New Banach spaces-based mixed finite element methods for the coupled poroelasticity and heat equations*

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Abstract

In this paper we introduce and analyze a Banach spaces-based approach yielding a fully-mixed finite element method for numerically solving the coupled poroelasticity and heat equations, which describe the interaction between the fields of deformation and temperature. A non-symmetric pseudostress tensor is utilized to redefine the constitutive equation for the total stress, which is an extension of Hooke's law to account for thermal effects. The resulting continuous formulation, posed in suitable Banach spaces, consists of a coupled system of three saddle point-type problems, each with right-hand terms that depend on data and the unknowns of the other two. The well-posedness of it is analyzed by means of a fixed-point strategy, so that the classical Banach theorem, along with the Babuška–Brezzi theory in Banach spaces, allow to conclude, under a smallness assumption on the data, the existence of a unique solution. The discrete analysis is conducted in a similar manner, utilizing the Brouwer and Banach theorems to demonstrate both the existence and uniqueness of the discrete solution. The rates of convergence of the resulting Galerkin method are then presented. Finally, a number of numerical tests are shown to validate the aforementioned statement and demonstrate the good performance of the method.

Key words: Thermo-poroelasticity, porous media, mixed finite element methods, analysis in Banach spaces.

Mathematics subject classifications (2000): 65N30, 65J25, 74F05, 74F10.

1 Introduction

Scope. The relationship between the flow of a viscous fluid and the deformation of an elastic solid within a porous medium is described by the poroelasticity equations, which were initially introduced in the early works [34] and [7, 8]. While porous materials are commonly associated with objects such as rocks and clays, they also encompass a broader range of materials, including biological tissues, foams, and even paper products. Moreover, in applications such as the underground disposal of radioactive waste, geothermal energy production, and oil extraction from deep, high-temperature,

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high-pressure reservoirs, temperature plays a crucial role. Therefore, to study these phenomena, we focus on the coupling between poroelasticity and heat equations. The resulting system, a slightly modified version of the thermo-poroelastic problem [11, 12, 13], is non-linear and strongly coupled. The set of equations consists of the steady-state balance of linear momentum for the mixture and mass balance for the fluid content (using the modified Darcy law) and a convection-diffusion equation depending on the Darcy seepage velocity and the total stress. In terms of numerical solvability, a wide variety of techniques have been developed to simulate the poroelasticity problem, both by itself [10] and when coupled with other equations. These include couplings with chemotaxis [4], elasticity [2], Stokes [9, 33] and diffusion [31]. The thermo-poroelasticity problem has also been recently addressed in [11, 12, 13, 35, 36, 37]. These references include primal formulations [35], a combination of primal and mixed formulations [11, 37], discontinuous Galerkin methods [3], a fully-mixed formulation [12], and a mixed-primal-characteristics finite element method [37]. The introduction of additional variables of physical relevance is a common approach to solving problems that involve couplings and nonlinearities. Consequently, mixed methods are strongly justified in such a scenario. A recent approach to this method consists of defining the corresponding variational formulation in terms of Banach spaces instead of the usual Hilbertian framework without augmentation. It is important to note that, although augmented methods allow the recovery of a Hilbertian framework, they increase the cost of the computational implementation of the Galerkin scheme. Therefore, an analysis based on Banach spaces has the advantage of studying the problem in its purest form. Another advantage of this method lies in the relaxation of assumptions that must be made about the data, source terms, and eventual solutions of the system. Consequently, the unknowns are now associated with the natural spaces that result from the testing and integration by parts procedures; formulations of the models become simpler and more faithful to the original physical models; momentum-conservative schemes can be acquired; and additional unknowns can be calculated through postprocessing formulas. As a non-exhaustive list of contributions taking advantage of the use of Banach frameworks for solving the aforementioned kinds of problems, we refer to [14, 16, 17, 20, 23], and among the different models considered there, we find elasticity, Brinkman–Forchheimer, Poisson–Nernst–Planck, Navier-Stokes, chemotaxis/Navier–Stokes, Boussinesq, coupled flow-transport, and fluidized beds. For the coupled poroelasticity and heat equations, however, no mixed methods with the aforementioned benefits have, up to our knowledge, been developed yet. As motivated by the preceding discussion, the goal of this paper is to develop a Banach spaces-based formulation leading to new mixed finite element methods for the poroelasticity-heat model.

The manuscript is organized as follows. The rest of this section collects some preliminary notations, definitions, and results to be utilized throughout the paper. In Section 2, we describe the model of interest. In particular, we reformulate it in terms of the non-symmetric pseudostress tensor. In Section 3 we derive the fully-mixed variational formulation of the problem by splitting the analysis according to the three equations forming the coupled model. Suitable integration by parts formulae jointly with the Cauchy–Schwarz and Hölder inequalities are crucial for determining the right Lebesgue and related spaces to which the unknowns and corresponding test functions are required to belong. In Section 4, a fixed-point strategy is adopted to analyze the solvability of the continuous formulation. The Babuška–Brezzi theory in Banach spaces is employed to study the corresponding uncoupled problems, and then the classical Banach theorem is applied to conclude the existence of a unique solution. An analog fixed-point approach to that of Section 4 is utilized in Section 5 to study the well-posedness of the associated Galerkin scheme. Finally, numerical results showing how well the method works and confirming the theoretical rates of convergence given in Section 5, are presented in Section 6.

Preliminaries. Throughout the paper Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star-shaped with respect to a ball, and whose outward unit normal at its boundary Γ is denoted $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in [1,+\infty)$, and Sobolev spaces

$\mathrm{W}^{\ell,t}(\Omega)$ and $\mathrm{W}_0^{\ell,t}(\Omega)$, with $\ell \geqslant 0$, whose corresponding norms and seminorms, either for the scalar, vector, or tensorial version, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{\ell,t;\Omega}$, and $|\cdot|_{\ell,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and that when $t = 2$, we simply write $\mathrm{H}^\ell(\Omega)$ instead of $\mathrm{W}^{\ell,2}(\Omega)$, with its norm and seminorm denoted by $\|\cdot\|_{\ell;\Omega}$ and $|\cdot|_{\ell;\Omega}$, respectively. Now, letting $t,\, t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we let $\mathrm{W}^{1/t',t}(\Gamma)$ and $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the trace space of $\mathrm{W}^{1,t}(\Omega)$ and its dual, respectively, and denote the duality pairing between them by $\langle\cdot,\cdot\rangle$. In particular, when $t = t' = 2$, we simply write $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ instead of $\mathrm{W}^{1/2,2}(\Gamma)$ and $\mathrm{W}^{-1/2,2}(\Gamma)$, respectively.

Given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be its vector and tensorial counterparts. Furthermore, for any vector fields $\boldsymbol{v} = (v_i)_{i=1,n}$ and $\boldsymbol{w} = (w_i)_{i=1,n}$, we set the gradient and divergence operators as

$$\nabla \boldsymbol{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n} \quad \text{and} \quad \mathrm{div}(\boldsymbol{v}) := \sum_{j=1}^n \frac{\partial v_j}{\partial x_j}\,.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}\,, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^n \tau_{ii}\,, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij}\zeta_{ij}\,, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}\,, \tag{1.1}$$

where $\mathbb{I}$ stands for the identity tensor of $\mathbb{R} := \mathrm{R}^{n\times n}$. On the other hand, for each $t \in [1,+\infty)$, we introduce the Banach spaces

$$\begin{aligned}
\mathbf{H}(\mathrm{div}_t;\Omega) &:= \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\Big\}\,,\\
\mathbf{H}^t(\mathrm{div}_t;\Omega) &:= \Big\{\boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\Big\}\,,
\end{aligned}$$

and

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^t(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}\,,$$

which are endowed with the natural norms

$$\begin{aligned}
\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} &:= \|\boldsymbol{\tau}\|_{0;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega)\,,\\
\|\boldsymbol{\tau}\|_{t,\mathrm{div}_t;\Omega} &:= \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\,,
\end{aligned}$$

and

$$\|\boldsymbol{\tau}\|_{t,\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\,.$$

Then, we recall that, proceeding as in [25, eq. (1.43), Section 1.3.4] (see also [15, Section 4.1] and [21, Section 3.1]), one can prove that for each $t \in \begin{cases} (1,+\infty] \text{ in } \mathrm{R}^2\,,\\ [\frac{6}{5},+\infty] \text{ in } \mathrm{R}^3\,,\end{cases}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},v) \in \mathbf{H}(\mathrm{div}_t;\Omega)\times \mathrm{H}^1(\Omega)\,, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. In turn, given $t,\, t' \in (1,+\infty)$ conjugate to each other, there also holds (cf. [24, Corollary B.57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},v) \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega)\,, \tag{1.3}$$

and analogously

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \boldsymbol{v}\rangle = \int_\Omega \Big\{\boldsymbol{\tau} : \nabla \boldsymbol{v} + \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\times \mathbf{W}^{1,t'}(\Omega)\,, \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ denotes in (1.3) (resp. (1.4)) the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ (resp. $\mathbf{W}^{-1/t,t}(\Gamma)$) and $\mathrm{W}^{1/t,t'}(\Gamma)$ (resp. $\mathbf{W}^{1/t,t'}(\Gamma)$).

2 Governing equations and boundary conditions

We consider a homogeneous porous medium constituted by a mixture of incompressible grains and interstitial fluid. The domain of interest $\Omega \subset \mathrm{R}^n, n = 2,3$, is assumed bounded. For a given body force $\mathbf{f}$ and given source terms f and g neglecting convective, gravitational, and inertial terms, we will concentrate the discussion on the following Biot's equations coupled with a stationary convection-diffusion equation modeling the heat of the mixture:

$$\boldsymbol{\sigma} \;=\; 2\mu\boldsymbol{e}(\boldsymbol{u}) + \lambda\mathrm{div}(\boldsymbol{u})\mathbb{I} - (\alpha p + \beta\theta)\mathbb{I}, \quad -\mathbf{div}(\boldsymbol{\sigma}) \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,, \tag{2.1a}$$

$$\chi\, p + \alpha\mathrm{div}(\boldsymbol{u}) - \mathrm{div}(\boldsymbol{w}) = f\,, \quad \boldsymbol{w} = \frac{\kappa}{\eta}\nabla p \quad \text{in} \quad \Omega\,, \tag{2.1b}$$

$$\theta + \boldsymbol{w}\cdot\nabla\theta - \mathrm{div}(\mathcal{D}(\boldsymbol{\sigma})\nabla\theta) = g \quad \text{in} \quad \Omega\,, \tag{2.1c}$$

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}}\,, \quad p = p_{\mathrm{D}} \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \Gamma\,, \tag{2.1d}$$

where the tensor $\boldsymbol{\sigma}$ is a generalized Hooke's law, extended to include thermal effects, $\boldsymbol{u}$ is the unknown vector of displacement of the solid particles, p is the bulk pressure of the fluid, $\boldsymbol{w}$ is the Darcy's seepage velocity and θ is the temperature distribution. The remaining terms are the infinitesimal strain tensor $\boldsymbol{e}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{t}})$, the permeability of the porous solid κ, the Lamé constants of the solid (moduli of dilation and shear, respectively) λ and μ, the constrained specific storage coefficient $\chi > 0$, the Biot-Willis parameter $\alpha \in (0,1]$, the scaling of active stress that indicates a two-way coupling between diffusion and motion β, the viscosity of the pore fluid η, and the stress-dependent diffusivity accounting for an altered diffusion acting in the poroelastic domain $\mathcal{D} : \mathbb{R} \to \mathrm{R}$.

Observe that tensor $\boldsymbol{\sigma}$ is symmetric since $\boldsymbol{e}(\boldsymbol{u})$ and $\mathbb{I}$ are both symmetric. In order to avoid the weak imposition of the symmetry of $\boldsymbol{\sigma}$, we reformulate (2.1) in terms of the pseudostress $\boldsymbol{\rho}$ (non-symmetric stress), defined by

$$\boldsymbol{\rho} := \mu\nabla\boldsymbol{u} + (\mu+\lambda)\mathrm{div}(\boldsymbol{u})\mathbb{I} - (\alpha p + \beta\theta)\mathbb{I} \quad \text{in} \quad \Omega\,. \tag{2.2}$$

Now, by applying trace to (2.2), we can express $\mathrm{div}(\boldsymbol{u})$ in terms of $\boldsymbol{\rho}$, p and θ, namely

$$\mathrm{div}(\boldsymbol{u}) = \gamma(\lambda)\left(\mathrm{tr}(\boldsymbol{\rho}) + n(\alpha p + \beta\theta)\right)\,, \tag{2.3}$$

with the parameter-dependent coefficient

$$\gamma(\lambda) \;:=\; (n\lambda + (n+1)\mu)^{-1}\,. \tag{2.4}$$

While this coefficient depends also on μ and n, only its dependence on λ and its relation with other model parameters will be important when we analyze the formulation in the quasi-incompressibility limit. Replacing the obtained expression for $\mathrm{div}(\boldsymbol{u})$ into (2.2) and using (1.1), we can equivalently rewrite the equations in (2.1a) in terms of $\boldsymbol{\rho}$ as follows

$$\frac{1}{\mu}\boldsymbol{\rho}^{\mathrm{d}} + \frac{\gamma(\lambda)}{n}\mathrm{tr}(\boldsymbol{\rho})\mathbb{I} - \nabla\boldsymbol{u} = -\gamma(\lambda)(\alpha p + \beta\theta)\mathbb{I}\,, \quad -\mathbf{div}(\boldsymbol{\rho}) = \mathbf{f} \quad \text{in} \quad \Omega\,.$$

Note that for the second equation above, we have used the fact that $\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{div}(\boldsymbol{\rho})$, which can be corroborated by taking divergence to the first equation of (2.1a) and to (2.2), respectively. Moreover, replacing (2.3) into the first equation of (2.1b), we obtain

$$c_1(\lambda)p - \mathrm{div}(\boldsymbol{w}) = f - c_2(\lambda)\mathrm{tr}(\boldsymbol{\rho}) - c_3(\lambda)\theta\,,$$

where we have used the following parameter-dependent coefficients

$$c_1(\lambda) \;:=\; \chi \;+ n\,\alpha^2\,\gamma(\lambda)\,, \quad c_2(\lambda) \;:=\; \alpha\,\gamma(\lambda)\,, \quad \text{and} \quad c_3(\lambda) \;:=\; n\,\alpha\,\beta\,\gamma(\lambda)\,. \tag{2.5}$$

Again, we stress here the dependence on λ only. Next we reformulate (2.1c) in terms of $\boldsymbol{\rho}$ within the diffusivity function $\mathcal{D}$, it is necessary to establish the function that maps $\boldsymbol{\sigma}$ to the triple $(\boldsymbol{\rho}, p, \theta)$. In this regard, from (2.1a), we have

$$2\mu \boldsymbol{e}(\boldsymbol{u}) = \boldsymbol{\rho} + \boldsymbol{\rho}^{\mathtt{t}} - 2(\mu+\lambda)\mathrm{div}(\boldsymbol{u})\mathbb{I} + 2(\alpha p + \beta\theta)\mathbb{I}\,, \tag{2.6}$$

and thus, we deduce from (2.2), along with (2.3) and (2.6), that the original stress tensor $\boldsymbol{\sigma}$ can be expressed in terms of the pseudostress $\boldsymbol{\rho}$, pressure p and temperature θ, through the linear mapping

$$\mathcal{C}(\boldsymbol{\rho}, p, \theta) := \boldsymbol{\rho} + \boldsymbol{\rho}^{\mathtt{t}} - \gamma(\lambda)\Big((2\mu+\lambda)\mathrm{tr}(\boldsymbol{\rho}) + (2n-1)(\alpha p + \beta\theta)\Big)\mathbb{I} = \boldsymbol{\sigma}\,. \tag{2.7}$$

Consequently, we can recast the original stress-dependent diffusivity $\mathcal{D}$ by a function $\mathcal{K}$ depending on $\boldsymbol{\rho}$, p and θ defined by

$$\mathcal{K}(\boldsymbol{\rho}, p, \theta) := \mathcal{D}(\mathcal{C}(\boldsymbol{\rho}, p, \theta))\,. \tag{2.8}$$

Finally, the model equations in (2.1) are restated, equivalently, on the unknowns $\boldsymbol{\rho}$, p and θ by the coupled system:

$$\frac{1}{\mu}\boldsymbol{\rho}^{\mathtt{d}} + \frac{\gamma(\lambda)}{n}\mathrm{tr}(\boldsymbol{\rho})\mathbb{I} - \nabla\boldsymbol{u} = -\gamma(\lambda)(\alpha p + \beta\theta)\mathbb{I}\,, \quad -\mathbf{div}(\boldsymbol{\rho}) = \mathbf{f} \quad \text{in} \quad \Omega\,, \tag{2.9a}$$

$$c_1(\lambda)p - \mathrm{div}(\boldsymbol{w}) = f - c_2(\lambda)\mathrm{tr}(\boldsymbol{\rho}) - c_3(\lambda)\theta\,, \quad \frac{\eta}{\kappa}\boldsymbol{w} - \nabla p = \mathbf{0} \quad \text{in} \quad \Omega\,, \tag{2.9b}$$

$$\theta + \boldsymbol{w}\cdot\nabla\theta - \mathrm{div}\big(\mathcal{K}(\boldsymbol{\rho}, p, \theta)\nabla\theta\big) = g \quad \text{in} \quad \Omega\,, \tag{2.9c}$$

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}}\,, \quad p = p_{\mathrm{D}} \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \Gamma\,. \tag{2.9d}$$

Throughout this work, we suppose that, $\mathcal{K} : \mathbb{R} \times \mathrm{R} \times \mathrm{R} \to \mathbb{R}$ is a function of class C^1 and uniformly positive definite, meaning the latter that there exists $\varkappa_0 > 0$ such that

$$\mathcal{K}(\boldsymbol{\tau}, q, \xi)\boldsymbol{s}\cdot\boldsymbol{s} > \varkappa_0\,|\boldsymbol{s}|^2, \qquad \forall\,(\boldsymbol{\tau}, q, \xi) \in \mathbb{R} \times \mathrm{R} \times \mathrm{R}\,. \tag{2.10}$$

We also require uniform boundedness and Lipschitz continuity of $\mathcal{K}$, that is that there exist positive constants $\varkappa_1$, $\varkappa_2$ and $L_{\mathcal{K}}$, such that

$$\varkappa_1 \leqslant \mathcal{K}(\boldsymbol{\tau}, q, \xi) \leqslant \varkappa_2 \quad \text{and} \quad |\mathcal{K}(\boldsymbol{\tau}, q, \xi) - \mathcal{K}(\boldsymbol{\tau}_0, q_0, \xi_0)| \leqslant L_{\mathcal{K}}\,|(\boldsymbol{\tau}, q, \xi) - (\boldsymbol{\tau}_0, q_0, \xi_0)|\,, \tag{2.11}$$

for all $(\boldsymbol{\tau}, q, \xi)$, $(\boldsymbol{\tau}_0, q_0, \xi_0) \in \mathbb{R}\times\mathrm{R}\times\mathrm{R}$. It is pertinent to mention here that one of the main consequences of introducing the new variable $\boldsymbol{\rho}$ is that (2.9c) becomes nonlinear with respect to θ unlike (2.1c). Furthermore, it is easily seen from (2.7) and (2.8) that sufficient conditions for (2.11) are given by analogue conditions for $\mathcal{D}$, that is by the existence of positive constants δ_1, δ_2, and $L_{\mathcal{D}}$, such that

$$\delta_1 \leqslant \mathcal{D}(\boldsymbol{\tau}) \leqslant \delta_2 \quad \text{and} \quad |\mathcal{D}(\boldsymbol{\zeta}) - \mathcal{D}(\boldsymbol{\tau})| \leqslant L_{\mathcal{D}}\,|\boldsymbol{\zeta} - \boldsymbol{\tau}| \qquad \forall\,\boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbb{R}\,.$$

3 Mixed weak formulation

In this section, we derive a mixed formulation of the system (2.9). To this end, we treat each variational formulation of (2.9a), (2.9b) and (2.9c) independently, ending up with three systems whose coupling is carried out via a fixed-point iteration strategy.

3.1 Mixed formulation of the poroelasticity equations

In what follows, we are going to address the mixed formulation for the poroelasticity equations in (2.9a) for a given pressure p and temperature θ, which are going to be determined by (2.9b) and (2.9c), respectively. The poroelasticity equations defined for the non-symmetric pseudostress $\boldsymbol{\rho}$ and velocity $\boldsymbol{u}$ unknowns are given by

$$\begin{gathered}\frac{1}{\mu}\boldsymbol{\rho}^{\mathrm{d}} + \frac{1}{n}\gamma(\lambda)\operatorname{tr}(\boldsymbol{\rho})\mathbb{I} - \nabla\boldsymbol{u} = -\gamma(\lambda)(\alpha p + \beta\theta)\mathbb{I} \quad \text{in} \quad \Omega\,, \\ -\mathbf{div}(\boldsymbol{\rho}) = \mathbf{f} \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}} \quad \text{on} \quad \Gamma\,.\end{gathered} \tag{3.1}$$

We notice that in order to properly couple the equations (2.9), we need to be able to control the following expression associated with the heat equation

$$\int_\Omega \big(\mathcal{K}(\boldsymbol{\zeta},\widetilde{p},\vartheta) - \mathcal{K}(\boldsymbol{\zeta}_0,\widetilde{p}_0,\vartheta_0)\big)\,\boldsymbol{t}\cdot\boldsymbol{s}\,,$$

where $(\boldsymbol{\zeta},\widetilde{p},\vartheta)$ and $(\boldsymbol{\zeta}_0,\widetilde{p}_0,\vartheta_0)$ belong to the same space in which we will seek the unknowns $(\boldsymbol{\rho},p,\theta)$, and the functions $\boldsymbol{t}$ and $\boldsymbol{s}$ are generic vectors that belong to the same space than $\nabla\theta$. In this regard, and employing the Lipschitz-continuity property of $\mathcal{K}$ (cf. (2.11)), straightforward applications of Cauchy–Schwarz and Hölder inequalities yield

$$\begin{aligned}&\left|\int_\Omega \big(\mathcal{K}(\boldsymbol{\zeta},\widetilde{p},\vartheta) - \mathcal{K}(\boldsymbol{\zeta}_0,\widetilde{p}_0,\vartheta_0)\big)\,\boldsymbol{t}\cdot\boldsymbol{s}\right| \\ &\qquad \leqslant L_{\mathcal{K}}\Big(\|\boldsymbol{\zeta}-\boldsymbol{\zeta}_0\|_{0,2j;\Omega} + \|\widetilde{p}-\widetilde{p}_0\|_{0,2j;\Omega} + \|\vartheta-\vartheta_0\|_{0,2j;\Omega}\Big)\,\|\boldsymbol{t}\|_{0,2k;\Omega}\,\|\boldsymbol{s}\|_{0;\Omega}\,,\end{aligned} \tag{3.2}$$

where $j,\,k \in (1,+\infty)$ are conjugate to each other. The latter inequality makes sense for $\boldsymbol{\zeta},\,\boldsymbol{\zeta}_0 \in \mathbb{L}^{2j}(\Omega)$, $\widetilde{p}$, $\widetilde{p}_0$, ϑ and $\vartheta_0 \in \mathrm{L}^{2j}(\Omega)$, and $\boldsymbol{t} \in \mathbf{L}^{2k}(\Omega)$. In this way, the above leads us to initially look for $\boldsymbol{\rho}$ in the space $\mathbb{L}^r(\Omega)$, $p \in \mathrm{L}^r(\Omega)$ and θ initially in $\mathrm{L}^r(\Omega)$, with $r := 2j$. The specific choice of r will be discussed later on, so that meanwhile we consider a generic r and let $s \in (1,2)$ be its respective conjugate. In turn, a suitable bounding of $\|\boldsymbol{t}\|_{0,2k;\Omega}$ in (3.2) for a particular $\boldsymbol{t}$ will also be explained subsequently by means of a regularity argument.

With the preliminary choice of the space to which $\boldsymbol{\rho}$ belongs established above, it follows now from the first equation of (3.1) that $\boldsymbol{u}$ should be initially sought in $\mathbf{W}^{1,r}(\Omega)$. Thus, in order to derive the variational formulation for the poroelasticity equations, we need to invoke a suitable integration by parts formula. Indeed, applying (1.4) with $t = s$ and $t' = r$ to $\boldsymbol{u} \in \mathbf{W}^{1,r}(\Omega)$, for which we assume from now on that $\boldsymbol{u}_{\mathrm{D}}$ belongs to $\mathbf{W}^{1/s,r}(\Gamma)$, we find that

$$\int_\Omega \nabla\boldsymbol{u}\cdot\boldsymbol{\tau} \;=\; -\int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \;+\; \langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{u}_{\mathrm{D}}\rangle_\Gamma\,,$$

so that, the testing of the first equation of (3.1) against $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$ gives

$$\frac{1}{\mu}\int_\Omega \boldsymbol{\rho}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{\gamma(\lambda)}{n}\int_\Omega \operatorname{tr}(\boldsymbol{\rho})\operatorname{tr}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{u}_{\mathrm{D}}\rangle_\Gamma - \gamma(\lambda)\int_\Omega (\alpha\,p + \beta\theta)\operatorname{tr}(\boldsymbol{\tau})\,. \tag{3.3}$$

Here, we notice that the second term on the right-hand side of (3.3) does indeed make sense for p and θ initially in $\mathrm{L}^r(\Omega)$. In fact, thanks to Hölder's inequality we have

$$\int_\Omega p\operatorname{tr}(\boldsymbol{\tau}) \leqslant n^{1/r}\|p\|_{0,r;\Omega}\|\boldsymbol{\tau}\|_{0,s;\Omega}\,, \qquad \int_\Omega \theta\operatorname{tr}(\boldsymbol{\tau}) \leqslant n^{1/r}\|\theta\|_{0,r;\Omega}\|\boldsymbol{\tau}\|_{0,s;\Omega}\,. \tag{3.4}$$

As a result, the third term on the left-hand side of (3.3) implies that it is sufficient to consider $\boldsymbol{u}$ in $\mathbf{L}^r(\Omega)$. Additionally, when testing the second equation of (3.1) against $\boldsymbol{v} \in \mathbf{L}^s(\Omega)$, we obtain

$$\int_\Omega \boldsymbol{v} \cdot \mathbf{div}(\boldsymbol{\rho}) = -\int_\Omega \mathbf{f} \cdot \boldsymbol{v}\,, \tag{3.5}$$

which makes sense when $\mathbf{div}(\boldsymbol{\rho}) \in \mathbf{L}^r(\Omega)$ and $\mathbf{f} \in \mathbf{L}^r(\Omega)$, the latter being assumed in what follows, and thus from now on we seek $\boldsymbol{\rho}$ in $\mathbb{H}^r(\mathbf{div}_r;\Omega)$. In addition, we notice that for each $t \in (1,+\infty)$ there holds the decomposition

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) \;=\; \mathbb{H}_0^t(\mathbf{div}_t;\Omega) \oplus \mathrm{R}\mathbb{I}, \quad \text{with} \quad \mathbb{H}_0^t(\mathbf{div}_t;\Omega) \;:=\; \left\{\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\right\}. \tag{3.6}$$

Note that replacing $\boldsymbol{\tau}$ by the identity tensor $\mathbb{I}$ in (3.3) and using that the deviator of $\mathbb{I}$ is the null tensor, we get an expression for the integral of the trace of $\boldsymbol{\rho}$, this is

$$\int_\Omega \mathrm{tr}(\boldsymbol{\rho}) = \frac{1}{\gamma(\lambda)} \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\nu} - n \int_\Omega (\alpha p + \beta\theta)\,. \tag{3.7}$$

Now, using the decomposition (3.6) with $t = r$, we have that $\boldsymbol{\rho} = \boldsymbol{\rho}_0 + c\mathbb{I}$ with unique $\boldsymbol{\rho}_0 \in \mathbb{H}_0^r(\mathbf{div}_t;\Omega)$ and constant $c \in \mathrm{R}$, which thanks to (3.7), can be computed by

$$c \;:=\; \frac{1}{n|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\rho}) \;=\; \frac{1}{n\gamma(\lambda)|\Omega|} \int_\Gamma \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{\nu} - \frac{1}{|\Omega|} \int_\Omega (\alpha p + \beta\theta)\,. \tag{3.8}$$

Hence, c can be obtained once the pressure and temperature are known, and in order to fully attain the explicit knowledge of the unknown $\boldsymbol{\rho}$, it only remains to find its $\mathbb{H}_0^r(\mathbf{div}_r;\Omega)$-component $\boldsymbol{\rho}_0$. On the other hand, (3.6) also applies to each $\boldsymbol{\tau}$ in $\mathbb{H}^s(\mathbf{div}_s;\Omega)$ with unique decomposition $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\mathbb{I}$, for $\boldsymbol{\tau}_0 \in \mathbb{H}_0^s(\mathbf{div}_s;\Omega)$ and respective constant $d \in \mathrm{R}$.

