Banach spaces-based mixed finite element methods for a steady sedimentation-consolidation system*

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Abstract

We introduce and analyze two Banach spaces-based new mixed finite element methods for a flow and transport model commonly encountered in sedimentation-consolidation processes, whose governing equations are given by the Brinkman flow with variable viscosity, coupled with a nonlinear advection–diffusion equation. The first variational formulation is based on a mixed approach for the Brinkman problem (written in terms of Cauchy stress and bulk velocity of the mixture) and the usual primal weak form for the transport equation. In turn, the second variational formulation arises from the introduction of the gradient of the solids volume fraction and the total (diffusive plus advective) flux for the concentration as new unknowns, which yields a momentum-conserving fully–mixed approach as the resulting system of equations. The respective continuous and discrete formulations are equivalently reformulated as fixed-point operator equations, whose solvability is established by combining the Schauder, Banach, and Brouwer theorems, with, among others, the Babuška–Brezzi theory and a recently introduced theory for perturbed saddle-point problems, both in Banach spaces, along with suitable regularity assumptions, Sobolev embeddings, and Rellich–Kondrachov compactness theorems. The mixed–primal and fully-mixed Galerkin schemes employ the classical Raviart–Thomas, piecewise continuous, and piecewise discontinuous polynomial approximations, for their corresponding unknowns. Next, Strang-type inequalities are utilized to rigorously derive optimal error estimates in the natural norms, which, combined with the approximation properties of the chosen finite element spaces yield optimal rates of convergence with respect to the mesh size. Finally, several numerical results illustrating the performance of both schemes and confirming the theoretical convergence rates are presented.

Key words: Brinkman equations, nonlinear transport problem, momentum conservation, fixed-point theory, sedimentation-consolidation process, finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 76R05, 76D07, 65N15

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1 Introduction

Our interest in this paper is the development of new numerical methods for the steady state of a class of sedimentation-consolidation processes. More precisely, in order to introduce the model problem, we let $\Omega \subseteq \mathrm{R}^n$, $n = \{2,3\}$, be a bounded domain with polyhedral boundary Γ and respective outward unit normal vector $\boldsymbol{\nu}$, which contains a solid phase suspended and subject to transport into an immiscible incompressible fluid. The flow of the fluid is influenced by gravity and by the local fluctuations of the volume fraction solids. The overall process is determined by the coupling of the Brinkman flow with variable viscosity and a nonlinear advection-diffusion equation, whose system of partial differential equations reads as

$$\begin{aligned}
&\boldsymbol{\sigma} = \mu(\phi)\,\nabla\mathbf{u} - p\,\mathbb{I}, \quad \boldsymbol{K}^{-1}\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) = \phi\,\boldsymbol{f}, \quad \mathrm{div}(\mathbf{u}) = 0, && \text{in} \quad \Omega,\\
&\varrho\,\phi - \mathrm{div}\big(\vartheta(\phi)\,\nabla\phi - \phi\,\mathbf{u} - f_{\mathrm{bk}}(\phi)\,\boldsymbol{k}\big) = g, && \text{in} \quad \Omega,\\
& && \int_\Omega p = 0,
\end{aligned} \tag{1.1}$$

subject to the Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{on} \quad \Gamma, \quad \phi = 0 \quad \text{on} \quad \Gamma.$$

The quantities of interest are the Cauchy fluid pseudo-stress $\boldsymbol{\sigma}$, the average velocity of the mixture $\mathbf{u}$, the fluid pressure p, and the volumetric fraction of the solids ϕ (which, for simplicity, will be simply called "concentration"). Here $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $g \in \mathrm{L}^2(\Omega)$ and $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ are given functions, with the latter, due to the incompressibility of the flow, satisfying the compatibility condition $\int_\Gamma \mathbf{u}_{\mathrm{D}} \cdot \boldsymbol{\nu} = 0$. Notice that the driving force per unit volume of the mixture is assumed to depend linearly on the local fluctuations of the concentration ϕ. The parameter ϱ is a positive constant representing the porosity of the medium, the permeability tensor $\boldsymbol{K} \in \mathbb{C}(\bar{\Omega}) := [C(\bar{\Omega})]^{n\times n}$ and its inverse are symmetric and assumed to be uniformly positive definite, and $\boldsymbol{k}$ is a constant vector pointing in the direction of gravity. We assume that the kinematic effective viscosity μ, the one-directional Kynch batch flux density function f_{bk} describing hindered settling, and the diffusion or sediment compressibility ϑ, are nonlinear scalar functions of the concentration ϕ. Furthermore, we restrict ourselves to suitably bounded Lipschitz functions μ, ϑ, and f_{bk}. More precisely, we suppose that there exist positive constants μ_1, μ_2, ϑ_1, ϑ_2, and γ_f, such that

$$\mu_1 \leqslant \mu(s) \leqslant \mu_2, \quad \vartheta_1 \leqslant \vartheta(s) \leqslant \vartheta_2, \quad \text{and} \quad 0 \leqslant f_{\mathrm{bk}}(s) \leqslant \gamma_f \qquad \forall\, s \in \mathrm{R}; \tag{1.2}$$

and positive constants L_μ, L_ϑ, and L_f, such that for each s, $t \in \mathrm{R}$ there hold

$$\begin{gathered}
|\mu(s) - \mu(t)| \leqslant L_\mu\,|s-t|, \quad |\vartheta(s) - \vartheta(t)| \leqslant L_\vartheta\,|s-t|, \quad \text{and}\\
|f_{\mathrm{bk}}(s) - f_{\mathrm{bk}}(t)| \leqslant L_f\,|s-t|.
\end{gathered} \tag{1.3}$$

The study of system (1.1) is of paramount importance as it models a great variety of natural processes arising in engineering applications, including fluidized beds, solid-liquid separation, purification in wastewater treatment, clot formation within the blood, macroscopic biofilm characterization, etcetera. Understanding and predicting the behavior of this problem is not an easy task due to the strong interaction of velocity and solids volume fraction via the Cauchy stress tensor and forcing term, the nonlinear structure of the overall coupled Brinkman flow and transport problem, and the saddle-point structure of the flow problem (to be seen later on), among others. These challenges are not only reflected within the solvability analysis of the governing equations but also during the development of appropriate schemes for numerical approximation and the derivation of corresponding stability results and error bounds.

Our problem of interest and related ones have received considerable attention during the past years. The solvability of the (time-dependent) sedimentation-consolidation problem was addressed in [12] for the case of large fluid viscosity. Since then, different mixed variational formulations and their associated Galerkin schemes have been introduced and studied, including [3, 4, 5, 7, 8, 21, 22]. The majority (if not all) of them rely on a fixed-point strategy to establish the well-posedness of both the continuous and discrete formulation. In each of these cases the corresponding fixed-point operator is comprised by a suitable composition of operators, each one of them defined by a subset of equations of the overall full system, thus allowing for the analysis of decoupled problems based upon Babuška–Brezzi or Lax–Milgram-type theorems. The solvability of the continuous and discrete fixed-point equations is addressed by a combination of further regularity assumptions, Sobolev embeddings, Rellich–Kondrachov compactness theorems, and the Schauder and Brouwer fixed-point theorems.

