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DIVERGENCE-CONFORMING METHODS FOR TRANSIENT DOUBLY-DIFFUSIVE FLOWS: A PRIORI AND A POSTERIORI ERROR ANALYSIS

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Abstract. The analysis of the double-diffusion model and $\mathbf{H}(\mathrm{div})$-conforming method introduced in [Bürger, Méndez, Ruiz-Baier, SINUM (2019), 57:1318–1343] is extended to the time-dependent case. In addition, the efficiency and reliability analysis of residual-based *a posteriori* error estimators for the steady, semi-discrete, and fully discrete problems is established. The resulting methods are applied to simulate the sedimentation of small particles in salinity-driven flows. The method consists of Brezzi-Douglas-Marini approximations for velocity and compatible piecewise discontinuous pressures, whereas Lagrangian elements are used for concentration and salinity distribution. Numerical tests confirm the properties of the proposed family of schemes and of the adaptive strategy guided by the *a posteriori* error indicator.

1. Introduction and problem formulation

1.1. **Scope.** A number of physical problems of relevance in industrial applications involve coupled incompressible flow and double-diffusion transport. We are interested in numerical schemes for the approximation of a class of coupled equations that arise as models of sedimentation of small particles under the effect of salinity of the ambient fluid. The governing model can be stated as follows (cf., e.g., [19, 33]):

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \mathbf{div}\big(\nu(c)\boldsymbol{\nabla}\boldsymbol{u}\big) - \frac{1}{\rho_{\mathrm{m}}}\nabla p + \frac{\rho}{\rho_{\mathrm{m}}}\boldsymbol{g}, \tag{1.1a}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{1.1b}$$

$$\partial_t s + \boldsymbol{u}\cdot\nabla s = \frac{1}{\mathrm{Sc}}\nabla^2 s, \tag{1.1c}$$

$$\partial_t c + (\boldsymbol{u} - v_{\mathrm{p}}\boldsymbol{e}_z)\cdot\nabla c = \frac{1}{\tau\mathrm{Sc}}\nabla^2 c, \tag{1.1d}$$

posed on a spatial domain $\Omega\subset\mathbb{R}^d$, $d=2$ or $d=3$, where $t\in(0,t_{\mathrm{end}}]$ is time, $\boldsymbol{u}$ is the fluid velocity, ν is the concentration-dependent viscosity, ρ_{m} is the mean density of the fluid, p is the fluid pressure, ρ is density, $\boldsymbol{g}$ is the gravity acceleration, s is the salinity concentration, c is the concentration of solid particles, $\mathrm{Sc}=\nu_{\mathrm{ref}}/\kappa_{\mathrm{s}}$ is the Schmidt number, where κ_{s} is the diffusivity of salinity, ν_{ref} is a reference viscosity in the absence of solid particles, and $\tau=\kappa_{\mathrm{s}}/\kappa_{\mathrm{c}}$ is the inverse of the diffusivity ratio, where κ_{c} is the diffusivity of solid particles, and $\boldsymbol{e}_z$ is the upward-pointing unit vector. We relate the densities through a linearised equation of state

$$\rho = \rho_{\mathrm{m}}(\alpha s + \beta c).$$

Again as in [19, 33], the solid particles are assumed mono-sized with radius r, and settle at dimensionless velocity $v_{\mathrm{p}} = v_{\mathrm{St}}/(\nu_{\mathrm{ref}}g')^{1/3}$, where $v_{\mathrm{St}} = 2r^2(\rho_{\mathrm{p}}-\rho_{\mathrm{m}})g/(9\rho_{\mathrm{m}}\nu_{\mathrm{ref}})$ is the Stokes velocity (settling velocity of a single particle in an unbounded fluid). The coupling mechanisms between flow and transport are only due to advection for concentration and salinity (where the advecting velocity for concentration, $\boldsymbol{u}-v_{\mathrm{p}}\boldsymbol{e}_z$, is also divergence-free), and through the concentration-dependent viscosity. Further details are provided in later parts of the paper.

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To put the paper further into the proper perspective, we mention that there exists an abundant body of literature devoted to constructing accurate finite element and related schemes for doubly-diffusive flows. Some recent contributions include variational multiscale stabilised schemes, least-squares methods, divergence-conforming mixed methods, volume-averaging discretisations, spectral elements, vorticity-based finite element formulations, and similar methods applied to, e.g., flows with heat and mass transport [1], reactive Boussinesq flows [2], nonlinear advection-reaction-diffusion in the context of bioconvective flows [10, 31], cross-diffusion and boundary layer effects in doubly-diffusive Navier-Stokes-Brinkman equations [18, 20] and in Darcy-Brinkman equations [39], or phase change models [21, 38, 40, 41]; where the list is far from exhaustive.

The solvability analysis for the continuous and discrete problems usually follows energy and fixed-point schemes as done for classical Boussinesq equations, and this is also the approach we follow here. The discretisation in space uses an interior penalty divergence-conforming method for the flow equations (in this case, Brezzi-Douglas-Marini (BDM) elements of degree $k \geq 1$ for the velocity and discontinuous elements of degree $k-1$ for the pressure following [11, 29]), combined with Lagrangian elements for the diffusive quantities, and the development stands as a natural extension of the formulation in [18] to the transient case. As such, it also features exactly divergence-free velocity approximations ensuring local conservativity and energy stability, and the error estimates of velocity are pressure-robust. The chosen time discretisation is the backward differentiation formula of degree 2 (BDF2), which for $k = 2$ gives a method of order 2 in space and time. Existence of discrete solutions is established by the Brouwer fixed-point theory similarly as in [18], and the error analysis in the semi-discrete and fully-discrete settings is adapted from the theory of [5] for the Boussinesq equations.

In many applications where double-diffusion effects occur, complicated flow patterns exist in zones far from boundary layers and sufficiently refined meshes are needed essentially in the whole spatial domain. However, for salinity-driven settling of solid particles that result in mathematical models such as (1.1), many of the flow features are clustered near zones of high-gradients of concentration, which is where the typical *plumes* are observed [19, 31]. This motivates the use of adaptive mesh refinement guided by *a posteriori* error indicators. For instance, in the context of phase change models there are some results based on error-related metric change [21, 34] and on goal-oriented adaptivity [41]. Regarding the design and rigorous analysis of residual-based *a posteriori* error estimators for flow-transport couplings, the literature is predominantly focused on the stationary case (see, e.g., [3, 6, 7, 9, 13, 22, 37, 42] and the references therein). Only a few results are available for the time-dependent regime, from which we mention the adaptive mixed method for Richards equation in porous media [15], the remeshing scheme based on goal-oriented adaptivity for solidification problems advanced in [14], the collection of adaptive schemes for reactive flow discussed in [16] and for heat transfer in [30]. However, none of these theoretical frameworks is directly applicable to (1.1) using divergence-conforming approximations.

The *a posteriori* error analysis we advance here is of residual type, and its analysis uses ideas from the abstract results related to spatial estimators for discontinuous Galerkin schemes applied to parabolic problems in [24]. The approach hinges on a decomposition of the discrete solution into a conforming and a non-conforming contribution, along with a reconstruction technique (see also [32]). This has also been exploited for the construction of *a posteriori* estimators of time-dependent Stokes and Navier-Stokes equations [12, 43]. Our *a posteriori* error analysis is divided into three parts. In first part, we present the error estimator for the steady coupled problem. In second part, we extend the *a posteriori* error estimation to the semi-discrete method, and finally we present the *a posteriori* error estimator for the unsteady coupled problem. We restrict that part of the analysis to the simpler backward Euler method. To the best of our knowledge, the *a posteriori* error estimation advanced in this paper is the first comprehensive study targeted for transient doubly-diffusive flows.

The remainder of this paper is organised as follows. In what is left of this section we outline the weak formulation of (1.1) and state the stability the continuous problem. In Section 2 we introduce the divergence-conforming method in fully discrete form, show existence of discrete solutions using fixed-point arguments, and rigorously establish *a priori* error estimates. Section 3 is devoted to the construction and analysis of efficiency and reliability for a residual-based *a posteriori* error estimator tailored for the stationary problem. In turn, these upper and lower bounds are used to establish properties of a second family of estimators for the transient case, and addressed in Sections 4 and 5. In Section 6 we collect numerical tests that verify the theoretical convergence rates predicted by the *a priori* error analysis, confirm the robustness of the proposed *a posteriori* error estimators, and illustrate the advantages of adaptive methods in the simulation of doubly-diffusive flows.

1.2. **Preliminaries.** Let Ω be an open and bounded domain in $\mathbb{R}^d$, $d = 2, 3$ with Lipschitz boundary $\Gamma = \partial\Omega$. We denote by $L^p(\Omega)$ and $W^{r,p}(\Omega)$ the usual Lebesgue and Sobolev spaces with respective norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{r,p}(\Omega)}$. If $p = 2$ we write $H^r(\Omega)$ in place of $W^{r,p}(\Omega)$, and denote the corresponding norm by $\|\cdot\|_{r,\Omega}$, ($\|\cdot\|_{0,\Omega}$

for $H^0(\Omega) = L^2(\Omega)$). The space $L^2_0(\Omega)$ denotes the restriction of $L^2(\Omega)$ to functions with zero mean value over Ω. For $r \geq 0$, we write the H^r-seminorm as $|\cdot|_{r,\Omega}$ and we denote by $(\cdot,\cdot)_\Omega$ the usual inner product in $L^2(\Omega)$. Spaces of vector-valued functions (in dimension d) are denoted in bold face, i.e., $\mathbf{H}^r(\Omega) = [H^r(\Omega)]^d$, and we use the vector-valued Hilbert spaces

$$\mathbf{H}(\mathrm{div};\Omega) := \big\{\boldsymbol{w} \in \mathbf{L}^2(\Omega) : \mathrm{div}\,\boldsymbol{w} \in L^2(\Omega)\big\},$$
$$\mathbf{H}_0(\mathrm{div};\Omega) := \big\{\boldsymbol{w} \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{w}\cdot\boldsymbol{n}_{\partial\Omega} = 0 \text{ on } \partial\Omega\big\},$$
$$\mathbf{H}_0(\mathrm{div}^0;\Omega) := \big\{\boldsymbol{w} \in \mathbf{H}_0(\mathrm{div};\Omega) : \mathrm{div}\,\boldsymbol{w} = 0 \text{ in } \Omega\big\},$$

where $\boldsymbol{n}_{\partial\Omega}$ denotes the outward normal on $\partial\Omega$; and we endow these spaces with the norm $\|\cdot\|_{\mathrm{div},\Omega}$ defined by $\|\boldsymbol{w}\|^2_{\mathrm{div},\Omega} := \|\boldsymbol{w}\|^2_{0,\Omega} + \|\mathrm{div}\,\boldsymbol{w}\|^2_{0,\Omega}$. We denote by $L^s(0,t_{\mathrm{end}};W^{m,p}(\Omega))$ the Banach space of all L^s-integrable functions from $[0,t_{\mathrm{end}}]$ into $W^{m,p}(\Omega)$, with norm

$$\|v\|_{L^s(0,t_{\mathrm{end}};W^{m,p}(\Omega))} = \begin{cases} \left(\displaystyle\int_0^{t_{\mathrm{end}}} \|v(t)\|^s_{W^{m,p}(\Omega)}\,\mathrm{d}t\right)^{1/s} & \text{if } 1 \leq s < \infty, \\ \mathrm{ess\,sup}_{t\in[0,t_{\mathrm{end}}]}\|v(t)\|_{W^{m,p}(\Omega)} & \text{if } s = \infty. \end{cases}$$

1.3. **Additional assumptions and weak formulation.** As in, e.g., [22], we assume that viscosity is a Lipschitz continuous and uniformly bounded function of concentration, i.e.,

$$\big|\nu(c_1) - \nu(c_2)\big| \leq L_\nu|c_1 - c_2| \quad \text{and} \quad \nu_1 \leq \nu(c) \leq \nu_2,$$

for any $c, c_1, c_2 \in \mathbb{R}$, and where L_ν, ν_1, ν_2 are positive constants.

For simplicity of notation in presenting the analysis we will restrict the weak form to the case of homogeneous Dirichlet boundary conditions for velocity, concentration, and salinity. Testing each equation in problem (1.1) against suitable functions and integrating by parts whenever adequate, gives the following weak formulation: For all $t \in (0,t_{\mathrm{end}}]$, find $(\boldsymbol{u},p,s,c) \in \mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)\times H^1_0(\Omega)\times H^1_0(\Omega)$ such that

$$(\partial_t\boldsymbol{u},\boldsymbol{v})_\Omega + a_1(c;\boldsymbol{u},\boldsymbol{v}) + c_1(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = F(s,c,\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbf{H}^1_0(\Omega), \tag{1.2a}$$
$$b(\boldsymbol{u},q) = 0 \quad \text{for all } q \in L^2_0(\Omega), \tag{1.2b}$$
$$(\partial_t s,\varphi)_\Omega + a_2(s,\varphi) + c_2(\boldsymbol{u};s,\varphi) = 0 \quad \text{for all } \varphi \in H^1_0(\Omega), \tag{1.2c}$$
$$(\partial_t c,\psi)_\Omega + \frac{1}{\tau}a_2(c,\psi) + c_2(\boldsymbol{u} - v_{\mathrm{p}}\boldsymbol{e}_z;c,\psi) = 0 \quad \text{for all } \psi \in H^1_0(\Omega), \tag{1.2d}$$

where the bilinear and trilinear forms $a_1 : H^1_0(\Omega)\times\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_0(\Omega) \to \mathbb{R}$, $a_2 : H^1_0(\Omega)\times H^1_0(\Omega) \to \mathbb{R}$, $b : \mathbf{H}^1_0(\Omega)\times L^2_0(\Omega) \to \mathbb{R}$, $c_1 : \mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_0(\Omega) \to \mathbb{R}$, $c_2 : \mathbf{H}^1_0(\Omega)\times H^1_0(\Omega)\times H^1_0(\Omega) \to \mathbb{R}$, as well as the linear functional $F : \mathbf{H}^1_0(\Omega) \to \mathbb{R}$, are defined as follows for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbf{H}^1_0(\Omega)$, $q \in L^2_0(\Omega)$, and $\varphi,\psi \in H^1_0(\Omega)$:

$$a_1(c;\boldsymbol{u},\boldsymbol{v}) := \big(\nu(c)\boldsymbol{\nabla}\boldsymbol{u},\boldsymbol{\nabla}\boldsymbol{v}\big)_\Omega, \quad c_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) := \big((\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{u},\boldsymbol{v}\big)_\Omega, \quad F(s,c,\boldsymbol{v}) = (\alpha s + \beta c)(\boldsymbol{g},\boldsymbol{v})_\Omega,$$
$$b(\boldsymbol{v},q) := \frac{1}{\rho_{\mathrm{m}}}(q,\mathrm{div}\,\boldsymbol{v})_\Omega, \quad a_2(\varphi,\psi) := \frac{1}{\mathrm{Sc}}(\nabla\varphi,\nabla\psi)_\Omega, \quad c_2(\boldsymbol{v};\varphi,\psi) := \big((\boldsymbol{v}\cdot\nabla)\varphi,\psi\big)_\Omega,$$

1.4. **Stability of the continuous problem.** We begin with he following auxiliary result

Lemma 1.1. *For $d = 2$ the following inequality holds:*

$$\|\boldsymbol{v}\|^2_{\mathbf{L}^4(\Omega)} \leq \sqrt{2}\|\boldsymbol{v}\|_{0,\Omega}|\boldsymbol{v}|_{1,\Omega}.$$

The variational forms defined above are continuous for all $\boldsymbol{u},\boldsymbol{v},\in \mathbf{H}^1_0(\Omega)$, $q \in L^2_0(\Omega)$, and $\varphi,\psi \in H^1_0(\Omega)$:

$$\big|a_1(\cdot,\boldsymbol{u},\boldsymbol{v})\big| \leq C_a\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega}, \qquad \big|a_2(\varphi,\psi)\big| \leq \hat{C}_a\|\varphi\|_{1,\Omega}\|\psi\|_{1,\Omega}, \tag{1.3a}$$
$$\big|b(\boldsymbol{v},q)\big| \leq C_b\|\boldsymbol{v}\|_{1,\Omega}\|q\|_{0,\Omega}, \tag{1.3b}$$
$$\big|c_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})\big| \leq C_c\|\boldsymbol{w}\|_{1,\Omega}\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega}, \qquad \big|c_2(\boldsymbol{u};\varphi,\psi)\big| \leq \hat{C}_c\|\boldsymbol{u}\|_{1,\Omega}\|\varphi\|_{1,\Omega}\|\psi\|_{1,\Omega}. \tag{1.3c}$$

We also recall (from [26, Chapter I, Lemma 3.1], for instance) the following Poincaré-Friedrichs inequality:

$$\|u\|_{0,\Omega} \leq C_p|u|_{1,\Omega} \qquad \text{for all } u \in H^1_0(\Omega). \tag{1.4}$$

Next, using (1.4) readily gives the coercivity of a_2 and also, for a fixed concentration, that of a_1, i.e.,

$$a_1(\cdot,\boldsymbol{v},\boldsymbol{v}) \geq \alpha_a\|\boldsymbol{v}\|^2_{1,\Omega} \qquad \text{for all } \boldsymbol{v} \in \mathbf{H}^1_0(\Omega), \tag{1.5a}$$
$$a_2(\varphi,\varphi) \geq \hat{\alpha}_a\|\varphi\|^2_{1,\Omega} \qquad \text{for all } \varphi \in H^1_0(\Omega). \tag{1.5b}$$

Using the definition and characterisation of the kernel of $b(\cdot,\cdot)$

$$\mathbf{X} := \big\{\boldsymbol{v} \in \mathbf{H}_0^1(\Omega) \,:\, b(\boldsymbol{v}, q) = 0 \;\forall q \in L_0^2(\Omega)\big\} = \big\{\boldsymbol{v} \in \mathbf{H}_0^1(\Omega) \,:\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\big\},$$

and using integration by parts we can readily observe that

$$c_1(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) = 0 \text{ and } c_2(\boldsymbol{w}; \varphi, \varphi) = 0 \qquad \text{for all } \boldsymbol{w} \in \mathbf{X}, \boldsymbol{v} \in \mathbf{H}^1(\Omega), \varphi \in H^1(\Omega). \tag{1.6}$$

It is well known that the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (see, e.g., [35]):

$$\sup_{\boldsymbol{v} \in \mathbf{H}_0^1(\Omega)\setminus\{\mathbf{0}\}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta \|q\|_{0,\Omega} \quad \text{for all } q \in L_0^2(\Omega).$$

Finally, for $\boldsymbol{v} \in \mathbf{W}^{1,\infty}(\Omega)$ and $\varphi \in W^{1,\infty}(\Omega)$ there exists an embedding constant $C_\infty > 0$ such that

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C_\infty \|\boldsymbol{v}\|_{\mathbf{W}^{1,\infty}(\Omega)} \quad \text{and} \quad \|\varphi\|_{1,\Omega} \leq C_\infty \|\varphi\|_{W^{1,\infty}(\Omega)}.$$

Lemma 1.2 (Stability)**.** *If $\boldsymbol{g} \in L^2(t, t_{\text{end}}; \mathbf{L}^2(\Omega))$, $\boldsymbol{u}_0 \in \mathbf{L}^2(\Omega)$ and $s_0, c_0 \in L^2(\Omega)$, then, for any solution $\boldsymbol{u}, s, c$ of (1.2) and for $t \in (0, t_{\text{end}}]$, there exists a constant $\gamma > 0$ such that*

$$\|\boldsymbol{u}\|_{L^2(0,t;\mathbf{H}^1(\Omega))} + \|s\|_{L^2(0,t;H^1(\Omega))} + \|c\|_{L^2(0,t;H^1(\Omega))} \leq \gamma\big(\|\boldsymbol{u}_0\|_{0,\Omega} + \|s_0\|_{0,\Omega} + \|c_0\|_{0,\Omega}\big),$$

where γ depends on $\eta_1, \mathrm{Sc}, \rho, \rho_{\mathrm{m}}, C_p$, $\|g\|_{\infty,\Omega}$, α and β.

Proof. We can take $\boldsymbol{u}$ on $\mathbf{X}$ and due to the inf-sup condition we can solve an equivalent reduced problem where $b(\cdot,\cdot)$ is removed from the variational form (1.2). Setting $\boldsymbol{v} = \boldsymbol{u}$ and using (1.6), (1.5a), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{0,\Omega}^2 + \alpha_a \|\boldsymbol{u}\|_{1,\Omega}^2 \leq \|\boldsymbol{g}\|_{\infty,\Omega} C\big(\|s\|_{0,\Omega} + \|c\|_{0,\Omega}\big)\|\boldsymbol{u}\|_{0,\Omega}.$$

Now we use Young's inequality with $\varepsilon = \alpha_a/4$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{0,\Omega}^2 + \frac{\alpha_a}{2}\|\boldsymbol{u}\|_{1,\Omega}^2 \leq \frac{C\|g\|_{\infty,\Omega}^2}{\alpha_a}\big(\|s\|_{0,\Omega}^2 + \|c\|_{0,\Omega}^2\big).$$

Integrating this equation between 0 and t yields, in particular

$$\|\boldsymbol{u}(\cdot,t)\|_{0,\Omega}^2 + \alpha_a \int_0^t \|\boldsymbol{u}(\cdot,z)\|_{1,\Omega}^2 \,\mathrm{d}z \leq \|\boldsymbol{u}(\cdot,0)\|_{0,\Omega} + \frac{C}{\alpha_a}\int_0^t \|c(\cdot,z)\|_{0,\Omega}^2 \,\mathrm{d}z + \frac{C}{\alpha_a}\int_0^t \|s(\cdot,z)\|_{0,\Omega}^2 \,\mathrm{d}z. \tag{1.7}$$

Analogously, using (1.5b) and (1.6) on (1.2c) and (1.2d), after integrating between 0 and t we find that

$$\|s(\cdot,t)\|_{0,\Omega}^2 + 2\hat{\alpha}_a \int_0^t \|s(\cdot,z)\|_{1,\Omega}^2 \,\mathrm{d}z \leq \|s(\cdot,0)\|_{0,\Omega}^2, \tag{1.8}$$

$$\|c(\cdot,t)\|_{0,\Omega}^2 + 2\hat{\alpha}_a \int_0^t \|c(\cdot,z)\|_{1,\Omega}^2 \,\mathrm{d}z \leq \|c(\cdot,0)\|_{0,\Omega}^2. \tag{1.9}$$

Finally, we derive the sought result from (1.7), (1.8) and (1.9). □

A problem similar to (1.2) has been studied in [4]. Assuming that F belongs to $L^2(0, t_{\text{end}}; \mathbf{H}^{-1}(\Omega))$, that the initial velocity $\boldsymbol{u}_0$ belongs to $\mathbf{L}^2(\Omega)$ and the initial data for the coupled species (s, c in the context of our problem) belongs to $L^2(\Omega)$, the authors showed existence of a solution by using the Galerkin method and applying the Cauchy-Lipschitz theorem, and proved its uniqueness in two dimensions. Such analysis can be applied to (1.2) by noting that F is a Lipschitz-continuous function, and assuming the initial data belongs to appropriate spaces. This is, however, not the focus of the paper.

2. Finite element discretisation and *a priori* error bounds

We discretise space by a family of regular partitions, denoted $\mathcal{T}_h$, of $\Omega \subset \mathbb{R}^d$ into simplices K (triangles in 2D or tetrahedra in 3D) of diameter h_K. We label by K^- and K^+ the elements adjacent to an edge, while h_e stands for the maximum diameter of the edge. If $\boldsymbol{v}$ and w are smooth vector and scalar fields defined on $\mathcal{T}_h$, then $(\boldsymbol{v}^\pm, w^\pm)$ denote the traces of $(\boldsymbol{v}, w)$ on e that are the extensions from the interior of K^+ and K^-, respectively. Let $\boldsymbol{n}_e$ denote the outward unit normal vector to e on K. The average $\{\!\{\cdot\}\!\}$ and jump $[\![\cdot]\!]$ operators are defined as

$$\{\!\{\boldsymbol{v}\}\!\} := (\boldsymbol{v}^- + \boldsymbol{v}^+)/2, \quad \{\!\{w\}\!\} := (w^- + w^+)/2, \quad [\![\boldsymbol{v}]\!] := (\boldsymbol{v}^- - \boldsymbol{v}^+), \quad [\![w]\!] := (w^- - w^+),$$

whereas for boundary jumps and averages we adopt the conventions $\{\!\{\boldsymbol{v}\}\!\} = [\![\boldsymbol{v}]\!] = \boldsymbol{v}$, and $\{\!\{w\}\!\} = [\![w]\!] = w$. In addition, ∇_h will denote the broken gradient operator.