Therefore, we reformulate our problem in terms of $\boldsymbol{\rho}_0$ instead. To do so, we replace $\boldsymbol{\rho} = \boldsymbol{\rho}_0 + c\mathbb{I}$ into (3.3) and (2.9b), denote $\boldsymbol{\rho}_0$ simply by $\boldsymbol{\rho}$ and substitute $\mathcal{K}(\boldsymbol{\rho}, p, \theta)$ by $\mathcal{K}(\boldsymbol{\rho} + c\mathbb{I}, p, \theta)$ in the heat equation (2.9c). Furthermore, we observe that testing (3.3) against $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0^s(\mathbf{div}_s;\Omega)$, which together with the above, leads us to consider the following Banach spaces

$$\mathbb{X}_2 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega)\,, \quad \mathbf{M}_1 := \mathbf{L}^r(\Omega)\,, \quad \mathbb{X}_1 := \mathbb{H}_0^s(\mathbf{div}_s;\Omega)\,, \quad \mathbf{M}_2 := \mathbf{L}^s(\Omega)\,,$$

so that, given p, $\theta \in \mathrm{L}^r(\Omega)$, and gathering (3.3) and (3.5), we arrive at the following mixed formulation for the poroelasticity equations (2.9a): Find $(\boldsymbol{\rho}, \boldsymbol{u}) \in \mathbb{X}_2 \times \mathbf{M}_1$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\rho}, \boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau}, \boldsymbol{u}) \quad &= \quad \mathbf{F}_{p,\theta}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_1\,, \\ \mathbf{b}_2(\boldsymbol{\rho}, \boldsymbol{v}) \quad &= \quad \mathbf{G}(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \mathbf{M}_2\,, \end{aligned} \tag{3.9}$$

where the bilinear forms $\mathbf{a} : \mathbb{X}_2 \times \mathbb{X}_1 \to \mathrm{R}$ and $\mathbf{b}_i : \mathbb{X}_i \times \mathbf{M}_i \to \mathrm{R}$, with $i \in \{1, 2\}$, are defined by

$$\begin{aligned} \mathbf{a}(\boldsymbol{\rho}, \boldsymbol{\tau}) \;&:=\; \frac{1}{\mu} \int_\Omega \boldsymbol{\rho}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{\gamma(\lambda)}{n} \int_\Omega \mathrm{tr}(\boldsymbol{\rho})\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\, (\boldsymbol{\rho}, \boldsymbol{\tau}) \in \mathbb{X}_2 \times \mathbb{X}_1\,, \\ \mathbf{b}_i(\boldsymbol{\tau}, \boldsymbol{v}) \;&:=\; \int_\Omega \boldsymbol{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \qquad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{X}_i \times \mathbf{M}_i\,. \end{aligned}$$

In turn, given q, ϑ in $\mathrm{L}^r(\Omega)$, the linear functionals $\mathbf{F}_{q,\vartheta} : \mathbb{X}_1 \to \mathrm{R}$ and $\mathbf{G} : \mathbf{M}_2 \to \mathrm{R}$, are defined by

$$\mathbf{F}_{q,\vartheta}(\boldsymbol{\tau}) \;:=\; \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{u}_{\mathrm{D}} \rangle_\Gamma - \gamma(\lambda) \int_\Omega (\alpha\, q + \beta\, \vartheta)\, \mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_1\,, \tag{3.10a}$$

$$\mathbf{G}(\boldsymbol{v}) \;:=\; -\int_\Omega \mathbf{f} \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbf{M}_2\,. \tag{3.10b}$$

Next, it is easily seen that $\mathbf{a}$, $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{G}$ are bounded. In fact, applying Hölder's inequality, we find that there exist positive constants, denoted and given by

$$\|\mathbf{a}\| := \frac{2}{\mu}, \quad \|\mathbf{b}_i\| := 1 \quad \text{and} \quad \|\mathbf{G}\| = \|\mathbf{f}\|_{0,r;\Omega}, \tag{3.11}$$

such that

$$\begin{aligned}
|\mathbf{a}(\boldsymbol{\rho},\boldsymbol{\tau})| &\leqslant \|\mathbf{a}\|\|\boldsymbol{\rho}\|_{\mathbb{X}_2}\|\boldsymbol{\tau}\|_{\mathbb{X}_1} && \forall\,(\boldsymbol{\rho},\boldsymbol{\tau}) \in \mathbb{X}_2 \times \mathbb{X}_1\,,\\
|\mathbf{b}_i(\boldsymbol{\tau},\boldsymbol{v})| &\leqslant \|\mathbf{b}\|\|\boldsymbol{\tau}\|_{\mathbb{X}_i}\|\boldsymbol{v}\|_{\mathbf{M}_i} && \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{X}_i \times \mathbf{M}_i\,,\\
|\mathbf{G}(\boldsymbol{v})| &\leqslant \|\mathbf{G}\|\|\boldsymbol{v}\|_{\mathbf{M}_2} && \forall\,\boldsymbol{v} \in \mathbf{M}_2\,.
\end{aligned}$$

Regarding the boundedness of the functional $\mathbf{F}_{p,\theta}$, where p and θ are initially in $\mathrm{L}^r(\Omega)$, we will establish this in the forthcoming Section 3.3, where the range of r will be determined for each unknown.

3.2 Mixed formulation of the perturbed Darcy problem

Continuing with the weak formulation of (2.9), we are going to focus now on the perturbed Darcy equation (2.9b) including the boundary condition of the pressure, for a given $\boldsymbol{\rho} \in \mathbb{X}_2$ and $p,\theta \in \mathrm{L}^r(\Omega)$. Following the derivation done in Section 3.1, we use decomposition (3.6) together with the definition of c (cf. (3.8)), and replace $\mathrm{tr}(\boldsymbol{\rho})$ by $\mathrm{tr}(\boldsymbol{\rho} + c\mathbb{I})$ into (2.9b), so that the perturbed Darcy problem describing the velocity $\boldsymbol{w}$ and pressure p is then given by

$$\begin{gathered}
\frac{\eta}{\kappa}\boldsymbol{w} - \nabla p = \mathbf{0} \quad \text{in} \quad \Omega\,,\\
\mathrm{div}(\boldsymbol{w}) - c_1(\lambda)p = c_2(\lambda)\mathrm{tr}(\boldsymbol{\rho} + c\mathbb{I}) + c_3(\lambda)\theta - f \quad \text{in} \quad \Omega\,,\\
p = p_{\mathrm{D}} \quad \text{on} \quad \Gamma\,,
\end{gathered} \tag{3.12}$$

where the constant c multiplying $\mathbb{I}$ on the right-hand side of the second equation is defined by (3.8), and depends on p and θ. Next, given $t \in (1,\infty)$, we consider the zero mean mapping $m : \mathrm{L}^t(\Omega) \to \mathrm{L}^t_0(\Omega)$ defined by

$$m(q) := q - \frac{1}{|\Omega|}\int_\Omega q \qquad \forall\, q \in \mathrm{L}^t(\Omega)\,. \tag{3.13}$$

Then, replacing (3.8) and using the notation $q_0 := m(q) \in \mathrm{L}^t_0(\Omega)$, the second equation of (3.12) can be written as

$$\mathrm{div}(\boldsymbol{w}) \,-\, \chi\, p \,-\, n\,\alpha^2\,\gamma(\lambda)\,p_0 \;=\; c_2(\lambda)\,\mathrm{tr}(\boldsymbol{\rho}) \,+\, c_3(\lambda)\,\theta_0 \,+\, \frac{\alpha}{|\Omega|}\int_\Gamma \boldsymbol{u}_{\mathrm{D}}\cdot\boldsymbol{\nu} - f\,. \tag{3.14}$$

Prior to addressing the weak formulation of (3.12), we notice that in order to properly couple (3.14) to equation (2.9c), we need to be able to control the expression

$$\int_\Omega (\boldsymbol{w}\cdot\nabla\theta)\vartheta\,,$$

which arises later on (cf. (3.25)) when dealing with the variational formulation of the heat equation. Here ϑ is a function belonging to the same space in which we will seek the temperature θ. Applying generalized Hölder's inequality to the triple product present in the above integral, we get

$$\left|\int_\Omega (\boldsymbol{w}\cdot\nabla\theta)\vartheta\right| \leqslant \|\boldsymbol{w}\|_{0,2j;\Omega}\,\|\nabla\theta\|_{0;\Omega}\,\|\vartheta\|_{0,2k;\Omega}\,, \tag{3.15}$$

where $j,k \in (1,+\infty)$ are conjugate to each other, and the inequality holds true for $\boldsymbol{w} \in \mathbf{L}^r(\Omega)$, $\nabla\theta \in \mathbf{L}^2(\Omega)$, and $\vartheta \in \mathrm{L}^\rho(\Omega)$, with $(r,\rho) := (2j,2k)$. Considering that θ is initially taken from $\mathrm{L}^r(\Omega)$, we have to require that $r \leqslant \rho$, a condition that will be satisfied when determining the range for ρ, so that for now we consider $\rho \in (2, +\infty)$, and let ϱ be its respective conjugate.

Having chosen $\mathbf{L}^r(\Omega)$ as the preliminary space for $\boldsymbol{w}$, (3.12) tentatively suggests to look for p in $\mathrm{W}^{1,r}(\Omega)$. In this way, testing the first equation of (3.12) against $\boldsymbol{z} \in \mathbf{H}^s(\mathrm{div}_s;\Omega)$, and applying (1.3) together with the Dirichlet boundary condition for p, we obtain

$$\frac{\eta}{\kappa}\int_\Omega \boldsymbol{w}\cdot\boldsymbol{z} + \int_\Omega p\,\mathrm{div}(\boldsymbol{z}) \;=\; \langle \boldsymbol{z}\cdot\boldsymbol{\nu}, p_{\mathrm{D}}\rangle_\Gamma \qquad \forall\, \boldsymbol{z} \in \mathbf{H}^s(\mathrm{div}_s;\Omega)\,, \tag{3.16}$$

which requires to assume that $p_{\mathrm{D}} \in \mathrm{W}^{1/s,r}(\Gamma)$. Then, a straightforward application of Hölder's inequality in the second term on the left-hand side of (3.16) shows that it suffices to seek the pressure p in the space $\mathrm{L}^r(\Omega)$, which coincides with the space obtained in (3.4). On the other hand, testing (3.14) against an arbitrary function q belonging to a space to be determined, we formally get

$$\begin{aligned}\int_\Omega q\,\mathrm{div}(\boldsymbol{w}) - \chi\int_\Omega p\,q - n\alpha^2\gamma(\lambda)\int_\Omega p_0\,q \\ = c_2(\lambda)\int_\Omega q\,\mathrm{tr}(\boldsymbol{\rho}) + c_3(\lambda)\int_\Omega \theta_0\,q + \frac{\alpha}{n|\Omega|}\int_\Gamma \boldsymbol{u}_{\mathrm{D}}\cdot\boldsymbol{\nu}\int_\Omega q - \int_\Omega f\,q\,.\end{aligned} \tag{3.17}$$

Since we will look for p in $\mathrm{L}^r(\Omega)$, a direct application of the Hölder's inequality implies that the second term on the left-hand side of (3.17) makes sense if q is considered in $\mathrm{L}^s(\Omega)$. Consequently, the remaining terms of (3.17) make sense if $\mathrm{div}(\boldsymbol{w})$ and f belong to $\mathrm{L}^r(\Omega)$, and then $\boldsymbol{w}$ must be sought in $\mathbf{H}^r(\mathrm{div}_r;\Omega)$. In this way, we define the following spaces

$$\mathbf{X}_2 := \mathbf{H}^r(\mathrm{div}_r;\Omega)\,,\quad \mathbf{X}_1 := \mathbf{H}^s(\mathrm{div}_s;\Omega)\,,\quad \mathrm{M}_1 := \mathrm{L}^r(\Omega) \quad\text{and}\quad \mathrm{M}_2 := \mathrm{L}^s(\Omega)\,. \tag{3.18}$$

Then, given $(\boldsymbol{\rho},\theta) \in \mathbb{X}_2 \times \mathrm{L}^\rho(\Omega)$, the mixed formulation for the perturbed Darcy equation reduces to: Find $(\boldsymbol{w},p) \in \mathbf{X}_2 \times \mathrm{M}_1$ such that

$$\begin{aligned}\mathbf{c}(\boldsymbol{w},\boldsymbol{z}) + \mathbf{d}_1(\boldsymbol{z},p) \qquad\qquad &= \boldsymbol{\mathcal{F}}(\boldsymbol{z}) \qquad \forall\, \boldsymbol{z} \in \mathbf{X}_1\,,\\ \mathbf{d}_2(\boldsymbol{w},q) \qquad\qquad - \mathbf{e}(p,q) &= \boldsymbol{\mathcal{G}}_{\boldsymbol{\rho},\theta}(q) \qquad \forall\, q \in \mathrm{M}_2\,,\end{aligned} \tag{3.19}$$

where the bilinear forms $\mathbf{c} : \mathbf{X}_2 \times \mathbf{X}_1 \to \mathrm{R}$, $\mathbf{d}_i : \mathbf{X}_i \times \mathrm{M}_i \to \mathrm{R}$, $i \in \{1,2\}$, and $\mathbf{e} : \mathrm{M}_1 \times \mathrm{M}_2 \to \mathrm{R}$, which are independent of $\boldsymbol{\rho}$ and θ, are defined by

$$\mathbf{c}(\boldsymbol{w},\boldsymbol{z}) := \frac{\eta}{\kappa}\int_\Omega \boldsymbol{w}\cdot\boldsymbol{z} \qquad \forall\,(\boldsymbol{w},\boldsymbol{z}) \in \mathbf{X}_2 \times \mathbf{X}_1\,, \tag{3.20a}$$

$$\mathbf{d}_i(\boldsymbol{z},q) := \int_\Omega q\,\mathrm{div}(\boldsymbol{z}) \qquad \forall\,(\boldsymbol{z},q) \in \mathbf{X}_i \times \mathrm{M}_i\,, \tag{3.20b}$$

and

$$\mathbf{e}(p,q) := \chi\int_\Omega pq + n\alpha^2\gamma(\lambda)\int_\Omega p_0 q \qquad \forall\,(p,q) \in \mathrm{M}_1 \times \mathrm{M}_2\,. \tag{3.20c}$$

Furthermore, the functionals $\boldsymbol{\mathcal{F}} : \mathbf{X}_1 \to \mathrm{R}$ and $\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta} : \mathrm{M}_2 \to \mathrm{R}$, for each $(\boldsymbol{\zeta},\vartheta) \in \mathbb{X}_2 \times \mathrm{L}^\rho(\Omega)$, are defined by

$$\boldsymbol{\mathcal{F}}(\boldsymbol{z}) \;:=\; \langle \boldsymbol{z}\cdot\boldsymbol{\nu}, p_{\mathrm{D}}\rangle_\Gamma \qquad \forall\, \boldsymbol{z} \in \mathbf{X}_1\,, \quad\text{and} \tag{3.21a}$$

$$\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}(q) := c_2(\lambda)\int_\Omega q\,\mathrm{tr}(\boldsymbol{\zeta}) + c_3(\lambda)\int_\Omega \vartheta_0\,q + \frac{\alpha}{n|\Omega|}\int_\Gamma \boldsymbol{u}_{\mathrm{D}}\cdot\boldsymbol{\nu}\int_\Omega q - \int_\Omega fq \qquad \forall q \in \mathrm{M}_2\,. \tag{3.21b}$$

In addition, the bilinear forms $\mathbf{c}$, $\mathbf{d}_i$, $i \in \{1,2\}$ and $\mathbf{e}$ are all bounded. Finally, applying Cauchy–Schwarz and Hölder inequalities, we find that there exist positive constants, given by

$$\|\mathbf{c}\| \;:=\; \frac{\eta}{\kappa}\,,\quad \|\mathbf{d}_i\| \;:=\; 1\,,\quad \|\mathbf{e}\| \;:=\; \max\left\{\chi\,, n\,\alpha^2\,\gamma(\lambda)\right\}\,, \tag{3.22}$$

such that

$$
\begin{aligned}
|\mathbf{c}(\boldsymbol{w},\boldsymbol{z})| &\leqslant \|\mathbf{c}\|\,\|\boldsymbol{w}\|_{\mathbf{X}_2}\,\|\boldsymbol{z}\|_{\mathbf{X}_1} && \forall\,(\boldsymbol{w},\boldsymbol{z})\in\mathbf{X}_2\times\mathbf{X}_1\,,\\
|\mathbf{d}_i(\boldsymbol{z},q)| &\leqslant \|\mathbf{d}_i\|\,\|\boldsymbol{z}\|_{\mathbf{X}_i}\,\|q\|_{\mathrm{M}_i} && \forall\,(\boldsymbol{z},q)\in\mathbf{X}_i\times\mathrm{M}_i\,,\quad i\in\{1,2\},\\
|\mathbf{e}(p,q)| &\leqslant \|\mathbf{e}\|\,\|p\|_{\mathrm{M}_1}\,\|q\|_{\mathrm{M}_2} && \forall\,(p,q)\in\mathrm{M}_1\times\mathrm{M}_2\,.
\end{aligned}
$$

The boundedness of $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}$ will be proven later in the next section.

3.3 Mixed formulation of the heat equation

We treat now the mixed formulation of (2.9c) for a given $\boldsymbol{\rho}\in\mathbb{X}_2$ and $\boldsymbol{w}\in\mathbf{X}_2$. For this purpose, we define two auxiliary unknowns, the gradient of the temperature and the term contained in the argument of the divergence operator in (2.9c), this is

$$
\widetilde{\boldsymbol{t}} := \nabla\theta \quad\text{and}\quad \widetilde{\boldsymbol{\sigma}} := \mathcal{K}(\boldsymbol{\rho},p,\theta)\widetilde{\boldsymbol{t}}\,. \tag{3.23}
$$

Then, replacing these variables, the heat equation (2.9c) describing the temperature θ can be written as

$$
\begin{gathered}
\widetilde{\boldsymbol{t}} = \nabla\theta,\quad \widetilde{\boldsymbol{\sigma}} = \mathcal{K}(\boldsymbol{\rho},p,\theta)\widetilde{\boldsymbol{t}} \quad\text{and}\quad \theta + \boldsymbol{w}\cdot\widetilde{\boldsymbol{t}} - \operatorname{div}(\widetilde{\boldsymbol{\sigma}}) = g \quad\text{in}\quad \Omega\,,\\
\theta = 0 \quad\text{on}\quad \Gamma\,.
\end{gathered} \tag{3.24}
$$

Now, testing the third equation of (3.24) against an arbitrary function ϑ belonging to a space to be determined, we formally get

$$
\int_\Omega \theta\vartheta + \int_\Omega \boldsymbol{w}\cdot\widetilde{\boldsymbol{t}}\vartheta - \int_\Omega \vartheta\operatorname{div}(\widetilde{\boldsymbol{\sigma}}) = \int_\Omega g\vartheta\,. \tag{3.25}
$$

Next, proceeding as in (3.15), we notice that applying generalized Hölder's inequality to the triple product in the second term on the left-hand side of (3.25) we get

$$
\left|\int_\Omega \boldsymbol{w}\cdot\widetilde{\boldsymbol{t}}\vartheta\right| \leqslant \|\boldsymbol{w}\|_{0,r;\Omega}\|\widetilde{\boldsymbol{t}}\|_{0;\Omega}\|\vartheta\|_{0,\rho;\Omega}\,,
$$

whence we can look for $\widetilde{\boldsymbol{t}}\in\mathbf{L}^2(\Omega)$ and $\vartheta\in\mathrm{L}^\rho(\Omega)$. In addition, performing similar calculations as before but over the first term on the left-hand side of (3.25), for $\rho = 2k > 2$, we obtain

$$
\left|\int_\Omega \theta\vartheta\right| \leqslant \|\theta\|_{0;\Omega}\|\vartheta\|_{0;\Omega} \leqslant |\Omega|^{\frac{\rho-2}{\rho}}\|\theta\|_{0,\rho;\Omega}\|\vartheta\|_{0,\rho;\Omega}\,,
$$

and in consequence θ can be sought in the same space as ϑ, its associated test function, which is $\mathrm{L}^\rho(\Omega)$. In light of this, the data g will be considered in $\mathrm{L}^\varrho(\Omega)$. Furthermore, a direct application of Hölder's inequality yields the third term on the left-hand side of (3.25) to be bounded as follows

$$
\left|\int_\Omega \vartheta\operatorname{div}(\widetilde{\boldsymbol{\sigma}})\right| \leqslant \|\vartheta\|_{0,\rho;\Omega}\|\operatorname{div}(\widetilde{\boldsymbol{\sigma}})\|_{0,\varrho;\Omega}\,,
$$

where, recalling that ϱ is the conjugate of ρ, we observe that this term makes sense as long as $\operatorname{div}(\widetilde{\boldsymbol{\sigma}})\in\mathrm{L}^\varrho(\Omega)$. Moreover, since $\widetilde{\boldsymbol{t}}\in\mathbf{L}^2(\Omega)$ and $\mathcal{K}$ is bounded (cf. (2.11)), we can test the second equation of (3.23) against $\widetilde{\boldsymbol{s}}$ in $\mathbf{L}^2(\Omega)$, that is

$$
-\int_\Omega \widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\boldsymbol{s}} + \int_\Omega \mathcal{K}(\boldsymbol{\rho},p,\theta)\widetilde{\boldsymbol{t}}\cdot\widetilde{\boldsymbol{s}} = 0 \quad \forall\,\widetilde{\boldsymbol{s}}\in\mathbf{L}^2(\Omega)\,, \tag{3.26}
$$

where, from the first term, we obtain that $\widetilde{\boldsymbol{\sigma}}$ must be searched in $\mathbf{L}^2(\Omega)$, and more specifically in $\mathbf{H}(\mathrm{div}_{\varrho};\Omega)$ according to the preceding discussion.

Now, we observe that from the first equation of (3.24) we need $\theta \in \mathrm{H}^1(\Omega)$, but since $\theta \in \mathrm{L}^\rho(\Omega)$ this condition will be valid if $\mathrm{H}^1(\Omega)$ is continuously embedded in $\mathrm{L}^\rho(\Omega)$. The latter is guaranteed for $\rho \in [1,+\infty)$ when $n = 2$, which is always satisfied in the two-dimensional case, and for $\rho \in [1,6]$ when $n = 3$. Furthermore, in order to prove an inf-sup condition associated to $\boldsymbol{w}$ we are going to apply, e.g. [27, Theorem 3.2], which requires that $r \in [4/3,4]$ when $n = 2$ and $r \in [3/2,3]$ when $n = 3$. On the other hand, since $r > 2$ (see Section 3.1), the respective lower bounds are already satisfied, and we only need to verify the upper ones. We readily observe that since $r = 2\rho/(\rho-2)$, for $n = 2$, $r \leqslant 4$ if only if $\rho \geqslant 4$, whereas for $n = 3$, $r \leqslant 3$ if only if $\rho \geqslant 6$. Thus, intersecting the above with the previous restrictions on ρ, we find that when $n = 2$ we require $\rho \geqslant 4$, and when $n = 3$ the only possible choice is $\rho = 6$. Therefore, we conclude that the feasible ranges for (r,ρ) and their respective conjugates, (s,ϱ), are given by

$$\left\{\begin{array}{llll} r \in (2,4] & \text{and} & s \in [4/3,2) & \text{if } n = 2\,, \\ r = 3 & \text{and} & s = 3/2 & \text{if } n = 3\,, \end{array}\right. \qquad \left\{\begin{array}{llll} \rho \in [4,+\infty) & \text{and} & \varrho \in (1,4/3] & \text{if } n = 2\,, \\ \rho = 6 & \text{and} & \varrho = 6/5 & \text{if } n = 3\,. \end{array}\right. \tag{3.27}$$

Then, bearing in mind that $\widetilde{\boldsymbol{t}}$ and θ belong to $\mathbf{L}^2(\Omega)$ and $\mathrm{L}^\rho(\Omega)$, respectively, we test the first equation of (3.23) against a $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{\varrho};\Omega)$ and applying (1.2), we formally get

$$\int_\Omega \widetilde{\boldsymbol{t}}\cdot\widetilde{\boldsymbol{\tau}} + \int_\Omega \theta\,\mathbf{div}(\widetilde{\boldsymbol{\tau}}) = 0 \qquad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{\varrho};\Omega)\,. \tag{3.28}$$

Consequently, taking into account the foregoing discussion, we introduce the following spaces and notation to be used in our formulation:

$$\mathbf{H}_1 := \mathrm{L}^\rho(\Omega)\,, \quad \mathbf{H}_2 := \mathbf{L}^2(\Omega)\,, \quad \mathbf{H} := \mathbf{H}_1 \times \mathbf{H}_2\,, \quad \mathbf{Q} := \mathbf{H}(\mathrm{div}_{\varrho};\Omega)\,,$$
$$\vec{\theta} := (\theta,\widetilde{\boldsymbol{t}})\,, \quad \vec{\vartheta} := (\vartheta,\widetilde{\boldsymbol{s}}) \in \mathbf{H}\,.$$

Finally, suitably gathering (3.25), (3.26) and (3.28), for a given $\vec{\boldsymbol{p}} := (\boldsymbol{\rho},\boldsymbol{w},p) \in \mathbb{X}_2 \times \mathbf{X}_2 \times \mathrm{M}_1$, we arrive at the following mixed formulation for the heat equation: Find $(\vec{\theta},\widetilde{\boldsymbol{\sigma}}) := \big((\theta,\widetilde{\boldsymbol{t}}),\widetilde{\boldsymbol{\sigma}}\big) \in \mathbf{H}\times\mathbf{Q}$ such that

$$\begin{array}{lcll} a_{\vec{\boldsymbol{p}},\theta}(\vec{\theta},\vec{\vartheta}) + b(\vec{\vartheta},\widetilde{\boldsymbol{\sigma}}) & = & F(\vec{\vartheta}) & \forall\,\vec{\vartheta} := (\vartheta,\widetilde{\boldsymbol{s}}) \in \mathbf{H}\,, \\ b(\vec{\theta},\widetilde{\boldsymbol{\tau}}) & = & 0 & \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \end{array} \tag{3.29}$$

where, given $\vec{\boldsymbol{q}} = (\boldsymbol{\zeta},\mathbf{z},q) \in \mathbb{X}_2 \times \mathbf{X}_2 \times \mathrm{M}_1$ and $\xi \in \mathbf{H}_1$, $a_{\vec{\boldsymbol{q}},\xi} : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$ and $b : \mathbf{H}\times\mathbf{Q} \to \mathrm{R}$ are the bilinear forms defined by

$$a_{\vec{\boldsymbol{q}},\xi}(\vec{\theta},\vec{\vartheta}) := \int_\Omega \theta\vartheta + \int_\Omega \mathcal{K}\big(\boldsymbol{\zeta},q,\xi\big)\,\widetilde{\boldsymbol{t}}\cdot\widetilde{\boldsymbol{s}} + \int_\Omega \mathbf{z}\cdot\widetilde{\boldsymbol{t}}\vartheta \qquad \forall\,\vec{\theta},\,\vec{\vartheta} \in \mathbf{H}\,, \tag{3.30a}$$

$$b(\vec{\vartheta},\widetilde{\boldsymbol{\tau}}) := -\int_\Omega \widetilde{\boldsymbol{\tau}}\cdot\widetilde{\boldsymbol{s}} - \int_\Omega \vartheta\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\,(\vec{\vartheta},\,\widetilde{\boldsymbol{\tau}}) \in \mathbf{H}\times\mathbf{Q}\,. \tag{3.30b}$$

It is important to notice that, since $a_{\vec{\boldsymbol{p}},\theta}$ involves the function $\mathcal{K}$ in its definition, which in turn depends on θ, the term $a_{\vec{\boldsymbol{p}},\theta}(\vec{\theta},\vec{\vartheta})$ is nonlinear. Additionally, the functional $F : \mathbf{H} \to \mathrm{R}$ is given by

$$F(\vec{\vartheta}\,) := \int_\Omega g\vartheta \qquad \forall\,\vec{\vartheta} = (\vartheta,\widetilde{\boldsymbol{s}}) \in \mathbf{H}\,.$$

Next, it is easily seen that, given $\vec{\boldsymbol{q}} \in \mathbb{X}_2 \times \mathbf{X}_2 \times \mathrm{M}_1$ and $\xi \in \mathbf{H}_1$, $a_{\vec{\boldsymbol{q}},\xi}$, b, and F are bounded. In fact, endowing $\mathbf{H}$ and $\mathbf{Q}$ with the norms

$$\|\vec{\vartheta}\|_{\mathbf{H}} := \|\vartheta\|_{0,\rho;\Omega} + \|\widetilde{\boldsymbol{s}}\|_{0;\Omega} \quad \forall\,\vec{\vartheta} \in \mathbf{H}\,, \qquad \|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} := \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_{\varrho};\Omega} \quad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,,$$

and applying the Cauchy–Schwarz and Hölder inequalities, we find that there exist positive constants, denoted and given by

$$\|a\| := \max\{|\Omega|^{(\rho-2)/2\rho}, \varkappa_2\}\,, \quad \|b\| := 1\,, \quad \text{and} \quad \|F\| = \|g\|_{0,\varrho;\Omega}\,, \tag{3.31}$$

such that

$$\begin{aligned} |a_{\vec{\mathbf{q}},\xi}(\vec{\theta},\vec{\vartheta})| &\leqslant \big(\|a\| + \|\mathbf{z}\|_{0,r;\Omega}\big)\|\vec{\theta}\,\|_{\mathbf{H}}\|\vec{\vartheta}\,\|_{\mathbf{H}} && \forall\,\vec{\theta},\,\vec{\vartheta}\in\mathbf{H}\,,\\ |b(\vec{\vartheta},\widetilde{\boldsymbol{\sigma}})| &\leqslant \|b\|\|\vec{\vartheta}\,\|_{\mathbf{H}}\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} && \forall\,(\vec{\vartheta},\widetilde{\boldsymbol{\tau}})\in\mathbf{H}\times\mathbf{Q}\,,\\ |F(\vec{\vartheta}\,)| &\leqslant \|F\|\|\vec{\vartheta}\,\|_{\mathbf{H}} && \forall\,\vec{\vartheta}\in\mathbf{H}\,. \end{aligned} \tag{3.32}$$