In particular, a modified formulation of (1.1) based on Stokes flow, in which the Brinkman term $\boldsymbol{K}^{-1}\,\mathbf{u}$ is not considered and the kinematic effective viscosity is assumed to depend explicitly on the gradient of the concentration, was studied in [3, 7, 8, 28]. An augmented dual-mixed formulation for the flow (in which the pressure is eliminated) and the usual primal formulation for the transport equation were studied in [3], whose unknowns given by the Cauchy stress tensor, the velocity of the fluid and the concentration were looked for in suitable Hilbert spaces. The need for augmentation was later circumvented in [7], resulting in the Cauchy stress tensor and velocity being now sought in the (non-Hilbert) Banach spaces $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ (the space of tensors in $[\mathrm{L}^2(\Omega)]^{n\times n}$ whose row-wise distributional divergence lies in $[\mathrm{L}^{4/3}(\Omega)]^n$) and $\mathbf{L}^4(\Omega)$, respectively; whereas the concentration is kept in the usual space $\mathrm{H}^1(\Omega)$. The fully-mixed approach has been studied in both its augmented [28] and non-augmented forms [7], in which the gradient of the concentration and the total (diffusive plus advective) flux for the concentration are introduced, thus leading to an overall six-field formulation.

The approach presented in [3] was expanded in [4] to address our model of interest, again by considering an augmented dual-mixed formulation for the flow and the usual primal formulation for the transport equation, which, due to the nonlinear diffusivity depending on the concentration and not on its gradient, made the analysis require one more further regularity assumption than its predecessor. In [22], a non-augmented four-field formulation was introduced upon enriching the dual-mixed formulation for the flow by introducing the gradient of the velocity as an unknown, which, due to the incompressibility condition, needs to be sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$ (the space of trace-free tensors in $[\mathrm{L}^2(\Omega)]^{n\times n}$) in order to yield an overall stable system.

It is worth noting that models of sedimentation-consolidation share structural similarities with the Boussinesq, Navier–Stokes Brinkman, and related models. Several mixed formulations have been proposed for them, such as those found in [14, 15, 16, 17, 20, 29]. For instance, the mixed finite element method for the Boussinesq problem developed in [20] introduces the gradient of velocity as an auxiliary unknown in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$. In this reference, novel (and certainly non-trivial) continuous discrete and inf-sup conditions were proved, which were instrumental in [7, 8] to prove the stability of the stress-velocity pair coming from the flow equations, and in [8] for the concentration-gradient of concentration-total flux triple stability coming from the transport equation.

According to the bibliographic discussion above, the objective of this paper is to continue the development of non-augmented Banach spaces-based numerical methods by introducing two new mixed finite element methods for problem (1.1): a mixed-primal formulation, and a fully-mixed formulation. The analysis of each of them hinges on an only-recently-available Babuška–Brezzi-type theory for perturbed saddle-point problems in Banach spaces [23]. As in [7], the mixed–primal variational formulation seeks stresses in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, velocity in $\mathbf{L}^4(\Omega)$, and solids volume fraction in $\mathrm{H}^1(\Omega)$, while the fully–mixed formulation look for the stress and velocity in the same spaces, but the solids volume in $\mathrm{L}^4(\Omega)$, the concentration gradient in $\mathbf{L}^2(\Omega)$, and the total flux in $\mathbf{H}(\mathrm{div}_{4/3};\Omega)$. The resulting mixed–primal Galerkin scheme employs Raviart–Thomas approximations of order k for the stress, piecewise discontinuous polynomials of degree $\leqslant k$ for the velocity and piecewise continuous polynomials of degree $k+1$ for the volume fraction; the resulting fully-mixed Galerkin scheme, which yields momentum conservation properties in an approximate sense, employs the same approximation spaces for the stress and velocity as in the mixed-primal scheme, Raviart-Thomas approximations of order k for the total flux, and piecewise discontinuous polynomials of degree $\leqslant k$ for the solids volume fraction and its gradient. In turn and along the lines of the already cited references, the solvability analyses of the continuous formulations are based upon classical fixed-point theorems, suitable further regularity assumptions, the Lax–Milgram theorem, the new Babuška–Brezzi-type theory [23], Sobolev embeddings, and Rellich–Kondrachov compactness theorems. Under sufficiently small data, we are also able to prove uniqueness of solutions. The well-posedness of the discrete problem relies on the Brouwer's fixed-point theorem and similar arguments to those employed in the continuous analysis. Next, the combination of Strang-type estimates for the transport equations and the fluid flow equations yield the corresponding Céa estimate for the total error. Optimal a priori error bounds for the Galerkin solution are provided.

The rest of this paper is organized as follows. Section 1.1 introduces some standard notation for functional spaces and differential operators that will be used throughout the paper. In Section 1.2 we rewrite system (1.1) by eliminating the pressure, and setting it up for the subsequent derivation of the mixed-primal and fully-mixed variational formulations. Next, in Section 2 we derive the aforementioned mixed-primal and fully-mixed variational formulations. Section 3 is devoted to the solvability analysis of the continuous variational formulations via a fixed-point strategy and a classical fixed-point theorem. The corresponding Galerkin schemes are introduced in Section 4, which are analyzed by means of the discrete analogue of the theory used in Section 2. In Section, 5, we derive the corresponding a priori error estimates and the associated rates of convergence. Finally, in Section 6, we test the performance of our methods with numerical examples in 2D and 3D, which confirm the theoretical rates and the approximate momentum conservation properties.

1.1 Preliminaries

We recall the standard notation for Lebesgue spaces $\mathrm{L}^t(\Omega)$, $t \in (1,+\infty)$, with norm $\|\cdot\|_{0,t;\Omega}$, and for Sobolev spaces $\mathrm{H}^s(\Omega)$, $s \geqslant 0$, endowed with the norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $\mathrm{H}^{1/2}(\Gamma)$ stands for the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any functional space. In what follows $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. In turn, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$ we set the gradient and divergence operators as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n} \quad \text{and} \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\, \zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n} \operatorname{tr}(\boldsymbol{\tau})\, \mathbb{I}\,.$$

Furthermore, given $t \in (1, +\infty)$, we introduce the Banach space

$$\mathbf{H}(\mathrm{div}_t; \Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega) \Big\}\,, \tag{1.4}$$

and its tensor version

$$\mathbb{H}(\mathbf{div}_t; \Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \Big\}\,,$$

equipped, respectively, with the usual norms

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \Big\{ \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega}^2 \Big\}^{1/2}, \quad \text{and}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \Big\{ \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega}^2 \Big\}^{1/2}.$$

In the remainder of the paper we will consider the above definition for $t = 4/3$. Finally, for any pair of normed spaces, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ we provide the product space $X \times Y$ with the natural norm $\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y \quad \forall\, (x,y) \in X \times Y$.

