New mixed finite element methods for the coupled Stokes and Poisson–Nernst–Planck equations in Banach spaces*

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Abstract

In this paper we employ a Banach spaces-based framework to introduce and analyze new mixed finite element methods for the numerical solution of the coupled Stokes and Poisson–Nernst–Planck equations, which is a nonlinear model describing the dynamics of electrically charged incompressible fluids. The pressure of the fluid is eliminated from the system (though computed afterwards via a postprocessing formula) thanks to the incompressibility condition and the incorporation of the fluid pseudostress as an auxiliary unknown. In turn, besides the electrostatic potential and the concentration of ionized particles, we use the electric field (rescaled gradient of the potential) and total ionic fluxes as new unknowns. The resulting fully mixed variational formulation in Banach spaces can be written as a coupled system consisting of two saddle-point problems, each one with nonlinear source terms depending on the remaining unknowns, and a perturbed saddle-point problem with linear source terms, which is in turn additionally perturbed by a bilinear form. The well-posedness of the continuous formulation is a consequence of a fixed-point strategy in combination with the Banach theorem, the Babuška–Brezzi theory, the solvability of abstract perturbed saddle-point problems, and the Banach–Nečas–Babuška theorem. For this we also employ smallness assumptions on the data. An analogous approach, but using now both the Brouwer and Banach theorems, and invoking suitable stability conditions on arbitrary finite element subspaces, is employed to conclude the existence and uniqueness of solution for the associated Galerkin scheme. A priori error estimates are derived, and examples of discrete spaces that fit the theory, include, e.g., Raviart–Thomas elements of order k along with piecewise polynomials of degree $\leqslant k$. In addition, the latter yield approximate local conservation of momentum for all three equations involved. Finally, rates of convergence are specified and several numerical experiments confirm the theoretical error bounds. These tests also illustrate the aforementioned balance-preserving properties and the applicability of the proposed family of methods.

Key words: Poisson-Nernst-Planck, Stokes, fixed point theory, finite element methods.

Mathematics subject classifications (2000): 35J66, 65J15, 65N12, 65N15, 65N30, 47J26, 76D07.

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1 Introduction

Fluid mixtures with electrically charged ions are critical for many industrial processes and natural phenomena. Notable examples of current interest are efficient energy storage and electrodialysis cells, design of nanopore sensors, electro-osmotic water purification techniques, and even drug delivery in biological tissues [40]. One of the most well-known models for liquid electrolytes is the Poisson–Nernst–Planck / Stokes system. It describes the isothermal dynamics of the molar concentration of a number of charged species within a solvent. This classical model is valid for the regime of relatively small Reynolds numbers and it is written in terms of the concentrations, the barycentric velocity of the mixture, the pressure of the mixture, and the electrostatic potential. The system is strongly coupled and the set of equations consist of the transport equations for each dilute component of the electrolyte, a diffusion equation for the electrostatic equilibrium, the momentum balance for the mixture (including a force exerted by the electric field), and mass conservation.

Solving these systems lends itself difficult due to coupling nonlinearities of different nature. Numerical methods for incompressible flow equations coupled with Poisson–Nernst–Planck equations that are based on finite element schemes in primal formulation (also including stabilized and goal-adaptive methods) can be found in [3, 20, 32, 35, 37, 39], finite differences in e.g. [34], finite volume schemes in [38], spectral elements in [36], and also for virtual element methods in [17]. Regarding formulations using mixed methods, the first works addressing Stokes/PNP systems are relatively recent [27, 28], where a stabilized mixed method is employed for the Poisson problem, whereas the usual primal approach is applied to the Stokes, Navier-Stokes and Nernst-Planck equations, and all them within a Hilbertian framework. Mixed variational formulations are particularly interesting when direct discrete approximations of further variables of physical relevance are required. A recent approach to mixed methods consists in defining the corresponding variational settings in terms of Banach spaces instead of the usual Hilbertian framework, and without augmentation. As a consequence, the unknowns belong now to the natural spaces that are originated after carrying out the respective testing and integration by parts procedures, simpler and closer to the original physical model formulations arise, momentum conservative schemes can be obtained, and even other unknowns can be computed by postprocessing formulae. As a non-exhaustive list of contributions taking advantage of the use of Banach frameworks for solving the aforementioned kind of problems, we refer to [4, 7, 9, 10, 11, 13, 14, 25, 26, 29], and among the different models considered there, we find Poisson, Brinkman–Forchheimer, Darcy–Forchheimer, Navier–Stokes, chemotaxis/Navier–Stokes, Boussinesq, coupled flow-transport, and fluidized beds. Nevertheless, and up to our knowledge, no mixed methods with the described advantages seem to have been developed so far for the coupled Stokes and Poisson–Nernst–Planck equations.

As motivated by the previous discussion, the goal of this paper is to develop a Banach spaces-based formulation yielding new mixed finite element methods for, precisely, the coupled Stokes and Poisson–Nernst–Planck equations. The main novelties with respect to [27, 28] refer to the use of mixed methods for each one of the equations involved, the setting of the resulting variational formulation within a Banach framework, and the no need of incorporating any additional stabilization term. The rest of the manuscript is organized as follows. Required notations and basic definitions are collected at the end of this introductory section. In Section 2 we describe the model of interest and introduce the additional variables to be employed. The mixed variational formulation is deduced in Section 3. After some preliminaries, the respective analysis is split according to the three equations forming the whole system. In particular, the right spaces to which the trial and test functions must belong are derived in each case by applying suitable integration by parts formulae jointly with the Cauchy–Schwarz and Hölder inequalities. In Section 4 we utilize a fixed-point approach to study the solvability of the continuous formulation. The Babuška–Brezzi theory and recent results on perturbed saddle-point problems, both in Banach spaces, along with the Banach–Nečas–Babuška theorem, are utilized to prove that the corresponding uncoupled problems are well-posed. The classical Banach fixed-point theorem is then applied to conclude the existence of a unique solution. In Section 5 we proceed analogously to Section 4 and, under suitable stability assumptions on the discrete spaces employed, show existence and then uniqueness of solution for the Galerkin scheme by applying the Brouwer and Banach theorems, respectively. A priori error estimates are also derived here. Next, in Section 6 we define explicit finite element subspaces satisfying those conditions, and provide the associated rates of convergence. Finally, several numerical examples confirming the latter, showing the good performance of the method, and illustrating the approximate local conservation of momentum, are reported in Section 7.

Preliminary notations

Throughout the paper, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star shaped with respect to a ball, and whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{l,t}(\Omega)$ and $\mathrm{W}_0^{l,t}(\Omega)$, with $l \geqslant 0$ and $t \in [1,+\infty)$, whose corresponding norms, either for the scalar and vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and if $t = 2$ we write $\mathrm{H}^l(\Omega)$ instead of $\mathrm{W}^{l,2}(\Omega)$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. In addition, letting t, $t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we denote by $\mathrm{W}^{1/t',t}(\Gamma)$ the trace space of $\mathrm{W}^{1,t}(\Omega)$, and let $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the dual of $\mathrm{W}^{1/t',t}(\Gamma)$ endowed with the norms $\|\cdot\|_{-1/t',t';\Gamma}$ and $\|\cdot\|_{1/t',t;\Gamma}$, respectively. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$ will be employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong. Furthermore, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. Also, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$ we set the gradient and divergence operators, respectively, as $\nabla\mathbf{v} := \left(\dfrac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}$ and $\operatorname{div}(\mathbf{v}) := \displaystyle\sum_{j=1}^{n} \dfrac{\partial v_j}{\partial x_j}$. Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product operators, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} = (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n} \boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I}\,.$$

2 The model problem

We consider the nonlinear system given by the coupled Stokes and Poisson–Nernst–Planck equations, which constitute an electrohydrodinamic model describing the stationary flow of a Newtonian and incompressible fluid occupying the domain $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$, with polygonal (resp. polyhedral) boundary Γ in R^2 (resp. R^3). Under the assumption of isothermal properties, equal molar volumes and molar masses for each species, the behavior of the system is determined by the concentrations ξ_1 and ξ_2 of ionized particles, and by the electric current field $\boldsymbol{\varphi}$. Mathematically speaking, and firstly regarding the fluid, we look for the barycentric velocity $\mathbf{u}$ and the pressure p of the mixture, such that $(\mathbf{u}, p)$ is solution to the Stokes equations

$$\begin{gathered} -\mu\Delta\mathbf{u} + \nabla p \,=\, -\,(\xi_1 - \xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} \,+\, \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_{\Omega} p = 0\,, \end{gathered} \tag{2.1}$$

where μ is the constant viscosity, ε is the dielectric coefficient, also known as the electric conductivity coefficient, $\mathbf{f}$ is a source term, $\mathbf{g}$ is the Dirichlet datum for $\mathbf{u}$ on Γ, and the null mean value of p has been incorporated as a uniqueness condition for this unknown. In addition, $\boldsymbol{\varphi}$, ξ_1 and ξ_2 solve the Poisson–Nernst–Planck equations, which depend on the velocity $\mathbf{u}$ and are given by

$$\begin{gathered} \boldsymbol{\varphi} \,=\, \varepsilon\,\nabla\chi \quad \text{in} \quad \Omega\,, \quad -\operatorname{div}(\boldsymbol{\varphi}) \,=\, (\xi_1 - \xi_2) \,+\, f \quad \text{in} \quad \Omega\,, \\ \chi \,=\, g \quad \text{on} \quad \Gamma\,, \end{gathered} \tag{2.2}$$

where χ is the electrostatic potential, and for each $i \in \{1,2\}$

$$\begin{aligned} \xi_i - \operatorname{div}\big(\kappa_i(\nabla \xi_i + q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}) - \xi_i\,\mathbf{u}\big) &= f_i \quad \text{in} \quad \Omega\,, \\ \xi_i &= g_i \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.3}$$

where κ_1 and κ_2 are the diffusion coefficients, $q_i := \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = 2 \end{cases}$, f, f_1, and f_2 are external source/sink terms, and g, g_1 and g_2 are Dirichlet data for χ, ξ_1 and ξ_2, respectively, on Γ. The systems (2.2) and (2.3) correspond to the Poisson and Nernst–Planck equations, respectively. We end the description of the model by remarking that ε, κ_1, and κ_2 are all assumed to be bounded above and below, which means that there exist positive constants ε_0, ε_1, $\underline{\kappa}$, and $\bar{\kappa}$, such that

$$\varepsilon_0 \leqslant \varepsilon(\mathbf{x}) \leqslant \varepsilon_1 \quad \text{and} \quad \underline{\kappa} \leqslant \kappa_i(\mathbf{x}) \leqslant \bar{\kappa} \quad \text{for almost all } \mathbf{x} \in \Omega\,, \quad \forall\, i \in \{1,2\}\,. \tag{2.4}$$

We stress that in order to solve (2.3), $\mathbf{u}$ and $\boldsymbol{\varphi}$ are needed. In turn, (2.1) requires ξ_1, ξ_2 and $\boldsymbol{\varphi}$, whereas (2.2) makes use of ξ_1 and ξ_2. This multiple coupling is illustrated through the graph provided in Figure 2.1, where the vertexes represent the aforementioned equations and the arrows, properly labeled with the unknowns involved, show the respective dependence relationships.

Furthermore, since we are interested in employing a fully mixed variational formulation for the coupled model (2.1) – (2.3), we introduce the auxiliary variables of pseudostress

$$\boldsymbol{\sigma} := \mu\,\nabla \mathbf{u} - p\,\mathbb{I} \quad \text{in} \quad \Omega\,, \tag{2.5}$$

and, for each $i \in \{1,2\}$, the total (diffusive, cross-diffusive, and advective) ionic fluxes

$$\boldsymbol{\sigma}_i := \kappa_i\,(\nabla \xi_i + q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}) - \xi_i\,\mathbf{u} \quad \text{in} \quad \Omega\,. \tag{2.6}$$

Thus, applying the matrix trace in (2.5) and using the incompressibility condition, we deduce that

$$p = -\frac{1}{n}\,\operatorname{tr}(\boldsymbol{\sigma})\,, \tag{2.7}$$

so that, incorporating the latter expression into (2.5), p is eliminated from the system (2.1) - (2.3), which can then be rewritten in terms of the unknowns $\boldsymbol{\sigma}$, $\mathbf{u}$, $\boldsymbol{\varphi}$, χ, $\boldsymbol{\sigma}_i$ and ξ_i, $i \in \{1,2\}$, as

$$\begin{gathered} \frac{1}{\mu}\boldsymbol{\sigma}^{\mathrm{d}} = \nabla \mathbf{u} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) = (\xi_1 - \xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} - \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0\,, \\ \frac{1}{\varepsilon}\boldsymbol{\varphi} = \nabla \chi \quad \text{in} \quad \Omega\,, \quad -\operatorname{div}(\boldsymbol{\varphi}) = (\xi_1 - \xi_2) + f \quad \text{in} \quad \Omega\,, \\ \chi = g \quad \text{on} \quad \Gamma\,, \\ \frac{1}{\kappa_i}\boldsymbol{\sigma}_i := \nabla \xi_i + q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi} - \kappa_i^{-1}\,\xi_i\,\mathbf{u} \quad \text{in} \quad \Omega\,, \\ \xi_i - \operatorname{div}(\boldsymbol{\sigma}_i) = f_i \quad \text{in} \quad \Omega\,, \quad \xi_i = g_i \quad \text{on} \quad \Gamma\,, \quad i \in \{1,2\}\,. \end{gathered} \tag{2.8}$$

We notice here that the uniqueness condition for p has been rewritten equivalently as the null mean value constraint for $\operatorname{tr}(\boldsymbol{\sigma})$.

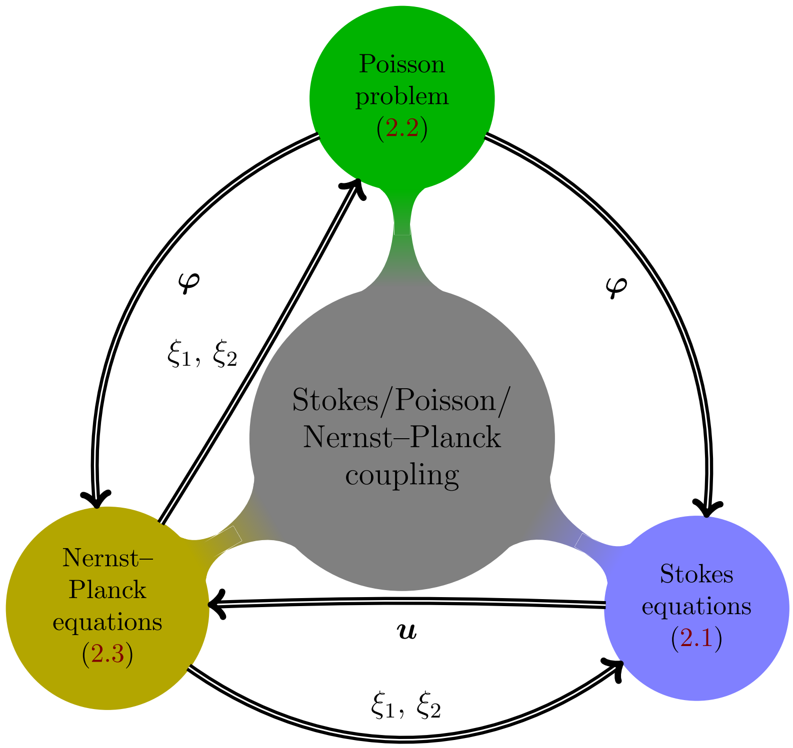

Figure 2.1: Illustrative graph of the coupling mechanisms connecting the three sub-problems (2.1), (2.2) and (2.3).

3 The fully mixed formulation

In this section we derive a suitable Banach spaces-based variational formulation for (2.8) by splitting the analysis in four sections. The first one collects some preliminary discussions and known results, and the remaining three deal with each one of the pairs of equations forming the whole nonlinear coupled system (2.8), namely Stokes, Poisson, and Nernst-Planck.

3.1 Preliminaries

We begin by noticing that there are three key expressions in (2.8) that need to be looked at carefully before determining the right spaces where the unknowns must be sought, namely $(\xi_1 - \xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi}$, $q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}$ and $\kappa_i^{-1}\,\xi_i\,\mathbf{u}$ in the first and fifth rows of (2.8). More precisely, ignoring the bounded above and below functions ε^{-1} and κ_i^{-1}, as well as the constant q_i, and given test functions $\mathbf{v}$ and $\boldsymbol{\tau}_i$ associated with $\mathbf{u}$ and $\boldsymbol{\sigma}_i$, respectively, straightforward applications of the Cauchy–Schwarz and Hölder inequalities yield

$$\left|\int_\Omega (\xi_1 - \xi_2)\,\boldsymbol{\varphi}\cdot\mathbf{v}\right| \,\leqslant\, \|\xi_1 - \xi_2\|_{0,2\ell;\Omega}\,\|\boldsymbol{\varphi}\|_{0,2j;\Omega}\,\|\mathbf{v}\|_{0,\Omega}\,, \tag{3.1a}$$

$$\left|\int_\Omega \xi_i\,\boldsymbol{\varphi}\cdot\boldsymbol{\tau}_i\right| \,\leqslant\, \|\xi_i\|_{0,2\ell;\Omega}\,\|\boldsymbol{\varphi}\|_{0,2j;\Omega}\,\|\boldsymbol{\tau}_i\|_{0,\Omega}\,, \tag{3.1b}$$

and similarly

$$\left|\int_\Omega \xi_i\,\mathbf{u}\cdot\boldsymbol{\tau}_i\right| \,\leqslant\, \|\xi_i\|_{0,2\ell;\Omega}\,\|\mathbf{u}\|_{0,2j;\Omega}\,\|\boldsymbol{\tau}_i\|_{0,\Omega}\,, \tag{3.1c}$$

where ℓ, $j \in (1,+\infty)$ are conjugate to each other. In this way, denoting

$$\rho := 2\ell\,, \quad \varrho := \frac{2\ell}{2\ell-1} \text{ (conjugate of } \rho)\,, \quad r := 2j\,, \quad \text{and} \quad s := \frac{2j}{2j-1} \text{ (conjugate of } r)\,, \tag{3.2}$$

it follows that the above expressions make sense for $\xi_i \in \mathrm{L}^\rho(\Omega)$, $\boldsymbol{\varphi}$, $\mathbf{u} \in \mathbf{L}^r(\Omega)$, and $\mathbf{v}$, $\boldsymbol{\tau}_i \in \mathbf{L}^2(\Omega)$. The specific choice of ℓ, and hence of j, ρ, r and the respective conjugates ϱ and s, will be addressed later on, so that meanwhile we consider generic values for the indexes defined in (3.2).

Having set the above preliminary choice for the space to which $\boldsymbol{\varphi}$ belongs, we deduce from the first equation in the third row of (2.8) that χ should be initially sought in $\mathrm{W}^{1,r}(\Omega)$. In turn, using that $\mathrm{H}^1(\Omega)$ is embedded in $\mathrm{L}^t(\Omega)$ for $t \in [1,+\infty)$ in R^2 (resp. $t \in [1,6]$ in R^3), and for reasons that will become clear below, the unknowns ξ_i, $i \in \{1,2\}$, and $\mathbf{u}$ are initially sought in $\mathrm{H}^1(\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively, certainly assuming that ρ and r verify the indicated ranges, namely ρ, $r \in (2,+\infty)$ in R^2, and ρ, $r \in (2,6]$ in R^3. Note that in terms of ℓ the latter constraint becomes $\ell \in [\frac{3}{2},3]$, which yields $\rho \in [3,6]$. Equivalently, $j \in [\frac{3}{2},3]$ and $r \in [3,6]$, though going through the respective intervals in the opposite direction to ℓ and ρ, respectively.

In turn, in order to derive the variational formulation of (2.8), we need to invoke a couple of integration by parts formulae, for which, given $t \in (1,+\infty)$, we first introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega) \Big\}\,, \tag{3.3a}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \Big\}\,, \tag{3.3b}$$

$$\mathbf{H}^t(\mathrm{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega) \Big\}\,, \tag{3.3c}$$

which are endowed with the natural norms defined, respectively, by

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega)\,, \tag{3.4a}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,, \tag{3.4b}$$

$$\|\boldsymbol{\tau}\|_{t,\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\,. \tag{3.4c}$$

Then, proceeding as in [22, eq. (1.43), Section 1.3.4] (see also [8, Section 4.1] and [13, Section 3.1]), it is easy to show that for each $t \geqslant \frac{2n}{n+2}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v \rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_t;\Omega)\times \mathrm{H}^1(\Omega)\,, \tag{3.5}$$

and analogously

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{v} \rangle = \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega)\times \mathbf{H}^1(\Omega)\,, \tag{3.6}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, as well as between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Furthermore, given t, $t' \in (1,+\infty)$ conjugate to each other, there also holds (cf. [19, Corollary B. 57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v \rangle_\Gamma = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega)\,, \tag{3.7}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$.

3.2 The Stokes equations

Let us first notice that, applying (3.6) with $t = s$ to $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_s;\Omega)$ and $\mathbf{u} \in \mathbf{H}^1(\Omega)$, and using the Dirichlet boundary condition on $\mathbf{u}$, for which we assume from now on that $\mathbf{g} \in \mathbf{H}^{1/2}(\Omega)$, we obtain

$$\int_\Omega \boldsymbol{\tau} : \nabla \mathbf{u} \,=\, -\int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \,+\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g} \rangle,$$

and thus, the testing of the first equation in the first row of (2.8) against $\boldsymbol{\tau}$ yields

$$\frac{1}{\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \,+\, \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g} \rangle. \tag{3.8}$$

Note from the second term on the left-hand side of (3.8) that, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^s(\Omega)$, it actually suffices to look for $\mathbf{u}$ in $\mathbf{L}^r(\Omega)$, which is coherent with a previous discussion on the space to which this unknown should belong. In addition, testing the second equation in the first row of (2.8) against $\mathbf{v} \in \mathbf{L}^r(\Omega)$, for which we require that $\mathbf{f} \in \mathbf{L}^s(\Omega)$, we get

$$\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) \,=\, \int_\Omega (\xi_1 - \xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} \cdot \mathbf{v} - \int_\Omega \mathbf{f} \cdot \mathbf{v}\,, \tag{3.9}$$

which makes sense for $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^s(\Omega)$. Hence, due to the last equation in the second row of (2.8), it follows that we should look for $\boldsymbol{\sigma}$ in $\mathbb{H}_0(\mathbf{div}_s;\Omega)$, where

$$\mathbb{H}_0(\mathbf{div}_s;\Omega) \,:=\, \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_s;\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0 \right\}.$$

Moreover, it is easily seen that there holds the decomposition

$$\mathbb{H}(\mathbf{div}_s;\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_s;\Omega) \oplus \mathrm{R}\mathbb{I}\,, \tag{3.10}$$

and that the incompressibility of the fluid forces the compatibility condition on $\mathbf{g}$ given by

$$\int_\Gamma \mathbf{g} \cdot \boldsymbol{\nu} \,=\, 0\,.$$

As a consequence of the above, we realize that imposing (3.8) for each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_s;\Omega)$ is equivalent to doing it for each $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_s;\Omega)$. Furthermore, since $r > 2$ it follows that $\mathbf{L}^r(\Omega)$ is embedded in $\mathbf{L}^2(\Omega)$, which, along with the estimate (3.1a), confirms that the first term on the right-hand side of (3.9) is also well-defined. In this way, denoting from now on $\boldsymbol{\xi} := (\xi_1, \xi_2)$, and joining (3.8) and (3.9), we arrive at the following mixed variational formulation for the Stokes equations (given by the first two rows of (2.8)): Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) \,&=\, \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \mathbf{H}\,, \\ \mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) \,&=\, \mathbf{G}_{\boldsymbol{\xi},\boldsymbol{\varphi}}(\mathbf{v}) \qquad \forall\,\mathbf{v} \in \mathbf{Q}\,, \end{aligned} \tag{3.11}$$

where

$$\mathbf{H} \,:=\, \mathbb{H}_0(\mathbf{div}_s;\Omega)\,, \qquad \mathbf{Q} \,:=\, \mathbf{L}^r(\Omega)\,, \tag{3.12}$$

and the bilinear forms $\mathbf{a} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$ and $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$, and the functional $\mathbf{F} : \mathbf{H} \longrightarrow \mathrm{R}$, are defined, respectively, as

$$\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau} \in \mathbf{H}\,, \tag{3.13a}$$

$$\mathbf{b}(\boldsymbol{\tau},\mathbf{v}) \,:=\, \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H} \times \mathbf{Q}\,, \tag{3.13b}$$

$$\mathbf{F}(\boldsymbol{\tau}) \,:=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g} \rangle \qquad \forall\,\boldsymbol{\tau} \in \mathbf{H}\,, \tag{3.13c}$$

whereas, given $\boldsymbol{\eta} := (\eta_1, \eta_2) \in \mathbf{L}^\rho(\Omega)$ and $\boldsymbol{\phi} \in \mathbf{L}^r(\Omega)$, the functional $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} : \mathbf{Q} \longrightarrow \mathrm{R}$ is set as

$$\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\mathbf{v}) \,:=\, \int_\Omega (\eta_1 - \eta_2)\, \varepsilon^{-1} \boldsymbol{\phi} \cdot \mathbf{v} - \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{Q}\,. \tag{3.13d}$$

It is readily seen that, endowing $\mathbf{H}$ with the corresponding norm from (3.4b), that is

$$\|\boldsymbol{\tau}\|_{\mathbf{H}} \,:=\, \|\boldsymbol{\tau}\|_{\mathbf{div}_s;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}\,, \tag{3.14}$$

and recalling that $\|\cdot\|_{0,r;\Omega}$ is that of $\mathbf{Q}$, the bilinear forms $\mathbf{a}$ and $\mathbf{b}$, and the linear functionals $\mathbf{F}$ and $\mathrm{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}$, are all bounded. Indeed, applying the Cauchy–Schwarz and Hölder inequalities, noting that $\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} \leqslant \|\boldsymbol{\tau}\|_{0,\Omega}$ for all $\boldsymbol{\tau} \in \mathbf{H}$, invoking the identity (3.6) along with the continuous injection $\mathbf{i}_r : \mathbf{H}^1(\Omega) \to \mathbf{L}^r(\Omega)$, using (3.1a) together with the fact that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{(r-2)/2r}\,\|\cdot\|_{0,r;\Omega}$, and bounding ε^{-1} according to (2.4), we deduce the existence of positive constants, denoted and given as

$$\begin{gathered} \|\mathbf{a}\| \,:=\, \frac{1}{\mu}\,, \qquad \|\mathbf{b}\| \,:=\, 1\,, \qquad \|\mathbf{F}\| \,:=\, \big(1 + \|\mathbf{i}_r\|\big)\, \|\mathbf{g}\|_{1/2,\Gamma}\,, \\ \text{and} \quad \|\mathbf{G}\| \,:=\, \max\big\{\varepsilon_0^{-1}\,|\Omega|^{(r-2)/2r}, 1\big\}\,, \end{gathered} \tag{3.15}$$

such that

$$\begin{aligned} &|\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau})| \,\leqslant\, \|\mathbf{a}\|\,\|\boldsymbol{\zeta}\|_{\mathbf{H}}\,\|\boldsymbol{\tau}\|_{\mathbf{H}} \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbf{H}\,, \\ &|\mathbf{b}(\boldsymbol{\tau},\mathbf{v})| \,\leqslant\, \|\mathbf{b}\|\,\|\boldsymbol{\tau}\|_{\mathbf{H}}\,\|\mathbf{v}\|_{\mathbf{Q}} \qquad \forall\, (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H} \times \mathbf{Q}\,, \\ &|\mathbf{F}(\boldsymbol{\tau})| \,\leqslant\, \|\mathbf{F}\|\,\|\boldsymbol{\tau}\|_{\mathbf{H}} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}\,, \quad \text{and} \\ &|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\mathbf{v})| \,\leqslant\, \|\mathbf{G}\| \Big\{ \|\eta_1 - \eta_2\|_{0,\rho;\Omega}\,\|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,s;\Omega} \Big\}\, \|\mathbf{v}\|_{\mathbf{Q}} \qquad \forall\, \mathbf{v} \in \mathbf{Q}\,. \end{aligned} \tag{3.16}$$

We end this section by emphasizing, according to the previous discussion, that the introduction of the pseudostress $\boldsymbol{\sigma}$ as an auxiliary unknown leads to the derivation of simple postprocessing formulae for the pressure p (cf. (2.7)) and the velocity gradient $\nabla\mathbf{u}$ (cf. first eq. in the first row of (2.8)). In addition, it allows us to seek the velocity $\mathbf{u}$ in a Lebesgue space, which is certainly less regular, whence the corresponding finite element subspace, not requiring any continuity property, can be chosen cheaper and easier to implement.

