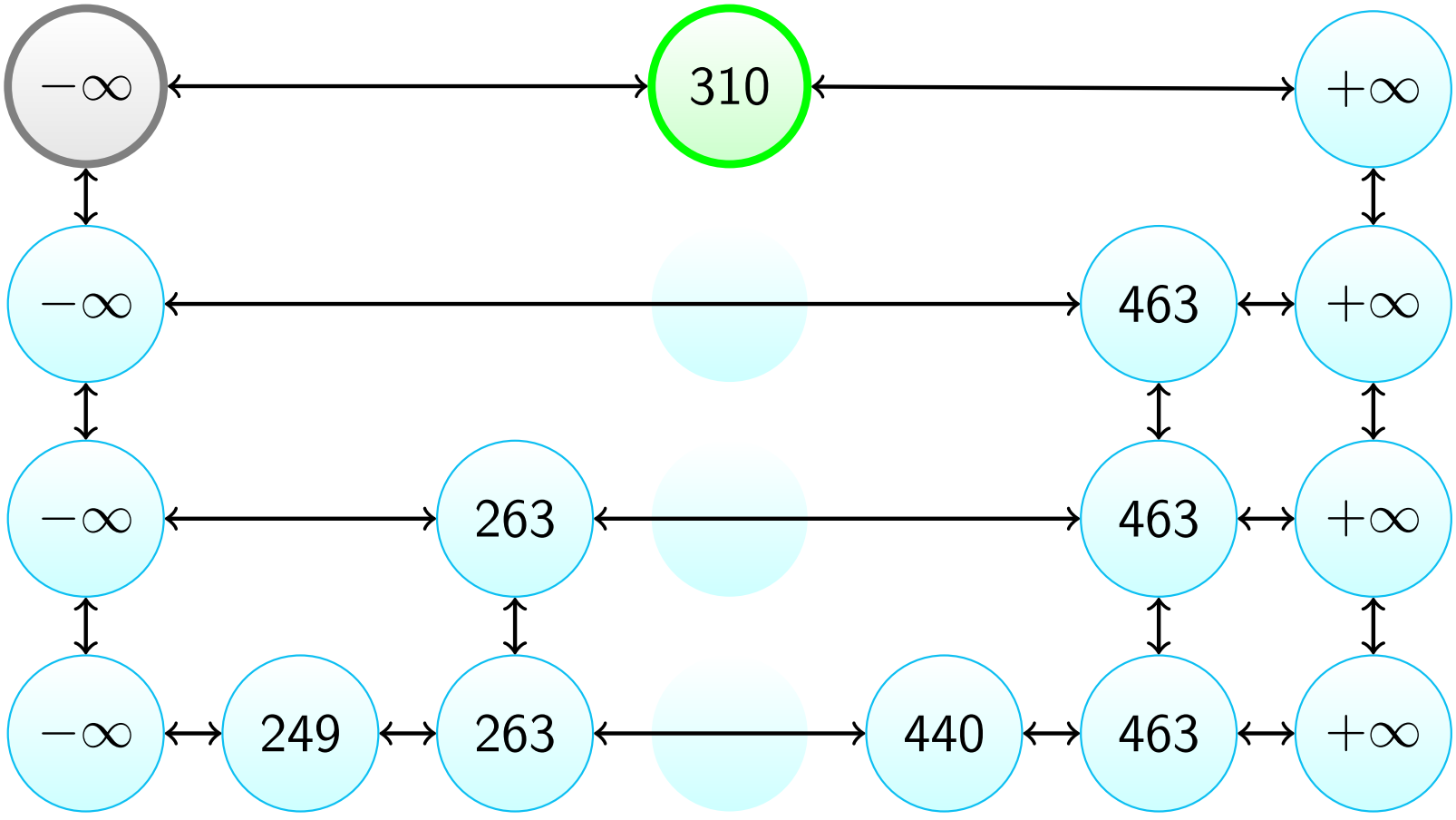

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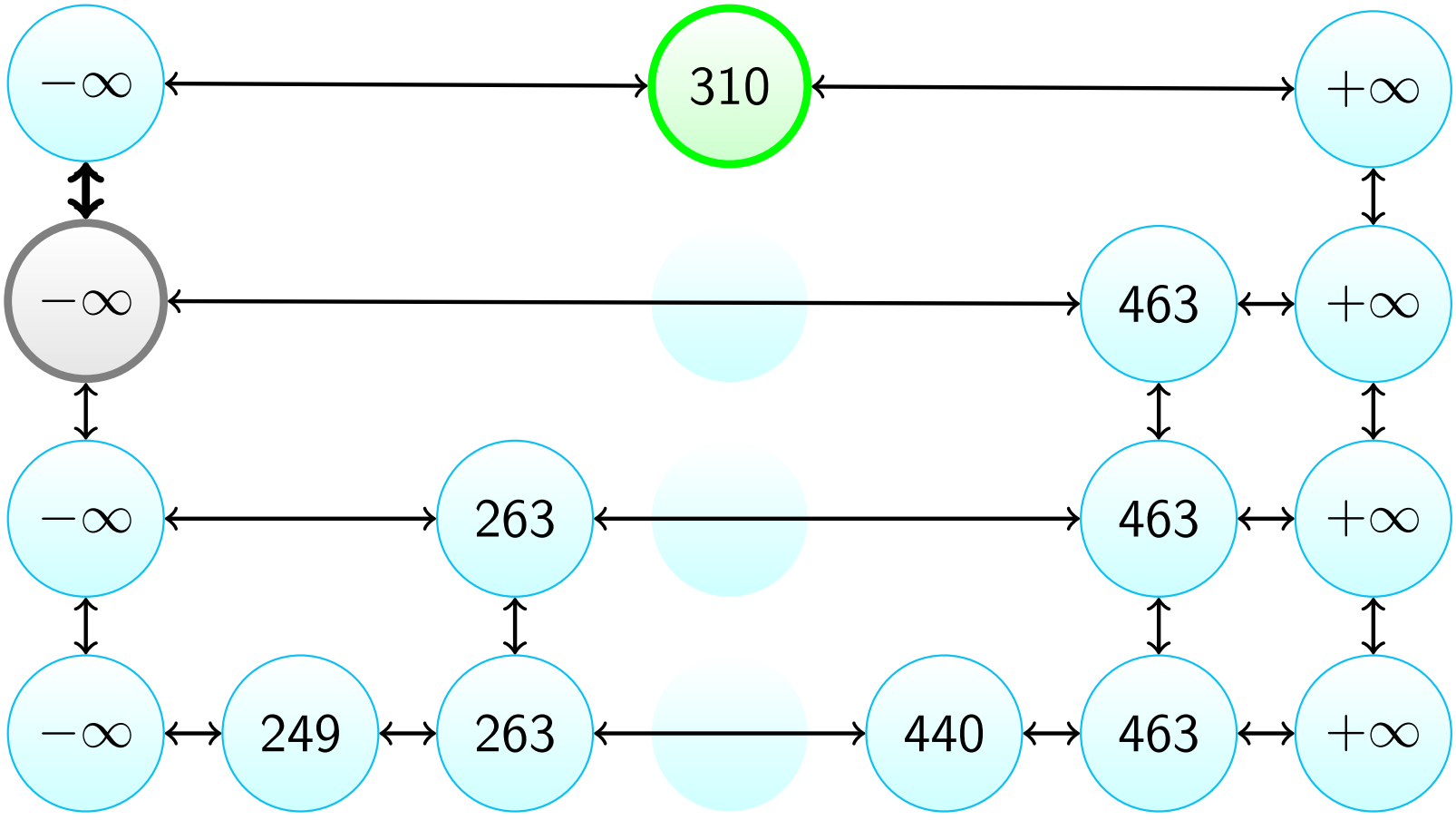

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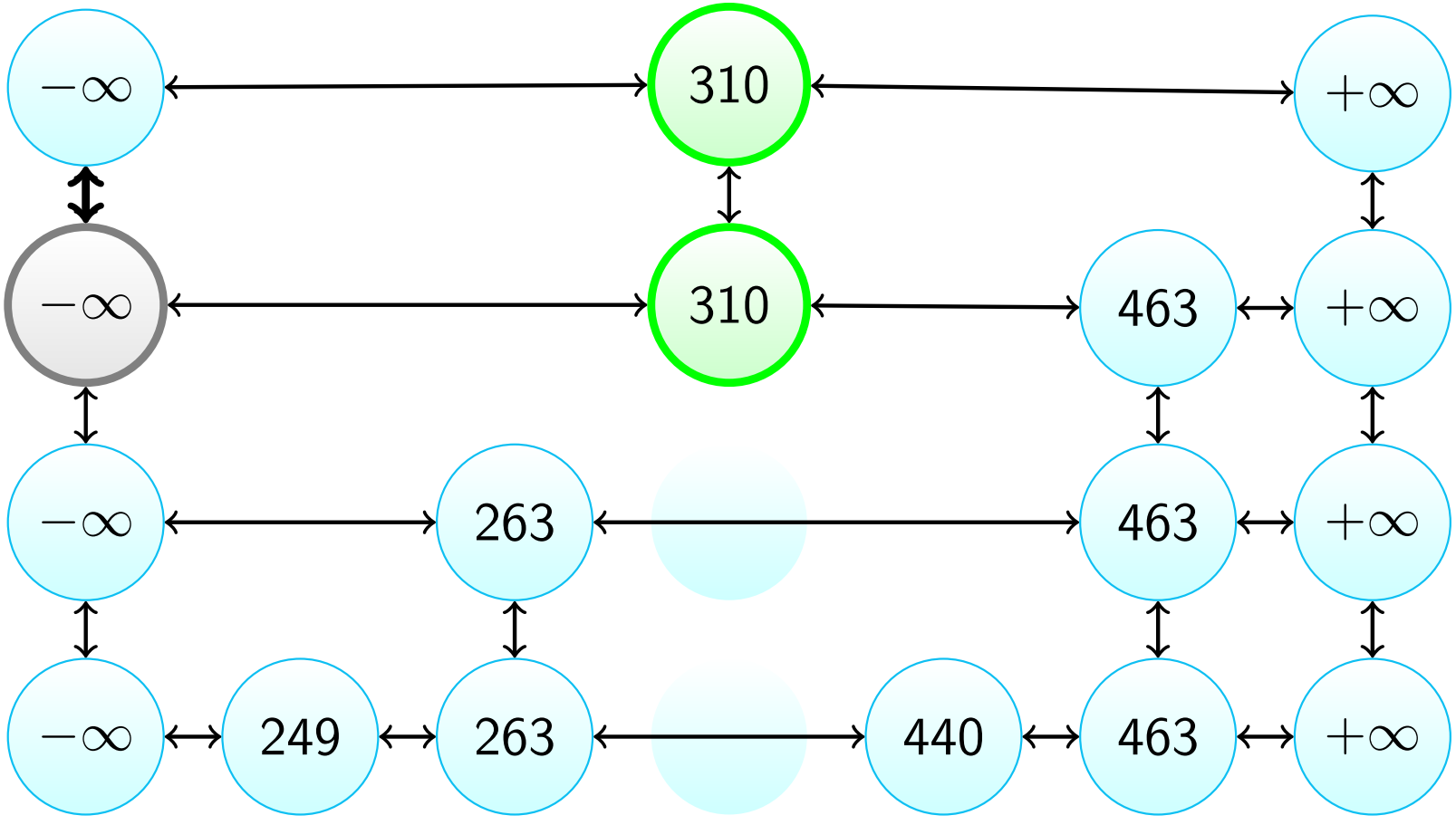