For $k \geq 1$ and a mesh $\mathcal{T}_h$ on Ω, let us consider the discrete spaces (see e.g. [17])

$$\mathbf{V}_h \coloneqq \left\{ \boldsymbol{v}_h \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{v}_h|_K \in [\mathcal{P}_k(K)]^{\mathrm{d}} \quad \forall K \in \mathcal{T}_h \right\},$$
$$\mathcal{Q}_h \coloneqq \left\{ q_h \in L^2(\Omega) : q_h|_K \in \mathcal{P}_{k-1}(K) \quad \forall K \in \mathcal{T}_h \right\},$$
$$\mathcal{M}_h \coloneqq \left\{ s_h \in C(\bar{\Omega}) : l_h|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathcal{T}_h \right\}, \qquad \mathcal{M}_{h,0} \coloneqq \mathcal{M}_h \cap H_0^1(\Omega),$$

which, in particular, satisfy $\mathrm{div}\,\mathbf{V}_h \subset \mathcal{Q}_h$ (cf. [29]). Here $\mathcal{P}_k(K)$ denotes the local space spanned by polynomials of degree up to k and $\mathbf{V}_h$ is the space of divergence-conforming BDM elements. We then state the following semi-discrete Galerkin formulation for problem (1.2): Find $(\boldsymbol{u}_h, p_h, s_h, c_h) \in \mathbf{V}_h \times \mathcal{Q}_h \times \mathcal{M}_h \times \mathcal{M}_h$ such that:

$$\begin{aligned}
(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h)_\Omega + a_1^h(c_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + c_1^h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) &= F(s_h, c_h, \boldsymbol{v}_h) \quad \text{for all } \boldsymbol{v}_h \in \mathbf{V}_h,\\
b(\boldsymbol{u}_h, q_h) &= 0 \quad \text{for all } q_h \in \mathcal{Q}_h,\\
(\partial_t s_h, \varphi_h)_\Omega + a_2(s_h, \varphi_h) + c_2(\boldsymbol{u}_h; s_h, \varphi_h) &= 0 \quad \text{for all } \varphi_h \in \mathcal{M}_h,\\
(\partial_t c_h, \psi_h)_\Omega + \frac{1}{\tau} a_2(c_h, \psi_h) + c_2(\boldsymbol{u}_h - v_{\mathrm{p}} \boldsymbol{e}_z; c_h, \psi_h) &= 0 \quad \text{for all } \psi_h \in \mathcal{M}_h.
\end{aligned} \tag{2.1}$$

The discrete versions of the trilinear forms $a_1^h(\cdot;\cdot,\cdot)$ and $c_1^h(\cdot;\cdot,\cdot)$ are defined using a symmetric interior penalty and an upwind approach, respectively (see, e.g., [11, 29]):

$$\begin{aligned}
a_1^h(c_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \coloneqq \int_\Omega \nu(c_h) \boldsymbol{\nabla}_h(\boldsymbol{u}_h) : \boldsymbol{\nabla}_h(\boldsymbol{v}_h) + \sum_{e \in \mathcal{E}_h} \int_e \Big(&-\{\!\{\nu(c_h)\boldsymbol{\nabla}_h(\boldsymbol{u}_h)\boldsymbol{n}_e\}\!\} \cdot [\![\boldsymbol{v}_h]\!]\\
&- \{\!\{\nu(c_h)\boldsymbol{\nabla}_h(\boldsymbol{v}_h)\boldsymbol{n}_e\}\!\} \cdot [\![\boldsymbol{u}_h]\!] + \frac{a_0}{h_e}\nu(c_h)[\![\boldsymbol{u}_h]\!] \cdot [\![\boldsymbol{v}_h]\!]\Big),\\
c_1^h(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \coloneqq \int_\Omega (\boldsymbol{w}_h \cdot \nabla)\boldsymbol{u}_h \cdot \boldsymbol{v}_h + \sum_{K \in \mathcal{T}_h} \int_{\partial K \setminus \Gamma} &\hat{\boldsymbol{w}}_h^{\mathrm{up}}(\boldsymbol{u}_h) \cdot \boldsymbol{v}_h,
\end{aligned} \tag{2.2}$$

where the upwind flux is defined as $\hat{\boldsymbol{w}}_h^{\mathrm{up}}(\boldsymbol{u}_h) \coloneqq \frac{1}{2}(\boldsymbol{w}_h \cdot \boldsymbol{n}_k - |\boldsymbol{w}_h \cdot \boldsymbol{n}_K|)(\boldsymbol{u}_h^e - \boldsymbol{u}_h)$, and $\boldsymbol{u}_h^e$ is the trace of $\boldsymbol{u}_h$ taken from within the exterior of K.

We partition the interval $[0, t_{\mathrm{end}}]$ into N subintervals $[t_{n-1}, t_n]$ of length Δt. We use the implicit BDF2 scheme where all first-order time derivatives are approximated using the centred operator

$$\partial_t \boldsymbol{u}_h(t^{n+1}) \approx \frac{1}{\Delta t}\left(\frac{3}{2}\boldsymbol{u}_h^{n+1} - 2\boldsymbol{u}_h^n + \frac{1}{2}\boldsymbol{u}_h^{n-1}\right), \tag{2.3}$$

(similarly for $\partial_t c$) and for the first time step a first-order backward Euler method is used from t^0 to t^1, starting from the interpolates $\boldsymbol{u}_h^0, s_h^0$ of the initial data. In what follows, we define the difference operator

$$\mathcal{D}y^{n+1} \coloneqq 3y^{n+1} - 4y^n + y^{n-1}$$

for any quantity indexed by the time step n. For instance, (2.3) can be written as $\partial_t \boldsymbol{u}_h(t^{n+1}) \approx \frac{1}{2\Delta t}\mathcal{D}\boldsymbol{u}_h^{n+1}$.

The resulting set of nonlinear equations is solved by an iterative Newton-Raphson method with exact Jacobian. Hence the complete discrete system is given by

$$\begin{aligned}
\frac{1}{3}\left(\mathcal{D}\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right)_\Omega &= \frac{2}{3}\Delta t\big(-a_1^h(c_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) - c_1^h(\boldsymbol{u}_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^{n+1}) + F(s_h^{n+1}, c_h^{n+1}, \boldsymbol{v}_h)\big)\\
&\qquad \text{for all } \boldsymbol{v}_h \in \mathbf{V}_h,\\
b(\boldsymbol{u}_h^{n+1}, q_h) &= 0 \quad \text{for all } q_h \in \mathcal{Q}_h,\\
\frac{1}{3}\left(\mathcal{D}s_h^{n+1}, \varphi_h\right)_\Omega &= \frac{2}{3}\Delta t\big(-a_2(s_h^{n+1}, \varphi_h) - c_2(\boldsymbol{u}_h^{n+1}; s_h, \varphi_h)\big) \quad \text{for all } \varphi_h \in \mathcal{M}_h,\\
\frac{1}{3}\left(\mathcal{D}c_h^{n+1}, \psi_h\right)_\Omega &= \frac{2}{3}\Delta t\left(-\frac{1}{\tau}a_2(c_h^{n+1}, \psi_h) - c_2(\boldsymbol{u}_h^{n+1} - v_{\mathrm{p}}\boldsymbol{e}_z; c_h^{n+1}, \psi_h)\right) \quad \text{for all } \psi_h \in \mathcal{M}_h.
\end{aligned} \tag{2.4}$$

For the subsequent analysis, we introduce for $r \geq 0$ the broken $\mathbf{H}^r(\mathcal{T}_h)$ space

$$\mathbf{H}^r(\mathcal{T}_h) = \left\{ \boldsymbol{v} \in \mathbf{L}^2(\Omega) : \boldsymbol{v}|_K \in \mathbf{H}^r(K), K \in \mathcal{T}_h \right\},$$

as well as the following mesh-dependent broken norms

$$\|\boldsymbol{v}\|_{*,\mathcal{T}_h}^2 \coloneqq \sum_{K \in \mathcal{T}_h} \|\boldsymbol{\nabla}_h(\boldsymbol{v})\|_{\mathbf{L}^2(K)}^2 + \sum_{e \in \mathcal{E}_h} \frac{1}{h_e}\|[\![\boldsymbol{v}]\!]\|_{\mathbf{L}^2(e)}^2,$$

$$\|\boldsymbol{v}\|_{1,\mathcal{T}_h}^2 := \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\boldsymbol{v}\|_{*,\mathcal{T}_h}^2 \quad \text{for all } \boldsymbol{v} \in \mathbf{H}^1(\mathcal{T}_h), \quad \|\boldsymbol{v}\|_{2,\mathcal{T}_h}^2 := \|\boldsymbol{v}\|_{1,\mathcal{T}_h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |\boldsymbol{v}|_{H^2(K)}^2 \quad \text{for all } \boldsymbol{v} \in \mathbf{H}^2(\mathcal{T}_h),$$

where the stronger norm $\|\cdot\|_{2,\mathcal{T}_h}$ is used to show continuity. This norm is equivalent to $\|\cdot\|_{1,\mathcal{T}_h}$ on $\mathbf{H}^1(\mathcal{T}_h)$ (see [11]). Finally, adapting the argument used in [28, Proposition 4.5] we have the discrete Sobolev embedding: for $r = 2, 4$ there exists a constant $C_{\text{emb}} > 0$ such that

$$\|\boldsymbol{v}\|_{\mathbf{L}^r(\Omega)} \le C_{\text{emb}} \|\boldsymbol{v}\|_{1,\mathcal{T}_h} \qquad \text{for all } \boldsymbol{v} \in \mathbf{H}^1(\mathcal{T}_h). \tag{2.5}$$

Using these norms, we can establish continuity of the trilinear and bilinear forms involved. The proof follows from [11, Section 4].

Lemma 2.1. *The following properties hold:*

$$\begin{aligned} \left|a_1^h(\cdot,\boldsymbol{u},\boldsymbol{v})\right| &\le C\|\boldsymbol{u}\|_{2,\mathcal{T}_h}\|\boldsymbol{v}\|_{1,\mathcal{T}_h} && \text{for all } \boldsymbol{u} \in \mathbf{H}^2(\mathcal{T}_h),\ \boldsymbol{v} \in \mathbf{V}_h,\\ \left|a_1^h(\cdot,\boldsymbol{u},\boldsymbol{v})\right| &\le \tilde{C}_a\|\boldsymbol{u}\|_{1,\mathcal{T}_h}\|\boldsymbol{v}\|_{1,\mathcal{T}_h} && \text{for all } \boldsymbol{u},\boldsymbol{v} \in \mathbf{V}_h,\\ \left|b(\boldsymbol{v},q)\right| &\le \tilde{C}_b\|\boldsymbol{v}\|_{1,\mathcal{T}_h}\|q\|_{0,\Omega} && \text{for all } \boldsymbol{v} \in \mathbf{H}^1(\mathcal{T}_h),\ q \in L^2(\Omega), \end{aligned}$$

and for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbf{H}^1(\mathcal{T}_h)$ and $\varphi,\psi \in H^1(\Omega)$, there holds

$$\left|c_2(\boldsymbol{w};\varphi,\psi)\right| \le \tilde{C}\|\boldsymbol{w}\|_{1,\mathcal{T}_h}\|\varphi\|_{1,\Omega}\|\psi\|_{1,\Omega}. \tag{2.6a}$$

Moreover, for $c_1, c_2 \in H^1(\Omega)$, $\boldsymbol{u} \in \mathbf{C}^1(\mathcal{T}_h) \cap \mathbf{H}_0^1(\Omega)$ and $\boldsymbol{v} \in \mathbf{V}_h$, there holds

$$\left|a_1^h(c_1;\boldsymbol{u},\boldsymbol{v}) - a_1^h(c_2;\boldsymbol{u},\boldsymbol{v})\right| \le \tilde{C}_{\text{Lip}}\|c_1 - c_2\|_{1,\Omega}\|\boldsymbol{u}\|_{\mathbf{W}^{1,\infty}(\mathcal{T}_h)}\|\boldsymbol{v}\|_{1,\mathcal{T}_h}, \tag{2.7}$$

where the constant $\tilde{C}_{\text{Lip}} > 0$ is independent of h (cf. [18]). Note that while the coercivity of the form $a_2(\cdot,\cdot)$ in the discrete setting is readily implied by (1.5b), there also holds (cf. [29, Lemma 3.2])

$$a_1^h(\cdot,\boldsymbol{v},\boldsymbol{v}) \ge \tilde{\alpha}_a\|\boldsymbol{v}\|_{1,\mathcal{T}_h}^2 \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h, \tag{2.8}$$

provided that the stabilisation parameter $a_0 > 0$ in (2.2) is sufficiently large and independent of the mesh size.

Let $\boldsymbol{w} \in \mathbf{H}_0(\text{div}^0;\Omega)$, then due to the skew-symmetric form of the operators c_1^h and c_2, and the positivity of the non-linear upwind term of c_1^h, we can write

$$c_1^h(\boldsymbol{w};\boldsymbol{u},\boldsymbol{u}) \ge 0 \quad \text{for all } \boldsymbol{u} \in \mathbf{V}_h, \tag{2.9a}$$

$$c_2^h(\boldsymbol{w};\psi_h,\psi_h) = 0 \quad \text{for all } \psi_h \in \mathcal{M}_h, \tag{2.9b}$$

as well as the following relation (which is based on (2.5) and follows by the same method as in [28]): For any $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{u} \in \mathbf{H}^2(\mathcal{T}_h)$ there holds for all $\boldsymbol{v} \in \mathbf{V}_h$

$$\left|c_1^h(\boldsymbol{w}_1;\boldsymbol{u},\boldsymbol{v})\right| - \left|c_1^h(\boldsymbol{w}_2;\boldsymbol{u},\boldsymbol{v})\right| \le \tilde{C}_c\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{1,\mathcal{T}_h}\|\boldsymbol{v}\|_{1,\mathcal{T}_h}\|\boldsymbol{u}\|_{1,\mathcal{T}_h}. \tag{2.10}$$

We also have

$$F(\psi,\phi,\boldsymbol{v}) \le C_f\big(\|\psi\|_{0,\Omega} + \|\phi\|_{0,\Omega}\big)\|\boldsymbol{v}\|_{0,\Omega} \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h.$$

Finally, we recall from [29] the following discrete inf-sup condition for $b(\cdot,\cdot)$, where $\tilde{\beta}$ is independent of h:

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{1,\mathcal{T}_h}} \ge \tilde{\beta}\|q_h\|_{0,\Omega} \quad \text{for all } q_h \in \mathcal{Q}_h. \tag{2.11}$$

We will use the following algebraic relation: for any real numbers a^{n+1}, a^n, a^{n-1} and defining $\Lambda a^n := a^{n+1} - 2a^n + a^{n-1}$, we have

$$2(3a^{n+1} - 4a^n + a^{n-1}, a^n) = |a^{n+1}|^2 + |2a^{n+1} - a^n|^2 + |\Lambda a^n|^2 - |a^n|^2 - |2a^n - a^{n-1}|^2. \tag{2.12}$$

Theorem 2.1. *Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}) \in \mathbf{V}_h \times \mathcal{Q}_h \times \mathcal{M}_{h,0} \times \mathcal{M}_{h,0}$ be a solution of problem (2.4). Then the following bounds are satisfied, where C_1, C_2 and C_3 are constants that are independent of h and Δt:*

$$\begin{aligned}
&\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda \boldsymbol{u}_h^j\|_{0,\Omega}^2 + \sum_{j=1}^n \Delta t \|\boldsymbol{u}_h^{j+1}\|_{1,\mathcal{T}_h}^2 \\
&\quad \le C_1\big(\|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2 + \|c_h^1\|_{0,\Omega}^2 + \|2c_h^1 - c_h^0\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^1\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|_{0,\Omega}^2\big), \\
&\|s_h^{n+1}\|_{0,\Omega}^2 + \|2s_h^{n+1} - s_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda s_h^j\|_{0,\Omega}^2 + \sum_{j=1}^n \Delta t \|s_h^{j+1}\|_{1,\Omega}^2 \le C_2\big(\|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2\big), \\
&\|c_h^{n+1}\|_{0,\Omega}^2 + \|2c_h^{n+1} - c_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda c_h^j\|_{0,\Omega}^2 + \sum_{j=1}^n \Delta t \|c_h^{j+1}\|_{1,\Omega}^2 \le C_3\big(\|c_h^1\|_{0,\Omega}^2 + \|2c_h^1 - c_h^0\|_{0,\Omega}^2\big).
\end{aligned} \tag{2.13}$$

Proof. First we take $\boldsymbol{v}_h = 4\boldsymbol{u}_h^{n+1}$ and $q_h = 4p_h^{n+1}$ in the first and second equation of (2.4), respectively and apply (2.12), (2.8) and (2.9a) to deduce the estimate

$$\begin{aligned}
&\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{0,\Omega}^2 + \|\Lambda \boldsymbol{u}_h^n\|_{0,\Omega}^2 + 4\Delta t \tilde{\alpha}_a \|\boldsymbol{u}_h^{n+1}\|_{1,\mathcal{T}_h}^2 \\
&\quad \le 4\Delta t C_f (\|s_h^{n+1}\|_{0,\Omega} + \|c_h^{n+1}\|_{0,\Omega}) \|\boldsymbol{u}_h^{n+1}\|_{0,\Omega} + \|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{0,\Omega}^2.
\end{aligned}$$

Using Young's inequality with $\varepsilon = \tilde{\alpha}_a/2$ and summing over n we can assert that

$$\begin{aligned}
&\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda \boldsymbol{u}_h^j\|_{0,\Omega}^2 + 2\tilde{\alpha}_a \sum_{j=1}^n \Delta t \|\boldsymbol{u}_h^{j+1}\|_{1,\mathcal{T}_h}^2 \\
&\quad \le \frac{C}{\tilde{\alpha}_a} \sum_{j=1}^n \Delta t \|s_h^j\|_{0,\Omega}^2 + \frac{C}{\tilde{\alpha}_a} \sum_{j=1}^n \Delta t \|c_h^j\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^1\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|_{0,\Omega}^2.
\end{aligned} \tag{2.14}$$

Similarly in the third equation of (2.4), we take $\varphi_h = 4s_h^{n+1}$ and use property (2.9b) and relation (2.12) to deduce the inequality

$$\|s_h^{n+1}\|_{0,\Omega}^2 + \|2s_h^{n+1} - s_h^n\|_{0,\Omega}^2 + \|\Lambda s_h^n\|_{0,\Omega}^2 + 4\alpha_2 \Delta t \|s_h^{n+1}\|_{1,\Omega}^2 \le \|s_h^n\|_{0,\Omega}^2 + \|2s_h^n - s_h^{n-1}\|_{0,\Omega}^2.$$

Hence, summing over n, we get

$$\|s_h^{n+1}\|_{0,\Omega}^2 + \|2s_h^{n+1} - s_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda s_h^j\|_{0,\Omega}^2 + 4\hat{\alpha}_a \sum_{j=1}^n \Delta t \|s_h^{j+1}\|_{1,\Omega}^2 \le \|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2.$$

We proceed in the same way taking $\psi_h = 4c_h^{n+1}$ in the fourth equation of (2.4), to get the third result. We get the first result by substituting bounds for c_h and s_h into (2.14). □

Theorem 2.2 (Existence of discrete solutions)**.** *The problem (2.4) admits at least one solution*

$$(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}) \in \mathbf{V}_h \times \mathcal{Q}_h \times \mathcal{M}_{h,0} \times \mathcal{M}_{h,0}.$$

The proof of Theorem 2.2 makes use of Brouwer's fixed-point theorem in the following form (given by [25, Corollary 1.1, Chapter IV]):

Theorem 2.3 (Brouwer's fixed-point theorem)**.** *Let H be a finite-dimensional Hilbert space with scalar product $(\cdot,\cdot)_H$ and corresponding norm $\|\cdot\|_H$. Let $\Phi\colon H \to H$ be a continuous mapping for which there exists $\mu > 0$ such that $(\Phi(u), u)_H \ge 0$ for all $u \in H$ with $\|u\|_H = \mu$. Then there exists $u \in H$ such that $\Phi(u) = 0$ and $\|u\|_H \le \mu$.*

Proof of Theorem 2.2. To simplify the proof we introduce the following constants:

$$\begin{aligned}
C_u &:= C_1\big(\|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2 + \|c_h^1\|_{0,\Omega}^2 + \|2c_h^1 - c_h^0\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^1\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|_{0,\Omega}^2\big), \\
C_s &:= C_2\big(\|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2\big), \qquad C_c := C_3\big(\|c_h^1\|_{0,\Omega}^2 + \|2c_h^1 - c_h^0\|_{0,\Omega}^2\big).
\end{aligned}$$

We proceed by induction on $n \ge 2$. We define the mapping

$$\Phi : \mathbf{V}_h \times \mathcal{Q}_h \times \mathcal{M}_{h,0} \times \mathcal{M}_{h,0} \to \mathbf{V}_h \times \mathcal{Q}_h \mathcal{M}_{h,0} \times \mathcal{M}_{h,0}$$

using the relation

$$\begin{aligned}
&\big(\Phi(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}), (\boldsymbol{v}_h, q_h, \varphi_h, \psi_h)\big)_\Omega \\
&= \frac{(\mathcal{D}\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)_\Omega}{2\Delta t} + a_1^h\big(c_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\big) + c_1^h\big(\boldsymbol{u}_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\big) + b(\boldsymbol{v}_h, p_h^{n+1}) - b(\boldsymbol{u}_h^{n+1}, q_h) - F\big(s_h^{n+1}, c_h^{n+1}, \boldsymbol{v}_h\big)
\end{aligned}$$

$$+\frac{(\mathcal{D}s_h^{n+1},\varphi_h)_\Omega}{2\Delta t}+a_2\big(s_h^{n+1},\varphi_h\big)+c_2\big(\boldsymbol{u}_h^{n+1};s_h^{n+1},\varphi_h\big)+\frac{(\mathcal{D}c_h^{n+1},\psi_h)_\Omega}{2\Delta t}+\frac{a_2(c_h^{n+1},\psi_h)}{\tau}$$
$$+c_2\big(\boldsymbol{u}_h-v_{\mathrm{p}}\boldsymbol{e}_z;c_h^{n+1},\psi_h\big).$$

Note this map is well-defined and continuous on $\mathbf{V}_h\times\mathcal{Q}_h\times\mathcal{M}_{h,0}\times\mathcal{M}_{h,0}$. On the other hand, if we take

$$(\boldsymbol{v}_h,q_h,\varphi_h,\psi_h)=(\boldsymbol{u}_h^{n+1},p_h^{n+1},s_h^{n+1},c_h^{n+1}),$$

and employ (2.9a), (2.9b) and (2.8), we obtain

$$\begin{aligned}&\big(\Phi(\boldsymbol{u}_h^{n+1},p_h^{n+1},s_h^{n+1},c_h^{n+1}),(\boldsymbol{u}_h^{n+1},p_h^{n+1},s_h^{n+1},c_h^{n+1})\big)_\Omega\\&\geq-\frac{1}{2\Delta t}\|4\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\|_{0,\Omega}\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}+\tilde{\alpha}_a\|\boldsymbol{u}_h^{n+1}\|_{1,\mathcal{T}_h}^2-C_f\big(\|s_h^{n+1}\|_{0,\Omega}+\|c_h^{n+1}\|_{0,\Omega}\big)\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}\\&\quad-\frac{1}{2\Delta t}\|4s_h^n-s_h^{n-1}\|_{0,\Omega}\|s_h^{n+1}\|_{0,\Omega}+\hat{\alpha}_a\|s_h^{n+1}\|_{1,\Omega}^2-\frac{1}{2\Delta t}\|4c_h^n-c_h^{n-1}\|_{0,\Omega}\|c_h^{n+1}\|_{0,\Omega}+\frac{\hat{\alpha}_a}{\tau}\|c_h^{n+1}\|_{1,\Omega}^2.\end{aligned}$$

Next, using (2.13) we deduce that

$$\begin{aligned}&\big(\Phi(\boldsymbol{u}_h^{n+1},p_h^{n+1},s_h^{n+1},c_h^{n+1}),(\boldsymbol{u}_h^{n+1},p_h^{n+1},s_h^{n+1},c_h^{n+1})\big)_\Omega\\&\geq\frac{\tilde{\alpha}_a}{2}\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2+\hat{\alpha}_a\|s_h^{n+1}\|_{0,\Omega}^2+\frac{\hat{\alpha}_a}{\tau}\|c_h^{n+1}\|_{0,\Omega}^2-\frac{5}{2\Delta t}C_u\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}-C_f(C_u+C_c)^{1/2}\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}\\&\quad-\frac{5}{2\Delta t}C_s\|s_h^{n+1}\|_{0,\Omega}-\frac{5}{2\Delta t}C_c\|c_h^{n+1}\|_{0,\Omega}.\end{aligned}$$

Then, setting

$$C_R=\min\left\{\tilde{\alpha}_a,\frac{\hat{\alpha}_a}{\tau}\right\},\quad C_r=\sqrt{2}\max\left\{\frac{5}{2\Delta t}C_u,C_f(C_u+C_c)^{1/2},\frac{5}{2\Delta t}C_s,\frac{5}{2\Delta t}C_c\right\},$$

we may apply the inequality $a+b\leq\sqrt{2}(a^2+b^2)^{1/2}$, valid for all $a,b\in\mathbb{R}$, to obtain

$$\begin{aligned}&\big(\Phi(\boldsymbol{u}_h^{n+1},p_h^{n+1},s_h^{n+1},c_h^{n+1}),(\boldsymbol{u}_h^{n+1},p_h^{n+1},s_h^{n+1},c_h^{n+1})\big)_\Omega\\&\quad\geq C_R\big(\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2+\|s_h^{n+1}\|_{0,\Omega}^2+\|c_h^{n+1}\|_{0,\Omega}^2\big)-C_r\big(\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2+\|s_h^{n+1}\|_{0,\Omega}^2+\|c_h^{n+1}\|_{0,\Omega}^2\big)^{1/2}.\end{aligned}$$

Hence, the right-hand side is nonnegative on a sphere of radius $r:=C_r/C_R$. Consequently, by Theorem 2.3, there exists a solution to the fixed-point problem $\Phi(\boldsymbol{u}_h^{n+1},p_h^{n+1},s_h^{n+1},c_h^{n+1})=0$. □

Let us denote by $\mathcal{I}_h:C(\bar{\Omega})\to\mathcal{M}_h$ the nodal interpolator with respect to a unisolvent set of Lagrangian interpolation nodes associated with $\mathcal{M}_h$. $\Pi_h\,\boldsymbol{u}$ will denote the BDM projection of $\boldsymbol{u}$, and $\mathcal{L}_h\,p$ is the L^2-projection of p onto $\mathcal{Q}_h$. Under usual assumptions, the following approximation properties hold (see [29]):

$$\|\boldsymbol{u}-\Pi_h\,\boldsymbol{u}\|_{1,\mathcal{T}_h}\leq C^*h^{k+1}\|\boldsymbol{u}\|_{k+1,\Omega},\quad\|c-\mathcal{I}_h\,c\|_{1,\Omega}\leq C^*h^k\|c\|_{k+1,\Omega},\quad\|p-\mathcal{L}_h\,p\|_{0,\Omega}\leq C^*h^k\|p\|_{k,\Omega}.\tag{2.15}$$

The following development follows the structure adopted in [5].

