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A POSTERIORI ERROR ANALYSIS FOR A VISCOUS FLOW-TRANSPORT PROBLEM *

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Abstract. In this paper we develop an *a posteriori* error analysis for an augmented mixed-primal finite element approximation of a stationary viscous flow and transport problem. The governing system corresponds to a scalar, nonlinear convection-diffusion equation coupled with a Stokes problem with variable viscosity, and it serves as a prototype model for sedimentation-consolidation processes and other phenomena where the transport of species concentration within a viscous fluid is of interest. The solvability of the continuous mixed-primal formulation along with *a priori* error estimates for a finite element scheme using Raviart−Thomas spaces of order k for the stress approximation, and continuous piecewise polynomials of degree $\leq k+1$ for both velocity and concentration, have been recently established in [M. Alvarez *et al., ESAIM: M2AN* **49** (2015) 1399–1427]. Here we derive two efficient and reliable residual-based a posteriori error estimators for that scheme: for the first estimator, and under suitable assumptions on the domain, we apply a Helmholtz decomposition and exploit local approximation properties of the Clément interpolant and Raviart−Thomas operator to show its reliability. On the other hand, its efficiency follows from inverse inequalities and the localization arguments based on triangle-bubble and edge-bubble functions. Secondly, an alternative error estimator is proposed, whose reliability can be proved without resorting to Helmholtz decompositions. Our theoretical results are then illustrated *via* some numerical examples, highlighting also the performance of the scheme and properties of the proposed error indicators.

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1. Introduction

We have recently analyzed in [2] the solvability of a three-field flow-transport problem given by the coupling of a scalar nonlinear convection-diffusion problem with the Stokes equations where the viscosity depends on the distribution of the solution to the transport problem. There, an augmented mixed-primal variational formulation was proposed, where the Cauchy stresses are sought in $\mathbb{H}(\mathbf{div};\Omega)$, the velocity is in $\mathbf{H}^1(\Omega)$, and the solution to the transport problem has $\mathrm{H}^1(\Omega)$ regularity. The associated numerical scheme employed Raviart$-$Thomas spaces of order k for the Cauchy stress, whereas the velocity and a coupled scalar field (*e.g.* concentration as in [3], or temperature) were approximated with continuous piecewise polynomials of degree $\leq k+1$. Optimal *a priori* error estimates were also derived.

Our goal in this paper is to propose reliable and efficient residual-based *a posteriori* error estimators for the coupled flow-transport problem studied in [2]. Estimators of this kind are typically used to guide adaptive mesh refinement in order to guarantee an adequate convergence behavior of the Galerkin approximations, even under the eventual presence of singularities. The global estimator $\boldsymbol{\theta}$ depends on local estimators θ_T defined on each element T of a given mesh $\mathcal{T}_h$. Then, $\boldsymbol{\theta}$ is said to be efficient (resp. reliable) if there exists a constant $C_{\mathtt{eff}}>0$ (resp. $C_{\mathtt{rel}}>0$), independent of meshsizes, such that

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} + \text{h.o.t.} \leq \|error\| \leq C_{\mathtt{rel}}\,\boldsymbol{\theta} + \text{h.o.t.},$$

where h.o.t. is a generic expression denoting one or several terms of higher order. A number of *a posteriori* error estimators specifically targeted for non-viscous (*e.g.*, Darcy) flow coupled with transport problems area available in the literature (see, *e.g.* [11, 18, 28, 32, 35]). However, only a couple of contributions deal with *a posteriori* error analysis for coupled viscous flow-transport problems. In particular, we mention the reactive flow equations studied in [12] and the adaptive finite element method for heat transfer in incompressible fluid flow proposed in [29], which is based on dual weighted residual error estimation.

In contrast, here we apply a Helmholtz decomposition, local approximation properties of the Clément interpolant and Raviart$-$Thomas operator, and known estimates from [4, 19, 22, 24, 25], to prove the reliability of a residual-based estimator. Then, inverse inequalities, the localization technique based on triangle-bubble and edge-bubble functions imply the efficiency of the estimator. An alternative (also reliable and efficient) residual-based *a posteriori* error estimator is proposed, where the Helmholtz decomposition is not employed in the corresponding proof of reliability. The remainder of this paper is structured as follows. In Section 2, we first recall from [2] the model problem and a corresponding augmented mixed-primal formulation as well as the associated Galerkin scheme. In Section 3, we derive a reliable and efficient residual-based *a posteriori* error estimator for our Galerkin scheme. A second estimator is introduced and studied in Section 4. Finally, in Section 5 we provide some numerical results confirming the reliability and efficiency of the estimators, and illustrating the good performance of the associated adaptive algorithm for the augmented mixed-primal finite element method.

2. A coupled viscous flow-transport problem

Let us denote by $\Omega \subseteq \mathrm{R}^n$, $n=2,3$ a given bounded domain with polyhedral boundary $\Gamma = \bar{\Gamma}_{\mathrm{D}} \cup \bar{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$ and $|\Gamma_{\mathrm{D}}|, |\Gamma_{\mathrm{N}}| > 0$, and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. By $\mathbf{M}, \mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M. We recall that the space

$$\mathbb{H}(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\},$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}$$

is a Hilbert space. As usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes both the Euclidean norm in R^n and the Frobenius norm in $\mathrm{R}^{n\times n}$.

2.1. The three-field formulation

The following system of partial differential equations describes the stationary state of the transport of species ϕ in an immiscible fluid occupying the domain Ω (*cf.* [2]):

$$\begin{gathered}\frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathrm{d}} = \nabla\boldsymbol{u} \quad\text{in}\quad \Omega, \quad -\mathbf{div}\boldsymbol{\sigma} = \boldsymbol{f}\phi \quad\text{in}\quad \Omega,\\ \widetilde{\boldsymbol{\sigma}} = \vartheta(|\nabla\phi|)\,\nabla\phi - \phi\boldsymbol{u} - \gamma(\phi)\boldsymbol{k} \quad\text{in}\quad \Omega, \quad -\operatorname{div}\widetilde{\boldsymbol{\sigma}} = g \quad\text{in}\quad \Omega,\end{gathered} \tag{2.1}$$

$$\begin{gathered}\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}} \quad\text{on}\quad \Gamma_{\mathrm{D}}, \quad \boldsymbol{\sigma\nu} = \mathbf{0} \quad\text{on}\quad \Gamma_{\mathrm{N}},\\ \phi = \phi_{\mathrm{D}} \quad\text{on}\quad \Gamma_{\mathrm{D}}, \quad\text{and}\quad \widetilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu} = 0 \quad\text{on}\quad \Gamma_{\mathrm{N}},\end{gathered} \tag{2.2}$$

where the sought quantities are the Cauchy fluid stress $\boldsymbol{\sigma}$, the local volume-average velocity of the fluid $\boldsymbol{u}$, and the local concentration of species ϕ. In this model, the kinematic effective viscosity, μ; the diffusion coefficient, ϑ; and the one-directional flux function describing hindered settling, γ; depend nonlinearly on ϕ. In turn, $\boldsymbol{k}$ is a vector pointing in the direction of gravity, and $\boldsymbol{f}\in\mathbf{L}^\infty(\Omega)$, $\boldsymbol{u}_{\mathrm{D}}\in\mathbf{H}^{1/2}(\Gamma_{\mathrm{D}})$, $g\in\mathrm{L}^2(\Omega)$ are given functions. For sake of the subsequent analysis, the Dirichlet datum for the concentration will be assumed homogeneous, that is $\phi_{\mathrm{D}}=0$, ϑ is assumed of class C^1, and we suppose that there exist positive constants μ_1, μ_2, γ_1, γ_2, ϑ_1, ϑ_2, L_μ and L_γ, such that

$$\mu_1 \le \mu(s) \le \mu_2 \quad\text{and}\quad \gamma_1 \le \gamma(s) \le \gamma_2 \qquad \forall\, s\in\mathrm{R}, \tag{2.3}$$

$$\vartheta_1 \le \vartheta(s) \le \vartheta_2 \quad\text{and}\quad \vartheta_1 \le \vartheta(s) + s\,\vartheta'(s) \le \vartheta_2 \qquad \forall\, s\ge 0, \tag{2.4}$$

$$|\mu(s)-\mu(t)| \le L_\mu\,|s-t| \qquad \forall\, s,\, t\in\mathrm{R}, \tag{2.5}$$

$$|\gamma(s)-\gamma(t)| \le L_\gamma\,|s-t| \qquad \forall\, s,\, t\in\mathrm{R}. \tag{2.6}$$

2.2. The augmented mixed-primal formulation

The homogeneous Neumann boundary condition for $\boldsymbol{\sigma}$ on Γ_N and the Dirichlet datum for ϕ (*cf.* second and third relations of (2.2), respectively) suggest the introduction of the spaces

$$\begin{aligned}\mathbb{H}_N(\mathbf{div};\Omega) &:= \Big\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega): \quad \boldsymbol{\tau\nu}=\mathbf{0} \quad\text{on}\quad \Gamma_{\mathrm{N}}\Big\},\\ \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) &:= \Big\{\psi\in\mathrm{H}^1(\Omega): \quad \psi=0 \quad\text{on}\quad \Gamma_{\mathrm{D}}\Big\}.\end{aligned}$$

Also, due to the generalized Poincaré inequality, there exists $c_p>0$, depending only on Ω and Γ_{D}, such that

$$\|\psi\|_{1,\Omega} \le c_p\,|\psi|_{1,\Omega} \qquad \forall\,\psi\in\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega). \tag{2.7}$$

An augmented mixed-primal formulation for problem (2.1) reads as follows: find $(\boldsymbol{\sigma},\boldsymbol{u},\phi)\in\mathbb{H}_N(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$\begin{aligned}B_\phi((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) &= F_\phi(\boldsymbol{\tau},\boldsymbol{v}) \quad \forall(\boldsymbol{\tau},\boldsymbol{v})\in\mathbb{H}_N(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega),\\ A_{\boldsymbol{u}}(\phi,\psi) &= G_\phi(\psi) \qquad \forall\,\psi\in\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\end{aligned} \tag{2.8}$$

where

$$
\begin{aligned}
B_\phi((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) \;:=&\; \int_\Omega \frac{1}{\mu(\phi)}\,\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}\boldsymbol{\tau} - \int_\Omega \boldsymbol{v}\cdot\mathbf{div}\boldsymbol{\sigma} \\
&+ \kappa_1 \int_\Omega \left(\nabla\boldsymbol{u} - \frac{1}{\mu(\phi)}\,\boldsymbol{\sigma}^{\mathrm{d}}\right) : \nabla\boldsymbol{v} + \kappa_2 \int_\Omega \mathbf{div}\boldsymbol{\sigma}\cdot\mathbf{div}\boldsymbol{\tau} + \kappa_3 \int_{\Gamma_{\mathrm{D}}} \boldsymbol{u}\cdot\boldsymbol{v},
\end{aligned} \tag{2.9}
$$

$$
F_\phi(\boldsymbol{\tau},\boldsymbol{v}) \;:= \langle\boldsymbol{\tau\nu},\boldsymbol{u}_{\mathrm{D}}\rangle_{\Gamma_{\mathrm{D}}} + \int_\Omega \boldsymbol{f}\phi\cdot\boldsymbol{v} - \kappa_2 \int_\Omega \boldsymbol{f}\phi\cdot\mathbf{div}\boldsymbol{\tau} + \kappa_3 \int_{\Gamma_{\mathrm{D}}} \boldsymbol{u}_{\mathrm{D}}\cdot\boldsymbol{v}, \tag{2.10}
$$

$$
A_{\boldsymbol{u}}(\phi,\psi) \;:= \int_\Omega \vartheta(|\nabla\phi|)\,\nabla\phi\cdot\nabla\psi - \int_\Omega \phi\,\boldsymbol{u}\cdot\nabla\psi \qquad \forall\,\phi,\,\psi\in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega), \tag{2.11}
$$

$$
G_\phi(\psi) \;:= \int_\Omega \gamma(\phi)\,\boldsymbol{k}\cdot\nabla\psi + \int_\Omega g\psi \qquad \forall\,\psi\in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega),
$$

where κ_i, $i\in\{1,2,3\}$, are the stabilization parameters specified in ([2], Lem. 4.1). Further details yielding the weak formulation (2.8) can be found in ([2], Sect. 3.1), whereas its solvability follows from the fixed point strategy developed in ([2], Thm. 3.13).

2.3. The augmented mixed-primal finite element method

We denote by $\mathcal{T}_h$ a regular partition of Ω into triangles T (resp. tetrahedra T in R^3) of diameter h_T, and meshsize $h := \max\left\{h_T : T\in\mathcal{T}_h\right\}$. In addition, given an integer $k\geq 0$, the space $\mathrm{P}_k(T)$ contains polynomial functions on T of degree $\leq k$, and we define the corresponding local Raviart$-$Thomas space of order k as $\mathbf{RT}_k(T) := \mathbf{P}_k(T)\oplus \mathrm{P}_k(T)\,\boldsymbol{x}$, where, according to the notations described in Section 1, $\mathbf{P}_k(T) = [\mathrm{P}_k(T)]^n$, and $\boldsymbol{x}\in\mathrm{R}^n$. Then, the Galerkin scheme associated to (2.8) is as follows: find $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\phi_h)\in\mathbb{H}^{\boldsymbol{\sigma}}_h\times\mathbf{H}^{\boldsymbol{u}}_h\times\mathrm{H}^\phi_h$ such that

$$
\begin{aligned}
B_{\phi_h}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\tau}_h,\boldsymbol{v}_h)) \;&=\; F_{\phi_h}(\boldsymbol{\tau}_h,\boldsymbol{v}_h) \quad \forall(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\in\mathbb{H}^{\boldsymbol{\sigma}}_h\times\mathbf{H}^{\boldsymbol{u}}_h,\\
A_{\boldsymbol{u}_h}(\phi_h,\psi_h) \;&=\; \int_\Omega \gamma(\phi_h)\,\boldsymbol{k}\cdot\nabla\psi_h + \int_\Omega g\psi_h \qquad \forall\,\psi_h\in\mathrm{H}^\phi_h,
\end{aligned} \tag{2.12}
$$

where the involved finite element spaces are defined as:

$$
\begin{aligned}
\mathbb{H}^{\boldsymbol{\sigma}}_h \;&:=\; \Big\{\boldsymbol{\tau}_h\in\mathbb{H}_N(\mathbf{div};\Omega) : \quad \mathbf{c}^{\mathrm{t}}\,\boldsymbol{\tau}_h|_T\in\mathbf{RT}_k(T) \quad \forall\,\mathbf{c}\in\mathrm{R}^n, \quad \forall\,T\in\mathcal{T}_h\Big\},\\
\mathbf{H}^{\boldsymbol{u}}_h \;&:=\; \Big\{\boldsymbol{v}_h\in\mathbf{C}(\overline{\Omega}) : \quad \boldsymbol{v}_h|_T\in\mathbf{P}_{k+1}(T) \quad \forall\,T\in\mathcal{T}_h\Big\},\\
\mathrm{H}^\phi_h \;&:=\; \Big\{\psi_h\in C(\overline{\Omega})\cap\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) : \quad \psi_h|_T\in\mathrm{P}_{k+1}(T) \quad \forall\,T\in\mathcal{T}_h\Big\}.
\end{aligned}
$$

The solvability analysis and *a priori* error bounds for (2.12) are established in ([2], Sect. 5).

3. A residual-based *a posteriori* error estimator

In this section we derive a reliable and efficient residual-based *a posteriori* error estimator for the Galerkin scheme (2.12). The analysis will be restricted to the two-dimensional case, with the discrete spaces introduced in Section 2. However, we point out that a straightforward extension of our analysis to 3D does also apply since the key estimate given by the stability of the corresponding Helmholtz decomposition (see (3.36) below for our 2D case) follows from the technique suggested in ([26], Lem. 4.3) and the results provided in ([10], Thms. 2.17 and 3.12, and Cor. 3.16).