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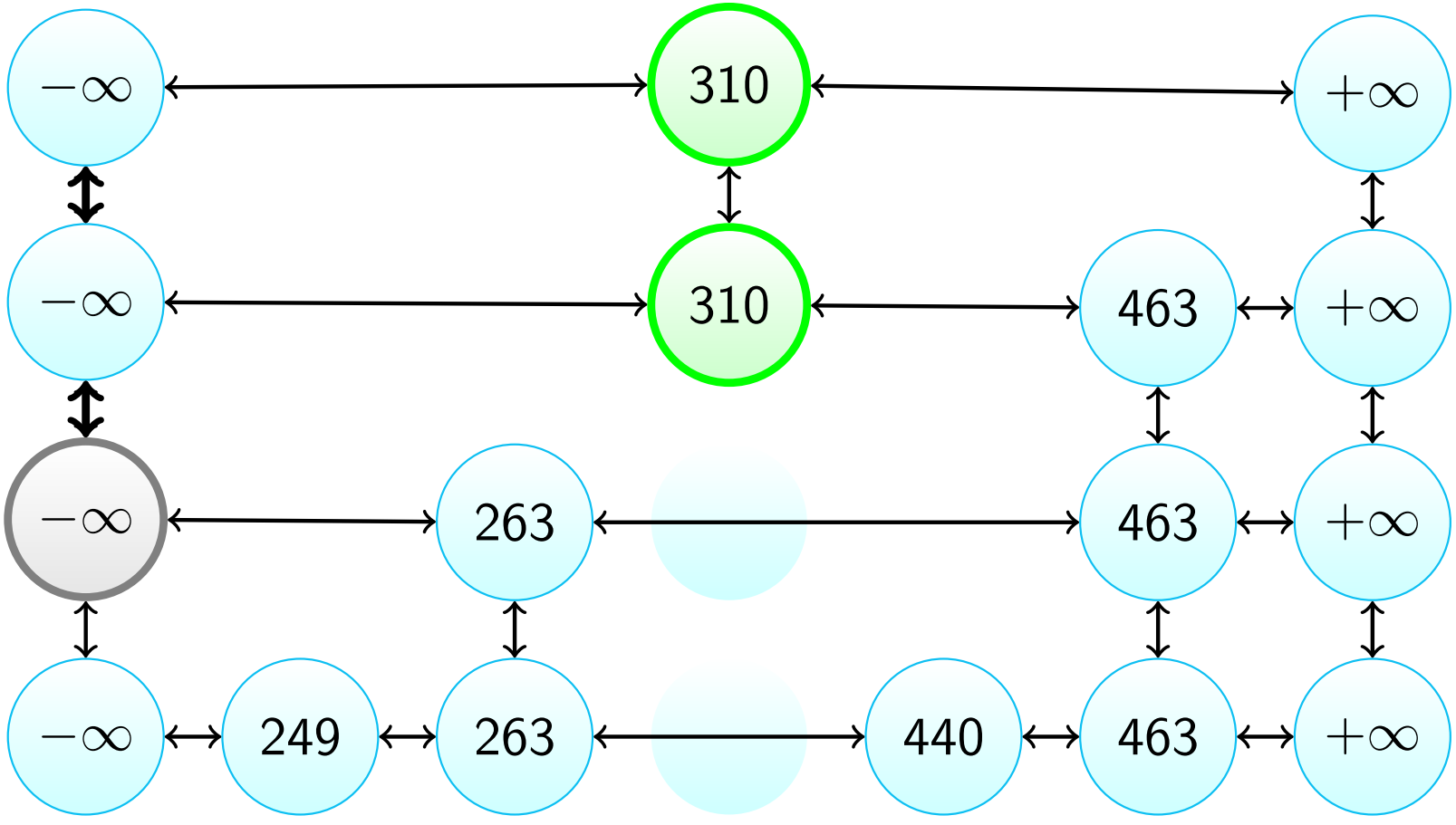

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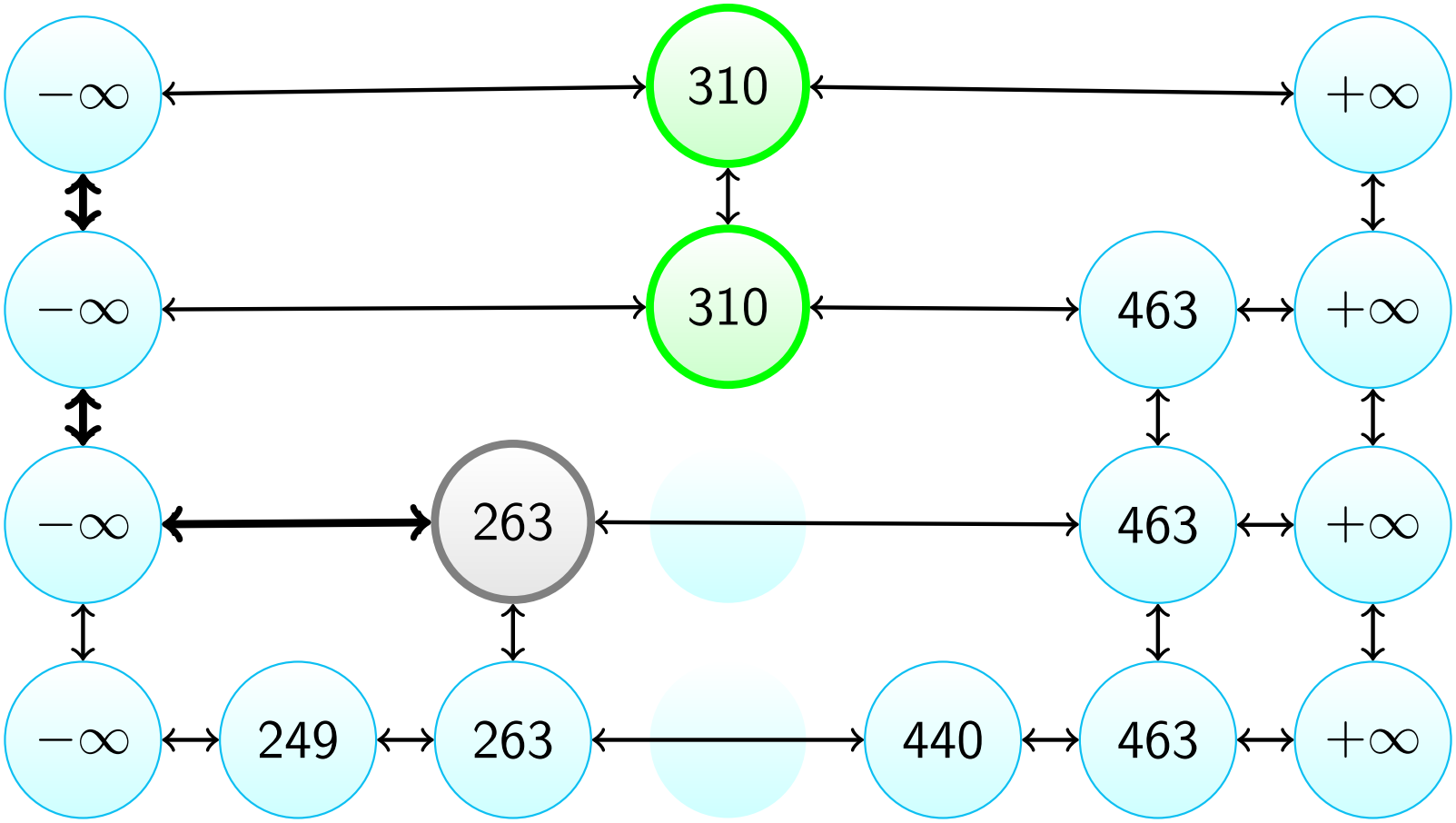

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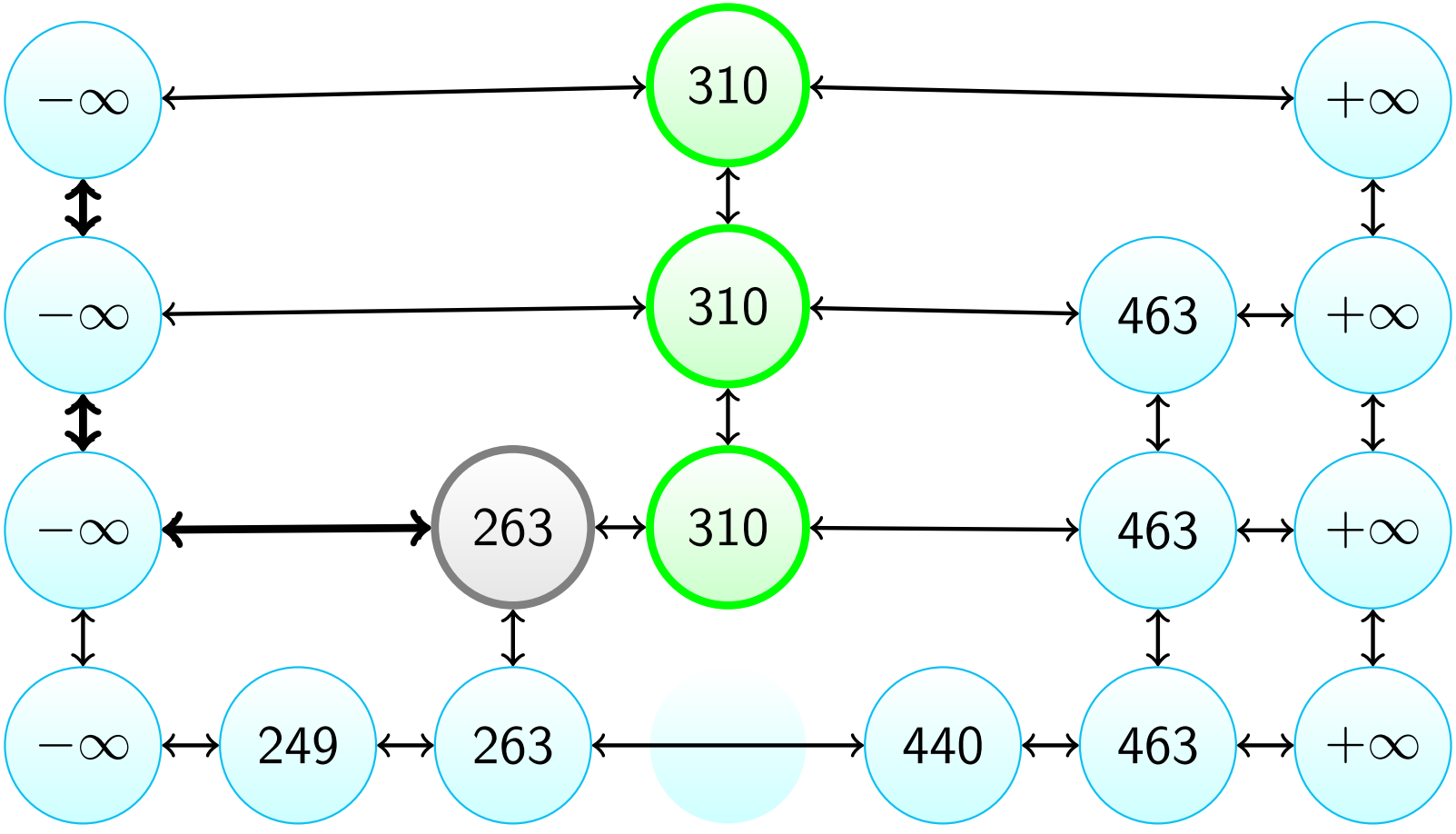

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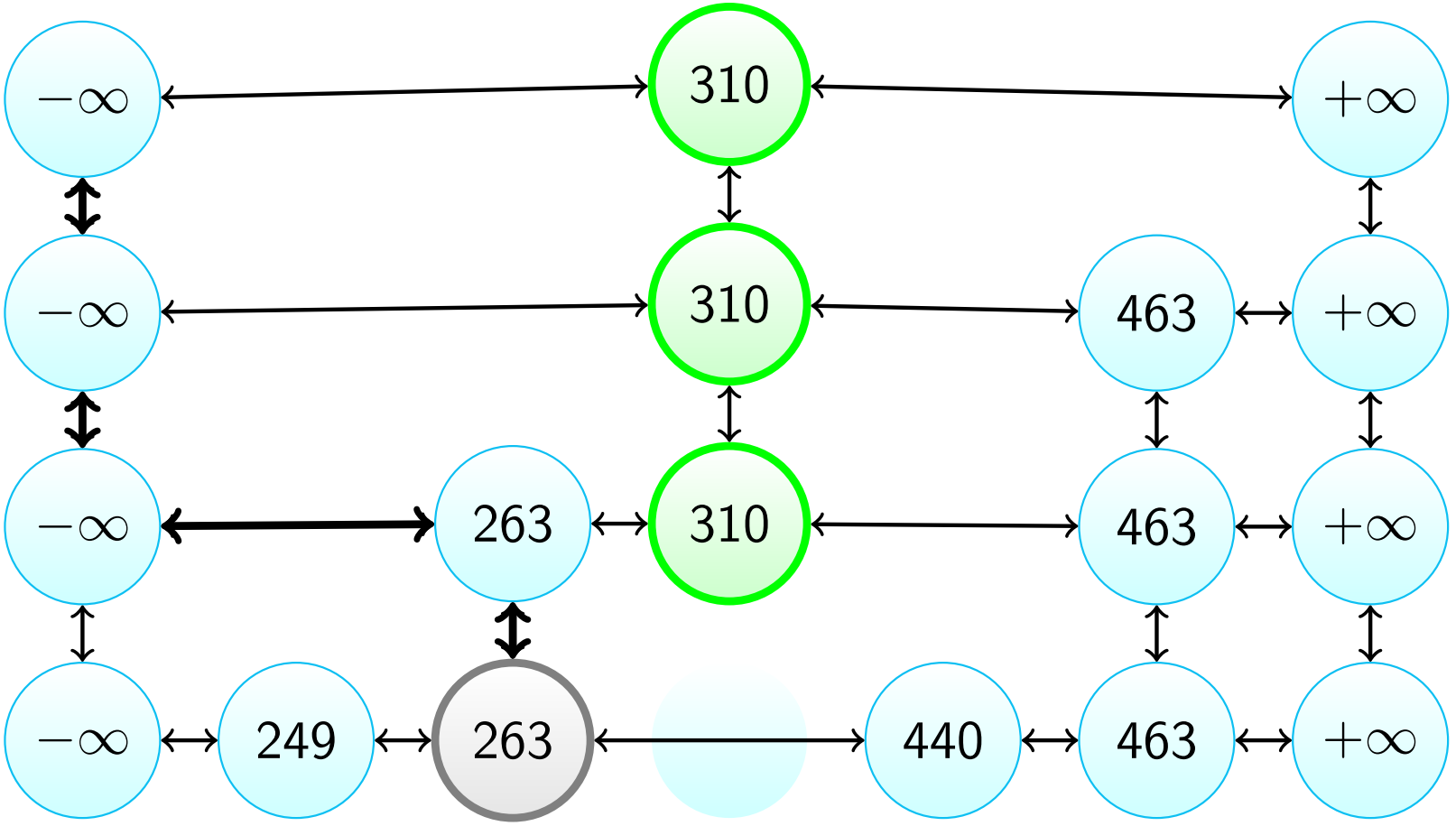