1.2 System rewrite

We start by observing that the first and third equations in the first row of (1.1) can be equivalently written as

$$\boldsymbol{\sigma} = \mu(\phi)\, \nabla \mathbf{u} - p\, \mathbb{I} \quad \text{and} \quad p = -\frac{1}{n} \operatorname{tr}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,, \tag{1.5}$$

which allow us to eliminate the pressure p from the first equation of the first row of (1.1). As a consequence, we arrive at the following coupled system equivalent to (1.1):

$$\begin{aligned}
&\frac{1}{\mu(\phi)}\, \boldsymbol{\sigma}^{\mathrm{d}} = \nabla \mathbf{u} \quad \text{in} \quad \Omega\,, \quad \boldsymbol{K}^{-1}\, \mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) = \phi\, \boldsymbol{f} \quad \text{in} \quad \Omega\,, \\
&\varrho\, \phi - \mathrm{div}\big(\vartheta(\phi)\, \nabla\phi - \phi\, \mathbf{u} - f_{\mathrm{bk}}(\phi)\, \boldsymbol{k}\big) = g \quad \text{in} \quad \Omega\,, \\
&\mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{on} \quad \Gamma\,, \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0\,.
\end{aligned} \tag{1.6}$$

We stress that the incompressibility condition is implicitly present in the first equation of (1.6), relating $\boldsymbol{\sigma}$ and $\mathbf{u}$. In addition, the uniqueness condition for p, originally given by $\int_\Omega p = 0$, is now stated as $\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0$, which certainly follows from the postprocessed formula for p provided by the second equation in (1.5).

In turn, in order to obtain a fully–mixed formulation, we introduce the unknowns $\mathbf{t} = \nabla\phi$ and the total (diffusive plus advective) flux for concentration $\boldsymbol{\eta}$, which is explicitly defined below, thus obtaining the following coupled system:

$$\begin{aligned}
&\frac{1}{\mu(\phi)}\, \boldsymbol{\sigma}^{\mathrm{d}} = \nabla \mathbf{u} \quad \text{in} \quad \Omega\,, \quad \boldsymbol{K}^{-1}\, \mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) = \phi\, \boldsymbol{f} \quad \text{in} \quad \Omega\,, \\
&\mathbf{t} = \nabla\phi \quad \text{in} \quad \Omega\,, \quad \boldsymbol{\eta} = \vartheta(\phi)\, \mathbf{t} - \phi\, \mathbf{u} - f_{\mathrm{bk}}(\phi)\, \boldsymbol{k} \quad \text{in} \quad \Omega\,, \quad \varrho\, \phi - \mathrm{div}(\boldsymbol{\eta}) = g \quad \text{in} \quad \Omega\,, \\
&\mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{on} \quad \Gamma\,, \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0\,.
\end{aligned} \tag{1.7}$$

2 The weak formulations

The purpose of this section is to introduce the announced weak formulations of our original model (1.1). More precisely, in Section 2.1 we proceed similarly as in [7] to derive the mixed–primal variational formulation arising from (1.6), whereas in Section 2.2, partially inspired by [8], we derive the fully–mixed variational formulation that emerges from (1.7).

2.1 The mixed–primal approach

We begin by considering the transport equation (cf. first row of (1.6)), whose Dirichlet boundary condition for ϕ motivates the introduction of the space

$$\mathrm{H}_0^1(\Omega) \,:=\, \Big\{\psi \in \mathrm{H}^1(\Omega) : \quad \psi \,=\, 0 \quad \text{on} \quad \Gamma\Big\}\,.$$

Recall that, thanks to the Poincaré inequality, there exists a positive constant c_P, depending only on Ω, such that

$$\|\psi\|_{1,\Omega} \,\leqslant\, c_P\,|\psi|_{1,\Omega} \qquad \forall\,\psi \in \mathrm{H}_0^1(\Omega)\,. \tag{2.1}$$

Moreover, the continuous injection i_4 of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^4(\Omega)$ (cf. [1, Theorem 4.12], [30, Theorem 1.3.4]) yields

$$\|\psi\|_{0,4;\Omega} \,\leqslant\, \|\mathrm{i}_4\|\,\|\psi\|_{1,\Omega} \qquad \forall\,\psi \in \mathrm{H}^1(\Omega)\,. \tag{2.2}$$

We now look at the equilibrium equation $\varrho\,\phi \,-\, \mathrm{div}\big(\vartheta(\phi)\,\nabla\phi \,-\, \phi\,\mathbf{u} \,-\, f_{\mathrm{bk}}(\phi)\,\boldsymbol{k}\big) \,=\, g$ in the second row of (1.6). In fact, given $\mathbf{u}$ living in a suitable space to be specified later, we multiply by $\psi \in \mathrm{H}_0^1(\Omega)$ and integrate by parts to deduce that the primal formulation for the concentration becomes: Find $\phi \in \mathrm{H}_0^1(\Omega)$ such that

$$A_{\mathbf{u}}(\phi,\psi) \,=\, G_\phi(\psi) \qquad \forall\,\psi \in \mathrm{H}_0^1(\Omega)\,, \tag{2.3}$$

where $A_{\mathbf{u}} : \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega) \to \mathrm{R}$ is the semilinear form given by

$$A_{\mathbf{u}}(\phi,\psi) \,:=\, \int_\Omega \vartheta(\phi)\,\nabla\phi\cdot\nabla\psi \,-\, \int_\Omega \phi\,\mathbf{u}\cdot\nabla\psi \,+\, \int_\Omega \varrho\,\phi\,\psi \qquad \forall\,\phi,\,\psi \in \mathrm{H}_0^1(\Omega)\,, \tag{2.4}$$

and $G_\phi : \mathrm{H}_0^1(\Omega) \to \mathrm{R}$ is the functional defined as

$$G_\phi(\psi) \,:=\, \int_\Omega f_{\mathrm{bk}}(\phi)\,\boldsymbol{k}\cdot\nabla\psi \,+\, \int_\Omega g\,\psi \qquad \forall\,\psi \in \mathrm{H}_0^1(\Omega)\,. \tag{2.5}$$

Regarding $A_{\mathbf{u}}$, and in order to address later on the analysis of (2.3), we note that using (1.2), Cauchy–Schwarz's inequality, and (2.2), we find that

$$\left|\int_\Omega \varrho\,\varphi\,\psi\right| \,\leqslant\, \varrho\,\|\varphi\|_{1,\Omega}\,\|\psi\|_{1,\Omega} \qquad \forall\,\varphi,\,\psi \in \mathrm{H}^1(\Omega)\,, \tag{2.6}$$

$$\left|\int_\Omega \vartheta(\varphi)\,\nabla\varphi\cdot\nabla\psi\right| \,\leqslant\, \vartheta_2\,\|\varphi\|_{1,\Omega}\,|\psi|_{1,\Omega} \qquad \forall\,\varphi,\,\psi \in \mathrm{H}^1(\Omega)\,, \quad \text{and} \tag{2.7}$$

$$\left|\int_\Omega \varphi\,\mathbf{v}\cdot\nabla\psi\right| \,\leqslant\, \|\mathrm{i}_4\|\,\|\varphi\|_{1,\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,|\psi|_{1,\Omega} \qquad \forall\,\varphi,\,\psi \in \mathrm{H}^1(\Omega)\,, \quad \forall\,\mathbf{v} \in \mathbf{L}^4(\Omega)\,. \tag{2.8}$$

In particular, (2.8) shows that considering $\mathbf{u} \in \mathbf{L}^4(\Omega)$ ensures that $A_{\mathbf{u}}$ is well-defined, and hence from now on we look for this unknown in that space. In turn, it is easy to see from (2.5) and (1.2) that the functional G_ϕ is bounded independently of ϕ with a norm satisfying

$$\|G_\phi\| \,\leqslant\, \gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,\Omega}\,.$$