Now, given a suitably chosen $r>0$ (see [2] for details), we define the balls

$$
W := \{\phi\in\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) : \quad \|\phi\|_{1,\Omega}\leq r\} \quad \text{and} \quad W_h := \{\phi_h\in\mathrm{H}^\phi_h : \quad \|\phi_h\|_{1,\Omega}\leq r\}, \tag{3.1}
$$

and throughout the rest of the paper we let $(\boldsymbol{\sigma}, \boldsymbol{u}, \phi) \in \mathbb{H}_N(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ with $\phi \in W$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\phi}$ with $\phi_h \in W_h$ be the solutions of the continuous and discrete formulations (2.8) and (2.12), respectively. In addition, we set

$$H := \mathbb{H}_N(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega), \qquad \|(\boldsymbol{\tau}, \boldsymbol{v})\|_H := \|\boldsymbol{\tau}\|_{\mathbf{div};\Omega} + \|\boldsymbol{v}\|_{1,\Omega} \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in H,$$

and recall from ([2], Thms. 3.13 and 4.7) that the following *a priori* estimates hold

$$\begin{aligned} \|(\boldsymbol{\sigma}, \boldsymbol{u})\|_H &\le C_S \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\phi\|_{1,\Omega}\, \|\boldsymbol{f}\|_{\infty,\Omega} \right\}, \\ \|(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H &\le C_S \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\phi_h\|_{1,\Omega}\, \|\boldsymbol{f}\|_{\infty,\Omega} \right\}, \end{aligned} \tag{3.2}$$

where C_S is a positive constant independent of ϕ and ϕ_h.

3.1. The local error indicator

Given $T \in \mathcal{T}_h$, we let $\mathcal{E}_h(T)$ be the set of its edges, and let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$. Then we write $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_{\mathrm{D}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{N}})$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : e \subseteq \Omega\}$, $\mathcal{E}_h(\Gamma_{\mathrm{D}}) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_{\mathrm{D}}\}$ and $\mathcal{E}_h(\Gamma_{\mathrm{N}}) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_{\mathrm{N}}\}$. Also, for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\boldsymbol{\nu}_e := (\nu_1, \nu_2)^{\mathtt{t}}$, and let $\boldsymbol{s}_e := (-\nu_2, \nu_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Then, given $e \in \mathcal{E}_h(\Omega)$ and $\tau \in \mathbf{L}^2(\Omega)$ such that $\tau|_T \in [C(T)]^2$ on each $T \in \mathcal{T}_h$, we let $[\![\tau \cdot \boldsymbol{\nu}_e]\!]$ be the corresponding jump across e, that is, $[\![\tau \cdot \boldsymbol{\nu}_e]\!] := (\tau|_T - \tau|_{T'})|_e \cdot \boldsymbol{\nu}_e$, where T and T' are the triangles of $\mathcal{T}_h$ having e as a common edge. Similarly, given $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$ such that $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$ on each $T \in \mathcal{T}_h$, we let $[\![\boldsymbol{\tau} \boldsymbol{s}_e]\!]$ be the corresponding jump across e, that is, $[\![\boldsymbol{\tau} \boldsymbol{s}_e]\!] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e\, \boldsymbol{s}_e$. If no confusion arises, we will simple write $\boldsymbol{s}$ and $\boldsymbol{\nu}$ instead of $\boldsymbol{s}_e$ and $\boldsymbol{\nu}_e$, respectively. The curl operator applied to scalar, vector and tensor valued fields v, $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)$ and $\boldsymbol{\tau} := (\tau_{ij})$, respectively, will be denoted as

$$\mathbf{curl}(v) := \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\boldsymbol{\varphi}) := \begin{pmatrix} \mathbf{curl}(\varphi_1)^{\mathtt{t}} \\ \mathbf{curl}(\varphi_2)^{\mathtt{t}} \end{pmatrix}, \quad \text{and} \quad \mathrm{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Then, we let $\widetilde{\boldsymbol{\sigma}}_h := \vartheta(|\nabla \phi_h|)\nabla \phi_h - \phi_h \boldsymbol{u}_h - \gamma(\phi_h)\boldsymbol{k}$ and define for each $T \in \mathcal{T}_h$ a local error indicator θ_T as follows

$$\begin{aligned} \theta_T^2 := {}& \|\boldsymbol{f}\phi_h + \mathbf{div}\boldsymbol{\sigma}_h\|_{0,T}^2 + \left\| \nabla \boldsymbol{u}_h - \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\|_{0,T}^2 + h_T^2\, \|g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \\ & + h_T^2 \left\| \mathrm{curl} \left\{ \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\} \right\|_{0,T}^2 + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Omega)} h_e \left\| \left[\!\!\left[\frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}}\, \boldsymbol{s} \right]\!\!\right] \right\|_{0,e}^2 \\ & + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Omega)} h_e\, \|[\![\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}_e]\!]\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Gamma_{\mathrm{N}})} h_e\, \|\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}\|_{0,e}^2 \\ & + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Gamma_{\mathrm{D}})} \|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Gamma_{\mathrm{D}})} h_e \left\| \frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}\boldsymbol{s}} - \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}}\, \boldsymbol{s} \right\|_{0,e}^2. \end{aligned} \tag{3.3}$$

The residual character of each term defining θ_T^2 is clear, and hence, proceeding as usual, a *global* residual error estimator can be defined as:

$$\boldsymbol{\theta} := \left\{ \sum_{T \in \mathcal{T}_h} \theta_T^2 \right\}^{1/2}. \tag{3.4}$$

Note that the last term defining θ_T^2 requires that $\left.\frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}\boldsymbol{s}}\right|_e \in \mathbf{L}^2(e)$ for each $e \in \mathcal{E}_h(\Gamma_{\mathrm{D}})$. This is ensured below by assuming that $\boldsymbol{u}_D \in \mathbf{H}_0^1(\Gamma_{\mathrm{D}})$.

3.2. Reliability

The following theorem constitutes the main result of this section

Theorem 3.1. *Assume that Ω is a connected domain and that $\Gamma_{\rm N}$ is the boundary of a convex part of Ω, that is Ω can be extended to a convex domain B such that $\overline{\Omega} \subseteq B$ and $\Gamma_{\rm N} \subseteq \partial B$ (see Fig. 1 below). In addition, assume that $\boldsymbol{u}_D \in \mathbf{H}^1_0(\Gamma_{\rm D})$ and that for some $\varepsilon \in (0,1)$ (when $n=2$) or some $\varepsilon \in (1/2,1)$(when $n=3$) there holds*

$$C_3\,|\boldsymbol{k}| \,+\, C_6\,\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_{\rm D}} \,+\, C_7\,\|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{1}{2}. \tag{3.5}$$

where C_3, C_6 and C_7 are the constants given below in (3.22). Then, there exists a constant $C_{\tt rel} > 0$, which depends only on parameters, $\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_{\rm D}}$, $\|\boldsymbol{f}\|_{\infty,\Omega}$, and other constants, all them independent of h, such that

$$\|\phi - \phi_h\|_{1,\Omega} \,+\, \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H \,\leq\, C_{\tt rel}\,\boldsymbol{\theta}. \tag{3.6}$$

We begin the proof of (3.6) with the upper bounds derived in the following two subsections.

3.2.1. A preliminary estimate for $\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H$

In order to simplify the subsequent writing, we first introduce the following constants

$$C_0 := \frac{1}{\alpha}, \quad C_1 \,:=\, 2\,C_0\,C_\varepsilon\,\widetilde{C}_\varepsilon\,\widetilde{C}_S(r)\frac{L_\mu(1+\kappa_1^2)^{1/2}}{\mu_1^2}, \quad C_2 \,:= C_0\,(1+\kappa_2^2)^{1/2} \,+\, r\,C_1, \tag{3.7}$$

where $\widetilde{C}_S(r)$ and C_ε, $\widetilde{C}_\varepsilon$ are defined in ([2], *cf.* (3.22)) and ([2], Lem. 3.9 and Thm. 3.13), respectively.

Lemma 3.2. *Let $\boldsymbol{\theta}_0^2 \,:=\, \sum_{T\in\mathcal{T}_h} \theta_{0,T}^2$, where for each $T \in \mathcal{T}_h$ we set*

$$\theta_{0,T}^2 \,:=\, \|\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h\|_{0,T}^2 \,+\, \left\|\nabla\boldsymbol{u}_h \,-\, \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\tt d}\right\|_{0,T}^2 \,+\, \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_{\rm D})} \|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{0,e}^2. \tag{3.8}$$

Then there exists $\bar{C} > 0$, depending on C_0, κ_1, κ_3, and the trace operator in $\mathbf{H}^1(\Omega)$, such that

$$\begin{aligned} &\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H \\ &\quad\leq\; \bar{C}\left\{\boldsymbol{\theta}_0 + \|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}\right\} \,+\, \left\{C_1\,\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_{\rm D}} + C_2\,\|\boldsymbol{f}\|_{\infty,\Omega}\right\}\|\phi - \phi_h\|_{1,\Omega}, \end{aligned} \tag{3.9}$$

where C_1 and C_2 are given by (3.7), and $E_h \in \mathbb{H}_N(\mathbf{div},\Omega)'$, defined for each $\boldsymbol{\zeta} \in \mathbb{H}_N(\mathbf{div},\Omega)$ by

$$E_h(\boldsymbol{\zeta}) := \langle\boldsymbol{\zeta}\,\boldsymbol{\nu},\boldsymbol{u}_D\rangle_{\Gamma_{\rm D}} \,-\, \int_\Omega \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\tt d} : \boldsymbol{\zeta} \,-\, \int_\Omega \boldsymbol{u}_h\cdot\mathbf{div}\boldsymbol{\zeta} \,-\, \kappa_2\int_\Omega(\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h)\cdot\mathbf{div}\boldsymbol{\zeta}, \tag{3.10}$$

satisfies

$$E_h(\boldsymbol{\zeta}_h) = 0 \qquad \forall\,\boldsymbol{\zeta}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}. \tag{3.11}$$

Proof. We first deduce from the H-ellipticity of B_ϕ (*cf.* [2], Lem. 3.4) that there holds the global inf-sup condition

$$\alpha\,\|(\boldsymbol{\tau},\boldsymbol{v})\|_H \,\leq\, \sup_{\substack{(\boldsymbol{\zeta},\boldsymbol{w})\in \mathrm{H}\\(\boldsymbol{\zeta},\boldsymbol{w})\neq\mathbf{0}}} \frac{B_\phi((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\zeta},\boldsymbol{w}))}{\|(\boldsymbol{\zeta},\boldsymbol{w})\|_H} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in H, \tag{3.12}$$

where α is the constant of ellipticity, which depends only on $\mu_1,\mu_2,\Omega,\Gamma_{\rm N}$ and $\Gamma_{\rm D}$ (see [2], Lem. 3.4). Then, applying (3.12) to the error $(\boldsymbol{\tau},\boldsymbol{v}) := (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{u}-\boldsymbol{u}_h)$, we find that

$$\alpha\,\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H \,\leq\, \sup_{\substack{(\boldsymbol{\zeta},\boldsymbol{w})\in \mathrm{H}\\(\boldsymbol{\zeta},\boldsymbol{w})\neq\mathbf{0}}} \frac{F_\phi(\boldsymbol{\zeta},\boldsymbol{w}) - B_\phi((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\zeta},\boldsymbol{w}))}{\|(\boldsymbol{\zeta},\boldsymbol{w})\|_H}. \tag{3.13}$$

Next, using the definitions of B_ϕ (*cf.* (2.9)) and F_ϕ (*cf.* (2.10)), and adding and subtracting suitable terms, we can write

$$\begin{aligned} F_\phi(\boldsymbol{\zeta},\boldsymbol{w}) \,-\, B_\phi((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\zeta},\boldsymbol{w})) \,&=\, F_{\phi_h}(\boldsymbol{\zeta},\boldsymbol{w}) \,-\, B_{\phi_h}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\zeta},\boldsymbol{w})) \\ +\quad B_{\phi_h}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\zeta},\boldsymbol{w})) \,&-\, B_\phi((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\zeta},\boldsymbol{w})) \,+\, F_\phi(\boldsymbol{\zeta},\boldsymbol{w}) \,-\, F_{\phi_h}(\boldsymbol{\zeta},\boldsymbol{w}). \end{aligned} \tag{3.14}$$

In this way, employing the estimate for $|B_{\phi_h}(\,\cdot,(\boldsymbol{\zeta},\boldsymbol{w})) - B_\phi(\,\cdot,(\boldsymbol{\zeta},\boldsymbol{w}))|$ (see [2], Eq. (3.29)) and $|F_\phi(\boldsymbol{\zeta},\boldsymbol{w}) - F_{\phi_h}(\boldsymbol{\zeta},\boldsymbol{w})|$ (see [2], Eq. (3.28)), we deduce from (3.13) and (3.14) that

$$\begin{aligned} \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H \,&\le\, C_0\,\|F_{\phi_h}(\,\cdot\,) - B_{\phi_h}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),\,\cdot\,)\|_{H'} \\ &+ \Big\{ C_1\,\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_{\mathrm{D}}} \,+\, C_2\,\|\boldsymbol{f}\|_{\infty,\Omega} \Big\}\,\|\phi-\phi_h\|_{1,\Omega}, \end{aligned} \tag{3.15}$$

where, bearing in mind (3.10), there holds

$$F_{\phi_h}(\boldsymbol{\zeta},\boldsymbol{w}) \,-\, B_{\phi_h}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\zeta},\boldsymbol{w})) \,=\, E_h(\boldsymbol{\zeta}) \,+\, \widehat{E}_h(\boldsymbol{w}) \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{w})\in H, \tag{3.16}$$

with $\widehat{E}_h \in \mathbf{H}^1(\Omega)'$ defined for each $\boldsymbol{w}\in\mathbf{H}^1(\Omega)$ by

$$\widehat{E}_h(\boldsymbol{w}) \,:=\,=\, \int_\Omega (\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h)\cdot\boldsymbol{w} \,+\, \kappa_1\int_\Omega\left(\nabla\boldsymbol{u}_h \,-\, \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathrm{d}}\right) : \nabla\boldsymbol{w} \,+\, \kappa_3\int_{\Gamma_{\mathrm{D}}}(\boldsymbol{u}_D \,-\, \boldsymbol{u}_h)\cdot\boldsymbol{w}.$$

Then, applying the Cauchy−Schwarz inequality we readily deduce the existence of a constant $\widehat{c}>0$, depending on κ_1, κ_3, and the trace operator in $\mathbf{H}^1(\Omega)$, such that

$$\|\widehat{E}_h\|_{\mathbf{H}^1(\Omega)'} \,\le\, \widehat{c}\,\boldsymbol{\theta}_0,$$

which, together with (3.15) and (3.16), imply the main inequality (3.9). Moreover, using the fact that

$$F_{\phi_h}(\boldsymbol{\zeta}_h,\boldsymbol{w}_h) - B_{\phi_h}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)) \,=\, 0 \qquad \forall\,(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\in H_h,$$

and taking in particular $\boldsymbol{w}_h = \mathbf{0}$, we deduce (3.11), which completes the proof. □

Observe, according to (3.11), that for each $\boldsymbol{\zeta}\in\mathbb{H}_N(\mathbf{div},\Omega)$ there holds

$$E_h(\boldsymbol{\zeta}) \,=\, E_h(\boldsymbol{\zeta}-\boldsymbol{\zeta}_h) \qquad \forall\,\boldsymbol{\zeta}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}},$$

and hence the upper bound of $\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}$ to be derived below (see Sect. 3.2.3) will employ the foregoing expression with a suitable choice of $\boldsymbol{\zeta}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}$.

3.2.2. A preliminary estimate for $\|\phi-\phi_h\|_{1,\Omega}$

We begin with the following results.

Lemma 3.3. *The nonlinear operator* $\mathbf{A}_{\boldsymbol{u}} : \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) \to [\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)]'$ *induced by* $A_{\boldsymbol{u}}$ *(cf. (2.11)), that is*

$$[\mathbf{A}_{\boldsymbol{u}}(\phi),\psi] \,:=\, \int_\Omega \vartheta(|\nabla\phi|)\,\nabla\phi\cdot\nabla\psi \,-\, \int_\Omega \phi\,\boldsymbol{u}\cdot\nabla\psi \qquad \forall\,\psi\in\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega), \tag{3.17}$$

where $[\cdot,\cdot]$ *is the duality pairing between* $\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ *and* $[\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)]'$, *is Gâteaux differentiable in* $\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$.