3.3 The electrostatic potential equations

We begin the derivation of the mixed formulation for the Poisson equation by testing the first equation in the third row of (2.8) against $\boldsymbol{\psi} \in \mathbf{H}^s(\mathrm{div}_s;\Omega)$. In this way, applying (3.7) with $t = s$ and $t' = r$ to the given $\boldsymbol{\psi}$ and $\chi \in \mathrm{W}^{1,r}(\Omega)$, and employing the Dirichlet boundary condition on χ, for which we assume that $g \in \mathrm{W}^{1/s,r}(\Gamma)$, we get

$$\int_\Omega \frac{1}{\varepsilon}\,\boldsymbol{\varphi}\cdot\boldsymbol{\psi} \,+\, \int_\Omega \chi\,\mathrm{div}(\boldsymbol{\psi}) \,=\, \langle \boldsymbol{\psi}\cdot\boldsymbol{\nu}, g\rangle_\Gamma\,. \tag{3.17}$$

In turn, testing the second equation in the third row of (2.8) against $\lambda \in \mathrm{L}^s(\Omega)$, which requires to assume that $f \in \mathrm{L}^r(\Omega)$, we obtain

$$\int_\Omega \lambda\,\mathrm{div}(\boldsymbol{\varphi}) \,=\, -\int_\Omega \lambda\,(\xi_1 - \xi_2) \,-\, \int_\Omega f\,\lambda\,, \tag{3.18}$$

which certainly makes sense for $\mathrm{div}(\boldsymbol{\varphi}) \in \mathrm{L}^r(\Omega)$. Thus, recalling from (3.1a) and (3.1b) that $\boldsymbol{\varphi}$ must belong to $\mathbf{L}^r(\Omega)$, it follows from the above that this unknown should be sought then in $\mathbf{H}^r(\mathrm{div}_r;\Omega)$. Furthermore, bearing in mind from (3.1a) - (3.1c) that ξ_1 and ξ_2 must belong to $\mathrm{L}^\rho(\Omega)$, we notice that in order for the first term on the right-hand side of (3.18) to make sense, we require that $\rho \geqslant r$, which is assumed from now on.

Therefore, placing together (3.17) and (3.18), we obtain the following mixed variational formulation for the electrostatic potential equations (given by the third and fourth rows of (2.8)): Find $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ such that

$$\begin{aligned} a(\boldsymbol{\varphi}, \boldsymbol{\psi}) + b_1(\boldsymbol{\psi}, \chi) &= \mathrm{F}(\boldsymbol{\psi}) \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \\ b_2(\boldsymbol{\varphi}, \lambda) &= \mathrm{G}_{\boldsymbol{\xi}}(\lambda) \qquad \forall\, \lambda \in \mathrm{M}_2\,, \end{aligned} \tag{3.19}$$

where

$$\mathrm{X}_2 := \mathbf{H}^r(\mathrm{div}_r;\Omega)\,, \quad \mathrm{M}_1 := \mathrm{L}^r(\Omega)\,, \quad \mathrm{X}_1 := \mathbf{H}^s(\mathrm{div}_s;\Omega)\,, \quad \mathrm{M}_2 := \mathrm{L}^s(\Omega)\,, \tag{3.20}$$

and the bilinear forms $a : \mathrm{X}_2 \times \mathrm{X}_1 \to \mathrm{R}$ and $b_i : \mathrm{X}_i \times \mathrm{M}_i \to \mathrm{R}$, $i \in \{1,2\}$, and the functional $\mathrm{F} : \mathrm{X}_1 \to \mathrm{R}$, are defined, respectively, as

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) := \int_\Omega \frac{1}{\varepsilon}\, \boldsymbol{\phi} \cdot \boldsymbol{\psi} \qquad \forall\, (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathrm{X}_2 \times \mathrm{X}_1\,, \tag{3.21a}$$

$$b_i(\boldsymbol{\psi}, \lambda) := \int_\Omega \lambda\, \mathrm{div}(\boldsymbol{\psi}) \qquad \forall\, (\boldsymbol{\psi}, \lambda) \in \mathrm{X}_i \times \mathrm{M}_i\,, \tag{3.21b}$$

$$\mathrm{F}(\boldsymbol{\psi}) := \langle \boldsymbol{\psi} \cdot \boldsymbol{\nu}, g \rangle_\Gamma \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \tag{3.21c}$$

whereas, given $\boldsymbol{\eta} := (\eta_1, \eta_2) \in \mathbf{L}^\rho(\Omega)$, the functional $\mathrm{G}_{\boldsymbol{\eta}} : \mathrm{M}_2 \to \mathrm{R}$ is defined by

$$\mathrm{G}_{\boldsymbol{\eta}}(\lambda) := -\int_\Omega \lambda\, (\eta_1 - \eta_2) - \int_\Omega f\, \lambda \qquad \forall\, \lambda \in \mathrm{M}_2\,. \tag{3.21d}$$

We end this section by establishing the boundedness of a, b_i, $i \in \{1,2\}$, F, and $\mathrm{G}_{\boldsymbol{\eta}}$, for which we recall that the norms of X_1 and X_2 are defined by (3.4c) with $t = s$ and $t = r$, respectively, whereas those of M_1 and M_2 are certainly given by $\|\cdot\|_{0,r;\Omega}$ and $\|\cdot\|_{0,s;\Omega}$, respectively. Then, employing again the Cauchy–Schwarz and Hölder inequalities, bounding ε^{-1} according to (2.4), and using that $\|\cdot\|_{0,r;\Omega} \leqslant |\Omega|^{(\rho-r)/\rho r}\, \|\cdot\|_{0,\rho;\Omega}$, which follows from the fact that $\rho \geqslant r$, we find that there exist positive constants

$$\|a\| := \frac{1}{\varepsilon_0}\,, \qquad \|b_1\| = \|b_2\| := 1\,, \quad \text{and} \quad \|\mathrm{G}\| := \max\left\{1, |\Omega|^{(\rho-r)/\rho r}\right\}, \tag{3.22}$$

such that

$$\begin{aligned} |a(\boldsymbol{\phi}, \boldsymbol{\psi})| &\leqslant \|a\|\, \|\boldsymbol{\phi}\|_{\mathrm{X}_2}\, \|\boldsymbol{\psi}\|_{\mathrm{X}_1} \qquad \forall\, (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathrm{X}_2 \times \mathrm{X}_1\,, \\ |b_i(\boldsymbol{\psi}, \lambda)| &\leqslant \|b_i\|\, \|\boldsymbol{\psi}\|_{\mathrm{X}_i}\, \|\lambda\|_{\mathrm{M}_i} \qquad \forall\, (\boldsymbol{\psi}, \lambda) \in \mathrm{X}_i \times \mathrm{M}_1\,, \quad \forall\, i \in \{1,2\}\,, \quad \text{and} \\ |\mathrm{G}_{\boldsymbol{\eta}}(\lambda)| &\leqslant \|\mathrm{G}\| \left\{ \|\eta_1 - \eta_2\|_{0,\rho;\Omega} + \|f\|_{0,r;\Omega} \right\} \|\lambda\|_{0,s;\Omega} \qquad \forall\, \lambda \in \mathrm{M}_2\,. \end{aligned} \tag{3.23}$$

Regarding the boundedness of F, we need to apply [19, Lemma A.36], which, along with the surjectivity of the trace operator mapping $\mathrm{W}^{1,r}(\Omega)$ onto $\mathrm{W}^{1/s,r}(\Gamma)$, yields the existence of a fixed positive constant C_r, such that for the given $g \in \mathrm{W}^{1/s,r}(\Gamma)$, there exists $v_g \in \mathrm{W}^{1,r}(\Omega)$ satisfying $v_g|_\Gamma = g$ and

$$\|v_g\|_{1,r;\Omega} \leqslant C_r\, \|g\|_{1/s,r;\Gamma}\,.$$

Hence, employing (3.7) with $(t, t') = (s, r)$ and $(\boldsymbol{\tau}, v) = (\boldsymbol{\psi}, v_g)$, and then using Hölder's inequality, we arrive at

$$|\mathrm{F}(\boldsymbol{\psi})| \leqslant \|\mathrm{F}\|\, \|\boldsymbol{\psi}\|_{\mathrm{X}_1} \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \tag{3.24}$$

with

$$\|\mathrm{F}\| := C_r\, \|g\|_{1/s,r;\Gamma}\,. \tag{3.25}$$

3.4 The ionized particles concentration equations

We now deal with the Nernst-Planck equations, that is the fifth and sixth rows of (2.8), for which we proceed analogously as we did for the Stokes equations in Section 3.2. More precisely, applying (3.5) with $t = \varrho$ to $\boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$ and $\xi_i \in \mathrm{H}^1(\Omega)$, and using the Dirichlet boundary condition on ξ_i, for which we assume from now on that $g_i \in \mathrm{H}^{1/2}(\Gamma)$, we obtain

$$\int_\Omega \nabla\xi_i \cdot \boldsymbol{\tau}_i \;=\; -\int_\Omega \xi_i \,\mathrm{div}(\boldsymbol{\tau}_i) \;+\; \langle \boldsymbol{\tau}_i \cdot \boldsymbol{\nu}, g_i \rangle\,,$$

so that the testing of the equation in the fifth row of (2.8) against $\boldsymbol{\tau}_i$, yields

$$\int_\Omega \frac{1}{\kappa_i}\boldsymbol{\sigma}_i \cdot \boldsymbol{\tau}_i \;+\; \int_\Omega \xi_i \,\mathrm{div}(\boldsymbol{\tau}_i) \;-\; \int_\Omega \Big\{ q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi} \;-\; \kappa_i^{-1}\,\xi_i\,\mathbf{u} \Big\} \cdot \boldsymbol{\tau}_i \;=\; \langle \boldsymbol{\tau}_i \cdot \boldsymbol{\nu}, g_i \rangle\,. \tag{3.26}$$

Since $\mathrm{div}(\boldsymbol{\tau}_i) \in \mathrm{L}^\varrho(\Omega)$, we notice from the second term on the left-hand side of (3.26) that it suffices to look for ξ_i in $\mathrm{L}^\rho(\Omega)$, which, similarly as for Stokes, is coherent with a previous discussion on where to seek this unknown. In fact, as already commented, the corresponding estimates (3.1b) and (3.1c) confirm that the third term on the left-hand side of (3.26) is well-defined as well. We end this derivation by testing the first equation of the sixth row of (2.8) against a function in the same space to which ξ_i belongs, that is $\eta_i \in \mathrm{L}^\rho(\Omega)$, which gives

$$\int_\Omega \eta_i \,\mathrm{div}(\boldsymbol{\sigma}_i) \;-\; \int_\Omega \xi_i\,\eta_i \;=\; -\int_\Omega f_i\,\eta_i\,. \tag{3.27}$$

We remark that the above requires to assume that both f_i and $\mathrm{div}(\boldsymbol{\sigma}_i)$ belong to $\mathrm{L}^\varrho(\Omega)$, which is coherent with the fact that ξ_i is sought in $\mathrm{L}^\rho(\Omega)$ since, being $\rho > 2$, it follows that $\rho > \varrho$, and hence $\mathrm{L}^\rho(\Omega) \subseteq \mathrm{L}^\varrho(\Omega)$. Consequently, we arrive at the following mixed variational formulation for the ionized particles concentration equations: Find $(\boldsymbol{\sigma}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ such that

$$\begin{aligned} a_i(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i, \xi_i) \;-\; c_{\boldsymbol{\varphi},\mathbf{u}}(\boldsymbol{\tau}_i, \xi_i) \;&=\; \mathrm{F}_i(\boldsymbol{\tau}_i) \qquad \forall\,\boldsymbol{\tau}_i \in \mathrm{H}_i\,,\\ c_i(\boldsymbol{\sigma}_i, \eta_i) \;-\; d_i(\xi_i, \eta_i) \;&=\; \mathrm{G}_i(\eta_i) \qquad \forall\,\eta_i \in \mathrm{Q}_i\,, \end{aligned} \tag{3.28}$$

where

$$\mathrm{H}_i \;:=\; \mathbf{H}(\mathrm{div}_\varrho;\Omega)\,, \qquad \mathrm{Q}_i \;:=\; \mathrm{L}^\rho(\Omega)\,, \tag{3.29}$$

and the bilinear forms $a_i : \mathrm{H}_i \times \mathrm{H}_i \to \mathrm{R}$, $c_i : \mathrm{H}_i \times \mathrm{Q}_i \to \mathrm{R}$, and $d_i : \mathrm{Q}_i \times \mathrm{Q}_i \to \mathrm{R}$, and the functionals $\mathrm{F}_i : \mathrm{H}_i \longrightarrow \mathrm{R}$ and $\mathrm{G}_i : \mathrm{Q}_i \longrightarrow \mathrm{R}$, are defined, respectively, as

$$a_i(\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) \;:=\; \int_\Omega \frac{1}{\kappa_i}\,\boldsymbol{\zeta}_i \cdot \boldsymbol{\tau}_i \qquad \forall\,(\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) \in \mathrm{H}_i \times \mathrm{H}_i\,, \tag{3.30a}$$

$$c_i(\boldsymbol{\tau}_i, \eta_i) \;:=\; \int_\Omega \eta_i \,\mathrm{div}(\boldsymbol{\tau}_i) \qquad \forall\,(\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,, \tag{3.30b}$$

$$d_i(\vartheta_i, \eta_i) \;:=\; \int_\Omega \vartheta_i\,\eta_i \qquad \forall\,(\vartheta_i, \eta_i) \in \mathrm{Q}_i \times \mathrm{Q}_i\,, \tag{3.30c}$$

$$\mathrm{F}_i(\boldsymbol{\tau}_i) \;:=\; \langle \boldsymbol{\tau}_i \cdot \boldsymbol{\nu}, g_i \rangle \qquad \forall\,\boldsymbol{\tau}_i \in \mathrm{H}_i\,, \tag{3.30d}$$

$$\mathrm{G}_i(\eta_i) \;:=\; -\int_\Omega f_i\,\eta_i \qquad \forall\,\eta_i \in \mathrm{Q}_i\,, \tag{3.30e}$$

whereas, given $(\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q} = \mathbf{H}^r(\mathrm{div}_r;\Omega) \times \mathbf{L}^r(\Omega)$, the bilinear form $c_{\boldsymbol{\phi},\mathbf{v}} : \mathrm{H}_i \times \mathrm{Q}_i \to \mathrm{R}$ is set as

$$c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \eta_i) \;:=\; \int_\Omega \Big\{ q_i\,\eta_i\,\varepsilon^{-1}\boldsymbol{\phi} \;-\; \kappa_i^{-1}\,\eta_i\,\mathbf{v} \Big\} \cdot \boldsymbol{\tau}_i \qquad \forall\,(\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,. \tag{3.30f}$$

Similarly to the analysis at the end of Section 3.2 (cf. (3.15) and (3.16)), we conclude here that a_i, c_i, d_i, F_i, G_i, and $c_{\boldsymbol{\phi},\mathbf{v}}$ are all bounded with the norm defined by (3.4a) with $t = \varrho$ for H_i, and certainly the norm $\|\cdot\|_{0,\rho;\Omega}$ for Q_i. Indeed, applying the Cauchy–Schwarz and Hölder inequalities, bounding both ε^{-1} and κ_i^{-1} according to (2.4), noting that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{(\rho-2)/2\rho}\|\cdot\|_{0,\rho;\Omega}$, invoking the identity (3.5) and the continuous injection $\mathrm{i}_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$, and utilizing (3.1b) and (3.1c), we find that there exist positive constants

$$
\begin{gathered}
\|a_i\| \,:=\, \frac{1}{\underline{\kappa}}\,, \quad \|c_i\| \,:=\, 1\,, \quad \|d_i\| \,:=\, |\Omega|^{(\rho-2)/\rho}\,, \quad \|\mathrm{F}_i\| \,:=\, \big(1 + \|\mathrm{i}_\rho\|\big)\,\|g_i\|_{1/2,\Gamma}\,, \\
\|\mathrm{G}_i\| \,:=\, \|f_i\|_{0,\varrho;\Omega}\,, \qquad \text{and} \quad \|c\| \,:=\, \max\big\{\varepsilon_0^{-1}, \underline{\kappa}^{-1}\big\}\,,
\end{gathered}
\tag{3.31}
$$

such that

$$
\begin{aligned}
&|a_i(\boldsymbol{\zeta}_i,\boldsymbol{\tau}_i)| \,\leqslant\, \|a_i\|\,\|\boldsymbol{\zeta}_i\|_{\mathrm{H}_i}\,\|\boldsymbol{\tau}_i\|_{\mathrm{H}_i} \qquad \forall\,(\boldsymbol{\zeta}_i,\boldsymbol{\tau}_i) \in \mathrm{H}_i \times \mathrm{H}_i\,, \\
&|c_i(\boldsymbol{\tau}_i,\eta_i)| \,\leqslant\, \|c_i\|\,\|\boldsymbol{\tau}_i\|_{\mathrm{H}_i}\,\|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,, \\
&|d_i(\vartheta_i,\eta_i)| \,\leqslant\, \|d_i\|\,\|\vartheta_i\|_{\mathrm{Q}_i}\,\|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\,(\vartheta_i,\eta_i) \in \mathrm{Q}_i \times \mathrm{Q}_i\,, \\
&|\mathrm{F}_i(\boldsymbol{\tau}_i)| \,\leqslant\, \|\mathrm{F}_i\|\,\|\boldsymbol{\tau}_i\|_{\mathrm{H}_i} \qquad \forall\,\boldsymbol{\tau}_i \in \mathrm{H}_i\,, \\
&|\mathrm{G}_i(\eta_i)| \,\leqslant\, \|\mathrm{G}_i\|\,\|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\,\eta_i \in \mathrm{Q}_i\,, \quad \text{and} \\
&|c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i,\eta_i)| \,\leqslant\, \|c\|\,\Big\{\|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\mathbf{v}\|_{0,r;\Omega}\Big\}\,\|\eta_i\|_{0,\rho;\Omega}\,\|\boldsymbol{\tau}_i\|_{0,\Omega} \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,.
\end{aligned}
\tag{3.32}
$$

Throughout the rest of the paper we will use indistinctly either $\|\boldsymbol{\eta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}$ or $\|\boldsymbol{\eta}\|_{0,\rho;\Omega}$, where

$$
\|\boldsymbol{\eta}\|_{0,\rho;\Omega} \,:=\, \|\eta_1\|_{0,\rho;\Omega} \,+\, \|\eta_2\|_{0,\rho;\Omega} \qquad \forall\,\boldsymbol{\eta} := (\eta_1,\eta_2) \in \mathrm{Q}_1 \times \mathrm{Q}_2\,.
$$

Summarizing, and putting together (3.11), (3.19), and (3.28), we find that, under the assumptions that $\mathbf{f} \in \mathbf{L}^s(\Omega)$, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, $f \in \mathrm{L}^r(\Omega)$, $g \in \mathrm{W}^{1/s,r}(\Gamma)$, $f_i \in \mathrm{L}^\varrho(\Omega)$, $g_i \in \mathrm{H}^{1/2}(\Gamma)$, and $\rho \geqslant r$, the mixed variational formulation of (2.8) reduces to: Find $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{H} \times \mathbf{Q}$, $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2 \times \mathrm{M}_1$, and $(\boldsymbol{\sigma}_i,\xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$, $i \in \{1,2\}$, such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau} \in \mathbf{H}\,, \\
\mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) &= \mathbf{G}_{\boldsymbol{\xi},\boldsymbol{\varphi}}(\mathbf{v}) && \forall\,\mathbf{v} \in \mathbf{Q}\,, \\
a(\boldsymbol{\varphi},\boldsymbol{\psi}) + b_1(\boldsymbol{\psi},\chi) &= \mathrm{F}(\boldsymbol{\psi}) && \forall\,\boldsymbol{\psi} \in \mathrm{X}_1\,, \\
b_2(\boldsymbol{\varphi},\lambda) &= \mathrm{G}_{\boldsymbol{\xi}}(\lambda) && \forall\,\lambda \in \mathrm{M}_2\,, \\
a_i(\boldsymbol{\sigma}_i,\boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i,\xi_i) \,-\, c_{\boldsymbol{\varphi},\mathbf{u}}(\boldsymbol{\tau}_i,\xi_i) &= \mathrm{F}_i(\boldsymbol{\tau}_i) && \forall\,\boldsymbol{\tau}_i \in \mathrm{H}_i\,, \\
c_i(\boldsymbol{\sigma}_i,\eta_i) \,-\, d_i(\xi_i,\eta_i) &= \mathrm{G}_i(\eta_i) && \forall\,\eta_i \in \mathrm{Q}_i\,.
\end{aligned}
\tag{3.33}
$$

Notice here that the second, fourth, and sixth rows of (3.33) constitute the conservation of momentum for each respective equation. We will refer again to this subject from the discrete point of view later on in Sections 5.1 and 6.2.

We end this section by stressing that, as compared with previously studied Banach spaces-based mixed formulations for other coupled nonlinear models (see, e.g. [4], [13], [25], and [26]), the main novelty of the analysis to be developed for (3.33) has to do with the occurrence in its last two rows of the perturbed saddle point scheme in Banach spaces represented by the bilinear forms a_i, c_i, and d_i. Indeed, up to our knowledge, the present one constitutes the first work applying the theoretical results provided recently in [15] to perform the continuous and discrete analyses of a problem showing that structure.

4 The continuous solvability analysis

In this section we proceed as in several related previous contributions (see, e.g. [9] and the references therein), and employ a fixed-point strategy to address the solvability of (3.33).

4.1 The fixed-point strategy

In order to rewrite (3.33) as an equivalent fixed point equation, we introduce suitable operators associated with each one of the three problems forming the whole nonlinear coupled system. Indeed, we first let $\widehat{\mathrm{T}} : (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2 \to \mathbf{Q}$ be the operator defined by

$$\widehat{\mathrm{T}}(\boldsymbol{\eta}, \boldsymbol{\phi}) := \widehat{\mathbf{u}} \qquad \forall\, (\boldsymbol{\eta}, \boldsymbol{\phi}) \in (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2\,,$$

where $(\widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be confirmed below) of problem (3.11) (equivalently, the first two rows of (3.33)) with $(\boldsymbol{\eta}, \boldsymbol{\phi})$ instead of $(\boldsymbol{\xi}, \boldsymbol{\varphi})$, that is

$$\begin{aligned} \mathbf{a}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \widehat{\mathbf{u}}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad &&\forall\, \boldsymbol{\tau} \in \mathbf{H}\,, \\ \mathbf{b}(\widehat{\boldsymbol{\sigma}}, \mathbf{v}) &= \mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\mathbf{v}) \qquad &&\forall\, \mathbf{v} \in \mathbf{Q}\,. \end{aligned} \tag{4.1}$$

In turn, we let $\bar{\mathrm{T}} : \mathrm{Q}_1 \times \mathrm{Q}_2 \to \mathrm{X}_2$ be the operator given by

$$\bar{\mathrm{T}}(\boldsymbol{\eta}) := \bar{\boldsymbol{\varphi}} \qquad \forall\, \boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2\,,$$

where $(\bar{\boldsymbol{\varphi}}, \bar{\chi}) \in \mathrm{X}_2 \times \mathrm{M}_1$ is the unique solution (to be confirmed below) of problem (3.19) (equivalently, the third and fourth rows of (3.33)) with $\boldsymbol{\eta}$ instead of $\boldsymbol{\xi}$, that is

$$\begin{aligned} a(\bar{\boldsymbol{\varphi}}, \boldsymbol{\psi}) + b_1(\boldsymbol{\psi}, \bar{\chi}) &= \mathrm{F}(\boldsymbol{\psi}) \qquad &&\forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \\ b_2(\bar{\boldsymbol{\varphi}}, \lambda) &= \mathrm{G}_{\boldsymbol{\eta}}(\lambda) \qquad &&\forall\, \lambda \in \mathrm{M}_2\,. \end{aligned} \tag{4.2}$$

Similarly, for each $i \in \{1, 2\}$, we let $\widetilde{\mathrm{T}}_i : \mathrm{X}_2 \times \mathbf{Q} \to \mathrm{Q}_i$ be the operator defined by

$$\widetilde{\mathrm{T}}_i(\boldsymbol{\phi}, \mathbf{v}) := \widetilde{\xi}_i \qquad \forall\, (\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q}\,,$$

where $(\widetilde{\boldsymbol{\sigma}}_i, \widetilde{\xi}_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ is the unique solution (to be confirmed below) of problem (3.28) (equivalently, the fifth and sixth rows of (3.33)) with $(\boldsymbol{\phi}, \mathbf{v})$ instead of $(\boldsymbol{\varphi}, \mathbf{u})$, that is

$$\begin{aligned} a_i(\widetilde{\boldsymbol{\sigma}}_i, \boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i, \widetilde{\xi}_i) \,-\, c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \widetilde{\xi}_i) &= \mathrm{F}_i(\boldsymbol{\tau}_i) \qquad &&\forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,, \\ c_i(\widetilde{\boldsymbol{\sigma}}_i, \eta_i) \,-\, d_i(\widetilde{\xi}_i, \eta_i) &= \mathrm{G}_i(\eta_i) \qquad &&\forall\, \eta_i \in \mathrm{Q}_i\,, \end{aligned} \tag{4.3}$$

so that we can define the operator $\widetilde{\mathrm{T}} : \mathrm{X}_2 \times \mathbf{Q} \to (\mathrm{Q}_1 \times \mathrm{Q}_2)$ as:

$$\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{v}) := \big(\widetilde{\mathrm{T}}_1(\boldsymbol{\phi}, \mathbf{v}), \widetilde{\mathrm{T}}_2(\boldsymbol{\phi}, \mathbf{v})\big) = (\xi_1, \xi_2) =: \widetilde{\boldsymbol{\xi}} \qquad \forall\, (\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q}\,. \tag{4.4}$$

Finally, defining the operator $\mathbf{T} : (\mathrm{Q}_1 \times \mathrm{Q}_2) \to (\mathrm{Q}_1 \times \mathrm{Q}_2)$ as

$$\mathbf{T}(\boldsymbol{\eta}) := \widetilde{\mathrm{T}}\big(\bar{\mathrm{T}}(\boldsymbol{\eta}), \widehat{\mathrm{T}}(\boldsymbol{\eta}, \bar{\mathrm{T}}(\boldsymbol{\eta}))\big) \qquad \forall\, \boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2\,, \tag{4.5}$$

we observe that solving (3.33) is equivalent to seeking a fixed point of $\mathbf{T}$, that is: Find $\boldsymbol{\xi} \in \mathrm{Q}_1 \times \mathrm{Q}_2$ such that

$$\mathbf{T}(\boldsymbol{\xi}) = \boldsymbol{\xi}\,. \tag{4.6}$$

4.2 Well-posedness of the uncoupled problems

In this section we establish the well-posedness of the problems (4.1), (4.2), and (4.3), defining the operators $\widehat{\mathrm{T}}$, $\bar{\mathrm{T}}$, and $\widetilde{\mathrm{T}}_i$, respectively. To this end, we apply the Babuška–Brezzi theory in Banach spaces for the general case (cf. [5, Theorem 2.1, Corollary 2.1, Section 2.1]), and for a particular one [19, Theorem 2.34], as well as a recently established result for perturbed saddle point formulations in Banach spaces (cf. [15, Theorem 3.4]) along with the Banach–Nečas–Babuška Theorem (also known as the generalized Lax–Milgram Lemma) (cf. [19, Theorem 2.6]).