On the other hand, testing the first equation of (1.6) by $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, integrating by parts, and using the Dirichlet boundary condition for $\mathbf{u}$ (cf. third row of (1.6)) and the identity $\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau} = \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}$, we obtain

$$\int_\Omega \frac{1}{\mu(\phi)}\,\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \,+\, \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_{\mathrm{D}}\rangle_\Gamma\,, \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)\,, \tag{2.9}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Note that the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ guarantees that $\boldsymbol{\tau}\,\boldsymbol{\nu}$ belongs to $\mathbf{H}^{-1/2}(\Gamma)$ when $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and that there exists a positive constant $c(\Omega)$, depending only on Ω, such that (see [20, Section 3.1])

$$\|\boldsymbol{\tau}\,\boldsymbol{\nu}\|_{-1/2,\Gamma} \,\leqslant\, c(\Omega)\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}\,. \tag{2.10}$$

Therefore, looking for the unknown $\boldsymbol{\sigma}$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well, we realize that the momentum equation $\boldsymbol{K}^{-1}\,\mathbf{u} \,-\, \mathbf{div}(\boldsymbol{\sigma}) \,=\, \phi\,\boldsymbol{f}$ can be weakly imposed as

$$-\int_\Omega \boldsymbol{K}^{-1}\,\mathbf{u}\cdot\mathbf{v} \,+\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,=\, -\int_\Omega \phi\,\boldsymbol{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in \mathbf{L}^4(\Omega)\,. \tag{2.11}$$

Moreover, the null mean value of $\mathrm{tr}(\boldsymbol{\sigma})$ stated in the last equation of (1.6) suggests that $\boldsymbol{\sigma}$ must be actually sought in $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:=\, \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,=\, 0\right\}.$$

In this way, given $\phi \in \mathrm{H}^1_0(\Omega)$, we collect (2.9) and (2.11) to arrive at first glance at the following mixed formulation of the Brinkman flow: Find $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ such that

$$\begin{aligned} \mathbf{a}_\phi(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) \,&=\, \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)\,,\\ \mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) \,-\, \mathbf{c}(\mathbf{u},\mathbf{v}) \,&=\, \mathbf{G}_\phi(\mathbf{v}) \qquad \forall\,\mathbf{v} \in \mathbf{L}^4(\Omega)\,, \end{aligned} \tag{2.12}$$

where the bilinear forms $\mathbf{a}_\phi : \mathbb{H}(\mathbf{div}_{4/3};\Omega) \times \mathbb{H}(\mathbf{div}_{4/3};\Omega) \to \mathrm{R}$, $\mathbf{b} : \mathbb{H}(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega) \to \mathrm{R}$, and $\mathbf{c} : \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega) \to \mathrm{R}$, and the linear functionals $\mathbf{F} : \mathbb{H}(\mathbf{div}_{4/3};\Omega) \to \mathrm{R}$ and $\mathbf{G}_\phi : \mathbf{L}^4(\Omega) \to \mathrm{R}$ are defined as

$$\mathbf{a}_\phi(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \int_\Omega \frac{1}{\mu(\phi)}\,\boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}\,, \qquad \mathbf{b}(\boldsymbol{\tau},\mathbf{v}) \,:=\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \qquad \mathbf{c}(\mathbf{w},\mathbf{v}) \,:=\, \int_\Omega \boldsymbol{K}^{-1}\,\mathbf{w}\cdot\mathbf{v} \tag{2.13}$$

$$\mathbf{F}(\boldsymbol{\tau}) \,:=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_{\mathrm{D}}\rangle_\Gamma \qquad \text{and} \qquad \mathbf{G}_\phi(\mathbf{v}) \,:=\, -\int_\Omega \phi\,\boldsymbol{f}\cdot\mathbf{v} \tag{2.14}$$

for $\boldsymbol{\zeta},\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ and $\mathbf{v},\mathbf{w} \in \mathbf{L}^4(\Omega)$. Moreover, using (1.2), the Cauchy–Schwarz and Hölder inequalities, and the continuous injection $\mathbf{i}_{4/3}$ of $\mathbf{L}^4(\Omega)$ into $\mathbf{L}^{4/3}(\Omega)$, with $\|\mathbf{i}_{4/3}\| \,=\, |\Omega|^{1/2}$, it follows that

$$\begin{aligned} |\mathbf{a}_\phi(\boldsymbol{\zeta},\boldsymbol{\tau})| \,&\leqslant\, \frac{1}{\mu_1}\,\|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}\,,\\ |\mathbf{b}(\boldsymbol{\zeta},\mathbf{v})| \,&\leqslant\, \|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega}\,\|\mathbf{v}\|_{0,4;\Omega},\\ |\mathbf{c}(\mathbf{w},\mathbf{v})| \,&\leqslant\, \|\mathbf{i}_{4/3}\|\,\|\boldsymbol{K}^{-1}\|_{\infty,\Omega}\,\|\mathbf{w}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,, \end{aligned} \tag{2.15}$$

which establishes the boundedness of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, with constants

$$\|\mathbf{a}_\phi\| \,=\, \frac{1}{\mu_1}\,, \quad \|\mathbf{b}\| \,=\, 1, \quad \text{and} \quad \|\mathbf{c}\| \,:=\, \|\mathbf{i}_{4/3}\|\,\|\boldsymbol{K}^{-1}\|_{\infty,\Omega}. \tag{2.16}$$

In turn, employing the duality between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$ along with the estimate (2.10), and then applying Cauchy–Schwarz's inequality and the estimate from (2.2), we deduce that

$$|\mathbf{F}(\boldsymbol{\tau})| \,\leqslant\, c(\Omega)\,\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}\,, \qquad |\mathbf{G}_\phi(\mathbf{v})| \,\leqslant\, \|\mathrm{i}_4\|\,\|\boldsymbol{f}\|_{0,\Omega}\,\|\phi\|_{1,\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,, \tag{2.17}$$

which says that $\mathbf{F}$ and $\mathbf{G}_\phi$ are also bounded, with

$$\|\mathbf{F}\| \,\leqslant\, c(\Omega)\,\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \qquad \text{and} \qquad \|\mathbf{G}_\phi\| \,\leqslant\, \|\mathrm{i}_4\|\,\|\boldsymbol{f}\|_{0,\Omega}\,\|\phi\|_{1,\Omega}\,. \tag{2.18}$$

Furthermore, thanks to the compatibility condition $\int_\Gamma \mathbf{u}_{\mathrm{D}}\cdot\boldsymbol{\nu} = 0$ and the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\oplus \mathrm{R}\mathbb{I}\,,$$

it is easily shown that imposing the first equation of (2.12) against $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing so against $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. In this way, (2.12) reduces equivalently to: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)$ such that

$$\begin{aligned}
\mathbf{a}_\phi(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) \,&=\, \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\
\mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) \,-\, \mathbf{c}(\mathbf{u},\mathbf{v}) \,&=\, \mathbf{G}_\phi(\mathbf{v}) \qquad \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,.
\end{aligned} \tag{2.19}$$