Lemma 2.2. *Assume that $\boldsymbol{u}\in\mathbf{H}^2(\Omega)$, $p\in L^2(\Omega)$ and $s,c\in H^1(\Omega)$. Then we have*

$$(\partial_t\boldsymbol{u},\boldsymbol{v})_\Omega+a_1^h(c;\boldsymbol{u},\boldsymbol{v})+c_1^h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)=F(\boldsymbol{v})\quad\text{for all }\boldsymbol{v}\in\mathbf{V}_h,\tag{2.16a}$$
$$b(\boldsymbol{u},q)=0\quad\text{for all }q\in\mathcal{Q}_h,\tag{2.16b}$$
$$(\partial_t s,\varphi)_\Omega+a_2(s,\varphi)+c_2(\boldsymbol{u};s,\varphi)=0\quad\text{for all }\varphi\in\mathcal{M}_h,\tag{2.16c}$$
$$(\partial_t c,\psi)_\Omega+\frac{1}{\tau}a_2(c,\psi)+c_2(\boldsymbol{u}-v_{\mathrm{p}}\boldsymbol{e}_z;c,\psi)=0\quad\text{for all }\psi\in\mathcal{M}_h.\tag{2.16d}$$

Proof. Since we assume $\boldsymbol{u}\in\mathbf{H}^2(\Omega)$, integration by parts yields the required result. See also [11]. The third and fourth equations are a straightforward result from the continuous form. □

Now we decompose the errors as follows:

$$\boldsymbol{u}_h-\boldsymbol{u}=E_{\boldsymbol{u}}+\xi_{\boldsymbol{u}}=(\Pi_h\,\boldsymbol{u}-\boldsymbol{u})+(\boldsymbol{u}_h-\Pi_h\,\boldsymbol{u}),\quad p_h-p=E_p+\xi_p=(\mathcal{L}_h\,p-p)+(p_h-\mathcal{L}_h\,p),$$
$$s_h-s=E_s+\xi_s=(\mathcal{I}_h\,s-s)+(s_h-\mathcal{I}_h\,s),\quad c_h-c=E_c+\xi_c=(\mathcal{I}_h\,c-c)+(c_h-\mathcal{I}_h\,c).$$

Assuming that $\boldsymbol{u}_h^0=\Pi_h\,\boldsymbol{u}(0)$, $s_h^0=\mathcal{I}_h\,s(0)$ and $c_h^0=\mathcal{I}_h\,c(0)$, we will use also the notation $E_{\boldsymbol{u}}^n=(\boldsymbol{u}(t_n)-\Pi_h\,\boldsymbol{u}(t_n))$ and $\xi_{\boldsymbol{u}}^n=(\Pi_h\,\boldsymbol{u}(t_n)-\boldsymbol{u}_h^n)$, and similar notation for other variables. Since for the first time iteration of system (2.1) we adopt a backward Euler scheme, an error estimate is required for this step.

Theorem 2.4. *Let us assume that*

$$\boldsymbol{u} \in L^\infty(0,t_{\rm end};\mathbf{H}_0^{k+1}(\Omega)\cap \mathbf{C}^1(\mathcal{T}_h)),\quad \boldsymbol{u}' \in L^\infty(0,t_{\rm end};\mathbf{H}^k(\Omega)),\quad \boldsymbol{u}'' \in L^\infty(0,t_{\rm end};\mathbf{L}^2(\Omega)),$$
$$p \in L^\infty(0,t_{\rm end};H^2(\Omega)),\quad s \in L^\infty(0,t_{\rm end};H_0^{k+1}(\Omega)),\quad s' \in L^\infty(0,t_{\rm end};H^k(\Omega)),\quad s'' \in L^\infty(0,t_{\rm end};L^2(\Omega)),$$
$$c \in L^\infty(0,t_{\rm end};H_0^{k+1}(\Omega)),\quad c' \in L^\infty(0,t_{\rm end};H^k(\Omega)),\quad c'' \in L^\infty(0,t_{\rm end};L^2(\Omega)),$$

and also that $\|\boldsymbol{u}\|_{L^\infty(0,t_{\rm end};\mathbf{W}^{1,\infty}(\Omega))} < M$ *for a sufficiently small* $M>0$ *(a precise condition can be found in Theorem 2.5). Then there exist positive constants* C_u^1, C_s^1, C_c^1, *independently of* h *and* Δt, *such that*

$$\frac{1}{4}\|\xi_{\boldsymbol{u}}^1\|_{0,\Omega}^2 + \frac{1}{2}\Delta t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}\|_{1,\mathcal{T}_h}^2 \le C_u^1(h^{2k}+\Delta t^4),$$
$$\frac{1}{4}\|\xi_s^1\|_{0,\Omega}^2 + \frac{1}{2}\Delta t\hat{\alpha}_a\|\xi_s\|_{1,\Omega}^2 \le C_s^1(h^{2k}+\Delta t^4),$$
$$\frac{1}{4}\|\xi_c^1\|_{0,\Omega}^2 + \frac{1}{2}\Delta t\hat{\alpha}_a\|\xi_c^1\|_{0,\Omega}^2 \le C_c^1(h^{2k}+\Delta t^4).$$

Proof. As in the continuous case, we define the discrete kernel of the bilinear form $b(\cdot,\cdot)$ as

$$\mathbf{X}_h := \{\, \boldsymbol{v}_h \in \mathbf{V}_h : b(\boldsymbol{v}_h, q_h) = 0\ \forall q_h \in \mathcal{Q}_h \,\} = \{\, \boldsymbol{v}_h \in \mathbf{V}_h : \operatorname{div}\boldsymbol{v}_h = 0 \text{ in } \Omega \,\},$$

and relying on the inf-sup condition (2.11), we can continue with an equivalent discrete problem without pressure.

Taking into account the assumed regularity for $\boldsymbol{u}$, we have for all $\boldsymbol{x}$, a $\gamma(\boldsymbol{x}) \in (0,1)$ such that

$$\boldsymbol{u}(0) = \boldsymbol{u}(\Delta t) - \Delta t\boldsymbol{u}'(\Delta t) + \frac{1}{2}\Delta t^2\boldsymbol{u}''(\Delta t\gamma),$$

then $\boldsymbol{u}$ satisfies the following error equation

$$\begin{aligned}\|\xi_{\boldsymbol{u}}^1\|_{0,\Omega}^2 + \Delta t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}^1\|_{1,\mathcal{T}_h}^2 \le{}& \big(\Pi_h\,\boldsymbol{u}(\Delta t) - \boldsymbol{u}(\Delta t) + \boldsymbol{u}_h^0 - \boldsymbol{u}(0), \xi_{\boldsymbol{u}}^1\big)_\Omega \\ &+ \Delta t\big(a_1^h(c_h^1;\Pi_h\,\boldsymbol{u}(\Delta t),\xi_{\boldsymbol{u}}^1) - a_1^h(c^1;\boldsymbol{u}(\Delta t),\xi_{\boldsymbol{u}}^1)\big) \\ &- \Delta t\big(c_1^h(\boldsymbol{u}_h^1;\boldsymbol{u}_h^1,\xi_{\boldsymbol{u}}^1) - c_1^h(\boldsymbol{u}(\Delta t),\boldsymbol{u}(\Delta t),\xi_{\boldsymbol{u}}^1)\big) \\ &+ \Delta t\big(F(s_h^1,c_h^1,\xi_{\boldsymbol{u}}^1) - F(s(\Delta t),c(\Delta t),\xi_{\boldsymbol{u}}^1)\big),\end{aligned}$$

which results after choosing $\xi_{\boldsymbol{u}}^1$ as test function in the first equation of the reduced form of Lemma 2.2 and system (2.1), performing an Euler scheme step, subtracting both equations, and adding $\pm a_1^h(c_h^1;\Pi_h\,\boldsymbol{u}(\Delta t),\xi_{\boldsymbol{u}}^1)$. Now, invoking the approximation estimates (2.15), Young's inequality, and the stability properties, we get

$$\begin{aligned}\frac{1}{4}\|\xi_{\boldsymbol{u}}^1\|_{0,\Omega}^2 + \frac{1}{4}\Delta t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}^1\|_{1,\mathcal{T}_h}^2 \le{}& Ch^{2k}\Delta t\Big(\|\boldsymbol{u}(\Delta t)\|_{k+1,\Omega}^2 + \|\boldsymbol{u}(0)\|_{k+1,\Omega}^2 + \|c(\Delta t)\|_{k+1,\Omega}^2\Big) \\ &+ C\Delta t^4(\|\boldsymbol{u}''\|_{L^\infty(0,t_{\rm end};\mathbf{L}^2(\Omega))}^2) + \frac{4\tilde{C}_{\rm Lip}^2M^2}{\tilde{\alpha}_a}\Delta t\|\xi_c^1\|_{1,\Omega}^2 \\ &+ \Delta tC_f^2\|\xi_c^1\|_{1,\Omega}^2 + \Delta tC_f^2\|\xi_s^1\|_{1,\Omega}^2.\end{aligned} \tag{2.17}$$

Next, we choose ξ_s^1 as test function in (2.16c) and system (2.1); we follow the same steps as before, adding to the sum of both equations the term $\pm a_2(\mathcal{I}_h\,s^1,\xi_c^1)$ to obtain

$$\begin{aligned}\frac{1}{4}\|\xi_s^1\|_{0,\Omega}^2 + \frac{1}{2}\Delta t\hat{\alpha}_a\|\xi_s^1\|_{1,\Omega}^2 \le{}& C\Delta th^{2k}\Big(\|\boldsymbol{u}(\Delta t)\|_{k+1,\Omega}^2 + \|s(\Delta t)\|_{k+1,\Omega}^2 + \|s(0)\|_{k+1,\Omega}^2\Big) \\ &+ C\Delta t^4(\|s''\|_{L^\infty(0,t_{\rm end};L^2(\Omega))}^2) + \frac{4\tilde{C}_c^2}{\hat{\alpha}_a}\Delta t\|s\|_{L^\infty(0,t_{\rm end};H^1(\Omega))}^2\|\xi_{\boldsymbol{u}}\|_{1,\mathcal{T}_h}^2.\end{aligned} \tag{2.18}$$

In the same way, choosing ξ_c^1 as test function in (2.16d) and in (2.1) we obtain

$$\begin{aligned}\frac{1}{4}\|\xi_c^1\|_{0,\Omega}^2 + \frac{1}{2}\Delta t\hat{\alpha}_a\|\xi_c^1\|_{1,\Omega}^2 \le{}& C\Delta th^{2k}\Big(\|\boldsymbol{u}(\Delta t)\|_{k+1,\Omega}^2 + \|c(\Delta t)\|_{k+1,\Omega}^2 + \|c(0)\|_{k+1,\Omega}^2 + v_{\rm p}\Big) \\ &+ C\Delta t^4(\|c''\|_{L^\infty(0,t_{\rm end};L^2(\Omega))}^2) + \frac{4\tilde{C}_c^2}{\hat{\alpha}_a}\Delta t\|c\|_{L^\infty(0,t_{\rm end};H^1(\Omega))}^2\|\xi_{\boldsymbol{u}}\|_{1,\mathcal{T}_h}^2.\end{aligned} \tag{2.19}$$

Now, from (2.17) we deduce that

$$\Delta t\|\xi_{\boldsymbol{u}}\|_{1,\mathcal{T}_h}^2 \le C(h^{2k}+\Delta t^4) + \frac{16\tilde{C}_{\rm Lip}^2M^2}{\tilde{\alpha}_a^2}\Delta t\|\xi_c^1\|_{1,\Omega}^2 + \Delta t\frac{4C_f^2}{\tilde{\alpha}_a}\|\xi_c^1\|_{1,\Omega}^2 + \Delta t\frac{4C_f^2}{\tilde{\alpha}_a}\|\xi_s^1\|_{1,\Omega}^2. \tag{2.20}$$

We insert the previous identity into (2.19) and consider M and Δt sufficiently small such that the terms multiplying $\|\xi_c\|^2_{1,\Omega}$, can be absorbed into the left-hand side of the inequality, to get

$$\frac{1}{4}\|\xi_c^1\|^2_{0,\Omega} + \frac{1}{4}\Delta t\hat{\alpha}_a\|\xi_c\|^2_{1,\Omega} \le C_c^1(h^{2k} + \Delta t^4) + \Delta t\frac{4C_f^2}{\tilde{\alpha}_a}\|\xi_s^1\|^2_{1,\Omega}. \tag{2.21}$$

Substituting this result back into (2.20) and then into (2.18), get us the second estimate. The first estimate follows by directly substituting (2.21) into (2.17). □

Theorem 2.5. *Let $(\boldsymbol{u}, p, s, c)$ be the solution of (1.2) under the assumptions of Section 1.4, and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ be the solution of (2.4). Suppose that*

$$\boldsymbol{u} \in L^\infty(0, t_{\text{end}}; \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0^1(\Omega)), \quad c \in L^\infty(0, t_{\text{end}}; H^{k+1}(\Omega) \cap H_0^1(\Omega)),$$
$$\boldsymbol{u}' \in L^\infty(0, t_{\text{end}}; \mathbf{H}^k(\Omega)), \quad \boldsymbol{u}^{(3)} \in L^2(0, t_{\text{end}}; \mathbf{L}^2(\Omega))$$

and $\|\boldsymbol{u}\|_{L^\infty(0,t_{\text{end}};\mathbf{H}^1(\Omega))} < M$ for a sufficiently small constant $M > 0$. Then there exist positive constants $C, \gamma_1 \ge 0$ independent of h and Δt such that for all $m + 1 \le N$,

$$\|\xi_{\boldsymbol{u}}^{m+1}\|^2_{0,\Omega} + \|2\xi_{\boldsymbol{u}}^{m+1} - \xi_{\boldsymbol{u}}^m\|^2_{0,\Omega} + \sum_{n=1}^m \|\Lambda\xi_{\boldsymbol{u}}^n\|^2_{0,\Omega} + \sum_{n=1}^m \Delta t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{1,\mathcal{T}_h} \le C(\Delta t^4 + h^{2k}) + \sum_{n=1}^m \gamma_1\Delta t\|\xi_c^{n+1}\|^2_{1,\Omega}.$$

Proof. We appeal to the reduced form of the problem again, taking $\boldsymbol{u}_h \in \mathbf{X}_h$ and $\boldsymbol{u} \in \mathbf{X}$, then we choose as test function $\boldsymbol{v}_h = \xi_{\boldsymbol{u}}^{n+1}$ in the first equation of (2.4) and insert the terms

$$\pm\frac{(\mathcal{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t}, \quad \pm\frac{(\mathcal{D}\Pi_h\,\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t}, \quad \pm a_1^h\big(c_h^{n+1}; \Pi_h\,\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big)_\Omega.$$

Hence we get

$$\begin{aligned} &-\frac{(\mathcal{D}\xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t} - \frac{(\mathcal{D}E_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t} + \frac{(\mathcal{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t} + a_1^h\big(c_h^{n+1}; \xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big) \\ &+ a_1^h\big(c_h^{n+1}; \Pi_h\,\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big) + c_1^h\big(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big) = F\big(s_h^{n+1}, c_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big). \end{aligned} \tag{2.22}$$

We consider (2.16a) (see Lemma 2.2) at $t = t_{n+1}$ and $\boldsymbol{v} = \xi_{\boldsymbol{u}}^{n+1}$. Inserting the term $\pm(\mathcal{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega/(2\Delta t)$, we readily deduce the identity

$$\frac{(\mathcal{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_{1,\Omega}}{2\Delta t} + a_1^h\big(c^{n+1}; \boldsymbol{u}(t_n), \xi_{\boldsymbol{u}}^{n+1}\big) + c_1^h\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big) = F\big(s(t_{n+1}), c(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big). \tag{2.23}$$

We can then subtract (2.23) from (2.22) and multiply both sides by $4\Delta t$, yielding $I_1 + I_2 + \cdots + I_7 = 0$, with

$$\begin{aligned} &I_1 := 2\big(\mathcal{D}\xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big)_\Omega, \quad I_2 := 4\Delta t a_1^h(c_h^{n+1}; \xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_\Omega, \quad I_3 := 4\Delta t\left(\boldsymbol{u}'(t_{n+1}) - \frac{1}{2\Delta t}\mathcal{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\right)_\Omega, \\ &I_4 := 2\big(\mathcal{D}E_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big)_\Omega, \quad I_5 := 4\Delta t\left(a_1^h\big(c^{n+1}; \boldsymbol{u}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big) - a_1^h\big(c_h^{n+1}; \Pi_h\,\boldsymbol{u}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big)\right), \\ &I_6 := 4\Delta t\big(c_1^h\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big) - c_1^h(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1})\big), \\ &I_7 := 4\Delta t\big(F(s(t_{n+1}), c(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) - F(s_h^{n+1}, c_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1})\big). \end{aligned}$$

For the first term, using (2.12) we can assert that

$$I_1 = \|\xi_{\boldsymbol{u}}^{n+1}\|^2_{0,\Omega} + \|2\xi_{\boldsymbol{u}}^{n+1} - \xi_{\boldsymbol{u}}^n\|^2_{0,\Omega} + \|\Lambda\xi_{\boldsymbol{u}}^{n+1}\|^2_{0,\Omega} - \|\xi_{\boldsymbol{u}}^n\|^2_{0,\Omega} - \|2\xi_{\boldsymbol{u}}^n - \xi_{\boldsymbol{u}}^{n-1}\|^2_{0,\Omega}.$$

Using the ellipticity stated in (2.8), we readily get

$$I_2 \ge 4\Delta t\tilde{\alpha_a}\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{1,\mathcal{T}_h}.$$

By using Taylor's formula with integral remainder we have

$$\left|\boldsymbol{u}'(t_{n+1}) - \frac{1}{2\Delta t}\mathcal{D}\boldsymbol{u}(t_{n+1})\right| = \frac{\Delta t^{3/2}}{2\sqrt{3}}\|\boldsymbol{u}^{(3)}\|_{L^2(t^{n-1},t^{n+1};\mathbf{L}^2(\Omega))},$$

then by combining Cauchy-Schwarz and Young's inequality, we obtain the bound

$$|I_3| \le \frac{\Delta t^4}{24\varepsilon_1}\|\boldsymbol{u}^{(3)}\|^2_{L^2(t_{n-1},t_{n+1};\mathbf{L}^2(\Omega))} + \frac{\Delta t\varepsilon_1}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{1,\mathcal{T}_h}.$$

Now we insert $\pm 4\Delta t E'_{\boldsymbol{u}}(t_{n+1})$ into the fourth term, which leads to

$$I_4 = -4\Delta t\big(E'_{\boldsymbol{u}}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big)_\Omega + \left(E'_{\boldsymbol{u}}(t_{n+1}) - \frac{\mathcal{D}E_{\boldsymbol{u}}^{n+1}}{2\Delta t}, \xi_{\boldsymbol{u}}^{n+1}\right)_\Omega.$$

Proceeding as before and using (2.15) on the first term of I_4, we get

$$|I_4| \leq \frac{C}{2\varepsilon_2} h^{2k} \|\boldsymbol{u}'\|^2_{\mathbf{L}^\infty(0,t_{\text{end}};\mathbf{H}^k(\Omega))} + \frac{\Delta t \varepsilon_2}{2} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \frac{\Delta t^4 C}{2\varepsilon_3} \|\boldsymbol{u}^{(3)}\|^2_{L^2(0,t_{\text{end}};\mathbf{L}^2(\Omega))} + \frac{\Delta t \varepsilon_3}{2} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h}.$$

Again we insert $\pm a_1^h(\mathcal{I}_h\, c^{n+1}; \boldsymbol{u}(t_{n+1}), \xi^{n+1}_{\boldsymbol{u}})$ and $\pm a_1^h(c_h^{n+1}; \boldsymbol{u}(t_{n+1}), \xi^{n+1}_{\boldsymbol{u}})$. Then by (2.15), (2.7), Lemma 2.1 and Young's inequality we immediately have

$$|I_5| \leq \frac{8\tilde{C}^2_{\text{Lip}} M^2 C^* \Delta t}{\varepsilon_4} h^{2k} \|c\|^2_{L^\infty(0,t_{\text{end}};H^{k+1}(\Omega))} + \frac{\varepsilon_4}{2} \Delta t \|\xi^{n+1}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} + \frac{8\tilde{C}^2_{\text{Lip}} M^2 \Delta t}{\varepsilon_5} \|\xi^{n+1}_c\|^2_{1,\Omega} + \frac{\varepsilon_5}{2} \Delta t \|\xi^{n+1}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} + \frac{8\tilde{C}^2_a C^* \Delta t}{\varepsilon_6} h^{2k} \|c\|^2_{L^\infty(0,t_{\text{end}};H^{k+1}(\Omega))} + \frac{\varepsilon_6}{2} \Delta t \|\xi^{n+1}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h}.$$

Adding and subtracting suitable terms within I_6 yields

$$I_6 = \tilde{I}_6 - 4\delta t c_1^h\big(\boldsymbol{u}_h^{n+1}, \xi^{n+1}_{\boldsymbol{u}}, \xi^{n+1}_{\boldsymbol{u}}\big),$$

where we define

$$\begin{aligned}\tilde{I}_6 := -4\Delta t\big(&c_1^h(\boldsymbol{u}(t_{n+1}), \Pi_h\, \boldsymbol{u}(t_{n+1}), \xi^{n+1}_{\boldsymbol{u}}) - c_1^h(\Pi_h\, \boldsymbol{u}(t_{n+1}), \Pi_h\, \boldsymbol{u}(t_{n+1}), \xi^{n+1}_{\boldsymbol{u}}) \\ &+ c_1^h(\Pi_h\, \boldsymbol{u}(t_{n+1}), \Pi_h\, \boldsymbol{u}(t_{n+1}), \xi^{n+1}_{\boldsymbol{u}}) - c_1^h(\Pi_h\, \boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi^{n+1}_{\boldsymbol{u}}) \\ &+ c_1^h(\Pi_h\, \boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi^{n+1}_{\boldsymbol{u}}) - c_1^h(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi^{n+1}_{\boldsymbol{u}})\big).\end{aligned}$$