Regarding the boundedness of $\mathbf{F}_{q,\vartheta}$, $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}$ (cf. (3.10a),(3.21a) and (3.21b), respectively), we observe that knowing already that $(q,\vartheta)\in \mathrm{L}^r(\Omega)\times\mathrm{L}^\rho(\Omega)$, with r and ρ within the ranges stipulated by (3.27), invoking the identity (1.3), the continuous injections $\mathbf{i}_r : \mathbf{H}^1(\Omega)\to\mathbf{L}^r(\Omega)$ and $i_\rho : \mathrm{H}^1(\Omega)\to \mathrm{L}^\rho(\Omega)$, the definitions of the constants $c_2(\lambda)$ and $c_3(\lambda)$ (cf. (2.5)), and employing the Cauchy–Schwarz and Hölder inequalities, we can conclude that there exist positive constants $C_{\mathbf{F}}$, $C_{\boldsymbol{\mathcal{F}}}$, and $C_{\boldsymbol{\mathcal{G}}}$, depending on n, r, ρ, $\|\mathbf{i}_r\|$, $\|i_\rho\|$, $|\Omega|$, α and β, so that letting

$$\begin{aligned} \|\mathbf{F}_{q,\vartheta}\| &:= C_{\mathbf{F}}\Big\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\Big(\|q\|_{0,r;\Omega} + \|\vartheta\|_{0,\rho;\Omega}\Big)\Big\}\,,\\ \|\boldsymbol{\mathcal{F}}\| &:= C_{\boldsymbol{\mathcal{F}}}\,\|p_{\mathrm{D}}\|_{1/s,r;\Gamma}\,, \quad \text{and}\\ \|\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}\| &:= C_{\boldsymbol{\mathcal{G}}}\Big\{\|f\|_{0,r;\Omega} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\Big(\|\boldsymbol{\zeta}\|_{\mathbb{X}_2} + \|\vartheta\|_{0,\rho;\Omega}\Big)\Big\}\,, \end{aligned}$$

there holds

$$\begin{aligned} |\mathbf{F}_{q,\vartheta}(\boldsymbol{\tau})| &\leqslant \|\mathbf{F}_{q,\vartheta}\|\,\|\boldsymbol{\tau}\|_{\mathbb{X}_1} && \forall\,\boldsymbol{\tau}\in\mathbb{X}_1\,,\\ |\boldsymbol{\mathcal{F}}(\boldsymbol{z})| &\leqslant \|\boldsymbol{\mathcal{F}}\|\,\|\boldsymbol{z}\|_{\mathbf{X}_1} && \forall\,\boldsymbol{z}\in\mathbf{X}_1\,, \quad \text{and}\\ |\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}(q)| &\leqslant \|\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}\|\,\|q\|_{\mathrm{M}_2} && \forall\,q\in\mathrm{M}_2\,. \end{aligned} \tag{3.33}$$

3.4 The coupled fully-mixed formulation

Following the derivations presented in the previous sections, the fully-mixed formulation for (2.9) reduces to gathering (3.9), (3.19) and (3.29), that is: Find $(\boldsymbol{\rho},\boldsymbol{u})\in\mathbb{X}_2\times\mathbf{M}_1$, $(\boldsymbol{w},p)\in\mathbf{X}_2\times\mathrm{M}_1$ and $(\vec{\theta},\widetilde{\boldsymbol{\sigma}}) := \big((\theta,\widetilde{\boldsymbol{t}}),\widetilde{\boldsymbol{\sigma}}\big)\in\mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\rho},\boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau},\boldsymbol{u}) \quad &= \mathbf{F}_{p,\theta}(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbb{X}_1\,,\\ \mathbf{b}_2(\boldsymbol{\rho},\boldsymbol{v}) \quad &= \mathbf{G}(\boldsymbol{v}) && \forall\,\boldsymbol{v}\in\mathbf{M}_2\,,\\ \mathbf{c}(\boldsymbol{w},\boldsymbol{z}) + \mathbf{d}_1(\boldsymbol{z},p) \quad &= \boldsymbol{\mathcal{F}}(\boldsymbol{z}) && \forall\,\boldsymbol{z}\in\mathbf{X}_1\,,\\ \mathbf{d}_2(\boldsymbol{w},q) \quad - \mathbf{e}(p,q) &= \boldsymbol{\mathcal{G}}_{\boldsymbol{\rho},\theta}(q) && \forall\,q\in\mathrm{M}_2\,,\\ a_{\vec{\boldsymbol{p}},\theta}(\vec{\theta},\vec{\vartheta}) + b(\vec{\vartheta},\widetilde{\boldsymbol{\sigma}}) \quad &= F(\vec{\vartheta}) && \forall\,\vec{\vartheta}\in\mathbf{H}\,,\\ b(\vec{\theta},\widetilde{\boldsymbol{\tau}}) \quad &= 0 && \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{Q}\,, \end{aligned} \tag{3.34}$$

where $\vec{\boldsymbol{p}} = (\boldsymbol{\rho},\boldsymbol{w},p)\in\mathbb{X}_2\times\mathbf{X}_2\times\mathrm{M}_1$.

4 The continuous solvability analysis

In this section, we will first use the Babuška–Brezzi theory in Banach spaces (cf. [6, Theorem 2.1, Corollary 2.1, Section 2.1] for the general case, and [24, Theorem 2.34] for a particular one) to address the well-posedness of each one of the decoupled problems arising from (3.9), (3.19), and (3.29). Then, we proceed similarly as in [21] and [29] (see also [15], [30], and some references therein), and adopt a fixed-point strategy to analyze the solvability of the fully coupled system (3.34).

4.1 The decoupled poroelasticity equations

We begin by introducing the operator $\mathbf{S} : \mathrm{M}_1 \times \mathbf{H}_1 \to \mathbb{X}_2$ defined by

$$\mathbf{S}(q, \vartheta) := \boldsymbol{\rho} \qquad \forall\, (q, \vartheta) \in \mathrm{M}_1 \times \mathbf{H}_1 \,,$$

where $(\boldsymbol{\rho}, \boldsymbol{u}) \in \mathbb{X}_2 \times \mathbf{M}_1$ is the unique solution (to be confirmed below) of the mixed formulation arising from (3.9) after replacing (p, θ) by (q, ϑ), that is

$$\begin{aligned} \mathbf{a}(\boldsymbol{\rho}, \boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau}, \boldsymbol{u}) &= \mathbf{F}_{q,\vartheta}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_1 \,, \\ \mathbf{b}_2(\boldsymbol{\rho}, \boldsymbol{v}) &= \mathbf{G}(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \mathbf{M}_2 \,. \end{aligned} \tag{4.1}$$

In order to prove that (4.1) is well-posed (equivalently, that $\mathbf{S}$ is well-defined), we notice that (4.1) has the same bilinear forms of [28, eq. (3.15)]. Then, assuming that the Lamé parameter is sufficiently large, namely $\lambda > M$, where M is specified in [28, Lemma 3.4], we can establish that the operator $\mathbf{S}$ is well defined. Indeed, letting $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}$, $\boldsymbol{\beta}_1$, and $\boldsymbol{\beta}_2$ be the constants yielding the continuous inf-sup conditions for $\mathbf{a}$, $\mathbf{b}_1$, and $\mathbf{b}_2$ (cf. [28, Lemmas 3.4 and 3.5]), we have the following result.

Lemma 4.1. *Let r and s be within the range of values stipulated by (3.27), and assume that $\lambda > M$. Then, for each $(q, \vartheta) \in \mathrm{M}_1 \times \mathbf{H}_1$ there exists a unique $(\boldsymbol{\rho}, \boldsymbol{u}) \in \mathbb{X}_2 \times \mathbf{M}_1$ solution of (4.1), and hence one can define $\mathbf{S}(q, \vartheta) := \boldsymbol{\rho}$. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending on $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}$, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $C_{\mathbf{F}}$, and μ, such that*

$$\|\mathbf{S}(q, \vartheta)\| \,=\, \|\boldsymbol{\rho}\|_{\mathbb{X}_2} \,\leqslant\, C_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \gamma(\lambda)\Big(\|q\|_{0,r;\Omega} + \|\vartheta\|_{0,\rho;\Omega} \Big) \right\}. \tag{4.2}$$

Proof. Thanks to the fact that $\mathbb{X}_i$ and $\mathbf{M}_i$, with $i = \{1, 2\}$, are reflexive Banach spaces, along with the boundedness of all the forms and functionals involved, and the inf-sup conditions provided by [28, Lemmas 3.4 and 3.5], the proof reduces to a direct application of [6, Theorem 2.1, Corollary 2.1]. In particular, the a priori estimate (4.2) follows from [6, Corollary 2.1, eq. (2.15)]. Note that the dependence of the constant $C_{\mathbf{S}}$ on μ is due to $\|\mathbf{a}\|$ (cf. (3.11)). □

Regarding the a priori estimate for the component $\boldsymbol{u} \in \mathbf{M}_1$ of the unique solution of (4.1), we recall that, given $(q, \vartheta) \in \mathrm{M}_1 \times \mathbf{H}_1$, the second inequality in [6, Corollary 2.1] yields

$$\|\boldsymbol{u}\|_{\mathbf{M}_1} \leqslant \bar{C}_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \gamma(\lambda)\Big(\|q\|_{0,r;\Omega} + \|\vartheta\|_{0,\rho;\Omega} \Big) \right\},$$

where $\bar{C}_{\mathbf{S}}$ is a positive constant which depends principally on $C_{\mathbf{F}}$, $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}$, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$.

4.2 The decoupled perturbed Darcy problem

As in Section 4.1, we now introduce the operator $\boldsymbol{\Xi} : \mathbb{X}_2 \times \mathbf{H}_1 \to \mathbf{X}_2 \times \mathrm{M}_1$ defined by

$$\boldsymbol{\Xi}(\boldsymbol{\zeta}, \vartheta) \,=\, \big(\Xi_1(\boldsymbol{\zeta}, \vartheta), \Xi_2(\boldsymbol{\zeta}, \vartheta)\big) \,:=\, (\boldsymbol{w}, p) \qquad \forall\, (\boldsymbol{\zeta}, \vartheta) \in \mathbb{X}_2 \times \mathrm{L}^{\rho}(\Omega)\,,$$

where $(\boldsymbol{w}, p) \in \mathbf{X}_2 \times \mathrm{M}_1$ is the unique solution (to be confirmed below) of the mixed formulation arising from (3.19) after replacing $(\boldsymbol{\rho}, \theta)$ by $(\boldsymbol{\zeta}, \vartheta)$, that is

$$\begin{aligned} \mathbf{c}(\boldsymbol{w}, \boldsymbol{z}) \;+ \mathbf{d}_1(\boldsymbol{z}, p) \qquad\qquad &= \boldsymbol{\mathcal{F}}(\boldsymbol{z}) \qquad \forall\, \boldsymbol{z} \in \mathbf{X}_1 \,, \\ \mathbf{d}_2(\boldsymbol{w}, q) \qquad\qquad - \mathbf{e}(p, q) &= \boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}(q) \quad\; \forall\, q \in \mathrm{M}_2 \,. \end{aligned} \tag{4.3}$$

We observe that (4.3) has a perturbed saddle point structure over Banach spaces, but the fact that the trial and test spaces are different prevent us from using, e.g. [22, Theorem 3.1], and therefore an additional treatment is needed. Then, proceeding as in [20, Section 3.2.3], we first employ the Babška–Brezzi theory in Banach spaces (cf. [6, Theorem 2.1, Corollary 2.1, Section 2.1]) to analyze part of (4.3), and then apply the Banach–Nečas–Babuška theorem (cf. [24, Theorem 2.6]) to conclude the well-posedness of the whole problem. According to this, we now let $\widetilde{\boldsymbol{A}} : (\mathbf{X}_2 \times \mathrm{M}_1) \times (\mathbf{X}_1 \times \mathrm{M}_2) \to \mathrm{R}$ be the bounded bilinear form arising from (4.3) after adding the left-hand sides of its equations, but without including $\mathbf{e}$, that is

$$\widetilde{\boldsymbol{A}}((\boldsymbol{w}, p), (\boldsymbol{z}, q)) := \mathbf{c}(\boldsymbol{w}, \boldsymbol{z}) + \mathbf{d}_1(\boldsymbol{z}, p) + \mathbf{d}_2(\boldsymbol{w}, q) \quad \forall\, (\boldsymbol{w}, p) \in \mathbf{X}_2 \times \mathrm{M}_1 \;\; \forall\, (\boldsymbol{z}, q) \in \mathbf{X}_1 \times \mathrm{M}_2 \,, \tag{4.4}$$

and aim to prove next that $\widetilde{\boldsymbol{A}}$ satisfies global continuous inf-sup conditions with respect to both its first and second component. Note that the boundedness of $\widetilde{\boldsymbol{A}}$ follows from those of $\mathbf{c}$, $\mathbf{d}_1$ and $\mathbf{d}_2$ (cf.(3.20a) and (3.20b)). The verification of the aforementioned properties of $\widetilde{\boldsymbol{A}}$ is equivalent to establishing that the bilinear forms $\mathbf{c}$, $\mathbf{d}_1$ and $\mathbf{d}_2$ verify the hypotheses of [6, Theorem 2.1], which we address in what follows. Firstly, according to the definitions of $\mathbf{X}_i$ and M_i (cf. (3.18)), the kernel of the operators $\mathbf{d}_i$, $i \in \{1, 2\}$, are given by

$$\mathcal{V}_1 := \Big\{ \boldsymbol{z} \in \mathbf{H}^s(\mathrm{div}_s; \Omega) : \quad \mathrm{div}(\boldsymbol{z}) = 0 \Big\} \quad \text{and} \quad \mathcal{V}_2 := \Big\{ \boldsymbol{z} \in \mathbf{H}^r(\mathrm{div}_r; \Omega) : \quad \mathrm{div}(\boldsymbol{z}) = 0 \Big\} \,.$$

The two subsequent lemmas, akin to those previously stated and demonstrated in [20] and [29], establish the inf-sup conditions required by [6, Theorem 2.1] for the bilinear forms $\mathbf{c}$ (cf. (3.20a)), and $\mathbf{d}_1, \mathbf{d}_2$ (cf. (3.20b)), respectively.

Lemma 4.2. *Assume that r and s satisfy the particular range specified by (3.27). Then, there exists a positive constant $\boldsymbol{\alpha}_{\mathbf{c}}$ such that*

$$\sup_{\substack{\boldsymbol{z} \in \mathcal{V}_1 \\ \boldsymbol{z} \neq \mathbf{0}}} \frac{\mathbf{c}(\boldsymbol{w}, \boldsymbol{z})}{\|\boldsymbol{z}\|_{\mathbf{X}_1}} \geqslant \boldsymbol{\alpha}_{\mathbf{c}} \|\boldsymbol{w}\|_{\mathbf{X}_2} \qquad \forall\, \boldsymbol{w} \in \mathcal{V}_2 \,,$$

and

$$\sup_{\boldsymbol{w} \in \mathcal{V}_2} \mathbf{c}(\boldsymbol{w}, \boldsymbol{z}) > 0 \qquad \forall\, \boldsymbol{z} \in \mathbf{X}_1, \quad \boldsymbol{z} \neq \mathbf{0} \,.$$

Proof. The proof follows a similar approach as in [20, Lemma 3.4], leading to $\boldsymbol{\alpha}_{\mathbf{c}} = \frac{\eta}{\kappa \|\mathrm{D}_s\|}$, with D_s being the bounded linear operator introduced in [20, Lemma 3.3].

□

The continuous inf-sup conditions for the bilinear forms $\mathbf{d}_i$, $i \in \{1, 2\}$ are presented next.

Lemma 4.3. *For each $i \in \{1, 2\}$ there exists a positive constant $\widetilde{\boldsymbol{\beta}}_i$ such that*

$$\sup_{\substack{\boldsymbol{z} \in \mathbf{X}_i \\ \boldsymbol{z} \neq \mathbf{0}}} \frac{\mathbf{d}_i(\boldsymbol{z}, q)}{\|\boldsymbol{z}\|_{\mathbf{X}_i}} \geqslant \widetilde{\boldsymbol{\beta}}_i \|q\|_{\mathrm{M}_i} \qquad \forall\, q \in \mathrm{M}_i \,.$$

Proof. A proof of this lemma can be done by slightly modifying that of [29, Lemma 2.7], considering Dirichlet boundary conditions of the auxiliary problems instead. □

According to Lemmas 4.2 and 4.3, the required hypotheses of [6, Theorem 2.1, Section 2.1] are satisfied, and hence the a priori estimation provided by [6, Corollary 2.1, Section 2.1] imply the existence of a positive constant $\boldsymbol{\alpha}_{\boldsymbol{A}}$, depending only on $\boldsymbol{\alpha}_{\mathbf{c}}$, $\widetilde{\boldsymbol{\beta}}_1$, $\widetilde{\boldsymbol{\beta}}_2$ and $\|\mathbf{c}\|$, such that

$$\sup_{\substack{(\boldsymbol{z},q)\in\mathbf{X}_1\times\mathrm{M}_2\\(\boldsymbol{z},q)\neq\mathbf{0}}}\frac{\widetilde{\boldsymbol{A}}((\boldsymbol{w},p),(\boldsymbol{z},q))}{\|(\boldsymbol{z},q)\|_{\mathbf{X}_1\times\mathrm{M}_2}}\geqslant\boldsymbol{\alpha}_{\boldsymbol{A}}\|(\boldsymbol{w},p)\|_{\mathbf{X}_2\times\mathrm{M}_1}\qquad\forall\,(\boldsymbol{w},p)\in\mathbf{X}_2\times\mathrm{M}_1\,,\tag{4.5a}$$

$$\sup_{\substack{(\boldsymbol{w},p)\in\mathbf{X}_2\times\mathrm{M}_1\\(\boldsymbol{w},p)\neq\mathbf{0}}}\frac{\widetilde{\boldsymbol{A}}((\boldsymbol{w},p),(\boldsymbol{z},q))}{\|(\boldsymbol{w},p)\|_{\mathbf{X}_2\times\mathrm{M}_1}}\geqslant\boldsymbol{\alpha}_{\boldsymbol{A}}\|(\boldsymbol{z},q)\|_{\mathbf{X}_1\times\mathrm{M}_2}\qquad\forall\,(\boldsymbol{z},q)\in\mathbf{X}_1\times\mathrm{M}_2\,.\tag{4.5b}$$

Therefore, we let $\boldsymbol{A}:(\mathbf{X}_2\times\mathrm{M}_1)\times(\mathbf{X}_1\times\mathrm{M}_2)\to\mathrm{R}$ be the bounded and linear operator arising from (4.3) after adding the full left-hand sides of its equations, that is

$$\begin{gathered}\boldsymbol{A}((\boldsymbol{w},p),(\boldsymbol{z},q))=\mathbf{c}(\boldsymbol{w},\boldsymbol{z})+\mathbf{d}_1(\boldsymbol{z},p)+\mathbf{d}_2(\boldsymbol{w},q)-\mathbf{e}(p,q)\\\forall\,(\boldsymbol{w},p)\in\mathbf{X}_2\times\mathrm{M}_1\,,\quad\forall\,(\boldsymbol{z},q)\in\mathbf{X}_1\times\mathrm{M}_2\,.\end{gathered}\tag{4.6}$$

Having introduced this operator, we realize that solving (4.3) for a given pair $(\boldsymbol{\zeta},\vartheta)\in\mathbb{X}_2\times\mathbf{H}_1$, is equivalent to: Find $(\boldsymbol{w},p)\in\mathbf{X}_2\times\mathrm{M}_1$ such that

$$\boldsymbol{A}((\boldsymbol{w},p),(\boldsymbol{z},q))=\boldsymbol{\mathcal{F}}(\boldsymbol{z})+\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}(q)\qquad\forall\,(\boldsymbol{z},q)\in\mathbf{X}_1\times\mathrm{M}_2\,.$$

We notice that, thanks to the boundedness of $\widetilde{\boldsymbol{A}}$ and $\mathbf{e}$, the operator $\boldsymbol{A}$ is bounded as well. Thus bearing in mind (4.6), employing (4.5a) and the boundedness of $\mathbf{e}$ (cf. (3.22)), we have

$$\sup_{\substack{(\boldsymbol{z},q)\in\mathbf{X}_1\times\mathrm{M}_2\\(\boldsymbol{z},q)\neq\mathbf{0}}}\frac{\boldsymbol{A}((\boldsymbol{w},p),(\boldsymbol{z},q))}{\|(\boldsymbol{z},q)\|_{\mathbf{X}_1\times\mathrm{M}_2}}\geqslant\Big\{\boldsymbol{\alpha}_{\boldsymbol{A}}-\|\mathbf{e}\|\Big\}\|(\boldsymbol{w},p)\|_{\mathbf{X}_2\times\mathrm{M}_1}\qquad\forall\,(\boldsymbol{w},p)\in\mathbf{X}_2\times\mathrm{M}_1\,.$$

Then, assuming that the data satisfy

$$\|\mathbf{e}\|\;:=\;\max\big\{\chi\,,n\,\alpha^2\gamma(\lambda)\big\}\leqslant\frac{\boldsymbol{\alpha}_{\boldsymbol{A}}}{2}\,,\tag{4.7}$$

we arrive at the global inf-sup condition for the perturbed Darcy problem

$$\sup_{\substack{(\boldsymbol{z},q)\in\mathbf{X}_1\times\mathrm{M}_2\\(\boldsymbol{z},q)\neq\mathbf{0}}}\frac{\boldsymbol{A}((\boldsymbol{w},p),(\boldsymbol{z},q))}{\|(\boldsymbol{z},q)\|_{\mathbf{X}_1\times\mathrm{M}_2}}\geqslant\frac{\boldsymbol{\alpha}_{\boldsymbol{A}}}{2}\|(\boldsymbol{w},p)\|_{\mathbf{X}_2\times\mathrm{M}_1}\qquad\forall\,(\boldsymbol{w},p)\in\mathbf{X}_2\times\mathrm{M}_1\,.\tag{4.8}$$

Similarly, but employing now (4.5b) instead of (4.5a), and under the same assumption (4.7), we obtain the second desired inf-sup condition for $\boldsymbol{A}$, this is

$$\sup_{\substack{(\boldsymbol{w},p)\in\mathbf{X}_2\times\mathrm{M}_1\\(\boldsymbol{w},p)\neq\mathbf{0}}}\frac{\boldsymbol{A}((\boldsymbol{w},p),(\boldsymbol{z},q))}{\|(\boldsymbol{w},p)\|_{\mathbf{X}_2\times\mathrm{M}_1}}\geqslant\frac{\boldsymbol{\alpha}_{\boldsymbol{A}}}{2}\|(\boldsymbol{z},q)\|_{\mathbf{X}_1\times\mathrm{M}_2}\qquad\forall\,(\boldsymbol{z},q)\in\mathbf{X}_1\times\mathrm{M}_2\,.\tag{4.9}$$

We are now in position to establish the well-posedness of the operator $\boldsymbol{\Xi}$, equivalently the existence and uniqueness of solution of (4.3).

Lemma 4.4. *Let r and s be within the range of values stipulated by (3.27), and assume that the data fulfill condition (4.7). Then, for each $(\boldsymbol{\zeta},\vartheta) \in \mathbb{X}_2 \times \mathbf{H}_1$ there exists a unique $(\boldsymbol{w},p) \in \mathbf{X}_2 \times \mathrm{M}_1$ solution of (4.3), and hence one can define $\boldsymbol{\Xi}(\boldsymbol{\zeta},\vartheta) := (\boldsymbol{w},p) \in \mathbf{X}_2 \times \mathrm{M}_1$. Moreover, there exists a positive constant $C_{\boldsymbol{\Xi}}$, depending on $\boldsymbol{\alpha_A}$, $C_{\boldsymbol{\mathcal{F}}}$, and $C_{\boldsymbol{\mathcal{G}}}$, such that*

$$\begin{aligned}\|\boldsymbol{\Xi}(\boldsymbol{\zeta},\vartheta)\|_{\mathbf{X}_2\times\mathrm{M}_1} \;&=\; \|\boldsymbol{w}\|_{\mathbf{X}_2} + \|p\|_{\mathrm{M}_1} \\ &\leqslant\; C_{\boldsymbol{\Xi}}\Big\{\|p_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\Big(\|\boldsymbol{\zeta}\|_{\mathbb{X}_2} + \|\vartheta\|_{0,\rho;\Omega}\Big)\Big\}.\end{aligned} \tag{4.10}$$

Proof. Given $(\boldsymbol{\zeta},\vartheta) \in \mathbb{X}_2 \times \mathbf{H}_1$, thanks to the boundedness of $\boldsymbol{A}$, and the global inf-sup conditions (4.8) and (4.9), a direct application of [24, Theorem 2.6] provides the existence of a unique solution $(\boldsymbol{w},p) \in \mathbf{X}_2 \times \mathrm{M}_1$ to (4.3). The a priori estimate (cf. [24, Theorem 2.6, eq. (2.5)]) yields

$$\|\boldsymbol{\Xi}(\boldsymbol{\zeta},\vartheta)\|_{\mathbf{X}_2\times\mathrm{M}_1} = \|\boldsymbol{w}\|_{\mathbf{X}_2} + \|p\|_{\mathrm{M}_1} \leqslant \frac{2}{\boldsymbol{\alpha_A}}\Big\{\|\boldsymbol{\mathcal{F}}\| + \|\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}\|\Big\},$$

which, together with the expressions for $\|\boldsymbol{\mathcal{F}}\|$, $\|\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}\|$ given in (3.33) imply (4.10). □

4.3 The decoupled heat equation

We now introduce the operator $\boldsymbol{\Pi} : \mathbb{X}_2 \times (\mathbf{X}_2 \times \mathrm{M}_1) \times \mathbf{H}_1 \to \mathbf{H}$ defined by

$$\boldsymbol{\Pi}(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi) \;=\; \big(\boldsymbol{\Pi}_1(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi),\boldsymbol{\Pi}_2(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi)\big) := \vec{\theta} = (\theta,\widetilde{\boldsymbol{t}}),$$

for all $(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi) = \big(\boldsymbol{\zeta},(\mathbf{z},q),\xi\big) \in \mathbb{X}_2 \times (\mathbf{X}_2 \times \mathrm{M}_1) \times \mathbf{H}_1$, where $(\vec{\theta},\widetilde{\boldsymbol{\sigma}}) = \big((\theta,\widetilde{\boldsymbol{t}}),\widetilde{\boldsymbol{\sigma}}\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be confirmed below) of the problem arising from (3.29) after replacing $a_{\vec{\boldsymbol{p}},\theta}$, with $\vec{\boldsymbol{p}} = (\boldsymbol{\rho},\boldsymbol{\omega},p)$, by $a_{\vec{\boldsymbol{q}},\xi}$, with $\vec{\boldsymbol{q}} = (\boldsymbol{\zeta},\mathbf{z},q)$, that is

$$\begin{aligned} a_{\vec{\boldsymbol{q}},\xi}(\vec{\theta},\vec{\vartheta}) + b(\vec{\vartheta},\widetilde{\boldsymbol{\sigma}}) \quad &= \quad F(\vec{\vartheta}) \qquad \forall\, \vec{\vartheta} := (\vartheta,\widetilde{\boldsymbol{s}}) \in \mathbf{H}\,, \\ b(\vec{\theta},\widetilde{\boldsymbol{\tau}}) \quad &= \quad 0 \qquad\quad\;\; \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,. \end{aligned} \tag{4.11}$$

We recall from (3.32) that the bilinear form $a_{\vec{\boldsymbol{q}},\xi}$ (cf. (3.30a)) is bounded with constant $\|a\| + \|\mathbf{z}\|_{0,r;\Omega}$, which is independent of $\boldsymbol{\zeta}$, q and ξ. Furthermore, it is easy to see that the null space associated with the bilinear form b is given by (see, e.g. [21, eq. (3.35)] for the case $(\rho,\varrho) = (4,4/3)$)

$$\begin{aligned}\mathcal{V}_b \;:=&\; \Big\{(\vartheta,\widetilde{\boldsymbol{s}}) \in \mathbf{H}: \quad \int_\Omega \widetilde{\boldsymbol{\tau}}\cdot\widetilde{\boldsymbol{s}} + \int_\Omega \vartheta\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) = 0 \quad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\Big\} \\ =&\; \Big\{(\vartheta,\widetilde{\boldsymbol{s}}) \in \mathbf{H}: \quad \widetilde{\boldsymbol{s}} = \nabla\vartheta \quad \text{and} \quad \vartheta \in \mathrm{H}^1_0(\Omega)\Big\}.\end{aligned}$$

Then, following the same ideas as in [21, Lemma 3.6], we have to prove that $a_{\vec{\boldsymbol{q}},\xi}$ is $\mathcal{V}_b$-elliptic plus an inf-sup condition on b. To show the property of $a_{\vec{\boldsymbol{q}},\xi}$, we use the above characterization along with (2.10) and the continuous injection $i_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$. In this way, for each $\vec{\vartheta} = (\vartheta,\widetilde{\boldsymbol{s}}) \in \mathcal{V}_b$, we get

$$\begin{aligned} a_{\vec{\boldsymbol{q}},\xi}(\vec{\vartheta},\vec{\vartheta}) &\geqslant \|\vartheta\|^2_{0;\Omega} + \varkappa_0\,\|\widetilde{\boldsymbol{s}}\|^2_{0;\Omega} + \int_\Omega \mathbf{z}\cdot\widetilde{\boldsymbol{s}}\,\vartheta \;\geqslant\; \widetilde{\varkappa}_0\,\|\vartheta\|^2_{1;\Omega} + (\varkappa_0/2)\|\widetilde{\boldsymbol{s}}\|^2_{0;\Omega} + \int_\Omega \mathbf{z}\cdot\widetilde{\boldsymbol{s}}\vartheta \\ &\geqslant \widetilde{\varkappa}_0\,\|i_\rho\|^{-2}\|\vartheta\|^2_{0,\rho;\Omega} + (\varkappa_0/2)\|\widetilde{\boldsymbol{s}}\|^2_{0;\Omega} - \|\mathbf{z}\|_{0,r;\Omega}\|\widetilde{\boldsymbol{s}}\|_{0;\Omega}\|\vartheta\|_{0,\rho;\Omega} \\ &\geqslant \tfrac{1}{2}\,(2\varkappa - \|\mathbf{z}\|_{0,r;\Omega})\,\|\vec{\vartheta}\|^2\,, \end{aligned} \tag{4.12}$$

where the constants $\widetilde{\varkappa}_0$ and $\varkappa$ are given by

$$\widetilde{\varkappa}_0 := \min\Big\{\frac{\varkappa_0}{2},1\Big\} \quad \text{and} \quad \varkappa := \min\Big\{\widetilde{\varkappa}_0\,\|i_\rho\|^{-2},\frac{\varkappa_0}{2}\Big\}\,.$$

Thus, under the assumption $\|\mathbf{z}\|_{\mathbf{X}_2} \leqslant \alpha_A := \frac{2}{3}\,\varkappa$, the inequality (4.12) implies

$$a_{\vec{\boldsymbol{q}},\xi}(\vec{\vartheta},\vec{\vartheta}) \;\geqslant\; \alpha_A\,\|\vec{\vartheta}\|^2 \qquad \forall\,\vec{\vartheta} := (\vartheta,\widetilde{\boldsymbol{s}}) \in \mathcal{V}_b\,, \tag{4.13}$$

which establishes the $\mathcal{V}_b$-ellipticity of $a_{\vec{\boldsymbol{q}},\xi}$ with constant α_A.