4.2.1 Well-definedness of the operator $\widehat{\mathrm{T}}$

Here we apply [19, Theorem 2.34] to show that, given an arbitrary $(\boldsymbol{\eta},\boldsymbol{\phi}) \in \big(\mathrm{Q}_1\times\mathrm{Q}_2\big)\times\mathrm{X}_2$, (4.1) is well-posed, equivalently that $\widehat{\mathrm{T}}$ is well-defined. We remark that $(\boldsymbol{\eta},\boldsymbol{\phi})$ only influences the functional $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}$, and that the boundedness of all the bilinear forms and linear functionals defining (4.1), has already been established in (3.15) and (3.16). Hence, the discussion below just refers to the remaining hypotheses to be satisfied by $\mathbf{a}$ and $\mathbf{b}$. We begin by letting $\mathbb{V}$ be the kernel of the operator induced by $\mathbf{b}$, that is

$$\mathbb{V} := \Big\{\boldsymbol{\tau}\in\mathbf{H}: \qquad \mathbf{b}(\boldsymbol{\tau},\mathbf{v}) = 0 \quad \forall\,\mathbf{v}\in\mathbf{Q}\Big\},$$

which, according to the definitions of $\mathbf{H}$, $\mathbf{Q}$, and $\mathbf{b}$ (cf. (3.12), (3.13b)), along with the fact that $\mathbf{L}^s(\Omega)$ is isomorphic to the dual of $\mathbf{L}^r(\Omega)$, yields

$$\mathbb{V} := \Big\{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_s;\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}\Big\}. \tag{4.7}$$

Next, we recall that a slight modification of the proof of [22, Lemma 2.3] allows to prove that for each $t\geqslant\frac{2n}{n+2}$ (see, e.g., [7, Lemma 3.1] for the case $t = 4/3$, which is extensible almost verbatim for any t in the indicated range) there exists a constant C_t, depending only on Ω, such that

$$C_t\,\|\boldsymbol{\tau}\|_{0,\Omega}^2 \,\leqslant\, \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega}^2 \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega}^2 \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_t;\Omega)\,. \tag{4.8}$$

Then, assuming that $s\geqslant\frac{2n}{n+2}$, and using (4.8), we deduce from the definition of $\mathbf{a}$ (cf. (3.13a)), and similarly to [7, Lemma 3.2], that

$$\mathbf{a}(\boldsymbol{\tau},\boldsymbol{\tau}) \,\geqslant\, \boldsymbol{\alpha}\,\|\boldsymbol{\tau}\|_{\mathrm{div}_s;\Omega}^2 \qquad \forall\,\boldsymbol{\tau}\in\mathbb{V}\,, \tag{4.9}$$

with $\boldsymbol{\alpha} := C_s/\mu$. Hence, thanks to (4.9), it is straightforward to see that $\mathbf{a}$ satisfies the hypotheses specified in [19, Theorem 2.34, eq. (2.28)] with the foregoing constant $\boldsymbol{\alpha}$. In order to fulfill all the hypotheses of the latter theorem, and knowing from (3.15) and (3.16) that the boundedness of the corresponding bilinear forms and linear functionals has already been established, it only remains to show the continuous inf-sup condition for $\mathbf{b}$. Moreover, being this result already proved for the particular case $s = 4/3$ (cf. [7, Lemma 3.3] and [26, Lemma 3.5] for a closely related one), and arising no significant differences for an arbitrary $s\geqslant\frac{2n}{n+2}$, we provide below, and for sake of completeness, only the main aspects of its proof.

Indeed, given $\mathbf{v}\in\mathbf{Q} := \mathbf{L}^r(\Omega)$, we first recall from (3.2) that $r>2$, and set $\mathbf{v}_s := |\mathbf{v}|^{r-2}\,\mathbf{v}$, which is easily seen to satisfy

$$\mathbf{v}_s \,\in\, \mathbf{L}^s(\Omega) \quad \text{and} \quad \int_\Omega \mathbf{v}\cdot\mathbf{v}_s \,=\, \|\mathbf{v}\|_{0,r;\Omega}\,\|\mathbf{v}_s\|_{0,s;\Omega}\,.$$

In what follows, we make use of both, the Poincaré inequality, which refers to the existence of a positive constant c_{P}, depending on Ω, such that $c_{\mathrm{P}}\,\|\mathbf{z}\|_{1,\Omega}^2 \leqslant |\mathbf{z}|_{1,\Omega}^2 \quad \forall\,\mathbf{z}\in\mathbf{H}_0^1(\Omega)$, and the continuous injection $\mathbf{i}_r:\mathbf{H}^1(\Omega)\to\mathbf{L}^r(\Omega)$ for the indicated range of s. Then, we let $\mathbf{w}\in\mathbf{H}_0^1(\Omega)$ be the unique solution of: $\int_\Omega\nabla\mathbf{w}\cdot\nabla\mathbf{z} \,=\, -\int_\Omega\mathbf{v}_s\cdot\mathbf{z}$ for all $\mathbf{z}\in\mathbf{H}_0^1(\Omega)$, which is guaranteed by the classical Lax–Milgram Lemma, and notice, thanks to the corresponding continuous dependence estimate, that $\|\mathbf{w}\|_{1,\Omega} \,\leqslant\, \frac{\|\mathbf{i}_r\|}{c_{\mathrm{P}}}\,\|\mathbf{v}_s\|_{0,s;\Omega}$. Hence, defining $\boldsymbol{\zeta} := \nabla\mathbf{w}\in\mathbf{L}^2(\Omega)$, we deduce that $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{v}_s$ in Ω, so that $\boldsymbol{\zeta}\in\mathbb{H}(\mathbf{div}_s;\Omega)$, and $\|\boldsymbol{\zeta}\|_{\mathbf{div}_s;\Omega} \,\leqslant\, \big(1+\frac{\|\mathbf{i}_r\|}{c_{\mathrm{P}}}\big)\,\|\mathbf{v}_s\|_{0,s;\Omega}$. Finally, letting $\boldsymbol{\zeta}_0$ be the $\mathbb{H}_0(\mathbf{div}_s;\Omega)$-component of $\boldsymbol{\zeta}$, it is clear that $\mathbf{div}(\boldsymbol{\zeta}_0) = \mathbf{v}_s$ and that $\|\boldsymbol{\zeta}_0\|_{\mathbf{div}_s;\Omega} \,\leqslant\, \|\boldsymbol{\zeta}\|_{\mathbf{div}_s;\Omega}$, whence bounding by below with $\boldsymbol{\tau} := \boldsymbol{\zeta}_0\in\mathbf{H}$, and using the definition of $\mathbf{b}$ (cf. (3.13b)) along with the above identities and estimates, we conclude that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbf{H}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{H}}} \,\geqslant\, \boldsymbol{\beta}\,\|\mathbf{v}\|_{\mathbf{Q}} \qquad \forall\,\mathbf{v}\in\mathbf{Q}\,, \tag{4.10}$$

with $\boldsymbol{\beta} := \big(1+\frac{\|\mathbf{i}_r\|}{c_{\mathrm{P}}}\big)^{-1}$. The foregoing inequality (4.10) proves [19, Theorem 2.34, eq. (2.29)] and completes the hypotheses of this theorem.

Consequently, the well-definedness of the operator $\widehat{\mathrm{T}}$ is stated as follows.

Theorem 4.1. *For each $(\boldsymbol{\eta}, \boldsymbol{\phi}) \in (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2$ there exists a unique $(\widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ solution to (4.1), and hence one can define $\widehat{\mathrm{T}}(\boldsymbol{\eta}, \boldsymbol{\phi}) := \widehat{\mathbf{u}} \in \mathbf{Q}$. Moreover, there exists a positive constant $C_{\widehat{\mathrm{T}}}$, depending only on μ, $\|\mathbf{i}_r\|$, ε_0, $|\Omega|$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, and hence independent of $(\boldsymbol{\eta}, \boldsymbol{\phi})$, such that*

$$\|\widehat{\mathrm{T}}(\boldsymbol{\eta}, \boldsymbol{\phi})\|_{\mathbf{Q}} \;=\; \|\widehat{\mathbf{u}}\|_{\mathbf{Q}} \;\leqslant\; C_{\widehat{\mathrm{T}}} \Big\{ \|\mathbf{g}\|_{1/2,\Gamma} \;+\; \|\mathbf{f}\|_{0,s,\Omega} \;+\; \|\boldsymbol{\eta}\|_{0,\rho;\Omega} \, \|\boldsymbol{\phi}\|_{0,r;\Omega} \Big\} . \tag{4.11}$$

Proof. Given $(\boldsymbol{\eta}, \boldsymbol{\phi}) \in (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2$, a direct application of [19, Theorem 2.34] guarantees the existence of a unique $(\widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ solution to (4.1). Then, the corresponding a priori estimate in [19, Theorem 2.34, eq. (2.30)] gives

$$\|\widehat{\mathbf{u}}\|_{\mathbf{Q}} \;\leqslant\; \frac{1}{\boldsymbol{\beta}} \left(1 + \frac{\|\mathbf{a}\|}{\boldsymbol{\alpha}}\right) \|\mathbf{F}\|_{\mathbf{H}'} \;+\; \frac{\|\mathbf{a}\|}{\boldsymbol{\beta}^2} \left(1 + \frac{\|\mathbf{a}\|}{\boldsymbol{\alpha}}\right) \|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}\|_{\mathbf{Q}'} \,, \tag{4.12}$$

which, according to the identities and estimates given by (3.15) and (3.16), along with some algebraic manipulations, yields (4.11) and finishes the proof. □

Regarding the a priori bound for the component $\widehat{\boldsymbol{\sigma}}$ of the unique solution to (4.1), it follows from [19, Theorem 2.34, eq. (2.30)] that

$$\|\widehat{\boldsymbol{\sigma}}\|_{\mathbf{H}} \;\leqslant\; \frac{1}{\boldsymbol{\alpha}} \, \|\mathbf{F}\|_{\mathbf{H}'} \;+\; \frac{1}{\boldsymbol{\beta}} \left(1 + \frac{\|\mathbf{a}\|}{\boldsymbol{\alpha}}\right) \|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}\|_{\mathbf{Q}'} \,,$$

which yields the same inequality as (4.11), but with a different constant. Hence, choosing the largest of the respective constants, and still denoting it by $C_{\widehat{\mathrm{T}}}$, we can summarize the a priori estimates for $\widehat{\mathbf{u}}$ and $\widehat{\boldsymbol{\sigma}}$ by saying that both are given by the right-hand side of (4.11).

4.2.2 Well-definedness of the operator $\bar{\mathrm{T}}$

We now employ [5, Theorem 2.1, Section 2.1] to prove that, given an arbitrary $\boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$, (4.2) is well-posed, equivalently that $\bar{\mathrm{T}}$ is well-defined. Similarly as for Section 4.2.1, we first stress that $\boldsymbol{\eta}$ is utilized only to define the functional $\mathrm{G}_{\boldsymbol{\eta}}$, and that the boundedness of all the bilinear forms and functionals defining (4.2), was already established by (3.22) and (3.23). In this way, it only remains to show that a, b_1, and b_2 satisfy the corresponding hypotheses from [5, Theorem 2.1, Section 2.1]. To this end, and because of the evident similarities, we follow very closely the analysis in [9, Section 3.2.3], which, in turn, suitably adopts the approach from [25, Section 2.4.2]. Indeed, we begin by letting K_i be the kernel of the operator induced by the bilinear form b_i, for each $i \in \{1, 2\}$, that is

$$\mathrm{K}_i \;:=\; \Big\{ \boldsymbol{\psi} \in \mathrm{X}_i : \quad b_i(\boldsymbol{\psi}, \lambda) \;=\; 0 \quad \forall\, \lambda \in \mathrm{M}_i \Big\} , \tag{4.13}$$

which, according to the definitions of X_i and M_i (cf. (3.20)), and b_i (cf. (3.21b)), along again with the fact that $\mathbf{L}^r(\Omega)$ and $\mathbf{L}^s(\Omega)$ can be isomorphically identified with $\big(\mathbf{L}^s(\Omega)\big)'$ and $\big(\mathbf{L}^r(\Omega)\big)'$, respectively, gives

$$\mathrm{K}_1 \;:=\; \Big\{ \boldsymbol{\psi} \in \mathbf{H}^s(\mathrm{div}_s; \Omega) : \quad \mathrm{div}(\boldsymbol{\psi}) = 0 \quad \text{in} \quad \Omega \Big\} , \tag{4.14}$$

and

$$\mathrm{K}_2 \;:=\; \Big\{ \boldsymbol{\psi} \in \mathbf{H}^r(\mathrm{div}_r; \Omega) : \quad \mathrm{div}(\boldsymbol{\psi}) = 0 \quad \text{in} \quad \Omega \Big\} . \tag{4.15}$$

Next, in order to establish the inf-sup conditions required for the bilinear form a (cf. [5, eqs. (2.8) and (2.9)]), we resort to [9, Lemma 3.3], which is recalled below.

Lemma 4.2. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let $t, t' \in (1,+\infty)$ conjugate to each other with t (and hence t') lying in $\begin{cases} [4/3,4] & \text{if } n=2 \\ [3/2,3] & \text{if } n=3 \end{cases}$. Then, there exists a linear and bounded operator $D_t : \mathbf{L}^t(\Omega) \to \mathbf{L}^t(\Omega)$ such that*

$$\mathbf{div}(D_t(\mathbf{w})) \;=\; 0 \quad in \quad \Omega \qquad \forall\, \mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{4.16}$$

In addition, for each $\mathbf{z} \in \mathbf{L}^{t'}(\Omega)$ such that $\mathbf{div}(\mathbf{z}) = 0$ in Ω, there holds

$$\int_\Omega \mathbf{z} \cdot D_t(\mathbf{w}) \;=\; \int_\Omega \mathbf{z} \cdot \mathbf{w} \qquad \forall\, \mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{4.17}$$

Proof. It reduces to a minor modification of the proof of [25, Lemma 2.3], for which one needs to apply the well-posedness in $\mathrm{W}^{1,t}(\Omega)$ of a Poisson problem with homogeneous Dirichlet boundary conditions (see [23, Theorem 3.2] or [30, Theorems 1.1 and 1.3] for the vector version of it). The specified ranges for t and t' are precisely forced by the latter result. We omit further details and refer to the proof of [9, Lemma 3.3]. $\square$

We are now in position to prove the required hypotheses on a.

Lemma 4.3. *Assume that s (and hence r) satisfy the ranges specified in Lemma 4.2. Then, there exists a positive constant $\bar{\alpha}$ such that*

$$\sup_{\substack{\boldsymbol{\psi} \in \mathrm{K}_1 \\ \boldsymbol{\psi} \neq \mathbf{0}}} \frac{a(\boldsymbol{\phi}, \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{\mathrm{X}_1}} \;\geqslant\; \bar{\alpha}\, \|\boldsymbol{\phi}\|_{\mathrm{X}_2} \qquad \forall\, \boldsymbol{\phi} \in \mathrm{K}_2\,. \tag{4.18}$$

In addition, there holds

$$\sup_{\boldsymbol{\phi} \in \mathrm{K}_2} \, a(\boldsymbol{\phi}, \boldsymbol{\psi}) \;>\; 0 \qquad \forall\, \boldsymbol{\psi} \in \mathrm{K}_1, \quad \boldsymbol{\psi} \neq \mathbf{0}\,. \tag{4.19}$$

Proof. Being almost verbatim to that of [9, Lemma 3.4], we just proceed to sketch it. Indeed, given $\boldsymbol{\phi} \in \mathrm{K}_2$, we recall from (3.2) that $r > 2$ and set $\boldsymbol{\phi}_s := |\boldsymbol{\phi}|^{r-2}\, \boldsymbol{\phi}$, which belongs to $\mathbf{L}^s(\Omega)$ and satisfies

$$\int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{\phi}_s \;=\; \|\boldsymbol{\phi}\|_{0,r,\Omega}\, \|\boldsymbol{\phi}_s\|_{0,s,\Omega}\,. \tag{4.20}$$

In this way, bounding by below with $\boldsymbol{\psi} := D_s(\boldsymbol{\phi}_s)$, which, according to Lemma 4.2, belongs to K_1, and then using (4.17), (4.20), the boundedness of D_s, and the upper bound of ε (cf. (2.4)), we arrive at (4.18) with $\bar{\alpha} := \big(\|D_s\|\, \varepsilon_1\big)^{-1}$. On the other hand, given now $\boldsymbol{\psi} \in \mathrm{K}_1$, $\boldsymbol{\psi} \neq \mathbf{0}$, we define $\boldsymbol{\psi}_r := \begin{cases} |\boldsymbol{\psi}|^{s-2}\, \boldsymbol{\psi} & \text{if } \boldsymbol{\psi} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \boldsymbol{\psi} = \mathbf{0} \end{cases}$, which lies in $\mathbf{L}^r(\Omega)$ and satisfies $\displaystyle\int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{\psi}_r \;=\; \|\boldsymbol{\psi}\|_{0,s;\Omega}^s \;>\; 0$. Thus, bounding by below with $\boldsymbol{\phi} := D_r(\boldsymbol{\psi}_r) \in \mathrm{K}_2$, and proceeding similarly as for (4.18), we deduce (4.19) and conclude the proof. $\square$

Before continuing with the continuous inf-sup conditions for the bilinear forms b_i, $i \in \{1,2\}$, we now check the feasibility of the indexes employed so far, according to the different constraints that have arisen along the analysis. In fact, from the preliminary discussion provided in Section 3.1, we have the following initial ranges

$$\begin{cases} l,\, j \in (1,+\infty) \quad \text{and} \quad \rho,\, r \in (2,+\infty) & \text{if } n = 2\,, \\ l,\, j \in [3/2,3] \quad \text{and} \quad \rho,\, r \in [3,6] & \text{if } n = 3\,, \end{cases} \tag{4.21}$$

which, being added the request $\rho \geqslant r$, equivalently $l \geqslant j$, becomes

$$\begin{cases} l \in [2,+\infty)\,, \quad j \in (1,2]\,, \quad \rho \in [4,+\infty)\,, \quad r \in (2,4] & \text{if } n = 2\,, \\ l \in [2,3]\,, \quad j \in [3/2,2]\,, \quad \rho \in [4,6]\,, \quad r \in [3,4] & \text{if } n = 3\,. \end{cases} \tag{4.22}$$

Finally, imposing to r (and hence to s) the ranges required by Lemma 4.2, and guaranteeing that $s \geqslant \frac{2n}{n+2}$, we arrive at the final feasible choices

$$\begin{cases} l \in [2,+\infty)\,,\ j \in (1,2]\,, \quad \rho \in [4,+\infty)\,,\ \varrho \in (1,4/3]\,, \quad r \in (2,4]\,,\ s \in [4/3,2) \quad \text{if } n=2\,, \\ \qquad\qquad l = 3\,,\ j = 3/2\,, \quad \rho = 6\,,\ \varrho = 6/5\,, \quad r = 3\,,\ s = 3/2 \quad \text{if } n = 3\,. \end{cases} \tag{4.23}$$

In particular, the only possibility for the 3D case is obtained by intersecting the range for r specified in the second row of (4.22), that is $r \in [3,4]$, with the one required by Lemma 4.2, that is $r \in [3/2,3]$, which certainly yields $r = 3$. The respective conjugate becomes $s = 3/2$, which clearly verifies $s \geqslant \frac{2n}{n+2} = 6/5$. The occurrence of this unique way of choosing the exponents does not seem in principle to be coincidental since it has also arose in some related papers when a technical result like Lemma 4.2 (or a similar one), is employed (see, e.g. [25, eq. (2.20)], [9, eqs. (2.25) and (2.26)], and [16, Section 4.2]). However, this is not the case for the stress-assisted diffusion problem studied in [24], where the feasible ranges obtained in 3D are actually intervals (see [24, eqs. (3.70) and (3.71)]), and hence it is not possible to conclude a corresponding general rule.

Note that in (4.23) we have included the consequent ranges for $\varrho := \frac{\rho}{\rho-1}$ and $s := \frac{r}{r-1}$ as well. However, we remark that the above indexes are not chosen independently, but once l (or its conjugate j) is chosen, then all the remaining ones are fixed. In this regard, and extending a related comment made in Section 3.1, we stress here that in the 2D case the values of the feasible exponents r and ρ (equivalently, the indexes j and ℓ) vary in opposite directions, namely as r increases, ρ decreases, and conversely. Similarly, being (r,s) and (ρ,ϱ) conjugate pairs, as the first component of each increases, the second one decreases, and conversely. According to the above, and bearing in mind the spaces to which the unknowns belong (cf. (3.12), (3.20), and (3.29)), we deduce that as the regularities of $\boldsymbol{\sigma}$ and ξ_i increase, which means higher values for the exponents s and ρ, the ones of the remaining unknowns decrease, that is r and ϱ get smaller, and conversely. Consequently, no values yielding simultaneously either the least or the most regularity for each component of the solution are available, but only separately for each one of them. Certainly, the maximum or minimum regularity for a particular unknown in this latter case will not be achieved if the respective end of the corresponding interval is open.

We now go back to the well-definedness of $\bar{\mathrm{T}}$ by establishing the continuous inf-sup conditions for the bilinear forms b_i, $i \in \{1,2\}$. While the corresponding proofs are similar to those of [25, Lemma 2.7] and [9, Lemma 3.6], and very close to that of [24, Lemma 3.5], for sake of completeness we provide below the main details of them.

Lemma 4.4. *For each $i \in \{1,2\}$ there exists a positive constant $\bar{\beta}_i$ such that*

$$\sup_{\substack{\boldsymbol{\psi} \in \mathrm{X}_i \\ \boldsymbol{\psi} \neq \mathbf{0}}} \frac{b_i(\boldsymbol{\psi},\lambda)}{\|\boldsymbol{\psi}\|_{\mathrm{X}_i}} \geqslant \bar{\beta}_i\, \|\lambda\|_{\mathrm{M}_i} \qquad \forall\, \lambda \in \mathrm{M}_i\,. \tag{4.24}$$

Proof. We begin by noticing that the values of r and s specified in (4.23) are compatible with the range $[\frac{2n}{n+1}, \frac{2n}{n-1}]$ required by [24, Theorem 3.2], an existence result to be applied below. According to it, and since the pairs $(\mathrm{X}_1, \mathrm{M}_1)$ and $(\mathrm{X}_2,\mathrm{M}_2)$ result from each other exchanging r and s, it suffices to prove (4.24) either for $i = 1$ or for $i = 2$. In what follows we consider $i = 1$, so that, given $\lambda \in \mathrm{M}_1 := \mathrm{L}^r(\Omega)$, we set $\lambda_s := |\lambda|^{r-2}\,\lambda$, which belongs to $\mathrm{L}^s(\Omega)$ and satisfies $\int_\Omega \lambda\,\lambda_s = \|\lambda\|_{0,r;\Omega}\,\|\lambda_s\|_{0,s;\Omega}$. Thus, a straightforward application of the scalar version of [24, Theorem 3.2] yields the existence of a unique $z \in \mathrm{W}_0^{1,s}(\Omega)$ such that $\Delta z = \lambda_s$ in Ω, $z = 0$ on Γ. Moreover, the corresponding continuous dependence result reads $\|z\|_{1,s;\Omega} \leqslant \bar{C}_s\,\|\lambda_s\|_{0,s;\Omega}$, where $\bar{C}_s$ is a positive constant depending on s. Next, defining $\boldsymbol{\phi} := \nabla z \in \mathbf{L}^s(\Omega)$, it follows that $\operatorname{div}(\boldsymbol{\phi}) = \lambda_s$ in Ω, whence $\boldsymbol{\phi} \in \mathbf{H}^s(\operatorname{div}_s;\Omega) =: \mathrm{X}_1$, and there holds $\|\boldsymbol{\phi}\|_{\mathrm{X}_1} = \|\boldsymbol{\phi}\|_{s,\operatorname{div}_s;\Omega} \leqslant \big(1 + \bar{C}_s\big)\,\|\lambda_s\|_{0,s;\Omega}$. In this way, bounding by below with $\boldsymbol{\psi} := \boldsymbol{\phi} \in \mathrm{X}_1$, and bearing in mind the definition of b_1 (cf. (3.21b)) along with the foregoing identities and estimates, we arrive at (4.24) for $i = 1$ with $\beta_1 := \big(1 + \bar{C}_s\big)^{-1}$. The proof for $i = 2$ proceeds analogously, except for the fact that, given $\lambda \in \mathrm{M}_2 := \mathrm{L}^s(\Omega)$, and since $s < 2$, one needs to define $\lambda_r := \begin{cases} |\lambda|^{s-2}\,\lambda & \text{if } \lambda \neq 0\,, \\ 0 & \text{if } \lambda = 0\,. \end{cases}$ Further details are omitted. □

As a consequence of Lemmas 4.3 and 4.4, and the boundedness properties given by (3.22), (3.23), (3.24), and (3.25), we are able to conclude now that the operator $\bar{\mathrm{T}}$ is well-defined.