Proof. We begin by observing, thanks to simple computations and the C^1-regularity of ϑ, that for all $\widehat{\phi},\psi,\varphi\in\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$, with $\nabla\widehat{\phi}\neq\mathbf{0}$ there holds

$$\begin{aligned} \lim_{\epsilon\to 0}\frac{[\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi}+\epsilon\,\psi) - \mathbf{A}_{\boldsymbol{u}}(\widehat{\phi}),\varphi]}{\epsilon} \,&=\, \int_\Omega \vartheta'(|\nabla\widehat{\phi}|)\frac{\left(\nabla\widehat{\phi}\cdot\nabla\psi\right)}{|\nabla\widehat{\phi}|}\nabla\widehat{\phi}\cdot\nabla\varphi \\ &+ \int_\Omega \vartheta(|\nabla\widehat{\phi}|)\nabla\psi\cdot\nabla\varphi \,-\, \int_\Omega \psi\boldsymbol{u}\cdot\nabla\varphi, \end{aligned} \tag{3.18}$$

whereas for $\nabla\widehat{\phi} = \mathbf{0}$, we find that

$$\lim_{\epsilon\to 0} \frac{[\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi}+\epsilon\,\psi) - \mathbf{A}_{\boldsymbol{u}}(\widehat{\phi}),\varphi]}{\epsilon} \,=\, \int_\Omega \vartheta(0)\nabla\psi\cdot\nabla\varphi \,-\, \int_\Omega \psi\boldsymbol{u}\cdot\nabla\varphi. \tag{3.19}$$

In this way, the identities (3.18) and (3.19) show that $\mathbf{A}_{\boldsymbol{u}}$ is Gâteaux differentiable at $\widehat{\phi}$. Moreover, $\mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi})$ is the bounded linear operator of $\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ into $[\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)]'$ that can be identified with the bilinear form $\mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi}) : \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\times\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\to\mathrm{R}$ defined by

$$\mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi})(\psi,\varphi) \,:=\, \lim_{\epsilon\to 0} \frac{[\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi}+\epsilon\,\psi) - \mathbf{A}_{\boldsymbol{u}}(\widehat{\phi}),\varphi]}{\epsilon} \qquad \forall\,\psi,\varphi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega). \tag{3.20}$$

□

Lemma 3.4. *Let c_p and $c(\Omega)$ be the constants given by (2.7) and ([2], Eq. (3.5)), respectively, and let $\boldsymbol{u}\in\mathbf{H}^1(\Omega)$ be such that*

$$\|\boldsymbol{u}\|_{1,\Omega} < \frac{\vartheta_1}{2\,c_p\,c(\Omega)}.$$

Then, the family of Gâteaux derivates $\{\mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi})\}_{\widehat{\phi}\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)}$ is uniformly bounded and uniformly elliptic on $\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\times\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$. More precisely, there exist positive constants $\widetilde{\lambda},\widetilde{\alpha}$, depending only on ϑ_1,ϑ_2 (cf. (2.4)), $c(\Omega)$, and c_p, such that for all $\widehat{\phi},\varphi,\psi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, there holds

$$|\mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi})(\psi,\varphi)| \,\le\, \widetilde{\lambda}\,\|\psi\|_{1,\Omega}\,\|\varphi\|_{1,\Omega} \qquad \textit{and} \qquad \mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi})(\psi,\psi) \,\ge\, \widetilde{\alpha}\,\|\psi\|^2_{1,\Omega}.$$

Proof. It proceeds similarly to the proof of ([19], Lem. 5.1). □

As a consequence of the ellipticity of the family $\{\mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi})\}_{\widehat{\phi}\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)}$, we obtain the following global inf-sup condition

$$\widetilde{\alpha}\,\|\psi\|_{1,\Omega} \,\le\, \sup_{\substack{\varphi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\\ \varphi\neq 0}} \frac{\mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi})(\psi,\varphi)}{\|\varphi\|_{1,\Omega}} \qquad \forall\,\psi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega). \tag{3.21}$$

Next, similarly as before, we simplify the subsequent writing by introducing the following constants

$$\widetilde{C} \,:=\, \frac{1}{\widetilde{\alpha}}, \quad C_3 := \widetilde{C}\,L_\gamma, \quad C_4 := r\,c(\Omega)\,\bar{C}, \quad C_5 := r\,c(\Omega)\,\widetilde{C}, \quad C_6 \,:=\, C_1\,C_5 \qquad C_7 \,:=\, C_2\,C_5, \tag{3.22}$$

where $\bar{C}$ is the constant provided by Lemma 3.2.

Lemma 3.5. *Assume that*

$$C_3\,|\boldsymbol{k}| \,+\, C_6\,\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_\mathrm{D}} \,+\, C_7\,\|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{1}{2}. \tag{3.23}$$

Then, there exists $\widehat{C}>0$, depending on $\widetilde{C}$ and C_4 (cf. (3.22)), such that

$$\|\phi-\phi_h\|_{1,\Omega} \,\le\, \widehat{C}\,\Big\{\boldsymbol{\theta}_0 \,+\, \|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'} \,+\, \|\widetilde{E}_h\|_{\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)'}\Big\}, \tag{3.24}$$

where $\boldsymbol{\theta}_0$ and E_h are given in the statement of Lemmas 3.2 and (3.10), respectively, and $\widetilde{E}_h\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)'$, defined for each $\varphi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ by

$$\widetilde{E}_h(\varphi) \,:=\, \int_\Omega g\,\varphi \,-\, \int_\Omega \Big\{\vartheta(|\nabla\phi_h|)\,\nabla\phi_h \,-\, \phi_h\,\boldsymbol{u}_h \,-\, \gamma(\phi_h)\,\boldsymbol{k}\Big\}\cdot\nabla\varphi, \tag{3.25}$$

satisfies

$$\widetilde{E}_h(\varphi_h) \,=\, 0 \qquad \forall\,\varphi_h\in\mathrm{H}^\phi_h. \tag{3.26}$$

Proof. Since ϕ and ϕ_h belong to $\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, a straightforward application of the mean value theorem yields the existence of a convex combination of ϕ and ϕ_h, say $\widehat{\phi}_h \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, such that

$$\mathcal{D}\mathbf{A}_{\boldsymbol{u}}(\widehat{\phi}_h)(\phi - \phi_h, \varphi) \,=\, [\mathbf{A}_{\boldsymbol{u}}(\phi) - \mathbf{A}_{\boldsymbol{u}}(\phi_h), \varphi] \qquad \forall\, \varphi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega).$$

Next, applying (3.21) to the Galerkin error $\psi := \phi - \phi_h$, we find that

$$\widetilde{\alpha}\, \|\phi - \phi_h\|_{1,\Omega} \,\leq\, \sup_{\substack{\varphi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega) \\ \varphi \neq 0}} \frac{[\mathbf{A}_{\boldsymbol{u}}(\phi) - \mathbf{A}_{\boldsymbol{u}}(\phi_h), \varphi]}{\|\varphi\|_{1,\Omega}}. \tag{3.27}$$

Now, using the fact that $[\mathbf{A}_{\boldsymbol{u}}(\phi), \varphi] = [G_\phi, \varphi]$, the definition of $\mathbf{A}_{\boldsymbol{u}}$ (*cf.* (3.17)), and adding and subtracting suitable terms, it follows that

$$[\mathbf{A}_{\boldsymbol{u}}(\phi) - \mathbf{A}_{\boldsymbol{u}}(\phi_h), \varphi] \,=\, [G_{\phi_h} \,-\, \mathbf{A}_{\boldsymbol{u}_h}(\phi_h), \varphi] \,+\, [G_\phi \,-\, G_{\phi_h}, \varphi] \,+\, [\mathbf{A}_{\boldsymbol{u}_h}(\phi_h) \,-\, \mathbf{A}_{\boldsymbol{u}}(\phi_h), \varphi]. \tag{3.28}$$

In this way, applying the estimate for $|[G_\phi \,-\, G_{\phi_h}, \varphi]|$ (see [2], Eq. (5.5)) and $|[\mathbf{A}_{\boldsymbol{u}}(\phi_h) \,-\, \mathbf{A}_{\boldsymbol{u}_h}(\phi_h), \varphi]|$ (see [2], Eq. (5.6)), we deduce from (3.27) and (3.28) that

$$\|\phi - \phi_h\|_{1,\Omega} \,\leq\, \widetilde{C}\, \|G_{\phi_h} - \mathbf{A}_{\boldsymbol{u}_h}(\phi_h)\|_{[\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)]'} \,+\, \widetilde{C}\, L_\gamma\, |\boldsymbol{k}|\, \|\phi - \phi_h\|_{1,\Omega} \,+\, r\, c(\Omega)\, \widetilde{C}\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}. \tag{3.29}$$

Then, bounding $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}$ by the error estimate provided by (3.9) (*cf.* Lem. 3.2), and then employing (3.23), we arrive at

$$\|\phi - \phi_h\|_{1,\Omega} \,\leq\, 2\, \widetilde{C} \left\{ \|G_{\phi_h} - \mathbf{A}_{\boldsymbol{u}_h}(\phi_h)\|_{[\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)]'} \,+\, C_4 \left(\boldsymbol{\theta}_0 \,+\, \|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'} \right) \right\},$$

where, bearing in mind (3.25), there holds

$$[G_{\phi_h} - \mathbf{A}_{\boldsymbol{u}_h}(\phi_h), \varphi] \,=\, \widetilde{E}_h(\varphi) \qquad \forall\, \varphi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega).$$

Finally, using the fact that $[G_{\phi_h} - \mathbf{A}_{\boldsymbol{u}_h}(\phi_h), \varphi_h] = 0 \quad \forall \varphi_h \in \mathrm{H}^\phi_h$, we obtain (3.26) and the proof concludes. □

At this point we remark, similarly as we did at the end of Section 3.2.1, and thanks now to (3.26), that for each $\varphi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ there holds

$$\widetilde{E}_h(\varphi) \,=\, \widetilde{E}_h(\varphi - \varphi_h) \qquad \forall\, \varphi_h \in \mathrm{H}^\phi_h,$$

and therefore $\|\widetilde{E}_h\|_{\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)'}$ will be estimated below (see Sect. 3.2.3) by employing the foregoing expression with a suitable choice of $\varphi_h \in \mathrm{H}^\phi_h$.

3.2.3. A preliminary estimate for the total error

We now combine the inequalities provided by Lemmas 3.2 and 3.5 to derive a first estimate for the total error $\|\phi - \phi_h\|_{1,\Omega} + \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H$. To this end, we now introduce the constants

$$C(\boldsymbol{u}_D, \boldsymbol{f}) \,:=\, \widehat{C} \left\{ C_1\, \|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_\mathrm{D}} \,+\, C_2\, \|\boldsymbol{f}\|_{\infty,\Omega} \,+\, 1 \right\} \quad \text{and} \quad c(\boldsymbol{u}_D, \boldsymbol{f}) \,:=\, \bar{C} \,+\, C(\boldsymbol{u}_D, \boldsymbol{f}),$$

where $\bar{C}$ and $\widehat{C}$ are provided by Lemmas 3.2 and 3.5, respectively, and C_1 and C_2 are given by (3.7).

Theorem 3.6. *Assume that*

$$C_3\, |\boldsymbol{k}| \,+\, C_6\, \|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_\mathrm{D}} \,+\, C_7\, \|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{1}{2}.$$

Then there holds

$$\|\phi - \phi_h\|_{1,\Omega} \,+\, \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H \leq C(\boldsymbol{u}_D, \boldsymbol{f})\, \|\widetilde{E}_h\|_{\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)'} \,+\, c(\boldsymbol{u}_D, \boldsymbol{f}) \left\{ \boldsymbol{\theta}_0 + \|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'} \right\}. \tag{3.30}$$

Proof. It suffices to replace the upper bound for $\|\phi - \phi_h\|_{1,\Omega}$ given by (3.24) into the second term on the right hand side of (3.9), and then add the resulting estimate to the right hand side of (3.24). We omit further details. □

It is clear from (3.30) that, in order to obtain an explicit estimate for the total error, it only remains to derive suitable upper bounds for $\|\widetilde{E}_h\|_{\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)'}$ and $\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}$. This is precisely the purpose of the next subsection.

3.2.4. Upper bounds for $\|\widetilde{E}_h\|_{\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)'}$ and $\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}$

In what follows we make use of the Clément interpolation operator $\mathcal{I}_h : \mathrm{H}^1(\Omega) \to X_h$ (*cf.* [17]), where X_h is given by

$$X_h := \{v_h \in C(\overline{\Omega}) : v_h|_T \in \mathrm{P}_1(T) \quad \forall T \in \mathcal{T}_h\}.$$

The following Lemma establishes the local approximation properties of $\mathcal{I}_h$.

Lemma 3.7. *There exist constants $c_1, c_2 > 0$, independent of h, such that for all $v \in \mathrm{H}^1(\Omega)$ there hold*

$$\|v - \mathcal{I}_h(v)\|_{0,T} \le c_1 h_T \|v\|_{1,\Delta(T)} \quad \forall T \in \mathcal{T}_h, \tag{3.31}$$

and

$$\|v - \mathcal{I}_h(v)\|_{0,e} \le c_2 h_e^{1/2} \|v\|_{1,\Delta(e)} \quad \forall e \in \mathcal{E}_h, \tag{3.32}$$

where $\Delta(T)$ and $\Delta(e)$ are the union of all elements intersecting with T and e, respectively.

Proof. See [17]. □

We now recall from Subsection 3.1 that we have defined there

$$\widetilde{\boldsymbol{\sigma}}_h := \vartheta(|\nabla\phi_h|)\nabla\phi_h - \phi_h \boldsymbol{u}_h - \gamma(\phi_h)\boldsymbol{k}. \tag{3.33}$$

Then, the following lemma provides an upper bound for $\|\widetilde{E}_h\|_{\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)'}$.

Lemma 3.8. *Let* $\widetilde{\boldsymbol{\eta}}^2 := \sum_{T\in\mathcal{T}_h} \widetilde{\eta}_T^2$, *where for each $T \in \mathcal{T}_h$ we set*

$$\widetilde{\eta}_T^2 := h_T^2 \|g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Omega)} h_e \|[\![\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}_e]\!]\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_\mathrm{N})} h_e \|\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}\|_{0,e}^2.$$

Then there exists $c > 0$, independent of h, such that

$$\|\widetilde{E}_h\|_{\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)'} \le c\, \widetilde{\boldsymbol{\eta}}. \tag{3.34}$$

Proof. Given $\varphi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ we let $\varphi_h := \mathcal{I}_h(\varphi) \in \mathrm{H}^\phi_h$, and observe, according to (3.25), (3.26), and (3.33), that

$$\widetilde{E}_h(\varphi) = \widetilde{E}_h(\varphi - \varphi_h) = \sum_{T\in\mathcal{T}_h} \int_T g\,(\varphi - \varphi_h) - \sum_{T\in\mathcal{T}_h} \int_T \widetilde{\boldsymbol{\sigma}}_h \cdot \nabla(\varphi - \varphi_h).$$

Next, integrating by parts on each $T \in \mathcal{T}_h$ in the last term on the right hand side of the foregoing equation, we find that

$$\widetilde{E}_h(\varphi) = \sum_{T\in\mathcal{T}_h} \int_T (g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h)(\varphi - \varphi_h) - \sum_{e\in\mathcal{E}_h(\Omega)} \int_e (\varphi - \varphi_h)\, [\![\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}_e]\!] - \sum_{e\in\mathcal{E}_h(\Gamma_\mathrm{N})} \int_e (\varphi - \varphi_h)\, \widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu},$$

from which, applying Cauchy−Schwarz inequality, employing the approximation properties of the Clément operator given by (3.31) and (3.32), and performing some algebraic rearrangements, we readily conclude that

$$|\widetilde{E}_h(\varphi)| \le c\, \widetilde{\boldsymbol{\eta}}\, \|\varphi\|_{1,\Omega},$$

which yields (3.34) and finishes the proof. □

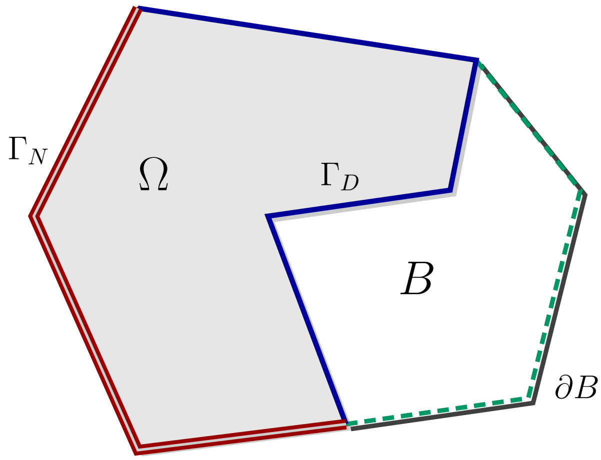

FIGURE 1. Extension of Ω to a convex domain B for the Helmholtz decomposition.

We now aim to provide an upper bound for $\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}$ (*cf.* (3.10)), which, being less straightforward than Lemma 3.8, requires several preliminary results and estimates. We begin by introducing the space

$$\mathbf{H}^1_{\Gamma_{\rm N}}(\Omega) := \Big\{\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega): \quad \boldsymbol{\varphi} = \mathbf{0} \quad \text{on} \quad \Gamma_{\rm N}\Big\},$$

and establishing a suitable Helmholtz decomposition of our space $\mathbb{H}_N(\mathbf{div},\Omega)$.