The inf-sup condition for the bilinear form b states that there exists a constant $\widetilde{\beta} > 0$ such that

$$\sup_{\substack{\vec{\vartheta}\in\mathbf{H}\\ \vec{\vartheta}\neq\mathbf{0}}} \frac{b(\vec{\vartheta},\widetilde{\boldsymbol{\tau}})}{\|\vec{\vartheta}\|_{\mathbf{H}}} \;\geqslant\; \widetilde{\beta}\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_\varrho;\Omega} \qquad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \tag{4.14}$$

which can be proved analogously to the case $(\rho,\varrho) = (4,4/3)$ provided in [21, Lemma 3.3, eq. (3.45)] since the present indexes ρ and ϱ are conjugate to each other as well.

The previous discussion allows us to establish the following lemma on the existence and uniqueness of solution of the decoupled system (4.11).

Lemma 4.5. *Let ρ and ϱ be within the range of values stipulated by (3.27). Then, for each $(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi) = \big(\boldsymbol{\zeta},(\mathbf{z},q),\xi\big) \in \mathbb{X}_2\times(\mathbf{X}_2\times \mathrm{M}_1)\times\mathbf{H}_1$ such that $\|\mathbf{z}\| \leqslant \alpha_A$, there exists a unique $(\vec{\theta},\widetilde{\boldsymbol{\sigma}}) = \big((\theta,\widetilde{\boldsymbol{t}}),\widetilde{\boldsymbol{\sigma}}\big) \in \mathbf{H}\times\mathbf{Q}$ solution of (4.11), and hence one can define $\mathbf{\Pi}(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi) := \vec{\theta}$. Moreover, there exist positive constants $C_{\mathbf{\Pi}}$ and $\bar{C}_{\mathbf{\Pi}}$, depending on α_A, $\widetilde{\beta}$, $|\Omega|$, ρ, and $\varkappa_2$, such that the following a priori estimates hold*

$$\|\mathbf{\Pi}(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi)\| = \|\vec{\theta}\|_{\mathbf{H}} \leqslant C_{\mathbf{\Pi}}\,\|g\|_{0,\varrho;\Omega}\,, \qquad \|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \leqslant \bar{C}_{\mathbf{\Pi}}\,\|g\|_{0,\varrho;\Omega}\,. \tag{4.15}$$

Proof. The proof is a consequence of the $\mathcal{V}_b$-ellipticity of $a_{\vec{\boldsymbol{q}},\xi}$ (cf. (4.13)), the inf-sup condition (4.14), and a direct application of [6, Theorem 2.1, Corollary 2.1]. Note that the dependence of the constants $C_{\mathbf{\Pi}}$ and $\bar{C}_{\mathbf{\Pi}}$ on $|\Omega|$, ρ, and $\varkappa_2$, is due to $\|a\|$ (cf. (3.31)) since $\|a_{\vec{\boldsymbol{q}},\xi}\|$, which is required by the abstract a priori estimates from [6, Corollary 2.1, eqs. (2.15) and (2.16)], is bounded above by $\|a\| + \|\mathbf{z}\|$. $\square$

4.4 Solvability of the fully-mixed formulation

In order to solve the fully-mixed coupled problem (3.34) we propose a fixed-point strategy based on the operators $\mathbf{S}$, $\mathbf{\Xi}$ and $\mathbf{\Pi}$, which correspond to the decoupled problems (4.1), (4.3) and (4.11), respectively. The coupling of the three problems can be analyzed in terms of the compose operator $\mathbf{T} : \mathbb{X}_2\times\mathbf{H}_1 \to \mathbb{X}_2\times\mathbf{H}_1$ given by

$$\mathbf{T}(\boldsymbol{\zeta},\vartheta) \;:=\; \Big(\mathbf{S}\big(\mathbf{\Xi}_2(\boldsymbol{\zeta},\vartheta),\vartheta\big),\,\mathbf{\Pi}_1\big(\mathbf{S}(\mathbf{\Xi}_2(\boldsymbol{\zeta},\vartheta),\vartheta),\mathbf{\Xi}(\boldsymbol{\zeta},\vartheta),\vartheta\big)\Big) \quad \forall\,(\boldsymbol{\zeta},\vartheta) \in \mathbb{X}_2\times\mathbf{H}_1\,. \tag{4.16}$$

The well-definedness of $\mathbf{S}$, $\mathbf{\Xi}$ and $\mathbf{\Pi}$, which was obtained from Lemmas 4.1, 4.4 and 4.5, respectively, implies the same property for the operator $\mathbf{T}$. Furthermore, due to the nonlinear character of $\mathbf{\Pi}$, the operator $\mathbf{T}$ becomes nonlinear as well. Then, we observe that solving (3.34) is equivalent to seeking a fixed-point of $\mathbf{T}$, that is: Find $(\boldsymbol{\rho},\theta) \in \mathbb{X}_2\times\mathbf{H}_1$ such that

$$\mathbf{T}(\boldsymbol{\rho},\theta) \;=\; (\boldsymbol{\rho},\theta)\,. \tag{4.17}$$

In what follows, we address the solvability of the nonlinear equation (4.17), equivalently of (3.34), by means of the Banach fixed-point theorem. For this purpose, given $\delta > 0$, we first introduce the ball

$$\mathcal{W}(\delta) \;:=\; \Big\{(\boldsymbol{\zeta},\vartheta) \in \mathbb{X}_2\times\mathbf{H}_1 : \quad \|(\boldsymbol{\zeta},\vartheta)\| := \|\boldsymbol{\zeta}\|_{\mathbb{X}_2} + \|\vartheta\|_{0,\rho;\Omega} \leqslant \delta\Big\}\,.$$

Now, given $(\boldsymbol{\zeta},\vartheta) \in \mathcal{W}(\delta)$, the definition of $\mathbf{T}$ yields

$$\|\mathbf{T}(\boldsymbol{\zeta},\vartheta)\| = \big\|\mathbf{S}\big(\mathbf{\Xi}_2(\boldsymbol{\zeta},\vartheta),\vartheta\big)\big\|_{\mathbb{X}_2} + \big\|\mathbf{\Pi}_1\big(\mathbf{S}(\mathbf{\Xi}_2(\boldsymbol{\zeta},\vartheta),\vartheta),\mathbf{\Xi}(\boldsymbol{\zeta},\vartheta),\vartheta\big)\big\|_{0,\rho;\Omega}\,,$$

from which, assuming (4.7) and the upper bound

$$\|\boldsymbol{\Xi}_1(\boldsymbol{\zeta},\vartheta)\|_{\mathbf{X}_2} \leqslant \alpha_A\,, \tag{4.18}$$

and bearing in mind the a priori estimates for $\mathbf{S}$, $\boldsymbol{\Xi}$ and $\boldsymbol{\Pi}$ (cf. (4.2), (4.10) and (4.15), respectively), we find that

$$\begin{aligned} \|\mathbf{T}(\boldsymbol{\zeta},\vartheta)\| \,\leqslant\, C_{\mathbf{T}}\Big\{ & \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \|p_{\mathrm{D}}\|_{1/s,r;\Omega} \\ & + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} \,+\, \gamma(\lambda)\left(\|\boldsymbol{\zeta}\|_{\mathbb{X}_2} + \|\vartheta\|_{0,\rho;\Omega}\right)\Big\}\,, \end{aligned} \tag{4.19}$$

where $C_{\mathbf{T}}$ is a positive constant depending on $C_{\mathbf{S}}$, $C_{\boldsymbol{\Xi}}$ and $C_{\boldsymbol{\Pi}}$. In turn, we deduce from the estimate for $\|\boldsymbol{\Xi}(\boldsymbol{\zeta},\vartheta)\|$ (cf. (4.10)) that a sufficient condition for the assumption (4.18) is given by

$$C_{\boldsymbol{\Xi}}\Big\{\|p_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\big(\|\boldsymbol{\zeta}\|_{\mathbb{X}_2} + \|\vartheta\|_{0,\rho;\Omega}\big)\Big\} \leqslant \alpha_A\,.$$

In this way, noting that certainly $\|\boldsymbol{\zeta}\|_{\mathbb{X}_2} + \|\vartheta\|_{0,\rho;\Omega} \,\leqslant\, \delta$ we conclude the following result.

Lemma 4.6. *Let ρ, ϱ, r and s be the real numbers within the range specified in (3.27), and $\lambda > M$. Assume that the data are sufficiently small so that (4.7) and the conditions*

$$C_{\boldsymbol{\Xi}}\Big\{\|p_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\,\delta\Big\} \,\leqslant\, \alpha_A\,, \quad \textit{and} \tag{4.20a}$$

$$C_{\mathbf{T}}\Big\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \|p_{\mathrm{D}}\|_{1/s,r;\Omega} + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} \,+\, \gamma(\lambda)\delta\Big\} \leqslant \delta\,, \tag{4.20b}$$

are satisfied. Then, the operator $\mathbf{T}$ maps the ball $\mathcal{W}(\delta)$ into itself, that is $\mathbf{T}(\mathcal{W}(\delta)) \subseteq \mathcal{W}(\delta)$.

We now aim to prove that the operator $\mathbf{T}$ is Lipschitz-continuous, for which, according to its definition (cf. (4.16)), it suffices to show that $\mathbf{S}$, $\boldsymbol{\Xi}$ and $\boldsymbol{\Pi}$ satisfy the same property. We begin with the corresponding result for $\mathbf{S}$.

Lemma 4.7. *Let r and s be within the range of values stipulated by (3.27), and $\lambda > M$. Then, with the same constant $C_{\mathbf{S}}$ from the a priori estimate (4.2) for $\mathbf{S}$ (cf. Lemma 4.1), there holds*

$$\|\mathbf{S}(q_1,\vartheta_1) - \mathbf{S}(q_2,\vartheta_2)\|_{\mathbb{X}_2} \,\leqslant\, C_{\mathbf{S}}\,\gamma(\lambda)\,\|(q_1,\vartheta_1) - (q_2,\vartheta_2)\|_{\mathrm{M}_1\times\mathbf{H}_1}\,, \tag{4.21}$$

for all (q_1,ϑ_1), $(q_2,\vartheta_2) \in \mathrm{M}_1 \times \mathbf{H}_1$.

Proof. Given (q_1,ϑ_1), $(q_2,\vartheta_2) \in \mathrm{M}_1 \times \mathbf{H}_1$, we let $\mathbf{S}(q_1,\vartheta_1) = \boldsymbol{\rho}_1 \in \mathbb{X}_2$ and $\mathbf{S}(q_2,\vartheta_2) = \boldsymbol{\rho}_2 \in \mathbb{X}_2$, where $(\boldsymbol{\rho}_1,\boldsymbol{u}_1)$ and $(\boldsymbol{\rho}_2,\boldsymbol{u}_2)$ in $\mathbb{X}_2 \times \mathbf{M}_1$ are the respective unique solutions of (4.1). Then, thanks to the linearity of this problem, it is straightforward to see that $(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \in \mathbb{X}_2 \times \mathbf{M}_1$ is the unique solution of (4.1) with $\mathbf{F}_{q_1,\vartheta_1} - \mathbf{F}_{q_2,\vartheta_2}$ and the null functional instead of $\mathbf{F}_{q,\vartheta}$ and $\mathbf{G}$, respectively. Consequently, noting from (3.10a) that

$$\big(\mathbf{F}_{q_1,\vartheta_1} - \mathbf{F}_{q_2,\vartheta_2}\big)(\boldsymbol{\tau}) \;=\; -\gamma(\lambda)\int_\Omega \big(\alpha(q_1-q_2) + \beta(\vartheta_1-\vartheta_2)\big)\,\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{X}_1\,,$$

the a priori estimate (4.2) yields

$$\|\mathbf{S}(q_1,\vartheta_1) - \mathbf{S}(q_2,\vartheta_2)\|_{\mathbb{X}_2} = \|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\|_{\mathbb{X}_2} \,\leqslant\, C_{\mathbf{S}}\,\gamma(\lambda)\left(\|q_1-q_2\|_{0,r;\Omega} + \|\vartheta_1-\vartheta_2\|_{0,\rho;\Omega}\right),$$

which ends the proof. $\square$

The Lipschitz continuity of the operator $\boldsymbol{\Xi}$ is addressed next.

Lemma 4.8. *Let r and s be within the range of values stipulated by (3.27), and assume that the data fulfills condition (4.7). Then, with the same constant $C_{\boldsymbol{\Xi}}$ from the a priori estimate (4.10) for $\boldsymbol{\Xi}$ (cf. Lemma 4.4), there holds*

$$\|\boldsymbol{\Xi}(\boldsymbol{\zeta}_1,\vartheta_1)-\boldsymbol{\Xi}(\boldsymbol{\zeta}_2,\vartheta_2)\|_{\mathbf{X}_2\times \mathrm{M}_1} \,\leqslant\, C_{\boldsymbol{\Xi}}\,\gamma(\lambda)\,\|(\boldsymbol{\zeta}_1,\vartheta_1)-(\boldsymbol{\zeta}_2,\vartheta_2)\|_{\mathbb{X}_2\times\mathbf{H}_1}\,, \tag{4.22}$$

for all $(\boldsymbol{\zeta}_1,\vartheta_1)$, $(\boldsymbol{\zeta}_2,\vartheta_2)\in\mathbb{X}_2\times\mathbf{H}_1$.

Proof. The proof follows in a similar fashion as the previous lemma. Given the two pairs of functions $(\boldsymbol{\zeta}_1,\vartheta_1)$, $(\boldsymbol{\zeta}_2,\vartheta_2)\in\mathbb{X}_2\times\mathbf{H}_1$, we let $\boldsymbol{\Xi}(\boldsymbol{\zeta}_1,\vartheta_1)=(\boldsymbol{w}_1,p_1)\in\mathbf{X}_2\times\mathrm{M}_1$ and $\boldsymbol{\Xi}(\boldsymbol{\zeta}_2,\vartheta_2)=(\boldsymbol{w}_2,p_2)\in\mathbf{X}_2\times\mathrm{M}_1$ in $\mathbf{X}_2\times\mathrm{M}_1$, where $(\boldsymbol{w}_1,p_1)$ and $(\boldsymbol{w}_2,p_2)$ are the respective solutions of (4.3). Then, thanks to the linearity of (4.3), we realize that $(\boldsymbol{w}_1-\boldsymbol{w}_2,p_1-p_2)\in\mathbf{X}_2\times\mathrm{M}_1$ is the unique solution of this problem when $\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}$ and $\boldsymbol{\mathcal{F}}$ are replaced by $\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta}_1,\vartheta_1}-\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta}_2,\vartheta_2}$ and the null functional, respectively. In this way, noting from (3.21b) that

$$\big(\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta}_1,\vartheta_1}-\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta}_2,\vartheta_2}\big)(q) \,=\, c_2(\lambda)\int_\Omega \mathrm{tr}(\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2)\,q \,+\, c_3(\lambda)\int_\Omega(\vartheta_{1,0}-\vartheta_{2,0})\,q$$

where $\vartheta_{i,0} \,=\, m(\vartheta_i)$ (cf. (3.13)), $i\in\{1,2\}$, the a priori estimate (4.10) gives

$$\begin{aligned}\|\boldsymbol{\Xi}(\boldsymbol{\zeta}_1,\vartheta_1)-\boldsymbol{\Xi}(\boldsymbol{\zeta}_2,\vartheta_2)\| \,&=\, \|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{\mathbf{X}_2}+\|p_1-p_2\|_{\mathrm{M}_1}\\ &\leqslant\, C_{\boldsymbol{\Xi}}\,\gamma(\lambda)\Big(\|\boldsymbol{\zeta}_1-\boldsymbol{\zeta}_2\|_{\mathbb{X}_2}+\|\vartheta_1-\vartheta_2\|_{0,\rho;\Omega}\Big)\,,\end{aligned}$$

which concludes the proof. □

It remains to establish the continuity of $\boldsymbol{\Pi}$, for which, following the approach from several previous works (see, e.g [26, 28, 29]), we assume from now on a regularity assumption on the solution of the problem defining this operator, namely

(**H.1**) there exists $\varepsilon \,\geqslant\, \frac{n}{r}$ and a positive constant $\widetilde{C}_\varepsilon$, such that for each $(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi)\in\mathbb{X}_2\times(\mathbf{X}_2\times\mathrm{M}_1)\times\mathbf{H}_1$, there hold $\boldsymbol{\Pi}(\boldsymbol{\zeta},\vec{\mathbf{z}},\xi):=\vec{\theta}=(\theta,\widetilde{\boldsymbol{t}})\,\in\mathrm{W}^{\varepsilon,\rho}(\Omega)\times\mathbf{H}^\varepsilon(\Omega)$, and

$$\|\vec{\theta}\| \,:=\, \|\theta\|_{\varepsilon,\rho;\Omega}+\|\widetilde{\boldsymbol{t}}\|_{\varepsilon;\Omega} \,\leqslant\, \widetilde{C}_\varepsilon\,\|g\|_{0,\varrho;\Omega}\,. \tag{4.23}$$

The aforementioned lower bound of ε is explained within the proof of Lemma 4.9 below, which provides the Lipschitz-continuity of $\boldsymbol{\Pi}$. In this regard, we recall here that for each $\varepsilon<\frac{n}{2}$ there holds $\mathbf{H}^\varepsilon(\Omega)\,\subset\,\mathbf{L}^{\varepsilon^*}(\Omega)$ with continuous injection

$$i_\varepsilon:\mathbf{H}^\varepsilon(\Omega)\to\mathbf{L}^{\varepsilon^*}(\Omega),\quad\text{ where }\quad \varepsilon^*=\frac{2n}{n-2\varepsilon}\,.$$

Note that the indicated lower and upper bounds for the additional regularity ε, which turn out to require that $\varepsilon\in[\frac{n}{r},\frac{n}{2})$, are compatible if and only if $r>2$, which is coherent with the range stipulated in (3.27). Thus, we have the following result.

Lemma 4.9. *Let ρ and ϱ be within the range of values stipulated by (3.27), and assume that the regularity condition (**H.1**) (cf. (4.23)) holds. Then, there exists a positive constant $L_{\boldsymbol{\Pi}}$, depending on $L_{\mathcal{K}}$, α_A, i_ε, $\widetilde{C}_\varepsilon$, $|\Omega|$, r, n, and ε, such that*

$$\|\boldsymbol{\Pi}(\boldsymbol{\zeta}_1,\vec{\mathbf{z}}_1,\xi_1)-\boldsymbol{\Pi}(\boldsymbol{\zeta}_2,\vec{\mathbf{z}}_2,\xi_2)\|_{\mathbf{H}} \,\leqslant\, L_{\boldsymbol{\Pi}}\,\|g\|_{0,\varrho;\Omega}\,\|(\boldsymbol{\zeta}_1,\vec{\mathbf{z}}_1,\xi_1)-(\boldsymbol{\zeta}_2,\vec{\mathbf{z}}_2,\xi_2)\|\,, \tag{4.24}$$

for all $(\boldsymbol{\zeta}_1,\vec{\mathbf{z}}_1,\xi_1)=\big(\boldsymbol{\zeta}_1,(\mathbf{z}_1,q_1),\xi_1\big)$, $(\boldsymbol{\zeta}_2,\vec{\mathbf{z}}_2,\xi_2)=\big(\boldsymbol{\zeta}_2,(\mathbf{z}_2,q_2),\xi_2\big)\in\mathbb{X}_2\times(\mathbf{X}_2\times\mathrm{M}_1)\times\mathbf{H}_1$, such that $\|\mathbf{z}_1\|_{\mathbf{X}_2}$, $\|\mathbf{z}_2\|_{\mathbf{X}_2}\,\leqslant\,\alpha_A$.

Proof. Given $(\boldsymbol{\zeta}_1, \vec{\mathbf{z}}_1, \xi_1)$, $(\boldsymbol{\zeta}_2, \vec{\mathbf{z}}_2, \xi_2) \in \mathbb{X}_2 \times (\mathbf{X}_2 \times \mathrm{M}_1) \times \mathbf{H}_1$ as indicated, we let $\vec{\theta}_1 := \mathbf{\Pi}(\boldsymbol{\rho}_1, \vec{\mathbf{z}}_1, \xi_1) \in \mathbf{H}$ and $\vec{\theta}_2 := \mathbf{\Pi}(\boldsymbol{\rho}_2, \vec{\mathbf{z}}_2, \xi_2) \in \mathbf{H}$, where $(\vec{\theta}_1, \widetilde{\boldsymbol{\sigma}}_1) = \big((\theta_1, \widetilde{\boldsymbol{t}}_1), \widetilde{\boldsymbol{\sigma}}_1\big) \in \mathbf{H} \times \mathbf{Q}$ and $(\vec{\theta}_2, \widetilde{\boldsymbol{\sigma}}_2) = \big((\theta_2, \widetilde{\boldsymbol{t}}_2), \widetilde{\boldsymbol{\sigma}}_2\big) \in \mathbf{H} \times \mathbf{Q}$ are the respective solutions of (4.11). Defining $\vec{\boldsymbol{q}}_1 := (\boldsymbol{\zeta}_1, \vec{\mathbf{z}}_1, q_1)$ and $\vec{\boldsymbol{q}}_2 := (\boldsymbol{\zeta}_2, \vec{\mathbf{z}}_2, q_2)$, it follows from the corresponding second equation of (4.11) that $\vec{\theta}_1 - \vec{\theta}_2 \in \mathcal{V}_b$, and then the $\mathcal{V}_b$-ellipticity of $a_{\vec{\boldsymbol{q}}_1, \xi_1}$ (cf. (4.13)) gives

$$\|\vec{\theta}_1 - \vec{\theta}_2\|_{\mathbf{H}}^2 \leqslant \frac{1}{\alpha_A} a_{\vec{\boldsymbol{q}}_1, \xi_1}(\vec{\theta}_1 - \vec{\theta}_2, \vec{\theta}_1 - \vec{\theta}_2)\,. \tag{4.25}$$

In turn, the evaluation at $\vec{\theta}_1 - \vec{\theta}_2$ of the two systems arising from (4.11) for the pairs $(\vec{\boldsymbol{q}}_1, \xi_1)$ and $(\vec{\boldsymbol{q}}_2, \xi_2)$, lead to

$$a_{\vec{\boldsymbol{q}}_1, \xi_1}(\vec{\theta}_1, \vec{\theta}_1 - \vec{\theta}_2) \;=\; F(\vec{\theta}_1 - \vec{\theta}_2) \quad \text{and} \quad a_{\vec{\boldsymbol{q}}_2, \xi_2}(\vec{\theta}_2, \vec{\theta}_1 - \vec{\theta}_2) \;=\; F(\vec{\theta}_1 - \vec{\theta}_2)\,,$$

from which we find that

$$\begin{aligned} a_{\vec{\boldsymbol{q}}_1, \xi_1}(\vec{\theta}_1 - \vec{\theta}_2, \vec{\theta}_1 - \vec{\vartheta}_2) &= a_{\vec{\boldsymbol{q}}_1, \xi_1}(\vec{\theta}_1, \vec{\theta}_1 - \vec{\theta}_2) - a_{\vec{\boldsymbol{q}}_1, \xi_1}(\vec{\theta}_2, \vec{\theta}_1 - \vec{\theta}_2) \\ &= a_{\vec{\boldsymbol{q}}_2, \xi_2}(\vec{\theta}_2, \vec{\theta}_1 - \vec{\theta}_2) - a_{\vec{\boldsymbol{q}}_1, \xi_1}(\vec{\theta}_2, \vec{\theta}_1 - \vec{\theta}_2) \\ &= \int_\Omega (\mathcal{K}(\boldsymbol{\zeta}_2, q_2, \xi_2) - \mathcal{K}(\boldsymbol{\zeta}_1, q_1, \xi_1)) \widetilde{\boldsymbol{t}}_2 \cdot (\widetilde{\boldsymbol{t}}_1 - \widetilde{\boldsymbol{t}}_2) + \int_\Omega (\mathbf{z}_2 - \mathbf{z}_1) \cdot \widetilde{\boldsymbol{t}}_2 (\theta_1 - \theta_2)\,. \end{aligned} \tag{4.26}$$

Next, invoking the Lipschitz-continuity of $\mathcal{K}$ (cf. (2.11)), and making use of the Cauchy–Schwarz and Hölder inequalities, we obtain

$$\begin{aligned} &\int_\Omega (\mathcal{K}(\boldsymbol{\zeta}_2, q_2, \xi_2) - \mathcal{K}(\boldsymbol{\zeta}_1, q_1, \xi_1))\, \widetilde{\boldsymbol{t}}_2 \cdot (\widetilde{\boldsymbol{t}}_1 - \widetilde{\boldsymbol{t}}_2) \\ &\qquad \leqslant L_{\mathcal{K}} \Big(\|\boldsymbol{\zeta}_2 - \boldsymbol{\zeta}_1\|_{0,2t';\Omega} + \|q_1 - q_2\|_{0,2t';\Omega} + \|\theta_1 - \theta_2\|_{0,2t';\Omega} \Big) \|\widetilde{\boldsymbol{t}}_2\|_{0,2t;\Omega} \|\widetilde{\boldsymbol{t}}_1 - \widetilde{\boldsymbol{t}}_2\|_{0;\Omega}\,, \end{aligned} \tag{4.27}$$

where $t, t' \in (1, +\infty)$ are conjugate to each other. Now, choosing t such that $2t = \varepsilon^*$, we get $2t' = \frac{n}{\varepsilon}$, which, according to the range stipulated for ε, yields $2t' \leqslant r$, and certainly $r \leqslant \rho$, so that the norm of the embedding of the respective Lebesgue spaces is given by $C_{r,\varepsilon} := |\Omega|^{\frac{r\varepsilon - n}{rn}}$. In this way, using additionally the continuity of i_ε along with the regularity assumption (4.23), the estimate (4.27) becomes

$$\begin{aligned} &\int_\Omega (\mathcal{K}(\boldsymbol{\zeta}_2, q_2, \xi_2) - \mathcal{K}(\boldsymbol{\zeta}_1, q_1, \xi_1)) \widetilde{\boldsymbol{t}}_2 \cdot (\widetilde{\boldsymbol{t}}_1 - \widetilde{\boldsymbol{t}}_2) \\ &\qquad \leqslant \widetilde{L}_{\mathbf{\Pi}}\, \|g\|_{0,\varrho;\Omega} \Big\{ \|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{\mathbb{X}_2} + \|q_1 - q_2\|_{\mathrm{M}_1} + \|\xi_1 - \xi_2\|_{0,\rho;\Omega} \Big\} \|\vec{\theta}_1 - \vec{\theta}_2\|_{\mathbf{H}}\,, \end{aligned} \tag{4.28}$$

where $\widetilde{L}_{\mathbf{\Pi}}$ depends on $L_{\mathcal{K}}$, $C_{r,\varepsilon}$, $\widetilde{C}_\varepsilon$, $\|i_\varepsilon\|$ and $|\Omega|$. In turn, bearing in mind the a priori estimation of $\widetilde{\boldsymbol{t}}_2$ (cf. (4.15)), the Cauchy–Schwarz and Hölder inequalities yield

$$\int_\Omega (\mathbf{z}_2 - \mathbf{z}_1) \cdot \widetilde{\boldsymbol{t}}_2 (\vartheta_1 - \vartheta_2) \leqslant C_{\mathbf{\Pi}}\, \|g\|_{0,\varrho;\Omega}\, \|\mathbf{z}_2 - \mathbf{z}_1\|_{\mathbf{X}_2}\, \|\vec{\theta}_1 - \vec{\theta}_2\|_{\mathbf{H}}\,. \tag{4.29}$$

Finally, replacing (4.28) and (4.29) back into (4.26), we deduce, along with (4.25), the required inequality (4.24) with $L_{\mathbf{\Pi}} := \frac{1}{\alpha_A} \max\{\widetilde{L}_{\mathbf{\Pi}}, C_{\mathbf{\Pi}}\}$, which ends the proof. $\square$

Now, we conclude that, under the hypotheses of Lemmas 4.7, 4.8 and 4.9, the compose operator $\mathbf{T}$ (cf. (4.16)) becomes Lipschitz-continuous within the ball $\mathcal{W}(\delta)$ of the space $\mathbb{X}_2 \times \mathrm{L}^\rho(\Omega)$. This is summarized in the next lemma.