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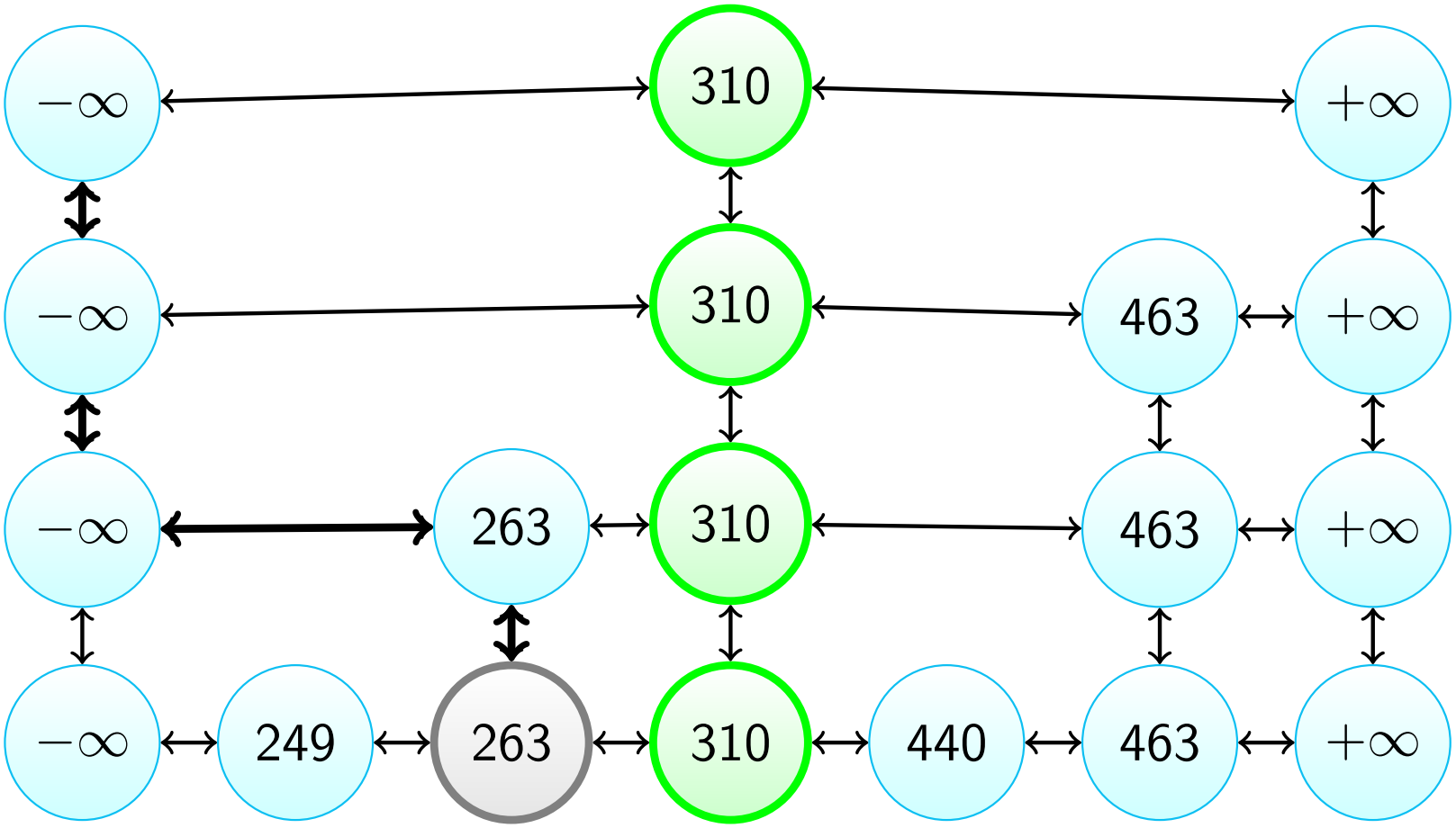

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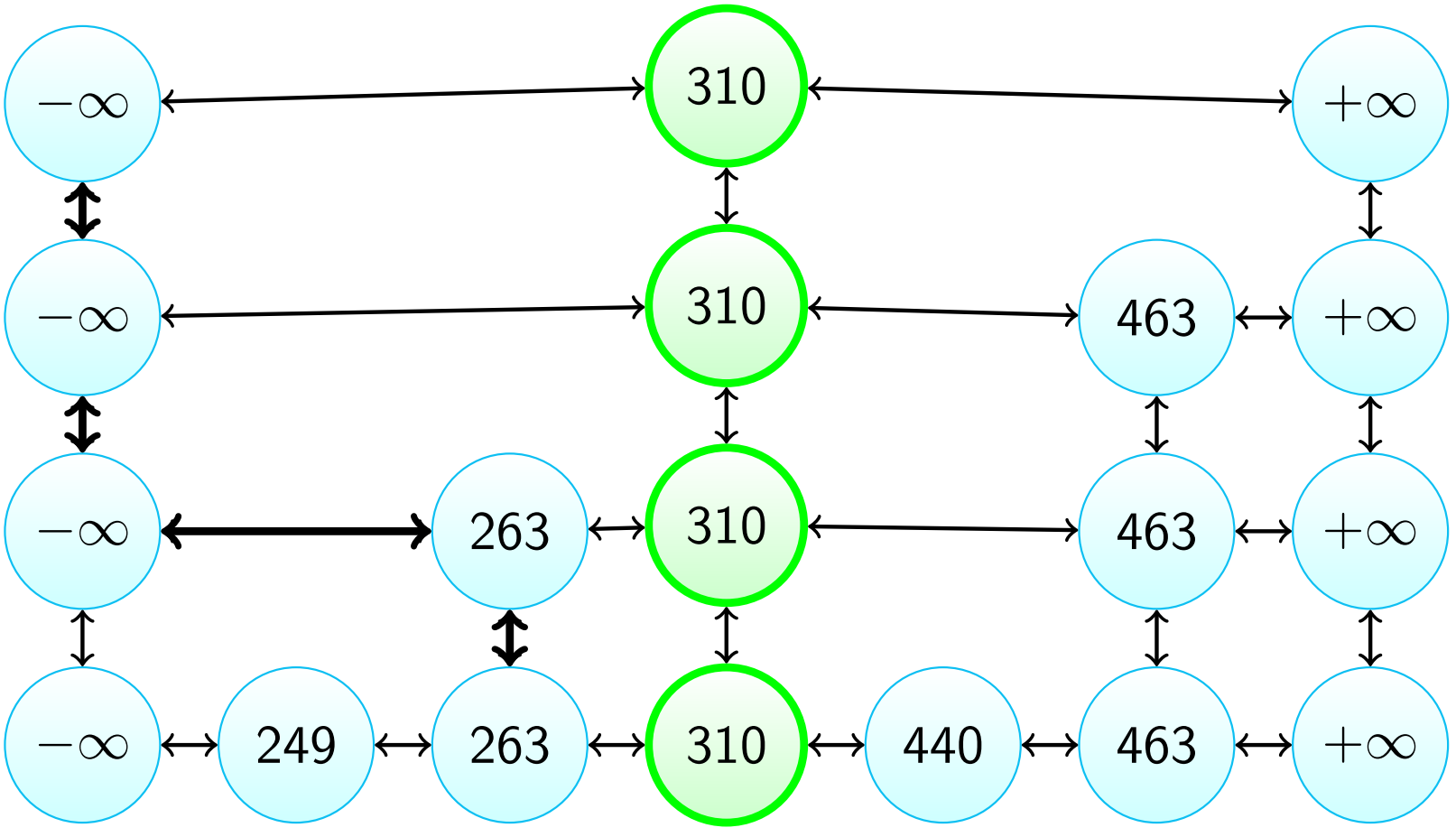
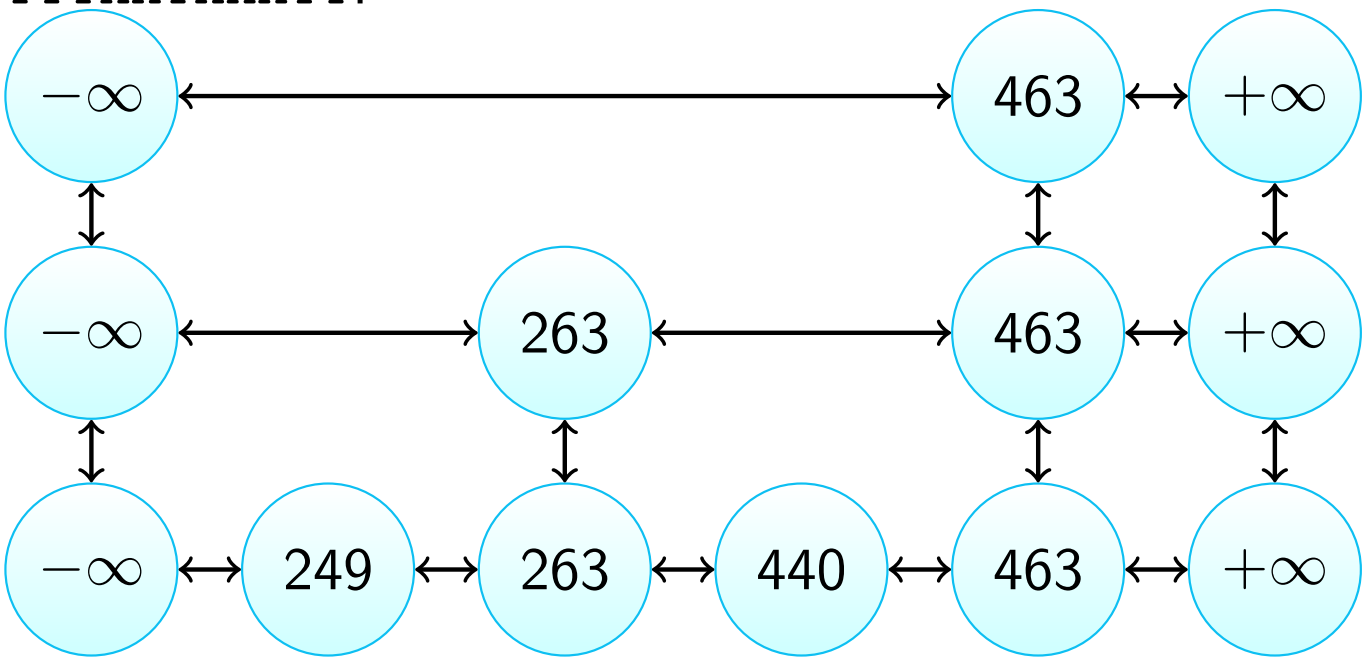


Adding

For example, if you were adding 118, and the coin flipper gave TTTHHTHHHTT:



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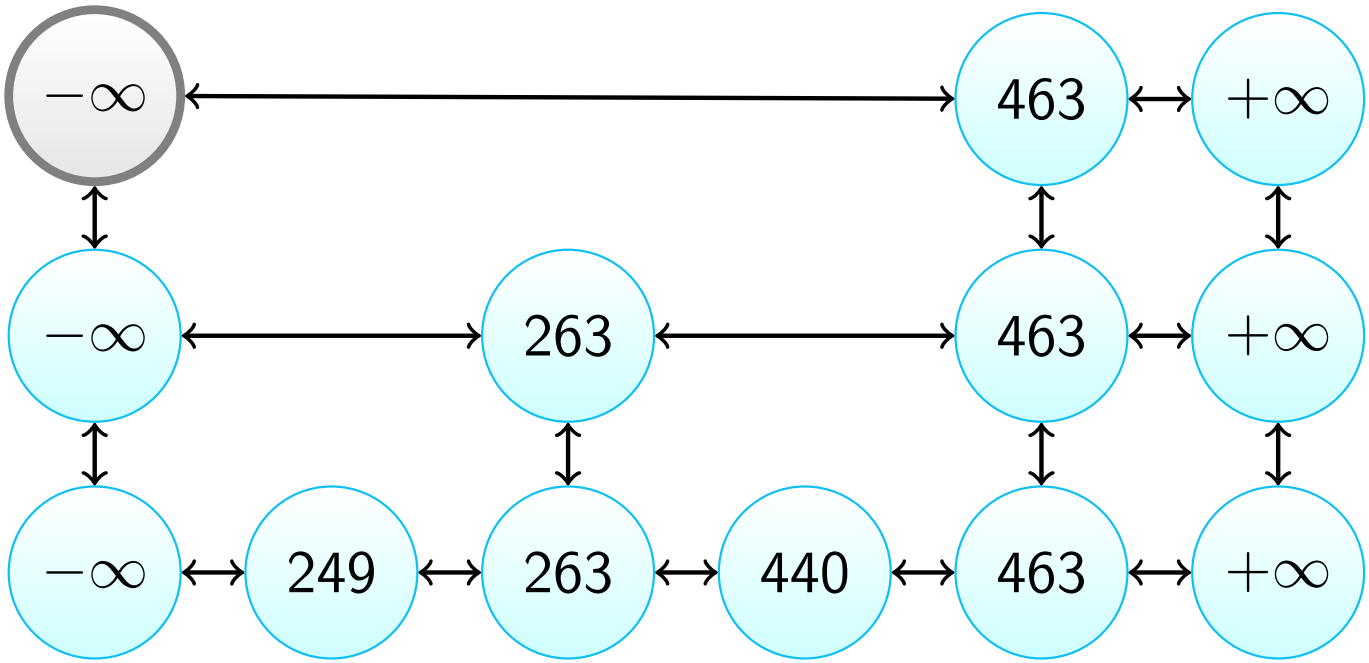
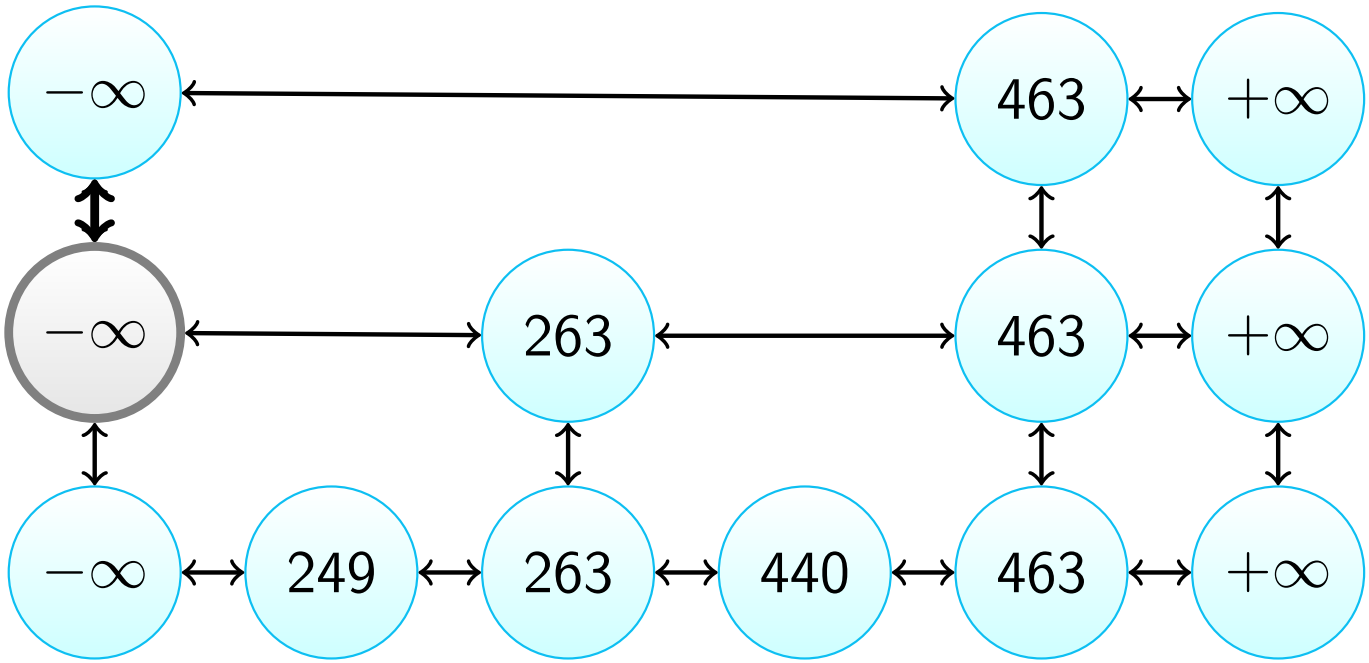


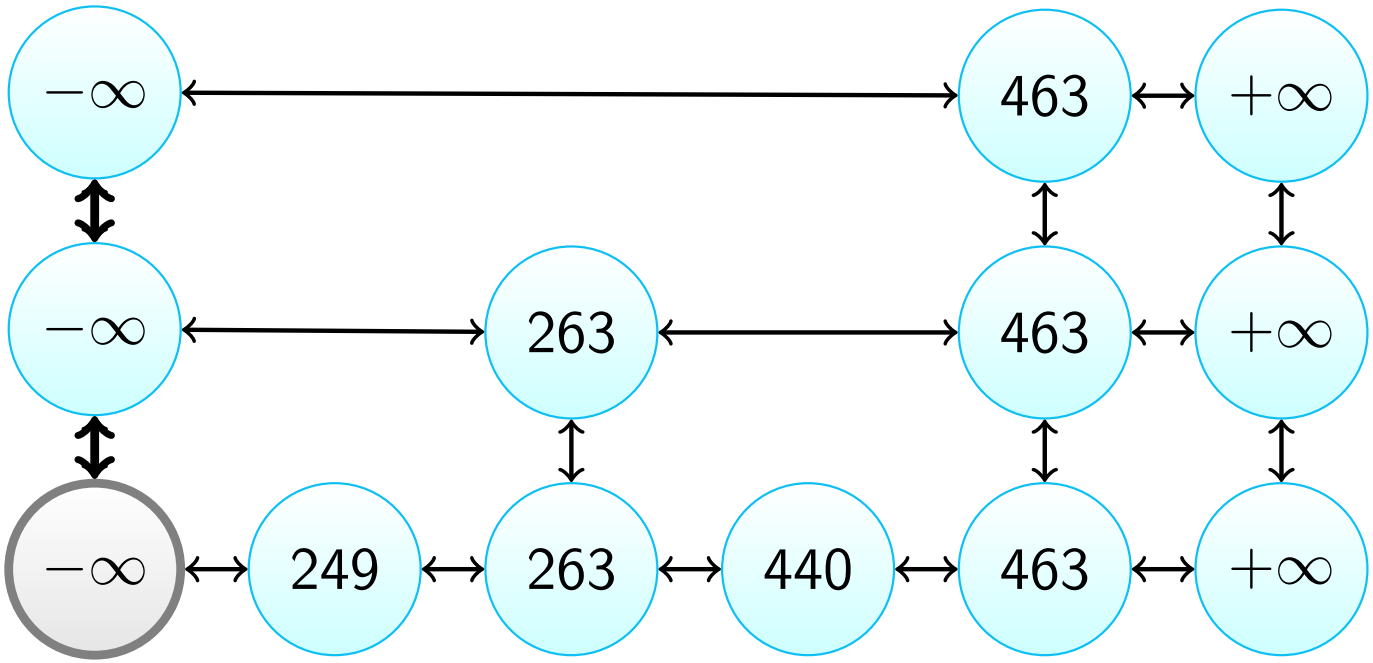
Adding

For example, if you were adding 118, and the coin flipper gave TTTHHTHHHTT:



Adding

For example, if you were adding 118, and the coin flipper gave TTTHHTHHHTT:



Adding

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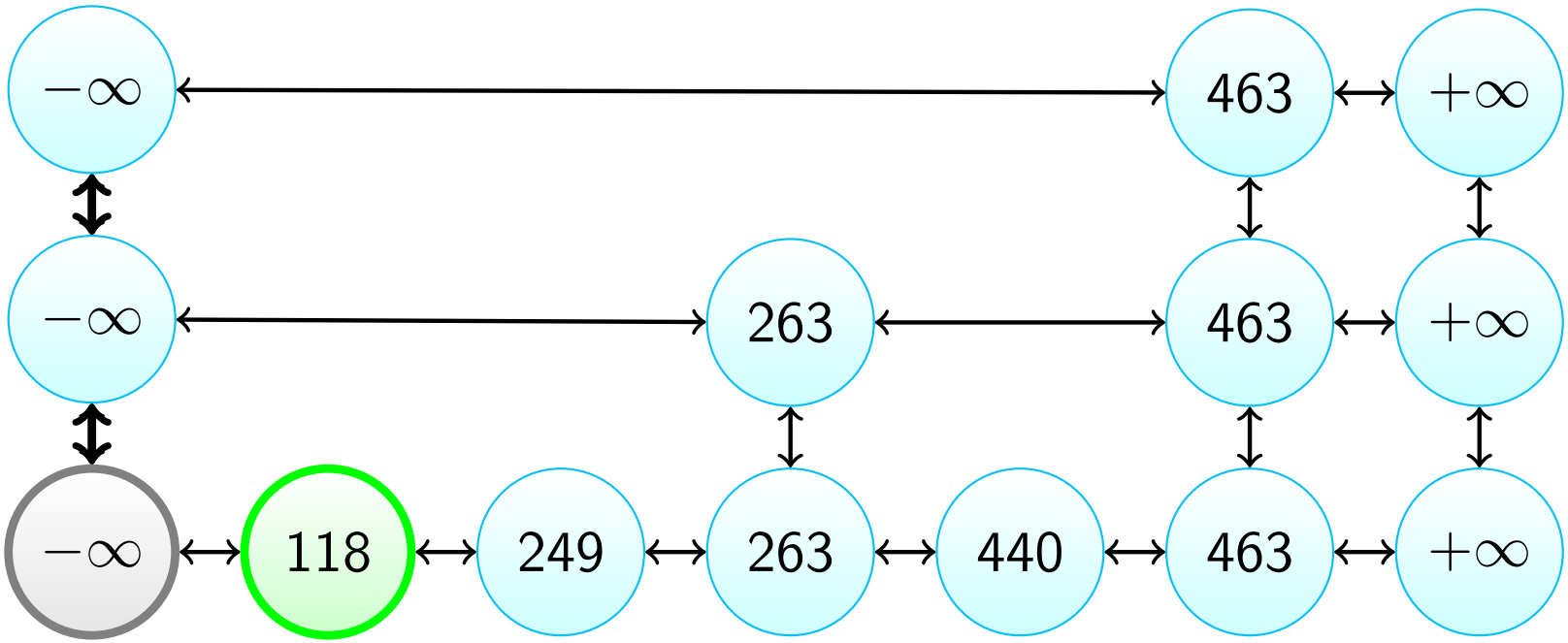
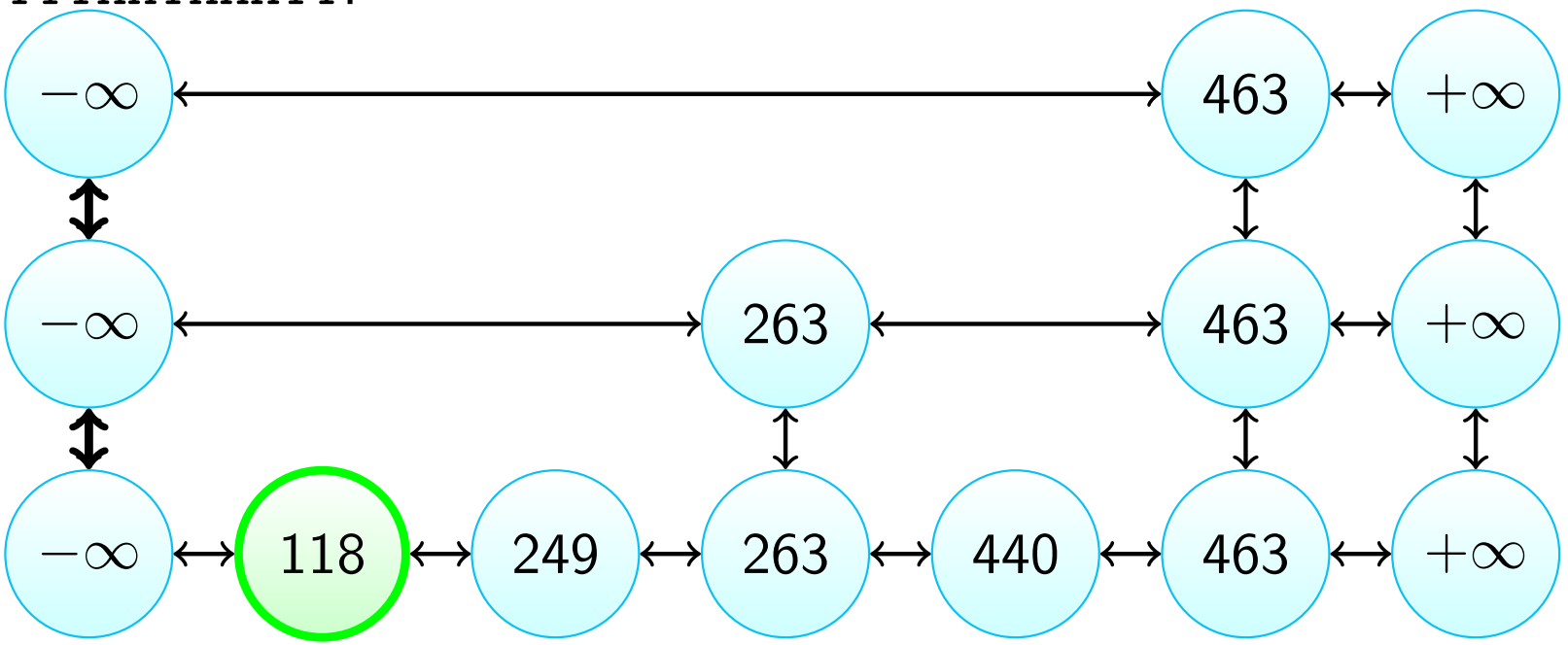


Adding

For example, if you were adding 118, and the coin flipper gave TTTHHTHHHTT:



Adding

procedure ADD(*data*)
    levels ← highest level the data will be on (based on coin flipper)
    while highest level $<$ *levels* **do**
        create another level
    end while
    ADD(*data*, *levels*, top-left phantom node, `NULL`)
end procedure

Adding

```
procedure ADD(data, levels, node, upperNode)
    if node is not valid then
        return
    else
        while data > node.next.data do
            node ← node.next
        end while
        if node.level <= levels then
            Add new node containing data after node
            node.up ← upperNode
            upperNode ← newly-added node
        end if
        ADD(data, levels, node.down, upperNode)
    end if
end procedure
```

Removing

- Follow the same steps as searching until you find the item in the skip list or go off of the skip list (in which case the data is not there).