Finally, gathering (2.19) and (2.3), we arrive at the following mixed–primal formulation for the coupled problem (1.6): Find $(\boldsymbol{\sigma},\mathbf{u},\phi)\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)\times\mathrm{H}^1_0(\Omega)$ such that

$$\begin{aligned}
\mathbf{a}_\phi(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) \,&=\, \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\
\mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) \,-\, \mathbf{c}(\mathbf{u},\mathbf{v}) \,&=\, \mathbf{G}_\phi(\mathbf{v}) \qquad \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,,\\
A_{\mathbf{u}}(\phi,\psi) \,&=\, G_\phi(\psi) \qquad \forall\,\psi\in\mathrm{H}^1_0(\Omega)\,.
\end{aligned} \tag{2.20}$$

2.2 The fully–mixed approach

In order to establish a variational formulation for the system (1.7), we start by noting that the variational formulation associated with the Brinkman flow equation (cf. first row of (1.7)) will remain essentially the same as in Section 2.1 (cf. (2.19)), the only difference being, as we explain below, that the given ϕ will belong now to $\mathrm{L}^4(\Omega)$ instead of the stronger space $\mathrm{H}^1_0(\Omega)$ as in Section 2.1. Indeed, proceeding as in [8, Section 3.1], we begin by testing the second equation of the second row of (1.7) against a suitable vector function $\mathbf{s}$, which yields

$$\int_\Omega \vartheta(\phi)\,\mathbf{t}\cdot\mathbf{s} \,-\, \int_\Omega \phi\,\mathbf{u}\cdot\mathbf{s} \,-\, \int_\Omega \boldsymbol{\eta}\cdot\mathbf{s} \,=\, \int_\Omega f_{\mathrm{bk}}(\phi)\,\boldsymbol{k}\cdot\mathbf{s}\,. \tag{2.21}$$

Then, knowing from (2.19) that we are looking for $\mathbf{u}$ in $\mathbf{L}^4(\Omega)$, we readily see that the second term on the left hand-side of (2.21) makes sense for $\phi\in\mathrm{L}^4(\Omega)$ and $\mathbf{s}\in\mathbf{L}^2(\Omega)$. In turn, the remaining terms of this equation are well-defined if both $\mathbf{t}$ and $\boldsymbol{\eta}$ belong to $\mathbf{L}^2(\Omega)$ as well. Furthermore, in order to handle the Dirichlet boundary condition for ϕ we assume originally that $\phi\in\mathrm{H}^1(\Omega)$, which is certainly contained in $\mathrm{L}^4(\Omega)$, and multiply the first equation of the second row of (1.7) against $\boldsymbol{\chi}\in\mathbf{H}(\mathrm{div}_t;\Omega)$ (cf. (1.4)), with $t\in(1,+\infty)$ to be chosen. In this way, integrating by parts the term involving $\nabla\phi$, and using that $\phi = 0$ on Γ, we obtain

$$-\int_\Omega \mathbf{t}\cdot\boldsymbol{\chi} \,-\, \int_\Omega \phi\,\mathrm{div}(\boldsymbol{\chi}) \,=\, 0\,. \tag{2.22}$$

It is clear that the first term of (2.22) is well-defined, whereas the second one makes sense even if we look for ϕ in $\mathrm{L}^4(\Omega)$, and choose $t = 4/3$ so that $\mathrm{div}(\boldsymbol{\chi}) \in \mathrm{L}^{4/3}(\Omega)$, and hence $\boldsymbol{\chi} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)$. Moreover, seeking $\boldsymbol{\eta}$ in this very same space, and assuming here that $g \in \mathrm{L}^{4/3}(\Omega)$, we can test the momentum equation (cf. third equation of the second row of (1.7)) against $\psi \in \mathrm{L}^4(\Omega)$, thus yielding

$$\int_\Omega \varrho\,\phi\,\psi \,-\, \int_\Omega \psi\,\mathrm{div}(\boldsymbol{\eta}) \,=\, \int_\Omega g\,\psi\,. \tag{2.23}$$

Consequently, given $\mathbf{u} \in \mathbf{L}^4(\Omega)$, and gathering (2.21), (2.22), and (2.23) we arrive at the following mixed formulation for the concentration: Find $(\phi,\mathbf{t},\boldsymbol{\eta}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ such that

$$\begin{array}{rcll} \mathcal{A}_{\phi,\mathbf{u}}\big((\phi,\mathbf{t}),(\psi,\mathbf{s})\big) \,+\, \mathcal{B}\big((\psi,\mathbf{s}),\boldsymbol{\eta}\big) & = & \mathcal{F}_\phi\big((\psi,\mathbf{s})\big) & \forall\,(\psi,\mathbf{s}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)\,, \\ \mathcal{B}\big((\phi,\mathbf{t}),\boldsymbol{\chi}\big) & = & 0 & \forall\,\boldsymbol{\chi} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,, \end{array} \tag{2.24}$$

where $\mathcal{A}_{\varphi,\mathbf{w}} : \big(\mathrm{L}^4(\Omega) \times \mathrm{L}^2(\Omega)\big) \times \big(\mathrm{L}^4(\Omega) \times \mathrm{L}^2(\Omega)\big) \to \mathrm{R}$, for each $(\varphi,\mathbf{w}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$, and $\mathcal{B} : \big(\mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)\big) \times \mathbf{H}(\mathrm{div}_{4/3};\Omega) \to \mathrm{R}$, are the bilinear forms defined by

$$\mathcal{A}_{\varphi,\mathbf{w}}\big((\phi,\mathbf{t}),(\psi,\mathbf{s})\big) := \int_\Omega \vartheta(\varphi)\,\mathbf{t}\cdot\mathbf{s} \,-\, \int_\Omega \phi\,\mathbf{w}\cdot\mathbf{s} \,+\, \int_\Omega \varrho\,\phi\,\psi\,, \tag{2.25}$$

$$\mathcal{B}\big((\psi,\mathbf{s}),\boldsymbol{\chi}\big) := -\int_\Omega \boldsymbol{\chi}\cdot\mathbf{s} \,-\, \int_\Omega \psi\,\mathrm{div}(\boldsymbol{\chi})\,, \tag{2.26}$$

for all $(\phi,\mathbf{t}),(\psi,\mathbf{s}) \in (\mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega))$ and for all $\boldsymbol{\chi} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, whereas, given $\varphi \in \mathrm{L}^4(\Omega)$, we let $\mathcal{F}_\varphi : \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \to \mathrm{R}$ be the linear functional given by

$$\mathcal{F}_\varphi\big((\psi,\mathbf{s})\big) \,:=\, \int_\Omega f_{\mathrm{bk}}(\varphi)\,\boldsymbol{k}\cdot\mathbf{s} \,+\, \int_\Omega g\,\psi \qquad \forall\,(\psi,\mathbf{s}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)\,. \tag{2.27}$$