Lemma 4.10. *Let ρ, ϱ, r and s be the real numbers within the range specified in* (3.27)*, and $\lambda > M$. In addition, assume that the regularity condition* ($\mathbf{H}.\mathbf{1}$) *(cf.* (4.23)*) holds, and that the data are sufficiently small so that* (4.7)*,* (4.20a)*, and* (4.20b) *are satisfied, that is*

$$\|\mathbf{e}\| := \max\left\{\chi\,, n\,\alpha^2\gamma(\lambda)\right\} \leqslant \frac{\boldsymbol{\alpha_A}}{2}\,,$$

$$C_{\boldsymbol{\Xi}}\left\{\|p_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\,\delta\right\} \leqslant \alpha_A\,, \quad \text{and}$$

$$C_{\mathbf{T}}\left\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \|p_{\mathrm{D}}\|_{1/s,r;\Omega} + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} + \gamma(\lambda)\,\delta\right\} \leqslant \delta\,.$$

Then, there exists a positive constant $L_{\mathbf{T}}$, depending on $C_{\mathbf{S}}$, $C_{\boldsymbol{\Xi}}$, and $L_{\boldsymbol{\Pi}}$, such that

$$\begin{aligned}&\|\mathbf{T}(\boldsymbol{\zeta}_1,\vartheta_1) - \mathbf{T}(\boldsymbol{\zeta}_2,\vartheta_2)\| \\ &\qquad \leqslant L_{\mathbf{T}}\Big(\gamma(\lambda)\big(\gamma(\lambda) + \|g\|_{0,\varrho;\Omega}\big) + \|g\|_{0,\varrho;\Omega}\Big)\,\|(\boldsymbol{\zeta}_1,\vartheta_1) - (\boldsymbol{\zeta}_2,\vartheta_2)\|_{\mathbb{X}_2\times\mathrm{L}^\rho(\Omega)}\,,\end{aligned} \tag{4.30}$$

for all $(\boldsymbol{\zeta}_1,\vartheta_1)\,,(\boldsymbol{\zeta}_2,\vartheta_2) \in \mathcal{W}(\delta)$.

Proof. It readily follows from the definition of the operator $\mathbf{T}$ (cf. (4.16)), and the estimates (4.21), (4.22), and (4.24). $\square$

We are now in position to formulate the main result of this section, which establishes the existence of a unique fixed-point of $\mathbf{T}$ (cf. (4.17)), equivalently, the existence and uniqueness of solution of the coupled system (3.34).

Theorem 4.11. *Let ρ, ϱ, r and s be the real numbers within the range specified in* (3.27)*, and $\lambda > M$. In addition, assume that the regularity condition* ($\mathbf{H}.\mathbf{1}$) *(cf.* (4.23)*) holds, and that the data are sufficiently small so that* (4.7)*,* (4.20a)*, and* (4.20b) *are satisfied. Besides, suppose that*

$$L_{\mathbf{T}}\Big(\gamma(\lambda)\,\big(\gamma(\lambda) + \|g\|_{0,\varrho;\Omega}\big) + \|g\|_{0,\varrho;\Omega}\Big) < 1\,, \tag{4.31}$$

where $L_{\mathbf{T}}$ is the positive constant from Lemma 4.10. Then, the operator $\mathbf{T}$ has a unique fixed-point $(\boldsymbol{\rho},\theta) \in \mathcal{W}(\delta)$. Equivalently, the coupled problem (3.34) *has a unique solution $(\boldsymbol{\rho},\boldsymbol{u}) \in \mathbb{X}_2 \times \mathbf{M}_1$, $(\boldsymbol{w},p) \in \mathbf{X}_2 \times \mathrm{M}_1$ and $(\vec{\theta},\widetilde{\boldsymbol{\sigma}}) \in \mathbf{H} \times \mathbf{Q}$, with $(\boldsymbol{\rho},\theta) \in \mathcal{W}(\delta)$. Moreover, there hold*

$$\begin{aligned}\|(\boldsymbol{\rho},\boldsymbol{u})\|_{\mathbb{X}_2\times\mathbf{M}_1} &\leqslant \widetilde{C}_{\mathbf{S}}\left\{\|u_D\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \|p_{\mathrm{D}}\|_{1/s,r;\Omega} + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} + \gamma(\lambda)\,\delta\right\},\\ \|(\boldsymbol{w},p)\|_{\mathbf{X}_2\times\mathrm{M}_1} &\leqslant \widetilde{C}_{\boldsymbol{\Xi}}\left\{\|u_D\|_{1/s,r;\Gamma} + \|p_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} + \gamma(\lambda)\,\delta\right\},\\ \|(\vec{\theta},\widetilde{\boldsymbol{\sigma}})\|_{\mathbf{H}\times\mathbf{Q}} &\leqslant \widetilde{C}_{\boldsymbol{\Pi}}\,(1+\delta)\,\|g\|_{0,\varrho;\Omega}\,,\end{aligned}$$

where $\widetilde{C}_{\mathbf{S}}$, $\widetilde{C}_{\boldsymbol{\Xi}}$, and $\widetilde{C}_{\boldsymbol{\Pi}}$ are positive constants depending on $C_{\mathbf{S}}$, $C_{\boldsymbol{\Xi}}$ and $C_{\boldsymbol{\Pi}}$.

Proof. Recall, from Lemma 4.6, that (4.20a) and (4.20b) guarantee that $\mathbf{T}$ maps $\mathcal{W}(\delta)$ into itself. Hence, in virtue of the equivalence between (3.34) and (4.17), and bearing in mind the Lipschitz-continuity of (4.30) (cf. Lemma 4.10) and the hypothesis (4.31), a straightforward application of the Banach fixed point Theorem implies the existence of a unique solution $(\boldsymbol{\rho},\theta) \in \mathcal{W}(\delta)$ of (3.34), and hence, the existence of a unique $(\boldsymbol{\rho},\boldsymbol{u}) \in \mathbb{X}_2 \times \mathbf{M}_1$, $(\boldsymbol{w},p) \in \mathbf{X}_2 \times \mathrm{M}_1$ and $(\vec{\theta},\widetilde{\boldsymbol{\sigma}}) \in \mathbf{H} \times \mathbf{Q}$ solution of (3.34). In addition, the a priori estimates follow straightforwardly from (4.2), (4.10) and (4.15), and bounding $\|\boldsymbol{\rho}\|_{\mathbb{X}_2}$ and $\|\theta\|_{\mathbf{X}_2}$ by δ. $\square$

We would like to end this section by emphasizing that the hypothesis $\lambda > M$ (as used in Sections 4.1 and 4.4) naturally hold true in the context of the nearly incompressible scenario. Consequently, we proceed by assuming that λ is sufficiently large, which, in turn, makes $\gamma(\lambda)$ to become sufficiently small (cf. (2.4)). In this way, considering diminutive values for χ, we ensure the feasibility of (4.7). A similar remark arises later on in the discrete analysis.

5 The discrete analysis

5.1 Preliminaries

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations $\mathcal{T}_h$ of the domain Ω made of triangles K in 2D (resp. tetrahedra K in 3D) with corresponding diameter $h_K > 0$. The meshsize h, which also stands for the sub-index, is defined by the largest diameter of the triangulation $\mathcal{T}_h$, that is $h := \max\{h_K : K \in \mathcal{T}_h\}$. Furthermore, we let $\mathrm{P}_\ell(S)$ (resp. $\bar{\mathrm{P}}_\ell(S)$) be the space of polynomials defined on $S \subset \Omega$ of degree $\leqslant \ell \in \mathbb{N}$ (resp. $= \ell$). The vector counterpart of $\mathrm{P}_\ell(S)$ is denoted by $\mathbf{P}_\ell(S) := [\mathrm{P}_\ell(S)]^n$. In turn, for a generic vector $\boldsymbol{x} \in \mathbf{R}^n$, we define the local Raviart–Thomas finite element space of order ℓ over $K \in \mathcal{T}_h$ as $\mathbf{RT}_\ell(K) := \mathbf{P}_\ell(K) \oplus \bar{\mathrm{P}}_\ell(K)\,\boldsymbol{x}$. Then, based on the above, we introduce the following global spaces

$$
\begin{aligned}
\mathrm{P}_\ell(\Omega) &:= \Big\{ w_h \in \mathrm{L}^2(\Omega) : \quad w_h|_K \in \mathrm{P}_\ell(K), \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbf{P}_\ell(\Omega) &:= \Big\{ \boldsymbol{w}_h \in \mathbf{L}^2(\Omega) : \quad \boldsymbol{w}_h|_K \in \mathbf{P}_\ell(K), \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbf{RT}_\ell(\Omega) &:= \Big\{ \boldsymbol{\tau}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbf{RT}_\ell(K), \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbb{RT}_\ell(\Omega) &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_{h,i}|_K \in \mathbf{RT}_\ell(K), \quad \forall\, i \in \{1,\dots,n\}, \quad \forall\, K \in \mathcal{T}_h \Big\},
\end{aligned}
$$

where $\boldsymbol{\tau}_{h,i}$ stands for the ith-row of a tensor $\boldsymbol{\tau}_h$. It is easy to see that for each $t \in [1,+\infty]$ there hold

$$
\begin{gathered}
\mathrm{P}_\ell(\Omega) \subseteq \mathrm{L}^t(\Omega), \quad \mathbf{P}_\ell(\Omega) \subseteq \mathbf{L}^t(\Omega), \quad \mathbf{RT}_\ell(\Omega) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega) \cap \mathbf{H}^t(\mathrm{div}_t;\Omega), \\
\text{and} \qquad \mathbb{RT}_\ell(\Omega) \subseteq \mathbb{H}(\mathrm{div}_t;\Omega) \cap \mathbb{H}^t(\mathrm{div}_t;\Omega).
\end{gathered}
$$

5.2 The discrete coupled system

In order to set the discrete version of (3.34), we now resort to the definitions from Section 5.1 to introduce the following sets of finite element subspaces, one for each decoupled problem:

$$
\mathbb{X}_{2,h} := \mathbb{H}_0^r(\mathbf{div}_r;\Omega) \cap \mathbb{RT}_\ell(\Omega), \quad \mathbb{X}_{1,h} := \mathbb{H}_0^s(\mathbf{div}_s;\Omega) \cap \mathbb{RT}_\ell(\Omega), \quad \mathbf{M}_{1,h} := \mathbf{P}_\ell(\Omega) =: \mathbf{M}_{2,h}, \tag{5.1a}
$$

$$
\mathbf{X}_{2,h} := \mathbf{RT}_\ell(\Omega), \quad \mathbf{X}_{1,h} := \mathbf{RT}_\ell(\Omega), \quad \mathrm{M}_{1,h} := \mathrm{P}_\ell(\Omega) =: \mathrm{M}_{2,h}, \tag{5.1b}
$$

$$
\mathbf{H}_{1,h} := \mathrm{P}_\ell(\Omega), \quad \mathbf{H}_{2,h} := \mathbf{P}_\ell(\Omega), \quad \mathbf{H}_h := \mathbf{H}_{1,h} \times \mathbf{H}_{2,h}, \quad \mathbf{Q}_h := \mathbf{RT}_\ell(\Omega). \tag{5.1c}
$$

Then, the Galerkin scheme associated with (3.34) reads: Find $(\boldsymbol{\rho}_h, \boldsymbol{u}_h) \in \mathbb{X}_{2,h} \times \mathbf{M}_{1,h}$, $(\boldsymbol{w}_h, p_h) \in \mathbf{X}_{2,h} \times \mathrm{M}_{1,h}$ and $(\vec{\theta}_h, \widetilde{\boldsymbol{\sigma}}_h) := \big((\theta_h, \widetilde{\boldsymbol{t}}_h), \widetilde{\boldsymbol{\sigma}}_h\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$
\begin{array}{llll}
\mathbf{a}(\boldsymbol{\rho}_h, \boldsymbol{\tau}_h) + \mathbf{b}_1(\boldsymbol{\tau}_h, \boldsymbol{u}_h) & & = \mathbf{F}_{p_h,\theta_h}(\boldsymbol{\tau}_h) & \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{1,h}, \\
\mathbf{b}_2(\boldsymbol{\rho}_h, \boldsymbol{v}_h) & & = \mathbf{G}(\boldsymbol{v}_h) & \forall\, \boldsymbol{v}_h \in \mathbf{M}_{2,h}, \\
\mathbf{c}(\boldsymbol{w}_h, \boldsymbol{z}_h) + \mathbf{d}_1(\boldsymbol{z}_h, p_h) & & = \boldsymbol{\mathcal{F}}(\boldsymbol{z}_h) & \forall\, \boldsymbol{z}_h \in \mathbf{X}_{1,h}, \\
\mathbf{d}_2(\boldsymbol{w}_h, q_h) & -\, \mathbf{e}(p_h, q_h) & = \boldsymbol{\mathcal{G}}_{\boldsymbol{\rho}_h,\theta_h}(q_h) & \forall\, q_h \in \mathrm{M}_{2,h}, \\
a_{\vec{\boldsymbol{p}}_h,\theta_h}(\vec{\theta}_h, \vec{\vartheta}_h) + b(\vec{\vartheta}_h, \widetilde{\boldsymbol{\sigma}}_h) & & = F(\vec{\vartheta}_h) & \forall\, \vec{\vartheta}_h \in \mathbf{H}_h, \\
b(\vec{\theta}_h, \widetilde{\boldsymbol{\tau}}_h) & & = 0 & \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h,
\end{array} \tag{5.2}
$$

where $\vec{\boldsymbol{p}}_h := (\boldsymbol{\rho}_h, \boldsymbol{w}_h, p_h) \in \mathbb{X}_{2,h} \times \mathbf{X}_{2,h} \times \mathrm{M}_{1,h}$.

For the solvability analysis of (5.2) we will adopt a discrete version of the fixed-point strategy developed in Section 4.4. To this end, we first use the analogues of the operators $\mathbf{S}$, $\boldsymbol{\Xi}$, and $\boldsymbol{\Pi}$ to introduce in the following section the corresponding discrete decoupled problems, and establish their well-posedness.

5.3 The discrete decoupled problems

We begin by letting $\mathbf{S}_h : \mathrm{M}_{1,h} \times \mathbf{H}_{1,h} \to \mathbb{X}_h$ be the operator defined by

$$\mathbf{S}_h(q_h, \vartheta_h) \,:=\, \boldsymbol{\rho}_h \qquad \forall\, (q_h, \vartheta_h) \in \mathrm{M}_{1,h} \times \mathbf{H}_{1,h}\,,$$

where $(\boldsymbol{\rho}_h, \boldsymbol{u}_h) \in \mathbb{X}_{2,h} \times \mathbf{M}_{1,h}$ is the unique solution (to be confirmed below) of the discrete formulation arising from the first and second rows of (5.2) after replacing (p_h, θ_h) by (q_h, ϑ_h), that is

$$\begin{aligned} \mathbf{a}(\boldsymbol{\rho}_h, \boldsymbol{\tau}_h) + \mathbf{b}_1(\boldsymbol{\tau}_h, \boldsymbol{u}_h) &= \mathbf{F}_{q_h,\vartheta_h}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{1,h}\,, \\ \mathbf{b}_2(\boldsymbol{\rho}_h, \boldsymbol{v}_h) &= \mathbf{G}(\boldsymbol{v}_h) && \forall\, \boldsymbol{v}_h \in \mathbf{M}_{2,h}\,. \end{aligned} \tag{5.3}$$

For the solvability analysis of (5.3), we first observe from (5.1a) that

$$\mathbf{div}(\mathbb{X}_{i,h}) \,\subseteq\, \mathbb{H}_{i,h} \qquad \forall i \in \{1,2\}\,,$$

whence the discrete kernels of $\mathbf{b}_1$ and $\mathbf{b}_2$ coincide, and are given by

$$\mathrm{K}_h^\ell \,:=\, \Big\{ \boldsymbol{\tau}_h \in \mathbb{RT}_\ell(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}_h) \,=\, \mathbf{0} \quad \text{and} \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) = 0 \Big\}\,.$$

Furthermore, since the bilinear forms involved in the mixed formulation of the poroelasticity equations coincide with those of [28, eq. (3.15)], and additionally the same finite element subspaces (cf. (5.1a)) are employed here, in what follows we proceed to simply use the results from [28]. In this way, given $t \in (1, +\infty)$, we consider the mesh size h_t^ℓ for which the usual $\mathrm{L}^2(\Omega)$-orthogonal projector satisfies the property stated in [28, eq. (5.21)]. Then, thanks to [28, Lemma 5.3], there exist positive constants $M_{\mathtt{d}}$ and $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}},\mathtt{d}}$ such that for each $\lambda > M_{\mathtt{d}}$ and for each $h \leqslant h_0 := \min\{h_r^\ell, h_s^\ell\}$ there hold

$$\begin{aligned} &\sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{K}_h^\ell \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{a}(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}_1}} \,\geqslant\, \boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}},\mathtt{d}}\, \|\boldsymbol{\zeta}_h\|_{\mathbb{X}_2} \qquad \forall\, \boldsymbol{\zeta}_h \in \mathbb{X}_{2,h}\,, \\ &\sup_{\boldsymbol{\zeta}_h \in \mathrm{K}_h^\ell} \mathbf{a}(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h) \,>\, 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathrm{K}_h^\ell\,, \quad \boldsymbol{\tau}_h \neq \mathbf{0}\,. \end{aligned} \tag{5.4}$$

In addition, the inf-sup conditions for the bilinear forms $\mathbf{b}_1$ and $\mathbf{b}_2$, proved in [28, Lemma 5.4], provide the existence of positive constants $\boldsymbol{\beta}_{1,\mathtt{d}}$ and $\boldsymbol{\beta}_{2,\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{X}_{i,h} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}_i(\boldsymbol{\tau}_h, \boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}_i}} \,\geqslant\, \boldsymbol{\beta}_{i,\mathtt{d}}\, \|\boldsymbol{v}_h\|_{\mathbf{M}_i} \qquad \forall\, \boldsymbol{v} \in \mathbf{M}_{i,h}\,, \quad \forall i \in \{1,2\}\,. \tag{5.5}$$

Thus, thanks to (5.4) and (5.5), we are in position to show next the discrete version of Lemma 4.1.

Lemma 5.1. *Let r and s be within the range of values stipulated by (3.27), and $\lambda > M_{\mathtt{d}}$. Then, for each $(q_h, \vartheta_h) \in \mathrm{M}_{1,h} \times \mathbf{H}_{1,h}$ there exists a unique $(\boldsymbol{\rho}_h, \boldsymbol{u}_h) \in \mathbb{X}_{2,h} \times \mathbf{M}_{1,h}$ solution of (5.3), and hence one can define $\mathbf{S}_h(q_h, \vartheta_h) := \boldsymbol{\rho}_h$. Moreover, there exists a positive constant $C_{\mathbf{S},\mathtt{d}}$, depending on $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}},\mathtt{d}}$, $\boldsymbol{\beta}_{1,\mathtt{d}}$, $\boldsymbol{\beta}_{2,\mathtt{d}}$, $C_{\mathbf{F}}$, and μ, and hence independent of h, such that for each $h \leqslant h_0 := \min\{h_r^\ell, h_s^\ell\}$ there holds*

$$\|\mathbf{S}_h(q_h, \vartheta_h)\| \,=\, \|\boldsymbol{\rho}_h\|_{\mathbb{X}_2} \,\leqslant\, C_{\mathbf{S},\mathtt{d}} \Big\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \gamma(\lambda) \Big(\|q_h\|_{0,r;\Omega} + \|\vartheta_h\|_{0,\rho;\Omega} \Big) \Big\}\,. \tag{5.6}$$

Proof. It follows from a direct application of the discrete Babuška–Brezzi theory in Banach spaces (cf. [6, Theorem 2.1, Corollary 2.1,]). Note that the dependence of the constant $C_{\mathbf{S},\mathtt{d}}$ on μ is due to $\|\mathbf{a}\|$ (cf. (3.11)). $\square$

We now let $\boldsymbol{\Xi}_h : \mathbb{X}_{2,h} \times \mathbf{H}_{1,h} \to \mathrm{M}_{1,h}$ be the operator defined by

$$\boldsymbol{\Xi}_h(\boldsymbol{\zeta}_h, \vartheta_h) \;=\; \big(\boldsymbol{\Xi}_{1,h}(\boldsymbol{\zeta}_h, \vartheta_h), \boldsymbol{\Xi}_{2,h}(\boldsymbol{\zeta}_h, \vartheta_h)\big) \;:=\; (\boldsymbol{w}_h, p_h) \qquad \forall\, (\boldsymbol{\zeta}_h, \vartheta_h) \in \mathbb{X}_{2,h} \times \mathbf{H}_{1,h}\,,$$

where $(\boldsymbol{w}_h, p_h) \in \mathbf{X}_{2,h} \times \mathrm{M}_{1,h}$ is the unique solution (to be confirmed below) of the discrete formulation arising from the third and fourth rows of (5.2) after replacing $(\boldsymbol{\rho}_h, \theta_h)$ by $(\boldsymbol{\zeta}_h, \vartheta_h)$, that is

$$\begin{array}{llll} \mathbf{c}(\boldsymbol{w}_h, \boldsymbol{z}_h) & + \mathbf{d}_1(\boldsymbol{z}_h, p_h) & = \boldsymbol{\mathcal{F}}(\boldsymbol{z}_h) & \forall\, \boldsymbol{z}_h \in \mathbf{X}_{1,h}\,, \\[2mm] \mathbf{d}_2(\boldsymbol{w}_h, q_h) & -\mathbf{e}(p_h, q_h) & = \boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta}_h,\vartheta_h}(q_h) & \forall\, q_h \in \mathrm{M}_{2,h}\,. \end{array} \tag{5.7}$$

Then, similarly as for (5.3), we first notice that

$$\mathrm{div}(\mathbf{X}_{i,h}) \;\subseteq\; \mathrm{M}_{i,h} \qquad \forall\, i \in \{1, 2\}\,,$$

which yields the discrete kernels of $\mathbf{d}_1$ and $\mathbf{d}_2$ to become

$$\mathcal{V}_h^\ell \;:=\; \Big\{\boldsymbol{z}_h \in \mathbf{RT}_\ell(\Omega) : \quad \mathrm{div}(\boldsymbol{z}_h) = 0\Big\}\,.$$

Knowing the above, the discrete version of Lemma 4.2 is now recalled from [20, Lemma 5.2].

Lemma 5.2. *Assume that r and s satisfy the particular range specified by* (3.27)*. Then, there exists a positive constant $\boldsymbol{\alpha}_{\mathbf{c},\mathtt{d}}$ such that*

$$\sup_{\substack{\boldsymbol{z}_h \in \mathcal{V}_h^\ell \\ \boldsymbol{z}_h \neq \mathbf{0}}} \frac{\mathbf{c}(\boldsymbol{w}_h, \boldsymbol{z}_h)}{\|\boldsymbol{z}_h\|_{\mathbf{X}_1}} \;\geqslant\; \boldsymbol{\alpha}_{\mathbf{c},\mathtt{d}}\, \|\boldsymbol{w}_h\|_{\mathbf{X}_2} \qquad \forall\, \boldsymbol{w}_h \in \mathcal{V}_h^\ell\,,$$

and

$$\sup_{\boldsymbol{w} \in \mathcal{V}_h^\ell} \mathbf{c}(\boldsymbol{w}_h, \boldsymbol{z}_h) \;>\; 0 \qquad \forall\, \boldsymbol{z}_h \in \mathbf{X}_{1,h}\,, \quad \boldsymbol{z}_h \neq \mathbf{0}\,.$$

Proof. It proceeds analogously to the proof of [29, Lemma 4.3]. However, for full details we refer to [19, Lemma 5.2], which is the preprint version of [20]. $\square$

On the other hand, the discrete inf-sup conditions for the bilinear forms $\mathbf{d}_1$ and $\mathbf{d}_2$, which can be found in [29, Lemma 5.3], state that for each $i \in \{1, 2\}$, there exists a positive constant $\widetilde{\boldsymbol{\beta}}_{i,\mathtt{d}}$ such that

$$\sup_{\substack{\boldsymbol{z}_h \in \mathbf{X}_{i,h} \\ \boldsymbol{z}_h \neq \mathbf{0}}} \frac{\mathbf{d}_i(\boldsymbol{z}_h, q_h)}{\|\boldsymbol{z}_h\|_{\mathbf{X}_{i,h}}} \;\geqslant\; \widetilde{\boldsymbol{\beta}}_{i,\mathtt{d}}\, \|q_h\|_{\mathrm{M}_i} \qquad \forall\, q_h \in \mathrm{M}_{i,h}\,. \tag{5.8}$$

Then, analogously to the continuous case, Lemma 5.2 and (5.8) imply that the bilinear form $\widetilde{\boldsymbol{A}}$ (cf. (4.4)) satisfies the global inf-sup conditions given by the discrete versions of (4.5a) and (4.5b), both with a positive constant $\boldsymbol{\alpha}_{\boldsymbol{A},\mathtt{d}}$ depending on $\boldsymbol{\alpha}_{\mathbf{c},\mathtt{d}}$, $\widetilde{\boldsymbol{\beta}}_{1,\mathtt{d}}$, $\widetilde{\boldsymbol{\beta}}_{2,\mathtt{d}}$, and $\|\mathbf{c}\|$, and hence independent of h. Moreover, using these inequalities, and proceeding analogously to the derivation of (4.8) and (4.9), which means assuming now the discrete version of (4.7), this is

$$\|\mathbf{e}\| = \max\big\{\chi, n\,\alpha^2\,\gamma(\lambda)\big\} \;\leqslant\; \frac{\boldsymbol{\alpha}_{\boldsymbol{A},\mathtt{d}}}{2}\,, \tag{5.9}$$

we arrive at the discrete global inf-sup conditions for the global operator $\boldsymbol{A}$ (cf. (4.6)), namely

$$\sup_{\substack{(\boldsymbol{z}_h,q_h)\in\mathbf{X}_{1,h}\times\mathrm{M}_{2,h}\\(\boldsymbol{z}_h,q_h)\neq\mathbf{0}}} \frac{\boldsymbol{A}((\boldsymbol{w}_h,p_h),(\boldsymbol{z}_h,q_h))}{\|(\boldsymbol{z}_h,q_h)\|_{\mathbf{X}_1\times\mathrm{M}_2}} \geqslant \frac{\boldsymbol{\alpha}_{\boldsymbol{A},\mathtt{d}}}{2}\,\|(\boldsymbol{w}_h,p_h)\|_{\mathbf{X}_2\times\mathrm{M}_1} \quad \forall\,(\boldsymbol{w}_h,p_h)\in\mathbf{X}_{2,h}\times\mathrm{M}_{1,h}\,, \tag{5.10a}$$

$$\sup_{\substack{(\boldsymbol{w}_h,p_h)\in\mathbf{X}_{2,h}\times\mathrm{M}_{1,h}\\(\boldsymbol{w}_h,p_h)\neq\mathbf{0}}} \frac{\boldsymbol{A}((\boldsymbol{w}_h,p_h),(\boldsymbol{z}_h,q_h))}{\|(\boldsymbol{w}_h,p_h)\|_{\mathbf{X}_2\times\mathrm{M}_1}} \geqslant \frac{\boldsymbol{\alpha}_{\boldsymbol{A},\mathtt{d}}}{2}\,\|(\boldsymbol{z}_h,q_h)\|_{\mathbf{X}_{1,h}\times\mathrm{M}_{2,h}} \quad \forall\,(\boldsymbol{z}_h,q_h)\in\mathbf{X}_{1,h}\times\mathrm{M}_{2,h}\,. \tag{5.10b}$$

Similarly as for the continuous analysis, we stress here that the fact that $\gamma(\lambda)$ approaches 0 as λ increases (cf. (2.4)), ensures the feasibility of (5.9) for sufficiently large λ and sufficiently small χ.

Having established (5.10a) and (5.10b), a straightforward application of the discrete version of the Banach–Nečas–Babuška theorem (cf. [24, Theorem 2.22]) allows to conclude the following result.