The bound (2.10) and (2.15) imply that

$$\begin{aligned}|\tilde{I}_6| &\leq 4\Delta t \tilde{C}_c\big(\|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} \|\Pi_h\, \boldsymbol{u}(t_{n+1})\|_{1,\mathcal{T}_h} + \|\Pi_h\, \boldsymbol{u}(t_{n+1})\|_{1,\mathcal{T}_h} \|E^{n+1}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} \|\xi^{n+1}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} \\ &\quad + \|E^{n+1}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} \|\boldsymbol{u}(t_{n+1})\|_{1,\mathcal{T}_h} \|\xi^{n+1}_{\boldsymbol{u}}\|_{1,\mathcal{T}_h}\big) \\ &\leq 4\Delta t \bigg(\tilde{C}_c C^* \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} \|\boldsymbol{u}\|^2_{L^\infty(0,t_{\text{end}};\mathbf{H}^1(\Omega))} \\ &\quad + \frac{2h^{2k} C \tilde{C}_c^2}{\varepsilon_7} \|\boldsymbol{u}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \|\boldsymbol{u}\|^2_{L^\infty(0,t_{\text{end}};\mathbf{H}^{k+1}_1(\Omega))} + \frac{\varepsilon_7}{8} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} \\ &\quad + \frac{2Ch^{2k} \tilde{C}_c^2}{\varepsilon_8} \|\boldsymbol{u}\|^2_{L^\infty(0,t_{\text{end}};\mathbf{H}^{k+1}(\Omega))} \|\boldsymbol{u}\|^2_{L^\infty(0,t_{\text{end}};\mathbf{H}^1(\Omega))} + \frac{\varepsilon_8}{8} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h}\bigg) \\ &\leq 4\Delta t \bigg(C^* \tilde{C}_c M \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \frac{h^{2k} C}{2\varepsilon_7} \|\boldsymbol{u}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \|\boldsymbol{u}\|^2_{L^\infty(0,t_{\text{end}};\mathbf{H}^{k+1}(\Omega))} \\ &\quad + \frac{\varepsilon_7}{8} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \frac{2Ch^{2k}}{\varepsilon_8} \|\boldsymbol{u}\|^2_{L^\infty(0,t_{\text{end}};\mathbf{H}^{k+1}(\Omega))} \|\boldsymbol{u}\|^2_{L^\infty(0,t_{\text{end}};\mathbf{H}^1(\Omega))} + \frac{\varepsilon_8}{8} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h}\bigg),\end{aligned}$$

where C^* is a positive constant coming from (2.15). We also have

$$\begin{aligned}|\tilde{I}_7| &\leq 4\Delta t \bigg(\frac{2C_f^2}{\varepsilon_9}(\|\xi^{n+1}_s\|^2_{0,\Omega} + \|E^{n+1}_s\|^2_{0,\Omega}) + \frac{2C_f^2}{\varepsilon_{10}}(\|\xi^{n+1}_c\|^2_{0,\Omega} + \|E^{n+1}_c\|^2_{0,\Omega}) \\ &\quad + \frac{\varepsilon_9}{8} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \frac{\varepsilon_{10}}{8} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h}\bigg) \\ &\leq 4\Delta t \bigg(\frac{2C_f^2}{\varepsilon_9}(\|\xi^{n+1}_s\|^2_{0,\Omega} + C^* h^{2k} \|s\|^2_{L^\infty(0,t_{\text{end}},H^{k+1}(\Omega)})) + \frac{2C_f^2}{\varepsilon_{10}}(\|\xi^{n+1}_c\|^2_{0,\Omega} + +C^* h^{2k} \|c\|^2_{L^\infty(0,t_{\text{end}},H^{k+1}(\Omega))}) \\ &\quad + \frac{\varepsilon_9}{8} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \frac{\varepsilon_{10}}{8} \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h}\bigg).\end{aligned}$$

Hence, by choosing $\varepsilon_i = 3\tilde{\alpha}_a/5$ for $i = 1, \ldots, 10$, collecting the above estimates, and summing over $1 \leq n \leq m$ for all $m + 1 \leq N$, we get

$$\begin{aligned}&\|\xi^{m+1}_{\boldsymbol{u}}\|^2_{0,\Omega} + \|2\xi^{m+1}_{\boldsymbol{u}} - \xi^m_{\boldsymbol{u}}\|^2_{0,\Omega} + \sum_{n=1}^m \|\Lambda \xi^n_{\boldsymbol{u}}\|^2_{0,\Omega} - 3\|\xi^1_{\boldsymbol{u}}\|^2_{0,\Omega} \\ &\quad + \sum_{n=1}^m \Delta t \tilde{\alpha}_a \|\xi^{n+1}_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} \leq C(\Delta t^4 + h^{2k}) + \frac{16 M^2 C_{\text{Lip}^2} \Delta t}{\tilde{\alpha}_a} \sum_{n=1}^m \|\xi^{n+1}_c\|^2_{1,\Omega},\end{aligned}$$

where

$$M \le \min\left\{\frac{\tilde{\alpha}_a}{4\tilde{C}_c C^*}, \frac{\sqrt{\hat{\alpha}_a}}{4C_{\mathrm{Lip}}\sqrt{2}}\right\}, \quad \gamma_1 = \frac{16M^2 C_{\mathrm{Lip}^2}}{\tilde{\alpha}_a} \le \frac{\hat{\alpha}_a\tilde{\alpha}_a}{2}.$$

Finally, Theorem 2.4 yields the desired result. □

Theorem 2.6. *Let $(\boldsymbol{u}, p, s, c)$ be the solution of (1.2) under the assumptions of Section 1.4, and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ be the solution of (2.4). If*

$$\boldsymbol{u} \in L^\infty(0, t_{\mathrm{end}}; \mathbf{H}_1^{k+1}(\Omega) \cap \mathbf{H}_0^1(\Omega)), \quad s \in L^\infty(0, t_{\mathrm{end}}; H^{k+1}(\Omega) \cap H_0^1(\Omega)),$$
$$s' \in L^\infty(0, t_{\mathrm{end}}; H^k(\Omega)), \quad s^{(3)} \in L^2(0, t_{\mathrm{end}}; L^2(\Omega)),$$

then there exist constants $C, \gamma_2 > 0$, independent of h and Δt, such that for all $m+1 \le N$

$$\|\xi_s^{m+1}\|_{0,\Omega}^2 + \|2\xi_s^{m+1} - \xi_s^m\|_{0,\Omega}^2 + \sum_{n=1}^m \|\Lambda\xi_s^{n+1}\|_{0,\Omega}^2 + \sum_{n=1}^m \Delta t\hat{\alpha}_a\|\xi_s^{n+1}\|_{1,\Omega}^2$$
$$\le C(\Delta t^4 + h^{2k}) + \sum_{n=1}^m \gamma_2 \Delta t \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathcal{T}_h}^2.$$

Proof. Proceeding similarly as for Theorem 2.5, we choose as test function $\varphi_h = \xi_s^{n+1}$ in the second equation of (2.4) and insert suitable additional terms to obtain the following identity (analogous to (2.22))

$$\begin{aligned} &-\frac{(\mathcal{D}\xi_s^{n+1}, \xi_s^{n+1})_\Omega}{2\Delta t} - \frac{(\mathcal{D}E_s^{n+1}, \xi_s^{n+1})_\Omega}{2\Delta t} - a_2(\xi_s^{n+1}, \xi_s^{n+1}) + \frac{(\mathcal{D}s(t_{n+1}), \xi_s^{n+1})_\Omega}{2\Delta t} \\ &+ a_2(\mathcal{I}_h\, s(t_{n+1}), \xi_s^{n+1}) + c_2(\boldsymbol{u}_h^{n+1}, s_h^{n+1}, \xi_s^{n+1}) = 0. \end{aligned} \tag{2.24}$$

From (2.16b), focusing on $t = t_{n+1}$, using $\varphi = \xi_s^{n+1}$ and proceeding as in the derivation of (2.23), we obtain

$$\frac{(\mathcal{D}s(t_{n+1}), \xi_s^{n+1})_\Omega}{2\Delta t} + a_2\big(s(t_{n+1}), \xi_s^{n+1}\big) + c_2\big(\boldsymbol{u}(t_{n+1}), s(t_{n+1}), \xi_s^{n+1}\big) = -\left(s'(t_{n+1}) - \frac{\mathcal{D}s(t_{n+1})}{2\Delta t}, \xi_s^{n+1}\right)_\Omega. \tag{2.25}$$

Next we subtract (2.25) from (2.24) and multiply both sides by $4\Delta t$. This yields $\hat{I}_1 + \cdots + \hat{I}_6 = 0$, where

$$\hat{I}_1 := 2(\mathcal{D}\xi_s^{n+1}, \xi_s^{n+1})_\Omega, \quad \hat{I}_2 := 4\Delta t a_2(\xi_s^{n+1}, \xi_s^{n+1}), \quad \hat{I}_3 := 4\Delta t\left(s'(t_{n+1}) - \frac{\mathcal{D}s(t_{n+1})}{2\Delta t}, \xi_s^{n+1}\right)_\Omega,$$
$$\hat{I}_4 := 2\big(\mathcal{D}E_s^{n+1}, \xi_s^{n+1}\big)_\Omega, \quad \hat{I}_5 := 4\Delta t a_2\big(E_s^{n+1}, \xi_s^{n+1}\big),$$
$$\hat{I}_6 := 4\Delta t\big(c_2\big(\boldsymbol{u}_h^{n+1}, s_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big) - c_2\big(\boldsymbol{u}(t_{n+1}), s(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big)\big).$$

For $\hat{I}_1$, $\hat{I}_2$ and $\hat{I}_3$ we use (2.12), (1.5b), and Taylor expansion along with Young's inequality, respectively, to obtain

$$\hat{I}_1 = \|\xi_s^{n+1}\|_{0,\Omega}^2 + \|2\xi_s^{n+1} - \xi_s^n\|_{0,\Omega}^2 + \|\Lambda\xi_s^{n+1}\|_{0,\Omega}^2 - \|\xi_s^n\|_{0,\Omega}^2 - \|2\xi_s^n - \xi_s^{n-1}\|_{0,\Omega}^2,$$
$$\hat{I}_2 \ge 4\Delta t\hat{\alpha}_a\|\xi_s^{n+1}\|_{1,\Omega}^2, \qquad |\hat{I}_3| \le \frac{\Delta t^4}{24\varepsilon_1}\|s^{(3)}\|_{L^2(t_{n-1},t_{n+1};L^2(\Omega))}^2 + \frac{\Delta t\varepsilon_1}{2}\|\xi_s^{n+1}\|_{1,\Omega}^2.$$

Inserting $\pm 4\Delta t E_s'(t_{n+1})$ into $\hat{I}_4$ and using (2.15) leads to the bound

$$|\hat{I}_4| \le \frac{C}{2\varepsilon_2}h^{2k}\|s'\|_{L^\infty(0,t_{\mathrm{end}};H^k(\Omega))}^2 + \frac{\Delta t\varepsilon_2}{2}\|\xi_s^{n+1}\|_{1,\Omega}^2 + \frac{\Delta t^4 C}{2\varepsilon_3}\|s^{(3)}\|_{L^2(0,t_{\mathrm{end}};L^2(\Omega))}^2 + \frac{\Delta t\varepsilon_3}{2}\|\xi_s^{n+1}\|_{1,\Omega}^2.$$

Employing again (2.15) in combination with (1.3a) we have

$$|\hat{I}_5| \le \frac{8\hat{C}_a^2\Delta t}{\varepsilon_4}h^{2k}\|s\|_{L^\infty(0,t_{\mathrm{end}};H^{k+1}(\Omega))}^2 + \frac{\Delta t\varepsilon_4}{2}\|\xi_s^{n+1}\|_{1,\Omega}^2.$$

In order to derive a bound for $\hat{I}_6$ we proceed as for the bound on I_7 in the proof of Theorem 2.5; namely adding and subtracting suitable terms in the definition of $\hat{I}_6$, defining $\tilde{I}_6$ in this case by

$$\hat{I}_6 = \tilde{I}_6 + 4\delta_t c_2\big(\boldsymbol{u}_h^{n+1}, \xi_s^{n+1}, \xi_s^{n+1}\big),$$

and applying (2.9b), (2.6a), (2.15) and Young's Inequality to the result, we get

$$\begin{aligned} |\tilde{I}_6| \le 4\Delta t\Bigg(&\frac{2\tilde{C}_c^2 C^{*2}}{\varepsilon_5}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathcal{T}_h}^2\|s\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}^2 + \frac{1}{8\varepsilon_5}\|\xi_s\|_{1,\Omega}^2 \\ &+ \frac{2h^{2k}\tilde{C}_c^2 C^{*2}}{\varepsilon_6}\|\boldsymbol{u}\|_{L^\infty(0,T;H^1(\Omega))}^2\|s\|_{L^\infty(0,t_{\mathrm{end}};H^{k+1}(\Omega))}^2 + \frac{\varepsilon_6}{8}\|\xi_s^{n+1}\|_{1,\Omega}^2 \end{aligned}$$

$$+\frac{2h^{2k}\tilde{C}^2C^{*2}}{\varepsilon_7}\|\boldsymbol{u}\|^2_{L^\infty(0,t_{\rm end};\mathbf{H}^{k+1}(\Omega))}\|s\|^2_{L^\infty(0,t_{\rm end};H^1(\Omega))}+\frac{\varepsilon_7}{8}\|\xi_s^{n+1}\|^2_{1,\Omega}\Big).$$

In this manner, and after choosing $\varepsilon_i = 6\hat{\alpha}_a/7$ for $i = 1,\dots,7$, we can collect the above estimates and sum over $1 \le n \le m$, for all $m+1 \le N$, to get

$$\|\xi_s^{m+1}\|^2_{0,\Omega}+\|2\xi_s^{m+1}-\xi_s^m\|^2_{0,\Omega}+\sum_{n=1}^m\|\Lambda\xi_s^n\|^2_{0,\Omega}+\sum_{n=1}^m\Delta t\hat{\alpha}_a\|\xi_s^{n+1}\|^2_{1,\Omega}-3\|\xi_s^1\|^2_{0,\Omega}$$
$$\le C(\Delta t^4+h^{2k})+\gamma_2\Delta t\sum_{n=1}^m\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{1,\mathcal{T}_h},\quad \text{where}\quad \gamma_2 := \frac{28\tilde{C}_c^2C^{*2}}{3\hat{\alpha}_a}\|s\|^2_{L^\infty(0,t_{\rm end};H^1(\Omega))}.$$

This concludes the proof. □

Theorem 2.7. *Let $(\boldsymbol{u},p,s,c)$ be the solution of (1.2) under the assumptions of Section 1.4, and $(\boldsymbol{u}_h,p_h,s_h,c_h)$ be the solution of (2.4). If*

$$\boldsymbol{u}\in L^\infty(0,t_{\rm end};\mathbf{H}^{k+1}(\Omega)\cap\mathbf{H}^1_0(\Omega)),\quad c\in L^\infty(0,t_{\rm end};H^{k+1}(\Omega)\cap H^1_0(\Omega)),$$
$$c'\in L^\infty(0,t_{\rm end};H^k(\Omega)),\quad c^{(3)}\in L^2(0,t_{\rm end};L^2(\Omega)),$$

then there exist constants $C,\gamma_3>0$ that are independent of h and Δt, such that for all $m+1\le N$

$$\|\xi_c^{m+1}\|^2_{0,\Omega}+\|2\xi_c^{m+1}-\xi_c^m\|^2_{0,\Omega}+\sum_{n=1}^m\|\Lambda\xi_c^{n+1}\|^2_{0,\Omega}+\sum_{n=1}^m\Delta t\hat{\alpha}_a\|\xi_c^{n+1}\|^2_{1,\Omega}$$
$$\le C(\Delta t^4+h^{2k})+\sum_{n=1}^m\gamma_3\Delta t\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{1,\mathcal{T}_h}.$$

Proof. It follows along the same lines of the proof of Theorem 2.6, with constant γ_3 given by

$$\gamma_3=\frac{28\tilde{C}_c^2C^{*2}}{3\hat{\alpha}_a}\|c\|^2_{L^\infty(0,t_{\rm end};H^1(\Omega))}.$$

□

Theorem 2.8. *Under the same assumptions of Theorems 2.5-2.7, there exist positive constants $\hat{\gamma}_u$, γ_s and γ_c independent of Δt and h, such that, for a sufficiently small Δt and all $m+1\le N$, there hold*

$$\left(\|\xi_{\boldsymbol{u}}^{m+1}\|^2_{0,\Omega}+\|2\xi_{\boldsymbol{u}}^{m+1}-\xi_{\boldsymbol{u}}^m\|^2_{0,\Omega}+\sum_{n=1}^m\big(\|\Lambda\xi_{\boldsymbol{u}}^n\|^2_{0,\Omega}+\Delta t\tilde{\alpha}_a\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{1,\mathcal{T}_h}\big)\right)^{1/2}\le\gamma_u(\Delta t^2+h^k),$$
$$\left(\|\xi_s^{m+1}\|^2_{0,\Omega}+\|2\xi_s^{m+1}-\xi_s^m\|^2_{0,\Omega}+\sum_{n=1}^m\big(\|\Lambda\xi_s^n\|^2_{0,\Omega}+\Delta t\hat{\alpha}_a\|\xi_s^{n+1}\|^2_{1,\Omega}\big)\right)^{1/2}\le\gamma_s(\Delta t^2+h^k),$$
$$\left(\|\xi_c^{m+1}\|^2_{0,\Omega}+\|2\xi_c^{m+1}-\xi_s^m\|^2_{0,\Omega}+\sum_{n=1}^m\big(\|\Lambda\xi_c^n\|^2_{0,\Omega}+\Delta t\hat{\alpha}_a\|\xi_c^{n+1}\|^2_{1,\Omega}\big)\right)^{1/2}\le\gamma_c(\Delta t^2+h^k).$$

Proof. From Theorems 2.5 and 2.7, since $\gamma_1\le\frac{\hat{\alpha}_a\tilde{\alpha}_a}{2}$ we have the estimate

$$\sum_{n=1}^m\Delta t\|\xi_{\boldsymbol{u}}^{n+1}\|^2_{1,\mathcal{T}_h}\le C(\Delta t^4+h^{2k})+\sum_{n=1}^m\Delta t\frac{\hat{\alpha}_a}{2}\|\xi_c^{n+1}\|^2_{1,\mathcal{T}_h},$$

which, after substitution back into Theorem 2.7, yields

$$\sum_{n=1}^m\Delta t\|\xi_c^{n+1}\|^2_{1,\Omega}\le C(\Delta t^4+h^{2k}).\tag{2.26}$$

The first bound follow by combining (2.26) and Theorem 2.5. The second and third bounds follow directly from the first bound and Theorems 2.6 and 2.7. □

Lemma 2.3. *Under the same assumptions of Theorem 2.8, we have*

$$\left(\sum_{n=1}^m\Delta t\|p(t_{n+1})-p_h^{n+1}\|^2_{0,\Omega}\right)^{1/2}\le\hat{\gamma}_p(\Delta t^2+h^k).$$

Proof. Owing to the inf-sup condition (2.11), there exists $\boldsymbol{w}_h \in \mathbf{X}_h^\perp$ such that

$$b\big(\boldsymbol{w}_h, p(t_{n+1}) - p_h^{n+1}\big) = \|p(t_{n+1}) - p_h^{n+1}\|_{0,\Omega}^2, \quad \|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} \leq \frac{1}{\tilde{\beta}}\|p(t_{n+1}) - p_h^{n+1}\|_{0,\Omega}. \tag{2.27}$$

From (2.4) and Lemma 2.2, proceeding as in the proof of Theorem 2.5, we obtain

$$\begin{aligned}
&\Delta t b(\boldsymbol{w}_h, p(t_{n+1}) - p_h^{n+1}) \\
&= -\Delta t \left(\boldsymbol{u}'(t_{n+1}) - \frac{1}{2\Delta t}\mathcal{D}\boldsymbol{u}_h^{n+1}, \boldsymbol{w}_h\right)_\Omega + \Delta t\left(a_1^h(c_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{w}_h) - a_1^h(c(t_{n+1}); \boldsymbol{u}(t_{n+1}), \boldsymbol{w}_h)\right) \\
&\quad + \Delta t\big(c_1^h(\boldsymbol{u}_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{w}_h) - c_1^h(\boldsymbol{u}(t_{n+1}); \boldsymbol{u}(t_{n+1}), \boldsymbol{w}_h)\big) \\
&\quad + \Delta t\big(F(s_h^{n+1}, c_h^{n+1}, \boldsymbol{w}_h) - F(s(t_{n+1}), c(t_{n+1}), \boldsymbol{w}_h)\big) \\
&\leq \frac{\Delta t^2}{2\sqrt{3}}\|\boldsymbol{u}^{(3)}\|_{L^2(t_{n-1},t_{n+1},\mathbf{L}^2(\Omega))}\sqrt{\Delta t}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} + \Delta t C_f\|\xi_s^{n+1}\|_{0,\Omega}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} \\
&\quad + \tilde{C}_a C^* h^k \Delta t\|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^{k+1}(\Omega))}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} + \tilde{C}_{\mathrm{lip}} M\Delta t\|\xi_c^{n+1}\|_{1,\Omega}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} \\
&\quad + \tilde{C}_{\mathrm{lip}}\Delta t M\|\xi_c^{n+1}\|_{1,\Omega}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} + \Delta t C\tilde{C}_c C^* C_\infty M\|\xi_c^{n+1}\|_{1,\Omega}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} \\
&\quad + 2\Delta t C^*\tilde{C}_c h^k\|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^1(\Omega))}\|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}},\mathbf{H}^{k+1}(\Omega))}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} + \Delta t C_f\|\xi_c^{n+1}\|_{0,\Omega}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} \\
&\quad + \Delta t C^* C_f h^k\|s\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h} + \Delta t C^* C_f h^k\|c\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}\|\boldsymbol{w}_h\|_{1,\mathcal{T}_h}.
\end{aligned}$$

Summing over $1 \leq n \leq m$ for all $m+1 \leq N$ and substituting back into (2.27), we obtain

$$\left(\sum_{n=1}^m \Delta t\|p(t_{n+1}) - p_h^{n+1}\|_{0,\Omega}^2\right)^{1/2} \leq \frac{C}{\tilde{\beta}}\left(\Delta t^2 + h^k + \left(\sum_{n=1}^m \Delta t\|\xi_c^{n+1}\|_{0,\Omega}^2\right)^{1/2} + \left(\sum_{n=1}^m \Delta t\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathcal{T}_h}^2\right)^{1/2}\right),$$

and the desired result readily follows from Theorem 2.8. □

We next proceed to derive and analyse *a posteriori* error estimators. We split the presentation into three cases of increasing complexity, starting with an estimator focusing on the steady coupled problem.

3. *A posteriori* error estimation for the stationary problem

Let us define the following nonlinear coupled problem in weak form, associated with the stationary version of the model equations. Find $(\boldsymbol{u}, p, s, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$a_1(c, \boldsymbol{u}, \boldsymbol{v}) + c_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbf{H}_0^1(\Omega) \tag{3.1a}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{3.1b}$$

$$a_2(s, \phi) + c_2(\boldsymbol{u}; s, \phi) = (f_1, \phi)_{0,\Omega} \quad \forall \phi \in H_0^1(\Omega), \tag{3.1c}$$

$$\frac{1}{\tau}a_2(c, \psi) + c_2(\boldsymbol{u} - v_p\boldsymbol{e}_z; c, \psi) = (f_2, \psi)_{0,\Omega} \quad \forall \psi \in H_0^1(\Omega), \tag{3.1d}$$

where $\boldsymbol{f} = (\rho/\rho_{\mathrm{m}})\boldsymbol{g} = (\alpha s + \beta c)\boldsymbol{g}$, $f_1 = 0$, and $f_2 = 0$. Let us also consider its discrete counterpart: Find $(\boldsymbol{u}_h, p_h, s_h, c_h) \in \mathbf{V}_h \times \mathcal{Q}_h \times \mathcal{M}_h \times \mathcal{M}_h$ such that

$$a_1(c_h, \boldsymbol{u}_h, \boldsymbol{v}) + c_1(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}_h, \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbf{V}_h \tag{3.2a}$$

$$b(\boldsymbol{u}_h, q) = 0 \quad \forall q \in \mathcal{Q}_h, \tag{3.2b}$$

$$a_2(s_h, \phi) + c_2(\boldsymbol{u}_h; s_h, \phi) = (f_1, \phi)_{0,\Omega} \quad \forall \phi \in \mathcal{M}_h \tag{3.2c}$$

$$\frac{1}{\tau}a_2(c_h, \psi) + c_2(\boldsymbol{u}_h - v_p\boldsymbol{e}_z; c_h, \psi) = (f_2, \psi)_{0,\Omega} \quad \forall \psi \in \mathcal{M}_h, \tag{3.2d}$$

where $\boldsymbol{f}_h = (\alpha s_h + \beta c_h)\boldsymbol{g}$, $f_1 = 0$, and $f_2 = 0$.