Lemma 3.9. *Assume that Ω is a connected domain and that $\Gamma_{\rm N}$ is contained in the boundary of a convex part of Ω, that is there exists a convex domain B such that $\overline{\Omega} \subseteq B$ and $\Gamma_{\rm N} \subseteq \partial B$ (see Fig. 1). Then, for each $\boldsymbol{\zeta} \in \mathbb{H}_N(\mathbf{div},\Omega)$, there exist $\boldsymbol{\tau} \in \mathbb{H}^1(\Omega)$ and $\boldsymbol{\chi} \in \mathbf{H}^1_{\Gamma_{\rm N}}(\Omega)$ such that*

$$\boldsymbol{\zeta} = \boldsymbol{\tau} + \underline{\mathbf{curl}}(\boldsymbol{\chi}) \quad \text{in} \quad \Omega, \tag{3.35}$$

and

$$\|\boldsymbol{\tau}\|_{1,\Omega} + \|\boldsymbol{\chi}\|_{1,\Omega} \leq C\,\|\boldsymbol{\zeta}\|_{\mathbf{div},\Omega}, \tag{3.36}$$

with a positive constant C independent of $\boldsymbol{\zeta}$.

Proof. Given $\boldsymbol{\zeta} \in \mathbb{H}_N(\mathbf{div},\Omega)$, we let $\boldsymbol{z} \in \mathbf{H}^2(B)$ be the unique weak solution of the boundary value problem:

$$\Delta \boldsymbol{z} = \begin{cases} \mathbf{div}\boldsymbol{\zeta} & \text{in} \quad \Omega \\ \dfrac{-1}{|B\backslash\overline{\Omega}|}\displaystyle\int_\Omega \mathbf{div}\boldsymbol{\zeta} & \text{in } B\backslash\overline{\Omega} \end{cases}, \qquad \nabla\boldsymbol{z}\,\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \partial B, \qquad \int_\Omega \boldsymbol{z} = \mathbf{0}. \tag{3.37}$$

Thanks to the elliptic regularity result of (3.37) we have that $\boldsymbol{z} \in \mathbf{H}^2(B)$ and

$$\|\boldsymbol{z}\|_{2,B} \leq c\,\|\mathbf{div}\boldsymbol{\zeta}\|_{0,\Omega}, \tag{3.38}$$

where $c > 0$ is independent of $\boldsymbol{z}$. In addition, it is clear that $\boldsymbol{\tau} := (\nabla \boldsymbol{z})|_\Omega \in \mathbb{H}^1(\Omega)$, $\mathbf{div}\boldsymbol{\tau} = \Delta\boldsymbol{z} = \mathbf{div}\boldsymbol{\zeta}$ in Ω, $\boldsymbol{\tau}\,\boldsymbol{\nu} = \mathbf{0}$ on ∂B (which certainly yields $\boldsymbol{\tau}\,\boldsymbol{\nu} = \mathbf{0}$ on $\Gamma_{\rm N}$), and

$$\|\boldsymbol{\tau}\|_{1,\Omega} \leq \|\boldsymbol{z}\|_{2,\Omega} \leq \|\boldsymbol{z}\|_{2,B} \leq c\,\|\mathbf{div}\boldsymbol{\zeta}\|_{0,\Omega}. \tag{3.39}$$

On the other hand, since $\mathbf{div}(\boldsymbol{\zeta}-\boldsymbol{\tau}) = \mathbf{0}$ in Ω, and Ω is connected, there exists $\boldsymbol{\chi} \in \mathbf{H}^1(\Omega)$ such that

$$\boldsymbol{\zeta}-\boldsymbol{\tau} = \underline{\mathbf{curl}}(\boldsymbol{\chi}) \qquad \text{in} \quad \Omega. \tag{3.40}$$

In turn, noting that $\mathbf{0} = (\boldsymbol{\zeta}-\boldsymbol{\tau})\boldsymbol{\nu} = \underline{\mathbf{curl}}(\boldsymbol{\chi})\boldsymbol{\nu} = \frac{\mathrm{d}\boldsymbol{\chi}}{\mathrm{d}\boldsymbol{s}}$ on Γ_{N}, we deduce that $\boldsymbol{\chi}$ is constant on Γ_{N}, and therefore $\boldsymbol{\chi}$ can be chosen so that $\boldsymbol{\chi} \in \mathbf{H}^1_{\Gamma_{\mathrm{N}}}(\Omega)$, which, together with (3.40), gives (3.35). In addition, from the equivalence between $\|\boldsymbol{\chi}\|_{1,\Omega}$ and $|\boldsymbol{\chi}|_{1,\Omega} = \|\underline{\mathbf{curl}}(\boldsymbol{\chi})\|_{0,\Omega}$ (which is a consequence of the generalized Poincaré inequality), and employing (3.39) and (3.35), we deduce that there exists a constant $\widetilde{c} > 0$ such that

$$\|\boldsymbol{\chi}\|_{1,\Omega} \le \widetilde{c}\,\|\boldsymbol{\zeta}\|_{\mathbf{div},\Omega}. \tag{3.41}$$

Finally, it is clear that (3.39) and (3.41) yield (3.36), which is the stability estimate for (3.35). □

We remark here, as already announced at the beginning of this section, that Lemma 3.9 also holds in the 3D case. The corresponding proof follows by combining the extension technique introduced in ([26], Lem. 4.3) with the approach suggested in the foregoing proof (*cf.* auxiliary problem (3.37)), and the results stated in ([10], Thms. 2.17 and 3.12, and Cor. 3.16).

We now consider the finite element subspace of $\mathbf{H}_{\Gamma_{\mathrm{N}}}(\Omega)$ given by

$$\mathbf{X}_{h,N} := \Big\{ \boldsymbol{\varphi}_h \in \mathbf{C}(\overline{\Omega}) : \quad \boldsymbol{\varphi}_h|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_h, \quad \boldsymbol{\varphi}_h = \mathbf{0} \text{ on } \Gamma_{\mathrm{N}} \Big\},$$

and introduce, analogously as before, the Clément interpolation operator $\boldsymbol{\mathcal{I}}_{h,N} : \mathbf{H}_{\Gamma_{\mathrm{N}}}(\Omega) \to \mathbf{X}_{h,N}$. In addition, we let $\Pi_h^k : \mathbb{H}^1(\Omega) \to \mathbb{H}_h^{\boldsymbol{\sigma}}$ be the Raviart−Thomas interpolation operator (see [13,31]), which, given $\bar{\boldsymbol{\tau}} \in \mathbb{H}^1(\Omega)$, is characterized by the identities:

$$\int_e \Pi_h^k(\bar{\boldsymbol{\tau}})\,\boldsymbol{\nu}\cdot\boldsymbol{p} = \int_e \bar{\boldsymbol{\tau}}\,\boldsymbol{\nu}\cdot\boldsymbol{p}, \quad \forall \text{ edge } e \in \mathcal{T}_h, \quad \forall \boldsymbol{p} \in \mathbf{P}_k(e), \quad \text{when } k \ge 0, \tag{3.42}$$

and

$$\int_T \Pi_h^k(\bar{\boldsymbol{\tau}}) : \boldsymbol{\rho} = \int_T \bar{\boldsymbol{\tau}} : \boldsymbol{\rho}, \quad \forall\, T \in \mathcal{T}_h, \quad \forall \boldsymbol{\rho} \in \mathbb{P}_{k-1}(T), \quad \text{when } k \ge 1, \tag{3.43}$$

where $\mathbf{P}_k(e) := [P_k(e)]^2$ and $\mathbb{P}_{k-1}(T) := [P_{k-1}(T)]^{2\times 2}$. It is easy to show, using (3.42) and (3.43), that (see, *e.g.* [21], Lem. 3.7, [30], Eq. (3.4.23))

$$\mathbf{div}(\Pi_h^k(\bar{\boldsymbol{\tau}})) = \mathcal{P}_h^k(\mathbf{div}\bar{\boldsymbol{\tau}}), \tag{3.44}$$

where $\mathcal{P}_h^k : \mathbf{L}^2(\Omega) \to \mathbf{Q}_h$ is the $\mathbf{L}^2(\Omega)$-orthogonal projector and

$$\mathbf{Q}_h := \big\{ \boldsymbol{v} \in \mathbf{L}^2(\Omega) : \quad \boldsymbol{v}|_T \in \mathbf{P}_k(T) \quad \forall\, T \in \mathcal{T}_h \big\}.$$

Note that $\mathcal{P}_h^k$ can also be identified with (P_h^k, P_h^k), where P_h^k is the orthogonal projector from $\mathrm{L}^2(\Omega)$ into Q_h with $\mathrm{Q}_h := \{ v \in \mathrm{L}^2(\Omega) : \quad v|_T \in P_k(T) \quad \forall\, T \in \mathcal{T}_h \}$. Furthermore, the following approximation properties hold (*cf.* [13,16,21,31]):

$$\|v - P_h^k(v)\|_{0,T} \le c\,h_T^m\,|v|_{m,T} \quad \forall T \in \mathcal{T}_h, \tag{3.45}$$

for each $v \in H^m(\Omega)$, with $0 \le m \le k+1$,

$$\|\bar{\boldsymbol{\tau}} - \Pi_h^k(\bar{\boldsymbol{\tau}})\|_{0,T} \le c\,h_T^m\,|\bar{\boldsymbol{\tau}}|_{m,T} \quad \forall T \in \mathcal{T}_h, \tag{3.46}$$

for each $\bar{\boldsymbol{\tau}} \in \mathbb{H}^m(\Omega)$, with $1 \le m \le k+1$,

$$\|\mathbf{div}(\bar{\boldsymbol{\tau}} - \Pi_h^k(\bar{\boldsymbol{\tau}}))\|_{0,T} \le c\,h_T^m\,|\mathbf{div}\bar{\boldsymbol{\tau}}|_{m,T} \quad \forall T \in \mathcal{T}_h, \tag{3.47}$$

for each $\bar{\boldsymbol{\tau}} \in \mathbb{H}^1(\Omega)$ such that $\mathbf{div}\bar{\boldsymbol{\tau}} \in \mathbf{H}^m(\Omega)$, with $0 \le m \le k+1$, and

$$\|\bar{\boldsymbol{\tau}}\boldsymbol{\nu} - \Pi_h^k(\bar{\boldsymbol{\tau}})\,\boldsymbol{\nu}\|_{0,e} \,\le\, c\, h_e^{1/2}\, \|\bar{\boldsymbol{\tau}}\|_{1,T_e} \quad \forall\, \text{edge}\, e \in \mathcal{T}_h, \tag{3.48}$$

for each $\bar{\boldsymbol{\tau}} \in \mathbb{H}^1(\Omega)$, where $T_e \in \mathcal{T}_h$ contains e on its boundary.

Then, given $\boldsymbol{\zeta} \in \mathbb{H}_N(\mathbf{div},\Omega)$ and its Helmholtz decomposition (3.35), we define $\boldsymbol{\chi}_h := \boldsymbol{\mathcal{I}}_{h,N}(\boldsymbol{\chi})$, and set

$$\boldsymbol{\zeta}_h \,:=\, \Pi_h^k(\boldsymbol{\tau}) \,+\, \underline{\mathbf{curl}}(\boldsymbol{\chi}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \tag{3.49}$$

as its associated discrete Helmholtz decomposition. It follows that

$$\boldsymbol{\zeta} - \boldsymbol{\zeta}_h \,=\, \boldsymbol{\tau} \,-\, \Pi_h^k(\boldsymbol{\tau}) \,+\, \underline{\mathbf{curl}}(\boldsymbol{\chi} - \boldsymbol{\chi}_h),$$

from which, using (3.44) and the fact that $\mathbf{div}\,\boldsymbol{\tau} = \Delta \boldsymbol{z} = \mathbf{div}\,\boldsymbol{\zeta}$ in Ω, yields

$$\mathbf{div}(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h) \,=\, \mathbf{div}(\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})) \,=\, (\mathbf{I} - \mathcal{P}_h^k)(\mathbf{div}\boldsymbol{\zeta}). \tag{3.50}$$

Hence, according to (3.10) and (3.11), and noting from (3.50) that

$$\int_\Omega \boldsymbol{u}_h \cdot \mathbf{div}(\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})) \,=\, \int_\Omega \boldsymbol{u}_h \cdot (\mathbf{I} \,-\, \mathcal{P}_h^k)(\mathbf{div}\,\boldsymbol{\zeta}) \,=\, 0,$$

we find that

$$E_h(\boldsymbol{\zeta}) \,=\, E_h(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h) \,=\, E_{h,1}(\boldsymbol{\tau}) \,+\, E_{h,2}(\boldsymbol{\chi}), \tag{3.51}$$

where

$$\begin{aligned} E_{h,1}(\boldsymbol{\tau}) \,:=\, & \langle (\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau}))\,\boldsymbol{\nu}, \boldsymbol{u}_D\rangle_{\Gamma_{\mathrm{D}}} \,-\, \int_\Omega \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}} : \big(\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\big) \\ & +\, \kappa_2 \int_\Omega (\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h)\cdot(\mathbf{I} - \mathcal{P}_h^k)(\mathbf{div}\,\boldsymbol{\tau})), \end{aligned} \tag{3.52}$$

and

$$E_{h,2}(\boldsymbol{\chi}) \,:=\, \langle \underline{\mathbf{curl}}(\boldsymbol{\chi} - \boldsymbol{\chi}_h)\,\boldsymbol{\nu}, \boldsymbol{u}_D\rangle_{\Gamma_{\mathrm{D}}} \,-\, \int_\Omega \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}} : \underline{\mathbf{curl}}(\boldsymbol{\chi} - \boldsymbol{\chi}_h). \tag{3.53}$$

It is now evident from (3.51) that, in order to estimate $\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}$, it only remains to bound $|E_{h,1}(\boldsymbol{\tau})|$ and $|E_{h,2}(\boldsymbol{\chi})|$ in terms of a multiple of $\|\boldsymbol{\zeta}\|_{\mathbf{div},\Omega}$, which is indeed the purpose of the following two lemmas.

Lemma 3.10. *Let* $\boldsymbol{\theta}_1^2 \,:=\, \sum_{T\in\mathcal{T}_h} \theta_{1,T}^2$, *where for each* $T \in \mathcal{T}_h$ *we set*

$$\theta_{1,T}^2 \,:=\, h_T^2 \left\|\nabla \boldsymbol{u}_h \,-\, \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\right\|_{0,T}^2 \,+\, \|\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{D}})} h_e\, \|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{0,e}^2.$$

Then there exists $c > 0$, independent of h, such that

$$|E_{h,1}(\boldsymbol{\tau})| \,\le\, c\,\boldsymbol{\theta}_1\, \|\boldsymbol{\zeta}\|_{\mathbf{div},\Omega}. \tag{3.54}$$

Proof. The analysis for the first two terms defining $E_{h,1}$ (*cf.* (3.52)) follows as in the proof of [24], Lem. 4.4, after replacing Γ by Γ_{D}, and then employing the characterization (3.42)−(3.43), the Cauchy−Schwarz inequality, the approximation properties (3.46) and (3.48), and the stability estimate (3.36). In turn, for the corresponding third term it suffices to see that

$$\begin{aligned} &\left|\int_\Omega (\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h)\cdot(\mathbf{I} - \mathcal{P}_h^k)(\mathbf{div}\,\boldsymbol{\tau}))\right| \\ &\qquad \le\quad \|\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h\|_{0,\Omega}\, \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \,\le\, \|\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h\|_{0,\Omega}\, \|\boldsymbol{\zeta}\|_{\mathbf{div},\Omega}, \end{aligned}$$

which concludes the proof. □

Lemma 3.11. *Assume that $\boldsymbol{u}_D \in \mathbf{H}_0^1(\Gamma_{\mathrm{D}})$, and let $\boldsymbol{\theta}_2^2 := \sum_{T\in\mathcal{T}_h} \theta_{2,T}^2$, where for each $T \in \mathcal{T}_h$ we set*

$$\theta_{2,T}^2 := h_T^2 \left\| \operatorname{curl}\left\{ \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\} \right\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Omega)} h_e \left\| \left[\!\left[\frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \boldsymbol{s} \right]\!\right] \right\|_{0,e}^2$$
$$+ \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{D}})} h_e \left\| \frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}\boldsymbol{s}} - \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \boldsymbol{s} \right\|_{0,e}^2 .$$

Then there exists $c > 0$, independent of h, such that

$$|E_{h,2}(\boldsymbol{\chi})| \le c\, \boldsymbol{\theta}_2 \, \|\boldsymbol{\zeta}\|_{\mathbf{div},\Omega}. \tag{3.55}$$