Bearing in mind the range of ϑ (cf. (1.2)), and applying the continuous injection of $\mathrm{L}^4(\Omega)$ into $\mathrm{L}^2(\Omega)$ (with boundedness constant $|\Omega|^{1/4}$) and the Cauchy–Schwarz and Hölder inequalities, we deduce that

$$\begin{array}{rcl} \big|\mathcal{A}_{\varphi,\mathbf{w}}\big((\phi,\mathbf{t}),(\psi,\mathbf{s})\big)\big| & \leqslant & \big(\vartheta_2 \,+\, \|\mathbf{w}\|_{0,4;\Omega} \,+\, |\Omega|^{1/2}\,\varrho\big)\,\|(\phi,\mathbf{t})\|\,\|(\psi,\mathbf{s})\|\,, \quad \text{and} \\ \big|\mathcal{B}\big((\psi,\mathbf{s}),\boldsymbol{\chi}\big)\big| & \leqslant & \|(\psi,\mathbf{s})\|\,\|\boldsymbol{\chi}\|_{\mathrm{div}_{4/3};\Omega}\,, \end{array} \tag{2.28}$$

which says that $\mathcal{A}_{\varphi,\mathbf{w}}$ and $\mathcal{B}$ are bounded with

$$\|\mathcal{A}_{\varphi,\mathbf{w}}\| \,\leqslant\, \vartheta_2 \,+\, \|\mathbf{w}\|_{0,4;\Omega} \,+\, |\Omega|^{1/2}\,\varrho \quad \text{and} \quad \|\mathcal{B}\| \,\leqslant\, 1\,. \tag{2.29}$$

In turn, considering the upper bound of f_{bk} (cf. (1.2)), and applying the Cauchy–Schwarz and Hölder inequalities, we have that

$$\|\mathcal{F}_\varphi\| \,\leqslant\, \gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,4/3;\Omega}\,. \tag{2.30}$$

In addition, it is important to note that the change in the space chosen for ϕ slightly modifies the bound of the operator $\mathbf{G}_\phi$ (cf. (2.17) and (2.18)). In fact, in this case we simply obtain that

$$\|\mathbf{G}_\phi\| \,\leqslant\, \|\boldsymbol{f}\|_{0,\Omega}\,\|\phi\|_{0,4;\Omega}\,. \tag{2.31}$$

Summarizing, the fully–mixed formulation for the coupled system (1.7) reduces to (2.19) and (2.24), that is: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$ and $(\phi, \mathbf{t}, \boldsymbol{\eta}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}(\mathrm{div}_{4/3}; \Omega)$ such that

$$\begin{aligned} \mathbf{a}_\phi(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) \;&=\; \mathbf{F}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)\,, \\ \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) \,-\, \mathbf{c}(\mathbf{u}, \mathbf{v}) \;&=\; \mathbf{G}_\phi(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,, \\ \mathcal{A}_{\phi,\mathbf{u}}\big((\phi, \mathbf{t}), (\psi, \mathbf{s})\big) \,+\, \mathcal{B}\big((\psi, \mathbf{s}), \boldsymbol{\eta}\big) \;&=\; \mathcal{F}_\phi\big((\psi, \mathbf{s})\big) && \forall\, (\psi, \mathbf{s}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)\,, \\ \mathcal{B}\big((\phi, \mathbf{t}), \boldsymbol{\chi}\big) \;&=\; 0 && \forall\, \boldsymbol{\chi} \in \mathbf{H}(\mathrm{div}_{4/3}; \Omega)\,. \end{aligned} \tag{2.32}$$

3 The continuous analysis

In this section we prove the well-posedness of the primal–mixed and fully–mixed formulations derived in Section 2.

3.1 The mixed–primal approach

In what follows we proceed similarly as in [4] (see also [3, 5, 7]), and employ a fixed-point strategy to analyze the solvability of the mixed–primal formulation (2.20). To this end, we first introduce the operator $\bar{\mathbf{S}} : \mathrm{H}_0^1(\Omega) \to \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$ given by

$$\bar{\mathbf{S}}(\varphi) \;=\; \big(\bar{\mathbf{S}}_1(\varphi), \bar{\mathbf{S}}_2(\varphi)\big) \;:=\; (\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}) \qquad \forall\, \varphi \in \mathrm{H}_0^1(\Omega)\,, \tag{3.1}$$

where $(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$ is the unique solution (to be confirmed below) of the problem arising from (2.19) when ϕ is replaced by the given φ, that is

$$\begin{aligned} \mathbf{a}_\varphi(\bar{\boldsymbol{\sigma}}, \boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau}, \bar{\mathbf{u}}) \;&=\; \mathbf{F}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)\,, \\ \mathbf{b}(\bar{\boldsymbol{\sigma}}, \mathbf{v}) \,-\, \mathbf{c}(\bar{\mathbf{u}}, \mathbf{v}) \;&=\; \mathbf{G}_\varphi(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,. \end{aligned} \tag{3.2}$$

In turn, we let $\widetilde{\mathbf{S}} : \mathrm{H}_0^1(\Omega) \times \mathbf{L}^4(\Omega) \to \mathrm{H}_0^1(\Omega)$ be the operator defined by

$$\widetilde{\mathbf{S}}(\varphi, \mathbf{w}) \;:=\; \widetilde{\phi} \qquad \forall\, (\varphi, \mathbf{w}) \,\in\, \mathrm{H}_0^1(\Omega) \times \mathbf{L}^4(\Omega)\,, \tag{3.3}$$

where $\widetilde{\phi} \in \mathrm{H}_0^1(\Omega)$ is the unique solution (to be confirmed below) of the problem obtained from (2.3) when $\mathbf{u}$, $\vartheta(\phi)$, and G_ϕ are replaced by $\mathbf{w}$, $\vartheta(\varphi)$, and G_φ, respectively, that is

$$A_{\varphi,\mathbf{w}}(\widetilde{\phi}, \widetilde{\psi}) \;=\; G_\varphi(\widetilde{\psi}) \qquad \forall\, \widetilde{\psi} \in \mathrm{H}_0^1(\Omega)\,, \tag{3.4}$$

where $A_{\varphi,\mathbf{w}} : \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega) \to \mathrm{R}$ is the bilinear form

$$A_{\varphi,\mathbf{w}}(\widetilde{\phi}, \widetilde{\psi}) \;:=\; \int_\Omega \vartheta(\varphi)\, \nabla\widetilde{\phi} \cdot \nabla\widetilde{\psi} \,-\, \int_\Omega \widetilde{\phi}\, \mathbf{w} \cdot \nabla\widetilde{\psi} \,+\, \int_\Omega \varrho\, \widetilde{\phi}\, \widetilde{\psi} \qquad \forall\, \widetilde{\phi}, \widetilde{\psi} \in \mathrm{H}_0^1(\Omega)\,, \tag{3.5}$$

and G_φ is the functional defined in (2.5), that is

$$G_\varphi(\widetilde{\psi}) \;:=\; \int_\Omega f\!\!\!\!\,\mathrm{bk}(\varphi)\, \boldsymbol{k} \cdot \nabla\widetilde{\psi} \,+\, \int_\Omega g\, \widetilde{\psi} \qquad \forall\, \widetilde{\psi} \in \mathrm{H}_0^1(\Omega)\,. \tag{3.6}$$

Then, we define the operator $\mathbf{S} : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ by

$$\mathbf{S}(\varphi) \;:=\; \widetilde{\mathbf{S}}(\varphi, \bar{\mathbf{S}}_2(\varphi)) \qquad \forall\, \varphi \,\in\, \mathrm{H}_0^1(\Omega)\,, \tag{3.7}$$

and realize that solving (2.20) is equivalent to seeking a fixed point of $\mathbf{S}$, that is, $\phi \in \mathrm{H}_0^1(\Omega)$ such that

$$\mathbf{S}(\phi) \,=\, \phi\,. \tag{3.8}$$

We find it important to emphasize here that, unlike (3.4), the approach in [3, eqs. (3.11)-(3.12)] and [4, eq. (3.12)] considered

$$A_{\mathbf{w}}(\widetilde{\phi},\widetilde{\psi}) \,=\, G_{\varphi}(\widetilde{\psi}) \qquad \forall\,\widetilde{\psi}\in \mathrm{H}_0^1(\Omega)\,,$$

where $A_{\mathbf{w}}$ is defined in (2.4), thus keeping the diffusivity as part of the nonlinearity. However, in the present setting, and for sake of simplicity of the subsequent analysis, we restrict ourselves to the linear problem (3.4).