Theorem 4.5. *For each $\boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$ there exists a unique $(\bar{\boldsymbol{\varphi}}, \bar{\chi}) \in \mathrm{X}_2 \times \mathrm{M}_1$ solution to (4.2), and hence one can define $\bar{\mathrm{T}}(\boldsymbol{\eta}) := \bar{\boldsymbol{\varphi}} \in \mathrm{X}_2$. Moreover, there exists a positive constant $C_{\bar{\mathrm{T}}}$, depending only on ε_0, C_r, $|\Omega|$, $\bar{\alpha}$, and $\bar{\beta}_2$, such that*

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta})\|_{\mathrm{X}_2} \,=\, \|\bar{\boldsymbol{\varphi}}\|_{\mathrm{X}_2} \,\leqslant\, C_{\bar{\mathrm{T}}} \left\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\eta}\|_{0,\rho;\Omega} \right\}. \tag{4.25}$$

Proof. Given $\boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$, a straightforward application of [5, Theorem 2.1, Section 2.1] implies the existence of a unique $(\bar{\boldsymbol{\varphi}}, \bar{\chi}) \in \mathrm{X}_2 \times \mathrm{M}_1$ solution to (4.2). In turn, the a priori estimate provided in [5, Corollary 2.1, Section 2.1, eq. (2.15)] establishes

$$\|\bar{\boldsymbol{\varphi}}\|_{\mathrm{X}_2} \,\leqslant\, \frac{1}{\bar{\alpha}} \|\mathrm{F}\|_{\mathrm{X}_1'} \,+\, \frac{1}{\bar{\beta}_2} \left(1 + \frac{\|a\|}{\bar{\alpha}}\right) \|\mathrm{G}_{\boldsymbol{\eta}}\|_{\mathrm{M}_2'}\,, \tag{4.26}$$

which, along with the aforementioned boundedness properties, yields (4.25) and ends the proof. □

Similarly as for $\widehat{\mathrm{T}}$, and employing now [5, Corollary 2.1, Section 2.1, eq. (2.16)], we observe that the a priori bound for the $\bar{\chi}$ component of the unique solution to (4.2) reduces to

$$\|\bar{\chi}\|_{\mathrm{M}_1} \,\leqslant\, \frac{1}{\bar{\beta}_1} \left(1 + \frac{\|a\|}{\bar{\alpha}}\right) \|\mathrm{F}\|_{\mathrm{X}_1'} \,+\, \frac{\|a\|}{\bar{\beta}_1\,\bar{\beta}_2} \left(1 + \frac{\|a\|}{\bar{\alpha}}\right) \|\mathrm{G}_{\boldsymbol{\eta}}\|_{\mathrm{M}_2'}\,,$$

which yields the same inequality as (4.25), but with a different constant, in particular depending additionally on $\bar{\beta}_1$. Therefore, as before, we still denote the largest of them by $C_{\bar{\mathrm{T}}}$, and simply say that the right hand-side of (4.25) constitutes the a priori estimate for both $\bar{\boldsymbol{\varphi}}$ and $\bar{\chi}$.

4.2.3 Well-definedness of the operator $\widetilde{\mathrm{T}}$

In this section we employ the solvability result for perturbed saddle point formulations in Banach spaces provided by [15, Theorem 3.4], along with the Banach–Nečas–Babuška Theorem (cf. [19, Theorem 2.6]), to show that, given an arbitrary $(\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q}$, (4.3) is well-posed for each $i \in \{1, 2\}$, equivalently that T_i is well-defined. Since this result was already derived in [15, Theorem 4.2] as an application of the abstract theory developed there, and more specifically of [15, Theorem 3.4], we just discuss in what follows the main aspects of its proof.

To begin with, we introduce the bilinear forms $\mathbf{A}$, $\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}} : (\mathrm{H}_i \times \mathrm{Q}_i) \times (\mathrm{H}_i \times \mathrm{Q}_i) \to \mathrm{R}$ given by

$$\mathbf{A}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big) \,:=\, a_i(\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) \,+\, c_i(\boldsymbol{\tau}_i, \vartheta_i) \,+\, c_i(\boldsymbol{\zeta}_i, \eta_i) \,-\, d_i(\vartheta_i, \eta_i)\,, \tag{4.27}$$

and

$$\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big) \,:=\, \mathbf{A}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big) \,-\, c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \vartheta_i)\,, \tag{4.28}$$

for all $(\boldsymbol{\zeta}_i, \vartheta_i)$, $(\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i$, and realize that (4.3) can be re-stated as: Find $(\widetilde{\boldsymbol{\sigma}}_i, \widetilde{\xi}_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ such that

$$\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}((\widetilde{\boldsymbol{\sigma}}_i, \widetilde{\xi}_i), (\boldsymbol{\tau}_i, \eta_i)) \,=\, \mathrm{F}_i(\boldsymbol{\tau}_i) + \mathrm{G}_i(\eta_i) \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,. \tag{4.29}$$

In this way, the proof reduces to show first that the bilinear forms forming part of $\mathbf{A}$ satisfy the hypotheses of [15, Theorem 3.4], and then to combine the consequence of this result with the effect of the extra term given by $c_{\boldsymbol{\phi},\mathbf{v}}(\cdot,\cdot)$, to conclude that $\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}$ satisfies a global inf-sup condition.

Indeed, it is clear from (3.30a), (3.30c), and the upper bound of κ_i (cf. (2.4)) that a_i and d_i are symmetric and positive semi-definite, which proves the assumption i) of [15, Theorem 3.4]. Next, bearing in mind the definitions of c_i (cf. (3.30b)) and the spaces H_i and Q_i (cf. (3.29)), and using again that $\mathrm{L}^\rho(\Omega)$ is isomorphic to the dual of $\mathrm{L}^\varrho(\Omega)$, we readily find that the null space V_i of the operator induced by c_i becomes

$$\mathrm{V}_i \,:=\, \Big\{ \boldsymbol{\tau}_i \in \mathrm{H}_i : \quad \mathrm{div}(\boldsymbol{\tau}_i) \,=\, 0 \Big\} , \tag{4.30}$$

and thus

$$a_i(\boldsymbol{\tau}_i, \boldsymbol{\tau}_i) \,\geqslant\, \frac{1}{\overline{\kappa}} \, \|\boldsymbol{\tau}_i\|^2_{0,\Omega} \,=\, \frac{1}{\overline{\kappa}} \, \|\boldsymbol{\tau}_i\|^2_{\mathrm{div}_\varrho;\Omega} \qquad \forall\, \boldsymbol{\tau}_i \in \mathrm{V}_i \,, \tag{4.31}$$

from which the assumption ii) of [15, Theorem 3.4], namely the continuous inf-sup condition for a_i, is clearly satisfied with constant $\widetilde{\alpha} := \overline{\kappa}^{-1}$.

In turn, while the continuous inf-sup condition for $\widetilde{c}_i$ was already established in [25, Lemma 2.9] (see also [15, Lemma 4.1]), for sake of clearness we provide below the main steps of its proof, which follows similarly to the one yielding the continuous inf-sup condition for $\mathbf{b}$ in the present Section 4.2.1. More precisely, given $\eta_i \in \mathrm{Q}_i := \mathrm{L}^\rho(\Omega)$, we set $\eta_{i,\varrho} := |\eta_i|^{\rho-2}\,\eta_i$, which uses from (4.23) that $\rho \geqslant 2$, and notice that there hold $\eta_{i,\varrho} \in \mathrm{L}^\varrho(\Omega)$ and $\int_\Omega \eta_i\, \eta_{i,\varrho} \,=\, \|\eta_i\|_{0,\rho;\Omega}\, \|\eta_{i,\varrho}\|_{0,\varrho;\Omega}$. Then, we let $\boldsymbol{\zeta}_i \,:=\, \nabla z \in \mathbf{L}^2(\Omega)$, where $z \in \mathrm{H}^1_0(\Omega)$ is the unique solution of the variational formulation: $\int_\Omega \nabla z \cdot \nabla w \,=\, -\int_\Omega \eta_{i,\varrho}\, w$ for all $w \in \mathrm{H}^1_0(\Omega)$, and deduce from the latter that $\mathrm{div}(\boldsymbol{\zeta}_i) \,=\, \eta_{i,\varrho}$ in Ω, which yields $\boldsymbol{\zeta}_i \in \mathrm{H}_i := \mathbf{H}(\mathrm{div}_\varrho;\Omega)$. In turn, denoting by c_{P} the positive constant guaranteeing the Poincaré inequality: $c_{\mathrm{P}}\, \|w\|^2_{1,\Omega} \,\leqslant\, |w|^2_{1,\Omega} \;\; \forall\, w \in \mathrm{H}^1_0(\Omega$, and letting again $\mathrm{i}_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$ be the continuous injection, we find that $\|z\|_{1,\Omega} \,\leqslant\, \frac{\|\mathrm{i}_\rho\|}{c_{\mathrm{P}}}\, \|\eta_{i,\varrho}\|_{0,\varrho;\Omega}$, and hence $\|\boldsymbol{\zeta}_i\|_{\mathrm{H}_i} \,\leqslant\, \big(1 + \frac{\|\mathrm{i}_\rho\|}{c_{\mathrm{P}}}\big)\, \|\eta_{i,\varrho}\|_{0,\varrho;\Omega}$. In this way, bounding by below with $\boldsymbol{\tau}_i := \boldsymbol{\zeta}_i \in \mathrm{H}_i$, recalling the definition of c_i (cf. (3.30b)), and employing the foregoing identities and estimates, we arrive at

$$\sup_{\substack{\boldsymbol{\tau}_i \in \mathrm{H}_i\\ \boldsymbol{\tau}_i \neq \mathbf{0}}} \frac{c_i(\boldsymbol{\tau}_i, \eta_i)}{\|\boldsymbol{\tau}_i\|_{\mathrm{H}_i}} \,\geqslant\, \widetilde{\beta}\, \|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\, \eta_i \in \mathrm{Q}_i \,, \tag{4.32}$$

with $\widetilde{\beta} := \big(1 + \frac{\|\mathrm{i}_\rho\|}{c_{\mathrm{P}}}\big)^{-1}$, thus confirming the verification of assumption iii) of [15, Theorem 3.4].

Consequently, having shown that a_i, c_i, and d_i verify all the hypotheses of [15, Theorem 3.4], we conclude that $\mathbf{A}$ satisfies the global inf-sup condition, which means that there exists a positive constant $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}$, depending only on $\|a_i\|$, $\|c_i\|$, $\widetilde{\alpha}$, and $\widetilde{\beta}$, such that

$$\sup_{\substack{(\boldsymbol{\tau}_i,\eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\\ (\boldsymbol{\tau}_i,\eta_i) \neq \mathbf{0}}} \frac{\mathbf{A}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big)}{\|(\boldsymbol{\tau}_i, \eta_i)\|_{\mathrm{H}_i \times \mathrm{Q}_i}} \,\geqslant\, \widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}\, \|(\boldsymbol{\zeta}_i, \vartheta_i)\|_{\mathrm{H}_i \times \mathrm{Q}_i} \qquad \forall\, (\boldsymbol{\zeta}_i, \vartheta_i) \in \mathrm{H}_i \times \mathrm{Q}_i \,. \tag{4.33}$$

Moreover, invoking the upper bound of $c_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (3.31), (3.32)), it follows from (4.28) and (4.33) that

$$\sup_{\substack{(\boldsymbol{\tau}_i,\eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\\ (\boldsymbol{\tau}_i,\eta_i) \neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big)}{\|(\boldsymbol{\tau}_i, \eta_i)\|_{\mathrm{H}_i \times \mathrm{Q}_i}} \,\geqslant\, \Big\{ \widetilde{\boldsymbol{\alpha}}_{\mathbf{A}} \,-\, \|c\| \, \Big(\|\boldsymbol{\phi}\|_{0,r,\Omega} + \|\mathbf{v}\|_{0,r,\Omega} \Big) \Big\}\, \|(\boldsymbol{\zeta}_i, \vartheta_i)\|_{\mathrm{H}_i \times \mathrm{Q}_i} \tag{4.34}$$

for all $(\boldsymbol{\zeta}_i, \vartheta_i) \in \mathrm{H}_i \times \mathrm{Q}_i$, from which, under the assumption that, say

$$\|\boldsymbol{\phi}\|_{0,r,\Omega} + \|\mathbf{v}\|_{0,r,\Omega} \,\leqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\, \|c\|} \,, \tag{4.35}$$

we conclude that

$$\sup_{\substack{(\boldsymbol{\tau}_i,\eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\\ (\boldsymbol{\tau}_i,\eta_i) \neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big)}{\|(\boldsymbol{\tau}_i, \eta_i)\|_{\mathrm{H}_i \times \mathrm{Q}_i}} \,\geqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2}\, \|(\boldsymbol{\zeta}_i, \vartheta_i)\|_{\mathrm{H}_i \times \mathrm{Q}_i} \qquad \forall\, (\boldsymbol{\zeta}_i, \vartheta_i) \in \mathrm{H}_i \times \mathrm{Q}_i \,. \tag{4.36}$$

Similarly, using the symmetry of $\mathbf{A}$ and (4.33), and assuming again (4.35), we find that

$$\sup_{\substack{(\boldsymbol{\zeta}_i,\vartheta_i)\in \mathrm{H}_i\times \mathrm{Q}_i\\ (\boldsymbol{\zeta}_i,\vartheta_i)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\boldsymbol{\zeta}_i,\vartheta_i),(\boldsymbol{\tau}_i,\eta_i)\big)}{\|(\boldsymbol{\zeta}_i,\vartheta_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i}} \,\geqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2}\,\|(\boldsymbol{\tau}_i,\eta_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i} \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i)\in \mathrm{H}_i\times\mathrm{Q}_i\,. \tag{4.37}$$

In this way, we are now in position of establishing that, for each $i\in\{1,2\}$, (4.3) is well-posed, which means, equivalently, that $\widetilde{\mathrm{T}}_i$ is well-defined.

Theorem 4.6. *For each $i\in\{1,2\}$, and for each $(\boldsymbol{\phi},\mathbf{v})\in \mathrm{X}_2\times\mathbf{Q}$ such that (4.35) holds, there exists a unique $(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)\in \mathrm{H}_i\times\mathrm{Q}_i$ solution to (4.3), and hence one can define $\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v}) := \widetilde{\xi}_i\in\mathrm{Q}_i$. Moreover, there exists a positive constant $C_{\widetilde{\mathrm{T}}}$, depending only on $\|\mathrm{i}_\rho\|$ and $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}$, such that*

$$\|\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v})\|_{\mathrm{Q}_i} \,=\, \|\widetilde{\xi}_i\|_{\mathrm{Q}_i} \,\leqslant\, \|(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i} \,\leqslant\, C_{\widetilde{\mathrm{T}}}\Big\{\|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\}\,. \tag{4.38}$$

Proof. Thanks to (4.36), (4.37), and the boundedness of F_i and G_i (cf. (3.31), (3.32)), the unique solvability of (4.3) follows from a straightforward application of [19, Theorem 2.6]. In turn, the a priori estimate given by [19, Theorem 2.6, eq. (2.5)] reads

$$\|(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i} \,\leqslant\, \frac{2}{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}\Big\{\|\mathrm{F}_i\|_{\mathrm{H}_i'} \,+\, \|\mathrm{G}_i\|_{\mathrm{Q}_i'}\Big\}\,,$$

which, along with the upper bounds for $\|\mathrm{F}_i\|_{\mathrm{H}_i'}$ and $\|\mathrm{G}_i\|_{\mathrm{Q}_i'}$ derived from (3.31) and (3.32), yields (4.38) with $C_{\widetilde{\mathrm{T}}} := \dfrac{2}{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}\big(1+\|\mathrm{i}_\rho\|\big)$. □

We end this section by observing from the definition of $\widetilde{\mathrm{T}}$ (cf. (4.4)) and the priori estimates given by (4.38) for each $i\in\{1,2\}$, that

$$\|\widetilde{\mathrm{T}}(\boldsymbol{\phi},\mathbf{v})\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \,:=\, \sum_{i=1}^{2}\|\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v})\|_{\mathrm{Q}_i} \,\leqslant\, C_{\widetilde{\mathrm{T}}}\sum_{i=1}^{2}\Big\{\|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\} \tag{4.39}$$

for each $(\boldsymbol{\phi},\mathbf{v})\in\mathrm{X}_2\times\mathbf{Q}$ satisfying (4.35).

4.3 Solvability analysis of the fixed-point scheme

Knowing that the operators $\widehat{\mathrm{T}}$, $\bar{\mathrm{T}}$, $\widetilde{\mathrm{T}}$, and hence $\mathbf{T}$ as well, are well defined, we now address the solvability of the fixed-point equation (4.5). For this purpose, and in order to finally apply the Banach Theorem, we first derive sufficient conditions under which $\mathbf{T}$ maps a closed ball of $\mathrm{Q}_1\times\mathrm{Q}_2$ into itself. Thus, letting δ be an arbitrary radius to be properly chosen later on, we define

$$\mathrm{W}(\delta) \,:=\, \Big\{\boldsymbol{\eta} \,:=\, (\eta_1,\eta_2)\in \mathrm{Q}_1\times\mathrm{Q}_2 : \quad \|\boldsymbol{\eta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \,\leqslant\, \delta\Big\}\,. \tag{4.40}$$

Then, given $\boldsymbol{\eta}\in\mathrm{W}(\delta)$, we have from the definition of $\mathbf{T}$ (cf. (4.5)) and the a priori estimate for $\widetilde{\mathrm{T}}$ (cf. (4.39)) that, under the assumption (cf. (4.35))

$$\mathcal{S}(\boldsymbol{\eta}) \,:=\, \|\bar{\mathrm{T}}(\boldsymbol{\eta})\|_{0,r,\Omega} + \|\widehat{\mathrm{T}}\big(\boldsymbol{\eta},\bar{\mathrm{T}}(\boldsymbol{\eta})\big)\|_{0,r,\Omega} \,\leqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\,\|c\|}\,, \tag{4.41}$$

there holds

$$\|\mathbf{T}(\boldsymbol{\eta})\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \,=\, \|\widetilde{\mathrm{T}}\big(\bar{\mathrm{T}}(\boldsymbol{\eta}),\widehat{\mathrm{T}}\big(\boldsymbol{\eta},\bar{\mathrm{T}}(\boldsymbol{\eta})\big)\big)\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \,\leqslant\, C_{\widetilde{\mathrm{T}}}\sum_{i=1}^{2}\Big\{\|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\}\,. \tag{4.42}$$

In turn, applying the a priori estimates for $\widehat{\mathrm{T}}$ (cf. (4.11)) and $\bar{\mathrm{T}}$ (cf. (4.25)), we find that

$$\begin{aligned}\mathcal{S}(\boldsymbol{\eta}) \,&\leqslant\, \big(1 + C_{\widehat{\mathrm{T}}}\,\|\boldsymbol{\eta}\|\big)\,\|\bar{\mathrm{T}}(\boldsymbol{\eta})\| \,+\, C_{\widehat{\mathrm{T}}}\,\Big\{\|\mathbf{g}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,s,\Omega}\Big\}\\ &\leqslant\, C_0\big(1+\|\boldsymbol{\eta}\|\big)\|\boldsymbol{\eta}\| + C_0\big(1+\|\boldsymbol{\eta}\|\big)\Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big\} + C_{\widehat{\mathrm{T}}}\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega}\Big\}\,,\end{aligned}$$

with $C_0 \,:=\, \max\big\{1, C_{\widehat{\mathrm{T}}}\big\}\,C_{\bar{\mathrm{T}}}$, so that, bounding $\|\boldsymbol{\eta}\|$ by δ, we deduce that a sufficient condition for (4.41) reduces to

$$C_0\big(1+\delta\big)\delta + C_0\big(1+\delta\big)\Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big\} + C_{\widehat{\mathrm{T}}}\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega}\Big\} \,\leqslant\, \frac{\tilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\,\|c\|}\,. \tag{4.43}$$

For instance, defining

$$\delta \,:=\, \min\Big\{1, \frac{\tilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{8C_0\|c\|}\Big\}\,, \tag{4.44}$$

letting $C_1 \,:=\, 2C_0$, and imposing

$$C_1\Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big\} + C_{\widehat{\mathrm{T}}}\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega}\Big\} \,\leqslant\, \frac{\tilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{4\,\|c\|}\,, \tag{4.45}$$

it is easily seen that (4.43) holds. We have therefore proved the following result.

Lemma 4.7. *Assume that δ and the data are sufficiently small so that there hold (4.43) and*

$$C_{\tilde{\mathrm{T}}}\,\sum_{i=1}^{2}\Big\{\|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\} \,\leqslant\, \delta\,. \tag{4.46}$$

Then, $\mathbf{T}\big(\mathrm{W}(\delta)\big) \subseteq \mathrm{W}(\delta)$. In particular, with the definition (4.44) of δ, and under the assumptions (4.45) and (4.46), the same conclusion is attained.

We now address the continuity properties of $\widehat{\mathrm{T}}$, $\bar{\mathrm{T}}$, $\tilde{\mathrm{T}}$, and hence of $\mathbf{T}$. We begin with that of $\widehat{\mathrm{T}}$.

Lemma 4.8. *There exists a positive constant $\mathrm{L}_{\widehat{\mathrm{T}}}$, depending only on ε_0, $|\Omega|$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\|\mathbf{a}\|$, such that*

$$\|\widehat{\mathrm{T}}(\boldsymbol{\eta},\boldsymbol{\phi}) - \widehat{\mathrm{T}}(\boldsymbol{\vartheta},\boldsymbol{\psi})\|_{\mathbf{Q}} \,\leqslant\, \mathrm{L}_{\widehat{\mathrm{T}}}\,\Big\{\|\boldsymbol{\eta}\|_{0,\rho,\Omega}\,\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{0,r,\Omega} \,+\, \|\boldsymbol{\eta}-\boldsymbol{\vartheta}\|_{0,\rho,\Omega}\,\|\boldsymbol{\psi}\|_{0,r,\Omega}\Big\} \tag{4.47}$$

for all $(\boldsymbol{\eta},\boldsymbol{\phi})$, $(\boldsymbol{\vartheta},\boldsymbol{\psi}) \in \big(\mathrm{Q}_1 \times \mathrm{Q}_2\big) \times \mathrm{X}_2$.

Proof. Given $(\boldsymbol{\eta},\boldsymbol{\phi})$, $(\boldsymbol{\vartheta},\boldsymbol{\psi}) \in \big(\mathrm{Q}_1 \times \mathrm{Q}_2\big) \times \mathrm{X}_2$, we let $\widehat{\mathrm{T}}(\boldsymbol{\eta},\boldsymbol{\phi}) \,:=\, \hat{\mathbf{u}}$ and $\widehat{\mathrm{T}}(\boldsymbol{\vartheta},\boldsymbol{\psi}) \,:=\, \hat{\mathbf{w}}$, where $(\hat{\boldsymbol{\sigma}},\hat{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ and $(\hat{\boldsymbol{\zeta}},\hat{\mathbf{w}}) \in \mathbf{H}\times\mathbf{Q}$ are the corresponding unique solutions of (4.1). Then, subtracting both systems, we obtain

$$\begin{aligned}\mathbf{a}(\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\zeta}},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\hat{\mathbf{u}} - \hat{\mathbf{w}}) \;&=\; 0 && \forall\,\boldsymbol{\tau} \in \mathbf{H}\,,\\ \mathbf{b}(\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\zeta}},\mathbf{v}) \;&=\; \big(\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}}\big)(\mathbf{v}) && \forall\,\mathbf{v} \in \mathbf{Q}\,,\end{aligned} \tag{4.48}$$

which says that $(\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\zeta}},\hat{\mathbf{u}} - \hat{\mathbf{w}}) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of a system like (4.1), but with $\mathbf{F} = \mathbf{0}$ and $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}}$ instead of just $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}$. Hence, similarly as for the derivation of (4.11), that is employing [19, Theorem 2.34, eq. (2.30)] (see also (4.12)), we deduce that

$$\|\widehat{\mathrm{T}}(\boldsymbol{\eta},\boldsymbol{\phi}) - \widehat{\mathrm{T}}(\boldsymbol{\vartheta},\boldsymbol{\psi})\|_{\mathbf{Q}} \,=\, \|\hat{\mathbf{u}} - \hat{\mathbf{w}}\|_{\mathbf{Q}} \,\leqslant\, \frac{\|\mathbf{a}\|}{\boldsymbol{\beta}^2}\,\Big(1 + \frac{\|\mathbf{a}\|}{\boldsymbol{\alpha}}\Big)\,\|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}}\|_{\mathbf{Q}'}\,. \tag{4.49}$$

In turn, it is clear from (3.13d), and then subtracting and adding $\boldsymbol{\psi}$ to the factor $\boldsymbol{\phi}$ in the first term, that for each $\mathbf{v} \in \mathbf{Q}$ there holds

$$\begin{aligned}(\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}})(\mathbf{v}) \,&=\, \int_\Omega \varepsilon^{-1} \Big\{ \big(\eta_1 - \eta_2\big)\, \boldsymbol{\phi} - \big(\vartheta_1 - \vartheta_2\big)\, \boldsymbol{\psi} \Big\} \cdot \mathbf{v} \\ &= \int_\Omega \varepsilon^{-1} \Big\{ \big(\eta_1 - \eta_2\big)\, \big(\boldsymbol{\phi} - \boldsymbol{\psi}\big) + \big((\eta_1 - \vartheta_1) - (\eta_2 - \vartheta_2)\big)\, \boldsymbol{\psi} \Big\} \cdot \mathbf{v}\,,\end{aligned}$$

from which, proceeding as for the boundedness of $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}$ (cf. (3.15), (3.16)), that is employing the lower bound of ε (cf. (2.4)), (3.1a), and the fact that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{(r-2)/2r} \|\cdot\|_{0,r;\Omega}$, we conclude that

$$\|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}}\|_{\mathbf{Q}'} \,\leqslant\, \varepsilon_0^{-1}\, |\Omega|^{(r-2)/2r} \Big\{ \|\boldsymbol{\eta}\|_{0,\rho,\Omega}\, \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{0,r,\Omega} \,+\, \|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{0,\rho,\Omega}\, \|\boldsymbol{\psi}\|_{0,r,\Omega} \Big\}\,. \tag{4.50}$$

In this way, replacing (4.50) back into (4.49), we arrive at (4.47) and finish the proof. $\square$

The next result establishes the continuity of $\bar{\mathrm{T}}$, whose proof follows similarly to that of Lemma 4.8.