Proof. We proceed similarly as in the proof of ([24], Lem. 4.3). In fact, using that $\underline{\mathbf{curl}}(\boldsymbol{\chi}-\boldsymbol{\chi}_h)\,\boldsymbol{\nu} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{s}}(\boldsymbol{\chi}-\boldsymbol{\chi}_h)$, noting that $\frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}\boldsymbol{s}} \in \mathbf{L}^2(\Gamma_{\mathrm{D}})$, and then integrating by parts on Γ_{D}, we find that

$$\langle \underline{\mathbf{curl}}(\boldsymbol{\chi}-\boldsymbol{\chi}_h)\,\boldsymbol{\nu}, \boldsymbol{u}_D \rangle_{\Gamma_{\mathrm{D}}} = -\left\langle \boldsymbol{\chi}-\boldsymbol{\chi}_h, \frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}\boldsymbol{s}} \right\rangle_{\Gamma_{\mathrm{D}}} = -\sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{D}})} \int_e (\boldsymbol{\chi}-\boldsymbol{\chi}_h)\cdot \frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}\boldsymbol{s}}.$$

On the other hand, integrating by parts on each $T \in \mathcal{T}_h$, we obtain that

$$\begin{aligned}
\int_\Omega \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} : \underline{\mathbf{curl}}(\boldsymbol{\chi}-\boldsymbol{\chi}_h) &= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \operatorname{curl}\left\{ \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\} \cdot (\boldsymbol{\chi}-\boldsymbol{\chi}_h) - \int_{\partial T} \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}}\, \boldsymbol{s}\cdot(\boldsymbol{\chi}-\boldsymbol{\chi}_h) \right\} \\
&= \sum_{T\in\mathcal{T}_h} \int_T \operatorname{curl}\left\{ \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\} \cdot (\boldsymbol{\chi}-\boldsymbol{\chi}_h) - \sum_{e\in\mathcal{E}_h(\Omega)} \int_e \left[\!\left[\frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \boldsymbol{s} \right]\!\right] \cdot (\boldsymbol{\chi}-\boldsymbol{\chi}_h) \\
&\quad - \sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{D}})} \int_e \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}}\, \boldsymbol{s}\cdot(\boldsymbol{\chi}-\boldsymbol{\chi}_h) - \sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{N}})} \int_e \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}}\, \boldsymbol{s}\cdot(\boldsymbol{\chi}-\boldsymbol{\chi}_h).
\end{aligned}$$

Then, replacing the above expressions on the right hand side of (3.53), and using the fact that $\boldsymbol{\chi}|_{\Gamma_{\mathrm{N}}} = \boldsymbol{\chi}_h|_{\Gamma_{\mathrm{N}}} = \mathbf{0}$, we deduce that

$$\begin{aligned}
E_{h,2}(\boldsymbol{\chi}) &= \sum_{e\in\mathcal{E}_h(\Omega)} \int_e \left[\!\left[\frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \boldsymbol{s} \right]\!\right] \cdot (\boldsymbol{\chi}-\boldsymbol{\chi}_h) - \sum_{T\in\mathcal{T}_h} \int_T \operatorname{curl}\left\{ \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\} \cdot (\boldsymbol{\chi}-\boldsymbol{\chi}_h) \\
&\quad - \sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{D}})} \int_e \left\{ \frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}\boldsymbol{s}} - \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\} \cdot (\boldsymbol{\chi}-\boldsymbol{\chi}_h).
\end{aligned}$$

Next, since $\boldsymbol{\chi}_h := \boldsymbol{\mathcal{I}}_{h,N}(\boldsymbol{\chi})$, the approximation properties of $\boldsymbol{\mathcal{I}}_{h,N}$ (*cf.* Lem. 3.7) yield

$$\|\boldsymbol{\chi}-\boldsymbol{\chi}_h\|_{0,T} \le c_1\, h_T\, \|\boldsymbol{\chi}\|_{1,\Delta(T)} \qquad \forall\, T \in \mathcal{T}_h, \tag{3.56}$$

and

$$\|\boldsymbol{\chi}-\boldsymbol{\chi}_h\|_{0,e} \le c_2\, h_e^{1/2}\, \|\boldsymbol{\chi}\|_{1,\Delta(e)} \qquad \forall\, e \in \mathcal{E}_h. \tag{3.57}$$

In this way, applying the Cauchy–Schwarz inequality to each term in the above expression for $E_{h,2}(\boldsymbol{\chi})$, and making use of (3.56), (3.57), and (3.36), together with the fact that the number of triangles in $\Delta(T)$ and $\Delta(e)$ are bounded, the proof is finished. □

As a consequence of Lemmas 3.10 and 3.11 we conclude the following upper bound for $\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}$.

Lemma 3.12. *There exists $c > 0$, independent of h, such that*

$$\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'} \;\leq\; c\Big\{\boldsymbol{\theta}_1 \,+\, \boldsymbol{\theta}_2\Big\}.$$

Proof. It follows straightforwardly from (3.51) and the upper bounds (3.54) and (3.55). □

We now observe that the terms $h_T^2\|\nabla \boldsymbol{u}_h - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\|_{0,T}^2$ and $h_e\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{0,e}^2$, which appear in the definition of $\theta_{1,T}^2$ (*cf.* Lem. 3.10), are dominated by $\|\nabla \boldsymbol{u}_h - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\|_{0,T}^2$ and $\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{0,e}^2$, respectively, which form part of $\theta_{0,T}^2$ (*cf.* (3.8)). In this way, the reliability estimate (3.6) (*cf.* Thm. 3.1) is a direct consequence of Theorem 3.6, the definition of $\boldsymbol{\theta}_0$ (*cf.* Lem. 3.2), and Lemmas 3.8, 3.10, 3.11, and 3.12.

We end this section by remarking that the assumption (3.5) on the data $\boldsymbol{k}$, $\boldsymbol{u}_D$, and $\boldsymbol{f}$, which, as shown throughout the foregoing analysis, is a key estimate to derive (3.6), is, unfortunately, unverifiable in practice. In fact, while the data are certainly known in advance, the constants C_3, C_6, and C_7 involved in that condition (*cf.* (3.22)), which in turn are expressed in terms of the previous constants C_1 and C_2 (*cf.* (3.7)), depend all on boundedness and regularity constants of operators, as well as on parameters, some of which are not explicitly calculable, and hence it is not possible to check whether (3.5) is indeed satisfied or not. This is, however, a quite common fact arising in the analysis of many nonlinear problems, and only in very particular cases (usually related to simple geometries of the domain) it could eventually be circumvented.

3.3. Efficiency

The following theorem is the main result of this section.

Theorem 3.13. *There exists a constant $\overline{C}_{\mathtt{eff}} > 0$, which depends only on parameters, $|\boldsymbol{k}|$, $\|\boldsymbol{u}_D\|_{1/2,\Gamma_{\mathrm{D}}}$, $\|\boldsymbol{f}\|_{\infty,\Omega}$, and other constants, all of them independent of h, such that*

$$\overline{C}_{\mathtt{eff}}\,\boldsymbol{\theta} \;\leq\; \|\phi - \phi_h\|_{1,\Omega} \,+\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \,+\, \left\|\frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathtt{d}} - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\right\|_{0,\Omega} \,+\, \text{h.o.t.} \tag{3.58}$$

where h.o.t. *stands for one or several terms of higher order. Moreover, under the assumption that $\boldsymbol{\sigma} \in \mathbb{L}^4(\Omega)$, there exists a constant $C_{\mathtt{eff}} > 0$, which depends only on parameters, $|\boldsymbol{k}|$, $\|\boldsymbol{u}_D\|_{1/2,\Gamma_{\mathrm{D}}}$, $\|\boldsymbol{f}\|_{\infty,\Omega}$, $\|\boldsymbol{\sigma}\|_{\mathbb{L}^4(\Omega)}$, and other constants, all them independent of h, such that*

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} \;\leq\; \|\phi - \phi_h\|_{1,\Omega} \,+\, \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H \,+\, \text{h.o.t.} \tag{3.59}$$

Throughout this and the following sections we assume for simplicity that the nonlinear functions μ, ϑ, and γ are such that $\dfrac{1}{\mu(\phi_h)}$, $\vartheta(|\nabla\phi_h|)$, $\gamma(\phi_h)$, and hence $\widetilde{\boldsymbol{\sigma}}_h$ as well, are all piecewise polynomials. The same is assumed for the data $\boldsymbol{u}_D$ and g. Otherwise, and if μ^{-1}, ϑ, γ, $\boldsymbol{u}_D$, and g are sufficiently smooth, higher order terms given by the errors arising from suitable polynomial approximations of these expressions and functions would appear in (3.58) and (3.59) (*cf.* Thm. 3.13), which explains the eventual h.o.t. in these inequalities. In this regard, we remark that (3.58) constitutes what we call a *quasi-efficiency* estimate for the global residual error estimator $\boldsymbol{\theta}$ (*cf.* (3.4)). Indeed, the *quasi-efficiency* concept refers here to the fact that the expression appearing on the right hand side of (3.58) is not exactly the error, but part of it plus the nonlinear term given by $\|\frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathtt{d}} - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\|_{0,\Omega}$. However, assuming additionally that $\boldsymbol{\sigma} \in \mathbb{L}^4(\Omega)$, we show at the end of this section that the latter can be bounded by $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} + \|\phi - \phi_h\|_{1,\Omega}$, thus yielding the efficiency estimate given by (3.59).

In order to prove (3.58) and (3.59), in what follows we derive suitable upper bounds for the ten terms defining the local error indicator θ_T^2 (*cf.* (3.3)). We first notice, using that $\boldsymbol{f}\phi = -\mathbf{div}\boldsymbol{\sigma}$ in Ω, that there holds

$$\begin{aligned}\|\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h\|_{0,T}^2 \;&\leq\; 2\,\|\boldsymbol{f}(\phi - \phi_h)\|_{0,T}^2 \,+\, 2\,\|\mathbf{div}(\boldsymbol{\sigma} \,-\, \boldsymbol{\sigma}_h)\|_{0,T}^2 \\ &\leq 2\,\|\boldsymbol{f}\|_{\infty,\Omega}^2\,\|\phi - \phi_h\|_{0,T}^2 \,+\, 2\,\|\mathbf{div}(\boldsymbol{\sigma} \,-\, \boldsymbol{\sigma}_h)\|_{0,T}^2.\end{aligned} \tag{3.60}$$

In addition, since $\nabla \boldsymbol{u} = \frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathtt{d}}$ in Ω, we find that

$$\left\| \nabla \boldsymbol{u}_h - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}} \right\|_{0,T}^2 \leq 2\, \|\nabla \boldsymbol{u} - \nabla \boldsymbol{u}_h\|_{0,T}^2 + 2 \left\| \frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathtt{d}} - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}} \right\|_{0,T}^2 . \tag{3.61}$$

Furthermore, employing that $\boldsymbol{u} = \boldsymbol{u}_D$ on Γ_{D} and applying the trace theorem, we obtain that

$$\sum_{e \in \mathcal{E}_h(\Gamma_{\mathrm{D}})} \|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{0,e}^2 = \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Gamma_{\mathrm{D}}}^2 \leq c_0^2\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}^2, \tag{3.62}$$

where c_0 is the norm of the trace operator in $\mathbf{H}^1(\Omega)$.

The upper bounds of the remaining seven terms, which depend on the mesh parameters h_T and h_e, will be derived next. We proceed as in [14,15] (see also [20]), and apply results ultimately based on inverse inequalities (see [16]) and the localization technique introduced in [34], which is based on triangle-bubble and edge-bubble functions. To this end, we now introduce further notations and preliminary results. In fact, given $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions, respectively (see [34], Eqs. (1.4) and (1.6)), which satisfy:

(i) $\psi_T \in P_3(T)$, $\mathrm{supp}(\psi_T) \subseteq T$, $\psi_T = 0$ on ∂T, and $0 \leq \psi_T \leq 1$ in T.
(ii) $\psi_e|_T \in P_2(T)$, $\mathrm{supp}(\psi_e) \subseteq \omega_e := \cup\{T' \in \mathcal{T}_h : e \in \mathcal{E}_h(T')\}$, $\psi_e = 0$ on $\partial T \backslash \{e\}$, and $0 \leq \psi_e \leq 1$ in ω_e.

We also recall from [33] that, given $k \in \mathbb{N} \cup \{0\}$, there exists a linear operator $L : C(e) \to C(T)$ that satisfies $L(p) \in P_k(T)$ and $L(p)|_e = p \quad \forall p \in P_k(e)$. A corresponding vectorial version of L, that is the component-wise application of L, is denoted by $\mathbf{L}$. Additional properties of ψ_T, ψ_e and L are collected in the following lemma.

Lemma 3.14. *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2, c_3, and c_4, depending only on k and the shape regularity of the triangulations (minimum angle condition), such that for each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h(T)$, there hold*

$$\begin{aligned} \|\psi_T\, q\|_{0,T}^2 \,&\leq\, \|q\|_{0,T}^2 \,\leq\, c_1\, \|\psi_T^{1/2}\, q\|_{0,T}^2 \qquad \forall\, q \in P_k(T), \\ \|\psi_e\, L(p)\|_{0,T}^2 \,&\leq\, \|p\|_{0,e}^2 \,\leq\, c_2\, \|\psi_e^{1/2}\, p\|_{0,e}^2 \qquad \forall\, p \in P_k(e), \\ c_3\, h_e\, \|p\|_{0,e}^2 \,&\leq\, \|\psi_e^{1/2}\, L(p)\|_{0,T}^2 \,\leq\, c_4\, h_e\, \|p\|_{0,e}^2 \qquad \forall\, p \in P_k(e). \end{aligned} \tag{3.63}$$

Proof. (see [33], Lem. 4.1). □

The following inverse estimate is also needed.

Lemma 3.15. *Let $l, m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then, there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h$ there holds*

$$|q|_{m,T} \leq c\, h_T^{l-m}\, |q|_{l,T} \qquad \forall\, q \in P_k(T). \tag{3.64}$$

Proof. (see [16], Thm. 3.2.6). □

The following Lemma is required for the terms involving the curl operator and the tangential jumps across the edges of $\mathcal{T}_h$. It proofs, which makes use of Lemmas 3.14 and 3.15, can be found in [14].

Lemma 3.16. *Let $\boldsymbol{\rho}_h \in \mathbb{L}^2(\Omega)$ be a piecewise polynomial of degree $k \geq 0$ on each $T \in \mathcal{T}_h$. In addition, let $\boldsymbol{\rho} \in \mathbb{L}^2(\Omega)$ be such that* $\mathrm{curl}(\boldsymbol{\rho}) = 0$ *on each $T \in \mathcal{T}_h$. Then, there exist $c, \widetilde{c} > 0$, independent of h, such that*

$$\|\mathrm{curl}(\boldsymbol{\rho}_h)\|_{0,T} \leq c\, h_T^{-1}\, \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,T} \qquad \forall\, T \in \mathcal{T}_h$$

and

$$\|[\![\boldsymbol{\rho}_h \boldsymbol{s}_e]\!]\|_{0,e} \leq \widetilde{c}\, h_e^{-1/2}\, \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\omega_e} \qquad \forall\, e \in \mathcal{E}_h.$$

Proof. For the first estimate we refer to ([14], Lem. 4.3), whereas the second one follows from a slight modification of the proof of ([14], Lem. 4.4). Further details are omitted. □

We now apply Lemma 3.16 to obtain upper bounds for two other terms defining θ_T^2.

Lemma 3.17. *There exist $\widetilde{c}_1, \widetilde{c}_2 > 0$, independent of h such that*

$$h_T^2 \left\| \operatorname{curl}\left\{ \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\} \right\|_{0,T}^2 \leq \widetilde{c}_1 \left\| \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^{\mathtt{d}} - \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\|_{0,T}^2 \qquad \forall\, T \in \mathcal{T}_h,$$

$$h_e \left\| \left[\!\left[\frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \boldsymbol{s} \right]\!\right] \right\|_{0,e}^2 \leq \widetilde{c}_2 \left\| \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^{\mathtt{d}} - \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\|_{0,\omega_e}^2 \qquad \forall\, e \in \mathcal{E}_h(\Omega).$$

Proof. It suffices to apply Lemma 3.16 to $\boldsymbol{\rho}_h := \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}}$ and $\boldsymbol{\rho} := \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^{\mathtt{d}} = \nabla \boldsymbol{u}$. □

Lemma 3.18. *There exists $\widetilde{c}_3 > 0$, independent of h, such that*

$$h_e \left\| \frac{\mathrm{d}\boldsymbol{u}_D}{\mathrm{d}\boldsymbol{s}} - \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \boldsymbol{s} \right\|_{0,e}^2 \leq \widetilde{c}_3 \left\| \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^{\mathtt{d}} - \frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}} \right\|_{0,T_e}^2 \qquad \forall\ e \in \mathcal{E}_h(\Gamma_{\mathrm{D}}). \tag{3.65}$$

Proof. We proceed similarly as in the proof of ([24], Lem. 4.15), by replacing $\mathbf{g}$, Γ, and $\frac{1}{\mu} \boldsymbol{\sigma}_h^{\mathtt{d}}$ by $\boldsymbol{u}_D$, Γ_{D}, and $\frac{1}{\mu(\phi_h)} \boldsymbol{\sigma}_h^{\mathtt{d}}$, respectively. □

Finally, it only remains to provide upper bounds for the three terms completing the definition of the local error indicator θ_T^2 (*cf.* (3.3)). This requires, however, the preliminary result given by the following *a priori* estimate for the error $\|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2$.