For each $K \in \mathcal{T}_h$ and each $e \in \mathcal{E}_h$ we define element-wise and edge-wise residuals as follows:

$$\mathbf{R}_K := \{\boldsymbol{f}_h + \nabla\cdot(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h) - \boldsymbol{u}_h\cdot\boldsymbol{\nabla}\boldsymbol{u}_h - (\rho_{\mathrm{m}})^{-1}\nabla p_h\}|_K,$$

$$R_{1,K} := \{f_1 + \mathrm{Sc}^{-1}\nabla^2 s_h - \boldsymbol{u}_h\cdot\nabla s_h\}|_K, \quad R_{2,K} := \{f_2 + (\tau\mathrm{Sc})^{-1}\nabla^2 c_h - (\boldsymbol{u}_h - v_p\boldsymbol{e}_z)\cdot\nabla c_h\}|_K,$$

$$\mathbf{R}_e := \begin{cases} \frac{1}{2}[\![((\rho_{\mathrm{m}})^{-1}p_h\mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h)\boldsymbol{n}]\!] & \text{for } e \in \mathcal{E}_h \setminus \Gamma, \\ 0 & \text{for } e \in \Gamma, \end{cases}$$

$$R_{1,e} := \begin{cases} \frac{1}{2}[\![(\mathrm{Sc}^{-1}\nabla s_h)\cdot \boldsymbol{n}]\!] & \text{for } e \in \mathcal{E}_h \setminus \Gamma, \\ 0 & \text{for } e \in \Gamma, \end{cases}, \qquad R_{2,e} := \begin{cases} \frac{1}{2}[\![((\tau\mathrm{Sc})^{-1}\nabla c_h)\cdot \boldsymbol{n}]\!] & \text{for } e \in \mathcal{E}_h \setminus \Gamma, \\ 0 & \text{for } e \in \Gamma. \end{cases}$$

Then we introduce the element-wise error estimator $\Psi_K^2 = \Psi_{R_K}^2 + \Psi_{e_K}^2 + \Psi_{J_K}^2$ with contributions defined as

$$\Psi_{R_K}^2 := h_K^2\big(\|\mathbf{R}_K\|_{0,K}^2 + \|R_{1,K}\|_{0,K}^2 + \|R_{2,K}\|_{0,K}^2\big),$$
$$\Psi_{e_K}^2 := \sum_{e\in\partial K} h_e\big(\|\mathbf{R}_e\|_{0,e}^2 + \|R_{1,e}\|_{0,e}^2 + \|R_{2,e}\|_{0,e}^2\big), \quad \Psi_{J_K}^2 := \sum_{e\in\partial K} h_e^{-1}\|[\![\boldsymbol{u}_h]\!]\|_{0,e}^2,$$

so a global *a posteriori* error estimator for the nonlinear coupled and steady problem (3.2) is

$$\Psi = \left(\sum_{K\in\mathcal{T}_h} \Psi_K^2\right)^{1/2}. \tag{3.3}$$

3.1. **Reliability.** Let us introduce the space

$$\tilde{X}(\mathcal{T}_h) = \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{div},\Omega) : \boldsymbol{v} \in \mathbf{H}_0^1(K)\ \forall K \in \mathcal{T}_h\}.$$

Then, for a fixed $(\tilde{\boldsymbol{u}},\tilde{c}) \in \tilde{X}(\mathcal{T}_h)\times H_0^1(\Omega)$, we define the bilinear form $\mathcal{A}_h^{(\tilde{\boldsymbol{u}},\tilde{c})}(\cdot,\cdot)$ as

$$\begin{aligned}\mathcal{A}_h^{(\tilde{\boldsymbol{u}},\tilde{c})}\big((\boldsymbol{u},p,s,c),(\boldsymbol{v},q,\phi,\psi)\big) &= \tilde{a}_1^h(\tilde{c},\boldsymbol{u},\boldsymbol{v}) + c_1^h(\tilde{\boldsymbol{u}};\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) + b(\boldsymbol{u},q) + a_2(s,\phi) \\ &\quad + c_2(\tilde{\boldsymbol{u}};s,\phi) + \frac{1}{\tau}a_2(c,\psi) + c_2(\tilde{\boldsymbol{u}} - v_p\boldsymbol{e}_z;c,\psi),\end{aligned}$$

for all $(\boldsymbol{u},p,s,c),(\boldsymbol{v},q,\phi,\psi) \in \mathbf{V}_h\times\mathcal{Q}_h\times\mathcal{M}_h\times\mathcal{M}_h$, where

$$\tilde{a}_1^h(\tilde{c},\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \big(\nu(\tilde{c})\boldsymbol{\nabla}_h(\boldsymbol{u}) : \boldsymbol{\nabla}_h(\boldsymbol{v})\big) + \sum_{e\in\mathcal{E}_h}\int_e\left(\frac{a_0}{h_e}\nu(\tilde{c})[\![\boldsymbol{u}]\!]\cdot[\![\boldsymbol{v}]\!]\right).$$

Note that $a_1^h(\tilde{c},\boldsymbol{u},\boldsymbol{v}) = \tilde{a}_1^h(\tilde{c},\boldsymbol{u},\boldsymbol{v}) + K_h(\tilde{c},\boldsymbol{u},\boldsymbol{v})$, where

$$K_h(\tilde{c},\boldsymbol{u},\boldsymbol{v}) := \sum_{e\in\mathcal{E}_h}\int_e\left(-\{\!\{\nu(\tilde{c})\boldsymbol{\nabla}_h(\boldsymbol{u})\boldsymbol{n}_e\}\!\}\cdot[\![\boldsymbol{v}]\!] - \{\!\{\nu(\tilde{c})\boldsymbol{\nabla}_h(\boldsymbol{v})\boldsymbol{n}_e\}\!\}\cdot[\![\boldsymbol{u}]\!]\right),$$

and we point out that $\mathcal{A}_h^{(\cdot,\cdot)}\big((\boldsymbol{u},p,s,c),(\boldsymbol{v},q,\phi,\psi)\big)$ is well-defined also for every $(\boldsymbol{u},p,s,c),(\boldsymbol{v},q,\phi,\psi) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$.

Theorem 3.1 (Global inf-sup stability)**.** *Let the pair $(\tilde{\boldsymbol{u}},\tilde{c}) \in \tilde{X}(\mathcal{T}_h)\times H_0^1(\Omega)$ satisfy $\|\tilde{\boldsymbol{u}}\|_{1,\mathcal{T}_h} < M$, for a sufficiently small $M > 0$. For any $(\boldsymbol{u},p,s,c) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$, there exists $(\boldsymbol{v},q,\phi,\psi) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$ with $|||(\boldsymbol{v},q,\phi,\psi)||| \le 1$ such that*

$$\mathcal{A}_h^{(\tilde{\boldsymbol{u}},\tilde{c})}\big((\boldsymbol{u},p,s,c),(\boldsymbol{v},q,\phi,\psi)\big) \ge C|||(\boldsymbol{u},p,s,c)|||,$$

where we define $|||(\boldsymbol{v},q,\phi,\psi)|||^2 := \|\boldsymbol{v}\|_{1,\mathcal{T}_h}^2 + \|q\|_{0,\Omega}^2 + \|\phi\|_{1,\Omega}^2 + \|\psi\|_{1,\Omega}^2$.

Proof. For any $(\boldsymbol{u},p,s,c) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$ there holds

$$\mathcal{A}_h^{(\tilde{\boldsymbol{u}},\tilde{c})}\big((\boldsymbol{u},p,s,c),(\boldsymbol{u},-p,s,c)\big) \ge \alpha_a\|\boldsymbol{u}\|_{1,\Omega}^2 + \tilde{\alpha}_a\|s\|_{1,\Omega}^2 + \frac{\tilde{\alpha}_a}{\tau}\|c\|_{1,\Omega}^2.$$

Applying the inf-sup condition, we get that for any $p \in L_0^2(\Omega)$, there exists a $\boldsymbol{v} \in \mathbf{H}_0^1(\Omega)$ such that $b(\boldsymbol{v},p) \ge \beta\|p\|_{0,\Omega}^2$ and $\|\boldsymbol{v}\|_{1,\Omega} \le \|p\|_{0,\Omega}$, where $\beta > 0$ is the inf-sup constant depending only on Ω. Then, we have

$$\begin{aligned}&\mathcal{A}_h(\tilde{u},\tilde{c})(\boldsymbol{u},p,s,c;v,0,0,0) \\ &= a_1(\tilde{c};\boldsymbol{u},\boldsymbol{v}) + c_1(\tilde{\boldsymbol{u}};\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) \ge \beta\|p\|_{0,\Omega}^2 - |a_1(\tilde{c};\boldsymbol{u},\boldsymbol{v})| - |c_1(\tilde{\boldsymbol{u}};\boldsymbol{u},\boldsymbol{v})| \\ &\ge \beta\|p\|_{0,\Omega}^2 - C_a\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} - C_c\|\tilde{\boldsymbol{u}}\|_{1,h}\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} \ge \beta\|p\|_{0,\Omega}^2 - 2C_a\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} \\ &\ge \beta\|p\|_{0,\Omega}^2 - 2C_a\|\boldsymbol{u}\|_{1,\Omega}\|p\|_{0,\Omega} \ge \left(\beta - \frac{1}{\epsilon}\right)\|p\|_{0,\Omega}^2 - \epsilon C_a^2\|\boldsymbol{u}\|_{1,\Omega}^2,\end{aligned}$$

where $\epsilon > 0$. Now, we introduce a $\delta > 0$ such that

$$\begin{aligned}\mathcal{A}_h^{(\tilde{\boldsymbol{u}},\tilde{c})}\big((\boldsymbol{u},p,s,c),(\boldsymbol{u}+\delta\boldsymbol{v},-p,s,c)\big) &= \mathcal{A}_h^{(\tilde{\boldsymbol{u}},\tilde{c})}(\boldsymbol{u},p,s,c,\boldsymbol{u},-p,s,c) + \delta\mathcal{A}_h^{(\tilde{\boldsymbol{u}},\tilde{c})}(\boldsymbol{u},p,s,c,\boldsymbol{v},0,0,0) \\ &\ge (\alpha_a - \delta\epsilon C_a^2)\|\boldsymbol{u}\|_{1,\Omega}^2 + \delta\left(\beta - \frac{1}{\epsilon}\right)\|p\|_{0,\Omega}^2 + \tilde{\alpha}\|s\|_{1,\Omega}^2 + \frac{\tilde{\alpha}}{\tau}\|c\|_{1,\Omega}^2.\end{aligned}$$

Choosing $\epsilon = 2/\beta$ and $\delta = \alpha_a/(2\epsilon C_a^2)$, we obtain

$$\begin{aligned}\mathcal{A}_h^{(\tilde{\boldsymbol{u}},\tilde{c})}(\boldsymbol{u},p,s,c,\boldsymbol{u}+\delta\boldsymbol{v},-p,s,c) &\geq \frac{\alpha_a}{2}\|\boldsymbol{u}\|_{1,\Omega}^2 + \frac{\beta}{2}\|p\|_{0,\Omega}^2 + \tilde{\alpha}\|s\|_{1,\Omega}^2 + \frac{\tilde{\alpha}}{\tau}\|c\|_{1,\Omega}^2 \\ &\geq \min\left\{\frac{\alpha_a}{2},\frac{\beta}{2},\tilde{\alpha},\frac{\tilde{\alpha}}{\tau}\right\}\big(\|\boldsymbol{u}\|_{1,\Omega}^2 + \|p\|_{0,\Omega}^2 + \|s\|_{1,\Omega}^2 + \|c\|_{1,\Omega}^2\big).\end{aligned}$$

Finally, using triangle inequality, the following relations hold:

$$\begin{aligned}|||(\boldsymbol{u}+\delta\boldsymbol{v},-p,s,c)|||^2 &= \|\boldsymbol{u}+\delta\boldsymbol{v}\|_{1,\Omega}^2 + \|p\|_{0,\Omega}^2 + \|s\|_{1,\Omega}^2 + \|c\|_{1,\Omega}^2 \\ &\leq 2\big(\|\boldsymbol{u}\|_{1,\Omega}^2 + \delta^2\|\boldsymbol{v}\|_{1,\Omega}^2\big) + \|p\|_{0,\Omega}^2 + \|s\|_{1,\Omega}^2 + \|c\|_{1,\Omega}^2 \\ &\leq \max\{2,(1+2\delta^2)\}\big(\|\boldsymbol{u}\|_{1,\Omega}^2 + \|p\|_{0,\Omega}^2 + \|s\|_{1,\Omega}^2 + \|c\|_{1,\Omega}^2\big).\end{aligned}$$

This concludes the proof. □

Next, we decompose the $\mathbf{H}(\mathrm{div})$-conforming velocity approximation uniquely into $\boldsymbol{u}_h = \boldsymbol{u}_h^c + \boldsymbol{u}_h^r$, where $\boldsymbol{u}_h^c \in \mathbf{V}_h^c$ and $\boldsymbol{u}_h^r \in (\mathbf{V}_h^c)^\perp$, and we note that $\boldsymbol{u}_h^r = \boldsymbol{u}_h - \boldsymbol{u}_h^c \in \mathbf{V}_h$.

Lemma 3.1. *There holds*

$$\|\boldsymbol{u}_h^r\|_{1,\mathcal{T}_h} \leq C_r\left(\sum_{K\in\mathcal{T}_h}\Psi_{J_K}^2\right)^{1/2}.$$

Proof. It follows straightforwardly from the decomposition $\boldsymbol{u}_h = \boldsymbol{u}_h^c + \boldsymbol{u}_h^r$ and from the edge residual. □

Lemma 3.2. *If $\|\boldsymbol{u}\|_{1,\infty} < M$, $\|s\|_\infty < M$ and $\|c\|_\infty < M$, then the following estimate holds:*

$$\begin{aligned}\frac{C}{2}|||(\boldsymbol{e}^{\boldsymbol{u}},e^p,e^s,e^c)||| &\leq \int_\Omega(\boldsymbol{f}-\boldsymbol{f}_h)\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{f}_h\cdot(\boldsymbol{v}-\boldsymbol{v}_h) + \int_\Omega f_1(\phi-\phi_h) + \int_\Omega f_2(\psi-\psi_h) + K_h(\boldsymbol{u}_h,\boldsymbol{v}_h) \\ &\quad - \mathcal{A}_h^{(\boldsymbol{u}_h,c_h)}(\boldsymbol{u}_h,p_h,s_h,c_h,\boldsymbol{v}-\boldsymbol{v}_h,q,\phi-\phi_h,\psi-\psi_h) + (1+C)C_r\left(\sum_{K\in\mathcal{T}_h}\Psi_{j_K}^2\right)^{1/2}.\end{aligned}$$

Proof. Using $\boldsymbol{u}_h = \boldsymbol{u}_h^c + \boldsymbol{u}_h^r$ and the triangle inequality imply

$$|||(\boldsymbol{e}^{\boldsymbol{u}},e^p,e^s,e^c)||| \leq |||(\boldsymbol{e}_c^{\boldsymbol{u}},e^p,e^s,e^c)||| + \|\boldsymbol{u}_h^r\|_{1,\mathcal{T}_h} \leq |||(\boldsymbol{e}_c^{\boldsymbol{u}},e^p,e^s,e^c)||| + C_r\left(\sum_{K\in\mathcal{T}_h}\Psi_{J_K}^2\right)^{1/2}.$$

Then, Theorem 3.1 gives

$$\begin{aligned}C|||(\boldsymbol{e}_c^{\boldsymbol{u}},e^p,e^s,e^c)||| &\leq \mathcal{A}_h^{(\boldsymbol{u}_h,c_h)}(\boldsymbol{e}^{\boldsymbol{u}},e^p,e^s,e^c;\boldsymbol{v},q,\phi,\psi) + \mathcal{A}_h^{(\boldsymbol{u}_h,c_h)}(\boldsymbol{u}_h^r,0,0,0;\boldsymbol{v},q,\phi,\psi) \\ &\leq \mathcal{A}_h^{(\boldsymbol{u}_h,c_h)}(\boldsymbol{e}^{\boldsymbol{u}},e^p,e^s,e^c;\boldsymbol{v},q,\phi,\psi) + C_r\left(\sum_{K\in\mathcal{T}_h}\Psi_{J_K}^2\right)^{1/2}.\end{aligned}$$

Owing to the relation

$$\begin{aligned}\mathcal{A}_h^{(\boldsymbol{u}_h,c_h)}(\boldsymbol{u},p,s,c;\boldsymbol{v},q,\phi,\psi) &= \mathcal{A}_h^{(\boldsymbol{u},c)}(\boldsymbol{u},p,s,c;\boldsymbol{v},q,\phi,\psi) - a_1(c;\boldsymbol{u},\boldsymbol{v}) + a_1(c_h;\boldsymbol{u},\boldsymbol{v}) \\ &\quad - c_1(e^{\boldsymbol{u}};\boldsymbol{u},\boldsymbol{v}) - c_2(e^{\boldsymbol{u}};s,\phi) - c_2(e^{\boldsymbol{u}};c,\psi),\end{aligned}$$

we then have

$$\begin{aligned}C|||(\boldsymbol{e}^{\boldsymbol{u}},e^p,e^s,e^c)||| &\leq C|||(\boldsymbol{e}_c^{\boldsymbol{u}},e^p,e^s,e^c)||| + C_r\left(\sum_{K\in\mathcal{T}_h}\Psi_{J_K}^2\right)^{1/2} \\ &\leq \mathcal{A}_h^{(\boldsymbol{u},c)}(\boldsymbol{u},p,s,c;\boldsymbol{v},q,\phi,\psi) - a_1(c;\boldsymbol{u},\boldsymbol{v}) + a_1(c_h,\boldsymbol{u},\boldsymbol{v}) - c_1(e^{\boldsymbol{u}};\boldsymbol{u},\boldsymbol{v}) - c_2(e^{\boldsymbol{u}};s,\phi) \\ &\quad - c_2(e^{\boldsymbol{u}};c,\psi) - \mathcal{A}_h(\boldsymbol{u}_h,c_h)(\boldsymbol{u}_h,p_h,s_h,c_h;\boldsymbol{v},q,\phi,\psi) + (1+C)C_r\left(\sum_{K\in\mathcal{T}_h}\Psi_{J_K}^2\right)^{1/2},\end{aligned}$$

while using the properties

$$\begin{gathered}\big|a_1(c;\boldsymbol{u},\boldsymbol{v}) - a_1(c_h;\boldsymbol{u},\boldsymbol{v})\big| \leq C_1\|c-c_h\|_1\|\boldsymbol{u}\|_{1,\infty}\|\boldsymbol{v}\|_1 \leq C_1M\|e^c\|_1, \\ c_1(e^{\boldsymbol{u}};\boldsymbol{u},\boldsymbol{v}) \leq C_2M\|e^{\boldsymbol{u}}\|_{1,\mathcal{T}_h}, \quad c_2(e^{\boldsymbol{u}};s,\phi) \leq C_3M\|e^{\boldsymbol{u}}\|_{1,\mathcal{T}_h}, \quad c_2(e^{\boldsymbol{u}};c,\psi) \leq C_4M\|e^{\boldsymbol{u}}\|_{1,\mathcal{T}_h}\end{gathered}$$

yields the bound

$$C|||(\boldsymbol{e}^{\boldsymbol{u}}, e^p, e^s, e^c)||| \leq \mathcal{A}_h^{(\boldsymbol{u},c)}(\boldsymbol{u}, p, s, c; \boldsymbol{v}, q, \phi, \psi) - \mathcal{A}_h(\boldsymbol{u}_h, c_h)(\boldsymbol{u}_h, p_h, s_h, c_h; \boldsymbol{v}, q, \phi, \psi)$$
$$+ (1+C)C_r \left(\sum_{K\in\mathcal{T}_h} \Psi_{J_K}^2 \right)^{1/2} - (C_1 + C_2 + C_3 + C_4)M|||(\boldsymbol{e}^{\boldsymbol{u}}, e^p, e^s, e^c)|||.$$

Moreover, we have

$$\frac{C}{2}|||(\boldsymbol{e}^{\boldsymbol{u}}, e^p, e^s, e^c)||| \leq \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_\Omega f_1\phi + \int_\Omega f_2\psi - \mathcal{A}_h^{(\boldsymbol{u}_h,c_h)}(\boldsymbol{u}_h, p_h, s_h, c_h; \boldsymbol{v}, q, \phi, \psi)$$
$$+ (1+C)C_r \left(\sum_{K\in\mathcal{T}_h} \Psi_{j_K}^2 \right)^{1/2}, \tag{3.4}$$

and we readily see that after stating the discrete problem as

$$a_1^h(c_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + c_1^h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) - \int_\Omega \boldsymbol{f}_h\cdot\boldsymbol{v}_h = 0, \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h,$$
$$a_2(s_h, \phi_h) + c_2(\boldsymbol{u}_h; s_h, \phi_h) - \int_\Omega f_1\phi_h = 0 \quad \forall \phi_h \in \mathcal{M}_h,$$
$$\frac{1}{\tau}a_2(c_h, \psi_h) + c_2(\boldsymbol{u}_h - v_p e_z; c_h, \psi_h) - \int_\Omega f_2\psi_h = 0 \quad \forall \psi_h \in \mathcal{M}_h,$$

and employing (3.4), the sought results follow. □

Lemma 3.3. *For $(\boldsymbol{v}, q, s, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$, there are $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $s_h \in \mathcal{M}_h$ and $c_h \in \mathcal{M}_h$ such that*

$$\int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h)\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{f}_h\cdot(\boldsymbol{v} - \boldsymbol{v}_h) + \int_\Omega f_1(\phi - \phi_h) + \int_\Omega f_2(\psi - \psi_h)$$
$$- \mathcal{A}_h^{(u_h,c_h)}(\boldsymbol{u}_h, p_h, s_h, c_h, \boldsymbol{v} - \boldsymbol{v}_h, q, \phi - \phi_h, \psi - \psi_h) \leq C\big(\Psi + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}\big)|||(\boldsymbol{v}, q, s, c)|||. \tag{3.5}$$

Proof. Using integration by parts gives

$$\int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h)\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{f}_h\cdot(\boldsymbol{v} - \boldsymbol{v}_h) + \int_\Omega f_1(\phi - \phi_h) + \int_\Omega f_2(\psi - \psi_h)$$
$$- \mathcal{A}_h^{(u_h,c_h)}(\boldsymbol{u}_h, p_h, s_h, c_h, \boldsymbol{v} - \boldsymbol{v}_h, q, \phi - \phi_h, \psi - \psi_h) = T_1 + \cdots + T_5, \tag{3.6}$$

where we define the terms

$$T_1 := \sum_{K\in\mathcal{T}_h} \int_K \left(\boldsymbol{f}_h + \nabla\cdot\big(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h\big) - \boldsymbol{u}_h\cdot\boldsymbol{\nabla}\boldsymbol{u}_h - \frac{1}{\rho_{\mathrm{m}}}\nabla p_h \right)\cdot(\boldsymbol{v} - \boldsymbol{v}_h)\,\mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h)\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},$$
$$T_2 := \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\left(\frac{1}{\rho_{\mathrm{m}}} p_h \mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h \right)\cdot\boldsymbol{n}_K \right)\cdot(\boldsymbol{v} - \boldsymbol{v}_h)\,\mathrm{d}S,$$
$$T_3 := \sum_{K\in\mathcal{T}_h} \int_{\partial K_{in}\setminus\Gamma} \boldsymbol{u}_h\cdot\boldsymbol{n}_K(\boldsymbol{u}_h - \boldsymbol{u}_h^e)\cdot(\boldsymbol{v} - \boldsymbol{v}_h)\,\mathrm{d}S,$$
$$T_4 := \sum_{K\in\mathcal{T}_h} \int_K \left(f_1 + \frac{1}{\mathrm{Sc}}\nabla^2 s_h - \boldsymbol{u}_h\cdot\nabla s_h \right)(\phi - \phi_h)\,\mathrm{d}\boldsymbol{x} + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\frac{1}{\mathrm{Sc}}\nabla s_h\cdot\boldsymbol{n}_K \right)(\phi - \phi_h)\,\mathrm{d}S,$$
$$T_5 := \sum_{K\in\mathcal{T}_h} \int_K \left(f_2 + \frac{1}{\tau\mathrm{Sc}}\nabla^2 c_h - (\boldsymbol{u}_h - v_p e^z)\cdot\nabla c_h \right)(\psi - \psi_h)\,\mathrm{d}\boldsymbol{x} + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\frac{1}{\tau\mathrm{Sc}}\nabla c_h\cdot\boldsymbol{n}_K \right)(\psi - \psi_h)\,\mathrm{d}S.$$

Applying the Cauchy-Schwarz inequality to T_1 implies

$$T_1 \leq \left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\mathbf{R}_K\|_{0,K}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} h_K^{-2} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,K}^2 \right)^{1/2} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}\|\boldsymbol{v}\|_{0,\Omega}$$
$$\leq \left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\mathbf{R}_K\|_{0,K}^2 \right)^{1/2} \bar{C}\|\boldsymbol{\nabla}\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}\|\boldsymbol{v}\|_{0,\Omega}.$$

Next, we rewrite T_2 in terms of a sum over interior edges and apply again the Cauchy-Schwarz inequality. Then

$$T_2 = \sum_{e\in\mathcal{E}_h}\int_e [\![((\rho_{\mathrm{m}})^{-1}p_h\mathbf{I}-\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h)\boldsymbol{n}]\!]\cdot(\boldsymbol{v}-\boldsymbol{v}_h)\,\mathrm{d}S$$
$$\leq \left(\sum_{e\in\mathcal{E}_h} h_e\|\mathbf{R}_e\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{0,e}^2\right)^{1/2} \leq \left(\sum_{e\in\mathcal{E}_h} h_e\|\mathbf{R}_e\|_{0,e}^2\right)^{1/2}\bar{C}\|\boldsymbol{\nabla}\boldsymbol{v}\|_{0,\Omega}. \tag{3.7}$$

Then, owing to the Cauchy-Schwarz inequality, it follows that

$$T_3 \leq \left(\sum_{e\in\mathcal{E}_h}\|[\![\boldsymbol{u}_h]\!]\|_{0,e}^2\right)^{1/2}\bar{C}\|\boldsymbol{\nabla}\boldsymbol{v}\|_{0,\Omega}. \tag{3.8}$$

Proceeding similarly, we may establish the following bounds for T_4 and T_5:

$$T_4 \leq \left(\left(\sum_{K\in\mathcal{T}_h} h_K^2\|R_{1,K}\|_{0,K}^2\right)^{1/2} + \left(\sum_{e\in\mathcal{E}_h} h_e\|R_{1,e}\|_{0,e}^2\right)^{1/2}\right)\bar{C}\|\nabla\phi\|_{0,\Omega},$$
$$T_5 \leq \left(\left(\sum_{K\in\mathcal{T}_h} h_K^2\|R_{2,K}\|_{0,K}^2\right)^{1/2} + \left(\sum_{e\in\mathcal{E}_h} h_e\|R_{2,e}\|_{0,e}^2\right)^{1/2}\right)\bar{C}\|\nabla\psi\|_{0,\Omega}.$$

Finally, (3.5) results as a combination of the bounds derived for T_1, T_2, T_3, T_4 and T_5. □

Theorem 3.2. *Let $(\boldsymbol{u},p,s,c)$ and $(\boldsymbol{u}_h,p_h,s_h,c_h)$ be the unique solutions to (3.1) and (3.2), respectively. Let Ψ be the a posteriori error estimator defined in (3.3). Then the following estimate holds:*

$$|||(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h,s-s_h,c-c_h)||| \leq C(\Psi+\|\boldsymbol{f}-\boldsymbol{f}_h\|_0), \tag{3.9}$$

where $C>0$ is a constant independent of h.