Removing

- Follow the same steps as searching until you find the item in the skip list or go off of the skip list (in which case the data is not there).
- Disconnect the node from the rest of the nodes on that level, and move down to the level below.

Removing

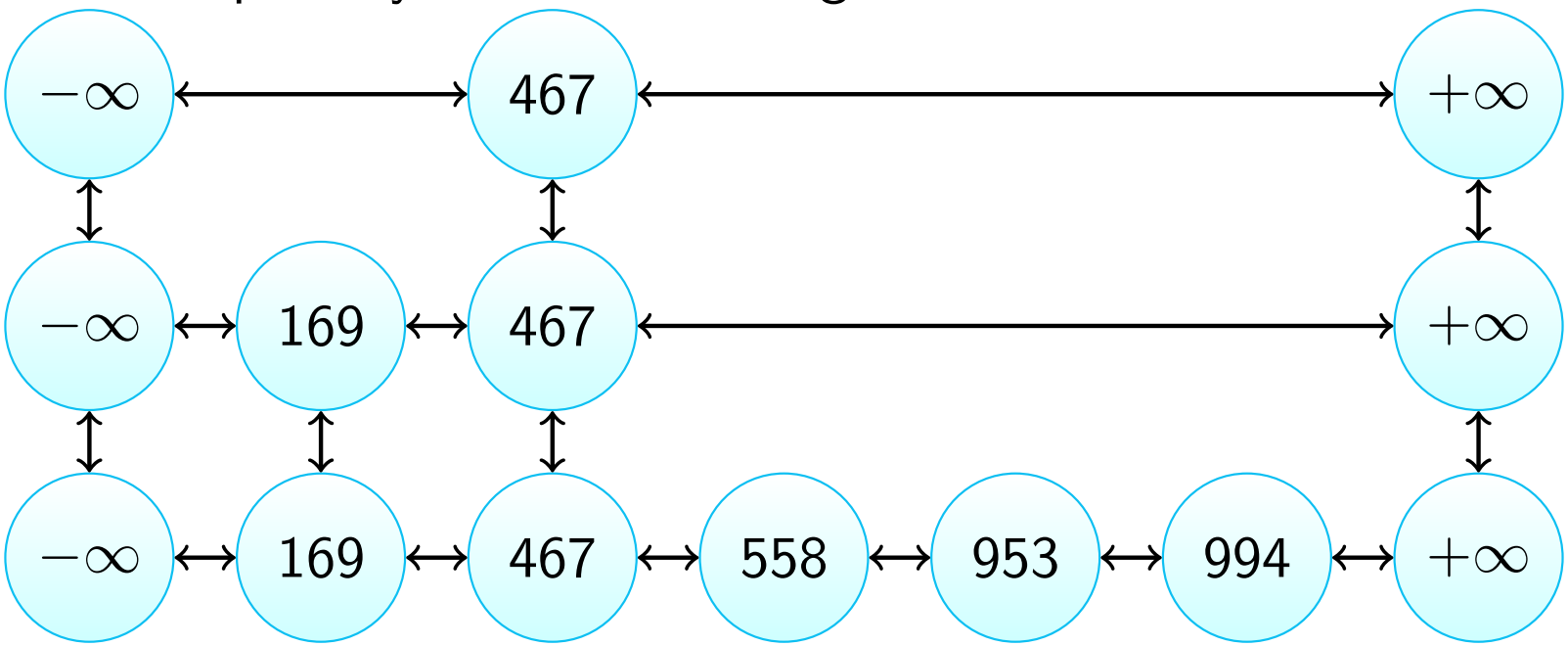
- Follow the same steps as searching until you find the item in the skip list or go off of the skip list (in which case the data is not there).
- Disconnect the node from the rest of the nodes on that level, and move down to the level below.
- Repeat the previous step until you go off of the skip list.

Removing

- Follow the same steps as searching until you find the item in the skip list or go off of the skip list (in which case the data is not there).
- Disconnect the node from the rest of the nodes on that level, and move down to the level below.
- Repeat the previous step until you go off of the skip list.
- Remove any empty levels in the skip list.

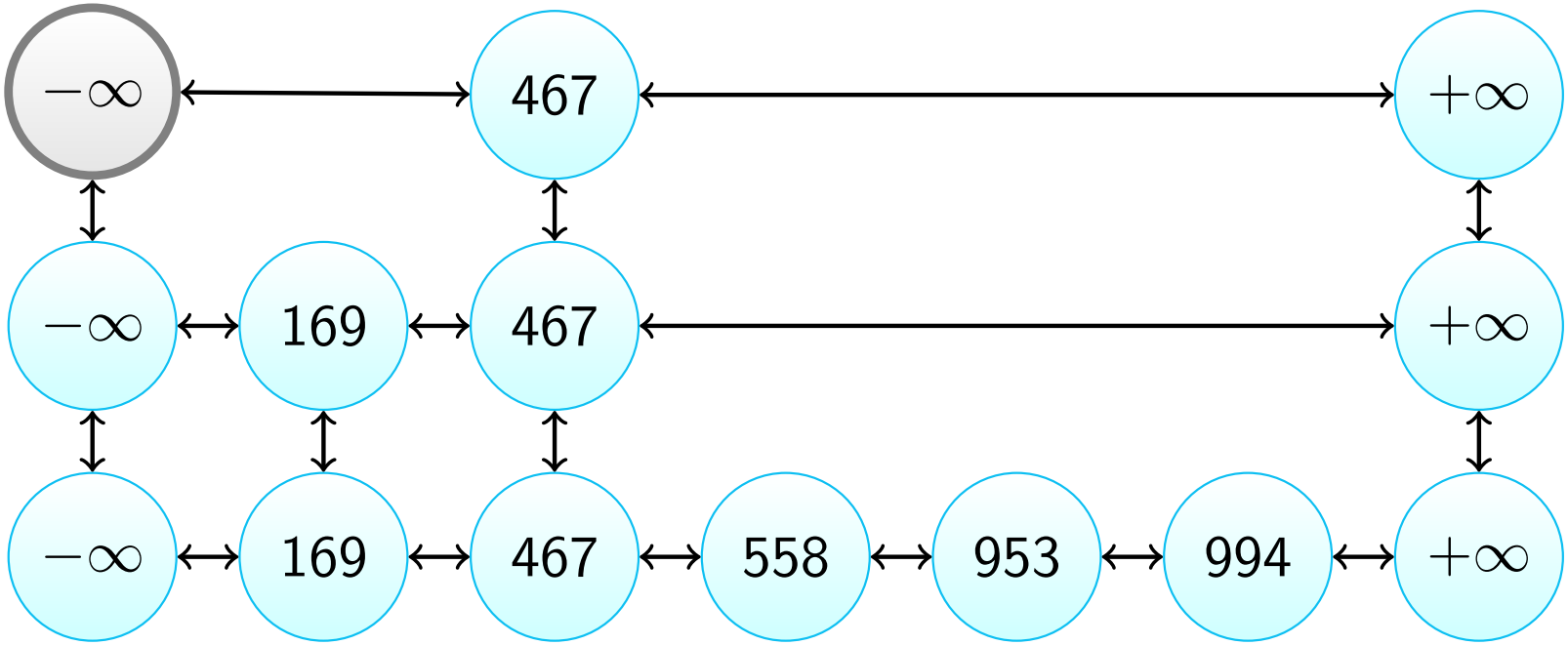
Removing

For example, if you were removing 467:



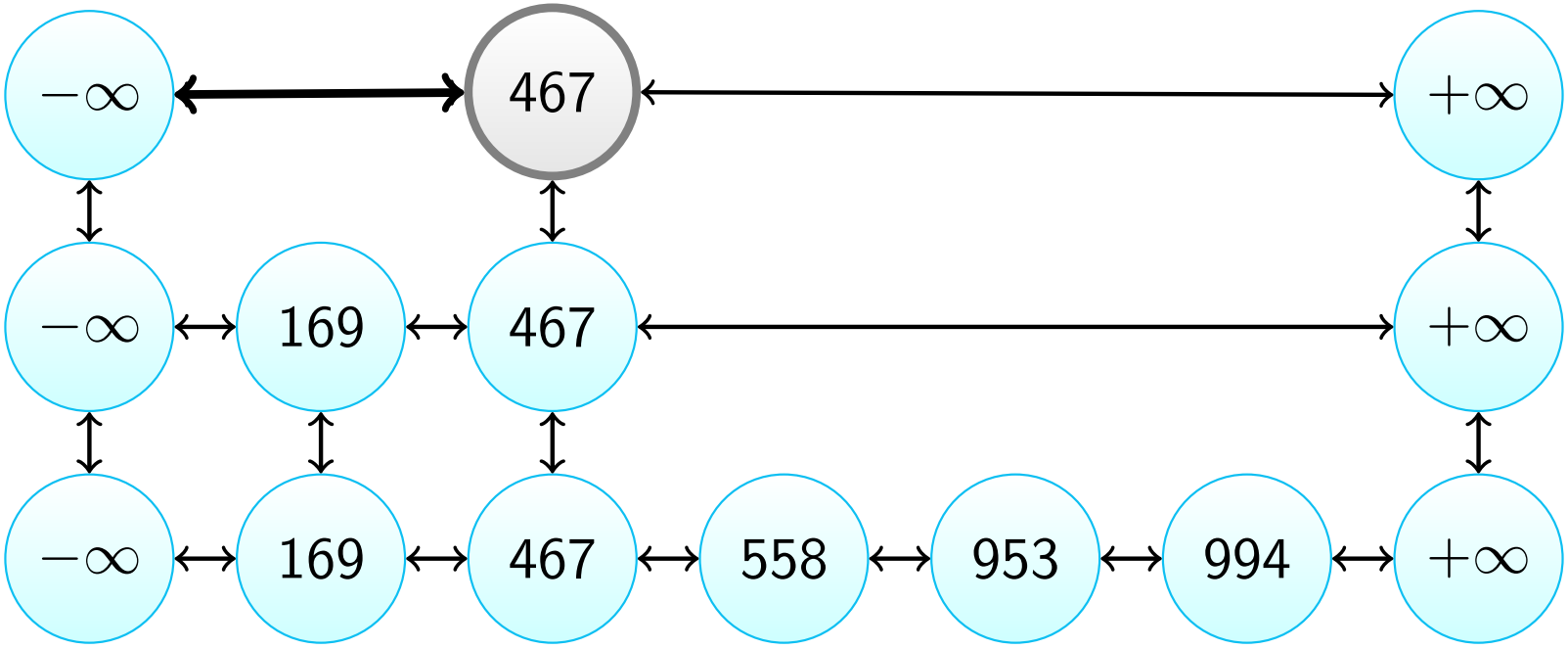
Removing

For example, if you were removing 467:



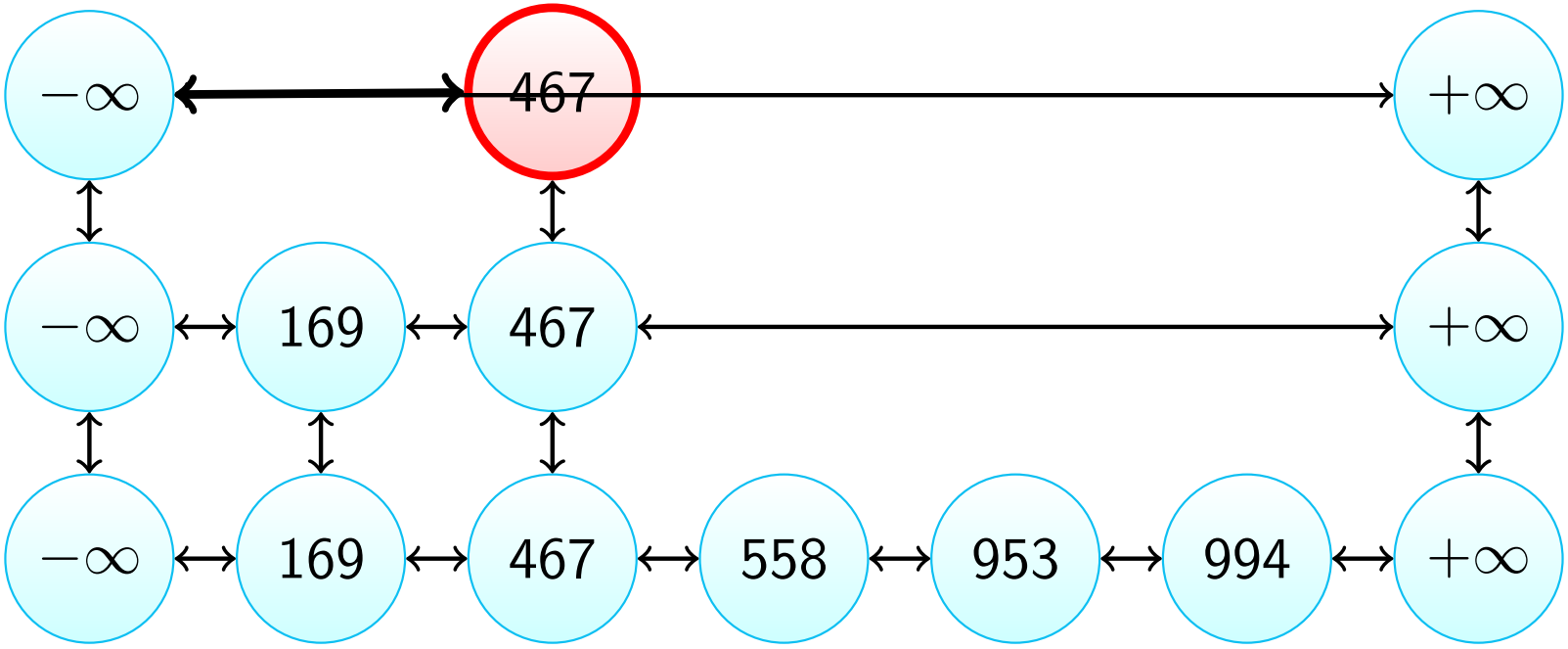
Removing

For example, if you were removing 467:



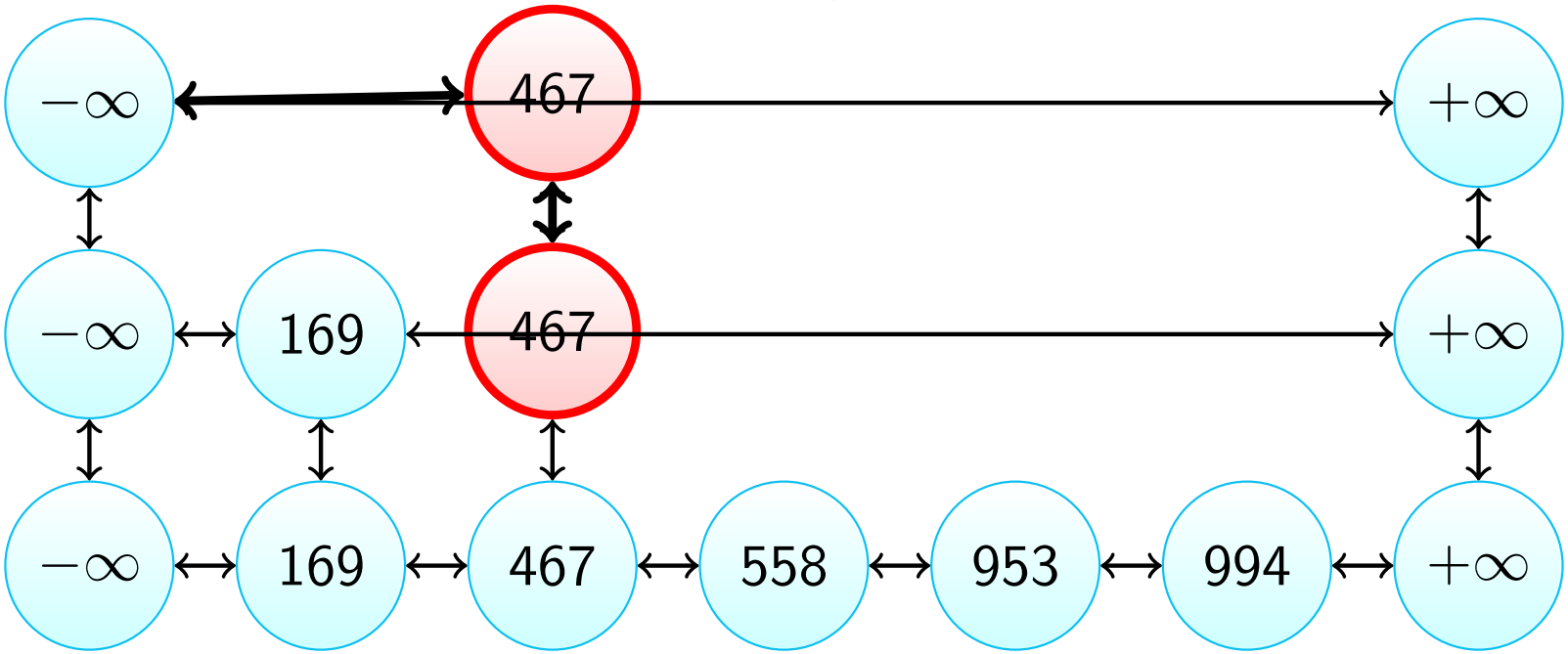
Removing

For example, if you were removing 467:



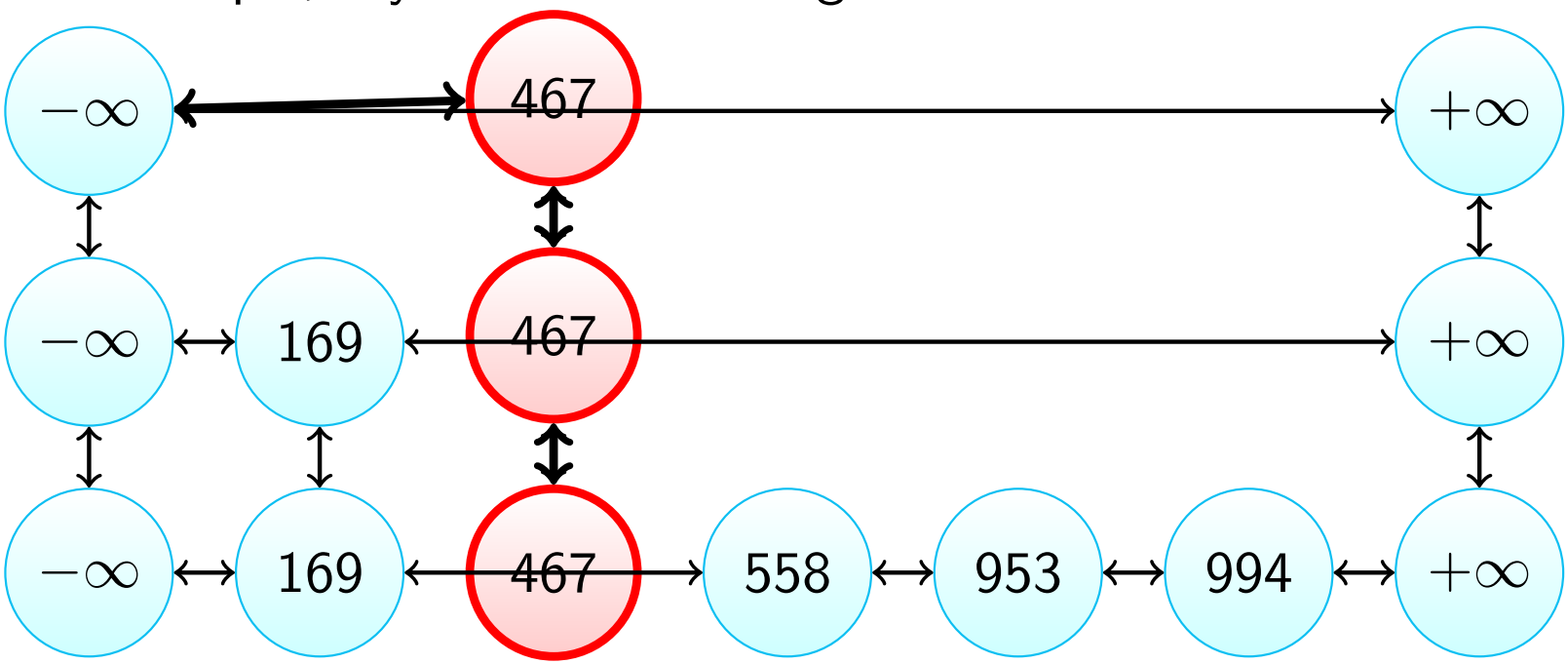
Removing

For example, if you were removing 467:



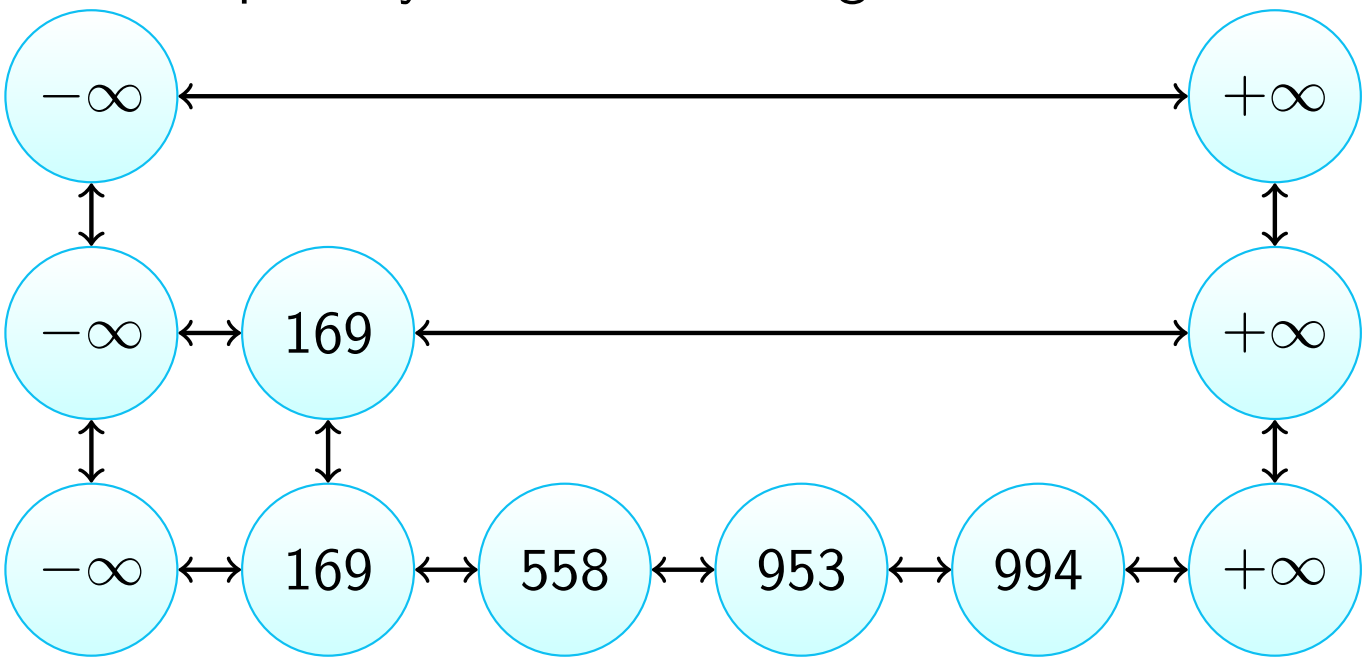
Removing

For example, if you were removing 467:



Removing

For example, if you were removing 467:



Removing

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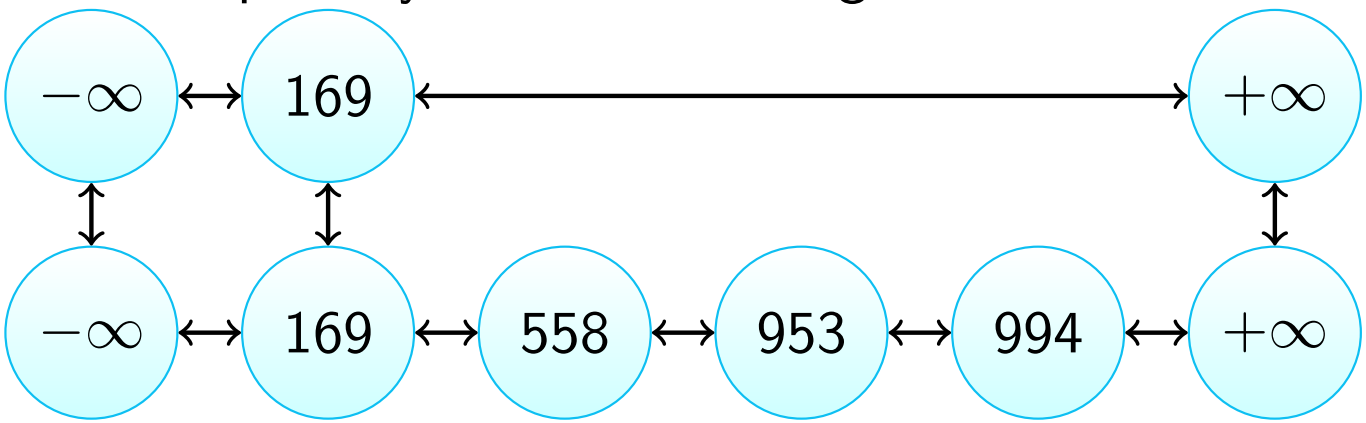
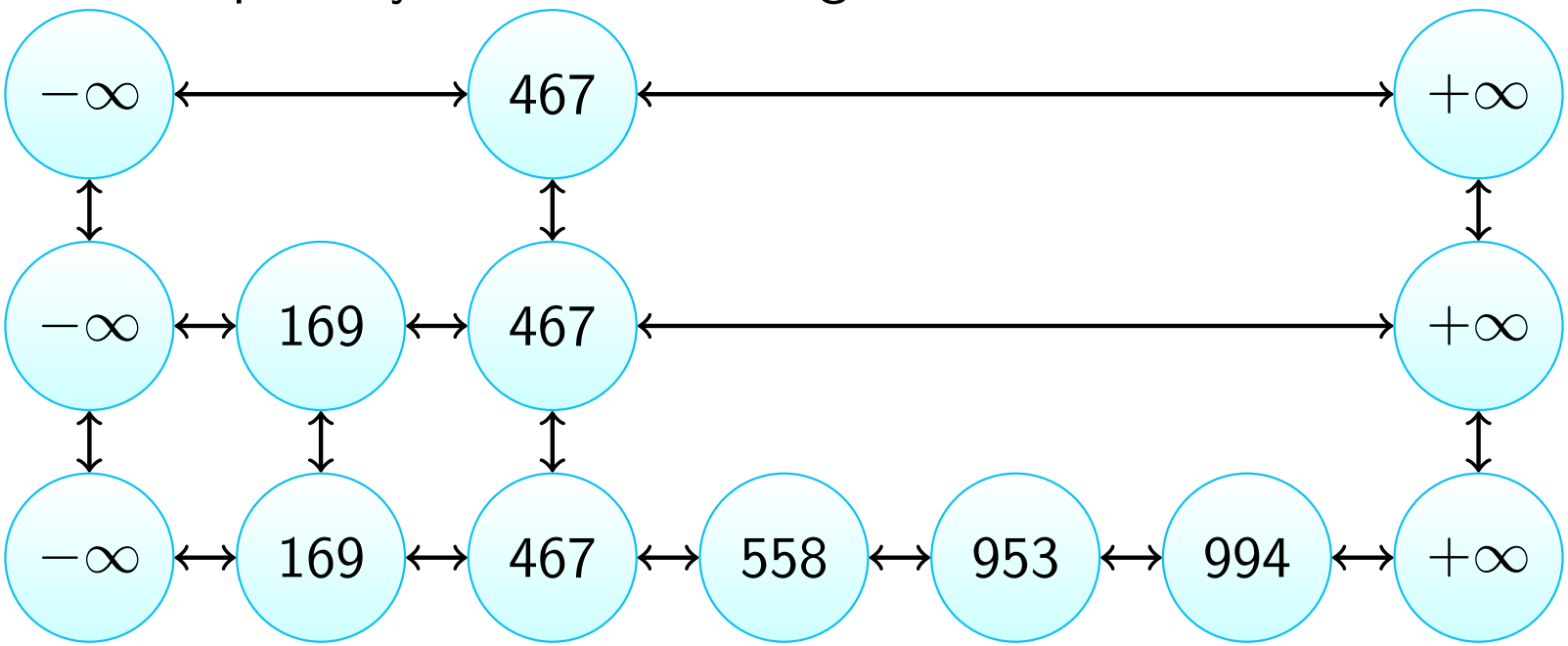


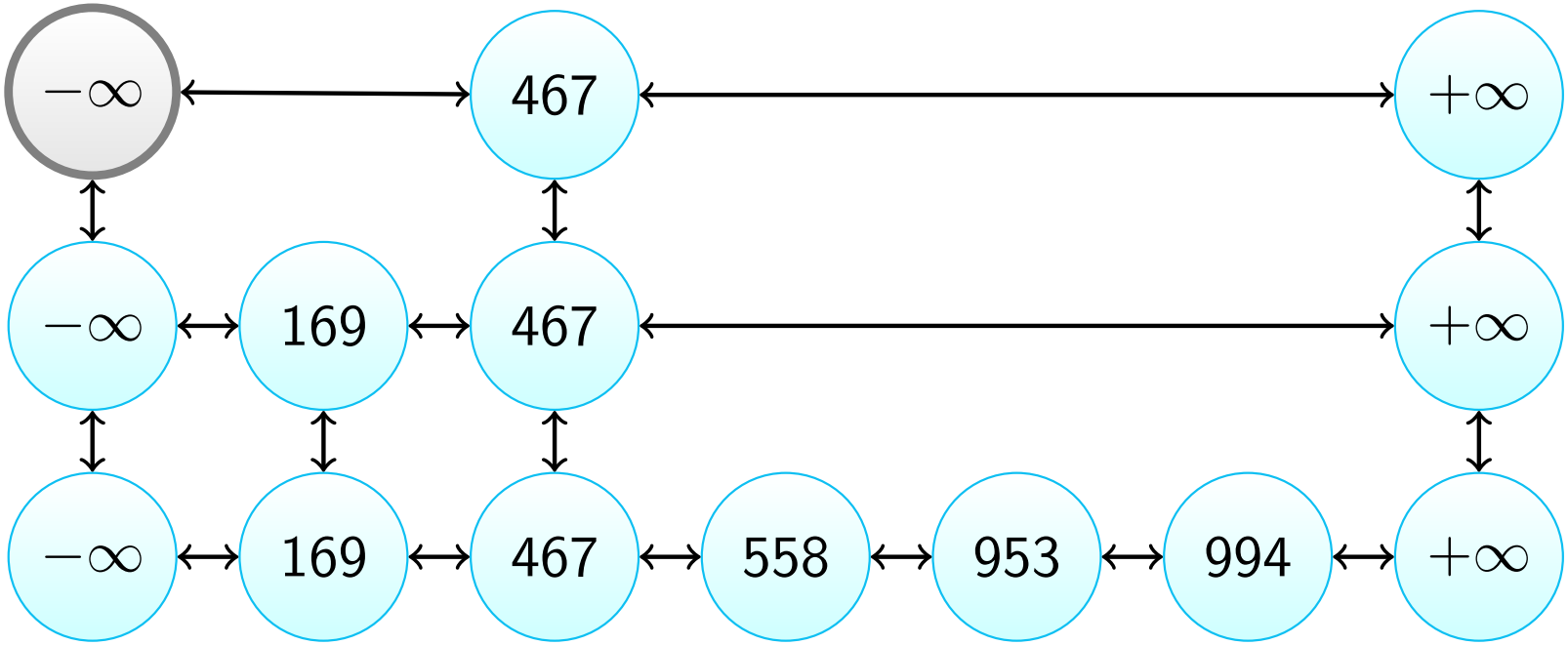
Removing

For example, if you were removing 169:



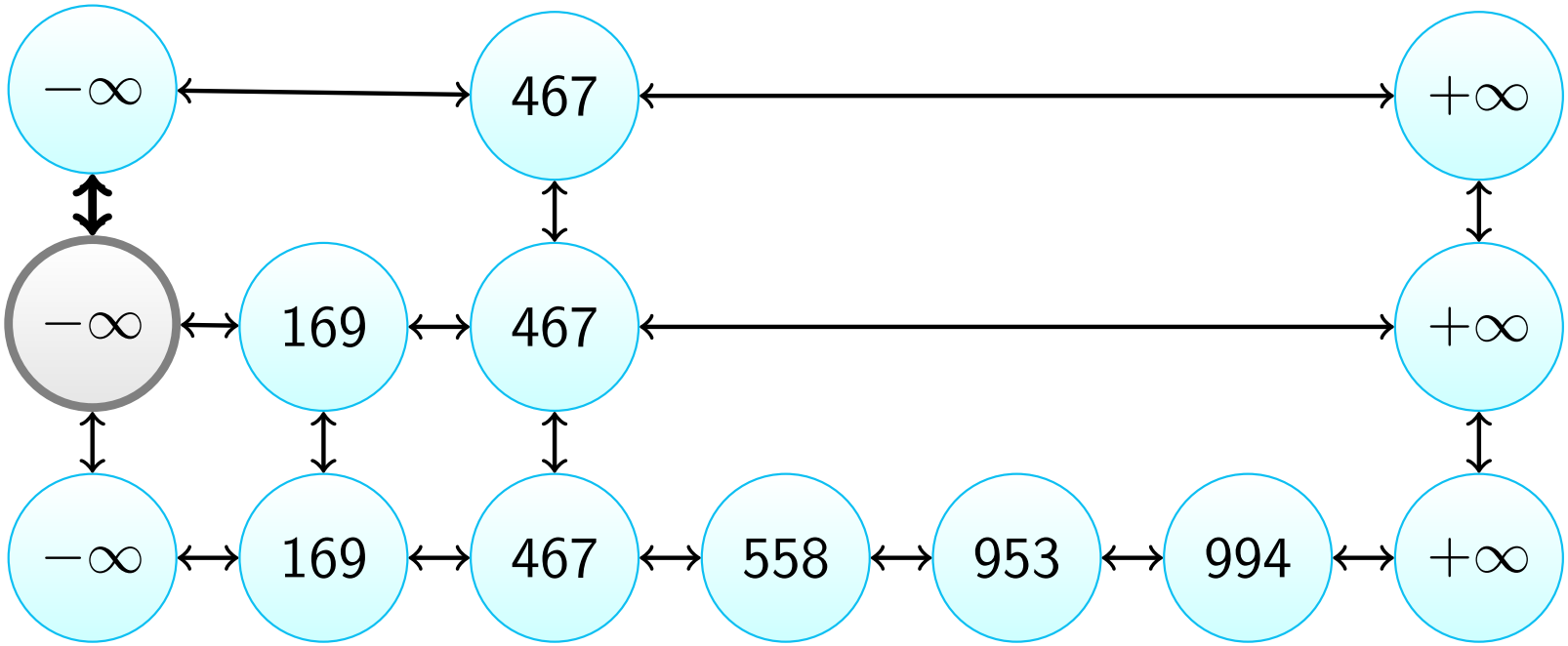
Removing

For example, if you were removing 169:



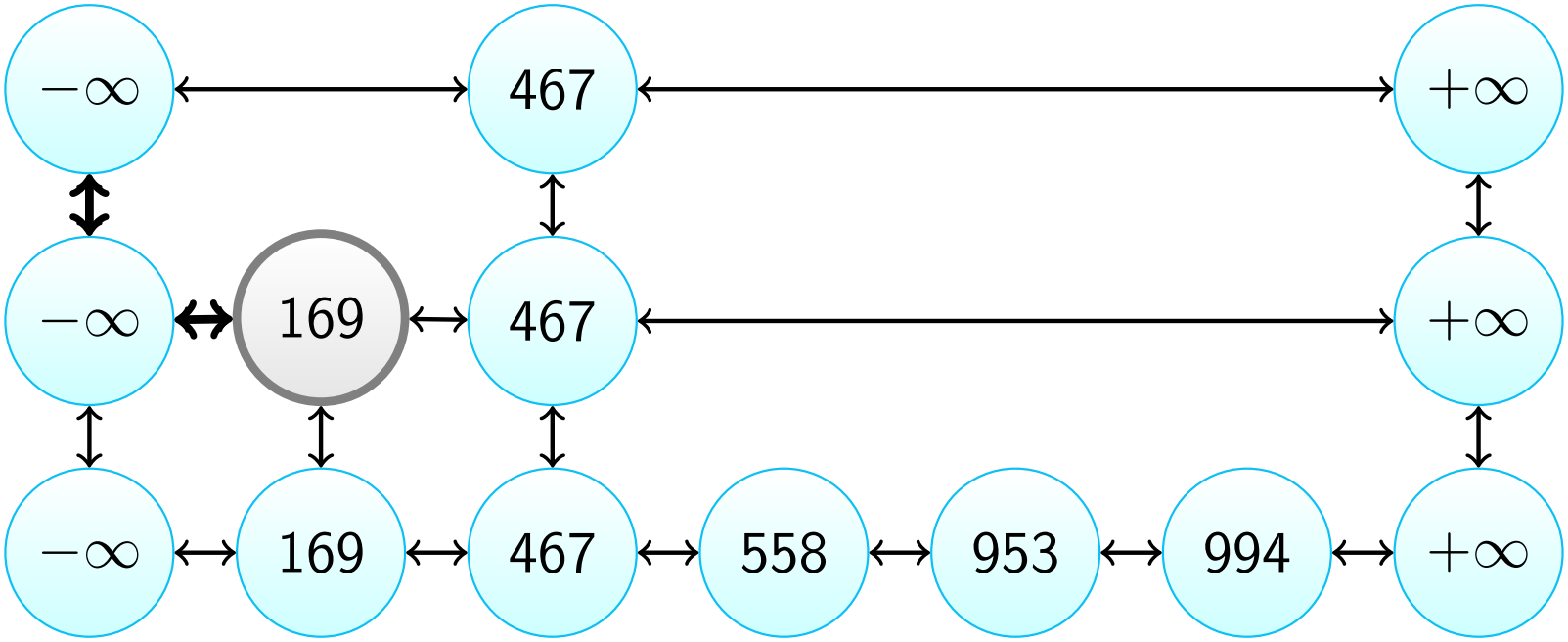
Removing

For example, if you were removing 169:



Removing

For example, if you were removing 169:



Removing

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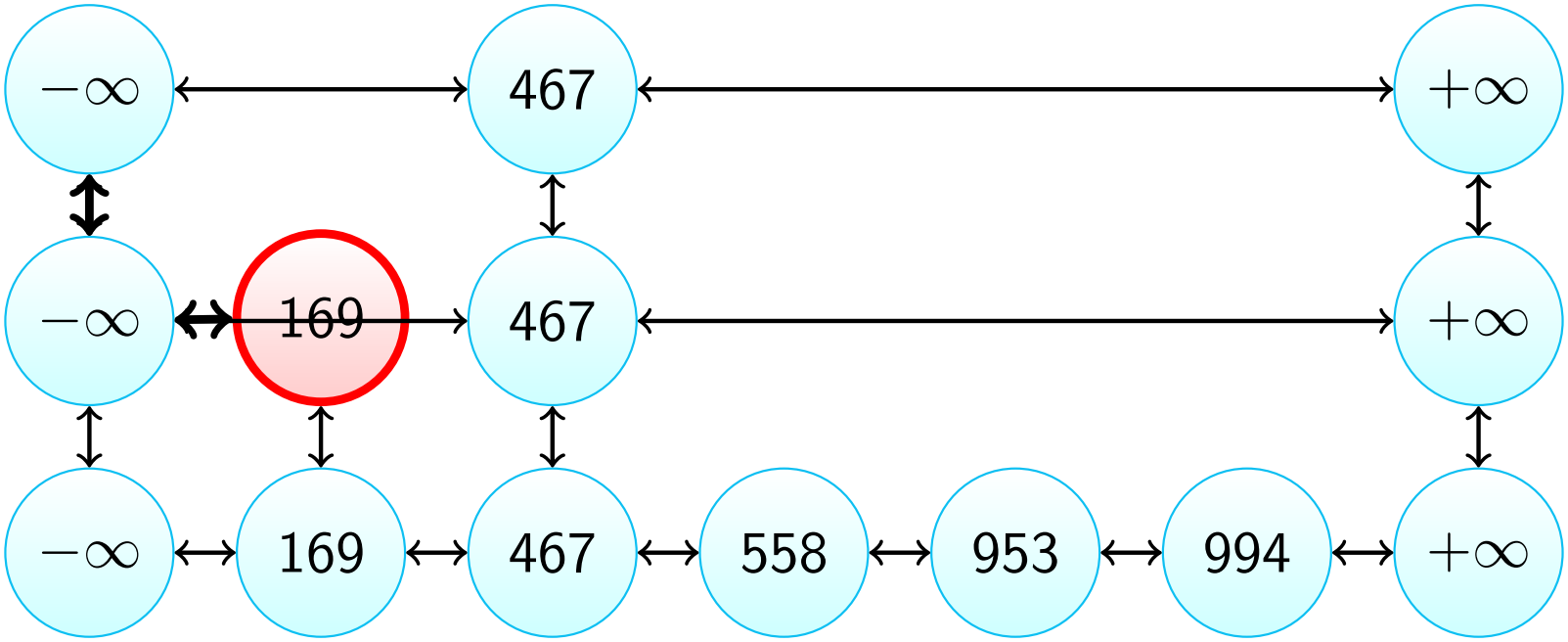

Removing

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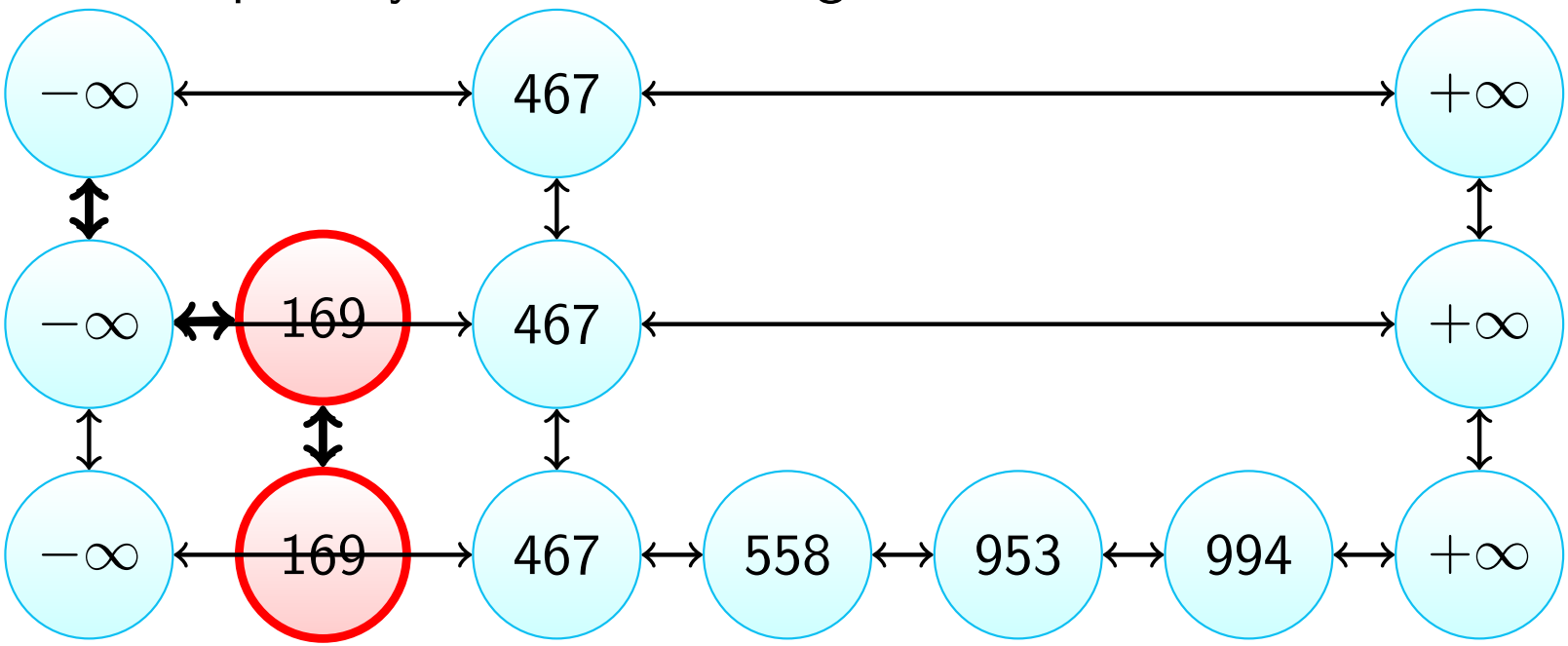

Removing

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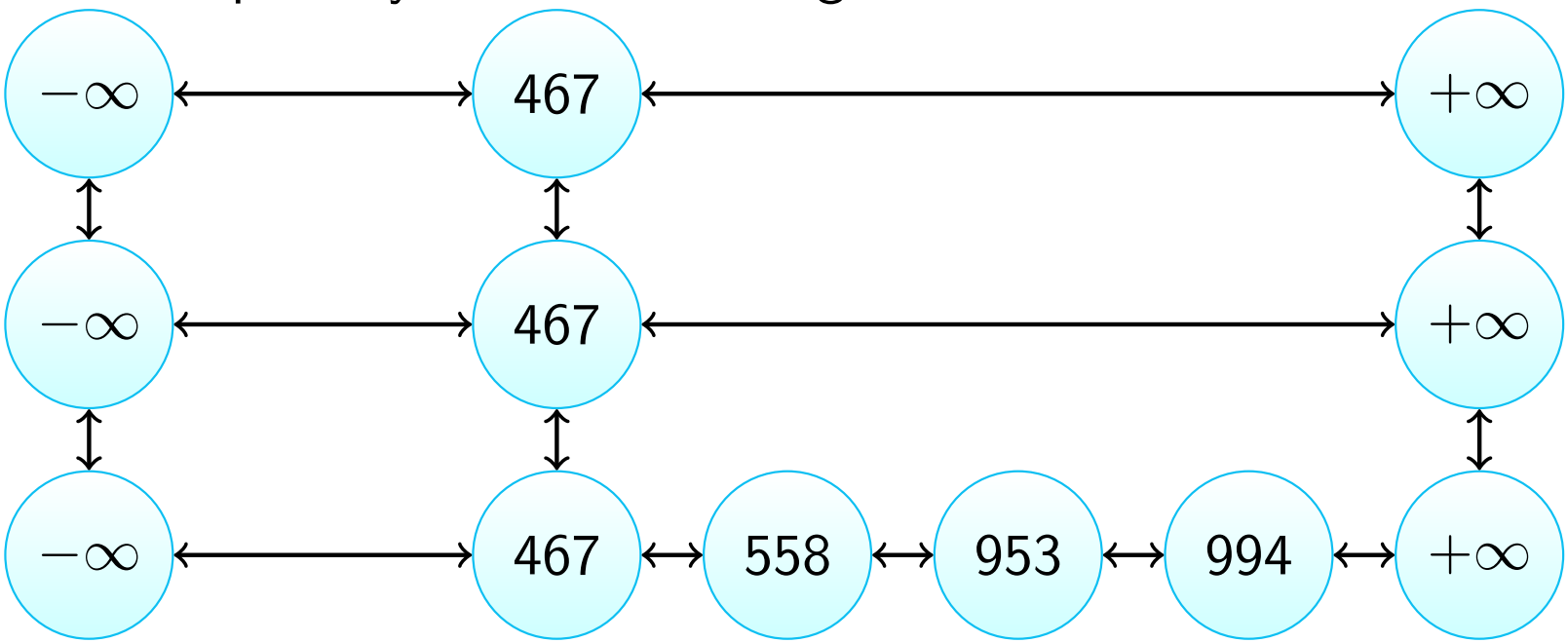

Removing

```
procedure REMOVE(data, node)
    if node is not valid then
        return
    else
        while data > node.next.data do
            node ← node.next
        end while
        if data = node.next.data then
            REMOVE(node.next)
            remove any empty levels
        else
            REMOVE(data, node.down)
        end if
    end if
end procedure
```

Removing

procedure REMOVE(*node*)
    while *node* is valid **do**
        disconnect/remove *node* from this level
        node $\leftarrow$ *node.down*
    end while
end procedure

Performance

- In the best case, a truly random coin flipper is used, the skip list has $\log n$ levels (where n is the number of data items), and each level has half of the items in the level below. In this case, adding, searching, and removing are $O(\log n)$.

Performance

- In the best case, a truly random coin flipper is used, the skip list has $\log n$ levels (where n is the number of data items), and each level has half of the items in the level below. In this case, adding, searching, and removing are $O(\log n)$.
- The average case for all three operations are $O(\log n)$ as well, because each level will contain roughly half of the items on the level below it.

Performance

- In the worst case, the coin flipper used is not truly random, or it just so happens that all of the items added into the skip list are at the same level. In this case, adding, searching, and removing are $O(n)$.

Performance

- In the worst case, the coin flipper used is not truly random, or it just so happens that all of the items added into the skip list are at the same level. In this case, adding, searching, and removing are $O(n)$.
- Note that unlike all of the data structures we've seen, because each item is stored multiple times in a skip list, the space complexity (the rate at which the space used increases) is $O(n \log n)$ in the worst case (on average, it is rougly $O(n)$). All of the other data structures we've seen have a space complexity of $O(n)$.