Lemma 4.9. *There exists a positive constant $\mathrm{L}_{\bar{\mathrm{T}}}$, depending only on $|\Omega|$, $\bar{\alpha}$, $\bar{\beta}_2$, and $\|a\|$, such that*

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta}) - \bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} \,\leqslant\, \mathrm{L}_{\bar{\mathrm{T}}}\, \|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{0,\rho;\Omega} \qquad \forall\, \boldsymbol{\eta},\, \boldsymbol{\vartheta} \in \mathrm{Q}_1 \times \mathrm{Q}_2\,. \tag{4.51}$$

Proof. Given $\boldsymbol{\eta}$, $\boldsymbol{\vartheta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$, we let $\bar{\mathrm{T}}(\boldsymbol{\eta}) := \bar{\boldsymbol{\varphi}}$ and $\bar{\mathrm{T}}(\boldsymbol{\vartheta}) := \bar{\boldsymbol{\phi}}$, where $(\bar{\boldsymbol{\varphi}}, \bar{\chi}) \in \mathrm{X}_2 \times \mathrm{M}_1$ and $(\bar{\boldsymbol{\phi}}, \bar{\omega}) \in \mathrm{X}_2 \times \mathrm{M}_1$ are the corresponding unique solutions of (4.2). Then, subtracting both systems, we get

$$\begin{aligned} a(\bar{\boldsymbol{\varphi}} - \bar{\boldsymbol{\phi}}, \boldsymbol{\psi}) + b_1(\boldsymbol{\psi}, \bar{\chi} - \bar{\omega}) \;&=\; 0 && \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \\ b_2(\bar{\boldsymbol{\varphi}} - \bar{\boldsymbol{\phi}}, \lambda) \;&=\; \big(\mathrm{G}_{\boldsymbol{\eta}} - \mathrm{G}_{\boldsymbol{\vartheta}}\big)(\lambda) && \forall\, \lambda \in \mathrm{M}_2\,,\end{aligned} \tag{4.52}$$

which states that $(\bar{\boldsymbol{\varphi}} - \bar{\boldsymbol{\phi}}, \bar{\chi} - \bar{\omega}) \in \mathrm{X}_2 \times \mathrm{M}_1$ is the unique solution of a problem like (4.2) with $\mathrm{G} = \mathbf{0}$ and $\mathrm{G}_{\boldsymbol{\eta}} - \mathrm{G}_{\boldsymbol{\vartheta}}$ instead of $\mathrm{G}_{\boldsymbol{\eta}}$. In this way, proceeding as for the derivation of (4.25), which means applying the a priori estimate given by [5, Corollary 2.1, Section 2.1, eq. (2.15)] (see also (4.26)), we find that

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta}) - \bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} \,=\, \|\bar{\boldsymbol{\varphi}} - \bar{\boldsymbol{\phi}}\|_{\mathrm{X}_2} \,\leqslant\, \frac{1}{\bar{\beta}_2} \left(1 + \frac{\|a\|}{\bar{\alpha}}\right) \|\mathrm{G}_{\boldsymbol{\eta}} - \mathrm{G}_{\boldsymbol{\vartheta}}\|_{\mathrm{M}_2'}\,. \tag{4.53}$$

Now, it is clear from (3.21d) that for each $\lambda \in \mathrm{M}_2$ there holds

$$\big(\mathrm{G}_{\boldsymbol{\eta}} - \mathrm{G}_{\boldsymbol{\vartheta}}\big)(\lambda) \,=\, \mathrm{G}_{\boldsymbol{\eta}-\boldsymbol{\vartheta}}(\lambda) \,=\, -\int_\Omega \lambda\, \big\{(\eta_1 - \vartheta_1) - (\eta_2 - \vartheta_2)\big\}\,,$$

from which, applying Hölder's inequality, as we did for the boundedness of $\mathrm{G}_{\boldsymbol{\eta}}$ (cf. (3.22), (3.23)), and using that $\|\cdot\|_{0,r;\Omega} \leqslant |\Omega|^{(\rho-r)/\rho r}\, \|\cdot\|_{0,\rho;\Omega}$, we deduce that

$$\|\mathrm{G}_{\boldsymbol{\eta}} - \mathrm{G}_{\boldsymbol{\vartheta}}\|_{\mathrm{M}_2'} \,\leqslant\, |\Omega|^{(\rho-r)/\rho r}\, \|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{0,\rho;\Omega}\,. \tag{4.54}$$

Finally, employing (4.54) in (4.53), we obtain (4.51) and conclude the proof. $\square$

It remains to prove the continuity of $\widetilde{\mathrm{T}}$, which is provided by the following lemma.

Lemma 4.10. *There exists a positive constant $\mathrm{L}_{\widetilde{\mathrm{T}}}$, depending only on ε_0, $\underline{\kappa}$, $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}$, and $C_{\widetilde{\mathrm{T}}}$, such that*

$$\|\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{v}) - \widetilde{\mathrm{T}}(\boldsymbol{\psi}, \mathbf{w})\|_{\mathrm{Q}_1 \times \mathrm{Q}_2} \,\leqslant\, \mathrm{L}_{\widetilde{\mathrm{T}}} \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \Big\}\, \|(\boldsymbol{\phi}, \mathbf{v}) - (\boldsymbol{\psi}, \mathbf{w})\|_{\mathrm{X}_2 \times \mathbf{Q}} \tag{4.55}$$

for all $(\boldsymbol{\phi}, \mathbf{v})$, $(\boldsymbol{\psi}, \mathbf{w}) \in \mathrm{X}_2 \times \mathbf{Q}$ satisfying (4.35).

Proof. Given $(\boldsymbol{\phi},\mathbf{v})$ and $(\boldsymbol{\psi},\mathbf{w})$ as indicated, we let, for each $i\in\{1,2\}$, $\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v}) := \widetilde{\xi}_i \in \mathrm{Q}_i$ and $\widetilde{\mathrm{T}}_i(\boldsymbol{\psi},\mathbf{w}) := \widetilde{\vartheta}_i \in \mathrm{Q}_i$, where $(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i) \in \mathrm{H}_i\times\mathrm{Q}_i$ and $(\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i) \in \mathrm{H}_i\times\mathrm{Q}_i$ are the corresponding unique solutions of (4.3), equivalently (cf. (4.29))

$$\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i),(\boldsymbol{\tau}_i,\eta_i)\big) \,=\, \mathrm{F}_i(\boldsymbol{\tau}_i)+\mathrm{G}_i(\eta_i) \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i)\in\mathrm{H}_i\times\mathrm{Q}_i\,, \tag{4.56}$$

and

$$\mathbf{A}_{\boldsymbol{\psi},\mathbf{w}}\big((\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big) \,=\, \mathrm{F}_i(\boldsymbol{\tau}_i)+\mathrm{G}_i(\eta_i) \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i)\in\mathrm{H}_i\times\mathrm{Q}_i\,. \tag{4.57}$$

It follows from (4.56) and (4.57), along with the definitions of the bilinear forms $\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (4.28)) and $c_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (3.30f)), that

$$\begin{aligned}\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)-(\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big) \,&=\, \mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i),(\boldsymbol{\tau}_i,\eta_i)\big) - \mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big)\\ &=\, \mathbf{A}_{\boldsymbol{\psi},\mathbf{w}}\big((\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big) - \mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big) \,=\, c_{\boldsymbol{\phi}-\boldsymbol{\psi},\mathbf{v}-\mathbf{w}}(\boldsymbol{\tau}_i,\widetilde{\vartheta}_i)\,,\end{aligned} \tag{4.58}$$

so that applying the global inf-sup condition (4.36) to $(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)-(\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i)$, and then using (4.58) and the boundedness of $c_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (3.31), (3.32)), we conclude that

$$\|\widetilde{\xi}_i-\widetilde{\vartheta}_i\|_{\mathrm{Q}_i} \,\leqslant\, \|(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)-(\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i} \,\leqslant\, \frac{2\,\|c\|}{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}\Big\{\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{0,r;\Omega}+\|\mathbf{v}-\mathbf{w}\|_{0,r;\Omega}\Big\}\,\|\widetilde{\vartheta}_i\|_{\mathrm{Q}_i}\,.$$

Next, invoking the a priori bound (4.38) for $\|\widetilde{\vartheta}_i\|_{\mathrm{Q}_i}$, the foregoing inequality yields

$$\|\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v})-\widetilde{\mathrm{T}}_i(\boldsymbol{\psi},\mathbf{w})\|_{\mathrm{Q}_i} \,\leqslant\, \frac{2\,\|c\|\,C_{\widetilde{\mathrm{T}}}}{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}\Big\{\|g_i\|_{1/2,\Gamma}\,+\,\|f_i\|_{0,\varrho;\Omega}\Big\}\,\|(\boldsymbol{\phi},\mathbf{v})-(\boldsymbol{\psi},\mathbf{w})\|_{\mathrm{X}_2\times\mathbf{Q}}\,,$$

from which, summing over $i\in\{1,2\}$, we arrive at (4.55) and end the proof. □

Having proved Lemmas 4.8, 4.9, and 4.10, we now aim to derive the continuity property of the fixed point operator $\mathbf{T}$. To this end, given $\boldsymbol{\eta},\,\boldsymbol{\vartheta}\in\mathrm{W}(\delta)$ (cf. (4.40)), we first recall from the definition of $\mathbf{T}$ (cf. (4.5)) and Theorem 4.6 that, in order to define $\mathbf{T}(\boldsymbol{\eta})$ and $\mathbf{T}(\boldsymbol{\vartheta})$, we need that the pairs $\big(\bar{\mathrm{T}}(\boldsymbol{\eta}),\widehat{\mathrm{T}}(\boldsymbol{\eta},\bar{\mathrm{T}}(\boldsymbol{\eta}))\big)$ and $\big(\bar{\mathrm{T}}(\boldsymbol{\vartheta}),\widehat{\mathrm{T}}(\boldsymbol{\vartheta},\bar{\mathrm{T}}(\boldsymbol{\vartheta}))\big)$ satisfy (4.35). Then, according to the discussion at the beginning of the present section, we know that a sufficient condition for the latter is given by (4.43), which we assume in what follows. Alternatively, and as indicated there as well, (4.44) and (4.45) are in turn sufficient for (4.43).

Thus, under the aforementioned assumption on δ and the data, a direct application of (4.55) (cf. Lemma 4.10) yields

$$\begin{aligned}\|\mathbf{T}(\boldsymbol{\eta})-\mathbf{T}(\boldsymbol{\vartheta})\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \,&=\, \|\widetilde{\mathrm{T}}\big(\bar{\mathrm{T}}(\boldsymbol{\eta}),\widehat{\mathrm{T}}(\boldsymbol{\eta},\bar{\mathrm{T}}(\boldsymbol{\eta}))\big)-\widetilde{\mathrm{T}}\big(\bar{\mathrm{T}}(\boldsymbol{\vartheta}),\widehat{\mathrm{T}}(\boldsymbol{\vartheta},\bar{\mathrm{T}}(\boldsymbol{\vartheta}))\big)\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\\ &\leqslant\, \mathrm{L}_{\widetilde{\mathrm{T}}}\sum_{i=1}^{2}\Big\{\|g_i\|_{1/2,\Gamma}+\|f_i\|_{0,\varrho,\Omega}\Big\}\Big\{\|\bar{\mathrm{T}}(\boldsymbol{\eta})-\bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2}+\|\widehat{\mathrm{T}}(\boldsymbol{\eta},\bar{\mathrm{T}}(\boldsymbol{\eta}))-\widehat{\mathrm{T}}(\boldsymbol{\vartheta},\bar{\mathrm{T}}(\boldsymbol{\vartheta}))\|_{\mathbf{Q}}\Big\}\,.\end{aligned} \tag{4.59}$$

In addition, employing now (4.51) (cf. Lemma 4.9) and (4.47) (cf. Lemma 4.8), we obtain

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta})-\bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} \,\leqslant\, \mathrm{L}_{\bar{\mathrm{T}}}\,\|\boldsymbol{\eta}-\boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\,, \tag{4.60}$$

and

$$\begin{aligned}&\|\widehat{\mathrm{T}}(\boldsymbol{\eta},\bar{\mathrm{T}}(\boldsymbol{\eta}))-\widehat{\mathrm{T}}(\boldsymbol{\vartheta},\bar{\mathrm{T}}(\boldsymbol{\vartheta}))\|_{\mathbf{Q}}\\ &\quad\leqslant\, \mathrm{L}_{\widehat{\mathrm{T}}}\Big\{\|\boldsymbol{\eta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\,\|\bar{\mathrm{T}}(\boldsymbol{\eta})-\bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2}\,+\,\|\boldsymbol{\eta}-\boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\,\|\bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2}\Big\}\,,\end{aligned} \tag{4.61}$$

respectively, whereas the a priori estimate for $\bar{\mathrm{T}}(\boldsymbol{\vartheta})$ (cf. (4.25), Theorem 4.5) states

$$\|\bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} \leqslant C_{\bar{\mathrm{T}}} \left\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \right\}. \tag{4.62}$$

In this way, using (4.60) in both (4.59) and (4.61), and then replacing the resulting (4.61) along with (4.62) in (4.59), as well as recalling that $\|\boldsymbol{\eta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}$ and $\|\boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}$ are bounded by δ, we deduce the existence of a positive constant $\mathrm{L}_{\mathbf{T}}$, depending only on $\mathrm{L}_{\widetilde{\mathrm{T}}}$, $\mathrm{L}_{\bar{\mathrm{T}}}$, $\mathrm{L}_{\widehat{\mathrm{T}}}$, and $C_{\bar{\mathrm{T}}}$, such that

$$\begin{aligned} &\|\mathbf{T}(\boldsymbol{\eta}) - \mathbf{T}(\boldsymbol{\vartheta})\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \\ &\leqslant \mathrm{L}_{\mathbf{T}} \Big(1 + \delta + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big) \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \Big\} \|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}, \end{aligned} \tag{4.63}$$

for all $\boldsymbol{\eta}$, $\boldsymbol{\vartheta} \in \mathrm{W}(\delta)$. We are thus in position to establish the main result of this section.

Theorem 4.11. *In addition to the hypotheses of Lemma 4.7, that is (4.43) and (4.46), or alternatively (4.44), (4.45), and (4.46), assume that*

$$\mathrm{L}_{\mathbf{T}} \Big(1 + \delta + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big) \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \Big\} < 1. \tag{4.64}$$

Then, the operator $\mathbf{T}$ has a unique fixed point $\boldsymbol{\xi} \in \mathrm{W}(\delta)$. Equivalently, the coupled problem (3.33) has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H} \times \mathbf{Q}$, $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$, and $(\boldsymbol{\sigma}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$, $i \in \{1,2\}$, with $\boldsymbol{\xi} := (\xi_1, \xi_2) \in \mathrm{W}(\delta)$. Moreover, there hold the following a priori estimates

$$\begin{aligned} \|(\boldsymbol{\sigma}, \mathbf{u})\|_{\mathbf{H}\times\mathbf{Q}} &\leqslant C_{\widehat{\mathrm{T}}} \Big\{ \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} + \|\boldsymbol{\xi}\|_{0,\rho;\Omega} \|\boldsymbol{\varphi}\|_{0,r;\Omega} \Big\}, \\ \|(\boldsymbol{\varphi}, \chi)\|_{\mathrm{X}_2\times\mathrm{M}_1} &\leqslant C_{\bar{\mathrm{T}}} \Big\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\xi}\|_{0,\rho;\Omega} \Big\}, \quad \textit{and} \\ \|(\sigma_i, \xi_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i} &\leqslant C_{\widetilde{\mathrm{T}}} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \Big\} \quad i \in \{1,2\}. \end{aligned} \tag{4.65}$$

Proof. We first recall that the assumptions of Lemma 4.7 guarantee that $\mathbf{T}$ maps $\mathrm{W}(\delta)$ into itself. Then, bearing in mind the Lipschitz-continuity of $\mathbf{T} : \mathrm{W}(\delta) \to \mathrm{W}(\delta)$ (cf. (4.63)) and the assumption (4.64), a straightforward application of the classical Banach theorem yields the existence of a unique fixed point $\boldsymbol{\xi} \in \mathrm{W}(\delta)$ of this operator, and hence a unique solution of (3.33). Finally, it is easy to see that the a priori estimates provided by (4.11) (cf. Theorems 4.1), (4.25) (cf. Theorem 4.5), and (4.38) (cf. Theorem 4.6) yield (4.65) and finish the proof. □

5 The Galerkin scheme

We now introduce the Galerkin scheme of the fully mixed variational formulation (3.33), analyze its solvability by applying a discrete version of the fixed point approach adopted in Section 4.1, and derive the corresponding a priori error estimate.

5.1 Preliminaries

We first let $\mathbf{H}_h$, $\mathbf{Q}_h$, $\mathrm{X}_{i,h}$, $\mathrm{M}_{i,h}$, $\mathrm{H}_{i,h}$,and $\mathrm{Q}_{i,h}$, $i \in \{1,2\}$, be arbitrary finite element subspaces of the spaces $\mathbf{H}$, $\mathbf{Q}$, X_i, M_i, H_i, and Q_i, $i \in \{1,2\}$, respectively. Hereafter, h denotes both the sub-index of each subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when

$n = 3$) of diameter h_K, so that $h := \max\{h_K : \quad K \in \mathcal{T}_h\}$. Explicit finite element subspaces satisfying the stability hypotheses to be introduced throughout the forthcoming analysis, will be defined later on in Section 6. Then, the Galerkin scheme associated with (3.33) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\boldsymbol{\varphi}_h, \chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $(\boldsymbol{\sigma}_{i,h}, \xi_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$, $i \in \{1,2\}$, such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) \quad &= \quad \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau} \in \mathbf{H}_h\,,\\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) \quad &= \quad \mathbf{G}_{\boldsymbol{\xi}_h, \boldsymbol{\varphi}_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,,\\
a(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + b_1(\boldsymbol{\psi}_h, \chi_h) \quad &= \quad \mathrm{F}(\boldsymbol{\psi}_h) && \forall\, \boldsymbol{\psi}_h \in \mathrm{X}_{1,h}\,,\\
b_2(\boldsymbol{\varphi}_h, \lambda_h) \quad &= \quad \mathrm{G}_{\boldsymbol{\xi}_h}(\lambda_h) && \forall\, \lambda_h \in \mathrm{M}_{2,h}\,,\\
a_i(\boldsymbol{\sigma}_{i,h}, \boldsymbol{\tau}_{i,h}) + c_i(\boldsymbol{\tau}_{i,h}, \xi_{i,h}) - c_{\boldsymbol{\varphi}_h, \mathbf{u}_h}(\boldsymbol{\tau}_{i,h}, \xi_{i,h}) \quad &= \quad \mathrm{F}_i(\boldsymbol{\tau}_{i,h}) && \forall\, \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}\,,\\
c_i(\boldsymbol{\sigma}_{i,h}, \eta_{i,h}) - d_i(\xi_{i,h}, \eta_{i,h}) \quad &= \quad \mathrm{G}_i(\eta_{i,h}) && \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}\,.
\end{aligned}
\tag{5.1}
$$

Similarly to the remark right after (3.33) in Section 3.4, we highlight here that the second, fourth, and sixth rows of (5.1) constitute the discrete conservation of momentum properties, which are actually satisfied in an approximate sense. At the end of Section 6.2 we describe them explicitly in terms of suitable projection operators.

In what follows, we adopt the discrete version of the strategy employed in Section 4.1 to analyse the solvability of (5.1). We now let $\widehat{\mathrm{T}}_h : (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h} \to \mathbf{Q}_h$ be the operator defined by

$$
\widehat{\mathrm{T}}_h(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) := \widehat{\mathbf{u}}_h \qquad \forall\, (\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \in (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h}\,,
$$

where $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed below) of the first two rows of (5.1) with $(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h)$ instead of $(\boldsymbol{\xi}_h, \boldsymbol{\varphi}_h)$, that is

$$
\begin{aligned}
\mathbf{a}(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \widehat{\mathbf{u}}_h) \quad &= \quad \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{H}_h\,,\\
\mathbf{b}(\widehat{\boldsymbol{\sigma}}_h, \mathbf{v}_h) \quad &= \quad \mathbf{G}_{\boldsymbol{\eta}_h, \boldsymbol{\phi}_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,.
\end{aligned}
\tag{5.2}
$$

In turn, we let $\bar{\mathrm{T}}_h : \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h} \to \mathrm{X}_{2,h}$ be the operator given by

$$
\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h) := \bar{\boldsymbol{\varphi}}_h \qquad \forall\, \boldsymbol{\eta}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}\,,
$$

where $(\bar{\boldsymbol{\varphi}}_h, \bar{\chi}_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ is the unique solution (to be confirmed below) of the third and fourth rows of (5.1) with $\boldsymbol{\eta}_h$ instead of $\boldsymbol{\xi}_h$, that is

$$
\begin{aligned}
a(\bar{\boldsymbol{\varphi}}_h, \boldsymbol{\psi}_h) + b_1(\boldsymbol{\psi}_h, \bar{\chi}_h) \quad &= \quad \mathrm{F}(\boldsymbol{\psi}_h) && \forall\, \boldsymbol{\psi}_h \in \mathrm{X}_{1,h}\,,\\
b_2(\bar{\boldsymbol{\varphi}}_h, \lambda_h) \quad &= \quad \mathrm{G}_{\boldsymbol{\eta}_h}(\lambda_h) && \forall\, \lambda_h \in \mathrm{M}_{2,h}\,.
\end{aligned}
\tag{5.3}
$$

Similarly, for each $i \in \{1,2\}$, we let $\widetilde{\mathrm{T}}_{i,h} : \mathrm{X}_{2,h} \times \mathbf{Q}_h \to \mathrm{Q}_{i,h}$ be the operator defined by

$$
\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h) := \widetilde{\xi}_{i,h} \qquad \forall\, (\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h\,,
$$

where $(\widetilde{\boldsymbol{\sigma}}_{i,h}, \widetilde{\xi}_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$ is the unique solution (to be confirmed below) of the fifth and sixth rows of (5.1) with $(\boldsymbol{\phi}_h, \mathbf{v}_h)$ instead of $(\boldsymbol{\varphi}_h, \mathbf{u}_h)$, that is

$$
\begin{aligned}
a_i(\widetilde{\boldsymbol{\sigma}}_{i,h}, \boldsymbol{\tau}_{i,h}) + c_i(\boldsymbol{\tau}_{i,h}, \widetilde{\xi}_{i,h}) - c_{\boldsymbol{\phi}_h, \mathbf{v}_h}(\boldsymbol{\tau}_{i,h}, \widetilde{\xi}_{i,h}) \quad &= \quad \mathrm{F}_i(\boldsymbol{\tau}_{i,h}) && \forall\, \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}\,,\\
c_i(\widetilde{\boldsymbol{\sigma}}_{i,h}, \eta_{i,h}) - d_i(\widetilde{\xi}_{i,h}, \eta_{i,h}) \quad &= \quad \mathrm{G}_i(\eta_{i,h}) && \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}\,,
\end{aligned}
\tag{5.4}
$$

so that we can define the operator $\widetilde{\mathrm{T}}_h : \mathrm{X}_{2,h} \times \mathbf{Q}_h \to (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h})$ as:

$$\widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h, \mathbf{v}_h) := \big(\widetilde{\mathrm{T}}_{1,h}(\boldsymbol{\phi}_h, \mathbf{v}_h), \widetilde{\mathrm{T}}_{2,h}(\boldsymbol{\phi}_h, \mathbf{v}_h)\big) = (\xi_{1,h}, \xi_{2,h}) =: \widetilde{\boldsymbol{\xi}}_h \qquad \forall\, (\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h\,. \tag{5.5}$$

Finally, defining the operator $\mathbf{T}_h : (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \to (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h})$ as

$$\mathbf{T}(\boldsymbol{\eta}_h) := \widetilde{\mathrm{T}}_h\big(\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h), \widehat{\mathrm{T}}_h\big(\boldsymbol{\eta}_h, \bar{\mathrm{T}}_h(\boldsymbol{\eta}_h)\big)\big) \qquad \forall\, \boldsymbol{\eta}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}\,, \tag{5.6}$$

we observe that solving (5.1) is equivalent to seeking a fixed point of $\mathbf{T}_h$, that is: Find $\boldsymbol{\xi}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}$ such that

$$\mathbf{T}_h(\boldsymbol{\xi}_h) = \boldsymbol{\xi}_h\,. \tag{5.7}$$

5.2 Discrete solvability analysis

In this section we proceed analogously to Sections 4.2 and 4.3 and establish the well-posedness of the discrete system (5.1) by means of the solvability study of the equivalent fixed point equation (5.7). In this regard, we emphasize in advance that, being the respective analysis very similar to that developed in the aforementioned sections, here we simply collect the main results and provide selected details of the corresponding proofs.

According to the above, we first aim to prove that the discrete operators $\widehat{\mathrm{T}}_h$, $\bar{\mathrm{T}}_h$, and $\widetilde{\mathrm{T}}_{i,h}$, $i \in \{1, 2\}$, and hence $\widetilde{\mathrm{T}}_h$ and $\mathbf{T}_h$, are all well-defined, which reduces, equivalently, to show that the problems (5.2), (5.3), and (5.4) are well-posed. To this end, we now apply the discrete versions of [19, Theorem 2.34], [5, Theorem 2.1, Section 2.1], and [15, Theorem 3.4], which are given by [19, Proposition 2.42], [5, Corollary 2.2, Section 2.2], and [15, Theorem 3.5], respectively. More precisely, following similar approaches from related works (see, e.g. [9, Section 4.2]), our analysis throughout the rest of this section is based on suitable hypotheses that need to be satisfied by the finite element subspaces utilized in (5.1), which are split according to the requirements of the associated decoupled problems. Explicit examples of discrete spaces verifying these assumptions will be specified later on in Section 6.