Lemma 5.3. *Let r and s be within the range of values specified by (3.27), and assume that the data satisfy (5.9). Then, for each $(\boldsymbol{\zeta}_h,\vartheta_h)\in\mathbb{X}_{2,h}\times\mathbf{H}_{1,h}$ there exists a unique $(\boldsymbol{w}_h,p_h)\in\mathbf{X}_{2,h}\times\mathrm{M}_{1,h}$ solution of (5.7), and hence one can define $\boldsymbol{\Xi}_h(\boldsymbol{\zeta}_h,\vartheta_h) = \big(\boldsymbol{\Xi}_{1,h}(\boldsymbol{\zeta}_h,\vartheta_h),\boldsymbol{\Xi}_{2,h}(\boldsymbol{\zeta}_h,\vartheta_h)\big) := (\boldsymbol{w}_h,p_h)\in \mathbf{X}_{2,h}\times\mathrm{M}_{1,h}$. Moreover, there exists a positive constant $C_{\boldsymbol{\Xi},\mathtt{d}}$, depending on $\boldsymbol{\alpha}_{\boldsymbol{A},\mathtt{d}}$, $C_{\boldsymbol{\mathcal{F}}}$, and $C_{\boldsymbol{\mathcal{G}}}$, and hence independent of h, such that*

$$\begin{aligned}\|\boldsymbol{\Xi}_h(\boldsymbol{\zeta}_h,\vartheta_h)\|_{\mathbf{X}_2\times\mathrm{M}_1} &= \|\boldsymbol{w}_h\|_{\mathbf{X}_2} + \|p_h\|_{\mathrm{M}_1}\\ &\leqslant C_{\boldsymbol{\Xi},\mathtt{d}}\Big\{\|p_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\Big(\|\boldsymbol{\zeta}_h\|_{\mathbb{X}_2} + \|\vartheta_h\|_{0,\rho;\Omega}\Big)\Big\}\,.\end{aligned} \tag{5.11}$$

Finally, we let $\boldsymbol{\Pi}_h : \mathbb{X}_{2,h}\times(\mathbf{X}_{2,h}\times\mathrm{M}_{1,h})\times\mathbf{H}_{1,h}\to\mathbf{H}_h$ be the operator defined by

$$\boldsymbol{\Pi}_h(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h,\xi_h) = \big(\boldsymbol{\Pi}_{1,h}(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h,\xi_h),\boldsymbol{\Pi}_{2,h}(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h,\xi_h)\big) := \vec{\theta}_h = (\theta_h,\widetilde{\boldsymbol{t}}_h)\,,$$

for all $(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h,\xi_h) = \big(\boldsymbol{\zeta}_h,(\mathbf{z}_h,q_h),\xi_h\big)\in\mathbb{X}_{2,h}\times(\mathbf{X}_{2,h}\times\mathrm{M}_{1,h})\times\mathbf{H}_{1,h}$, where $(\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h$ is the unique solution (to be confirmed below) of the discrete formulation arising from the fifth and sixth rows of (5.2) after replacing $a_{\vec{\boldsymbol{p}}_h,\theta_h}$, with $\vec{\boldsymbol{p}}_h := (\boldsymbol{\rho}_h,\boldsymbol{\omega}_h,p_h)$, by $a_{\vec{\boldsymbol{q}}_h,\xi_h}$, with $\vec{\boldsymbol{q}}_h := (\boldsymbol{\zeta}_h,\mathbf{z}_h,q_h)$, that is

$$\begin{aligned} a_{\vec{\boldsymbol{q}}_h,\xi_h}(\vec{\theta}_h,\vec{\vartheta}_h) + b(\vec{\vartheta}_h,\widetilde{\boldsymbol{\sigma}}_h) &= F(\vec{\vartheta}_h) && \forall\,\vec{\vartheta}_h := (\vartheta_h,\widetilde{\boldsymbol{s}}_h)\in\mathbf{H}_h\,,\\ b(\vec{\theta}_h,\widetilde{\boldsymbol{\tau}}_h) &= 0 && \forall\,\widetilde{\boldsymbol{\tau}}_h\in\mathbf{Q}_h\,.\end{aligned} \tag{5.12}$$

For the analysis of the Galerkin scheme (5.12), we proceed as in [21, Section 5.5] (see also [5, Section 4.3, Lemma 4.2] or [17, Section 5.3, eqs. (5.19), (5.20)]). More precisely, since the required results are already available in those references, in what follows we just describe the main aspects of the corresponding discussion, for which we first introduce the discrete kernel $\mathcal{V}^{\ell}_{b,h}$ of b (cf. (3.30b)), that is

$$\mathcal{V}^{\ell}_{b,h} := \Big\{\vec{\vartheta}_h := (\vartheta_h,\widetilde{\boldsymbol{s}}_h)\in\mathbf{H}_h : \quad b(\vec{\vartheta}_h,\widetilde{\boldsymbol{\tau}}_h) = 0 \quad \forall\,\widetilde{\boldsymbol{\tau}}_h\in\mathbf{Q}_h\Big\}\,,$$

and the subspace of $\mathbf{Q}_h$ given by

$$\mathcal{Z}^{\ell}_{b,h} := \Big\{\widetilde{\boldsymbol{\tau}}_h\in\mathbf{Q}_h : \quad \mathrm{div}(\widetilde{\boldsymbol{\tau}}_h) = 0 \quad \text{in} \quad \Omega\Big\}\,.$$

Then, applying the abstract result provided in [21, Lemma 5.1], one deduces that the existence of positive constants $\beta_{1,\mathtt{d}}$ and $\beta_{2,\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}}_h\in\mathbf{Q}_h\\ \widetilde{\boldsymbol{\tau}}_h\neq0}} \frac{\displaystyle\int_\Omega \vartheta_h\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}_h)}{\|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_\varrho;\Omega}} \geqslant \beta_{1,\mathtt{d}}\,\|\vartheta_h\|_{0,\rho;\Omega} \quad \forall\,\vartheta_h\in\mathbf{H}_{1,h}\,, \quad \text{and} \tag{5.13}$$

$$\sup_{\substack{\widetilde{\boldsymbol{s}}_h\in\mathbf{H}_{2,h}\\ \widetilde{\boldsymbol{s}}_h\neq 0}} \frac{\displaystyle\int_\Omega \widetilde{\boldsymbol{s}}_h\cdot\widetilde{\boldsymbol{\tau}}_h}{\|\widetilde{\boldsymbol{s}}_h\|_{0,\Omega}} \,\geqslant\, \beta_{2,\mathtt{d}}\,\|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_\varrho;\Omega} \quad \forall\,\widetilde{\boldsymbol{\tau}}_h\in\mathcal{Z}^\ell_{b,h}\,, \tag{5.14}$$

is equivalent to the existence of positive constants $\widetilde{\beta}_{\mathtt{d}}$ and $\widetilde{C}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\vartheta}_h\in\mathbf{H}_h\\ \vec{\vartheta}_h\neq 0}} \frac{b(\vec{\vartheta}_h,\widetilde{\boldsymbol{\tau}}_h)}{\|\vec{\vartheta}_h\|_{\mathbf{H}}} \,\geqslant\, \widetilde{\beta}_{\mathtt{d}}\,\|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_\varrho;\Omega} \quad \forall\,\widetilde{\boldsymbol{\tau}}_h\in\mathbf{Q}_h\,,\quad \text{and} \tag{5.15a}$$

$$\|\widetilde{\boldsymbol{s}}_h\|_{0,\Omega} \,\geqslant\, \widetilde{C}_{\mathtt{d}}\,\|\vartheta_h\|_{0,\rho;\Omega} \qquad \forall\,\vec{\vartheta}_h := (\vartheta_h,\widetilde{\boldsymbol{s}}_h)\in\mathcal{V}^\ell_{b,h}\,. \tag{5.15b}$$

The proof of (5.13) is basically provided at the last part of [21, Section 5.5] by noticing that it reduces to the vector version of [21, Lemma 5.5]. Actually, while the proof there is for $(\rho,\varrho) = (4,4/3)$, it can be extended almost verbatim to an arbitrary conjugate pair (ρ,ϱ) satisfying (3.27). In turn, it is readily seen that (5.14) follows from the fact that $\mathcal{Z}^\ell_{b,h}\subseteq\mathbf{H}_{2,h}$ (cf. [17, eq. (5.18)]). In this way, having already the discrete inf-sup condition (5.15a) for b, it only remains to employ (5.15b) to show the $\mathcal{V}^\ell_{b,h}$-ellipticity of $a_{\vec{\boldsymbol{q}}_h,\xi_h}$ for given $\vec{\boldsymbol{q}}_h = (\boldsymbol{\zeta}_h,\mathbf{z}_h,q_h)\in\mathbb{X}_{2,h}\times\mathbf{X}_{2,h}\times\mathrm{M}_{1,h}$ and $\xi_h\in\mathbf{H}_{1,h}$. Indeed, proceeding similarly to the first part of the derivation of (4.12), we have for each $\vec{\vartheta}_h := (\vartheta_h,\widetilde{\boldsymbol{s}}_h)\in\mathcal{V}^\ell_{b,h}$

$$\begin{aligned} a_{\vec{\boldsymbol{q}}_h,\xi_h}(\vec{\vartheta}_h,\vec{\vartheta}_h) \,&\geqslant\, (\varkappa_0/2)\,\widetilde{C}^2_{\mathtt{d}}\,\|\vartheta_h\|^2_{0,\rho;\Omega} \,+\, (\varkappa_0/2)\,\|\widetilde{\boldsymbol{s}}_h\|^2_{0;\Omega} \,-\, \|\mathbf{z}_h\|_{0,r;\Omega}\,\|\widetilde{\boldsymbol{s}}_h\|_{0;\Omega}\,\|\vartheta_h\|_{0,\rho;\Omega} \\ &\geqslant\, \frac{1}{2}\Big\{\varkappa_0\,\min\big\{1,\widetilde{C}^2_{\mathtt{d}}\big\} \,-\, \|\mathbf{z}_h\|_{0,r;\Omega}\Big\}\,\|\vec{\vartheta}_h\|^2\,, \end{aligned}$$

so that, under the constraint $\|\mathbf{z}_h\|_{0,r;\Omega} \,\leqslant\, \alpha_{A,\mathtt{d}} := \dfrac{1}{3}\,\varkappa_0\,\min\big\{1,\widetilde{C}^2_{\mathtt{d}}\big\}$, there holds

$$a_{\vec{\boldsymbol{q}}_h,\xi_h}(\vec{\vartheta}_h,\vec{\vartheta}_h) \,\geqslant\, \alpha_{A,\mathtt{d}}\,\|\vec{\vartheta}_h\|^2 \qquad \forall\,\vec{\vartheta}_h := (\vartheta_h,\widetilde{\boldsymbol{s}}_h)\in\mathcal{V}^\ell_{b,h}\,, \tag{5.16}$$

thus confirming the announced property of $a_{\vec{\boldsymbol{q}}_h,\xi_h}$.

Hence, the solvability of (5.12) and therefore the well-posedness of $\mathbf{\Pi}_h$ can be established in the following lemma.

Lemma 5.4. *Let ρ and ϱ be within the range of values stipulated by (3.27). Then, for each $(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h,\xi_h) = \big(\boldsymbol{\zeta}_h,(\mathbf{z}_h,q_h),\xi_h\big)\in\mathbb{X}_{2,h}\times(\mathbf{X}_{2,h}\times\mathrm{M}_{1,h})\times\mathbf{H}_{1,h}$ such that $\|\mathbf{z}_h\|\leqslant\alpha_{A,\mathtt{d}}$, there exists a unique $(\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h) = \big((\theta_h,\widetilde{\boldsymbol{t}}_h),\widetilde{\boldsymbol{\sigma}}_h\big)\in\mathbf{H}_h\times\mathbf{Q}_h$ solution of (5.12), and hence one can define $\mathbf{\Pi}_h(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h,\xi_h) = \vec{\theta}_h$. Moreover, there exist positive constants $C_{\mathbf{\Pi},\mathtt{d}}$ and $\bar{C}_{\mathbf{\Pi},\mathtt{d}}$, depending on $\alpha_{A,\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $|\Omega|$, ρ, and $\varkappa_2$, and hence independent of h, such that the following a priori estimates hold*

$$\|\mathbf{\Pi}_h(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h,\xi_h)\| = \|\vec{\theta}_h\|_{\mathbf{H}} \leqslant C_{\mathbf{\Pi},\mathtt{d}}\,\|g\|_{0,\varrho;\Omega}\,,\qquad \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \leqslant \bar{C}_{\mathbf{\Pi},\mathtt{d}}\,\|g\|_{0,\varrho;\Omega}\,. \tag{5.17}$$

Proof. The result is a consequence of the $\mathcal{V}^\ell_{b,h}$-ellipticity of $a_{\vec{\boldsymbol{q}}_h,\xi_h}$ (cf. (5.16)), the inf-sup condition (5.15a), and a direct application of, for instance, [24, Theorem 2.34, Proposition 2.42]. Note that the dependence of the constants $C_{\mathbf{\Pi},\mathtt{d}}$ and $\bar{C}_{\mathbf{\Pi},\mathtt{d}}$ on $|\Omega|$, ρ, and $\varkappa_2$, is due to $\|a\|$ (cf. (3.31)) since $\|a_{\vec{\boldsymbol{q}}_h,\xi_h}\|$, which is required by the theoretical estimates from [6, Corollary 2.2, eqs. (2.24) and (2.25)], is bounded above by $\|a\| + \|\mathbf{z}_h\|$. □

5.4 Solvability analysis of the discrete coupled system

The solvability analysis of the fully coupled discrete system (5.2) is performed in a similar fashion as in the continuous case by using a fixed-point strategy, but now applying the Brouwer theorem instead of the classical Banach one. Therefore, the structure and reasoning followed in this part, are going to resemble partially the ones of Section 4.4. We begin this analysis by defining the discrete fixed-point operator $\mathbf{T}_h : \mathbb{X}_{2,h} \times \mathbf{H}_{1,h} \to \mathbb{X}_{2,h} \times \mathbf{H}_{1,h}$ given by

$$\mathbf{T}_h(\boldsymbol{\zeta}_h, \vartheta_h) := \Big(\mathbf{S}_h\big(\boldsymbol{\Xi}_{2,h}(\boldsymbol{\zeta}_h, \vartheta_h), \vartheta_h\big), \boldsymbol{\Pi}_{1,h}\big(\mathbf{S}_h(\boldsymbol{\Xi}_{2,h}(\boldsymbol{\zeta}_h, \vartheta_h), \vartheta_h), \boldsymbol{\Xi}_h(\boldsymbol{\zeta}_h, \vartheta_h), \vartheta_h\big)\Big), \tag{5.18}$$

for all $(\boldsymbol{\zeta}_h, \vartheta_h) \in \mathbb{X}_{2,h} \times \mathbf{H}_{1,h}$. Then, showing existence of solution is equivalent to seeking a fixed-point to the operator $\mathbf{T}_h$, that is: Find $(\boldsymbol{\zeta}_h, \vartheta_h) \in \mathbb{X}_{2,h} \times \mathbf{H}_{1,h}$ such that

$$\mathbf{T}_h(\boldsymbol{\zeta}_h, \vartheta_h) = (\boldsymbol{\zeta}_h, \vartheta_h). \tag{5.19}$$

Now, given $\delta > 0$, we define the δ-ball in the finite-dimensional subspace $\mathbb{X}_{2,h} \times \mathbf{H}_{1,h}$ by

$$\mathcal{W}_h(\delta) := \Big\{(\boldsymbol{\zeta}_h, \vartheta_h) \in \mathbb{X}_{2,h} \times \mathbf{H}_{1,h} : \quad \|(\boldsymbol{\zeta}_h, \vartheta_h)\| := \|\boldsymbol{\zeta}_h\|_{\mathbb{X}_2} + \|\vartheta_h\|_{0,\rho;\Omega} \leqslant \delta\Big\},$$

where we conveniently choose $\delta := \alpha_{A,\mathtt{d}}$. Furthermore, assumption (5.9) applies to the discrete operator $\boldsymbol{\Xi}_{1,h}$ in the same way as (4.18) applies to $\boldsymbol{\Xi}_1$, this is

$$\|\boldsymbol{\Xi}_{1,h}(\boldsymbol{\zeta}_h, \vartheta_h)\|_{\mathbf{X}_2} \leqslant \alpha_{A,\mathtt{d}} \qquad \forall\, (\boldsymbol{\zeta}_h, \vartheta_h) \in \mathcal{W}_h(\delta). \tag{5.20}$$

Combining the estimates (5.6), (5.11), and (5.17), we obtain the discrete version of (4.19) as an priori bound for the operator $\mathbf{T}_h$, that is

$$\begin{aligned} \|\mathbf{T}_h(\boldsymbol{\zeta}_h, \vartheta_h)\| \leqslant C_{\mathbf{T},\mathtt{d}} \Big\{ & \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \|p_{\mathrm{D}}\|_{1/s,r;\Omega} \\ & + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} + \gamma(\lambda)\left(\|\boldsymbol{\zeta}_h\|_{\mathbb{X}_2} + \|\vartheta_h\|_{0,\rho;\Omega}\right)\Big\}, \end{aligned}$$

where $C_{\mathbf{T},\mathtt{d}}$ is a positive constant depending on $C_{\mathbf{S},\mathtt{d}}$, $C_{\boldsymbol{\Xi},\mathtt{d}}$ and $C_{\boldsymbol{\Pi},\mathtt{d}}$, and hence independent of h. In addition, taking into account the a priori estimate (5.11) with $(\boldsymbol{\zeta}_h, \vartheta_h) \in \mathcal{W}_h(\delta)$, we conclude that operator $\boldsymbol{\Xi}_{1,h}$ will satisfy assumption (5.20) if there holds

$$C_{\boldsymbol{\Xi},\mathtt{d}}\Big\{\|p_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\,\delta\Big\} \leqslant \alpha_{A,\mathtt{d}}.$$

Hence, the following lemma establishes the conditions under which the operator $\mathbf{T}_h$ maps the ball $\mathcal{W}_h(\delta)$ into itself, thus yielding the discrete analogue of Lemma 4.6.

Lemma 5.5. *Let ρ, ϱ, r and s be as specified in (3.27), and $\lambda > M$. Moreover, assume that $h \leqslant h_0 := \min\{h_r^\ell, h_s^\ell\}$, and that the data are sufficiently small so that (5.9) and the conditions*

$$C_{\boldsymbol{\Xi},\mathtt{d}}\Big\{\|p_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \gamma(\lambda)\,\delta\Big\} \leqslant \alpha_{A,\mathtt{d}}, \quad \textit{and} \tag{5.21a}$$

$$C_{\mathbf{T},\mathtt{d}}\Big\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \|p_{\mathrm{D}}\|_{1/s,r;\Omega} + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} + \gamma(\lambda)\,\delta\Big\} \leqslant \delta, \tag{5.21b}$$

are satisfied. Then, $\mathbf{T}_h(\mathcal{W}_h(\delta)) \subseteq \mathcal{W}_h(\delta)$.

The next two lemmas show, respectively, that the operators $\mathbf{S}_h$ and $\boldsymbol{\Xi}_h$ are Lipschitz-continuous.

Lemma 5.6. *Let r and s be within the range of values stipulated by (3.27), and $\lambda > M$. Then, with the same constant $C_{\mathbf{S},\mathtt{d}}$ from the a priori estimate (5.6) (cf. Lemma 5.1), for $h \leqslant h_0 := \min\{h_r^\ell, h_s^\ell\}$ there holds*

$$\|\mathbf{S}_h(q_{1,h}, \vartheta_{1,h}) - \mathbf{S}_h(q_{2,h}, \vartheta_{2,h})\|_{\mathbb{X}_2} \leqslant C_{\mathbf{S},\mathtt{d}}\,\gamma(\lambda)\,\|(q_{1,h}, \vartheta_{1,h}) - (q_{2,h}, \vartheta_{2,h})\|_{\mathrm{M}_1 \times \mathbf{H}_1}, \tag{5.22}$$

for all $(q_{1,h}, \vartheta_{1,h})$, $(q_{2,h}, \vartheta_{2,h}) \in \mathrm{M}_1 \times \mathbf{H}_{1,h}$.

Proof. It proceeds analogously to the proof of Lemma 4.7. We omit further details. □

Lemma 5.7. *Let r and s be within the range of values stipulated by (3.27), and assume that the data fulfills condition (5.9). Then, with the same constant $C_{\Xi,\mathtt{d}}$ from the a priori estimate (5.11) (cf. Lemma 5.3), there holds*

$$\|\Xi_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \Xi_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|_{\mathbf{X}_2\times \mathrm{M}_1} \leqslant C_{\Xi,\mathtt{d}}\,\gamma(\lambda)\,\|(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - (\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|_{\mathbb{X}_2\times \mathbf{H}_1}\,, \tag{5.23}$$

for all $(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h})$, $(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h}) \in \mathbb{X}_{2,h}\times \mathbf{H}_{1,h}$.

Proof. It proceeds analogously to the proof of Lemma 4.8. Further details are omitted. □

The next result shows the continuity of $\mathbf{\Pi}_h$. In this regard, we stress in advance that the obvious absence of a regularity assumption in the present discrete setting, stops us of proving a Lipschitz-continuity property of $\mathbf{\Pi}_h$.

Lemma 5.8. *Let ρ and ϱ be within the range of values stipulated by (3.27). Then, there exists a positive constant $L_{\mathbf{\Pi},\mathtt{d}}$, depending on $L_{\mathcal{K}}$, $\alpha_{A,\mathtt{d}}$, $|\Omega|$, r, and ρ, and hence independent of h, such that*

$$\begin{aligned}&\|\mathbf{\Pi}_h(\boldsymbol{\zeta}_{1,h},\vec{\boldsymbol{\omega}}_{1,h},\xi_{1,h}) - \mathbf{\Pi}_h(\boldsymbol{\zeta}_{2,h},\vec{\boldsymbol{\omega}}_{2,h},\xi_{2,h})\|_{\mathbf{H}}\\ &\qquad\leqslant L_{\mathbf{\Pi},\mathtt{d}}\,\|\mathbf{\Pi}_{2,h}(\boldsymbol{\zeta}_{2,h},\vec{\boldsymbol{\omega}}_{2,h},\xi_{2,h})\|_{0,\rho;\Omega}\,\|(\boldsymbol{\zeta}_{1,h},\vec{\boldsymbol{\omega}}_{1,h},\xi_{1,h}) - (\boldsymbol{\zeta}_{2,h},\vec{\boldsymbol{\omega}}_{2,h},\xi_{2,h})\|\,,\end{aligned} \tag{5.24}$$

for all $(\boldsymbol{\zeta}_{1,h},\vec{\mathbf{z}}_{1,h},\xi_{1,h}) = \big(\boldsymbol{\zeta}_{1,h},(\mathbf{z}_{1,h},q_{1,h}),\xi_{1,h}\big)$, $(\boldsymbol{\zeta}_{2,h},\vec{\mathbf{z}}_{2,h},\xi_{2,h}) = \big(\boldsymbol{\zeta}_{2,h},(\mathbf{z}_{2,h},q_{2,h}),\xi_{2,h}\big) \in \mathbb{X}_{2,h}\times (\mathbf{X}_{2,h}\times \mathrm{M}_{1,h})\times \mathbf{H}_{1,h}$, such that $\|\mathbf{z}_{1,h}\|_{\mathbf{X}_2}$, $\|\mathbf{z}_{2,h}\|_{\mathbf{X}_2} \leqslant \alpha_{A,\mathtt{d}}$.

Proof. The proof follows similarly to the one of Lemma 4.9, except for the fact, as already announced, that no regularity result can be applied. Indeed, given $(\boldsymbol{\zeta}_{1,h},\vec{\mathbf{z}}_{1,h},\xi_{1,h})$ and $(\boldsymbol{\zeta}_{2,h},\vec{\mathbf{z}}_{2,h},\xi_{2,h})$ as indicated, we let $\vec{\vartheta}_{1,h} := \mathbf{\Pi}_h(\boldsymbol{\zeta}_{1,h},\vec{\mathbf{z}}_{1,h},\xi_{1,h}) \in \mathbf{H}_h$ and $\vec{\vartheta}_{2,h} := \mathbf{\Pi}_h(\boldsymbol{\zeta}_{2,h},\vec{\mathbf{z}}_{2,h},\xi_{2,h}) \in \mathbf{H}_h$, where $(\vec{\vartheta}_{1,h},\widetilde{\boldsymbol{\sigma}}_{1,h}) \in \mathbf{H}_h\times \mathbf{Q}_h$ and $(\vec{\vartheta}_{2,h},\widetilde{\boldsymbol{\sigma}}_{2,h}) \in \mathbf{H}_h\times \mathbf{Q}_h$ are the respective solutions of (5.12). Defining $\vec{\boldsymbol{q}}_{1,h} := (\boldsymbol{\zeta}_{1,h},\vec{\mathbf{z}}_{1,h},q_{1,h})$ and $\vec{\boldsymbol{q}}_{2,h} := (\boldsymbol{\zeta}_{2,h},\vec{\mathbf{z}}_{2,h},q_{2,h})$, it follows from the corresponding second equation of (5.12) that $\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h} \in \mathcal{V}^{\ell}_{b,h}$, and then the $\mathcal{V}^{\ell}_{b,h}$-ellipticity of $a_{\vec{\mathbf{q}}_{1,h},\xi_{1,h}}$ (cf. (5.16)) yields

$$\|\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h}\|^2_{\mathbf{H}} \leqslant \frac{1}{\alpha_{A,\mathtt{d}}}\,a_{\vec{\mathbf{q}}_{1,h},\xi_{1,h}}(\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h},\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h})\,. \tag{5.25}$$

Then, proceeding analogously as for the derivation of (4.26), but now certainly employing $(\vec{\boldsymbol{q}}_{1,h},\xi_{1,h})$, $(\vec{\boldsymbol{q}}_{2,h},\xi_{2,h})$, and (5.12), we obtain

$$\begin{aligned}a_{\vec{\mathbf{q}}_{1,h},\xi_{1,h}}(\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h},\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h}) &= \int_\Omega (\mathbf{z}_{2,h} - \mathbf{z}_{1,h})\cdot \widetilde{\boldsymbol{t}}_{2,h}(\vartheta_{1,h} - \vartheta_{2,h})\\ &\quad + \int_\Omega (\mathcal{K}(\boldsymbol{\zeta}_{2,h},q_{2,h},\xi_{2,h}) - \mathcal{K}(\boldsymbol{\zeta}_{1,h},q_{1,h},\xi_{1,h}))\,\widetilde{\boldsymbol{t}}_{2,h}\cdot (\widetilde{\boldsymbol{t}}_{1,h} - \widetilde{\boldsymbol{t}}_{2,h})\,.\end{aligned} \tag{5.26}$$

Next, using the Lipschitz-continuity of $\mathcal{K}$ as in the estimate (3.2), recalling that $r = 2j$, $\rho = 2k$, and $r \leqslant \rho$, and noting that $\|\widetilde{\boldsymbol{t}}_{1,h} - \widetilde{\boldsymbol{t}}_{2,h}\|_{0;\Omega} \leqslant \|\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h}\|_{\mathbf{H}}$, we find that

$$\begin{aligned}\int_\Omega (\mathcal{K}(\boldsymbol{\zeta}_{2,h},q_{2,h},\xi_{2,h}) - \mathcal{K}(\boldsymbol{\zeta}_{1,h},q_{1,h},\xi_{1,h}))\,\widetilde{\boldsymbol{t}}_{2,h}\cdot (\widetilde{\boldsymbol{t}}_{1,h} - \widetilde{\boldsymbol{t}}_{2,h}) &\leqslant \widetilde{L}_{\mathcal{K}}\Big(\|\boldsymbol{\zeta}_{1,h} - \boldsymbol{\zeta}_{2,h}\|_{0,r;\Omega}\\ + \|q_{1,h} - q_{2,h}\|_{0,r;\Omega} + \|\xi_{1,h} - \xi_{2,h}\|_{0,\rho;\Omega}\Big)\,&\|\widetilde{\boldsymbol{t}}_{2,h}\|_{0,\rho;\Omega}\,\|\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h}\|_{\mathbf{H}}\,,\end{aligned} \tag{5.27}$$

where $\widetilde{L}_{\mathcal{K}}$ depends on $L_{\mathcal{K}}$, $|\Omega|$, r, and ρ. In turn, the Cauchy–Schwarz and Hölder inequalities, and the fact that

$$\|\vartheta_{1,h} - \vartheta_{2,h}\|_{0,\Omega} \,\leqslant\, |\Omega|^{(\rho-2)/\rho}\,\|\vartheta_{1,h} - \vartheta_{2,h}\|_{0,\rho;\Omega} \,\leqslant\, |\Omega|^{(\rho-2)/\rho}\,\|\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h}\|_{\mathbf{H}}\,,$$

yield

$$\int_\Omega (\mathbf{z}_{2,h} - \mathbf{z}_{1,h})\cdot \widetilde{\boldsymbol{t}}_{2,h}\,(\vartheta_{1,h} - \vartheta_{2,h}) \,\leqslant\, |\Omega|^{(\rho-2)/\rho}\,\|\mathbf{z}_{2,h} - \mathbf{z}_{1,h}\|_{0,r;\Omega}\,\|\widetilde{\boldsymbol{t}}_{2,h}\|_{0,\rho;\Omega}\,\|\vec{\vartheta}_{1,h} - \vec{\vartheta}_{2,h}\|_{\mathbf{H}}\,. \tag{5.28}$$

Finally, using (5.28) and (5.27) we can bound (5.26), so that the resulting estimate along with (5.25) and the fact that $\boldsymbol{\Pi}_{2,h}(\boldsymbol{\zeta}_{2,h},\vec{\mathbf{z}}_{2,h},\xi_{2,h}) = \widetilde{\boldsymbol{t}}_{2,h}$, imply (5.24) and conclude the proof. $\square$

Combining Lemmas 5.6, 5.7, and 5.9, we prove next that the operator $\mathbf{T}_h$ is continuous in the closed ball $\mathcal{W}_h(\delta)$ of the space $\mathbb{X}_{2,h}\times \mathbf{H}_{1,h}$. In order to simplify the corresponding statement and proof, we let $\widetilde{\mathbf{S}}_h$ and $\widetilde{\boldsymbol{\Pi}}_h := \big(\widetilde{\boldsymbol{\Pi}}_{1,h},\widetilde{\boldsymbol{\Pi}}_{2,h}\big)$ be the operators defined for each $(\boldsymbol{\zeta}_h,\vartheta_h)\in \mathbb{X}_{2,h}\times\mathbf{H}_{1,h}$ by

$$\widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_h,\vartheta_h) \,:=\, \mathbf{S}_h\big(\boldsymbol{\Xi}_{2,h}(\boldsymbol{\zeta}_h,\vartheta_h),\vartheta_h\big) \quad \text{and} \tag{5.29a}$$

$$\widetilde{\boldsymbol{\Pi}}_h(\boldsymbol{\zeta}_h,\vartheta_h) \,:=\, \boldsymbol{\Pi}_h\big(\widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_h,\vartheta_h),\boldsymbol{\Xi}_h(\boldsymbol{\zeta}_h,\vartheta_h),\vartheta_h\big)\,, \tag{5.29b}$$

so that $\widetilde{\boldsymbol{\Pi}}_{1,h}$ and $\widetilde{\boldsymbol{\Pi}}_{2,h}$ are obtained from (5.29b) by using, respectively, $\boldsymbol{\Pi}_{1,h}$ and $\boldsymbol{\Pi}_{2,h}$ instead of $\boldsymbol{\Pi}_h$.