Proof. It suffices to apply Lemmas 3.2 and 3.3. □

3.2. **Efficiency.** For each $K\in\mathcal{T}_h$, we can define the standard polynomial bubble function b_K. Then, for any polynomial function $\boldsymbol{v}$ on K, the following results hold:

$$\|b_K\boldsymbol{v}\|_{0,K} \leq C\|\boldsymbol{v}\|_{0,K}, \qquad \|\boldsymbol{v}\|_{0,K} \leq C\|b_K^{1/2}\boldsymbol{v}\|_{0,K}, \tag{3.10a}$$
$$\|\nabla(b_K\boldsymbol{v})\|_{0,K} \leq Ch_K^{-1}\|\boldsymbol{v}\|_{0,K}, \qquad \|b_K\boldsymbol{v}\|_{\infty,K} \leq Ch_K^{-1}\|\boldsymbol{v}\|_{0,K}, \tag{3.10b}$$

where C is a positive constant independent of K and $\boldsymbol{v}$.

Lemma 3.4. *The following estimates hold, where C is a positive constant:*

$$h_K\|\boldsymbol{R}_K\|_{0,K} \leq C\big(\|c-c_h\|_{1,K}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,K}+\|p-p_h\|_{0,K}+h_K\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}\big),$$
$$h_K\|R_{1,K}\|_{0,K} \leq C\big(\|s-s_h\|_{1,K}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,K}\big), \qquad h_K\|R_{2,K}\|_{0,K} \leq C\big(\|c-c_h\|_{1,K}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,K}\big),$$

Moreover, it also follows that

$$\Psi_K \leq C|||(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h,s-s_h,c-c_h)|||_K.$$

Proof. For each $K\in\mathcal{T}_h$, we define $\mathbf{W}_b = b_K\mathbf{R}_K$. Then, using (3.10), we have

$$\|\mathbf{R}_K\|_{0,K}^2 \leq \|b_K^{1/2}\mathbf{R}_K\|_{0,K}^2 = \int_K \mathbf{R}_K\cdot\mathbf{W}_b$$
$$= \int_K\left(\boldsymbol{f}_h+\nabla\cdot\big(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h\big)-(\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}_h-\frac{1}{\rho_{\mathrm{m}}}\nabla p_h\right)\cdot\mathbf{W}_b = T_1+T_2,$$

where

$$T_1 = \int_K\left(\big((\nu(c)-\nu(c_h))\boldsymbol{\nabla}\boldsymbol{u}+\nu(c_h)\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_h)\big):\nabla\mathbf{W}_b-\frac{1}{\rho_{\mathrm{m}}}(p-p_h)\nabla\cdot\mathbf{W}_b\right)+\int_K(\boldsymbol{f}_h-\boldsymbol{f})\cdot\mathbf{W}_b,$$
$$T_2 = \int_K\big(((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\nabla)\boldsymbol{u}+(\boldsymbol{u}_h\cdot\nabla)(\boldsymbol{u}-\boldsymbol{u}_h)\big)\cdot\mathbf{W}_b.$$

Using the Cauchy-Schwarz inequality and (3.10) we obtain

$$T_1 \leq C_1\big(\|c-c_h\|_{1,K}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,K}+\|p-p_h\|_{0,K}+h_K\|\boldsymbol{f}-\boldsymbol{f}_h\|\big)h_K^{-1}\|\mathbf{R}_K\|_{0,K},$$

$$T_2 \le C_2 \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K} h_K^{-1} \|\mathbf{R}_K\|_{0,K},$$

and combining these bounds leads to the first stated result. The other two bounds follow similarly. □

Let e denote an interior edge that is shared by two elements K and K'. Let ω_e be the patch which is the union of K and K'. Next, we define the edge bubble function ζ_e on e with the property that it is positive in the interior of the patch ω_e and zero on the boundary of the patch. From [36], the following results hold:

$$\|q\|_{0,e} \le C \|\zeta_e^{1/2} q\|_{0,e}, \tag{3.11a}$$

$$\|\zeta_e q\|_{0,K} \le C h_e^{1/2} \|q\|_{0,e}, \quad \|\nabla(\zeta_e q)\|_{0,K} \le C h_e^{-1/2} \|q\|_{0,e} \qquad \forall K \in \omega_e. \tag{3.11b}$$

Lemma 3.5. *The following estimates hold:*

$$h_e \|\boldsymbol{R}_e\|_{0,e}^2 \le C \sum_{K \in \omega_e} \big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}^2 + \|c - c_h\|_{1,K}^2 + \|p - p_h\|_{0,K}^2 + h_K^2 \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}^2 \big),$$

$$h_e \|R_{1,e}\|_{0,e}^2 \le C \sum_{K \in \omega_e} \big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}^2 + \|s - s_h\|_{1,K}^2 \big),$$

$$h_e \|R_{2,e}\|_{0,e}^2 \le C \sum_{K \in \omega_e} \big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}^2 + \|c - c_h\|_{1,K}^2 \big).$$

Moreover, we also have

$$\Psi_{e_K}^2 \le C \sum_{e \in \partial K} \sum_{K \in \omega_e} \big(|\!|\!| (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, s - s_h, c - c_h) |\!|\!|_K^2 + h_K^2 \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}^2 \big).$$

Proof. Let e be an interior edge and let us define a rescaling of the edge bubble function in the form

$$\boldsymbol{\vartheta}_e = \sum_{e \in \partial K} \frac{h_e}{2} \mathbf{R}_e \zeta_e.$$

Using (3.11) gives

$$\begin{aligned} h_e \|\mathbf{R}_e\|_{0,e}^2 &\le C \big([\![(\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h) \boldsymbol{\nabla} \boldsymbol{u}_h]\!], \boldsymbol{\vartheta}_e \big)_e \\ &\le C \big([\![(\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h) \boldsymbol{\nabla} \boldsymbol{u}_h]\!] - [\![(\rho_{\mathrm{m}})^{-1} p \mathbf{I} - \nu(c) \boldsymbol{\nabla} \boldsymbol{u}]\!], \boldsymbol{\vartheta}_e \big)_e. \end{aligned} \tag{3.12}$$

Using integration by parts on each element of patch ω_e implies

$$\begin{aligned} \big([\![(\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h) \boldsymbol{\nabla} \boldsymbol{u}_h]\!], \boldsymbol{\vartheta}_e \big)_e &= \sum_{K \in \omega_e} \int_K \Big(\nabla \cdot \big(\nu(c_h) \boldsymbol{\nabla} \boldsymbol{u}_h \big) - \nabla \cdot \big(\nu(c) \boldsymbol{\nabla} \boldsymbol{u} \big) + \frac{1}{\rho_{\mathrm{m}}} \nabla (p - p_h) \Big) \cdot \boldsymbol{\vartheta}_e \\ &\quad + \int_K \Big(\frac{1}{\rho_{\mathrm{m}}} (p - p_h) \mathbf{I} + \nu(c_h) \boldsymbol{\nabla} \boldsymbol{u}_h - \nu(c) \boldsymbol{\nabla} \boldsymbol{u} \Big) : \boldsymbol{\nabla} \boldsymbol{\vartheta}_e. \end{aligned}$$

Note that $(\boldsymbol{u}, p, s, c)$ solves the underlying problem, so we then have

$$\begin{aligned} \big([\![(\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h) \boldsymbol{\nabla} \boldsymbol{u}_h]\!], \boldsymbol{\vartheta}_e \big)_e &= \sum_{K \in \omega_e} \int_K \Big(\boldsymbol{f} + \nabla \cdot \big(\nu(c_h) \boldsymbol{\nabla} \boldsymbol{u}_h \big) - \boldsymbol{u}_h \cdot \boldsymbol{\nabla} \boldsymbol{u}_h - \frac{1}{\rho_{\mathrm{m}}} \nabla p_h \Big) \cdot \boldsymbol{\vartheta}_e \\ &\quad + \sum_{K \in \omega_e} \int_K (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{u}_h \cdot \boldsymbol{\nabla} \boldsymbol{u}_h) \cdot \boldsymbol{\vartheta}_e \\ &\quad + \sum_{K \in \omega_e} \int_K \Big(\frac{p - p_h}{\rho_{\mathrm{m}}} \mathbf{I} - \big(\nu(c) - \nu(c_h) \big) \boldsymbol{\nabla} \boldsymbol{u}_h - \nu(c) \boldsymbol{\nabla} (\boldsymbol{u} - \boldsymbol{u}_h) \Big) : \boldsymbol{\nabla} \boldsymbol{\vartheta}_e \\ &=: T_1 + T_2 + T_3. \end{aligned} \tag{3.13}$$

Next, applying the Cauchy-Schwarz inequality together with Lemma 3.4 and (3.11) gives

$$\begin{aligned} T_1 &\le C_1 \Bigg(\sum_{K \in \omega_e} h_K^2 \|\mathbf{R}_K\|_{0,K}^2 + h_K^2 \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}^2 \Bigg)^{1/2} \Bigg(\sum_{K \in \omega_e} h_K^{-2} \|\boldsymbol{\vartheta}_e\|_{0,K}^2 \Bigg)^{1/2} \\ &\le C_1 \Bigg(\sum_{K \in \omega_e} |\!|\!| (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, s - s_h, c - c_h) |\!|\!|_K^2 \Bigg)^{1/2} h_e^{1/2} \|\mathbf{R}_e\|_{0,e}, \end{aligned}$$

$$T_2 \leq C_2 \left(\sum_{K\in\omega_e} \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,K}^2 \right)^{1/2} h_e^{1/2} \|\mathbf{R}_e\|_{0,e},$$

$$T_3 \leq C_3 \left(\sum_{K\in\omega_e} |||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, s-s_h, c-c_h)|||_K^2 \right)^{1/2} h_e^{1/2} \|\mathbf{R}_e\|_{0,e}.$$

Combining the bounds of T_1, T_2 and T_3 with (3.12) and (3.13) implies the first stated result. Similarly, we can prove the other two bounds. □

Theorem 3.3. *Let $(\boldsymbol{u}, p, s, c)$ and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ be the unique solutions of problems (3.1) and (3.2), respectively. Let Ψ be defined as in (3.3). Then there exists a constant $C > 0$ that is independent of h such that*

$$\Psi \leq C \left(|||(\boldsymbol{u}-u_h, p-p_h, s-s_h, c-c_h)||| + \left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}^2 \right)^{1/2} \right).$$

Proof. Combining Lemmas 3.4 and 3.5 implies the stated result. □

4. *A posteriori* ERROR BOUND FOR THE SEMIDISCRETE METHOD

For each $t \in (0,T]$, let us consider the problem: find $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{c}, \tilde{s}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} a_1(c_h, \tilde{\boldsymbol{u}}, \boldsymbol{v}) + c_1(\boldsymbol{u}_h; \tilde{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}) &= (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \\ b(\tilde{\boldsymbol{u}}, q) &= 0, \quad \forall q \in L_0^2(\Omega), \\ a_2(\tilde{s}, \phi) + c_2(\boldsymbol{u}_h; \tilde{s}, \phi) &= (f_1, \phi), \quad \forall \phi \in H_0^1(\Omega), \\ \frac{1}{\tau} a_2(\tilde{c}, \psi) + c_2(\boldsymbol{u}_h - v_p \boldsymbol{e}_z; \tilde{c}, \psi) &= (f_2, \psi) \quad \forall \psi \in H_0^1(\Omega), \end{aligned}$$

where

$$\boldsymbol{f} = (\alpha s_h + \beta c_h)\mathbf{g} - \partial_t \boldsymbol{u}_h, \quad f_1 = -\partial_t s_h, \quad f_2 = -\partial_t c_h. \tag{4.1}$$

Also, for each $t \in (0,T]$, we write the discrete weak formulation: find $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{c}_h, \tilde{s}_h) \in C^{0,1}(0,T;\mathbf{V}_h) \times C^{0,0}(0,T;\mathcal{Q}_h) \times C^{0,1}(0,T;\mathcal{M}_h) \times C^{0,1}(0,T;\mathcal{M}_h)$ such that

$$a_1(c_h, \tilde{\boldsymbol{u}}_h, \boldsymbol{v}) + c_1(\boldsymbol{u}_h; \tilde{\boldsymbol{u}}_h, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{V}_h, \tag{4.2a}$$

$$b(\tilde{\boldsymbol{u}}_h, q) = 0, \quad \forall q \in \mathcal{Q}_h, \tag{4.2b}$$

$$a_2(\tilde{s}_h, \phi) + c_2(\boldsymbol{u}_h; \tilde{s}_h, \phi) = (f_1, \phi), \quad \forall \phi \in \mathcal{M}_h \tag{4.2c}$$

$$\frac{1}{\tau} a_2(\tilde{c}_h, \psi) + c_2(\boldsymbol{u}_h - v_p \boldsymbol{e}_z; \tilde{c}_h, \psi) = (f_2, \psi) \quad \forall \psi \in \mathcal{M}_h, \tag{4.2d}$$

where (4.1) remains in effect.

Lemma 4.1. *For each $t \in (0,T]$ and for all $(\boldsymbol{v}, q, \phi, \psi) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ we have*

$$\begin{aligned} (\partial_t e_{\boldsymbol{u}}, \boldsymbol{v}) + a_1(c, \rho_{\boldsymbol{u}}, \boldsymbol{v}) + c_1(\boldsymbol{u}; \rho_{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, p-\tilde{p}) &= a_1(c_h, \tilde{\boldsymbol{u}}, \boldsymbol{v}) - a_1(c, \tilde{\boldsymbol{u}}, \boldsymbol{v}) - c_1(e_{\boldsymbol{u}}; \tilde{\boldsymbol{u}}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \\ b(\boldsymbol{u}-\tilde{\boldsymbol{u}}, q) &= 0, \quad \forall q \in L_0^2(\Omega), \\ (\partial_t e_s, \phi) + a_2(\rho_s, \phi) + c_2(\boldsymbol{u}; \rho_s, \phi) &= -c_2(e_{\boldsymbol{u}}; \tilde{s}, \phi), \quad \forall \phi \in H_0^1(\Omega), \\ (\partial_t e_c, \psi) + \frac{1}{\tau} a_2(\rho_c, \psi) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; \rho_c, \psi) &= -c_2(e_{\boldsymbol{u}}; \tilde{c}, \psi) \quad \forall \psi \in H_0^1(\Omega), \end{aligned}$$

where $e_{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_h$, $e_s = s - s_h$, $e_c = c - c_h$, $\rho_{\boldsymbol{u}} = \boldsymbol{u} - \tilde{u}$, $\rho_s = s - \tilde{s}$ and $\rho_c = c - \tilde{c}$.

Next we introduce the error indicator Θ as

$$\Theta^2 = \|e_{\boldsymbol{u}}(0)\|_{0,\Omega}^2 + \|e_c(0)\|_{0,\Omega}^2 + \|e_s(0)\|_{0,\Omega}^2 + \int_0^T \Psi^2 + \int_0^T \Theta_2^2 + \max_{0\leq t\leq T} \Theta_3^2, \tag{4.3}$$

where

$$\Theta_2^2 = \sum_{e\in\mathcal{E}_h} h_e \|[\![\partial_t \boldsymbol{u}_h]\!]\|_{0,e}^2, \qquad \Theta_3^2 = \sum_{e\in\mathcal{E}_h} h_e \|[\![\boldsymbol{u}_h]\!]\|_{0,e}^2,$$

whereas Ψ is the global *a posteriori* error estimator for the steady problem with element and edge residual contributions defined in (3). In this case we now replace $\boldsymbol{f}$ and f_1, f_2 by (4.1).

Theorem 4.1. *Let $(\boldsymbol{u}, p, s, c)$ and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ be the solutions to (1.2) and (4.2), respectively. Let Θ be the a posteriori error estimator defined in (4.3). Then there exists $C > 0$, independent of h, such that*

$$\big(\|e_{\boldsymbol{u}}\|_\star^2 + \|e_s\|_\star^2 + \|e_c\|_\star^2\big)^{1/2} \le C\,\Theta,$$
$$\|\partial_t e_{\boldsymbol{u}} + \nabla(p - p_h)\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega))} + \|\partial_t e_s\|_{L^2(0,T;H^{-1}(\Omega))} + \|\partial_t e_c\|_{L^2(0,T;H^{-1}(\Omega))} \le C\,\Theta,$$

where

$$\|\boldsymbol{v}\|_\star^2 = \|\boldsymbol{v}\|^2_{L^\infty(0,T;L^2(\Omega))} + \int_0^T \|\boldsymbol{v}\|^2_{1,\mathcal{T}_h} dt, \quad \|\phi\|_\star^2 = \|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \int_0^T \|\phi\|_1^2 dt.$$

Proof. Choosing $\boldsymbol{v} = e^c_{\boldsymbol{u}}$, $q = p - \tilde{p}$, $\phi = e_s$ and $\psi = e_c$ in Lemma 4.1 gives

$$(\partial_t e_{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) + a_1(c, \rho_{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) + c_1(\boldsymbol{u}; \rho_{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) = a_1(c_h, \tilde{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) - a_1(c, \tilde{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) - c_1(e_{\boldsymbol{u}}; \tilde{\boldsymbol{u}}, e^c_{\boldsymbol{u}}),$$
$$(\partial_t e_s, e_s) + a_2(\rho_s, e_s) + c_2(\boldsymbol{u}; \rho_s, e_s) = -c_2(e_{\boldsymbol{u}}; \tilde{s}, e_s),$$
$$(\partial_t e_c, e_c) + \frac{1}{\tau} a_2(\rho_c, e_c) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; \rho_c, e_c) = -c_2(e_{\boldsymbol{u}}; \tilde{c}, e_c).$$

Moreover, there also holds

$$\begin{aligned}(\partial_t e^c_{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) + a_1(c, e^c_{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) + c_1(\boldsymbol{u}; e^c_{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) &= (\partial_t \boldsymbol{u}_{h,r}, e^c_{\boldsymbol{u}}) + a_1(c_h, \tilde{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) - a_1(c, \tilde{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) - c_1(e_{\boldsymbol{u}}; \tilde{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) \\ &\quad + a_1(c, \theta^c_{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) + c_1(\boldsymbol{u}; \theta^c_{\boldsymbol{u}}, e^c_{\boldsymbol{u}}) \\ (\partial_t e_s, e_s) + a_2(e_s, e_s) + c_2(\boldsymbol{u}; e_s, e_s) &= -c_2(e_{\boldsymbol{u}}; \tilde{s}, e_s) + a_2(\theta_s, e_s) + c_2(\boldsymbol{u}; \theta_s, e_s), \\ (\partial_t e_c, e_c) + \frac{1}{\tau} a_2(e_c, e_c) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; e_c, e_c) &= -c_2(e_{\boldsymbol{u}}; \tilde{c}, e_c) + \frac{1}{\tau} a_2(\theta_c, e_c) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; \theta_c, e_s),\end{aligned}$$

where $\theta^c_{\boldsymbol{u}} = \tilde{\boldsymbol{u}} - \boldsymbol{u}^c_h$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e^c_{\boldsymbol{u}}\|^2_{0,\Omega} + \alpha_a \|e^c_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} &\le \big(C_1 \|\theta^c_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} + M\|\boldsymbol{u}_{h,r}\|_{1,\mathcal{T}_h}\big)\|e^c_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} \\ &\quad + C_2 M \|e_s\|_{1,\Omega}\|e^c_{\boldsymbol{u}}\|_{1,\mathcal{T}_h} + C_3 M \|e^c_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \|\partial_t \boldsymbol{u}_{h,r}\|_{0,\Omega}\|e^c_{\boldsymbol{u}}\|_{0,\Omega}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\|e_s\|^2_{0,\Omega} + \tilde{\alpha}_a \|e_s\|^2_{1,\Omega} &\le \big(C_4 \|\theta_s\|_{1,\Omega} + M\|\boldsymbol{u}_{h,r}\|_{1,\mathcal{T}_h}\big)\|e_s\|_{1,\Omega} + C_5 M \|e_s\|_{1,\Omega}\|e^c_{\boldsymbol{u}}\|_{1,\Omega}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\|e_c\|^2_{0,\Omega} + \tilde{\alpha}_a \|e_c\|^2_{1,\Omega} &\le \big(C_6 \|\theta_c\|_{1,\Omega} + M\|\boldsymbol{u}_{h,r}\|_{1,\Omega}\big)\|e_c\|_{1,\Omega} + C_7 M \|e_c\|_{1,\Omega}\|e^c_{\boldsymbol{u}}\|_{1,\Omega}.\end{aligned}$$

Let us now suppose that $E_c := \|e^c_{\boldsymbol{u}}(T_0)\| = \|e_{\boldsymbol{u}}\|_{L^\infty(0,T;L^2(\Omega))}$, for some $T_0 \in [0, T]$. Then using Poincaré-Friedrichs's inequality, Young's inequality and then combining the three equations implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e^c_{\boldsymbol{u}}\|^2_{0,\Omega} + \|e^c_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \frac{\mathrm{d}}{\mathrm{d}t}\|e_c\|^2_{0,\Omega} + \|e_c\|_1^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|e_s\|^2_{0,\Omega} + \|e_s\|^2_{1,\Omega} \\ \le C\big(\|\theta^c_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \|\theta_s\|^2_{1,\Omega} + \|\theta_c\|^2_{1,\Omega} + \|\partial_t \boldsymbol{u}_{h,r}\|^2_{0,\Omega} + M\|\boldsymbol{u}_{h,r}\|^2_{1,\mathcal{T}_h}\big).\end{aligned}$$

Integrating with respect to t on $[0, T]$ and $[0, T_0]$ yields

$$\begin{aligned}\|e^c_{\boldsymbol{u}}\|^2_\star + \|e_c\|^2_\star + \|e_s\|^2_\star &\le \|e^c_{\boldsymbol{u}}(0)\|^2_{0,\Omega} + \|e_c(0)\|^2_{0,\Omega} + \|e_s(0)\|^2_{0,\Omega} \\ &\quad + C\left(\int_0^T \big(\|\theta^c_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \|\theta_s\|^2_{1,\Omega} + \|\theta_c\|^2_{1,\Omega}\big) + \int_0^T \|\partial_t \boldsymbol{u}_{h,r}\|^2_{0,\Omega} + M\int_0^T \|\boldsymbol{u}_{h,r}\|^2_{1,\mathcal{T}_h}\right),\end{aligned}$$

and we moreover have

$$\begin{aligned}\|e_{\boldsymbol{u}}\|^2_\star + \|e_c\|^2_\star + \|e_s\|^2_\star &\le \|e_{\boldsymbol{u}}(0)\|^2_{0,\Omega} + \|e_c(0)\|^2_{0,\Omega} + \|e_s(0)\|^2_{0,\Omega} \\ &\quad + C\left(\int_0^T \big(\|\theta_{\boldsymbol{u}}\|^2_{1,\mathcal{T}_h} + \|\theta_s\|^2_{1,\Omega} + \|\theta_c\|^2_{1,\Omega}\big) + \int_0^T \|\partial_t \boldsymbol{u}_{h,r}\|^2_{0,\Omega} + \|\boldsymbol{u}_{h,r}\|^2_\star\right),\end{aligned} \tag{4.4}$$

and as a result we can combine Theorem 3.2 and (4.4) to readily obtain the first stated result.