We begin by addressing the well-definedness of $\widehat{\mathrm{T}}_h$, for which we let $\mathbb{V}_h$ be the discrete kernel of $\mathbf{b}$, that is

$$\mathbb{V}_h := \Big\{\boldsymbol{\tau}_h \in \mathbf{H}_h : \quad \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h\Big\}\,, \tag{5.8}$$

and assume that

(**H.1**) there holds $\mathbf{div}(\mathbf{H}_h) \subseteq \mathbf{Q}_h$, and

(**H.2**) there exists a positive constant $\boldsymbol{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{H}}} \geqslant \boldsymbol{\beta}_{\mathtt{d}}\, \|\mathbf{v}_h\|_{\mathbf{Q}} \qquad \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,. \tag{5.9}$$

Then, according to the definition of $\mathbf{b}$ (cf. (3.13b)), it follows from (5.8) and (**H.1**) that

$$\mathbb{V}_h := \Big\{\boldsymbol{\tau}_h \in \mathbf{H}_h : \quad \mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}\Big\}\,, \tag{5.10}$$

which says that $\mathbb{V}_h$ is contained in the continuous kernel $\mathbb{V}$ (cf. (4.7)), and hence the discrete version of (4.9) is automatically satisfied, that is

$$\mathbf{a}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geqslant \boldsymbol{\alpha}_{\mathtt{d}}\, \|\boldsymbol{\tau}\|^2_{\mathrm{div}_s;\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h\,, \tag{5.11}$$

with $\boldsymbol{\alpha}_{\mathtt{d}} = \boldsymbol{\alpha} := C_s/\mu$. Recall here that C_s is the constant provided by inequality (4.8) with $t = s$. In this way, it is clear from (5.11) that $\mathbf{a}$ satisfies the hypotheses given by [19, Proposition 2.42, eq. (2.35)] with the constant $\boldsymbol{\alpha}_{\mathtt{d}}$, whereas (**H.2**) states that $\mathbf{b}$ fulfills [19, Proposition 2.42, eq. (2.36)] with the constant $\boldsymbol{\beta}_{\mathtt{d}}$. We are thus in position to establish next the following result.

Theorem 5.1. *For each $(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \in (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h}$ there exists a unique $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ solution to (5.2), and hence one can define $\widehat{\mathrm{T}}_h(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) := \widehat{\mathbf{u}}_h \in \mathbf{Q}_h$. Moreover, there exists a positive constant $C_{\widehat{\mathrm{T}},\mathtt{d}}$, depending only on μ, $\|\mathbf{i}_r\|$, ε_0, $|\Omega|$, $\boldsymbol{\alpha}_{\mathtt{d}}$, and $\boldsymbol{\beta}_{\mathtt{d}}$, and hence independent of $(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h)$, such that*

$$\|\widehat{\mathrm{T}}_h(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h)\|_{\mathbf{Q}} \,=\, \|\widehat{\mathbf{u}}_h\|_{\mathbf{Q}} \,\leqslant\, C_{\widehat{\mathrm{T}},\mathtt{d}} \Big\{ \|\mathbf{g}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,s,\Omega} \,+\, \|\boldsymbol{\eta}_h\|_{0,\rho;\Omega} \, \|\boldsymbol{\phi}_h\|_{0,r;\Omega} \Big\} . \tag{5.12}$$

Proof. Given $(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \in (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h}$, the existence of a unique solution to (5.2) follows from a straightforward application of [19, Proposition 2.42]. In turn, the corresponding a priori bound from [19, Theorem 2.34, eq. (2.30)] and the boundedness properties of $\mathbf{F}$ and $\mathbf{G}_{\boldsymbol{\eta}_h, \boldsymbol{\phi}_h}$ imply (5.12). □

Similarly as observed for the continuous operator $\widehat{\mathrm{T}}$, we remark here that the right-hand side of (5.12) can also be assumed as the respective a priori estimate for $\widehat{\boldsymbol{\sigma}}_h$.

Furthermore, for the well-definedness of $\bar{\mathrm{T}}_h$, we need to introduce the discrete kernels of b_1 and b_2, namely

$$\mathrm{K}_{1,h} \,:=\, \Big\{ \boldsymbol{\psi}_h \in \mathrm{X}_{1,h} : \quad b_1(\boldsymbol{\psi}_h, \lambda_h) = 0 \quad \forall\, \lambda_h \in \mathrm{M}_{1,h} \Big\} , \tag{5.13}$$

and

$$\mathrm{K}_{2,h} \,:=\, \Big\{ \boldsymbol{\psi}_h \in \mathrm{X}_{2,h} : \quad b_2(\boldsymbol{\psi}_h, \lambda_h) = 0 \quad \forall\, \lambda_h \in \mathrm{M}_{2,h} \Big\} , \tag{5.14}$$

respectively, and consider the following assumptions

(**H.3**) there exists a positive constant $\bar{\alpha}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\psi}_h \in \mathrm{K}_{1,h} \\ \boldsymbol{\psi}_h \neq \mathbf{0}}} \frac{a(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{X}_1}} \,\geqslant\, \bar{\alpha}_{\mathtt{d}} \, \|\boldsymbol{\phi}_h\|_{\mathrm{X}_2} \qquad \forall\, \boldsymbol{\phi}_h \in \mathrm{K}_{2,h} \, , \quad \text{and} \tag{5.15a}$$

$$\sup_{\boldsymbol{\phi}_h \in \mathrm{K}_{2,h}} a(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) \,>\, 0 \qquad \forall\, \boldsymbol{\psi}_h \in \mathrm{K}_{1,h}, \quad \boldsymbol{\psi}_h \neq \mathbf{0} . \tag{5.15b}$$

(**H.4**) for each $i \in \{1, 2\}$ there exists a positive constant $\bar{\beta}_{i,\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\psi}_h \in \mathrm{X}_{i,h} \\ \boldsymbol{\psi}_h \neq \mathbf{0}}} \frac{b_i(\boldsymbol{\psi}_h, \lambda_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{X}_i}} \,\geqslant\, \bar{\beta}_{i,\mathtt{d}} \, \|\lambda_h\|_{\mathrm{M}_i} \qquad \forall\, \lambda_h \in \mathrm{M}_{i,h} . \tag{5.16}$$

As a consequence of (**H.3**) and (**H.4**) we provide next the discrete version of Theorem 4.5.

Theorem 5.2. *For each $\boldsymbol{\eta}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}$ there exists a unique $(\bar{\boldsymbol{\varphi}}_h, \bar{\chi}_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ solution to (5.3), and hence one can define $\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h) := \bar{\boldsymbol{\varphi}}_h \in \mathrm{X}_{2,h}$. Moreover, there exists a positive constant $C_{\bar{\mathrm{T}},\mathtt{d}}$, depending only on ε_0, C_r, $|\Omega|$, $\bar{\alpha}_{\mathtt{d}}$, and $\bar{\beta}_{2,\mathtt{d}}$, such that*

$$\|\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h)\|_{\mathrm{X}_2} \,=\, \|\bar{\boldsymbol{\varphi}}_h\|_{\mathrm{X}_2} \,\leqslant\, C_{\bar{\mathrm{T}},\mathtt{d}} \Big\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\eta}_h\|_{0,\rho;\Omega} \Big\} . \tag{5.17}$$

Proof. Given $\boldsymbol{\eta}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}$, a direct application of [5, Corollary 2.2, Section 2.2] implies the existence of a unique solution to (5.3), whereas the a priori estimate provided in [5, Corollary 2.2, eq. (2.24)] and the boundedness properties of F and $\mathrm{G}_{\boldsymbol{\eta}_h}$ yield (5.17). □

Analogously as explained for the continuous operator $\bar{\mathrm{T}}$, here we can also assume that, except for a constant $C_{\bar{\mathrm{T}},\mathtt{d}}$ depending additionally on $\bar{\beta}_{1,\mathtt{d}}$, the a priori estimate for $\bar{\chi}_h$, which follows now from [5, Corollary 2.2, eq. (2.25)], is also given by the right-hand side of (5.17).

It remains to prove the well-definedness of $\widetilde{\mathrm{T}}_h := (\widetilde{\mathrm{T}}_{1,h}, \widetilde{\mathrm{T}}_{2,h})$, for which we first observe that, being a_i and c_i symmetric and positive semi-definite in the whole spaces H_i and Q_i, they certainly keep these properties in $\mathrm{H}_{i,h}$ and $\mathrm{Q}_{i,h}$, respectively, so that the assumption i) of [15, Theorem 3.5] is clearly satisfied. Next, given $i \in \{1,2\}$, we let $\mathrm{V}_{i,h}$ be the discrete kernel of c_i, that is

$$\mathrm{V}_{i,h} := \Big\{ \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h} : \quad c_i(\boldsymbol{\tau}_{i,h}, \eta_{i,h}) = 0 \quad \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h} \Big\}, \tag{5.18}$$

and consider the hypotheses

(**H.5**) for each $i \in \{1,2\}$ there holds $\operatorname{div}(\mathrm{H}_{i,h}) \subseteq \mathrm{Q}_{i,h}$, and

(**H.6**) there exists a positive constant $\widetilde{\beta}_{\mathtt{d}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h} \\ \boldsymbol{\tau}_{i,h} \neq \mathbf{0}}} \frac{c_i(\boldsymbol{\tau}_{i,h}, \eta_{i,h})}{\|\boldsymbol{\tau}_{i,h}\|_{\mathrm{H}_i}} \,\geqslant\, \widetilde{\beta}_{\mathtt{d}}\, \|\eta_{i,h}\|_{\mathrm{Q}_i} \qquad \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}\,. \tag{5.19}$$

It follows from (5.18), the definition of c_i (cf. (3.30b)), and (**H.5**) that

$$\mathrm{V}_{i,h} := \Big\{ \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h} : \quad \operatorname{div}(\boldsymbol{\tau}_{i,h}) = 0 \Big\}, \tag{5.20}$$

whence, similarly to the case of $\widehat{\mathrm{T}}_h$, $\mathrm{V}_{i,h}$ is contained in the continuous kernel V_i (cf. (4.30)) of c_i, thus yielding the discrete analogue of (4.31), that is

$$a_i(\boldsymbol{\tau}_{i,h}, \boldsymbol{\tau}_{i,h}) \,\geqslant\, \frac{1}{\bar{\kappa}}\, \|\boldsymbol{\tau}_{i,h}\|^2_{\operatorname{div}_\varrho;\Omega} \qquad \forall\, \boldsymbol{\tau}_{i,h} \in V_{i,h}\,. \tag{5.21}$$

In this way, it is clear from (5.21) that a_i satisfies the hypothesis ii) of [15, Theorem 3.5] with the constant $\widetilde{\alpha}_{\mathtt{d}} := \bar{\kappa}^{-1}$, whereas (**H.6**) constitutes itself the corresponding assumption iii). Consequently, a straightforward application of [15, Theorem 3.5] implies the discrete global inf-sup condition for $\mathbf{A}$ (cf. (4.27)) with a positive constant $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}$ depending only on $\|a_i\|$, $\|c_i\|$, $\widetilde{\alpha}_{\mathtt{d}}$, and $\widetilde{\beta}_{\mathtt{d}}$, and thus the same property is shared by $\mathbf{A}_{\boldsymbol{\phi}_h,\mathbf{v}_h}$ for each $(\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h$ satisfying the discrete version of (4.35), that is

$$\|\boldsymbol{\phi}_h\|_{0,r,\Omega} + \|\mathbf{v}_h\|_{0,r,\Omega} \,\leqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}}{2\,\|c\|}\,. \tag{5.22}$$

We are now in position of establishing the well-definedness of $\widetilde{\mathrm{T}}_{i,h}$ for each $i \in \{1,2\}$.

Theorem 5.3. *Given $i \in \{1,2\}$ and $(\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h$ such that (5.22) holds, there exists a unique $(\widetilde{\boldsymbol{\sigma}}_{i,h}, \widetilde{\xi}_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$ solution to (5.4), and hence one can define $\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h) := \widetilde{\xi}_{i,h} \in \mathrm{Q}_{i,h}$. Moreover, there exists a positive constant $C_{\widetilde{\mathrm{T}},\mathtt{d}}$, depending only on $\|\mathrm{i}_\rho\|$ and $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}$, such that*

$$\|\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h)\|_{\mathrm{Q}_i} \,=\, \|\widetilde{\xi}_{i,h}\|_{\mathrm{Q}_i} \,\leqslant\, \|(\widetilde{\boldsymbol{\sigma}}_{i,h}, \widetilde{\xi}_{i,h})\|_{\mathrm{H}_i \times \mathrm{Q}_i} \,\leqslant\, C_{\widetilde{\mathrm{T}},\mathtt{d}} \Big\{ \|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega} \Big\}\,. \tag{5.23}$$

Proof. It reduces to a direct application of [19, Theorem 2.22], whose corresponding a priori estimate, yielding (5.23), makes use of the boundedness of F_i and G_i (cf. (3.31) and (3.32)). $\square$

Analogously to the continuous case, it follows from the definition of $\widetilde{\mathrm{T}}_h$ (cf. (5.5)) and the a priori estimates given by (5.23) for each $i \in \{1,2\}$, that

$$\|\widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h,\mathbf{v}_h)\|_{\mathrm{Q}_1\times\mathrm{Q}_2} := \sum_{i=1}^{2} \|\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h,\mathbf{v}_h)\|_{\mathrm{Q}_i} \leqslant C_{\widetilde{\mathrm{T}},\mathtt{d}} \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big\} \tag{5.24}$$

for each $(\boldsymbol{\phi}_h,\mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h$ satisfying (5.22).

Having established that the discrete operators $\widehat{\mathrm{T}}_h$, $\bar{\mathrm{T}}_h$, $\widetilde{\mathrm{T}}_h$, and hence $\mathbf{T}_h$ (under the constraint imposed by (5.22)), are all well defined, we now proceed as in Section 4.3 to address the solvability of the corresponding fixed-point equation (5.7). Then, letting $\delta_{\mathtt{d}}$ be an arbitrary radius, we set

$$\mathrm{W}(\delta_{\mathtt{d}}) := \Big\{ \boldsymbol{\eta}_h := (\eta_{1,h},\eta_{2,h}) \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h} : \quad \|\boldsymbol{\eta}_h\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \leqslant \delta_{\mathtt{d}} \Big\}, \tag{5.25}$$

and, reasoning analogously to the derivation of Lemma 4.7 (cf. beginning of Section 4.3), we deduce that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_{\mathtt{d}})$ into itself under the discrete versions of (4.43) and (4.46), which, denoting $C_{0,\mathtt{d}} := \max\{1, C_{\widehat{\mathrm{T}},\mathtt{d}}\} C_{\bar{\mathrm{T}},\mathtt{d}}$, are given, respectively, by

$$C_{0,\mathtt{d}}(1+\delta_{\mathtt{d}})\delta_{\mathtt{d}} + C_{0,\mathtt{d}}(1+\delta_{\mathtt{d}})\Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big\} + C_{\widehat{\mathrm{T}},\mathtt{d}}\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega}\Big\} \leqslant \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}}{2\,\|c\|} \tag{5.26}$$

and

$$C_{\widetilde{\mathrm{T}},\mathtt{d}} \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big\} \leqslant \delta_{\mathtt{d}}\,. \tag{5.27}$$

Alternatively, the same conclusion is attained if, instead of (5.26), we define

$$\delta_{\mathtt{d}} := \min\Big\{1, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}}{8C_{0,\mathtt{d}}\|c\|}\Big\}, \tag{5.28}$$

and, letting $C_{1,\mathtt{d}} := 2C_{0,\mathtt{d}}$, impose

$$C_{1,\mathtt{d}}\Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big\} + C_{\widehat{\mathrm{T}},\mathtt{d}}\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega}\Big\} \leqslant \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}}{4\,\|c\|}\,. \tag{5.29}$$

Note, however, that only (5.26) is required for $\mathbf{T}_h$ to be well-defined. Furthermore, employing analogue arguments to those utilized in the proofs of Lemmas 4.8, 4.9, and 4.10, we are able to show the continuity properties of $\widehat{\mathrm{T}}_h$, $\bar{\mathrm{T}}_h$, and $\widetilde{\mathrm{T}}_h$, that is the discrete versions of (4.47), (4.51), and (4.55), which are exactly as the latter, but with corresponding constants denoted $\mathrm{L}_{\widehat{\mathrm{T}},\mathtt{d}}$, $\mathrm{L}_{\bar{\mathrm{T}},\mathtt{d}}$, and $\mathrm{L}_{\widetilde{\mathrm{T}},\mathtt{d}}$. Therefore, following an analogue procedure to the one that yielded (4.63), we deduce that, under the assumption (5.26), there exists a positive constant $\mathrm{L}_{\mathbf{T},\mathtt{d}}$, depending only on $\mathrm{L}_{\widetilde{\mathrm{T}},\mathtt{d}}$, $\mathrm{L}_{\bar{\mathrm{T}},\mathtt{d}}$, $\mathrm{L}_{\widehat{\mathrm{T}},\mathtt{d}}$, and $C_{\bar{\mathrm{T}},\mathtt{d}}$, such that

$$\begin{aligned} &\|\mathbf{T}_h(\boldsymbol{\eta}_h) - \mathbf{T}_h(\boldsymbol{\vartheta}_h)\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \\ &\quad \leqslant \mathrm{L}_{\mathbf{T},\mathtt{d}}\left(1 + \delta_{\mathtt{d}} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\right) \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \Big\} \|\boldsymbol{\eta}_h - \boldsymbol{\vartheta}_h\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\,, \end{aligned} \tag{5.30}$$

for all $\boldsymbol{\eta}_h$, $\boldsymbol{\vartheta}_h \in \mathrm{W}(\delta_{\mathtt{d}})$.

Consequently, we can establish next the main result of this section.

Theorem 5.4. *Assume that $\delta_{\mathtt{d}}$ and the data are sufficiently small so that (5.26) and (5.27) are satisfied, or alternatively that there holds (5.28), (5.29), and (5.27). Then, the operator $\mathbf{T}_h$ has a fixed point $\boldsymbol{\xi}_h \in \mathrm{W}(\delta_{\mathtt{d}})$. Equivalently, the coupled problem (5.1) has a solution $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\boldsymbol{\varphi}_h,\chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and*

$(\boldsymbol{\sigma}_{i,h}, \xi_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$, $i \in \{1,2\}$, *with* $\boldsymbol{\xi}_h := (\xi_{1,h}, \xi_{2,h}) \in \mathrm{W}(\delta_{\mathtt{d}})$. *Moreover, there hold the following a priori estimates*

$$
\begin{aligned}
\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,&\leqslant\, C_{\widehat{\mathrm{T}},\mathtt{d}} \left\{ \|\mathbf{g}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,s,\Omega} \,+\, \|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\,\|\boldsymbol{\varphi}_h\|_{0,r;\Omega} \right\}, \\
\|(\boldsymbol{\varphi}_h, \chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \,&\leqslant\, C_{\bar{\mathrm{T}},\mathtt{d}} \left\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\xi}_h\|_{0,\rho;\Omega} \right\}, \quad \textit{and} \\
\|(\sigma_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} \,&\leqslant\, C_{\tilde{\mathrm{T}},\mathtt{d}} \left\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \right\} \quad i \in \{1,2\}\,.
\end{aligned}
\tag{5.31}
$$

In addition, under the extra assumption

$$
\mathrm{L}_{\mathbf{T},\mathtt{d}} \Big(1 + \delta_{\mathtt{d}} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} \Big) \sum_{i=1}^{2} \left\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \right\} \,<\, 1\,,
\tag{5.32}
$$

the aforementioned solutions of (5.7) *and* (5.1) *are unique.*

Proof. As previously observed, the fact that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_{\mathtt{d}})$ into itself is consequence of (5.26) and (5.27), or alternatively of (5.28), (5.29), and (5.27). Then, the continuity of $\mathbf{T}_h$ (cf. (5.30)) and Brouwer's theorem (cf. [12, Theorem 9.9-2]) imply the existence of solution of (5.7), and hence of (5.1). In turn, under the additional hypothesis (5.32), the Banach fixed point theorem guarantees the uniqueness of solution. In either case, (4.11), (4.25), and (4.38) yield the a priori estimates (5.31) and conclude the proof. □

5.3 A priori error analysis

In this section we consider arbitrary finite element subspaces satisfying the assumptions specified in Section 5.2, and establish the Céa estimate for the Galerkin error

$$
\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\boldsymbol{\varphi}, \chi) - (\boldsymbol{\varphi}_h, \chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2} \|(\boldsymbol{\sigma}_i, \xi_i) - (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,,
\tag{5.33}
$$

where $\big((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\varphi}, \chi), (\boldsymbol{\sigma}_i, \xi_i)\big) \in \big(\mathbf{H} \times \mathbf{Q}\big) \times \big(\mathrm{X}_2 \times \mathrm{M}_1\big) \times \big(\mathrm{H}_i \times \mathrm{Q}_i\big)$, $i \in \{1,2\}$, is the unique solution of (3.33), and $\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\varphi}_h, \chi_h), (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\big) \in \big(\mathbf{H}_h \times \mathbf{Q}_h\big) \times \big(\mathrm{X}_{2,h} \times \mathrm{M}_{1,h}\big) \times \big(\mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}\big)$, $i \in \{1,2\}$, is a solution of (5.1). We proceed as in previous related works (see, e.g. [9]) by applying suitable Strang-type estimates to the pairs of associated continuous and discrete schemes arising from (3.33) and (5.1) after splitting them according to the three decoupled equations. Throughout the rest of this section, given a subspace Z_h of an arbitrary Banach space $\big(Z, \|\cdot\|_Z\big)$, we set

$$
\mathrm{dist}\big(z, Z_h\big) \,:=\, \inf_{z_h\in Z_h} \|z - z_h\|_Z \qquad \forall\, z \in Z\,.
$$

We begin the analysis by considering the first two rows of (3.33) and (5.1), so that, employing the estimates provided by [5, Proposition 2.1, Corollary 2.3, Theorem 2.3], we deduce the existence of a positive constant $\widehat{c}$, independent of h, such that

$$
\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, \widehat{c}\,\Big\{ \mathrm{dist}(\boldsymbol{\sigma}, \mathbf{H}_h) \,+\, \mathrm{dist}(\mathbf{u}, \mathbf{Q}_h) \,+\, \|\mathbf{G}_{\boldsymbol{\xi},\boldsymbol{\varphi}} - \mathbf{G}_{\boldsymbol{\xi}_h,\boldsymbol{\varphi}_h}\|_{\mathbf{Q}_h'} \Big\}\,.
\tag{5.34}
$$

Thus, proceeding analogously to the derivation of (4.50), we readily obtain

$$
\|\mathbf{G}_{\boldsymbol{\xi},\boldsymbol{\varphi}} - \mathbf{G}_{\boldsymbol{\xi}_h,\boldsymbol{\varphi}_h}\|_{\mathbf{Q}_h'} \,\leqslant\, \varepsilon_0^{-1}\,|\Omega|^{(r-2)/2r} \left\{ \|\boldsymbol{\xi}\|_{0,\rho,\Omega}\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,r,\Omega} \,+\, \|\boldsymbol{\varphi}_h\|_{0,r,\Omega}\,\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{0,\rho,\Omega} \right\},
\tag{5.35}
$$

which, substituted back in (5.34), yields

$$
\begin{aligned}
\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, c_{\widehat{\mathrm{T}}} \Big\{ &\mathrm{dist}(\boldsymbol{\sigma}, \mathbf{H}_h) \,+\, \mathrm{dist}(\mathbf{u}, \mathbf{Q}_h) \\
&+ \|\boldsymbol{\xi}\|_{0,\rho,\Omega}\,\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,r,\Omega} \,+\, \|\boldsymbol{\varphi}_h\|_{0,r,\Omega}\,\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{0,\rho,\Omega} \Big\},
\end{aligned}
\tag{5.36}
$$

with $c_{\widehat{\mathrm{T}}} := \widehat{c}\, \max\big\{1, \varepsilon_0^{-1}\, |\Omega|^{(r-2)/2r}\big\}$.

Next, employing the same estimates from [5, Proposition 2.1, Corollary 2.3, Theorem 2.3] to the context given by the third and fourth rows of (3.33) and (5.1), we find that there exists a positive constant $\bar{c}$, independent of h, such that

$$\|(\boldsymbol{\varphi},\chi) - (\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \,\leqslant\, \bar{c}\Big\{\mathrm{dist}(\boldsymbol{\varphi},\mathrm{X}_{2,h}) \,+\, \mathrm{dist}(\chi,\mathrm{M}_{1,h}) \,+\, \|\mathrm{G}_{\boldsymbol{\xi}} - \mathrm{G}_{\boldsymbol{\xi}_h}\|_{\mathrm{M}'_{2,h}}\Big\}\,. \tag{5.37}$$

In turn, proceeding as for the deduction of (4.54), we obtain

$$\|\mathrm{G}_{\boldsymbol{\xi}} - \mathrm{G}_{\boldsymbol{\xi}_h}\|_{\mathrm{M}'_{2,h}} \,\leqslant\, |\Omega|^{(\rho-r)/\rho r}\, \|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{0,\rho;\Omega}\,, \tag{5.38}$$

which, along with (5.37), gives

$$\|(\boldsymbol{\varphi},\chi) - (\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \,\leqslant\, c_{\bar{\mathrm{T}}}\Big\{\mathrm{dist}(\boldsymbol{\varphi},\mathrm{X}_{2,h}) \,+\, \mathrm{dist}(\chi,\mathrm{M}_{1,h}) \,+\, \|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{0,\rho;\Omega}\Big\}\,, \tag{5.39}$$

with $c_{\bar{\mathrm{T}}} := \bar{c}\, \max\big\{1, |\Omega|^{(\rho-r)/\rho r}\big\}$.