Lemma 5.9. *Let ρ, ϱ, r and s be the real numbers within the range specified in (3.27), and $\lambda > M$. Moreover, assume that $h \leqslant h_0 := \min\{h_r^\ell, h_s^\ell\}$, and that the data are sufficiently small so that there hold (5.9), (5.21a) and (5.21b). Then, there exists a positive constant $L_{\mathbf{T},\mathtt{d}}$, depending on $C_{\mathbf{S},\mathtt{d}}$, $C_{\boldsymbol{\Xi},\mathtt{d}}$, and $\gamma(\lambda)$, and hence independent of h, such that*

$$\begin{aligned}&\|\mathbf{T}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \mathbf{T}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|\\ &\qquad\leqslant\, L_{\mathbf{T},\mathtt{d}}\,\big(1 + L_{\boldsymbol{\Pi},\mathtt{d}}\,\|\widetilde{\boldsymbol{\Pi}}_{2,h}(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|_{0,\rho;\Omega}\big)\,\|(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - (\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|\,,\end{aligned} \tag{5.30}$$

for all $(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h})$, $(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\in\mathcal{W}_h(\delta)$.

Proof. Given $(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h})$, $(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\in\mathcal{W}_h(\delta)$, we first observe from (5.18), (5.29a), and (5.29b) that

$$\mathbf{T}_h(\boldsymbol{\zeta}_{i,h},\vartheta_{i,h}) \,=\, \big(\widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_{i,h},\vartheta_{i,h}),\widetilde{\boldsymbol{\Pi}}_{1,h}(\boldsymbol{\zeta}_{i,h},\vartheta_{i,h})\big) \qquad \forall\, i\in\{1,2\}\,,$$

which yields

$$\begin{aligned}&\|\mathbf{T}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \mathbf{T}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|\\ &\qquad\leqslant\, \|\widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\| \,+\, \|\widetilde{\boldsymbol{\Pi}}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h})\big) - \widetilde{\boldsymbol{\Pi}}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\big)\|\,.\end{aligned} \tag{5.31}$$

Then, employing (5.29b) and (5.24), we find that

$$\begin{aligned}&\|\widetilde{\boldsymbol{\Pi}}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h})\big) - \widetilde{\boldsymbol{\Pi}}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\big)\|\\ &\qquad\leqslant\, L_{\boldsymbol{\Pi},\mathtt{d}}\,\|\widetilde{\boldsymbol{\Pi}}_{2,h}(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|_{0,\rho;\Omega}\,\Big\{\|\widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|\\ &\qquad\quad +\, \|\boldsymbol{\Xi}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \boldsymbol{\Xi}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\| + \|\vartheta_{1,h} - \vartheta_{2,h}\|\Big\}\,,\end{aligned} \tag{5.32}$$

whereas (5.29a) and (5.22) imply

$$\begin{aligned}&\|\widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|\\ &\qquad\leqslant\, C_{\mathbf{S},\mathtt{d}}\,\gamma(\lambda)\,\Big\{\|\boldsymbol{\Xi}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \boldsymbol{\Xi}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\| + \|\vartheta_{1,h} - \vartheta_{2,h}\|\Big\}\,.\end{aligned} \tag{5.33}$$

In this way, replacing (5.33) back into (5.32) and (5.31), and the resulting (5.32) back into (5.31) as well, and performing minor algebraic manipulations, we arrive at

$$\begin{aligned} \|\mathbf{T}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \mathbf{T}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\| \,&\leqslant\, \big(1 + L_{\boldsymbol{\Pi},\mathtt{d}}\,|\widetilde{\boldsymbol{\Pi}}_{2,h}(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\|_{0,\rho;\Omega}\big)\,\big(1 + C_{\mathbf{S},\mathtt{d}}\,\gamma(\lambda)\big) \\ &\times\,\Big\{\|\boldsymbol{\Xi}_h(\boldsymbol{\zeta}_{1,h},\vartheta_{1,h}) - \boldsymbol{\Xi}_h(\boldsymbol{\zeta}_{2,h},\vartheta_{2,h})\| + \|\vartheta_{1,h} - \vartheta_{2,h}\|\Big\}\,. \end{aligned} \tag{5.34}$$

Finally, (5.34) and (5.23) give (5.30) with $L_{\mathbf{T},\mathtt{d}} := \big(1{+}C_{\mathbf{S},\mathtt{d}}\,\gamma(\lambda)\big)\,\big(1{+}C_{\boldsymbol{\Xi},\mathtt{d}}\,\gamma(\lambda)\big)$, and end the proof. □

The main result of this section, which establishes the existence of solution of the discrete fixed-point equation (5.19), or equivalently of the discrete coupled system (5.2), is presented now.

Theorem 5.10. *Let ρ, ϱ, r and s be the real numbers within the range specified in (3.27), and $\lambda > M$. Moreover, assume that $h \leqslant h_0 := \min\{h_r^\ell, h_s^\ell\}$, and that the data are sufficiently small so that there hold (5.9), (5.21a) and (5.21b). Then, the operator $\mathbf{T}_h$ has a fixed-point $(\boldsymbol{\rho}_h,\theta_h) \in \mathcal{W}_h(\delta)$. Equivalently, the coupled problem (5.2) has a solution $(\boldsymbol{\rho}_h,\boldsymbol{u}_h) \in \mathbb{X}_{2,h}\times\mathbf{M}_{1,h}$, $(\boldsymbol{w}_h,p_h) \in \mathbf{X}_{2,h}\times\mathrm{M}_{1,h}$, and $(\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h) \in \mathbf{H}_h\times\mathbf{Q}_h$, with $(\boldsymbol{\rho}_h,\theta_h) \in \mathcal{W}_h(\delta)$. Moreover, there hold*

$$\begin{aligned} \|(\boldsymbol{\rho}_h,\boldsymbol{u}_h)\|_{\mathbb{X}_2\times\mathbf{M}_1} \,&\leqslant\, \widetilde{C}_{\mathbf{S},\mathtt{d}}\Big\{\|u_D\|_{1/s,r;\Gamma} + \|\mathbf{f}\|_{0,r;\Omega} + \|p_{\mathrm{D}}\|_{1/s,r;\Omega} + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} + \gamma(\lambda)\delta\Big\}\,, \\ \|(\boldsymbol{w}_h,p_h)\|_{\mathbf{X}_2\times\mathrm{M}_1} \,&\leqslant\, \widetilde{C}_{\boldsymbol{\Xi},\mathtt{d}}\Big\{\|u_D\|_{1/s,r;\Gamma} + \|p_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,r;\Omega} + \|g\|_{0,\varrho;\Omega} + \gamma(\lambda)\delta\Big\}\,, \\ \|(\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,&\leqslant\, \widetilde{C}_{\boldsymbol{\Pi},\mathtt{d}}(1+\delta)\|g\|_{0,\varrho;\Omega}\,, \end{aligned}$$

where $\widetilde{C}_{\mathbf{S},\mathtt{d}}$, $\widetilde{C}_{\boldsymbol{\Xi},\mathtt{d}}$ and $\widetilde{C}_{\boldsymbol{\Pi},\mathtt{d}}$ are constants depending on $C_{\mathbf{S},\mathtt{d}}$, $C_{\boldsymbol{\Xi},\mathtt{d}}$ and $C_{\boldsymbol{\Pi},\mathtt{d}}$.

Proof. From the assumptions (5.21a) and (5.21b), and Lemma 5.5 we have that $\mathbf{T}_h$ maps $\mathcal{W}_h(\delta)$ into itself. Furthermore, bearing in mind the continuity of $\mathbf{T}_h$ (cf. Lemma 5.9), a straightforward application of the Brouwer Theorem implies the existence of a solution $(\boldsymbol{\rho}_h,\theta_h) \in \mathcal{W}_h(\delta)$ of (5.19), and hence, the existence of $(\boldsymbol{\rho}_h,\boldsymbol{u}_h) \in \mathbb{X}_{2,h}\times\mathbf{M}_{1,h}$, $(\boldsymbol{w}_h,p_h) \in \mathbf{X}_{2,h}\times\mathrm{M}_{1,h}$ and $(\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h) \in \mathbf{H}_h\times\mathbf{Q}_h$ solution of (5.2). Finally, the a priori estimates follows straightforwardly from (5.6), (5.11), and (5.17), and bounding $\|\boldsymbol{\rho}_h\|_{\mathbb{X}_2}$ and $\|\theta_h\|_{\mathbf{X}_2}$ by δ. □

5.5 A priori error analysis

The goal of this section is to establish an a priori error estimate for the Galerkin scheme (5.2). More precisely, we are interested in deriving the usual Céa estimate for the global error

$$\mathbf{E} \,:=\, \|(\boldsymbol{\rho},\boldsymbol{u}) - (\boldsymbol{\rho}_h,\boldsymbol{u}_h)\|_{\mathbb{X}_2\times\mathbf{M}_1} \,+\, \|(\boldsymbol{\omega},p) - (\boldsymbol{\omega}_h,p_h)\|_{\mathbf{X}_2\times\mathrm{M}_1} \,+\, \|(\vec{\theta},\widetilde{\boldsymbol{\sigma}}) - (\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,,$$

where $\big((\boldsymbol{\rho},\boldsymbol{u}),(\boldsymbol{w},p),(\vec{\theta},\widetilde{\boldsymbol{\sigma}})\big) \in (\mathbb{X}_2\times\mathbf{M}_1)\times(\mathbf{X}_2\times\mathrm{M}_1)\times(\mathbf{H}\times\mathbf{Q})$, with $(\boldsymbol{\rho},\theta) \in \mathcal{W}(\delta)$, is the unique solution of (3.34), which is guaranteed by Theorem 4.11, and $\big((\boldsymbol{\rho}_h,\boldsymbol{u}_h),(\boldsymbol{w}_h,p_h),(\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h)\big) \in (\mathbb{X}_{2,h}\times\mathbf{M}_{1,h})\times(\mathbf{X}_{2,h}\times\mathrm{M}_{1,h})\times(\mathbf{H}_h\times\mathbf{Q}_h)$, with $(\boldsymbol{\rho}_h,\theta_h) \in \mathcal{W}_h(\delta)$, is a solution of (5.2), which is guaranteed by Theorem 5.10. To this end, we proceed as in [20, Section 4.3] and apply suitable Strang estimates to each one of the three pairs of associated continuous and discrete formulations forming (3.34) and (5.2). Throughout the rest of this section, given a subspace Z_h of a generic Banach space $(Z,\|\cdot\|_Z)$, we set for each $z \in Z$

$$\mathrm{dist}(z,Z_h) \,:=\, \inf_{z_h\in Z_h}\|z - z_h\|_Z\,.$$

We begin the analysis by applying the Strang estimate provided by [6, Proposition 2.1, Corollary 2.3, Theorem 2.3] to the context given by the first and second rows of (3.34) and (5.2). In this way, we deduce the existence of a positive constant $\widehat{C}_{\mathbf{S}}$, depending on $\alpha_{\boldsymbol{\mathcal{A}},\mathtt{d}}$, $\boldsymbol{\beta}_{1,\mathtt{d}}$, $\boldsymbol{\beta}_{2,\mathtt{d}}$, $\|\mathbf{a}\|$, $\|\mathbf{b}_1\|$, and $\|\mathbf{b}_2\|$ (cf. (3.11), Section 5.3), such that there holds

$$\|(\boldsymbol{\rho},\boldsymbol{u}) - (\boldsymbol{\rho}_h,\boldsymbol{u}_h)\|_{\mathbb{X}_2\times\mathbf{M}_1} \,\leqslant\, \widehat{C}_{\mathbf{S}}\left\{\mathrm{dist}\big((\boldsymbol{\rho},\boldsymbol{u}),\mathbb{X}_{2,h}\times\mathbf{M}_{1,h}\big) \,+\, \|\mathbf{F}_{p,\theta} - \mathbf{F}_{p_h,\theta_h}\|_{\mathbb{X}'_{1,h}}\right\}. \tag{5.35}$$

Then, according to the definition of $\mathbf{F}_{q,\vartheta}$ (cf. (3.10a)), we have that

$$\big(\mathbf{F}_{p,\theta} - \mathbf{F}_{p_h,\theta_h}\big)(\boldsymbol{\tau}_h) \,=\, -\gamma(\lambda)\int_\Omega \big(\alpha\,(p-p_h) + \beta\,(\theta-\theta_h)\big)\,\mathrm{tr}(\boldsymbol{\tau}_h) \qquad \forall\,\boldsymbol{\tau}_h\in\mathbb{X}_{1,h}\,,$$

from which, applying Hölder's inequality, and using that $r \leqslant \rho$, we find that there exists a positive constant $\bar{C}_{\mathbf{F}}$, depending on n, r, ρ, $|\Omega|$, α, and β, such that

$$\|\mathbf{F}_{p,\theta} - \mathbf{F}_{p_h,\theta_h}\|_{\mathbb{X}'_{1,h}} \,\leqslant\, \bar{C}_{\mathbf{F}}\,\gamma(\lambda)\left\{\|p-p_h\|_{0,r;\Omega} + \|\theta-\theta_h\|_{0,\rho;\Omega}\right\}. \tag{5.36}$$

Next, we apply the classical first Strang's Lemma (cf. [24, Lemma 2.27]) to the context given by the third and fourth rows of (3.34) and (5.2). As a consequence, we obtain a positive constant $\widehat{C}_{\boldsymbol{\Xi}}$, depending on $\alpha_{\mathbf{A},\mathtt{d}}$, $\|\mathbf{c}\|$, $\|\mathbf{d}_1\|$, $\|\mathbf{d}_2\|$, and $\|\mathbf{e}\|$ (cf. (3.22), Section 5.3), such that there holds

$$\|(\boldsymbol{\omega},p) - (\boldsymbol{\omega}_h,p_h)\|_{\mathbf{X}_2\times\mathrm{M}_1} \,\leqslant\, \widehat{C}_{\boldsymbol{\Xi}}\left\{\mathrm{dist}\big((\boldsymbol{\omega},p),\mathbf{X}_{2,h}\times\mathrm{M}_{1,h}\big) \,+\, \|\boldsymbol{\mathcal{G}}_{\boldsymbol{\rho},\theta} - \boldsymbol{\mathcal{G}}_{\boldsymbol{\rho}_h,\theta_h}\|_{\mathrm{M}'_{2,h}}\right\}. \tag{5.37}$$

In this case, the definition of $\boldsymbol{\mathcal{G}}_{\boldsymbol{\zeta},\vartheta}$ (cf. (3.21b)) yields

$$\big(\boldsymbol{\mathcal{G}}_{\boldsymbol{\rho},\theta} - \boldsymbol{\mathcal{G}}_{\boldsymbol{\rho}_h,\theta_h}\big)(q_h) \,=\, c_2(\lambda)\int_\Omega \mathrm{tr}(\boldsymbol{\rho}-\boldsymbol{\rho}_h)\,q_h \,+\, c_3(\lambda)\int_\Omega (\theta_0-\theta_{h,0})\,q_h \qquad \forall\,q_h\in\mathrm{M}_{2,h}\,,$$

so that, employing again Hölder's inequality and the inequality $r \leqslant \rho$, and bearing in mind the definitions of the constants $c_2(\lambda)$ and $c_3(\lambda)$ (cf. (2.5)), we deduce that

$$\|\boldsymbol{\mathcal{G}}_{\boldsymbol{\rho},\theta} - \boldsymbol{\mathcal{G}}_{\boldsymbol{\rho}_h,\theta_h}\|_{\mathrm{M}'_{2,h}} \,\leqslant\, \bar{C}_{\boldsymbol{\mathcal{G}}}\,\gamma(\lambda)\left\{\|\boldsymbol{\rho}-\boldsymbol{\rho}_h\|_{0,r;\Omega} + \|\theta-\theta_h\|_{0,\rho;\Omega}\right\}, \tag{5.38}$$

where $\bar{C}_{\boldsymbol{\mathcal{G}}}$ is a positive constant depending on n, r, ρ, $|\Omega|$, α, and β.

Furthermore, we apply the Strang estimate provided by [21, Lemma 6.1] to the context given by the fifth and sixth rows of (3.34) and (5.2). As a result, we get a positive constant $\widehat{C}_{\boldsymbol{\Pi}}$, depending on $\alpha_{A,\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\|a_{\vec{p},\theta}\|$, $\|a_{\vec{p}_h,\theta_h}\|$, and $\|b\|$ (cf. (3.31), (3.32), Section 5.3), such that there holds

$$\|(\vec{\theta},\widetilde{\boldsymbol{\sigma}}) - (\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, \widehat{C}_{\boldsymbol{\Pi}}\left\{\mathrm{dist}\big((\vec{\theta},\widetilde{\boldsymbol{\sigma}}),\mathbf{H}_h\times\mathbf{Q}_h\big) \,+\, \|a_{\vec{\boldsymbol{p}},\theta}(\vec{\theta},\cdot) - a_{\vec{\boldsymbol{p}}_h,\theta_h}(\vec{\theta},\cdot)\|_{\mathbf{H}'_h}\right\}, \tag{5.39}$$

where $\vec{\boldsymbol{p}} = (\boldsymbol{\rho},\boldsymbol{\omega},p)$ and $\vec{\boldsymbol{p}}_h = (\boldsymbol{\rho}_h,\boldsymbol{\omega}_h,p_h)$. Note that, being $\|\boldsymbol{\omega}\|$ and $\|\boldsymbol{\omega}_h\|$ bounded by α_A and $\alpha_{A,\mathtt{d}}$, it turns out that $\|a_{\vec{p},\theta}\|$ and $\|a_{\vec{p}_h,\theta_h}\|$ are bounded by $\|a\| + \alpha_A$ and $\|a\| + \alpha_{A,\mathtt{d}}$, respectively. Now, according to the definition of $a_{\vec{\boldsymbol{q}},\xi}$ (cf. (3.30a)), we have for all $\vec{\vartheta}_h = (\vartheta_h,\widetilde{\boldsymbol{s}}_h)$

$$a_{\vec{\boldsymbol{p}},\theta}(\vec{\theta},\vec{\vartheta}_h) - a_{\vec{\boldsymbol{p}}_h,\theta_h}(\vec{\theta},\vec{\vartheta}_h) \,=\, \int_\Omega\Big\{\mathcal{K}(\boldsymbol{\rho},p,\theta) - \mathcal{K}(\boldsymbol{\rho}_h,p_h,\theta_h)\Big\}\,\widetilde{\boldsymbol{t}}\cdot\widetilde{\boldsymbol{s}}_h \,+\, \int_\Omega\big(\boldsymbol{\omega}-\boldsymbol{\omega}_h\big)\cdot\widetilde{\boldsymbol{t}}\,\vartheta_h\,. \tag{5.40}$$

Regarding the first term on the right hand side of (5.40), we proceed exactly as for the derivation of (4.28), so that, employing again the Lipschitz-continuity of $\mathcal{K}$ (cf. (2.11)), the Cauchy–Schwarz and Hölder inequalities, the fact that $r \leqslant \rho$, and the regularity assumption (**H.1**) (cf. (4.23)), we obtain with the same constant $\widetilde{L}_{\boldsymbol{\Pi}}$ from (4.28) that

$$\begin{aligned}&\left|\int_\Omega\Big\{\mathcal{K}(\boldsymbol{\rho},p,\theta) - \mathcal{K}(\boldsymbol{\rho}_h,p_h,\theta_h)\Big\}\,\widetilde{\boldsymbol{t}}\cdot\widetilde{\boldsymbol{s}}_h\right| \\ &\qquad\leqslant\, \widetilde{L}_{\boldsymbol{\Pi}}\,\|g\|_{0,\varrho;\Omega}\left\{\|\boldsymbol{\rho}-\boldsymbol{\rho}_h\|_{0,r;\Omega} + \|p-p_h\|_{0,r;\Omega} + \|\theta-\theta_h\|_{0,\rho;\Omega}\right\}\|\widetilde{\boldsymbol{s}}_h\|_{0,\Omega}\,.\end{aligned} \tag{5.41}$$

In turn, proceeding similarly to the deduction of (4.29), which means using the above mentioned classical inequalities, along with the a priori estimate (4.15), we can write with the same constant $C_{\mathbf{\Pi}}$ from (4.15) that

$$\left|\int_\Omega (\boldsymbol{\omega}-\boldsymbol{\omega}_h)\cdot\widetilde{\boldsymbol{t}}\,\vartheta_h\right| \leqslant C_{\mathbf{\Pi}}\,\|g\|_{0,\varrho;\Omega}\,\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{0,r;\Omega}\,\|\vartheta_h\|_{0,\rho;\Omega}\,. \tag{5.42}$$

Hence, utilizing the bounds provided by (5.42) and (5.41), we readily conclude from (5.40) that

$$\begin{aligned}&\|a_{\vec{\boldsymbol{p}},\theta}(\vec{\theta},\cdot)-a_{\vec{\boldsymbol{p}}_h,\theta_h}(\vec{\theta},\cdot)\|_{\mathbf{H}'_h}\\ &\quad\leqslant \bar{C}_a\,\|g\|_{0,\varrho;\Omega}\left\{\|\boldsymbol{\rho}-\boldsymbol{\rho}_h\|_{0,r;\Omega}+\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{0,r;\Omega}+\|p-p_h\|_{0,r;\Omega}+\|\theta-\theta_h\|_{0,\rho;\Omega}\right\},\end{aligned} \tag{5.43}$$

where $\bar{C}_a := \max\{\widetilde{L}_{\mathbf{\Pi}}, C_{\mathbf{\Pi}}\}$. In this way, replacing (5.43) back into (5.39), (5.38) back into (5.37), and (5.36) back into (5.35), and then adding the resulting inequalities, we arrive at

$$\begin{aligned}\mathbf{E} \leqslant\ & \widehat{C}_1\Big\{\mathrm{dist}\big((\boldsymbol{\rho},\boldsymbol{u}),\mathbb{X}_{2,h}\times\mathbf{M}_{1,h}\big)+\mathrm{dist}\big((\boldsymbol{\omega},p),\mathbf{X}_{2,h}\times\mathrm{M}_{1,h}\big)\\ &+\mathrm{dist}\big((\vec{\theta},\widetilde{\boldsymbol{\sigma}}),\mathbf{H}_h\times\mathbf{Q}_h\big)\Big\}+\Big\{\widehat{C}_2\,\gamma(\lambda)+\widehat{C}_3\,\|g\|_{0,\varrho;\Omega}\Big\}\,\mathbf{E}\,,\end{aligned} \tag{5.44}$$

where $\widehat{C}_1 := \max\{\widehat{C}_{\mathbf{S}},\widehat{C}_{\mathbf{\Xi}},\widehat{C}_{\mathbf{\Pi}}\}$, $\widehat{C}_2 := \max\{\widehat{C}_{\mathbf{S}}\,\bar{C}_{\mathbf{F}},\widehat{C}_{\mathbf{\Xi}}\,\bar{C}_{\boldsymbol{\mathcal{G}}}\}$, and $\widehat{C}_3 := \widehat{C}_{\mathbf{\Pi}}\,\bar{C}_a$.

The announced Céa estimate can be stated now.

Theorem 5.11. *In addition to the hypotheses of Theorems 4.11 and 5.10, assume that*

$$\widehat{C}_2\,\gamma(\lambda)+\widehat{C}_3\,\|g\|_{0,\varrho;\Omega}\leqslant\frac{1}{2}\,. \tag{5.45}$$

Then, denoting $\widehat{C} = 2\,\widehat{C}_1$, there holds

$$\begin{aligned}&\|(\boldsymbol{\rho},\boldsymbol{u})-(\boldsymbol{\rho}_h,\boldsymbol{u}_h)\|_{\mathbb{X}_2\times\mathbf{M}_1}+\|(\boldsymbol{\omega},p)-(\boldsymbol{\omega}_h,p_h)\|_{\mathbf{X}_2\times\mathrm{M}_1}+\|(\vec{\theta},\widetilde{\boldsymbol{\sigma}})-(\vec{\theta}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\ &\leqslant\widehat{C}\Big\{\mathrm{dist}\big((\boldsymbol{\rho},\boldsymbol{u}),\mathbb{X}_{2,h}\times\mathbf{M}_{1,h}\big)+\mathrm{dist}\big((\boldsymbol{\omega},p),\mathbf{X}_{2,h}\times\mathrm{M}_{1,h}\big)+\mathrm{dist}\big((\vec{\theta},\widetilde{\boldsymbol{\sigma}}),\mathbf{H}_h\times\mathbf{Q}_h\big)\Big\}.\end{aligned}$$

Proof. It readily follows after employing the assumption (5.45) in (5.44). □

We now aim to establish the associated rates of convergence of the Galerkin scheme (5.2), for which we collect approximation properties of the finite element subspaces that were introduced in Section 5.2. Indeed, thanks to the error estimates of the vector and tensor versions of the Raviart–Thomas interpolator (see, e.g. [29, Section 4.1, eq. (4.6)]), as well as of the scalar and vector versions of the L^2-type projector onto piecewise polynomial spaces (see, e.g. [24, Proposition 1.135]), and due to interpolation estimates of Sobolev spaces, there hold the following:

($\mathbf{AP}_h^{\boldsymbol{\rho}}$) there exists a positive constant C, independent of h, such that for each $k\in[1,\ell+1]$, and for each $\boldsymbol{\tau}\in\mathbb{W}^{k,r}(\Omega)\cap\mathbb{H}_0^r(\mathbf{div}_r;\Omega)$, with $\mathbf{div}(\boldsymbol{\tau})\in\mathbf{W}^{k,r}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau},\mathbb{X}_{2,h}) := \inf_{\boldsymbol{\tau}_h\in\mathbb{X}_{2,h}}\|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{r,\mathbf{div}_r;\Omega}\leqslant C\,h^k\left\{\|\boldsymbol{\tau}\|_{k,r;\Omega}+\|\mathbf{div}(\boldsymbol{\tau})\|_{k,r;\Omega}\right\},$$

($\mathbf{AP}_h^{\boldsymbol{u}}$) there exists a positive constant C, independent of h, such that for each $k\in[0,\ell+1]$, and for each $\boldsymbol{v}\in\mathbf{W}^{k,r}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{v},\mathbf{M}_{1,h}) := \inf_{\boldsymbol{v}_h\in\mathbf{M}_{1,h}}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{0,r;\Omega}\leqslant C\,h^k\,\|\boldsymbol{v}\|_{k,r;\Omega}\,,$$

$(\mathbf{AP}_h^{\boldsymbol{w}})$ there exists a positive constant C, independent of h, such that for each $k \in [1, \ell+1]$, and for each $\boldsymbol{z} \in \mathbf{W}^{k,r}(\Omega)$, with $\mathrm{div}(\boldsymbol{z}) \in \mathrm{W}^{k,r}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{z}, \mathbf{X}_{2,h}) := \inf_{\boldsymbol{z}_h \in \mathbf{X}_{2,h}} \|\boldsymbol{z} - \boldsymbol{z}_h\|_{r,\mathrm{div}_r;\Omega} \leqslant C\, h^k \left\{ \|\boldsymbol{z}\|_{k,r;\Omega} + \|\mathrm{div}(\boldsymbol{z})\|_{k,r;\Omega} \right\},$$

$(\mathbf{AP}_h^{p})$ there exists a positive constant C, independent of h, such that for each $k \in [0, \ell+1]$, and for each $q \in \mathrm{W}^{k,r}(\Omega)$, there holds

$$\mathrm{dist}(q, \mathrm{M}_{1,h}) := \inf_{q_h \in \mathrm{M}_{1,h}} \|q - q_h\|_{0,r;\Omega} \leqslant C\, h^k \|q\|_{k,r;\Omega},$$

$(\mathbf{AP}_h^{\theta})$ there exists a positive constant C, independent of h, such that for each $k \in [0, \ell+1]$, and for each $\vartheta \in \mathrm{W}^{k,\rho}(\Omega)$, there holds

$$\mathrm{dist}(\vartheta, \mathbf{H}_{1,h}) := \inf_{\vartheta_h \in \mathbf{H}_{1,h}} \|\vartheta - \vartheta_h\|_{0,\rho;\Omega} \leqslant C\, h^k \|\vartheta\|_{k,\rho;\Omega},$$

$(\mathbf{AP}_h^{\widetilde{\boldsymbol{t}}})$ there exists a positive constant C, independent of h, such that for each $k \in [0, \ell+1]$, and for each $\widetilde{\boldsymbol{s}} \in \mathbf{H}^k(\Omega)$, there holds

$$\mathrm{dist}(\widetilde{\boldsymbol{s}}, \mathbf{H}_{2,h}) := \inf_{\widetilde{\boldsymbol{s}}_h \in \mathbf{H}_{2,h}} \|\widetilde{\boldsymbol{s}} - \widetilde{\boldsymbol{s}}_h\|_{0,\Omega} \leqslant C\, h^k \|\widetilde{\boldsymbol{s}}\|_{k,\Omega},$$

$(\mathbf{AP}_h^{\widetilde{\boldsymbol{\sigma}}})$ there exists a positive constant C, independent of h, such that for each $k \in [1, \ell+1]$, and for each $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}^k(\Omega)$, with $\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \in \mathrm{W}^{k,\varrho}(\Omega)$, there holds

$$\mathrm{dist}(\widetilde{\boldsymbol{\tau}}, \mathbf{Q}_h) := \inf_{\widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h} \|\widetilde{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_\varrho;\Omega} \leqslant C\, h^k \left\{ \|\widetilde{\boldsymbol{\tau}}\|_{k,\Omega} + \|\mathrm{div}(\widetilde{\boldsymbol{\tau}})\|_{k,\varrho;\Omega} \right\}.$$

The rates of convergence of (5.2) are then stated as follows.