Our next aim is to show that the operators $\bar{\mathbf{S}}$ and $\widetilde{\mathbf{S}}$ are well-defined, which is equivalent to proving that the uncoupled problems (3.2) and (3.4), respectively, are well-posed. We begin with the linear problem (3.2), for which we apply the abstract result [23, Theorem 3.4], which establishes sufficient conditions for the well-posedness of perturbed saddle-point problems. Indeed, we first recall from (2.15) and (2.16) that the bilinear forms $\mathbf{a}_\varphi$, $\mathbf{b}$, and $\mathbf{c}$ (cf. (2.13)) are all bounded, and that $\mathbf{a}_\varphi$ and $\mathbf{c}$ are both symmetric and positive semi-definite, thus accomplishing assumption i) of [23, Theorem 3.4]. In turn, denoting by $\mathbb{V}$ the kernel of the bilinear form $\mathbf{b}$, and using that $\mathbf{L}^4(\Omega)$ is isomorphic to the dual space of $\mathbf{L}^{4/3}(\Omega)$, we readily find that

$$\mathbb{V} \,=\, \Big\{\boldsymbol{\tau}\in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) \,=\, 0 \quad \text{in} \quad \Omega\Big\}\,. \tag{3.9}$$

Then, we recall from [7, Lemma 3.1] that there exists a positive constant α, depending only on Ω and μ_2, such that for each $\phi\in \mathrm{H}_0^1(\Omega)$ there holds

$$\mathbf{a}_\phi(\boldsymbol{\tau},\boldsymbol{\tau}) \,\geqslant\, \alpha\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \quad \forall\,\boldsymbol{\tau}\in\mathbb{V}\,, \tag{3.10}$$

which, in particular, yields assumption ii) of [23, Theorem 3.4]. We stress that actually (3.10) is satisfied independently of the space where ϕ belongs, so that it also holds for $\phi\in \mathrm{L}^4(\Omega)$ (see [8, eq. (3.23)]). Now, regarding the inf-sup condition for $\mathbf{b}$, we refer to [13, Lemma 3.3], which states in fact the existence of a positive constant β, depending on n, c_P (cf. (2.1)) and $\|\mathrm{i}_4\|$ (cf. (2.2)), such that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}} \,\geqslant\, \beta\,\|\mathbf{v}\|_{0,4;\Omega} \quad \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,, \tag{3.11}$$

thus confirming assumption iii) of [23, Theorem 3.4].

Summarizing, we are now able to prove the well-definedness of the operator $\bar{\mathbf{S}}$.

Lemma 3.1 *For each $\varphi\in \mathrm{H}_0^1(\Omega)$ problem (3.2) has a unique solution $(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}})\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)$, and hence we can define $\bar{\mathbf{S}}(\varphi) := (\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}})$. Moreover, there exists a positive constant $C_{\bar{\mathbf{S}}}$, depending on μ_1, $\|\mathrm{i}_4\|$, $c(\Omega)$, $\|\mathbf{i}_{4/3}\|$, $\big\|\mathbf{K}^{-1}\big\|_{\infty,\Omega}$, α, and β, and hence independent of φ, such that*

$$\|\bar{\mathbf{S}}(\varphi)\| \,=\, \|(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}})\| \,\leqslant\, C_{\bar{\mathbf{S}}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\,\|\varphi\|_{1,\Omega}\Big\} \quad \forall\,\varphi\in \mathrm{H}_0^1(\Omega)\,. \tag{3.12}$$

Proof. According to the previous discussion, the proof follows from a straightforward application of [23, Theorem 3.4]. In particular, there exists a positive constant C, depending only on $\|\mathbf{a}_\phi\|$, $\|\mathbf{c}\|$, α, and β, such that

$$\|(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}})\| \,\leqslant\, C\Big\{\|\mathbf{F}\| \,+\, \|\mathbf{G}_\phi\|\Big\}\,. \tag{3.13}$$

Thus, the above inequality, along with (2.18), lead to (3.12), which finishes the proof. $\square$

We now state the well-posedness of (3.4), or equivalently, the well-definedness of operator $\widetilde{\mathbf{S}}$. In fact, we recall from [4, Lemma 3.4] the following result.

Lemma 3.2 *Let $\varphi \in \mathrm{H}_0^1(\Omega)$ and $\mathbf{w} \in \mathbf{L}^4(\Omega)$ such that $\|\mathbf{w}\|_{0,4;\Omega} \leqslant \frac{\vartheta_1}{2\,c_P\,\|\mathrm{i}_4\|}$ (cf. (1.2), (2.1), (2.8)). Then, problem (3.4) has a unique solution $\widetilde{\phi} \in \mathrm{H}_0^1(\Omega)$, whence we can define $\widetilde{\mathbf{S}}(\varphi, \mathbf{w}) := \widetilde{\phi}$. Moreover, letting $C_{\widetilde{\mathbf{S}}} := \frac{2\,c_P^2}{\vartheta_1}$, which is independent of $(\varphi, \mathbf{w})$, there holds*

$$\|\widetilde{\mathbf{S}}(\varphi, \mathbf{w})\|_{1,\Omega} \,=\, \|\widetilde{\phi}\|_{1,\Omega} \,\leqslant\, C_{\widetilde{\mathbf{S}}} \left\{ \gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,\Omega} \right\}. \tag{3.14}$$

At this point we highlight that within the proof of the previous lemma (cf. [4, Lemma 3.4]) it is shown that

$$A_{\varphi,\mathbf{w}}(\widetilde{\psi}, \widetilde{\psi}) \,\geqslant\, \alpha_A\,\|\widetilde{\psi}\|_{1,\Omega}^2 \qquad \forall\,\widetilde{\psi} \in \mathrm{H}_0^1(\Omega)\,, \tag{3.15}$$

with $\alpha_A := \frac{1}{C_{\widetilde{\mathbf{S}}}} = \frac{\vartheta_1}{2\,c_P^2}$, inequality that will prove useful for the subsequent discussion.

Having established that $\bar{\mathbf{S}}$ and $\widetilde{\mathbf{S}}$ are well-defined, the operator $\mathbf{S}$ is as well, and hence we address next the solvability of the fixed-point equation (3.8). To this end, we will apply the Schauder fixed-point theorem (see, e.g. [19, Theorem 9.12-1(b)]), and the classical Banach theorem.