On the other hand, integrating by parts in Lemma 4.1 yields

$$\begin{aligned}(\partial_t e_{\boldsymbol{u}} + \nabla(p - p_h), \boldsymbol{v}) &= -a_1(c, \rho_{\boldsymbol{u}}, \boldsymbol{v}) - c_1(\boldsymbol{u}; \rho_{\boldsymbol{u}}, \boldsymbol{v}) - b(\boldsymbol{v}, p_h - \tilde{p}) + a_1(c_h, \tilde{\boldsymbol{u}}, \boldsymbol{v}) \\ &\quad - a_1(c, \tilde{\boldsymbol{u}}, \boldsymbol{v}) - c_1(e_{\boldsymbol{u}}; \tilde{\boldsymbol{u}}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{H}^1_0(\Omega) \\ (\partial_t e_s, \phi) &= -a_2(\rho_s, \phi) - c_2(\boldsymbol{u}; \rho_s, \phi) - c_2(e_{\boldsymbol{u}}; \tilde{s}, \phi), \quad \forall \phi \in H^1_0(\Omega), \\ (\partial_t e_c, \psi) &= -\frac{1}{\tau} a_2(\rho_c, \psi) - c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; \rho_c, \psi) - c_2(e_{\boldsymbol{u}}; \tilde{c}, \psi) \quad \forall \psi \in H^1_0(\Omega).\end{aligned}$$

We apply Young's inequality and the definition of the dual norm. Then, we integrate in time the resulting expression. Finally, the second result is a consequence of Theorem 3.2 and (4.4). □

5. *A posteriori* ERROR ANALYSIS FOR THE FULLY DISCRETE METHOD

In this section, we develop an *a posteriori* error estimator for the fully discrete problem and focus the presentation on the simpler case of a time discretisation by the backward Euler method. For each time step k ($1 \le k \le N$), we define the (global in space) time indicator Ξ_k as

$$\Xi_k = (\Xi_{k,1}^2 + \Xi_{k,2}^2 + \Xi_{k,3}^2)^{1/2},$$

where

$$\Xi_{k,1}^2 := \tilde{\tau}_k\big(\|\boldsymbol{u}_h^k - I^k\boldsymbol{u}_h^{k-1}\|_{1,\mathcal{T}_{h,k}}^2 + h_e\tilde{\tau}_k^{-2}\|[\![I^k\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h^{n-1}]\!]\|_{0,e}^2 + h_e\tilde{\tau}_k^{-2}\|[\![\boldsymbol{u}_h^n - I^k\boldsymbol{u}_h^{n-1}]\!]\|_{0,e}^2\big),$$
$$\Xi_{k,2}^2 := \tilde{\tau}_k\|s_h^k - s_h^{k-1}\|_1^2, \qquad \Xi_{k,3}^2 := \tilde{\tau}_k\|c_h^k - c_h^{k-1}\|_1^2.$$

Here

$$\|\boldsymbol{u}_h^k\|_{1,\mathcal{T}_{h,k}}^2 = \sum_{K\in\mathcal{T}_{h,k}} \|\boldsymbol{\nabla}_h\boldsymbol{u}_h^k\|_{0,K} + \sum_{e\in\mathcal{E}_h}\frac{1}{h_e}\|[\![\boldsymbol{u}_h^k]\!]\|_{0,e}.$$

Next we define the accumulated time and spatial error indicators as

$$\Xi^2 = \sum_{k=1}^N \Xi_k^2, \qquad \Upsilon^2 = \sum_{k=1}^N \tilde{\tau}_k\big(\Upsilon_k^2(\boldsymbol{u}_h^k, p_h^k, s_h^k, c_h^k) + \Upsilon_k^2(I^k\boldsymbol{u}_h^{k-1}, I_p^k p_h^{k-1}, s_h^{k-1}, c_h^{k-1})\big), \tag{5.1}$$

where the terms Υ_k^2 are constructed with the *a posteriori* error estimator contributions defined as in the steady case (3), but at a given time step k. That is,

$$\Upsilon_k^2(\boldsymbol{u}_h^k, p_h^k, s_h^k, c_h^k) = \Upsilon_{K,k}^2 + \Upsilon_{e,k}^2 + \Upsilon_{J,k}^2,$$

with

$$\Upsilon_{K,k}^2 := h_K^2\big(\|\mathbf{R}_K^k\|_{0,K}^2 + \|R_{1,K}^k\|_{0,K}^2 + \|R_{2,K}^k\|_{0,K}^2\big),$$
$$\Upsilon_{e,k}^2 := \sum_{e\in\partial K} h_e\big(\|\mathbf{R}_e^k\|_{0,e}^2 + \|R_{1,e}^k\|_{0,e}^2 + \|R_{2,e}^k\|_{0,e}^2\big), \quad \Upsilon_{J,k}^2 := \sum_{e\in\partial K} h_e^{-1}\|[\![\boldsymbol{u}_h^k]\!]\|_{0,e}^2,$$

and

$$\mathbf{R}_K^k := \left\{-\frac{1}{\tilde{\tau}_k}(\boldsymbol{u}_h^k - I^k\boldsymbol{u}_h^{k-1}) + \frac{\rho}{\rho_{\mathrm{m}}}\boldsymbol{g} + \nabla\cdot(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h) - \boldsymbol{u}_h\cdot\boldsymbol{\nabla}\boldsymbol{u}_h - \frac{1}{\rho_{\mathrm{m}}}\nabla p_h\right\}\Big|_K,$$
$$R_{1,K}^k := \left\{-\frac{s_h^k - s_h^{k-1}}{\tilde{\tau}_k} + \frac{1}{\mathrm{Sc}}\nabla^2 s_h - \boldsymbol{u}_h\cdot\nabla s_h\right\}\Big|_K, \quad R_{1,e}^k := \begin{cases}\frac{1}{2}[\![(\mathrm{Sc}^{-1}\nabla s_h)\cdot\boldsymbol{n}]\!] & \text{for } e\in\mathcal{E}_h\setminus\Gamma,\\ 0 & \text{for } e\in\Gamma,\end{cases},$$
$$R_{2,K}^k := \left\{-\frac{c_h^k - c_h^{k-1}}{\tilde{\tau}_k} + \frac{1}{\tau\mathrm{Sc}}\nabla^2 c_h - (\boldsymbol{u}_h - v_p\boldsymbol{e}_z)\cdot\nabla c_h\right\}\Big|_K, \quad R_{2,e}^k := \begin{cases}\frac{1}{2}[\![((\tau\mathrm{Sc})^{-1}\nabla c_h)\cdot\boldsymbol{n}]\!] & \text{for } e\in\mathcal{E}_h\setminus\Gamma,\\ 0 & \text{for } e\in\Gamma.\end{cases}$$

For each time step k, we can split again the $\mathbf{H}(\mathrm{div})$-conforming discrete solution $\boldsymbol{u}_h^k$ into a conforming part $\boldsymbol{u}_{hc}^k$ and a non-conforming part $\boldsymbol{u}_{hr}^k$ such that $\boldsymbol{u}_h^k = \boldsymbol{u}_{h,c}^k + \boldsymbol{u}_{h,r}^k$. For each $t\in(t_{k-1},t_k]$, we introduce a linear interpolant $\boldsymbol{u}_h(t)$ in terms of t as

$$\boldsymbol{u}_h(t) := \frac{t_k - t}{\tilde{\tau}_k}I^k\boldsymbol{u}_h^{k-1} + \frac{t - t_k}{\tilde{\tau}_k}\boldsymbol{u}_h^k,$$

where $\{l_k, l_{k+1}\}$ is the standard linear interpolation basis defined on $[t^k, t^{k+1}]$. Similarly, we may introduce $\boldsymbol{u}_{h,c}(t)$ and $\boldsymbol{u}_{h,r}(t)$. Then, setting $\boldsymbol{e}_{\boldsymbol{u}_c} = \boldsymbol{u} - \boldsymbol{u}_{h,c}$, we have $\boldsymbol{e}^{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_h = \boldsymbol{e}^{\boldsymbol{u}_c} - \boldsymbol{u}_{h,r}$. For $t\in(t_{k-1},t_k)$, we define

$$\partial_t\boldsymbol{u}_{\tilde{\tau}}(t) := \frac{1}{\tilde{\tau}_k}(\boldsymbol{u}^k - \boldsymbol{u}^{k-1}),$$

and for all $t\in(t_{k-1},t_k)$, we consider the problem of finding $(\tilde{\boldsymbol{u}}^k, \tilde{p}^k, \tilde{s}^k, \tilde{c}^k)\in\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$ such that

$$\big(\partial_t\boldsymbol{u}_h(t), \boldsymbol{v}\big) + a_1(c_h, \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) + c_1(\boldsymbol{u}_h; \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}^k) = (\boldsymbol{f}^k, \boldsymbol{v}), \quad \forall\boldsymbol{v}\in\mathbf{H}_0^1(\Omega), \tag{5.2a}$$
$$b(\tilde{\boldsymbol{u}}^k, q) = 0, \quad \forall q\in L_0^2(\Omega), \tag{5.2b}$$
$$\big(\partial_t s_h(t), \phi\big) + a_2(\tilde{s}^k, \phi) + c_2(\boldsymbol{u}_h; \tilde{s}^k, \phi) = 0, \quad \forall\phi\in H_0^1(\Omega), \tag{5.2c}$$

$$(\partial_t c_h, \psi) + \frac{1}{\tau} a_2(\tilde{c}^k, \psi) + c_2(\boldsymbol{u}_h - v_p \boldsymbol{e}_z; \tilde{c}^k, \psi) = 0 \quad \forall \psi \in H_0^1(\Omega). \tag{5.2d}$$

Lemma 5.1. *The following estimates hold*

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2 + \alpha_1 \int_0^{t_n} \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t)\|_{1,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^s(t_n)\|_{0,\Omega}^2 + \alpha_2 \int_0^{t_n} \|e_{\tilde{\tau}}^s(t)\|_{1,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^c(t_n)\|_{0,\Omega}^2 + \alpha_3 \tau \int_0^{t_n} \|e_{\tilde{\tau}}^c(t)\|_{1,\Omega}^2 \\
&\le C(\Xi^2 + \Upsilon^2) + \frac{1}{2}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^s(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^c(0)\|_{0,\Omega}^2 + \frac{1}{2}\sum_{k=1}^{n-1}\big(\|\boldsymbol{u}(t_k) - I^{k+1}\tilde{\boldsymbol{u}}_{h,c}^k\|_{0,\Omega}^2 - \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2\big), \\
&\sum_{k=1}^{n}\big(\|\partial_t \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}} + \nabla(p - p_h)\|_{L^2(t_{k-1},t_k;\mathbf{H}^{-1}(\Omega))} + \|\partial_t e_{\tilde{\tau}}^s\|_{L^2(t_{k-1},t_k;H^{-1}(\Omega))} + \|\partial_t e_{\tilde{\tau}}^c\|_{L^2(t_{k-1},t_k;H^{-1}(\Omega))}\big) \\
&\le C(\Xi^2 + \Upsilon^2) + \frac{1}{2}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^s(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^c(0)\|_{0,\Omega}^2 + \frac{1}{2}\sum_{k=1}^{n-1}\big(\|\boldsymbol{u}(t_k) - I^{k+1}\tilde{\boldsymbol{u}}_{h,c}^k\|_{0,\Omega}^2 - \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2\big).
\end{aligned}$$

Proof. Combining (1.2) and (5.2) implies

$$\begin{aligned}
&\big(\partial_t(\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}}), \boldsymbol{v}\big) + a_1(c, \boldsymbol{u} - \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) + c_1(\boldsymbol{u}, \boldsymbol{u} - \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) + b(\boldsymbol{v}, p - \tilde{p}^k) \\
&\qquad + a_1(c - c_h, \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) + c_1(\boldsymbol{u} - \boldsymbol{u}_h, \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), && (5.3a)\\
&b(\boldsymbol{u} - \tilde{\boldsymbol{u}}^k, q) = 0, \quad \forall q \in L_0^2(\Omega), && (5.3b)\\
&\big(\partial_t e_{\tilde{\tau}}^s(t), \phi\big) - a_2(s - \tilde{s}^k, \phi) + c_2(\boldsymbol{u}; s - \tilde{s}^k, \phi) = c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{s}^k, \phi), \quad \forall \phi \in H_0^1(\Omega), && (5.3c)\\
&(\partial_t e_{\tilde{\tau}}^c, \psi) - \frac{1}{\tau} a_2(c - \tilde{c}^k, \psi) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; c - \tilde{c}^k, \psi) = c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{c}^k, \psi) \quad \forall \psi \in H_0^1(\Omega). && (5.3d)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(\partial_t \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{v}) + a_1(c, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{v}) + c_1(\boldsymbol{u}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{v}) + b(\boldsymbol{v}, p - p^k) &= (\partial_t \tilde{\boldsymbol{u}}_{h,r}, \boldsymbol{v}) - a_1(c - c_h, \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) - c_1(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}^k, \boldsymbol{v}) \\
&\quad - a_1(c, \tilde{\boldsymbol{u}}_{h,c} - \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) - c_1(\boldsymbol{u}, \tilde{\boldsymbol{u}}_{h,c} - \tilde{\boldsymbol{u}}^k, \boldsymbol{v}), \\
b(\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}}, q) &= 0, \\
\big(\partial_t e_{\tilde{\tau}}^s(t), \phi\big) - a_2(e_k^s, \phi) + c_2(\boldsymbol{u}; e_k^s, \phi) &= c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{s}^k, \phi) - a_2(s_h - \tilde{s}^k, \phi) + c_2(\boldsymbol{u}; s_h - \tilde{s}^k, \phi), \\
(\partial_t e_{\tilde{\tau}}^c, \psi) - \frac{1}{\tau} a_2(e_k^c, \psi) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; e_k^c, \psi) &= c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{c}^k, \psi) \\
&\quad - \frac{1}{\tau} a_2(c_h - \tilde{c}^k, \psi) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; c_h - \tilde{c}^k, \psi).
\end{aligned}$$

Choosing $\boldsymbol{v} = \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}$, $q = p - p^k$, $\phi = e_{\tilde{\tau}}^s$, $\psi = e_{\tilde{\tau}}^c$ and then combining the first two equations, we have

$$\begin{aligned}
&(\partial_t \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + a_1(c, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + c_1(\boldsymbol{u}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) - (\partial_t \tilde{\boldsymbol{u}}_{h,r}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + a_1(c - c_h, \tilde{\boldsymbol{u}}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) \\
&\quad + c_1(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + a_1(c, \tilde{\boldsymbol{u}}_{h,c} - \tilde{\boldsymbol{u}}^k, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + c_1(\boldsymbol{u}, \tilde{\boldsymbol{u}}_{h,c} - \tilde{\boldsymbol{u}}^k, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) = 0, \\
&(\partial_t e_{\tilde{\tau}}^s, e_{\tilde{\tau}}^s) + a_2(e_{\tilde{\tau}}^s, e_{\tilde{\tau}}^s) + c_2(\boldsymbol{u}; e_{\tilde{\tau}}^s, e_{\tilde{\tau}}^s) - c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{s}^k, e_{\tilde{\tau}}^s) + a_2(s_h - \tilde{s}^k, e_{\tilde{\tau}}^s) - c_2(\boldsymbol{u}; s_h - \tilde{s}^k, e_{\tilde{\tau}}^s) = 0, \\
&(\partial_t e_{\tilde{\tau}}^c, e_{\tilde{\tau}}^c) + \frac{1}{\tau} a_2(e_{\tilde{\tau}}^c, e_{\tilde{\tau}}^c) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; e_{\tilde{\tau}}^c, e_{\tilde{\tau}}^c) - c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{c}^k, e_{\tilde{\tau}}^c) + \frac{1}{\tau} a_2(c_h - \tilde{c}^k, e_{\tilde{\tau}}^c) \\
&\quad - c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; c_h - \tilde{c}^k, e_{\tilde{\tau}}^c) = 0.
\end{aligned}$$

These identities readily allow us to derive the following bounds:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{0,\Omega}^2 + \alpha_a \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{1,\Omega}^2 \le \big(\|\partial_t \tilde{\boldsymbol{u}}_{h,r}\|_{0,\Omega} + M\|c - c_h\|_{1,\Omega} + (1 + 2M)\|\boldsymbol{u}_{hc} - \tilde{\boldsymbol{u}}^k\|_{1,\Omega}\big)\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{1,\Omega} + M\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{1,\Omega}^2, \\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\tilde{\tau}}^s\|_{0,\Omega}^2 + \hat{\alpha}_a \|e_{\tilde{\tau}}^s\|_{1,\Omega}^2 \le \big(M\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\mathcal{T}_h} + (1 + M)\|s_h - \tilde{s}^k\|_{1,\Omega}\big)\|e_{\tilde{\tau}}^s\|_{1,\Omega}, \\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\tilde{\tau}}^c\|_{0,\Omega}^2 + \frac{\hat{\alpha}_a}{\tau}\|e_{\tilde{\tau}}^c\|_{1,\Omega}^2 \le \big(M\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\mathcal{T}_h} + (1 + M)\|c_h - \tilde{c}^k\|_{1,\Omega}\big)\|e_{\tilde{\tau}}^c\|_{1,\Omega}.
\end{aligned}$$

And owing to Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{0,\Omega}^2 + \alpha_1 \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{1,\Omega}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\tilde{\tau}}^s\|_{0,\Omega}^2 + \alpha_2 \|e_{\tilde{\tau}}^s\|_{1,\Omega}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\tilde{\tau}}^c\|_{0,\Omega}^2 + \alpha_3 \|e_{\tilde{\tau}}^c\|_{1,\Omega}^2 \\
&\qquad \le C_1\big(\|\tilde{\boldsymbol{u}}_{hc} - \tilde{\boldsymbol{u}}^k\|_{1,\mathcal{T}_h}^2 + \|s_h - \tilde{s}^k\|_{1,\Omega}^2 + \|c_h - \tilde{c}^k\|_{1,\Omega}^2\big) + \|\partial_t \tilde{\boldsymbol{u}}_{h,r}\|_{0,\Omega}^2 \\
&\qquad \le 2C_1\big(\|\tilde{\boldsymbol{u}}_{hc} - \tilde{\boldsymbol{u}}_{\tilde{\tau}}\|_{1,\mathcal{T}_h}^2 + \|s_h - \tilde{s}_{\tilde{\tau}}\|_{1,\Omega}^2 + \|c_h - \tilde{c}_{\tilde{\tau}}\|_{1,\Omega}^2
\end{aligned}$$

$$+ \|\tilde{\boldsymbol{u}}_{\tilde{\tau}} - \tilde{\boldsymbol{u}}^k\|^2_{1,\mathcal{T}_h} + \|\tilde{s}_{\tilde{\tau}} - \tilde{s}^k\|^2_{1,\Omega} + \|\tilde{c}_{\tilde{\tau}} - \tilde{c}^k\|^2_{1,\Omega}\big) + \|\partial_t \tilde{\boldsymbol{u}}_{h,r}\|^2_{0,\Omega}, \tag{5.4}$$

where $C_1 = \max\{\frac{1}{2\alpha_a}(1+2M)^2, \frac{1}{2\hat{\alpha}_a}(1+M)^2, \frac{\tau}{2\hat{\alpha}_a}(1+M)^2\}$, $\alpha_1 = \frac{\alpha}{2} - M - 2M^2$, $\alpha_2 = \hat{\alpha}_a/2$ and $\alpha_3 = \frac{\hat{\alpha}_a}{2\tau} - \frac{3M^2}{2}$. Moreover, we have

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(t_n)\|^2_{0,\Omega} + \alpha_1 \int_0^{t_n} \|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(t)\|^2_{1,\Omega} + \frac{1}{2}\|e^s_{\tilde{\tau}}(t_n)\|^2_{0,\Omega} + \alpha_2 \int_0^{t_n} \|e^s_{\tilde{\tau}}(t)\|^2_{1,\Omega} + \frac{1}{2}\|e^c_{\tilde{\tau}}(t_n)\|^2_{0,\Omega} + \alpha_3 \int_0^{t_n} \|e^c_{\tilde{\tau}}(t)\|^2_{1,\Omega} \\
&\quad \le \frac{1}{2}\|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(0)\|^2_{0,\Omega} + C\left(\sum_{k=1}^n \int_{t_{k-1}}^{t_k} (\|(\tilde{\boldsymbol{u}}_{\tilde{\tau}} - \tilde{\boldsymbol{u}}^k)\|^2_{1,\mathcal{T}_h} + \|(\tilde{s}_{\tilde{\tau}} - \tilde{s}^k)\|^2_{1,\Omega} + \|(\tilde{c}_{\tilde{\tau}} - \tilde{c}^k)\|^2_{1,\Omega}\right) \\
&\qquad + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \big(\|\tilde{\boldsymbol{u}}_{hc} - \tilde{\boldsymbol{u}}_{\tilde{\tau}}\|^2_{1,\mathcal{T}_h} + \|s_h - \tilde{s}^k\|^2_{1,\Omega} + \|c_h - \tilde{c}_{\tilde{\tau}}\|^2_{1,\Omega}\big) \\
&\qquad + \frac{1}{2}\sum_{k=1}^{n-1} \big(\|\boldsymbol{u}(t_k) - I^{k+1}\tilde{\boldsymbol{u}}^k_{h,c}\|^2_{0,\Omega} - \|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(t_n)\|^2_{0,\Omega}\big) + \|\partial_t \tilde{\boldsymbol{u}}_{h,r}\|^2_{0,\Omega}.
\end{aligned}$$

In light of the definition of $\boldsymbol{u}_{\tilde{\tau}}$, $s_{\tilde{\tau}}$ and $c_{\tilde{\tau}}$, we get

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} (\|\boldsymbol{\nabla}(\boldsymbol{u}_{\tilde{\tau}} - \boldsymbol{u}^k)\|^2_{0,\mathcal{T}_h} + \|\nabla(s_{\tilde{\tau}} - s^k)\|^2_{0,\Omega} + \|\nabla(c_{\tilde{\tau}} - c^k)\|^2_{0,\Omega}) \\
&\quad \le \tilde{\tau}_k\big(\|\boldsymbol{\nabla}(\boldsymbol{u}^k - \boldsymbol{u}^{k-1})\|^2_{0,\mathcal{T}_h} + \|\nabla(s^k - s^{k-1})\|^2_{0,\Omega} + \|\nabla(c^k - c^{k-1})\|^2_{0,\Omega}\big).
\end{aligned} \tag{5.5}$$

Then we can apply triangle inequality, which gives

$$\begin{aligned}
&\tilde{\tau}_k\big(\|\boldsymbol{u}^k - \boldsymbol{u}^{k-1}\|^2_{1,\Omega} + \|s^k - s^{k-1}\|^2_{1,\Omega} + \|c^k - c^{k-1}\|^2_{1,\Omega}\big) \\
&\le \Upsilon_k^2 + \tilde{\tau}_k\big(\|\boldsymbol{u}^k - \boldsymbol{u}^k_h\|^2_{1,\mathcal{T}_h} + \|s^k - s^k_h\|^2_{1,\Omega} + \|c^k - c^k_h\|^2_{1,\Omega}\big) \\
&\quad + \tilde{\tau}_k\big(\|\boldsymbol{u}^{k-1} - I_k\boldsymbol{u}^{k-1}_h\|^2_{1,\mathcal{T}_h} + \|s^{k-1} - s^{k-1}_h\|^2_{1,\Omega} + \|c^{k-1} - c^{k-1}_h\|^2_{1,\Omega}\big).
\end{aligned} \tag{5.6}$$

Combining the results with Theorem 3.2 implies that

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(t_n)\|^2_{0,\Omega} + \alpha_1 \int_0^{t_n} \|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(t)\|^2_{1,\Omega} + \frac{1}{2}\|e^s_{\tilde{\tau}}(t_n)\|^2 + \alpha_2 \int_0^{t_n} \|e^s_{\tilde{\tau}}(t)\|^2_{1,\Omega} + \frac{1}{2}\|e^c_{\tilde{\tau}}(t_n)\|^2 + \alpha_2 \int_0^{t_n} \|e^c_{\tilde{\tau}}(t)\|^2_{1,\Omega} \\
&\le C(\Xi^2 + \Upsilon^2) + \frac{1}{2}\|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(0)\|^2_{0,\Omega} + \frac{1}{2}\|e^s_{\tilde{\tau}}(0)\|^2_{0,\Omega} + \frac{1}{2}\|e^t_{\tilde{\tau}}(0)\|^2_{0,\Omega} + \frac{1}{2}\sum_{k=1}^{n-1}\big(\|\boldsymbol{u}(t_k) - I^{k+1}\tilde{\boldsymbol{u}}^k_{h,c}\|^2_{0,\Omega} - \|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(t_n)\|^2_{0,\Omega}\big).
\end{aligned} \tag{5.7}$$

Finally, applying integration by parts in (5.3a) yields

$$\begin{aligned}
\big(\partial_t \boldsymbol{e}^{\boldsymbol{u}}_{\tilde{\tau}} + \nabla(p - p_h), \boldsymbol{v}\big) &= -a_1(c, \boldsymbol{u} - \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) - c_1(\boldsymbol{u}, \boldsymbol{u} - \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) - b(\boldsymbol{v}, p_h - \tilde{p}^k) \\
&\quad - a_1(c - c_h, \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) - c_1(\boldsymbol{u} - \boldsymbol{u}_h, \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{H}^1_0(\Omega), \\
\big(\partial_t e^s_{\tilde{\tau}}(t), \phi\big) &= a_2(s - \tilde{s}^k, \phi) - c_2(\boldsymbol{u}; s - \tilde{s}^k, \phi) + c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{s}^k, \phi), \quad \forall \phi \in H^1_0(\Omega), \\
(\partial_t e^c_{\tilde{\tau}}, \psi) &= \frac{1}{\tau} a_2(c - \tilde{c}^k, \psi) - c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; c - \tilde{c}^k, \psi) + c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{c}^k, \psi) \quad \forall \psi \in H^1_0(\Omega).
\end{aligned}$$

Next we apply Young's inequality and the definition of the dual norm. Then, we integrate the whole expression in time between t_{k-1} and t_k for each $k = 1, 2, \ldots, n$ and sum the expression for each k. Finally, we use (5.4), (5.5), (5.6) and (5.7) to establish the second stated result. □