Theorem 5.12. *Let* $\big((\boldsymbol{\rho}, \boldsymbol{u}), (\boldsymbol{w}, p), (\vec{\theta}, \widetilde{\boldsymbol{\sigma}})\big) \in (\mathbb{X}_2 \times \mathbf{M}_1) \times (\mathbf{X}_2 \times \mathrm{M}_1) \times (\mathbf{H} \times \mathbf{Q})$, *with* $(\boldsymbol{\rho}, \theta) \in \mathcal{W}(\delta)$, *be the unique solution of* (3.34), *and let* $\big((\boldsymbol{\rho}_h, \boldsymbol{u}_h), (\boldsymbol{w}_h, p_h), (\vec{\theta}_h, \widetilde{\boldsymbol{\sigma}}_h)\big) \in (\mathbb{X}_{2,h} \times \mathbf{M}_{1,h}) \times (\mathbf{X}_{2,h} \times \mathrm{M}_{1,h}) \times (\mathbf{H}_h \times \mathbf{Q}_h)$, *with* $(\boldsymbol{\rho}_h, \theta_h) \in \mathcal{W}_h(\delta)$, *be a solution of* (5.2), *which is guaranteed by Theorems* 4.11 *and* 5.10, *respectively. Assume the hypotheses of Theorem* 5.11 *and that there exists* $k \in [1, \ell+1]$, *such that* $\boldsymbol{\rho} \in \mathbb{W}^{k,r}(\Omega) \cap \mathbb{H}_0^r(\mathbf{div}_r; \Omega)$, $\mathbf{div}(\boldsymbol{\rho}) \in \mathbf{W}^{k,r}(\Omega)$, $\boldsymbol{u} \in \mathbf{W}^{k,r}(\Omega)$, $\boldsymbol{w} \in \mathbf{W}^{k,r}(\Omega)$, $\mathrm{div}(\boldsymbol{w}) \in \mathrm{W}^{k,r}(\Omega)$, $p \in \mathrm{W}^{k,r}(\Omega)$, $\theta \in \mathrm{W}^{k,\rho}(\Omega)$, $\widetilde{\boldsymbol{t}} \in \mathbf{H}^k(\Omega)$, $\widetilde{\boldsymbol{\sigma}} \in \mathbf{H}^k(\Omega)$, *and* $\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{W}^{k,\varrho}(\Omega)$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned} & \|(\boldsymbol{\rho}, \boldsymbol{u}) - (\boldsymbol{\rho}_h, \boldsymbol{u}_h)\|_{\mathbb{X}_2 \times \mathbf{M}_1} + \|(\boldsymbol{\omega}, p) - (\boldsymbol{\omega}_h, p_h)\|_{\mathbf{X}_2 \times \mathrm{M}_1} + \|(\vec{\theta}, \widetilde{\boldsymbol{\sigma}}) - (\vec{\theta}_h, \widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H} \times \mathbf{Q}} \\ & \quad \leqslant C\, h^k \Big\{ \|\boldsymbol{\rho}\|_{k,r;\Omega} + \|\mathbf{div}(\boldsymbol{\rho})\|_{k,r;\Omega} + \|\boldsymbol{u}\|_{k,r;\Omega} + \|\boldsymbol{w}\|_{k,r;\Omega} + \|\mathrm{div}(\boldsymbol{w})\|_{k,r;\Omega} \\ & \qquad\quad + \|p\|_{k,r;\Omega} + \|\theta\|_{k,\rho;\Omega} + \|\widetilde{\boldsymbol{t}}\|_{k,\Omega} + \|\widetilde{\boldsymbol{\sigma}}\|_{k,\Omega} + \|\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\|_{k,\varrho;\Omega} \Big\}. \end{aligned} \tag{5.46}$$

Proof. It follows straightforwardly from Theorem 5.11 and the above approximation properties. □

6 Numerical examples

In this final section we present two sets of computational tests, first the verification of convergence with respect to manufactured solutions in 2D and 3D, and an application example pertaining to the flow through a deformable porous channel with obstacles and temperature gradient. In all cases we take the following indexes (according to (3.27), valid for both 2D and 3D) $r = 3$, $s = \frac{3}{2}$, $\rho = 6$, and $\varrho = \frac{6}{5}$. The numerical realization has been done using the finite element library FEniCS [1], selecting Newton–Raphson as nonlinear solver, with an incremental relative tolerance of 10^{-8}. The linear solves are done with the direct method MUMPS.

6.1 Example 1: convergence verification

The error history (investigating the error decay with respect to mesh refinement – in a sequence of successively refined regular grids) is done comparing approximate and closed-form exact solutions defined on the unit square domain $\Omega = (0,1)^2$. The mixed variables, forcing and source terms for the balance equations, and non-homogenous boundary data are taken in such a way that the manufactured primal unknowns are

$$\boldsymbol{u}(x,y) = \frac{1}{10}\begin{pmatrix}\sin(\pi xy)\\ \cos(\pi x)\cos(\pi y)\end{pmatrix}, \quad p(x,y) = \sin(\pi x)\sin(\pi y), \quad \theta(x,y) = \cos(x)\exp(-x-y).$$

The model constants assume the following simple values: $\mu = 1, \quad \lambda = 1, \quad \kappa = 1, \quad \alpha = 1, \quad \beta = 1, \quad \chi = 1, \quad \eta = 1$, whereas the stress-assisted diffusion coefficient is

$$\mathcal{D}(\boldsymbol{\sigma}) = D_0 + D_1\exp(-\mathrm{tr}(\boldsymbol{\sigma}^2))\,, \quad \text{with} \quad D_0 = 0.1 \quad \text{and} \quad D_1 = 0.01\,. \tag{6.1}$$

The error history associated with the proposed mixed finite element method on a sequence of successively refined partitions of the domain, are collected in Table 6.1. Absolute errors are computed for each variable in the following way

$$e(\boldsymbol{u}) = \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,r;\Omega}, \quad e(p) = \|p-p_h\|_{0,r;\Omega}, \quad e(\theta) = \|\theta-\theta_h\|_{0,\rho;\Omega}, \quad e(\boldsymbol{\rho}) = \|\boldsymbol{\rho}-\boldsymbol{\rho}_h\|_{r,\mathbf{div}_r;\Omega},$$
$$e(\boldsymbol{w}) = \|\boldsymbol{w}-\boldsymbol{w}_h\|_{r,\mathrm{div}_r;\Omega}, \quad e(\widetilde{\boldsymbol{t}}) = \|\widetilde{\boldsymbol{t}}-\widetilde{\boldsymbol{t}}_h\|_{0,\Omega}, \quad e(\widetilde{\boldsymbol{\sigma}}) = \|\widetilde{\boldsymbol{\sigma}}-\widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_\varrho;\Omega},$$

and we also tabulate rates of error decay computed as $r(\cdot) = \log(e(\cdot)/\tilde{e}(\cdot))[\log(h/\tilde{h})]^{-1}$, where $e, \tilde{e}$ denote errors generated on two consecutive meshes of sizes h and $\tilde{h}$, respectively. All results indicate optimal convergence of $O(h^{k+1})$ in all fields and for the two tested polynomial degrees, which coincides with the theoretical result proposed in Theorem 5.12. For this test we have also tabulated the loss of momentum and mass conservation by taking the ℓ^∞ norm of the corresponding residuals projected into the discrete spaces for displacement and pressure. More precisely, letting $\mathcal{P}_k$ and $\boldsymbol{\mathcal{P}}_k$ be the $\mathrm{L}^2(\Omega)$-type and $\mathbf{L}^2(\Omega)$-type orthogonal projectors, respectively, onto the scalar and vector piecewise polynomials of degree $\leqslant k$, we set

$$\mathtt{mom}_h := \|\boldsymbol{\mathcal{P}}_k[\mathbf{div}(\boldsymbol{\sigma}_h) + \mathbf{f}]\|_{\ell^\infty}, \quad \mathtt{mass}_h := \|\mathcal{P}_k[c_1(\lambda)\,p_h - \mathrm{div}(\boldsymbol{w}_h) + c_3(\lambda)\,\theta_h + c_2(\lambda)\,\mathrm{tr}(\boldsymbol{\rho}_h) - f]\|_{\ell^\infty},$$

which, according to the second and fourth equations of (5.2), are essentially zero at machine precision. The table also reports that a maximum of three iterations are needed by the Newton–Raphson method to reach a tolerance (either absolute or relative) of 10^{-8} on the residual. Sample approximate solutions for all fields, obtained with the method using $k = 0$, are plotted in Figure 6.1.

The convergence tests are also done in 3D, taking $\Omega = (0,1)^3$, the same model parameters as in the 2D case, and using the following manufactured primal solutions

$$\boldsymbol{u}(x,y,z) = \frac{1}{10}\begin{pmatrix}\sin(\pi xyz)\\ \cos(\pi x)\cos(\pi y)\cos(\pi z)\\ \sin(\pi x)\sin(\pi y)\sin(\pi z)\end{pmatrix}, \quad p(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z),$$
$$\theta(x,y,z) = \cos(xy)\exp(-x-y-z).$$

We report on the lowest-order case in Table 6.2 and Figure 6.2, allowing us to draw the same conclusions as in the 2D case.

Primal unknowns and discrete conservation

DoFs	h	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(p)$	$r(p)$	$e(\theta)$	$r(\theta)$	$\mathtt{mom}_h$	$\mathtt{mass}_h$
Errors and convergence rates for $k = 0$									
113	0.707	4.79e-02	$\star$	2.78e-01	$\star$	1.59e-01	$\star$	4.44e-16	2.11e-15
417	0.354	2.23e-02	1.11	1.50e-01	0.89	8.15e-02	0.96	1.52e-15	4.77e-15
1601	0.177	1.04e-02	1.10	7.65e-02	0.98	4.11e-02	0.99	5.12e-15	9.21e-15
6273	0.088	5.05e-03	1.04	3.84e-02	0.99	2.06e-02	1.00	1.48e-13	2.09e-14
24833	0.044	2.50e-03	1.01	1.92e-02	1.00	1.03e-02	1.00	2.14e-12	5.94e-13
98817	0.022	1.25e-03	1.00	9.61e-03	1.00	5.16e-03	1.00	1.26e-12	2.38e-13
Errors and convergence rates for $k = 1$									
337	0.707	1.22e-02	$\star$	8.91e-02	$\star$	1.48e-02	$\star$	6.46e-15	7.49e-15
1281	0.354	3.02e-03	2.02	2.29e-02	1.96	3.64e-03	2.02	1.17e-14	3.50e-14
4993	0.177	7.48e-04	2.01	5.83e-03	1.98	9.13e-04	1.99	3.08e-14	5.20e-14
19713	0.088	1.86e-04	2.01	1.46e-03	1.99	2.29e-04	2.00	7.41e-14	1.51e-13
78337	0.044	4.65e-05	2.00	3.66e-04	2.00	5.72e-05	2.00	1.49e-13	3.46e-13
312321	0.022	1.16e-05	2.00	9.16e-05	2.00	1.43e-05	2.00	1.66e-12	1.02e-12

Mixed unknowns and iteration count

DoFs	h	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(\boldsymbol{w})$	$r(\boldsymbol{w})$	$e(\widetilde{\boldsymbol{t}})$	$r(\widetilde{\boldsymbol{t}})$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	iter
Errors and convergence rates for $k = 0$										
113	0.707	2.14e+00	$\star$	6.50e+00	$\star$	1.50e-01	$\star$	1.08e-01	$\star$	3
417	0.354	1.16e+00	0.88	3.55e+00	0.87	7.38e-02	1.02	5.06e-02	1.09	3
1601	0.177	5.98e-01	0.96	1.81e+00	0.98	3.70e-02	1.00	2.81e-02	0.85	3
6273	0.088	3.01e-01	0.99	9.07e-01	0.99	1.86e-02	0.99	1.48e-02	0.93	3
24833	0.044	1.51e-01	1.00	4.54e-01	1.00	9.31e-03	1.00	7.48e-03	0.98	3
98817	0.022	7.54e-02	1.00	2.27e-01	1.00	4.65e-03	1.00	3.75e-03	1.00	3
Errors and convergence rates for $k = 1$										
337	0.707	6.93e-01	$\star$	1.99e+00	$\star$	3.31e-02	$\star$	6.14e-02	$\star$	3
1281	0.354	2.00e-01	1.79	5.12e-01	1.96	4.77e-03	2.79	1.44e-02	2.10	3
4993	0.177	5.21e-02	1.94	1.30e-01	1.98	1.27e-03	1.91	4.65e-03	1.63	3
19713	0.088	1.32e-02	1.98	3.26e-02	1.99	3.39e-04	1.91	1.33e-03	1.80	3
78337	0.044	3.31e-03	1.99	8.16e-03	2.00	8.82e-05	1.94	3.31e-04	2.01	3
312321	0.022	8.29e-04	2.00	2.04e-03	2.00	2.23e-05	1.98	8.33e-05	1.99	3

Table 6.1: Example 1 (2D). Error history for the primal unknowns together with discrete approximation of momentum and mass conservation (top table) and convergence of mixed unknowns together with Newton–Raphson iteration count with respect to mesh refinement (bottom table). The symbol $\star$ indicates that no convergence rate is computed at that refinement level.

6.2 Injection of fluid in a deformable porous channel with inclusions

To conclude this section, we investigate the flow patterns of infiltration of a poroelastic channel having an irregular array of eight circular cylinders that are maintained at a low temperature. The problem setup mimics the behaviour of sponge-like materials or soils in the presence of macro-pores, for example [18, 32]. The undeformed body occupies the rectangular domain $\Omega = (0, 1.6) \times (0, 1)$ (in m^2), which we discretize into an unstructured mesh of 55450 triangles.

We consider a simple time-dependent version of the model (2.1), where only the energy balance equation (2.1c) is modified to have $\partial_t \theta$. We use a backward Euler discretization in time, with constant time step $\Delta t = 1$ (in s) and an initial temperature of 10 degrees. In addition, the boundary conditions

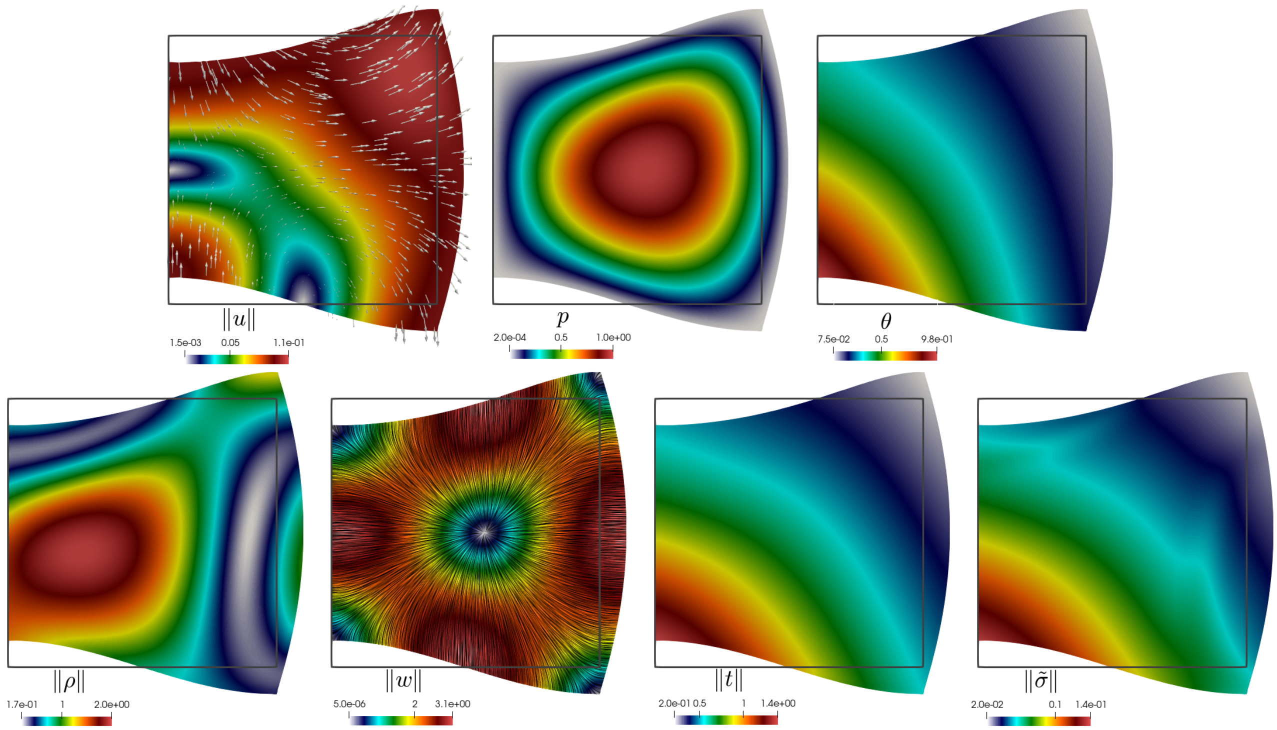

Figure 6.1: Example 1 (2D). Verification of convergence with respect to manufactured solutions. Approximate primal (top) and mixed (bottom) unknowns computed using the lowest-order scheme, and portrayed in the deformed configuration (the outline of the undeformed domain is also shown for reference).

Primal unknowns and discrete conservation

DoFs	h	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(p)$	$r(p)$	$e(\theta)$	$r(\theta)$	$\texttt{mom}_h$	$\texttt{mass}_h$
139	1.732	7.10e-02	$\star$	3.27e-01	$\star$	2.18e-01	$\star$	6.45e-16	4.00e-15
985	0.866	3.82e-02	0.89	2.23e-01	0.55	1.33e-01	0.71	1.16e-15	6.41e-15
7393	0.433	1.91e-02	1.00	1.17e-01	0.94	7.12e-02	0.90	2.72e-15	1.08e-14
57217	0.217	9.35e-03	1.03	5.96e-02	0.97	3.63e-02	0.97	7.76e-15	2.07e-14
450049	0.108	4.63e-03	1.01	3.00e-02	0.99	1.82e-02	0.99	1.99e-14	3.79e-14

Mixed unknowns and iteration count

DoFs	h	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(\boldsymbol{w})$	$r(\boldsymbol{w})$	$e(\tilde{\boldsymbol{t}})$	$r(\tilde{\boldsymbol{t}})$	$e(\tilde{\boldsymbol{\sigma}})$	$r(\tilde{\boldsymbol{\sigma}})$	$\texttt{iter}$
139	1.732	3.76e+00	$\star$	1.06e+01	$\star$	2.65e-01	$\star$	7.99e-02	$\star$	3
985	0.866	2.17e+00	0.79	7.55e+00	0.49	1.29e-01	1.04	4.67e-02	0.78	3
7393	0.433	1.16e+00	0.90	4.01e+00	0.91	6.56e-02	0.98	2.90e-02	0.69	3
57217	0.217	5.90e-01	0.97	2.05e+00	0.97	3.27e-02	1.00	1.65e-02	0.81	3
450049	0.108	2.96e-01	0.99	1.03e+00	0.99	1.63e-02	1.00	8.64e-03	0.94	3

Table 6.2: Example 1 (3D). Error history for the primal unknowns together with discrete approximation of momentum and mass conservation (top table) and convergence of mixed unknowns together with Newton–Raphson iteration count with respect to mesh refinement (bottom table). The symbol $\star$ indicates that no convergence rate is computed at that refinement level.

are of mixed type and do not coincide exactly with those analyzed in the manuscript. The left segment is considered an inflow boundary where we set zero displacements (as a natural boundary condition),

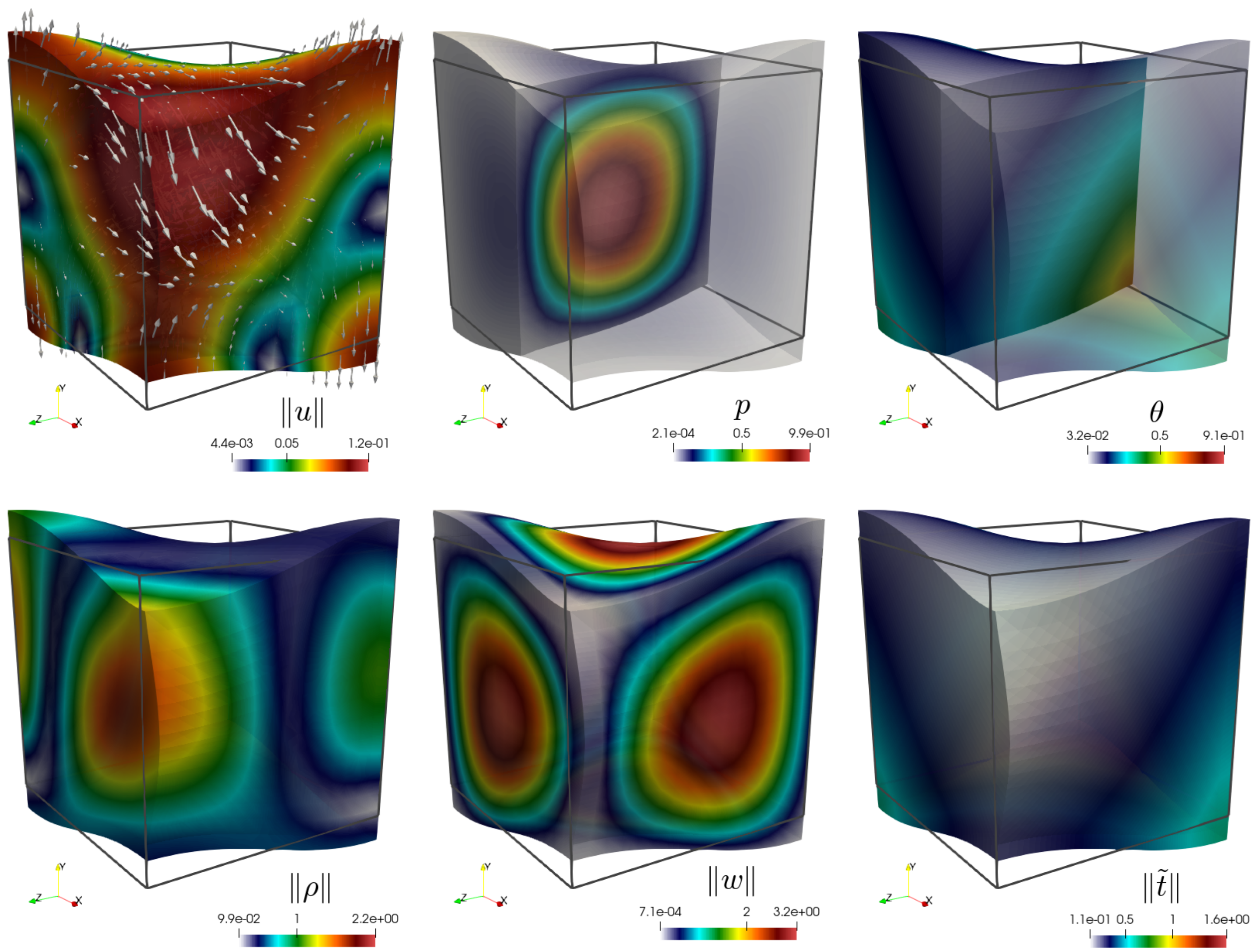

Figure 6.2: Example 1 (3D). Verification of convergence with respect to manufactured solutions. Approximate primal (top) and mixed (bottom) unknowns computed using the lowest-order scheme, and portrayed in the deformed configuration (the outline of the undeformed domain is also shown for reference).

a time-dependent parabolic profile as inflow of filtration flux (as an essential boundary condition), and a quadratic temperature profile (natural boundary condition)

$$\boldsymbol{u} = \mathbf{0}, \quad \boldsymbol{w}\cdot\boldsymbol{\nu} = \frac{t}{20}\mathrm{atan}(y[1-y])\,\mathrm{m/s}, \quad \theta = -74y^2 + 91y + 3 \text{ (in °C)} \qquad \text{on } \Gamma_{\mathrm{in}};$$

on the horizontal walls we approximate a zero-traction boundary condition with a zero normal pseudostress condition (imposed essentially), zero normal flux (essential), and a hot temperature on the top of the channel and cold on the bottom (natural boundary conditions)

$$\boldsymbol{\rho\nu} = \mathbf{0}, \quad \boldsymbol{w}\cdot\boldsymbol{\nu} = 0, \quad \theta = \theta_D, \qquad \text{on } \Gamma_{\mathrm{wall}},$$

(where θ_D is 3 degrees on the bottom and 20 degrees on the top); on the holes we impose

$$\boldsymbol{u} = \mathbf{0}, \quad \boldsymbol{w}\cdot\boldsymbol{\nu} = 0, \quad \theta = 3\,^\circ\mathrm{C}, \qquad \text{on } \Gamma_{\mathrm{cyl}};$$

and the boundary conditions are completed by prescribing zero traction (approximated by a zero normal pseudostress), a vanishing pressure (natural boundary condition), and a zero thermal flux on

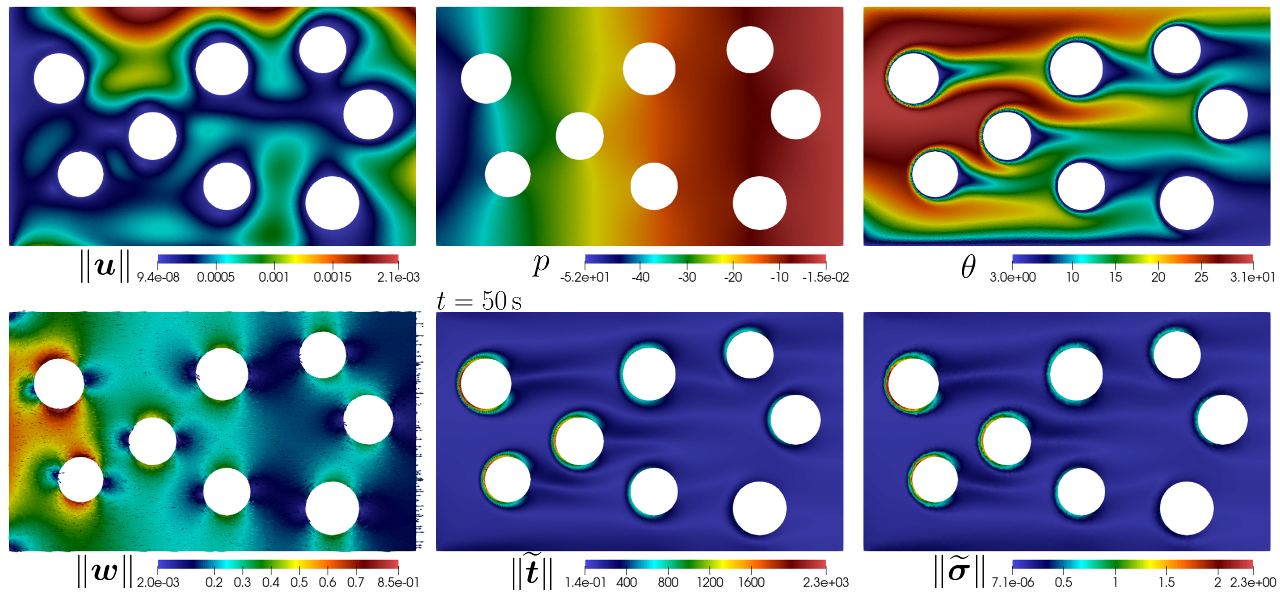

Figure 6.3: Example 2. Fluid injection using Biot–heat equations on a deformable channel with an array of cylinders, plotted on the undeformed configuration at time $t = 50\,\mathrm{s}$. Approximate solutions computed with a second-order method.

the outlet region (essentially imposed)

$$\boldsymbol{\rho\nu} = \mathbf{0}, \quad p = 0, \quad \widetilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu} = 0, \qquad \text{on } \Gamma_{\text{out}}.$$

We do not consider external volume forces nor fluid sources, therefore $\mathbf{f} = \mathbf{0}$, $f = g = 0$, the stress-assisted diffusion term is as in Example 1 (cf. (6.1)) with $D_0 = 10^{-3}$ and $D_1 = 10^{-4}$, and the remaining physical parameters are all constant and assuming the values

$$\mu = 210\,\mathrm{Pa},\ \lambda = 1800\,\mathrm{Pa},\ \eta = 10^{-3}\,\mathrm{Pa\,s},\ \kappa = 10^{-5}\,\mathrm{m}^2,\ \alpha = 0.9,\ \beta = 1.5,\ \chi = 10^{-2}\,\mathrm{Pa}.$$

The simulation runs until $t = 50\,\mathrm{s}$. The numerical solutions obtained with a second-order scheme (setting $k = 1$, for which the method consists of 667928 DoFs) are portrayed in Figure 6.3, showing snapshots of the deformed poroelastic region, filtration flux, and all other field variables at the final time. The expected injection patterns are seen in the flux plot, as well as the progressive heating of the fluid near the top plate.

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