Lemma 3.19. *There exists $C > 0$, depending on ϑ_1, ϑ_2, L_γ (cf. (2.4), (2.6)), and $|\boldsymbol{k}|$, such that*

$$\|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \leq C \Big\{ \|\phi - \phi_h\|_{1,T}^2 + \|\boldsymbol{u}(\phi - \phi_h)\|_{0,T}^2 + \|\phi_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,T}^2 \Big\}. \tag{3.66}$$

Proof. According to the definitions of $\widetilde{\boldsymbol{\sigma}}$ (*cf.* (2.1)) and $\widetilde{\boldsymbol{\sigma}}_h$ (*cf.* Sect. 3.1), and applying the triangle inequality, we obtain that

$$\begin{aligned} \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \leq 2 \Big\{ & \|\vartheta(|\nabla\phi|)\nabla\phi - \vartheta(|\nabla\phi_h|)\nabla\phi_h\|_{0,T}^2 + 2\,\|\boldsymbol{k}(\gamma(\phi) - \gamma(\phi_h))\|_{0,T}^2 \\ & + \quad 4\,\|\boldsymbol{u}(\phi - \phi_h)\|_{0,T}^2 + 4\,\|\phi_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,T}^2 \Big\}. \end{aligned} \tag{3.67}$$

We now recall from ([23], Thm. 3.8) that the nonlinear operator induced by the first term defining $\mathbf{A}_{\boldsymbol{u}}$ (*cf.* (3.17)) is Lipschitz-continuous with constant $L := \max\{\vartheta_2, 2\vartheta_2 - \vartheta_1\}$. In this way, applying the aforementioned Lipschitz continuity, but restricted to each triangle $T \in \mathcal{T}_h$ instead of Ω, and using the Lipschitz continuity assumption for γ (*cf.* (2.6)), we deduce from (3.67) that

$$\begin{aligned} \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \leq 2 \Big\{ & L^2\,\|\nabla\phi - \nabla\phi_h\|_{0,T}^2 + 2L_\gamma^2\,|\boldsymbol{k}|^2\,\|\phi - \phi_h\|_{0,T}^2 \\ & + \quad 4\,\|\boldsymbol{u}(\phi - \phi_h)\|_{0,T}^2 + 4\,\|\phi_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,T}^2 \Big\}, \end{aligned} \tag{3.68}$$

which readily yields (3.66) and ends the proof. □

Having proved the previous result we now establish the efficiency estimates given by the following three lemmas.

Lemma 3.20. *There exists $\widetilde{c}_4 > 0$, which depends only on L, L_γ, $|\boldsymbol{k}|$, and other constants, all them independent of h, such that*

$$h_T^2 \, \|g \, + \, \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \, \le \, \widetilde{c}_4 \left\{ \|\phi - \phi_h\|_{1,T}^2 \, + \, \|\boldsymbol{u}(\phi - \phi_h)\|_{0,T}^2 \, + \, \|\phi_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,T}^2 \right\}. \tag{3.69}$$

Proof. We proceed as in the proof of ([4], Lem. 4.4). In fact, given $T \in \mathcal{T}_h$ we first observe, using that $\mathrm{div}\, \widetilde{\boldsymbol{\sigma}} \, = \, -g$ in Ω, and integrating by parts, that

$$\|g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \, \le \, c_1 \, \|\psi_T^{1/2} \, (g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h)\|_{0,T}^2 \, = \, -c_1 \int_T \nabla\big(\psi_T \, (g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h)\big) \cdot (\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h).$$

Next, the Cauchy–Schwarz inequality, the inverse estimate (3.64), the fact that $0 \le \psi_T \le 1$, and the triangle inequality imply that

$$\|g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \, \le \, c_1 \, |\psi_T \, (g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h)|_{1,T} \, \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,T} \, \le \, C \, h_T^{-1} \, \|g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T} \, \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,T},$$

which gives

$$\|g + \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T} \, \le \, C \, h_T^{-1} \, \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}.$$

The foregoing inequality and (3.66) (*cf.* Lem. 3.19) imply (3.69) and complete the proof. □

Lemma 3.21. *There exists $\widetilde{c}_5 > 0$, which depends only on L, L_γ, $|\boldsymbol{k}|$, and other constants, all of them independent of h, such that for each $e \in \mathcal{E}_h(\Omega)$ there holds*

$$h_e \, \|[\![\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}_e]\!]\|_{0,e}^2 \, \le \, \widetilde{c}_5 \sum_{T \subseteq \omega_e} \left\{ \|\phi - \phi_h\|_{1,T}^2 \, + \, \|\boldsymbol{u}(\phi - \phi_h)\|_{0,T}^2 \, + \, \|\phi_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,T}^2 \right\}, \tag{3.70}$$

where ω_e is the union of the two triangles in $\mathcal{T}_h$ having e as an edge.

Proof. Proceeding analogously as in the proof of ([4], Lem. 4.5), we find that

$$h_e \, \|[\![\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}_e]\!]\|_{0,e}^2 \, \le \, c \sum_{T \subseteq \omega_e} \left\{ h_T^2 \, \|g \, + \, \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \, + \, \|\widetilde{\boldsymbol{\sigma}}_h \, - \, \widetilde{\boldsymbol{\sigma}}\|_{0,T}^2 \right\},$$

which, together with (3.66) and (3.69) (*cf.* Lems. 3.19 and 3.20), yields (3.70) and ends the proof.

□

Lemma 3.22. *There exists $\widetilde{c}_6 > 0$, which depends only on L, L_γ, $|\boldsymbol{k}|$, and other constants, all of them independent of h, such that for each $e \in \mathcal{E}_h(\Gamma_{\mathrm{N}})$ there holds*

$$h_e \, \|\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}\|_{0,e}^2 \, \le \, \widetilde{c}_6 \left\{ \|\phi - \phi_h\|_{1,T}^2 \, + \, \|\boldsymbol{u}(\phi - \phi_h)\|_{0,T}^2 \, + \, \|\phi_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,T}^2 \right\}, \tag{3.71}$$

where T is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. Following a similar reasoning to the proof of ([4], Lem. 4.6), we find that

$$h_e \, \|\widetilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\nu}\|_{0,e}^2 \, \le \, c \left\{ h_T^2 \, \|g \, + \, \mathrm{div}\, \widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \, + \, \|\widetilde{\boldsymbol{\sigma}}_h \, - \, \widetilde{\boldsymbol{\sigma}}\|_{0,T}^2 \right\},$$

which, thanks again to (3.66) and (3.69), provides (3.71) and ends the proof. □

In order to complete the global efficiency given by (3.58), we now need to estimate the terms $\|\boldsymbol{u}(\phi-\phi_h)\|_{0,T}^2$ and $\|\phi_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,T}^2$ appearing in the upper bounds provided by the last three lemmas. In fact, applying Cauchy−Schwarz's inequality, the compactness (and hence continuity) of the injections $i : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ and $\mathbf{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ (*cf.* [1], Thm. 6.3, [30], Thm. 1.3.5), and the *a priori* bound for $\|\boldsymbol{u}\|_{1,\Omega}$ given by (3.2), we find that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \|\boldsymbol{u}(\phi-\phi_h)\|_{0,T}^2 \;&\le\; \sum_{T\in\mathcal{T}_h} \|\boldsymbol{u}\|_{\mathbf{L}^4(T)}^2 \, \|\phi-\phi_h\|_{\mathrm{L}^4(T)}^2 \\ &\le\; \|\boldsymbol{u}\|_{\mathbf{L}^4(\Omega)}^2 \, \|\phi-\phi_h\|_{\mathrm{L}^4(\Omega)}^2 \;\le\; C\,\|\phi-\phi_h\|_{1,\Omega}^2,\end{aligned} \tag{3.72}$$

where C is a positive constant, independent of h, that depends only on $\|i\|$, $\|\mathbf{i}\|$, $\|\boldsymbol{u}_D\|_{1/2,\Gamma_{\mathrm{D}}}$, $\|\boldsymbol{f}\|_{\infty,\Omega}$, and r (*cf.* (3.1)). Similar arguments allow to establish the existence of another constant $C>0$, also independent of h, and depending now on $\|i\|$, $\|\mathbf{i}\|$, and r, such that

$$\sum_{T\in\mathcal{T}_h} \|\phi_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,T}^2 \;\le\; C\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}^2. \tag{3.73}$$

Consequently, it is not difficult to see that (3.58) follows straightforwardly from (3.60), (3.61), (3.62), Lemmas 3.17, 3.18, 3.20, 3.21, and 3.22, and the final estimates given by (3.72) and (3.73). Furthermore, adding and subtracting a suitable term, using the lower bound (*cf.* (2.3)) and the Lipschitz continuity (*cf.* (2.5)) of μ, and applying the boundedness of $\boldsymbol{\tau} \to \boldsymbol{\tau}^{\mathsf{d}}$, we find that

$$\left\| \frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathsf{d}} - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathsf{d}} \right\|_{0,\Omega} \;\le\; \frac{1}{\mu_1}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega} \;+\; \frac{L_\mu}{\mu_1^2}\,\|(\phi-\phi_h)\boldsymbol{\sigma}\|_{0,\Omega}, \tag{3.74}$$

from which, assuming now that $\boldsymbol{\sigma} \in \mathbb{L}^4(\Omega)$, and estimating $\|(\phi-\phi_h)\boldsymbol{\sigma}\|_{0,\Omega}$ almost verbatim as we derived (3.72) and (3.73), we arrive at (3.59), thus concluding the proof of Theorem 3.13.

4. A second *a posteriori* error estimator

In this section we introduce and analyze another *a posteriori* error estimator for our augmented mixed-primal finite element scheme (2.12), which is not based on the Helmholtz decomposition. More precisely, this second estimator arises simply from a different way of bounding $\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}$ in the preliminary estimate for the total error given by (3.30) (*cf.* Thm. 3.6). Then, with the same notations and discrete spaces introduced in Sections 2 and 3, we now set for each $T\in\mathcal{T}_h$ the local error indicator

$$\begin{aligned}\widetilde{\theta}_T^2 \;:=\; &\|\boldsymbol{f}\phi_h + \mathbf{div}\boldsymbol{\sigma}_h\|_{0,T}^2 + \left\|\nabla\boldsymbol{u}_h - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathsf{d}}\right\|_{0,T}^2 + h_T^2\,\|g + \mathrm{div}\,\widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2 \\ &+ \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Omega)} h_e\,\|[\![\widetilde{\boldsymbol{\sigma}}_h\cdot\boldsymbol{\nu}_e]\!]\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{N}})} h_e\,\|\widetilde{\boldsymbol{\sigma}}_h\cdot\boldsymbol{\nu}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{D}})} \|\boldsymbol{u}_D-\boldsymbol{u}_h\|_{0,e}^2,\end{aligned}$$

and define the following global residual error estimator

$$\widetilde{\boldsymbol{\theta}}^{\,2} \;:=\; \sum_{T\in\mathcal{T}_h} \widetilde{\theta}_T^2 \;+\; \|\boldsymbol{u}_D-\boldsymbol{u}_h\|_{1/2,\Gamma_{\mathrm{D}}}^2. \tag{4.1}$$

In what follows we establish *quasi-local* reliability and efficiency for the estimator $\widetilde{\boldsymbol{\theta}}$. The name *quasi-local* refers here to the fact that the last term defining $\widetilde{\boldsymbol{\theta}}$ can not be decomposed into local quantities associated to each triangle $T\in\mathcal{T}_h$ (unless it is either conveniently bounded or previously modified, as we will see below).

Theorem 4.1. *Assume that* $\boldsymbol{u}_D \in \mathbf{H}_0^1(\Gamma_{\mathrm{D}})$ *and*

$$C_3\,|\boldsymbol{k}| \;+\; C_6\,\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_{\mathrm{D}}} \;+\; C_7\,\|\boldsymbol{f}\|_{\infty,\Omega} \;<\; \frac{1}{2},$$

where C_3, C_6 and C_7 are the constants given in (3.22). Then, there exists a constant $\widetilde{C}_{\mathtt{rel}} > 0$, which depends only on $\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_{\mathrm{D}}}$, $\|\boldsymbol{f}\|_{\infty,\Omega}$ and other constants, all them independent of h, such that

$$\|\phi - \phi_h\|_{1,\Omega}^2 \,+\, \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H^2 \,\leq\, \widetilde{C}_{\mathtt{rel}}\,\widetilde{\boldsymbol{\theta}}^2. \tag{4.2}$$

Proof. As mentioned at the beginning of this section, the proof reduces basically to derive another upper bound for $\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'}$. Indeed, Integrating by parts the third term defining E_h (*cf.* (3.10)), and then using the homogeneous Neumann boundary condition on Γ_{N}, we find that for each $\boldsymbol{\zeta} \in \mathbb{H}_N(\mathbf{div},\Omega)$ there holds

$$E_h(\boldsymbol{\zeta}) \,=\, \langle \boldsymbol{\zeta}\,\boldsymbol{\nu}, \boldsymbol{u}_D - \boldsymbol{u}_h\rangle_{\Gamma_{\mathrm{D}}} \,+\, \int_\Omega \left(\nabla \boldsymbol{u}_h \,-\, \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\right) : \boldsymbol{\zeta} \,-\, \kappa_2 \int_\Omega (\boldsymbol{f}\phi_h \,+\, \mathbf{div}\boldsymbol{\sigma}_h)\cdot \mathbf{div}\boldsymbol{\zeta},$$

from which, applying the Cauchy−Schwarz inequality, we readily deduce that

$$\|E_h\|_{\mathbb{H}_N(\mathbf{div},\Omega)'} \,\leq\, C\left\{\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1/2,\Gamma_{\mathrm{D}}} \,+\, \left\|\nabla \boldsymbol{u}_h \,-\, \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\right\|_{0,\Omega} \,+\, \|\boldsymbol{f}\phi_h + \mathbf{div}\boldsymbol{\sigma}_h\|_{0,\Omega}\right\}, \tag{4.3}$$

where C is a positive constant independent of h. In this way, replacing (4.3) back into (3.30) (*cf.* Thm. 3.6), and employing again the upper bound for $\|\widetilde{E}_h\|_{\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)'}$ (*cf.* Lem. 3.8), and the definition of $\boldsymbol{\theta}_0$ (*cf.* Lem. 3.2), we obtain (4.2) and finish the proof. □

Theorem 4.2. *There exists a constant $C^*_{\mathtt{eff}} > 0$, which depends only on parameters, $|\boldsymbol{k}|$, $\|\boldsymbol{u}_D\|_{1/2,\Gamma_{\mathrm{D}}}$, $\|\boldsymbol{f}\|_{\infty,\Omega}$, and other constants, all them independent of h, such that*

$$C^*_{\mathtt{eff}}\,\widetilde{\boldsymbol{\theta}}^2 \,\leq\, \|\phi - \phi_h\|_{1,\Omega}^2 \,+\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}^2 \,+\, \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}^2 \,+\, \left\|\frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathtt{d}} - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\right\|_{0,\Omega}^2 \,+\, \text{h.o.t.} \tag{4.4}$$

where h.o.t. *stands for one or several terms of higher order. Moreover, assuming $\boldsymbol{\sigma} \in \mathbb{L}^4(\Omega)$, there exists a constant $\widetilde{C}_{\mathtt{eff}} > 0$, which depends only on parameters, $|\boldsymbol{k}|$, $\|\boldsymbol{u}_D\|_{1/2,\Gamma_{\mathrm{D}}}$, $\|\boldsymbol{f}\|_{\infty,\Omega}$, $\|\boldsymbol{\sigma}\|_{\mathbb{L}^4(\Omega)}$, and other constants, all them independent of h, such that*

$$\widetilde{C}_{\mathtt{eff}}\,\widetilde{\boldsymbol{\theta}}^2 \,\leq\, \|\phi - \phi_h\|_{1,\Omega}^2 \,+\, \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H^2 \,+\, \text{h.o.t.} \tag{4.5}$$

Proof. We simply observe, thanks to the trace theorem in $\mathbf{H}^1(\Omega)$, that there exists $c > 0$, depending on Γ_{D} and Ω, such that

$$\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1/2,\Gamma_{\mathrm{D}}}^2 \,\leq\, c\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}^2. \tag{4.6}$$

The rest of the arguments are contained in the proof of Theorem 3.13 (*cf.* Sect. 3.3), and hence we omit further details. □

At this point we remark that the eventual use of $\widetilde{\boldsymbol{\theta}}$ (*cf.* (4.1)) in an adaptive algorithm solving (2.12) would be discouraged by the non-local character of the expression $\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1/2,\Gamma_{\mathrm{D}}}^2$. In order to circumvent this situation, we now apply an interpolation argument and replace this term by a suitable upper bound, which yields a reliable and fully local *a posteriori* error estimate.