Furthermore, we now focus on the last two rows of (3.33) and (5.1), with the terms $c_{\boldsymbol{\varphi},\mathbf{u}}(\boldsymbol{\tau}_i,\xi_i)$ and $c_{\boldsymbol{\varphi}_h,\mathbf{u}_h}(\boldsymbol{\tau}_{i,h},\xi_{i,h})$ being considered as part of the respective functionals on the right-hand side. In this way, applying the estimate from [19, Lemma 2.27], we conclude that there exists a positive constant $\widetilde{c}$, independent of h, such that

$$\begin{aligned}
&\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\\
&\quad\leqslant\, \widetilde{c}\Big\{\mathrm{dist}(\boldsymbol{\sigma}_i,\mathrm{H}_{i,h}) \,+\, \mathrm{dist}(\xi_i,\mathrm{Q}_{i,h}) \,+\, \|c_{\boldsymbol{\varphi},\mathbf{u}}(\cdot,\xi_i) - c_{\boldsymbol{\varphi}_h,\mathbf{u}_h}(\cdot,\xi_{i,h})\|_{\mathrm{H}'_{i,h}}\Big\}\,.
\end{aligned} \tag{5.40}$$

Then, subtracting and adding $\xi_{i,h}$ to the second component of $c_{\boldsymbol{\varphi},\mathbf{u}}(\cdot,\xi_i)$, making use of the triangle inequality, bearing in mind the definition of $c_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (3.30f)), and employing its boundedness property (cf. (3.31), (3.32)), we get

$$\begin{aligned}
\|c_{\boldsymbol{\varphi},\mathbf{u}}(\cdot,\xi_i) - c_{\boldsymbol{\varphi}_h,\mathbf{u}_h}(\cdot,\xi_{i,h})\|_{\mathrm{H}'_{i,h}} \,&\leqslant\, \|c_{\boldsymbol{\varphi},\mathbf{u}}(\cdot,\xi_i - \xi_{i,h})\|_{\mathrm{H}'_{i,h}} \,+\, \|c_{\boldsymbol{\varphi}-\boldsymbol{\varphi}_h,\mathbf{u}-\mathbf{u}_h}(\cdot,\xi_{i,h})\|_{\mathrm{H}'_{i,h}}\\
&\leqslant\, \|c\|\,\Big\{\big(\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega}\big)\,\|\xi_i - \xi_{i,h}\|_{0,\rho;\Omega} + \|\xi_{i,h}\|_{0,\rho;\Omega}\,\big(\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,r;\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega}\big)\Big\}\,,
\end{aligned}$$

which, jointly with (5.40), and summing over $i \in \big\{1,2\big\}$, imply

$$\begin{aligned}
\sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} \,&\leqslant\, c_{\widetilde{\mathrm{T}}}\,\Big\{\sum_{i=1}^{2}\big(\mathrm{dist}(\boldsymbol{\sigma}_i,\mathrm{H}_{i,h}) \,+\, \mathrm{dist}(\xi_i,\mathrm{Q}_{i,h})\big)\\
&\quad + \big(\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega}\big)\,\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{0,\rho;\Omega}\\
&\quad + \|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\,\big(\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,r;\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega}\big)\Big\}\,,
\end{aligned} \tag{5.41}$$

with $c_{\widetilde{\mathrm{T}}} := \widetilde{c}\, \max\big\{1, \|c\|\big\}$.

For the rest of the analysis we introduce the partial error

$$\mathtt{E} \,:=\, \|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,,$$

and suitably combine the estimates (5.36), (5.39), and (5.41). More precisely, employing the right-hand side of (5.39) to bound $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,r;\Omega}$ in (5.36) and (5.41), adding the resulting inequalities, performing some algebraic manipulations, and then utilizing the a priori bounds for $\|\boldsymbol{\varphi}\|_{0,r;\Omega}$, $\|\boldsymbol{\varphi}_h\|_{0,r;\Omega}$, $\|\boldsymbol{\xi}\|_{0,\rho;\Omega}$, $\|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}$, and $\|\mathbf{u}\|_{0,r;\Omega}$

provided by Theorems 4.11 and 5.4, we find that there exists a positive constant C_e, depending on $c_{\widehat{\mathrm{T}}}$, $c_{\bar{\mathrm{T}}}$, $c_{\widetilde{\mathrm{T}}}$, δ, $\delta_{\mathtt{d}}$, $C_{\widehat{\mathrm{T}}}$, $C_{\bar{\mathrm{T}}}$, $C_{\widetilde{\mathrm{T}}}$, $C_{\bar{\mathrm{T}},\mathtt{d}}$, and $C_{\widetilde{\mathrm{T}},\mathtt{d}}$, and hence independent of h, such that

$$\begin{aligned}\mathtt{E} \,\leqslant\, C_e &\left\{\operatorname{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{H}_h\times\mathbf{Q}_h\big) + \operatorname{dist}\big((\boldsymbol{\varphi},\chi),\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big) + \sum_{i=1}^{2}\operatorname{dist}\big((\boldsymbol{\sigma}_i,\xi_i),\mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}\big)\right\} \\ &+ C_e\left\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \sum_{i=1}^{2}\big(\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\big)\right\}\mathtt{E}\,.\end{aligned} \tag{5.42}$$

Consequently, we are in position to establish the announced Céa estimate.

Theorem 5.5. *In addition to the hypotheses of Theorems 4.11 and 5.4, assume that*

$$C_e\left\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \sum_{i=1}^{2}\big(\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\big)\right\} \leqslant \frac{1}{2}\,. \tag{5.43}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\boldsymbol{\varphi},\chi) - (\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} \\ &\leqslant C\left\{\operatorname{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{H}_h\times\mathbf{Q}_h\big) + \operatorname{dist}\big((\boldsymbol{\varphi},\chi),\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big) + \sum_{i=1}^{2}\operatorname{dist}\big((\boldsymbol{\sigma}_i,\xi_i),\mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}\big)\right\}.\end{aligned} \tag{5.44}$$

Proof. Under the assumption (5.43), the a priori estimate for $\mathtt{E}$ follows from (5.42), which, along with (5.39), yield (5.44) and ends the proof. □

We end this section by remarking that (2.7) suggests the following postprocessed approximation for the pressure p

$$p_h \;=\; -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma}_h)\,, \tag{5.45}$$

for which it is easy to show that

$$\|p - p_h\|_{0,\Omega} \;\leqslant\; \frac{1}{\sqrt{n}}\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}\,. \tag{5.46}$$

Similarly, the first eq. in the first row of (2.8) suggests to approximate the velocity gradient as

$$\big(\widehat{\nabla\mathbf{u}}\big)_h \;:=\; \frac{1}{\mu}\,\boldsymbol{\sigma}_h^{\mathtt{d}}\,, \tag{5.47}$$

for which it is readily seen that

$$\|\nabla\mathbf{u} - \big(\widehat{\nabla\mathbf{u}}\big)_h\|_{0,\Omega} \;\leqslant\; \frac{1}{\mu}\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}\,. \tag{5.48}$$

6 Specific finite element subspaces

In this section we define explicit finite element subspaces satisfying the hypotheses (**H.1**) - (**H.6**) that were introduced in Section 5.2 for the well posedness of the Galerkin scheme (5.1), and provide the corresponding rates of convergence.

6.1 Preliminaries

In what follows we make use of the notations introduced at the beginning of Section 5.1. Thus, given an integer $k \geqslant 0$, for each $K \in \mathcal{T}_h$ we let $\mathrm{P}_k(K)$ and $\mathbf{P}_k(K)$ be the spaces of polynomials of degree $\leqslant k$ defined on K and its vector version, respectively. Similarly, letting $\mathbf{x}$ be a generic vector in R^n, $\mathbf{RT}_k(K) := \mathbf{P}_k(K) + \mathrm{P}_k(K)\mathbf{x}$ and $\mathbb{RT}_k(K)$ stand for the local Raviart-Thomas space of order k defined on K and its associated tensor counterpart. In addition, we let $\mathrm{P}_k(\mathcal{T}_h)$, $\mathbf{P}_k(\mathcal{T}_h)$, $\mathbf{RT}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, $\mathbf{RT}_k(K)$ and $\mathbb{RT}_k(K)$, respectively, that is

$$\mathrm{P}_k(\mathcal{T}_h) := \Big\{ v_h \in \mathrm{L}^2(\Omega) : \quad v_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

$$\mathbf{P}_k(\mathcal{T}_h) := \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

$$\mathbf{RT}_k(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

and

$$\mathbb{RT}_k(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,.$$

We notice here that for each $t \in (1, +\infty)$ there hold the inclusions $\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{P}_k(\mathcal{T}_h) \subseteq \mathbf{L}^t(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div}_t;\Omega)$, and $\mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}(\mathbf{div}_t;\Omega)$, which are employed below to introduce our specific finite element subspaces. Indeed, we now set

$$\begin{gathered}\mathbf{H}_h := \mathbb{RT}_k(\mathcal{T}_h) \cap \mathbb{H}_0(\mathbf{div}_s;\Omega)\,, \quad \mathbf{Q}_h := \mathbf{P}_k(\mathcal{T}_h)\,, \quad \mathrm{H}_{i,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{Q}_{i,h} := \mathrm{P}_k(\mathcal{T}_h)\,, \\ \mathrm{X}_{2,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{M}_{1,h} := \mathrm{P}_k(\mathcal{T}_h)\,, \quad \mathrm{X}_{1,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \text{and} \quad \mathrm{M}_{2,h} := \mathrm{P}_k(\mathcal{T}_h)\,.\end{gathered} \tag{6.1}$$

6.2 Verification of the hypotheses (H.1) - (H.6)

We begin by observing from (6.1) that (**H.1**) is trivially satisfied, whereas (**H.2**) was proved in [13, Lemma 5.5] (see, also, [7, Lemma 4.3]) for the particular case given by $r = 4$ and $s = 4/3$. In turn, a vector version of (**H.2**) was established in [25, Lemma 4.5] for $s \in (1, 2)$ in 2D (with local notation there given by ϱ instead of s). In both cases, the preliminary result provided by [13, Lemma 5.4] plays a key role in the respective proofs. While we could simply say, at least in 2D, that (**H.2**) follows basically from a direct extension of [25, Lemma 4.5], we provide its explicit proof below for sake of completeness. To this end, following [25, Section 4.1], we now introduce for each $t \in (1, +\infty)$ the space

$$\mathbf{H}_t := \Big\{ \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega) \cup \mathbf{H}(\mathrm{div}_t;\Omega) : \quad \boldsymbol{\tau}|_K \in \mathbf{W}^{1,t}(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

and let $\Pi_h^k : \mathbf{H}_t \to \mathbf{RT}_k(\mathcal{T}_h)$ be the global Raviart-Thomas interpolator (cf. [6, Section 2.5]). Then, we recall from [6, Proposition 2.5.2 and eq. (2.5.27)] the commuting diagram property

$$\mathrm{div}\big(\Pi_h^k(\boldsymbol{\tau})\big) = \mathcal{P}_h^k\big(\mathrm{div}(\boldsymbol{\tau})\big) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}_t\,, \tag{6.2}$$

where $\mathcal{P}_h^k : \mathrm{L}^1(\Omega) \to \mathrm{P}_k(\mathcal{T}_h)$ is the projector defined, for each $v \in \mathrm{L}^1(\Omega)$, as the unique element $\mathcal{P}_h^k(v) \in \mathrm{P}_k(\mathcal{T}_h)$ such that

$$\int_\Omega \mathcal{P}_h^k(v)\, q_h = \int_\Omega v\, q_h \qquad \forall\, q_h \in \mathrm{P}_k(\mathcal{T}_h)\,. \tag{6.3}$$

In turn, it follows from [19, Proposition 1.135] (see, also, [9, eq. (A.5)]) that there exists a positive constant $C_{\mathcal{P}}$, independent of h, such that for each $t \in (1, +\infty)$ there holds

$$\|\mathcal{P}_h^k(v)\|_{0,t;\Omega} \leqslant C_{\mathcal{P}}\, \|v\|_{0,t;\Omega} \qquad \forall\, v \in \mathrm{L}^t(\Omega)\,. \tag{6.4}$$

On the other hand, while here we could use again [13, Lemma 5.4], we prefer to resort to the slightly more general result provided by [9, Lemma A.2], thus giving a greater visibility to it, which establishes that, given an integer l such that $1 \leqslant l \leqslant k+1$, and given $t, p \in (1,+\infty)$, such that $p \leqslant t \leqslant \frac{np}{n-p}$ if $p < n$, or $p \leqslant t < +\infty$ if $p = n$, there exists a positive constant C, independent of h, such that

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \leqslant C\, h^{l+\frac{n}{t}-\frac{n}{p}}\, |\boldsymbol{\tau}|_{l,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{l,p}(\Omega)\,. \tag{6.5}$$

Note that for the first set of constraints on t and p, there holds $\frac{n}{t} - \frac{n}{p} \geqslant -1$, which yields $l + \frac{n}{t} - \frac{n}{p} \geqslant 0$, whereas for the second one, there holds $l + \frac{n}{t} - \frac{n}{p} = l - 1 + \frac{n}{t} \geqslant \frac{n}{t}$, thus proving that in any case the power of h in (6.5) is non-negative. In this way, it follows from (6.5) that, for $l = 1$, and under the specified ranges of t and p, there exists a positive constant C_Π, independent of h, such that (cf. [9, Lemma A.3])

$$\|\Pi_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \leqslant C_\Pi\, \|\boldsymbol{\tau}\|_{1,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{1,p}(\Omega)\,. \tag{6.6}$$

In particular, for $p < n$ and $t = 2$, the inequality $t \leqslant \frac{np}{n-p}$ becomes $p \geqslant \frac{2n}{n+2}$, so that for the resulting range of p, that is $p \in \big[\frac{2n}{n+2}, 2\big)$ in 2D, and $p \in \big[\frac{2n}{n+2}, 2\big]$ in 3D, we obtain

$$\|\Pi_h^k(\boldsymbol{\tau})\|_{0,\Omega} \leqslant C_\Pi\, \|\boldsymbol{\tau}\|_{1,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{1,p}(\Omega)\,. \tag{6.7}$$

Analogue identities and inequalities to those stated above are valid with the tensor and vector versions of Π_h^k and $\mathcal{P}_h^k$, which are denoted by $\boldsymbol{\Pi}_h^k$ and $\boldsymbol{\mathcal{P}}_h^k$, respectively.

We are now in position to prove that (**H.2**) holds.

Lemma 6.1. *Under the ranges for r and s specified by (4.23), there exists a positive constant $\boldsymbol{\beta}_{\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_s;\Omega}} \geqslant \boldsymbol{\beta}_{\mathtt{d}}\, \|\mathbf{v}_h\|_{0,r;\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,, \tag{6.8}$$

Proof. Given $\mathbf{v}_h \in \mathbf{Q}_h$, $\mathbf{v}_h \neq \mathbf{0}$, we set $\mathbf{v}_{h,s} := |\mathbf{v}_h|^{r-2}\, \mathbf{v}_h$, which belongs to $\mathbf{L}^s(\Omega)$, and notice that

$$\int_\Omega \mathbf{v}_h \cdot \mathbf{v}_{h,s} = \|\mathbf{v}_h\|_{0,r;\Omega}\, \|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{6.9}$$

Next, we let $\mathcal{O}$ be a bounded convex polygonal domain that contains $\bar{\Omega}$, and define

$$\mathbf{g} := \begin{cases} \mathbf{v}_{h,s} & \text{in} \quad \Omega\,, \\ \mathbf{0} & \text{in} \quad \mathcal{O}\backslash\bar{\Omega}\,, \end{cases} .$$

It is readily seen that $\mathbf{g} \in \mathbf{L}^s(\mathcal{O})$ and $\|\mathbf{g}\|_{0,s;\mathcal{O}} = \|\mathbf{v}_{h,s}\|_{0,s;\Omega}$. Then, applying the elliptic regularity result provided by [21, Corollary 1], we deduce that there exists a unique $\mathbf{z} \in \mathbf{W}^{2,s}(\mathcal{O}) \cap \mathbf{W}_0^{1,s}(\mathcal{O})$ such that: $\Delta\mathbf{z} = \mathbf{g}$ in $\mathcal{O}$, $\mathbf{z} = \mathbf{0}$ on $\partial\mathcal{O}$. Moreover, there exists a positive constant $C_{\mathtt{reg}}$, depending only on $\mathcal{O}$, such that

$$\|\mathbf{z}\|_{2,s;\mathcal{O}} \leqslant C_{\mathtt{reg}}\, \|\mathbf{g}\|_{0,s;\mathcal{O}} = C_{\mathtt{reg}}\, \|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{6.10}$$

Hence, defining $\boldsymbol{\zeta} := \nabla\mathbf{z}|_\Omega \in \mathbb{W}^{1,s}(\Omega)$, it follows that $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{v}_{h,s}$ in Ω, and, according to (6.10),

$$\|\boldsymbol{\zeta}\|_{1,s;\Omega} \leqslant \|\mathbf{z}\|_{2,s;\mathcal{O}} \leqslant C_{\mathtt{reg}}\, \|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{6.11}$$

Now, since the identity tensor $\mathbb{I}$ clearly belongs to $\mathbb{RT}_k(\mathcal{T}_h)$, we can let $\boldsymbol{\zeta}_h$ be the $\mathbb{H}_0(\mathbf{div}_s;\Omega)$-component (cf. (3.10)) of $\boldsymbol{\Pi}_h^k(\boldsymbol{\zeta})$, so that $\boldsymbol{\zeta}_h \in \mathbf{H}_h$. In this way, employing the analogue of (6.2), we find that

$$\mathbf{div}(\boldsymbol{\zeta}_h) = \mathbf{div}\big(\boldsymbol{\Pi}_h^k(\boldsymbol{\zeta})\big) = \boldsymbol{\mathcal{P}}_h^k\big(\mathbf{div}(\boldsymbol{\zeta})\big) = \boldsymbol{\mathcal{P}}_h^k\big(\mathbf{v}_{h,s}\big)\,, \tag{6.12}$$

which, along with the analogue of (6.4) for $t = s$, give

$$\|\mathbf{div}(\boldsymbol{\zeta}_h)\|_{0,s;\Omega} \,\leqslant\, C_{\boldsymbol{\mathcal{P}}}\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{6.13}$$

In turn, noting that the range for s (cf. (4.23)) fits into the one for p in (6.7), we can apply this inequality (with $p = s$) and the regularity estimate (6.11), to arrive at

$$\|\boldsymbol{\zeta}_h\|_{0,\Omega} \,\leqslant\, \|\boldsymbol{\Pi}_h^k(\boldsymbol{\zeta})\|_{0,\Omega} \,\leqslant\, C_{\boldsymbol{\Pi}}\,\|\boldsymbol{\zeta}\|_{1,s;\Omega} \,\leqslant\, C_{\boldsymbol{\Pi}}\,C_{\mathtt{reg}}\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,, \tag{6.14}$$

which, combined with (6.13), implies

$$\|\boldsymbol{\zeta}_h\|_{\mathbf{div}_s;\Omega} \,\leqslant\, \big(C_{\boldsymbol{\mathcal{P}}} + C_{\boldsymbol{\Pi}}\,C_{\mathtt{reg}}\big)\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{6.15}$$

Consequently, bounding below the supremum in (6.8) with $\boldsymbol{\zeta}_h$, and making use of (6.12), the analogue of (6.3), (6.9), and (6.15), we conclude the required discrete inf-sup condition with the constant $\boldsymbol{\beta}_{\mathtt{d}} := \big(C_{\boldsymbol{\mathcal{P}}} + C_{\boldsymbol{\Pi}}\,C_{\mathtt{reg}}\big)^{-1}$. $\square$

Furthermore, for the hypotheses (**H.3**) and (**H.4**), we first stress that (**H.3**) corresponds exactly to [9, (**H.5**)], and hence we omit most details and refer to [9, Section 5.2, Lemma 5.2]. We just make a few remarks here. First of all, we observe that the discrete kernels of the bilinear forms b_1 and b_2 coincide algebraically, which reduces to

$$\mathrm{K}_h^k \,:=\, \Big\{\boldsymbol{\psi}_h \in \mathbf{RT}_k(\mathcal{T}_h): \quad \mathrm{div}(\boldsymbol{\psi}_h) = 0 \quad \text{in} \quad \Omega\Big\}\,.$$

Then, we let $\Theta_h^k : \mathbf{L}^1(\Omega) \to \mathrm{K}_h^k$ be the projector defined similarly to (6.3), that is, given $\boldsymbol{\phi} \in \mathbf{L}^1(\Omega)$, $\Theta_h^k(\boldsymbol{\phi})$ is the unique element in K_h^k such that

$$\int_\Omega \Theta_h^k(\boldsymbol{\phi})\cdot\boldsymbol{\psi}_h \,=\, \int_\Omega \boldsymbol{\phi}\cdot\boldsymbol{\psi}_h \qquad \forall\,\boldsymbol{\psi}_h \in \mathrm{K}_h^k\,.$$

In this way, a quasi-uniform boundedness property of Θ_h^k in 2D (cf. [9, eq.(5.8)]), along with the properties of the operators D_t (cf. Lemma 4.2), play a key role in the proof of (**H.3**). Whether the aforementioned boundedness is satisfied or not in 3D is still an open problem, and hence, similarly to [9], the assumption (**H.3**) is the only aspect of the analysis in this section that does not hold in 3D. All the other conditions are valid in both 2D and 3D. Regarding (**H.4**), we remark that the discrete inf-sup conditions for b_1 and b_2, which adapt the continuous analysis from Lemma 4.4 to the present discrete setting, follow from slight modifications of the proofs of [25, Lemma 4.5] and [9, Lemma 5.3]. Further details are omitted here.

Finally, it is clear from (6.1) that (**H.5**) is trivially satisfied, whereas (**H.6**) was proved precisely by [25, Lemma 4.5]. Alternatively, for the discrete inf-sup condition for c_i we can proceed analogously to the proof of Lemma 6.1 by observing that the range of ϱ (cf. (4.23), recall that $\mathrm{H}_i := \mathbf{H}(\mathrm{div}_\varrho;\Omega)$) also fits into the one for p in (6.7), whence this inequality can be applied to $p = \varrho$ as well.

On the other hand, and as already announced in Section 5.1, we now observe that for the particular finite element subspaces introduced in Section 6.1 (cf. (6.1)), the discrete conservation of momentum properties (cf. second, fourth, and sixth rows of (5.1)) become

$$\begin{aligned}
\mathbf{div}(\boldsymbol{\sigma}_h) - \boldsymbol{\mathcal{P}}_h^k\big((\xi_{1,h}-\xi_{2,h})\,\varepsilon^{-1}\boldsymbol{\varphi}_h + \mathbf{f}\big) \;&=\; \mathbf{0} \quad \text{in} \quad \Omega\,,\\
\mathrm{div}(\boldsymbol{\varphi}_h) + (\xi_{1,h}-\xi_{2,h}) + \mathcal{P}_h^k(f) \;&=\; 0 \quad \text{in} \quad \Omega\,,\\
\xi_{i,h} - \mathrm{div}(\boldsymbol{\sigma}_{i,h}) - \mathcal{P}_h^k(f_i) \;&=\; 0 \quad \text{in} \quad \Omega\,,
\end{aligned} \tag{6.16}$$

so that, due to the presence of $\mathcal{P}_h^k$ and $\boldsymbol{\mathcal{P}}_h^k$, they are satisfied approximately.

6.3 The rates of convergence

Here we provide the rates of convergence of the Galerkin scheme (5.1) with the specific finite element subspaces introduced in Section 6.1, for which we previously collect the respective approximation properties. In fact, thanks to [19, Proposition 1.135] and its corresponding vector version, along with interpolation estimates of Sobolev spaces, those of $\mathbf{Q}_h$, $\mathrm{Q}_{i,h}$, and $\mathrm{M}_{1,h}$, are given as follows

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{v}, \mathbf{Q}_h\big) \,:=\, \inf_{\mathbf{v}_h \in \mathbf{Q}_h} \|\mathbf{v} \,-\, \mathbf{v}_h\|_{0,r;\Omega} \,\leqslant\, C\, h^l\, \|\mathbf{v}\|_{l,r;\Omega}\,,$$

$\Big(\mathbf{AP}_h^{\xi_i}\Big)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\eta_i \in \mathrm{W}^{l,\rho}(\Omega)$, there holds

$$\operatorname{dist}\big(\eta_i, \mathrm{Q}_{i,h}\big) \,:=\, \inf_{\eta_{i,h} \in \mathrm{Q}_{i,h}} \|\eta_i \,-\, \eta_{i,h}\|_{0,\rho;\Omega} \,\leqslant\, C\, h^l\, \|\eta_i\|_{l,\rho;\Omega}\,,$$

$\big(\mathbf{AP}_h^{\chi}\big)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\lambda \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}\big(\lambda, \mathrm{M}_{1,h}\big) \,:=\, \inf_{\lambda_h \in \mathrm{M}_{1,h}} \|\lambda \,-\, \lambda_h\|_{0,r;\Omega} \,\leqslant\, C\, h^l\, \|\lambda\|_{l,r;\Omega}\,.$$

Furthermore, from [25, eq. (4.6), Section 4.1] and its tensor version, which, as the foregoing ones, are derived in the classical way by using the Deny–Lions Lemma and the corresponding scaling estimates (cf. [19, Lemmas B.67 and 1.101]), we state next the approximation properties of $\mathbf{H}_h$ and $\mathrm{H}_{i,h}$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_s; \Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,s}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}, \mathbf{H}_h\big) \,:=\, \inf_{\boldsymbol{\tau}_h \in \mathbf{H}_h} \|\boldsymbol{\tau} \,-\, \boldsymbol{\tau}_h\|_{\mathbf{div}_s;\Omega} \,\leqslant\, C\, h^l\, \Big\{\|\boldsymbol{\tau}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{l,s;\Omega}\Big\}\,,$$

$\big(\mathbf{AP}_h^{\boldsymbol{\sigma}_i}\big)$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau}_i \in \mathbb{H}^l(\Omega)$ with $\operatorname{div}(\boldsymbol{\tau}_i) \in \mathbf{W}^{l,\varrho}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}_i, \mathrm{H}_{i,h}\big) \,:=\, \inf_{\boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}} \|\boldsymbol{\tau}_i \,-\, \boldsymbol{\tau}_{i,h}\|_{\operatorname{div}_\varrho;\Omega} \,\leqslant\, C\, h^l\, \Big\{\|\boldsymbol{\tau}_i\|_{l,\Omega} \,+\, \|\operatorname{div}(\boldsymbol{\tau}_i)\|_{l,\varrho;\Omega}\Big\}\,.$$

Finally, that of $X_{2,h}$, which we recall from [25, Section 4.5, $(\mathbf{AP}_h^{\mathbf{u}})$], becomes

$(\mathbf{AP}_{\mathbf{h}}^{\boldsymbol{\varphi}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\phi} \in \mathbf{W}^{l,r}(\Omega)$ with $\operatorname{div}(\boldsymbol{\phi}) \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\phi}, X_{2,h}\big) \,:=\, \inf_{\boldsymbol{\phi}_h \in X_{2,h}} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{r,\operatorname{div}_r;\Omega} \,\leqslant\, C\, h^l\, \Big\{\|\boldsymbol{\phi}\|_{l,r;\Omega} \,+\, \|\operatorname{div}(\boldsymbol{\phi})\|_{l,r;\Omega}\Big\}\,.$$

The rates of convergence of (5.1) are now provided by the following theorem.