Theorem 5.1. *Let $(\boldsymbol{u}, p, s, c)$ be the solution of (1.2), and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ the corresponding discrete solution. Let Ξ, Υ be the a posteriori error estimators defined in (5.1). Then the following reliability estimate holds:*

$$\begin{aligned}
&\big(\|\boldsymbol{e}^{\boldsymbol{u}}_{\tau}\|^2_\star + \|e^c_\tau\|^2_\star + \|e^s_\tau\|^2_\star\big)^{1/2} \\
&\quad \le C\left(\Xi^2 + \Upsilon^2 + \frac{1}{2}\|\boldsymbol{e}^{\boldsymbol{u}_c}_{\tilde{\tau}}(0)\|^2_{0,\Omega} + \frac{1}{2}\|e^s_{\tilde{\tau}}(0)\|^2_{0,\Omega} + \frac{1}{2}\|e^t_{\tilde{\tau}}(0)\|^2_{0,\Omega} + \sum_{k=1}^{N-1} \|u^k_{h,r} - I^{k+1}u^k_{h,r}\|^2_{0,\Omega}\right)^{1/2}, \\
&\sum_{k=1}^N \big(\|\partial_t \boldsymbol{e}^{\boldsymbol{u}}_{\tilde{\tau}} + \nabla(p - p_h)\|_{L^2(t_{k-1},t_k;\mathbf{H}^{-1}(\Omega))} + \|\partial_t e^s_{\tilde{\tau}}\|_{L^2(t_{k-1},t_k;H^{-1}(\Omega))} + \|\partial_t e^c_{\tilde{\tau}}\|_{L^2(t_{k-1},t_k;H^{-1}(\Omega))}\big)
\end{aligned}$$

1/h	$\mathtt{e}_{\boldsymbol{u}}$	rate	$\mathtt{e}_p$	rate	$\mathtt{e}_s$	rate	$\mathtt{e}_c$	rate	$\lVert\mathrm{div}\,\boldsymbol{u}_h\rVert_{\infty,\Omega}$
$\sqrt{2}$	1.3970	–	5.0910	–	0.03723	–	0.02511	–	2.19e-11
$2\sqrt{2}$	0.5651	1.306	1.9920	1.354	0.01098	1.762	0.00679	1.887	4.08e-12
$4\sqrt{2}$	0.1719	1.717	0.6402	1.637	0.00298	1.882	0.00171	1.990	1.00e-12
$8\sqrt{2}$	0.0456	1.914	0.1695	1.917	0.00080	1.904	0.00046	1.903	5.20e-13
$16\sqrt{2}$	0.0115	1.994	0.0412	2.039	0.00021	1.941	0.00012	1.962	2.23e-13

TABLE 6.1. Example 1. Experimental errors and convergence rates for the approximate solutions $\boldsymbol{u}_h$, p_h, s_h and c_h. The ℓ^∞-norm of the vector formed by the divergence of the discrete velocity computed at time t_{end} for each discretisation is shown in the last column.

$$\leq C\left(\Xi^2+\Upsilon^2+\frac{1}{2}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2+\frac{1}{2}\|e_{\tilde{\tau}}^{s}(0)\|_{0,\Omega}^2+\frac{1}{2}\|e_{\tilde{\tau}}^{t}(0)\|_{0,\Omega}^2+\sum_{k=1}^{N-1}\|u_{h,r}^k-I^{k+1}u_{h,r}^k\|_{0,\Omega}^2\right)^{1/2},$$

where we define

$$\|\boldsymbol{v}\|_\star^2 := \int_0^T \|\boldsymbol{v}\|_{1,\mathcal{T}_h}^2\,\mathrm{d}t, \quad \|\phi\|_\star^2 := \int_0^T \|\phi\|_{1,\Omega}^2\,\mathrm{d}t.$$

Proof. Using $\boldsymbol{u}_h^k=\boldsymbol{u}_{h,c}^k+\boldsymbol{u}_{h,r}^k$ together with the identity in [24, (5.59)-(5.60)] that in our context reads

$$\|\boldsymbol{u}(t_k)-I^{k+1}\tilde{\boldsymbol{u}}_{h,c}^k\|_{0,\Omega}^2-\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_k)\|_{0,\Omega}^2=\|\boldsymbol{u}_{h,r}^k-I^{k+1}\boldsymbol{u}_{h,r}^k\|_{0,\Omega}^2+\langle\boldsymbol{u}_{h,r}^k-I^{k+1}\boldsymbol{u}_{h,r}^k,e_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_k)\rangle,$$

we can invoke Lemma 5.1 and reuse the strategy applied in Theorem 4.1 to complete the proof. □

6. Numerical tests

We now present computational examples illustrating the properties of the numerical schemes. All numerical routines have been realised using the open-source finite element libraries FEniCS [8] and FreeFem++ [27].

6.1. **Example 1: accuracy verification against smooth solutions.** A known analytical solution example is used to verify theoretical convergence rates of the scheme. We choose $t_{\mathrm{end}}=2.0$ and $\Omega=(0,1)^2$. We take the parameter values $\nu=1.0$, $\rho=1.0$, $\rho_{\mathrm{m}}=1.5$, $\boldsymbol{g}=(0,-1)^T$, $\mathrm{Sc}=1.0$, $\tau=0.5$, $v_p=1.0$, $a_0=50$. Following the approach of manufactured solutions, we prescribe boundary data and additional external forces and adequate source terms so that the closed-form solutions to (1.1) are given by the smooth functions

$$\boldsymbol{u}(x,y,t)=\begin{pmatrix}(\sin(\pi x))^2(\sin(\pi y))^2\cos(\pi y)\sin(t)\\ -1/3\sin(2\pi x)(\sin(\pi y))^3\sin(t)\end{pmatrix},\quad p(x,y,t)=(x^4-y^4)\sin(t),$$
$$c(x,y,t)=\frac{1}{2}(1+\cos(\pi/4(xy)))\exp(-t),\quad s(x,y,t)=\frac{1}{2}(1+\sin(\pi/2(xy)))\exp(-t).$$

As $\boldsymbol{u}$ is prescribed everywhere on $\partial\Omega$, for sake of uniqueness we impose $p\in L_0^2(\Omega)$ through a real Lagrange multiplier approach. To verify the *a priori* error estimates, we introduce the discrete norms

$$|||\boldsymbol{u}|||_{0,\mathcal{T}_h}:=\left(\Delta t\sum_{n=1}^N\|\boldsymbol{u}_h^n\|_{1,\mathcal{T}_h}^2\right)^{1/2},\quad\text{and}\quad |||\chi|||_{0,k}:=\left(\Delta t\sum_{n=1}^N\|\chi_h^n\|_{k,\Omega}^2\right)^{1/2}.$$

The corresponding individual errors and convergence rates are computed as

$$\mathtt{e}_{\boldsymbol{u}}=|||\boldsymbol{u}-\boldsymbol{u}_h|||_{0,\mathcal{T}_h},\quad \mathtt{e}_p=|||p-p_h|||_{0,0},\quad \mathtt{e}_s=|||s-s_h|||_{0,1},\quad \mathtt{e}_c=|||c-c_h|||_{0,1},$$
$$\mathtt{rate}=\log(e_{(\cdot)}/\tilde{e}_{(\cdot)})[\log(\xi/\tilde{\xi})]^{-1},\xi=\{h,\Delta t\},\tag{6.1}$$

where $e,\tilde{e}$ denote errors generated on two consecutive pairs of mesh size and time step $(h,\Delta t)$, and $(\tilde{h},\tilde{\Delta t})$, respectively. Choosing $\Delta t=\sqrt{2}h$ and using scheme (2.4), the results in Table 6.1 confirm that the rates of convergence are optimal, coinciding with the theoretical bounds anticipated in Theorem 2.8.

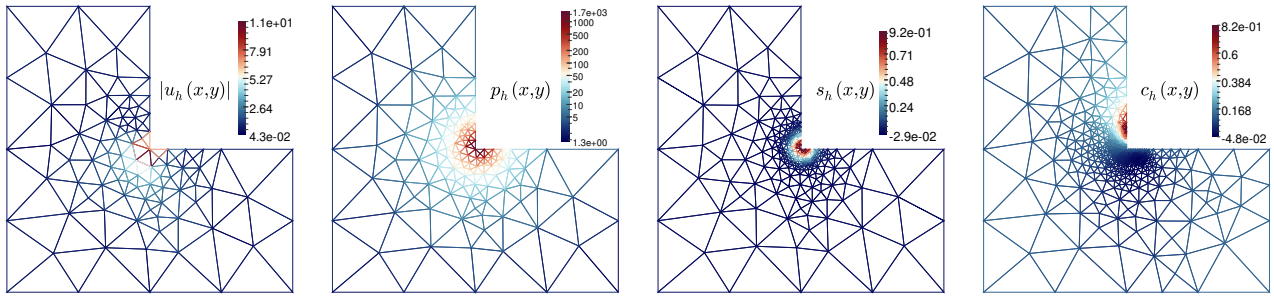

FIGURE 6.1. Example 2. Approximate velocity magnitude (after 3 refinement steps), pressure (after 4 refinement steps), concentration s (after 5 refinement steps), and distribution of c after 6 steps of adaptive refinement.

6.2. **Example 2: adaptive mesh refinement for the stationary problem.** The classical Dörfler strategy [23] is employed for the adaptive algorithm based on the steps of solving, estimating, marking, and refining. Estimation is performed by computing the error indicators and using them to select/mark elements that contribute the most to the error [30]. The marking is done following the bulk criterion of selecting sufficiently many elements so that they represent a given fraction of the total estimated error. That is, one refines all elements $K \in \mathcal{T}_h$ for which

$$\Psi_K \geq \gamma_{\text{ratio}} \max_{L \in \mathcal{T}_h} \Psi_L,$$

where $0 < \gamma_{\text{ratio}} < 1$ is a user-defined constant (that we tune in order to generate a similar number of degrees of freedom, or comparable errors, as those obtained under uniform refinement). And then the algorithm aims for equidistribution of the local error indicator in the updated mesh.

In the adaptive case, instead of (6.1) the convergence rates (for the spatial errors) are computed as

$$\texttt{rate} = -2\log(e_{(\cdot)}/\tilde{e}_{(\cdot)})[\log(\texttt{DoF}/\widetilde{\texttt{DoF}})]^{-1},$$

where $\texttt{DoF}$ and $\widetilde{\texttt{DoF}}$ are the number of degrees of freedom associated with each refinement level. The robustness of the global estimators is measured using the effectivity index (ratio between the total error and the indicator)

$$\texttt{eff}(\Psi) = \frac{\left\{\mathsf{e}_{\boldsymbol{u}}^2 + \mathsf{e}_p^2 + \mathsf{e}_s^2 + \mathsf{e}_c^2\right\}^{1/2}}{\Psi}.$$

We start with verifying the robustness of the *a posteriori* error estimator Ψ, and construct closed-form solutions to the stationary counterpart of the coupled problem (1.1). We consider concentration-dependent viscosity, model parameter values, and stabilisation constant as

$$\nu(c) = \frac{1}{10}(1 + \exp(-1/4\,c)), \quad \rho = 1, \quad \rho_{\text{m}} = 1.5, \quad \boldsymbol{g} = (0,-1)^T, \quad \text{Sc} = 1,$$
$$\tau = 0.5, \quad v_p = 1, \quad \alpha = 0.5, \quad \beta = 0.5, \quad a_0 = 5.$$

The considered exact solutions are defined on the L-shaped domain $\Omega = (-1,1)^2 \setminus (0,1)^2$

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p(x,y) = \frac{2 + \sin(xy)}{(x-0.02)^2 + (y-0.02)^2},$$
$$s(x,y) = \exp(-150(x-0.01)^2 - 150(y-0.01)^2), \quad c(x,y) = \frac{1}{10} + \frac{\cos(\pi x)\sin(\frac{\pi}{2}y)}{25((x-0.1)^2 + (y-0.1)^2)}.$$

These solutions exhibit a generic singularity towards the reentrant corner, and therefore one expects that the error decay is suboptimal when applying uniform mesh refinement. After solving the coupled stationary problem on sequences of uniformly and adaptively refined meshes, the aforementioned behaviour is indeed observed in Table 6.2, where the first part of the table shows deterioration of the convergence due to the high gradients of the exact solutions on the non-convex domain. The results shown in the bottom block of the table confirm that as more degrees of freedom are added, a restored error reduction rate is observed due to adaptive mesh refinement guided by the *a posteriori* error estimator Ψ. The second-last column of the table also indicates that the effectivity index oscillates under uniform refinement, while it is much more steady in the adaptive case. We tabulate as well the Newton-Raphson iteration count (needed to reach the relative residual tolerance of 1e-6), and this number is also systematically smaller for the adaptive case (about four steps in all instances) than for the uniform refinement case (up to seven nonlinear steps for certain refinement levels). As an example we plot

DoF	h	$\mathtt{e}_{\boldsymbol{u}}$	rate	$\mathtt{e}_p$	rate	$\mathtt{e}_s$	rate	$\mathtt{e}_c$	rate	$\lVert\operatorname{div}\boldsymbol{u}_h\rVert_{\infty,\Omega}$	eff(Ψ)	iter
Error history under uniform mesh refinement												
79	1	26.85	–	284.0	–	1.829	–	1.3724	–	2.22e-12	0.329	5
275	0.5	19.57	0.396	152.9	0.893	1.736	0.075	0.9903	0.499	3.65e-13	0.257	7
1027	0.25	29.59	-0.622	86.54	0.821	1.188	0.547	0.7004	0.441	1.07e-13	0.111	6
3971	0.125	12.53	0.924	72.47	0.256	0.642	0.886	0.4834	0.535	3.21e-14	0.104	7
15619	0.0625	7.561	0.805	53.41	0.440	0.457	0.491	0.2863	0.756	3.39e-14	0.171	7
Error history under adaptive mesh refinement												
79	1	26.85	–	284.0	–	2.829	–	1.3724	–	2.22e-12	0.329	4
275	0.5	15.92	0.824	154.8	0.973	1.725	0.937	0.9617	0.573	3.67e-13	0.261	3
943	0.5	10.78	1.172	90.23	0.878	0.822	0.863	0.6793	1.064	2.18e-13	0.260	4
1601	0.5	7.398	2.499	74.35	0.732	0.641	1.428	0.4398	1.642	2.28e-13	0.261	4
2363	0.5	2.139	2.265	53.35	1.706	0.461	1.569	0.2683	2.539	2.15e-13	0.257	3
4253	0.2877	3.420	1.394	29.41	2.027	0.235	2.295	0.1541	1.888	1.05e-13	0.258	4
11662	0.25	1.012	1.267	17.58	1.019	0.118	1.368	0.0873	1.126	1.07e-13	0.258	5
38174	0.1416	0.557	1.006	9.388	1.058	0.059	1.156	0.0464	1.063	9.01e-13	0.261	4

TABLE 6.2. Example 2. Experimental errors and convergence rates for the approximate solutions $\boldsymbol{u}_h$, p_h, s_h and c_h of the stationary problem under uniform and adaptive mesh refinement following the estimator Ψ. For the adaptive case we employ $\gamma_{\text{ratio}} = 1 \cdot 10^{-4}$.

in Figure 6.1 solutions on relative coarse meshes and display meshes generated with the adaptive algorithm, indicating significant refinement near the reentrant corner. Let us also remark that the boundary conditions for velocity have been imposed (here and in all other tests) essentially for the normal component, while the tangent component is fixed through a Nitsche's penalisation. For this example we use a constant $a_{\text{Nitsche}} = 10^3$.

6.3. **Example 3: robustness of the estimator for the transient problem.** Next we turn to the numerical verification of robustness of the *a posteriori* error estimator for the fully discrete approximations of the time-dependent coupled problem. We consider now the time interval $(0, 0.01]$ and choose $\Delta t = 0.002$. The closed-form solutions on the unit square domain are as follows

$$\boldsymbol{u}(x,y,t) = \sin(t)\begin{pmatrix}\cos(\pi x)\sin(\pi y)\\ -\sin(\pi x)\cos(\pi y)\end{pmatrix}, \quad p(x,y,t) = \cos(t)(x^4 - y^4),$$

$$c(x,y,t) = \frac{1}{2}(1+\cos(\pi/4(xy)))\exp(-t), \quad s(x,y,t) = \frac{1}{2}(1+\sin(\pi/2(xy)))\exp(-t).$$

Cumulative errors up to t_{final} are computed as

$$E_{\boldsymbol{u}} := \left(\Delta t \sum_{n=1}^{N} \lVert \boldsymbol{u}_h^n - \boldsymbol{u}(t^n)\rVert_{1,\mathcal{T}_h}^2\right)^{1/2}, \quad E_p := \left(\Delta t \sum_{n=1}^{N} \lVert p_h^n - p(t^n)\rVert_{0,\Omega}^2\right)^{1/2},$$

$$E_s := \left(\Delta t \sum_{n=1}^{N} \lVert s_h^n - s(t^n)\rVert_{1,\Omega}^2\right)^{1/2}, \quad E_c := \left(\Delta t \sum_{n=1}^{N} \lVert c_h^n - c(t^n)\rVert_{1,\Omega}^2\right)^{1/2},$$

and the resulting error history, after six steps of uniform mesh refinement, is collected in Table 6.3. To be consistent with the development in Section 5, the numerical verification in this set of tests was carried out using a backward Euler time discretisation. The *a posteriori* error estimator (5.1) is computed and the effectivity index is also tabulated, showing that the estimator is robust (and confirming the theoretical reliability bound as well as giving an heuristic indication of its efficiency). Note that in this case, since the mesh refinement is uniform, the auxiliary interpolation of the solutions at the last time step on the current mesh is not necessary. The average number of Newton-Raphson iterations required for convergence was 3.2.

6.4. **Example 4: simulation of salinity-driven flow instabilities.** To illustrate the behaviour of the model and the proposed method, we simulate a salinity-driven flow problem. Similar examples are found in [19] where direct numerical simulations (DNS) are applied to the version of (1.1) that has constant viscosity.

The configuration of layering in sedimentation is taken from [33]. We consider a rectangular domain with $L_x = 40$, $L_y = 300$ and an initial solid-particle concentration profile that is periodic in the horizontal direction, and periodic with Gaussian noise in the vertical direction:

$$s(x,y,z,0) = A_0 \exp(-z^2/\sigma^2) + A_1 \sin(x),$$

DoF	h	$E_{\boldsymbol{u}}$	rate	E_p	rate	E_s	rate	E_c	rate	eff(Υ)
59	0.7071	0.00228	–	0.02080	–	0.01811	–	0.00711	–	0.0859
195	0.3536	0.00081	1.504	0.01191	0.843	0.00989	0.872	0.00303	1.227	0.0970
707	0.1768	0.00034	1.217	0.00621	0.940	0.00486	1.025	0.00149	1.022	0.0973
2691	0.0884	0.00015	1.119	0.00313	0.984	0.00241	1.009	0.00074	1.008	0.0967
10499	0.0442	7.70e-05	1.053	0.00157	0.996	0.00120	1.004	0.00031	1.003	0.0961
41475	0.0221	3.78e-05	1.025	0.00078	0.999	0.00060	1.002	0.00018	1.001	0.0972
164867	0.0110	1.87e-05	1.012	0.00039	0.999	0.00030	1.001	9.26e-05	1.000	0.0966

TABLE 6.3. Example 3. Experimental errors and convergence rates for the approximate solutions of the transient problem under uniform mesh refinement following the estimator Υ.

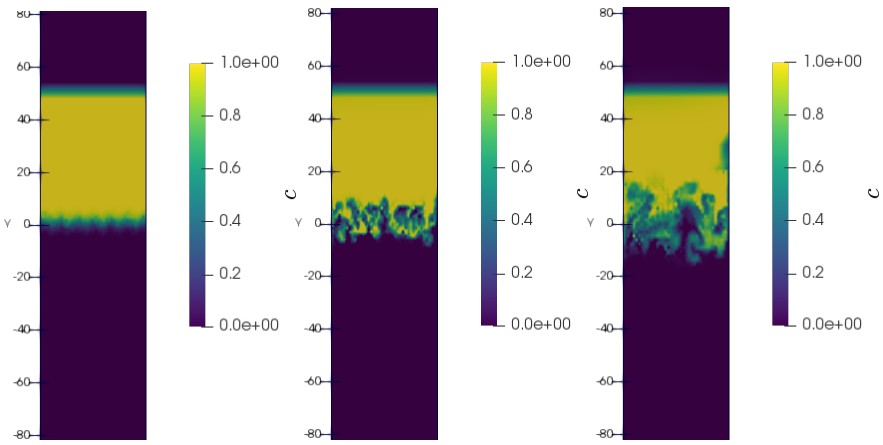

FIGURE 6.2. Example 4. Solid particle concentration profile at times $t = 0.0, 3.5$ and $t = 6.0$.

with initial amplitudes A_0, A_1 and width σ (see Figure 6.2). For the velocity field, we use non-slip boundary condition in all four walls and we choose $\Delta t = 0.1$. As discussed in [33], simulations at low density ratios are extremely costly because of the large Reynolds numbers of fingering convection. In consequence we choose an initial density ratio $R_0 = \alpha s_{0z}/\beta c_{0,z} \approx 4$, and we carry out the simulations on tall, thin domains. Apart from the specifications above, the remaining constant parameters needed in the model take the following values

$$A_0 = 2.86, \quad A_1 = 0.5, \quad \sigma = 0.35, \quad \nu = 1 \times 10^{-3}\,\mathrm{kg/m^3}, \quad g = 9.8\mathrm{m/s^2}, \quad \mathrm{Sc} = 7.0,$$
$$\tau = 25, \quad v_{\mathrm{p}} = 0.04\mathrm{m/s}, \quad \alpha = -2.0, \quad \beta = 0.5.$$

According to [33] a linear fingering instability occurs provided $1 < R_0 < \tau$, hence the instability shown in Figure 6.2 is expected.

6.5. **Example 5: adaptive simulation of exothermic flows.** To conclude this section, and to include an illustrative simulation exemplifying that the $\mathbf{H}(\mathrm{div})$-conforming scheme along with the *a posteriori* error estimator perform well for an applicative problem, we address the computation of exothermic flows that develop fingering instabilities. The problem configuration is adapted as a simplification of the problem solved in [31], where the fields c, s represent solutal concentration and temperature, respectively. The model assumes an additional drag term due to porosity so that the momentum equation is of Navier-Stokes-Brinkman type. The domain is the rectangular region $\Omega = (0, L) \times (0, H)$, and the initial solutal and temperature profiles are imposed as

$$c^0(x,y) = \begin{cases} 0.999 + 0.001\zeta_c & \text{if } H - \epsilon \le y \le H, \\ 0 & \text{otherwise,} \end{cases} \qquad s^0(x,y) = \begin{cases} 0.999 + 0.001\zeta_s & \text{if } H - \epsilon \le y \le H, \\ 0 & \text{otherwise} \end{cases},$$

where ζ_c, ζ_s are random fields uniformly distributed on $[0, 1]$. The geometric and model constants are $H = 1000$, $L = 2000$, $\Delta t = 20$, $t_{\mathrm{end}} = 1500$, $\nu = 1 + 0.25\zeta_\nu$, $\kappa = 1$, $1/\mathrm{Sc} = 8$, $1/(\tau\mathrm{Sc}) = 2.5$, $\rho_{\mathrm{m}} = 1$, $v_{\mathrm{p}} = 0$, $\alpha = 5$, $\beta = -1$.

Boundary conditions are of mixed type for solutal and temperature. Both fields are prescribed to 0 and 1 on the bottom and top of the domain, respectively; while on the vertical walls we impose zero-flux boundary conditions. The velocity is of slip type on the whole boundary, and therefore a zero-mean condition for the pressure is considered using a real Lagrange multiplier. The solution algorithm, differently from the previous tests, is based on an inner fixed-point iteration between an Oseen and a transport system, rather than an

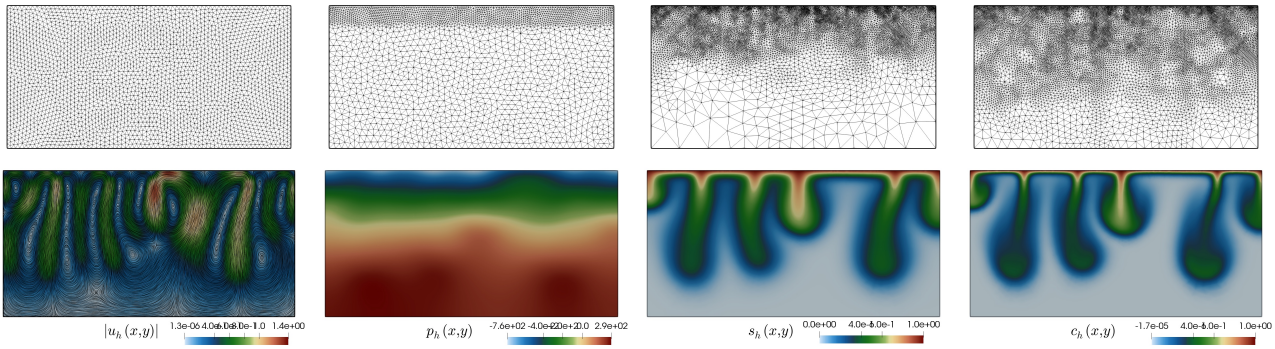

FIGURE 6.3. Example 5. Samples of adapted meshes at times $t = 0, 60, 600, 1200$ (top panels), and approximate solutions shown at time $t = 1500$ (bottom).

exact Newton-Raphson method. An initial coarse mesh of 5300 elements is constructed, and an adaptive mesh refinement (only one iteration) guided by the estimator (5.1) is applied at the end of each time step. Figure 6.3 shows snapshots of adapted meshes at different times, and also samples of solute concentration, temperature distribution, velocity, and pressure at the final time.

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