Theorem 4.3. *Assume that $\boldsymbol{u}_D \in \mathbf{H}_0^1(\Gamma_{\mathrm{D}})$ and that*

$$C_3\,|\boldsymbol{k}| \,+\, C_6\,\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_{\mathrm{D}}} \,+\, C_7\,\|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{1}{2},$$

where C_3, C_6 and C_7 are given in (3.22). In turn, let $\widehat{\boldsymbol{\theta}}^2 := \sum_{T\in\mathcal{T}_h} \widehat{\theta}_T^2$, where for each $T \in \mathcal{T}_h$ we set

$$\widehat{\theta}_T^2 := \|\boldsymbol{f}\phi_h + \mathbf{div}\boldsymbol{\sigma}_h\|_{0,T}^2 + \left\|\nabla\boldsymbol{u}_h - \frac{1}{\mu(\phi_h)}\boldsymbol{\sigma}_h^{\mathtt{d}}\right\|_{0,T}^2 + h_T^2\,\|g + \operatorname{div}\widetilde{\boldsymbol{\sigma}}_h\|_{0,T}^2$$
$$+ \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Omega)} h_e\,\|[\![\widetilde{\boldsymbol{\sigma}}_h\cdot\boldsymbol{\nu}_e]\!]\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{N}})} h_e\,\|\widetilde{\boldsymbol{\sigma}}_h\cdot\boldsymbol{\nu}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{D}})} \|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1,e}^2.$$

Then, there exists a constant $\widehat{C}_{\mathtt{rel}} > 0$, which depends only on parameters, $\|\boldsymbol{u}_D\|_{1/2+\varepsilon,\Gamma_{\mathrm{D}}}$, $\|\boldsymbol{f}\|_{\infty,\Omega}$, and other constants, all them independent of h, such that

$$\|\phi - \phi_h\|_{1,\Omega}^2 + \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H^2 \le \widehat{C}_{\mathtt{rel}}\,\widehat{\boldsymbol{\theta}}^2. \tag{4.7}$$

Proof. It reduces to bound $\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1/2,\Gamma_{\mathrm{D}}}$. In fact, since $H^{1/2}(\Gamma_{\mathrm{D}})$ is the interpolation space with index $1/2$ between $H^1(\Gamma_{\mathrm{D}})$ and $L^2(\Gamma_{\mathrm{D}})$, there exists a constant $c_D > 0$, depending on Γ_{D}, such that

$$\begin{aligned}\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1/2,\Gamma_{\mathrm{D}}}^2 &\le c_D\,\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{0,\Gamma_{\mathrm{D}}}\,\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1,\Gamma_{\mathrm{D}}}\\ &\le c_D\,\|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1,\Gamma_{\mathrm{D}}}^2 = c_D \sum_{e\in\mathcal{E}_h(\Gamma_{\mathrm{D}})} \|\boldsymbol{u}_D - \boldsymbol{u}_h\|_{1,e}^2,\end{aligned} \tag{4.8}$$

which, together with (4.2), implies (4.7) and finishes the proof. □

The same remark stated at the end of Section 3.2 concerning the assumption (3.5) (which is also required in Thm. 4.3) is valid here.

5. Numerical tests

This section serves to illustrate the properties of the estimators introduced in Sections 3 and 4. The domain of each example to be considered below is discretized into a series of nested uniform triangulations, where errors and experimental convergence rates will be computed as usual

$$e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega},\quad e(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega},\quad e(\phi) := \|\phi - \phi_h\|_{1,\Omega},$$
$$r(\boldsymbol{\sigma}) := \frac{\log(e(\boldsymbol{\sigma})/\hat{e}(\boldsymbol{\sigma}))}{\log(h/\hat{h})},\quad r(\boldsymbol{u}) := \frac{\log(e(\boldsymbol{u})/\hat{e}(\boldsymbol{u}))}{\log(h/\hat{h})},\quad r(\phi) := \frac{\log(e(\phi)/\hat{e}(\phi))}{\log(h/\hat{h})},$$

with e and $\hat{e}$ denoting errors associated to two consecutive meshes of sizes h and $\hat{h}$, respectively. In addition, the total error, the modified error suggested by (3.58) and (4.4), and the effectivity and quasi-effectivity indexes associated to a given global estimator η are defined, respectively, as

$$\mathtt{e} = \left\{[e(\boldsymbol{\sigma})]^2 + [e(\boldsymbol{u})]^2 + [e(\phi)]^2\right\}^{1/2},\qquad \mathtt{eff}(\eta) = \frac{\mathtt{e}}{\eta},$$
$$\mathtt{m} = \left\{[e(\boldsymbol{u})]^2 + [e(\phi)]^2 + \|\mathbf{div}\boldsymbol{\sigma} - \mathbf{div}\boldsymbol{\sigma}_h\|_{0,\Omega}^2 + \left\|\frac{\boldsymbol{\sigma}^{\mathtt{d}}}{\mu(\phi)} - \frac{\boldsymbol{\sigma}_h^{\mathtt{d}}}{\mu(\phi_h)}\right\|_{0,\Omega}^2\right\}^{1/2},\qquad \mathtt{qeff}(\eta) = \frac{\mathtt{m}}{\eta}\cdot$$

According to the coupling structure of the scheme (2.12), the linearization of the coupled problem can follow a Newton method solving the nonlinear transport problem, nested within a Picard iteration to establish the coupling with the Stokes problem. This procedure requires the computation of the Gâteaux derivative (3.20). When the residuals from Newton−Raphson and Picard iterations reach the tolerances ε_{N} = 1e-8 and ε_{P} = 1e-7,

TABLE 1. Test 1: Convergence history, average Newton iteration count, Picard steps to reach the desired tolerance, effectivity and quasi-effectivity indexes for the mixed-primal $\mathbf{RT}_k - \mathbf{P}_{k+1} - \mathrm{P}_{k+1}$ approximations of Cauchy stress, velocity, and concentration, with $k = 0, 1$.

h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(\phi)$	$r(\phi)$	i_N	i_P	$\mathtt{eff}(\boldsymbol{\theta})$	$\mathtt{qeff}(\boldsymbol{\theta})$	$\mathtt{eff}(\widetilde{\boldsymbol{\theta}})$	$\mathtt{qeff}(\widetilde{\boldsymbol{\theta}})$
Augmented $\mathbf{RT}_0 - \mathbf{P}_1 - \mathrm{P}_1$ scheme												
0.7071	99.1853	–	10.1168	–	1.5980	–	4	12	1.8570	1.8603	1.8740	1.8773
0.4714	83.1416	0.4351	8.6706	0.3804	1.1558	0.7990	3	15	1.5604	1.5627	1.5950	1.5973
0.2828	56.1085	0.7698	6.1721	0.6653	0.7191	0.9288	5	17	1.2142	1.2158	1.2474	1.2490
0.1571	31.7872	0.9667	2.9676	1.2458	0.4035	0.9828	4	16	1.0694	1.0707	1.1021	1.1034
0.0831	16.7731	1.0051	1.3190	1.2749	0.2136	0.9998	4	16	1.0088	1.0101	1.0409	1.0421
0.0428	8.5927	1.0083	0.6226	1.1316	0.1100	1.0009	4	16	0.9861	0.9873	1.0180	1.0193
0.0217	4.3466	1.0053	0.3071	1.0422	0.0558	1.0003	5	16	0.9777	0.9789	1.0097	1.0110
Augmented $\mathbf{RT}_1 - \mathbf{P}_2 - \mathrm{P}_2$ scheme												
0.7071	51.6128	–	7.1534	–	0.4563	–	5	13	1.0353	1.0370	1.1487	1.1506
0.4714	29.6423	1.3677	3.2044	1.9805	0.2157	1.8475	4	14	0.9638	0.9650	1.0549	1.0562
0.2828	15.2131	1.3058	1.2003	1.9222	0.0799	1.9439	4	16	0.9573	0.9580	1.0198	1.0205
0.1571	4.9972	1.8940	0.3289	2.2022	0.0249	1.9837	5	15	0.9525	0.9532	1.0088	1.0095
0.0831	1.4251	1.9726	0.0868	2.0947	0.0070	1.9944	5	16	0.9515	0.9522	1.0045	1.0052
0.0428	0.3799	1.9928	0.0225	2.0320	0.0018	1.9984	4	16	0.9488	0.9495	1.0003	1.0011
0.0217	0.0980	1.9983	0.0057	2.0070	0.0004	1.9996	5	16	0.9396	0.9403	0.9895	0.9902

respectively, the algorithms are terminated. The unsymmetric multi-frontal direct solver for sparse matrices (UMFPACK) is used to solve the linear systems appearing at each linearization step.

In a first example, the following exact solutions to system (2.1) are considered

$$\phi(x_1,x_2) = b - b\exp(-x_1(x_1-1)x_2(x_2-1)), \quad \boldsymbol{u}(x_1,x_2) = \begin{pmatrix} \sin(2\pi x_1)\cos(2\pi x_2) \\ -\cos(2\pi x_1)\sin(2\pi x_2) \end{pmatrix},$$
$$\boldsymbol{\sigma}(x_1,x_2) = \mu(\phi)\,\nabla\boldsymbol{u} - \mu(\phi)\frac{\partial u_1}{\partial x_1}\mathbb{I},$$

defined on the unit square $\Omega = (0,1)^2$ and satisfying the first and third conditions of (2.2) on the whole boundary $\Gamma_{\mathrm{D}} = \partial\Omega$. The data $\boldsymbol{u}_D, \boldsymbol{f}, g$ are constructed with these manufactured exact solutions, and the involved coefficients in the equations (and in the solutions) are $\boldsymbol{k} = (0,-1)^T$, $\mu(\phi) = (1-c\phi)^{-2}$, $\gamma(\phi) = c\phi(1-c\phi)^2$, $\vartheta(|\nabla\phi|) = m_1 + m_2(1+|\nabla\phi|^2)^{m_3/2-1}$, with $b = 15, c = m_1 = m_2 = 1/2, m_3 = 3/2$. These values imply $\mu_1 = 0.99$, $\mu_2 = 3.35$, and consequently the stabilization parameters adopt the values $\kappa_1 = \mu_1^2/\mu_2 = 0.2976$, $\kappa_2 = 1/\mu_2 = 0.2985$, and $\kappa_3 = \kappa_1/2 = 0.1488$.

The manufactured solutions on the considered (convex) domain are smooth, and the *a posteriori* error indicators show effectivity (and quasi-effectivity) indexes close to one in all studied cases. This behavior can be observed in Table 1, where errors in different norms indicate optimal convergence rates for the two lowest order methods ($k = 0, 1$). We also show the average number of Newton steps to achieve the tolerance ε_{N} and the total Picard iteration count at each refinement level. The subsequent examples will be restricted to the lowest order method $k = 0$.

Our second test focuses on the case where, under uniform mesh refinement, the convergence rates are affected by the loss of regularity of the exact solutions. The problem setting is as follows: the domain is taken as the non-convex *pacman*-shaped domain $\Omega = \{(x_1,x_2) \in \mathrm{R}^2 : x_1^2 + x_2^2 \leq 1\} \setminus (0,1)^2$, where an exact solution to (2.1) is given by the same velocity as in the previous test, while concentration and Cauchy stress now read

$$\begin{aligned} \phi(x_1,x_2) &= b - b\exp(x_1^2(x_1^2+x_2^2-1)), \\ \boldsymbol{\sigma}(x_1,x_2) &= \mu(\phi)\,\nabla\boldsymbol{u} - \left[\mu(\phi)\frac{\partial u_2}{\partial x_2} + \frac{x_2}{((x_1-a_1)^2+(x_2-a_2)^2)^2}\right]\mathbb{I}. \end{aligned} \tag{5.1}$$

Now the boundary is indeed split into $\Gamma_{\mathrm{N}} = (0,1)\times\{0\}$ (the horizontal segment of $\partial\Omega$) and $\Gamma_{\mathrm{D}} = \partial\Omega\setminus\Gamma_{\mathrm{N}}$ (the arch and vertical borders of the domain), and the only difference with respect to (2.1) is that a non-homogeneous concentration flux $\widetilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu} = j$ is imposed on Γ_{N}, where j is manufactured according to (5.1). In this case, the relevant term in the *a posteriori* error estimators will be evidently replaced by

$$\sum_{e\in\mathcal{E}_h(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{N}})} h_e\,\|\widetilde{\boldsymbol{\sigma}}_h\cdot\boldsymbol{\nu} - j\|_{0,e}^2,$$

whose estimation from below and above follows in a straightforward manner. For this example, the individual and total convergence rates are determined by the expression

$$r(\cdot) := -2\log(e(\cdot)/\hat{e}(\cdot))[\log(N/\hat{N})]^{-1},$$

where N and $\hat{N}$ denote the total number of degrees of freedom associated to each triangulation. Alternatively to the first test, here the Picard tolerance is set to $\varepsilon_{\mathrm{P}} = $ 1e-6, and no inner Newton linearization will be employed for the transport problem.

The viscosity, hindered settling and diffusivity functions μ,γ,ϑ are taken as in the first example with the parameters $a_1 = 0.1$, $a_2 = 0.5$, $b = 3$, $c = 4/3$. Notice that the isotropic part of the stress in (5.1) exhibits a singularity just outside the domain, at (a_1,a_2). With the chosen parameters, the concentration has a high gradient near Γ_{D}, and the viscosity vanishes whenever the concentration attains its maximum value. Therefore, and according to (5.1), high gradients are also expected in the stress approximation; and optimal convergence, especially in that field, is no longer evidenced under uniform mesh refinement (see first rows of Tab. 2). On the other hand, if an adaptive mesh refinement step (guided by the proposed residual error indicators) is applied, optimal convergence can be restored, as shown in the last two blocks of Table 2. This feature is also seen in Figure 3, where we plot the total errors $\mathtt{e},\mathtt{m}$ versus the degrees of freedom associated to each triangulation. Total errors under adaptive refinement exhibit a superconvergence whereas uniform refinement yields suboptimal rates. From the figure we also observe that the curves for $\mathtt{e}$ and $\mathtt{m}$ coincide for each algorithm.

Once the local and global error indicators are computed, the adaptation procedure uses the automatic `adaptmesh` tool (see particulars in *e.g.* [27]) to construct the next triangulation. The algorithm is based on an equi-distribution of the discrete *a posteriori* error indicators, where the diameter of each triangle in $\mathcal{T}_{h_{i+1}}$, which is contained in a generic element $T\in\mathcal{T}_{h_i}$ in the new step of the algorithm, is proportional to h_T times the ratio $\hat{\boldsymbol{\zeta}}_T/\boldsymbol{\zeta}_T$, where $\hat{\boldsymbol{\zeta}}$ denotes the mean value of an estimator over $\mathcal{T}_h$. Approximate solutions obtained after six adaptation steps are depicted in Figure 2, whereas a few adaptive meshes generated using the two proposed indicators are collected in Figure 4. At least for this particular configuration, the second *a posteriori* error estimator produces smaller errors but the convergence rates coincide with the ones obtained with the first indicator.