Theorem 6.2. *Let* $\big((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\varphi},\chi),(\boldsymbol{\sigma}_i,\xi_i)\big) \in \big(\mathbf{H}\times\mathbf{Q}\big)\times\big(\mathrm{X}_2\times\mathrm{M}_1\big)\times\big(\mathrm{H}_i\times\mathrm{Q}_i\big)$, $i\in\{1,2\}$ *be the unique solution of* (3.33) *with* $\boldsymbol{\xi} := (\xi_1,\xi_2)\in\mathrm{W}(\delta)$, *and let* $\big((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\varphi}_h,\chi_h),(\boldsymbol{\sigma}_{i,h},\xi_{i,h})\big) \in \big(\mathbf{H}_h\times\mathbf{Q}_h\big)\times\big(\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big)\times\big(\mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}\big)$, $i\in\{1,2\}$ *be a solution of* (5.1) *with* $\boldsymbol{\xi}_h := (\xi_{1,h},\xi_{2,h})\in\mathrm{W}(\delta_{\mathtt{d}})$, *which is guaranteed by Theorems* 4.11 *and* 5.4, *respectively. In turn, let* p, p_h, $\nabla\mathbf{u}$, *and* $\big(\widehat{\nabla\mathbf{u}}\big)_h$ *given by* (2.7), (5.45), *the first eq. in the first row of* (2.8), *and* (5.47), *respectively. Assume the hypotheses of Theorem* 5.5, *and that there exists* $l\in[1,k+1]$ *such that* $\boldsymbol{\sigma}\in\mathbb{H}^l(\Omega)\cap\mathbb{H}_0(\mathbf{div}_s;\Omega)$, $\mathbf{div}(\boldsymbol{\sigma})\in\mathbf{W}^{l,s}(\Omega)$, $\mathbf{u}\in\mathbf{W}^{l,r}(\Omega)$, $\boldsymbol{\varphi}\in\mathbf{W}^{l,r}(\Omega)$, $\mathrm{div}(\boldsymbol{\varphi})\in\mathrm{W}^{l,r}(\Omega)$, $\chi\in\mathrm{W}^{l,r}(\Omega)$, $\boldsymbol{\sigma}_i\in\mathbb{H}^l(\Omega)$, $\mathrm{div}(\boldsymbol{\sigma}_i)\in\mathbf{W}^{l,\varrho}(\Omega)$, *and* $\xi_i\in\mathrm{W}^{l,\rho}(\Omega)$, $i\in\{1,2\}$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$
\begin{aligned}
&\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|p-p_h\|_{0,\Omega} + \|(\boldsymbol{\varphi},\chi)-(\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1}\\
&+\|\nabla\mathbf{u}-\big(\widehat{\nabla\mathbf{u}}\big)_h\|_{0,\Omega} + \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i)-(\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\\
&\qquad\leqslant C\,h^l\Big\{\|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,s;\Omega} + \|\mathbf{u}\|_{l,r;\Omega} + \|\boldsymbol{\varphi}\|_{l,r;\Omega} + \|\mathrm{div}(\boldsymbol{\varphi})\|_{l,r;\Omega}\\
&\qquad+\|\chi\|_{l,r;\Omega} + \sum_{i=1}^{2}\big(\|\boldsymbol{\sigma}_i\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\sigma}_i)\|_{l,\varrho;\Omega} + \|\xi_i\|_{l,\rho;\Omega}\big)\Big\}\,.
\end{aligned}
$$

Proof. It follows straightforwardly from Theorem 5.5, (5.46), (5.48), and the above approximation properties. □

7 Computational results

We turn now to the numerical verification of the rates of convergence anticipated by Theorem 6.2. The following examples in 2D and 3D have been realized with the finite element library FEniCS [1]. The linearization of the nonlinear algebraic equations that arise after discretization is done using either a fixed-point Picard algorithm or an exact Newton–Raphson method (with the zero vector as initial guess and iterations are stopped once the absolute or relative residual drops below 10^{-8}) and the linear systems are solved with the multifrontal massively parallel sparse direct method MUMPS [2].

Example 1. Considering first the spatial domain $\Omega=(0,1)^3$ along with the arbitrarily chosen parameters

$$\mu=10^{-3},\quad \varepsilon=0.1,\quad \kappa_1=0.25,\quad \kappa_2=0.5,$$

we define the following manufactured exact solutions to (2.8)

$$
\begin{aligned}
\mathbf{u} &= \begin{pmatrix} \sin^2(\pi x)\sin(\pi y)\sin(2\pi z)\\ \sin(\pi x)\sin^2(\pi y)\sin(2\pi z)\\ -[\sin(2\pi x)\sin(\pi y)+\sin(\pi x)\sin(2\pi y)]\sin^2(\pi z)\end{pmatrix},\\
p &= x^4-\frac{1}{2}(y^4+z^4),\quad \xi_1=\exp(-xy+z),\\
\xi_2 &= \cos^2(xyz),\quad \chi=\sin(x)\cos(y)\sin(z),\quad \boldsymbol{\sigma}=\mu\nabla\mathbf{u}-p\mathbb{I},\\
\boldsymbol{\sigma}_i &= \kappa_i(\nabla\xi_i+q_i\xi_i\varepsilon^{-1}\boldsymbol{\varphi})-\xi_i\mathbf{u},\quad \boldsymbol{\varphi}=\varepsilon\nabla\chi,
\end{aligned}
$$

and construct forcing/source terms and non-homogeneous Dirichlet boundary conditions $\mathbf{f},\mathbf{g},f_i,g_i$ from these closed-form solutions. Using the lowest-order version of the finite element spaces defined in (6.1) (with polynomial degree $k=0$), we solve problem (5.1) on a sequence of six succesively refined regular meshes. The zero-mean pressure condition is enforced using a real Lagrange multiplier approach. At each refinement level we compute errors between approximate and smooth exact solutions using the norms in (5.33) and Theorem 6.2

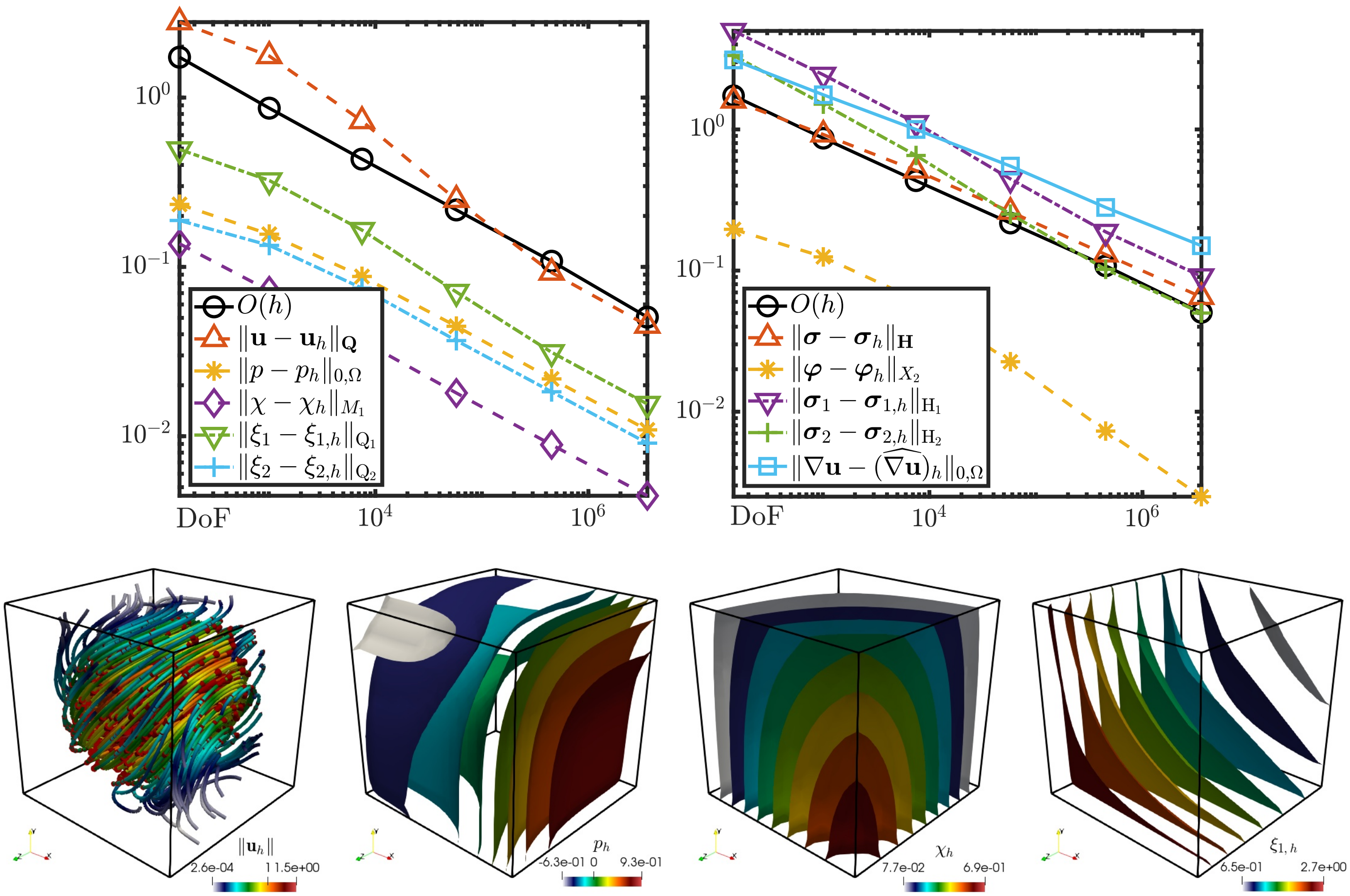

Figure 7.1: Example 1. Error history associated with the finite element family (6.1) with $k=0$ in 3D for primal variables (top left) and mixed variables (top right, including the velocity gradient using the postprocess in (5.47)), and samples of approximate primal variables (velocity streamlines $\mathbf{u}_h$, iso-surfaces of postprocessed pressure p_h, electrostatic potential χ_h, and positive ion concentration $\xi_{1,h}$; bottom plots). In all mesh refinements the number of Newton–Raphson iterations was 4.

(but we split their contribution coming from the error on each individual field variable). For this 3D accuracy test we consider the Banach spaces indexes specified in (4.23)

$$r=3, \quad s=3/2, \quad \rho=6, \quad \varrho=6/5.$$

The results of this convergence study are collected in Figure 7.1 (top panels), where we plot in log-log scale the error decay as the number of degrees of freedom increases. Apart from the electric field $\boldsymbol{\varphi}$ which converges with rate of approximately 1.5, all other variables exhibit an optimal rate of convergence. In the bottom panel of the figure we show approximate solutions for some of the field variables, which indicate well resolved profiles.

In addition, the balance-preserving properties (6.16) of the proposed mixed formulation are assessed by computing the quantities

$$\texttt{fluid}_h := \|\mathbf{div}(\boldsymbol{\sigma}_h) - \boldsymbol{\mathcal{P}}_h^k\big((\xi_{1,h}-\xi_{2,h})\,\varepsilon^{-1}\boldsymbol{\varphi}_h + \mathbf{f}\big)\|_{\ell^\infty},$$

$$\texttt{current}_h := \|\mathrm{div}(\boldsymbol{\varphi}_h) + (\xi_{1,h}-\xi_{2,h}) + \mathcal{P}_h^k(f)\|_{\ell^\infty},$$

$$\texttt{mass}_{i,h} := \|\xi_{i,h} - \mathrm{div}(\boldsymbol{\sigma}_{i,h}) - \mathcal{P}_h^k(f_i)\|_{\ell^\infty}.$$

DoF	h	e	r	$\texttt{fluid}_h$	$\texttt{current}_h$	$\texttt{mass}_{1,h}$	$\texttt{mass}_{2,h}$
145	1.732	4.51e+1	$\star$	2.37e-07	7.29e-17	1.83e-15	8.64e-16
1009	0.866	2.35e+1	0.88	8.61e-08	2.45e-16	4.14e-15	1.81e-15
7489	0.433	1.34e+1	0.91	6.07e-10	4.53e-16	5.10e-15	4.85e-15
57601	0.217	6.90e+0	0.96	1.27e-11	6.76e-16	1.45e-14	8.77e-15
451585	0.108	3.46e+0	1.00	1.04e-11	6.29e-15	1.47e-14	2.48e-11
3575809	0.051	1.72e+0	1.00	5.88e-11	4.20e-15	2.38e-15	2.95e-15

Table 7.1: Example 1. Total error, experimental rates of convergence, and ℓ^∞-norm of the projected residual of the momentum, potential, and ionic transport equations.

These values, for each refinement level, are collected in Table 7.1. We tabulate the total error

$$\begin{aligned}\mathtt{e} := \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \|p - p_h\|_{0,\Omega} \,+\, \|(\boldsymbol{\varphi}, \chi) - (\boldsymbol{\varphi}_h, \chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \\ +\, \|\nabla\mathbf{u} - \big(\widehat{\nabla\mathbf{u}}\big)_h\|_{0,\Omega} \,+\, \sum_{i=1}^{2} \|(\boldsymbol{\sigma}_i, \xi_i) - (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,,\end{aligned}$$

(as indicated by Theorem 6.2) as well as the rates of convergence computed as

$$\mathtt{r} = \log(\mathtt{e}/\widehat{\mathtt{e}})[\log(h/\widehat{h})]^{-1}\,,$$

where $\mathtt{e}$ and $\widehat{\mathtt{e}}$ denote errors produced on two consecutive meshes associated with mesh sizes h and $\widehat{h}$, respectively. From the last columns we see that the potential and transport balance equations are satisfied to machine precision while the error for the momentum balance is higher. This may be explained by the presence of the term $\boldsymbol{\varphi}_h$ on the right-hand side (which has a H(div)-component).

Example 2. In addition, and in order to illustrate the implementation of fixed-point solvers, we have realized numerically Picard versions of the linearization of (5.1). In case A we follow the fixed-point structure used in the analysis of Section 5.1, that is, solving sequentially problems

$$(5.2) \to (5.3) \to (5.4),$$

and iterating until the ℓ^2-norm of the vector containing the residual of the Picard iterates reaches 10^{-8}. Next, in case B we choose a different fixed-point splitting where we apply two modifications with respect to case A. First, in (5.4) instead of the linear functional for the second discrete electrostatic potential equation (discrete version of (3.21d)) we consider $G(\lambda_h) := -\int_\Omega f\lambda_h$ and the coupling term appears as a bilinear form contribution (and no longer as part of the linear functional), say

$$\widehat{g}(\lambda_h, (\xi_{1,h}, \xi_{2,h})) := \int_\Omega \lambda_h(\xi_{1,h} - \xi_{2,h}).$$

Secondly, with regards to the constitutive equation in the ionized particle equations, we swap the bilinearity in the flux definition (discrete version of (3.30f)) from $\xi_{i,h}$ to the pair $(\boldsymbol{\phi}_h, \mathbf{u}_h)$, that is, we consider

$$\widehat{c}_{\hat{\xi}_{i,h}}(\boldsymbol{\tau}_{i,h}, (\boldsymbol{\phi}_h, \mathbf{u}_h)) := \int_\Omega \Big\{q_i\,\hat{\xi}_i\,\varepsilon^{-1}\boldsymbol{\phi}_h \,-\, \kappa_i^{-1}\,\hat{\xi}_i\,\mathbf{u}_h\Big\}\cdot\boldsymbol{\tau}_i.$$

For both fixed-point cases we have taken as initial guess solution the zero vector. Moreover, we consider a 2D problem with manufactured solutions defined on $\Omega = (0,1)^2$

$$\mathbf{u} = \begin{pmatrix}\cos(\pi x)\sin(\pi y)\\ -\sin(\pi x)\cos(\pi y)\end{pmatrix}, \quad p = x^4 - y^4, \quad \chi = \sin(x)\cos(y), \quad \xi_1 = \exp(-xy), \quad \xi_2 = \cos^2(xy),$$

		case A			case B			case C		
DoF	h	`e`	`r`	`iter`	`e`	`r`	`iter`	`e`	`r`	`iter`
$k=0$										
221	0.500	2.55e+1	$\star$	100	2.55e+1	$\star$	13	2.55e+1	$\star$	4
841	0.250	1.30e+1	0.97	83	1.30e+1	0.97	8	1.30e+1	0.97	4
3281	0.125	6.39e+0	1.03	72	6.39e+0	1.03	8	6.39e+0	1.03	4
12961	0.062	3.15e+0	1.02	70	3.15e+0	1.02	9	3.15e+0	1.02	4
51521	0.031	1.57e+0	1.01	68	1.57e+0	1.01	9	1.57e+0	1.01	4
$k=1$										
681	0.500	4.33e+0	$\star$	68	4.33e+0	$\star$	9	4.33e+0	$\star$	4
2641	0.250	1.08e+0	2.00	68	1.08e+0	2.00	9	1.08e+0	2.00	4
10401	0.125	2.72e-01	2.00	68	2.72e-01	2.00	9	2.72e-01	2.00	4
41281	0.062	6.81e-02	2.00	68	6.81e-02	2.00	9	6.81e-02	2.00	4
164481	0.031	1.71e-02	2.00	77	1.71e-02	1.99	10	1.70e-02	2.00	4

Table 7.2: Example 2. Total error, experimental rates of convergence, and number of iterations required for two types of fixed-point methods as well as for Newton–Raphson linearization.

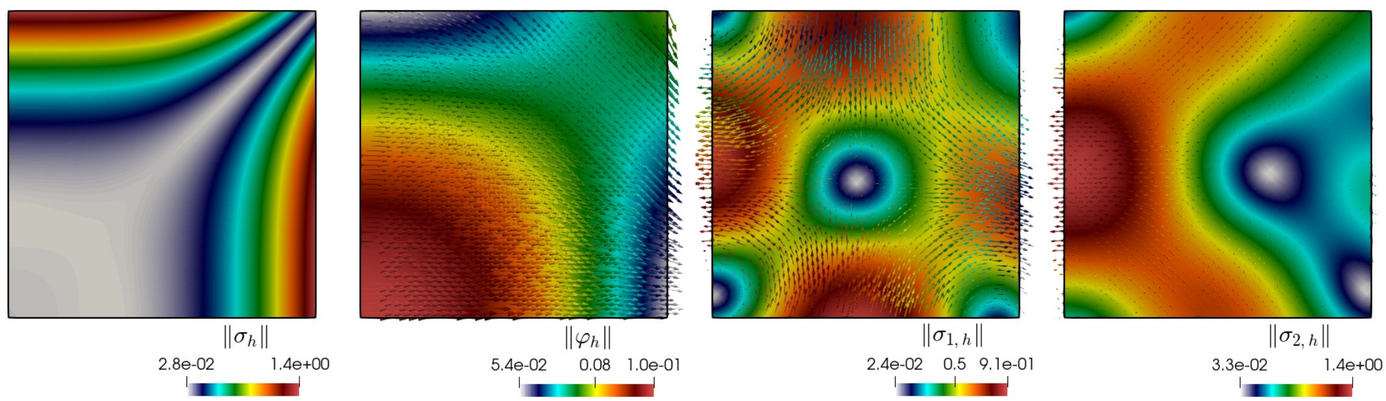

Figure 7.2: Example 2. Samples of approximate mixed variables (stress magnitude, electric field magnitude and arrows, and ionic fluxes) obtained with the fixed-point algorithm labelled case A, and for $k = 1$.

and take the same model constants as before. In 2D, and according to (4.23) we now choose

$$r = 4, \quad s = 4/3, \quad \rho = 4, \quad \varrho = 4/3.$$

We focus on the number of Picard iterations required in each case, displaying the obtained results in Table 7.2. While we confirm that all methods give exactly the same errors (and consequently also the same convergence rates, which are optimal in view of the theoretical bounds), from the number of fixed-point iterations we readily note that case B performs much better than case A, for the two polynomial degrees we tested $k = 0$, $k = 1$. This behaviour could be explained by the stability of different linearizations of advective nonlinearities and by the strength of the coupling for this particular choice of model parameters. We stress that the analysis of case B is, however, not at all straightforward since the decoupled linear electrostatic potential problem resulting from the first modification is no longer symmetric. For sake of reference we also tabulate total errors and number of nonlinear iterates obtained with the method we use also in Examples 1 and 3: an exact Newton–Raphson linearization (labelled here as case C). Needless to say, the latter is actually the one that one would employ in practical computations. Samples of the approximate solutions (only the mixed variables) computed with the method in case A are portrayed in Figure 7.2.

Example 3. We conclude this section with an application problem where we demonstrate the use of the mixed finite element scheme in simulating the transport process in an electrokinetic system with an ion-selective interface, where the development of an electroosmotic instability is expected. The problem configuration is adopted from [17, 18]. This system corresponds to a transient counterpart of (2.8) in the absence of external forces and sources ($\mathbf{f} = \mathbf{0}$, $f = f_i = 0$), where the following additional terms appear in the momentum and concentration equations (note also the different scaling of ε on the right-hand side of the momentum balance, required to match the adimensionalization in [18])

$$-\frac{1}{\mathrm{Sc}}\partial_t \mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) \;=\; (\xi_1 - \xi_2)\,\frac{1}{2\varepsilon^2}\boldsymbol{\varphi}, \quad -\partial_t \xi_i + \mathrm{div}(\boldsymbol{\sigma}_i) = 0.$$

The time derivatives are discretized using backward Euler's method. In the problem setup a boundary layer is present in the vicinity of the solid boundary (the bottom edge of the rectangular domain), and therefore we employ a graded mesh with a higher refinement close to the layer. For this problem we select the second-order family of finite element subspaces (setting $k = 1$ in Section 6.1), which gives for the chosen mesh 865201 degrees of freedom.

The physical properties of the system are as follows. The cation species is Na^+ having diffusivity $\kappa_1 = 1$ and the anion species is Cl^- with the same diffusivity $\kappa_2 = 1$. The dynamic viscosity of the mixture is $\mu = 1$. Initial conditions are given by $\mathbf{u} = \mathbf{0}$, and a 2% random perturbation on a linearly varying initial ionic concentrations $\xi_1 = \zeta(2 - y)$, $\xi_2 = \zeta x$, where ζ is a uniform random variable between 0.98 and 1. On the top boundary we set $\xi_1 = \xi_2 = 1$, $\mathbf{u} = \mathbf{0}$, and an applied voltage of $\chi = 120$. On the bottom boundary we impose $\chi = 0$, $\xi_1 = 2$, $\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nu} = 0$, and $\mathbf{u} = \mathbf{0}$. On the vertical walls we prescribe periodic boundary conditions. The other model parameters take the values $\varepsilon = 8 \cdot 10^{-6}$, Sc $=10^3$, and we use a timestep $\Delta t = 10^{-6}$. We plot snapshots of the anion concentration $\xi_{2,h}$ in Figure 7.3 at times $t = 10^{-4}, 10^{-3}$. We observe similar ionic patterns to those produced also in [31, 33].

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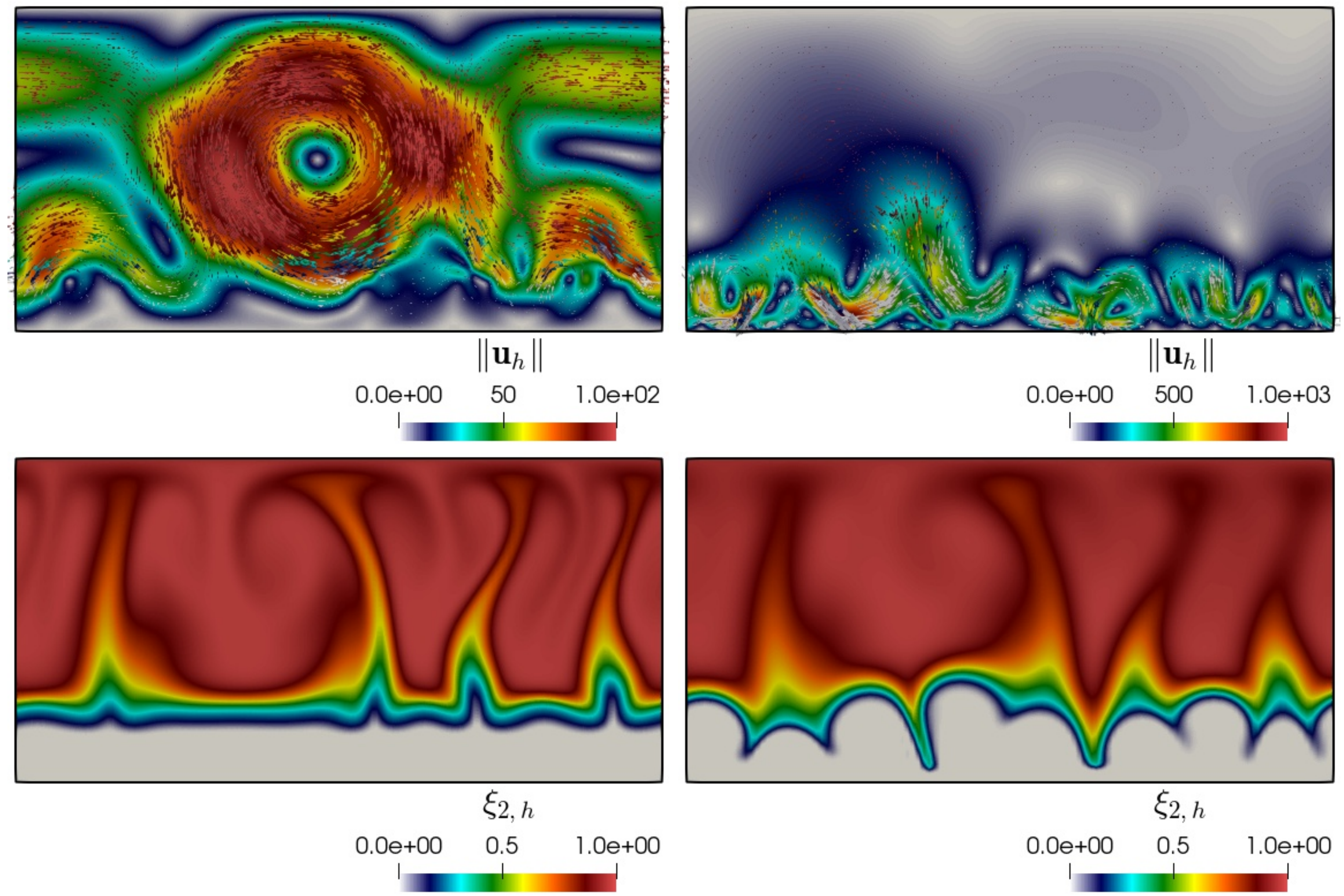

Figure 7.3: Example 3. Samples of approximate velocity (top) and anion concentration (bottom) at times $t = 10^{-4}$ and 10^{-3} (left and right, respectively), produced with the mixed method and using $k = 1$.

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