In our last example the assumptions on the diffusivity ϑ will not hold anymore: we allow ϑ to be constant and very small for any concentration below a so-called gel point $\phi\in[0,\phi_{\mathrm{c}}]$. This extension (whose limit case translates into a loss of ellipticity in the concentration equation) implies that for low volume fractions, one may expect shock-like fronts to develop (see *e.g.* the monograph [9] and the recent review [5]). Adaptive mesh refinement would then be highly appreciated in this particular case; not only to restitute optimal convergence orders, but also to resolve concentration profiles accurately without the need of refining the grid everywhere (even more important if higher-order schemes are used, or transient models are studied). The problem configuration corresponds to the so-called Boycott effect (*cf.* [6]), where the sedimentation-consolidation of small particles

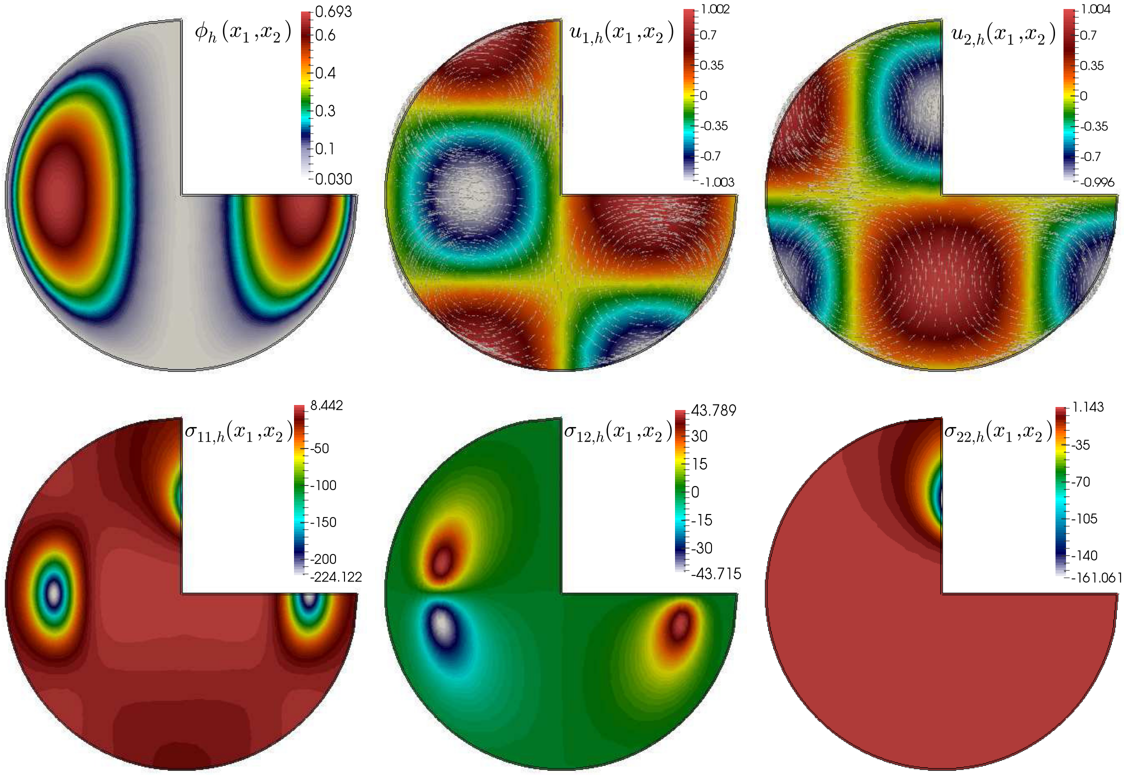

FIGURE 2. Test 2: Approximate solutions obtained with the lowest order method, after six steps of adaptive mesh refinement following the second indicator $\widetilde{\boldsymbol{\theta}}$. Concentration, velocity components, and stress components are depicted.

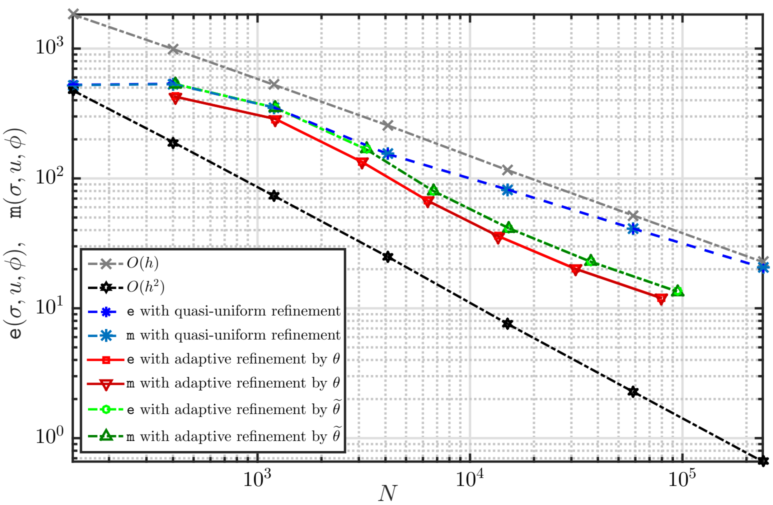

FIGURE 3. Test 2: log-log plot of the total errors *vs.* degrees of freedom associated to uniform and adaptive mesh refinements using the two proposed indicators.

TABLE 2. Test 2: Convergence history, Picard iteration count, effectivity and quasi-effectivity indexes for the mixed-primal approximation of the coupled problem under quasi-uniform, and adaptive refinement according to the indicators introduced in Sections 3, 4.

N	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(\phi)$	$r(\phi)$	i_P	$\mathtt{eff}(\boldsymbol{\theta})$	$\mathtt{qeff}(\boldsymbol{\theta})$	$\mathtt{eff}(\widetilde{\boldsymbol{\theta}})$	$\mathtt{qeff}(\widetilde{\boldsymbol{\theta}})$
Augmented $\mathbf{RT}_0 - \mathbf{P}_1 - \mathrm{P}_1$ scheme with quasi-uniform refinement											
135	497.2285	–	172.3239	–	1.6796	–	25	1.0930	1.0855	1.0736	1.0662
403	501.6796	-0.0174	189.3379	-0.1835	1.5883	0.1090	29	0.9688	0.9610	0.9654	0.9576
1191	325.2386	0.8264	136.1004	0.6295	0.7731	1.3728	28	1.0512	1.0490	1.0413	1.0391
4090	150.5401	1.2842	35.1355	2.2575	0.4022	1.0896	29	1.0112	1.0078	1.0096	1.0062
15074	81.0276	0.9336	12.4395	1.5649	0.1990	1.0606	28	1.0022	0.9988	1.0031	0.9997
58289	41.1328	1.0175	3.0515	2.1089	0.1012	1.0146	31	1.0004	0.9968	1.0031	0.9996
238705	20.5693	1.0063	0.8447	1.8650	0.0510	0.9956	29	0.9997	0.9962	1.0032	0.9997
Augmented $\mathbf{RT}_0 - \mathbf{P}_1 - \mathrm{P}_1$ scheme with adaptive refinement according to $\boldsymbol{\theta}$											
409	482.0538	–	229.0604	–	1.1557	–	21	1.1461	1.1431	–	–
1215	325.3517	0.7222	151.3681	0.7610	0.5798	1.2670	20	1.0059	1.0040	–	–
3108	160.3139	1.5071	47.6470	2.4614	0.4017	0.7817	19	1.0032	1.0007	–	–
6346	82.9351	1.8466	13.4484	3.5441	0.3298	0.5527	18	1.0060	1.0030	–	–
13629	44.6689	1.6201	4.6110	2.8026	0.2538	0.6851	21	0.9984	0.9949	–	–
31278	25.1967	1.3780	1.8416	2.2089	0.1908	0.6873	20	0.9941	0.9893	–	–
79064	15.0459	1.1118	0.8192	1.7468	0.1332	0.7739	19	0.9903	0.9849	–	–
Augmented $\mathbf{RT}_0 - \mathbf{P}_1 - \mathrm{P}_1$ scheme with adaptive refinement according to $\widetilde{\boldsymbol{\theta}}$											
409	482.0538	–	229.0604	–	1.1557	–	21	–	–	1.1219	1.1190
1206	318.9239	0.7641	148.2081	0.8052	0.5510	1.3700	19	–	–	0.9979	0.9962
3247	160.2352	1.3899	49.7563	2.2041	0.3399	0.9758	20	–	–	0.9986	0.9965
6703	79.3292	1.9399	12.8689	3.7315	0.3024	0.3221	21	–	–	1.0038	1.0011
15393	41.1173	1.5928	3.7409	2.9945	0.2553	0.4104	20	–	–	1.0073	1.0039
36869	22.9177	1.3296	1.2198	2.5490	0.1777	0.8247	20	–	–	1.0055	1.0008
94817	13.4813	1.1231	0.5302	1.7635	0.1289	0.6790	19	–	–	1.0058	1.0006

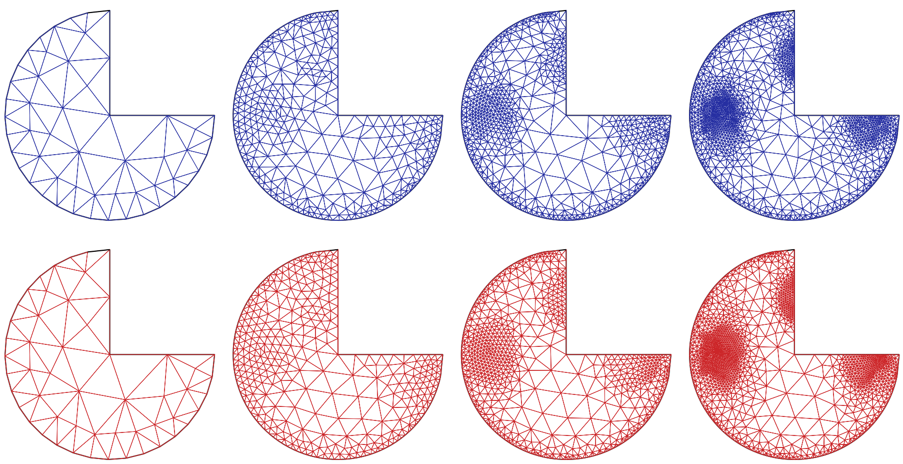

FIGURE 4. Test 2: From left to right, four snapshots of successively refined meshes according to the indicators $\boldsymbol{\theta}$ and $\widetilde{\boldsymbol{\theta}}$ (top and bottom panels, respectively).

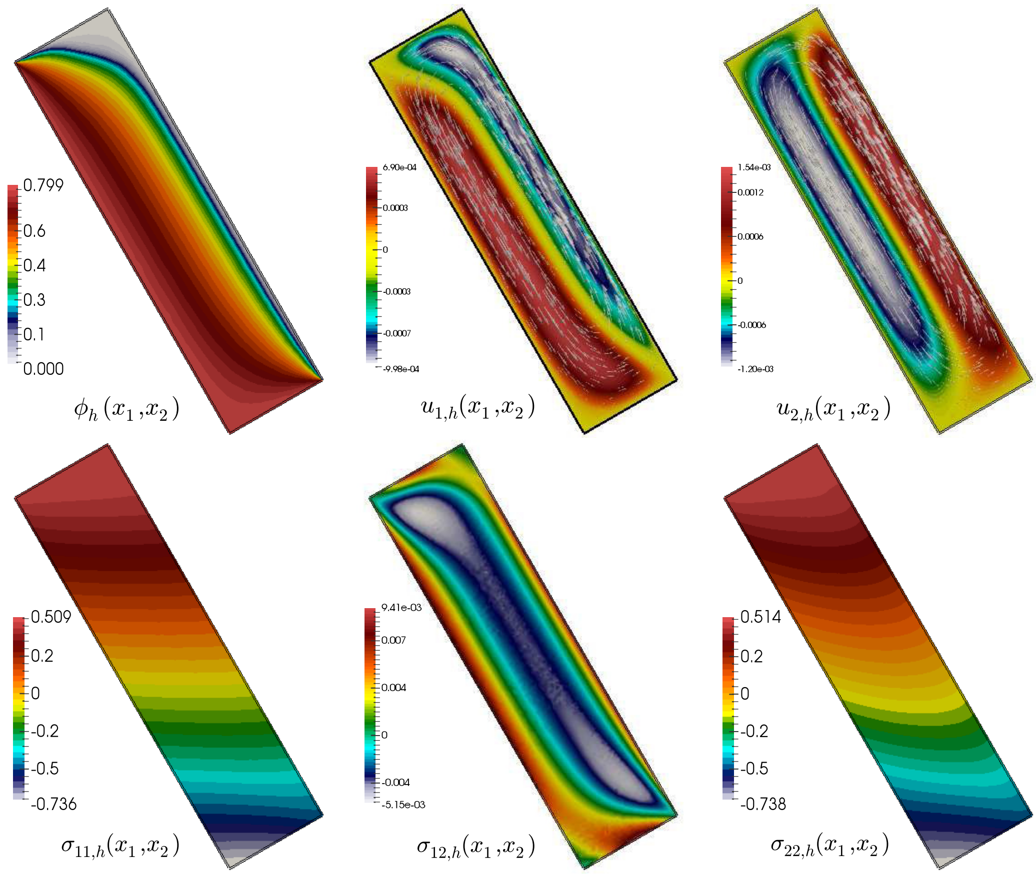

FIGURE 5. Test 3: Approximate solutions obtained with the lowest order method, after six steps of adaptive mesh refinement following the second indicator $\widetilde{\boldsymbol{\theta}}$. Concentration, velocity components, and stress components are depicted.

within an enclosure is enhanced by tilting the vessel (from the gravity direction), thus allowing the formation of recirculation zones carrying low concentration fluid along the underside of the inclined wall (see also [7]). The diffusivity function will be set to

$$\vartheta(\phi) = \begin{cases} \varepsilon & \text{for } \phi \leq \phi_{\mathrm{c}}, \\ \vartheta_0 \dfrac{\alpha}{\phi_{\mathrm{c}}} \left(\dfrac{\phi}{\phi_{\mathrm{c}}}\right)^{\alpha-1} & \text{otherwise,} \end{cases}$$

with $\alpha = 5, \vartheta_0 = 0.055, \varepsilon = 1e\text{-}6$. As computational domain we consider an inclined rectangle of height 1.5 m and width of 6 m forming an angle of $2\pi/3$ with the positive x_1-axis, which we initially discretize into a coarse mesh of 102 triangular elements. Viscosity is set as in the previous examples and the remaining coefficients are $\boldsymbol{f} = \Delta\rho\boldsymbol{k}$, $g = 0$, $\boldsymbol{u}_{\mathrm{D}} = \mathbf{0}$, $\phi_{\mathrm{D}} = \{0.8$ on the bottom and overside inclined wall; 0.0001 elsewhere in $\partial\Omega\}$, $\gamma(\phi) = \{\gamma_0\phi(1-\phi)^2$ if $\phi \geq \phi_{\mathrm{c}}$; 0 otherwise$\}$, $\phi_{\mathrm{c}} = 0.07$, $c = 2/3$, $\gamma_0 = 4.4\text{e-}3$, $\Delta\rho = 700$, The stabilization constants will depend on $\mu_1 = 1$ and $\mu_2 = 4.75$.

The numerical solutions are collected in Figure 5, where velocity shows a main circulation zone at the center of the domain, directing the flow towards the bottom along the lower inclined side and moving upwards on the opposite side. In addition, high concentration zones are located at the bottom of the vessel, while clear fluid

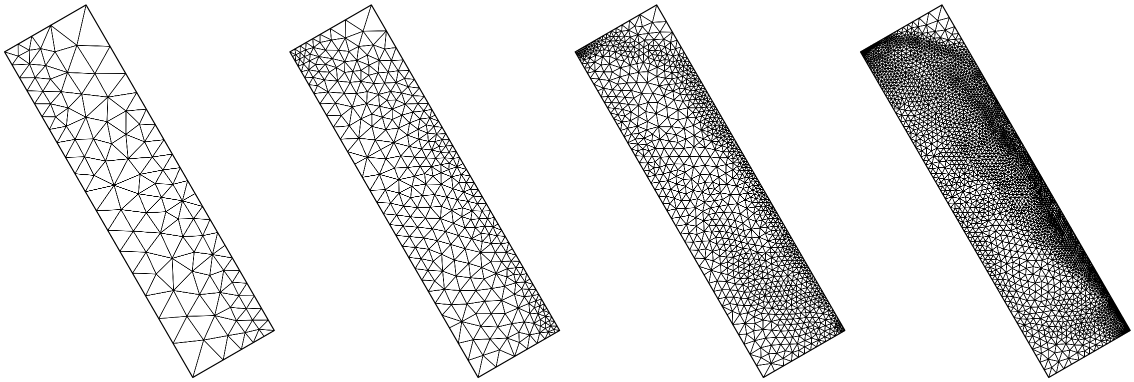

FIGURE 6. Test 3: From left to right, four snapshots of successively refined meshes according to the indicator $\widetilde{\boldsymbol{\theta}}$.

forms at the top. These flow patterns are in accordance with the observations in [7,8]. Four intermediate steps of mesh adaptation guided by the second *a posteriori* error estimator $\widetilde{\boldsymbol{\theta}}$ are displayed in Figure 6. We can see the capturing of the high concentration gradient and velocity boundary layer near the upper inclined side of the domain.

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