We begin by letting B be the closed ball of $\mathrm{H}_0^1(\Omega)$ with a given radius r, that is

$$B \,:=\, \left\{ \varphi \in \mathrm{H}_0^1(\Omega) : \quad \|\varphi\|_{1,\Omega} \,\leqslant\, r \right\}. \tag{3.16}$$

Then, we have the following result.

Lemma 3.3 *Assume that the data satisfy*

$$C_{\bar{\mathbf{S}}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, r\,\|\boldsymbol{f}\|_{0,\Omega} \right\} \,\leqslant\, \frac{\vartheta_1}{2\,c_P\,\|\mathrm{i}_4\|}\,, \quad \text{and} \tag{3.17}$$

$$C_{\widetilde{\mathbf{S}}} \left\{ \gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,\Omega} \right\} \,\leqslant\, r\,. \tag{3.18}$$

Then $\mathbf{S}(B) \subseteq B$.

Proof. The proof is carried out by mimicking the steps of the proof of [7, Lemma 3.7]. Indeed, it basically follows from the definition of $\mathbf{S}$ (cf. (3.7)), the a priori estimates (3.12) and (3.14), and the assumptions (3.17) and (3.18). Further details are omitted. $\square$

Our objective now is to prove that the fixed-point operator $\mathbf{S}$ is continuous, for which, according to its definition (3.7), it suffices to prove that $\bar{\mathbf{S}}$ and $\widetilde{\mathbf{S}}$ are. We begin with the corresponding result for $\bar{\mathbf{S}}$, which will require a further regularity assumption. This kind of hypotheses were originally introduced in [3, eq. (3.22)] and have been since utilized frequently, in its original or slightly modified forms, in subsequent works (see, e.g. [4, 7, 8]). More precisely, we suppose that $\mathbf{u}_D \in \mathbf{H}^{1/2+\varepsilon}(\Gamma)$ for some $\varepsilon \in (0,1)$ (when $n = 2$) or $\varepsilon \in (\frac{1}{2}, 1)$ (when $n = 3$), and that for each $\varphi \in \mathrm{H}_0^1(\Omega)$ with $\|\varphi\|_{1,\Omega} \leqslant r$, $r > 0$ given, there holds $\bar{\mathbf{S}}(\varphi) := (\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}) \in \big(\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,\cap\, \mathbb{H}^{\varepsilon}(\Omega)\big) \times \mathbf{W}^{\varepsilon,4}(\Omega)$, and

$$\|\bar{\boldsymbol{\sigma}}\|_{\varepsilon,\Omega} \,+\, \|\bar{\mathbf{u}}\|_{\varepsilon,4;\Omega} \,\leqslant\, \bar{C}_{\bar{\mathbf{S}}}(r) \left\{ \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega} \right\}, \tag{3.19}$$

with a constant $\bar{C}_{\bar{\mathbf{S}}}(r) > 0$, independent of φ, but depending on the upper bound r of its H^1-norm.

Having introduced the above assumption, and for further use along the paper, we now need to consider the continuous injections $\mathrm{i}_\varepsilon : \mathrm{H}^1(\Omega) \to \mathrm{L}^{n/\varepsilon}(\Omega)$ and $\mathrm{i}_{\widetilde{\varepsilon}} : \mathrm{H}^{\varepsilon}(\Omega) \to \mathrm{L}^{\widetilde{\varepsilon}}(\Omega)$, where

$$\widetilde{\varepsilon} \,:=\, \begin{cases} \frac{2}{1-\varepsilon}, & \text{if } n = 2\,, \\ \frac{6}{3-2\varepsilon}, & \text{if } n = 3\,. \end{cases} \tag{3.20}$$

Then, the continuity of $\bar{\mathbf{S}}$ is established as follows.

Lemma 3.4 *There exists a positive constant $L_{\bar{\mathbf{S}}}$, depending on μ_1, L_μ, $\|\boldsymbol{K}^{-1}\|_{\infty,\Omega}$, $\|\mathbf{i}_{4/3}\|$, $\|\mathrm{i}_4\|$, $\|\mathrm{i}_\varepsilon\|$, $\|\mathrm{i}_{\tilde{\varepsilon}}\|$, α, and β, (cf. (1.2), (1.3), (2.16), (3.10), and (3.11)), such that for all φ, $\psi \in \mathrm{H}^1_0(\Omega)$ there holds*

$$\|\bar{\mathbf{S}}(\varphi) \,-\, \bar{\mathbf{S}}(\psi)\| \,\leqslant\, L_{\bar{\mathbf{S}}} \left\{ \|\bar{\mathbf{S}}_1(\psi)\|_{\varepsilon,\Omega} \, \|\varphi \,-\, \psi\|_{0,n/\varepsilon;\Omega} \,+\, \|\boldsymbol{f}\|_{0,\Omega} \, \|\varphi \,-\, \psi\|_{0,4;\Omega} \right\}. \tag{3.21}$$

Proof. The proof is an adaptation of the one of [7, Lemma 3.7] to the present context of the perturbed saddle-point problem (3.2). First note that, letting $\mathbf{X} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$, the a priori estimate (3.13) for the solution of (3.2) with a given $\varphi \in \mathrm{H}^1_0(\Omega)$, is equivalent to the existence of a positive constant $\bar{C}$, depending only on $\|\mathbf{a}_\varphi\| = \frac{1}{\mu_1}$, $\|\mathbf{c}\| = \|\mathbf{i}_{4/3}\| \, \|\boldsymbol{K}^{-1}\|_{\infty,\Omega}$, α, and β, and hence independent of φ, such that there holds the global inf-sup condition (cf. [23, eq. (3.33)])

$$\|(\boldsymbol{\rho},\mathbf{z})\| \,\leqslant\, \bar{C} \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in\mathbf{X}\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{\mathbf{a}_\varphi(\boldsymbol{\rho},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},\mathbf{z}) \,+\, \mathbf{b}(\boldsymbol{\rho},\mathbf{v}) \,-\, \mathbf{c}(\mathbf{z},\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|} \qquad \forall\, (\boldsymbol{\rho},\mathbf{z}) \in \mathbf{X}\,, \tag{3.22}$$

Next, given φ, $\psi \in \mathrm{H}^1_0(\Omega)$, we let $(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}}) = \bar{\mathbf{S}}(\varphi)$ and $(\bar{\boldsymbol{\zeta}},\bar{\mathbf{w}}) = \bar{\mathbf{S}}(\psi)$, so that, according to the definition of $\bar{\mathbf{S}}$ (cf. Section (3.7)), they satisfy

$$\begin{aligned} \mathbf{a}_\varphi(\bar{\boldsymbol{\sigma}},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},\bar{\mathbf{u}}) &= \mathbf{F}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \\ \mathbf{b}(\bar{\boldsymbol{\sigma}},\mathbf{v}) \,-\, \mathbf{c}(\bar{\mathbf{u}},\mathbf{v}) &= \mathbf{G}_\varphi(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} \mathbf{a}_\psi(\bar{\boldsymbol{\zeta}},\boldsymbol{\tau}) \,+\, \mathbf{b}(\boldsymbol{\tau},\bar{\mathbf{w}}) &= \mathbf{F}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \\ \mathbf{b}(\bar{\boldsymbol{\zeta}},\mathbf{v}) \,-\, \mathbf{c}(\bar{\mathbf{w}},\mathbf{v}) &= \mathbf{G}_\psi(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,. \end{aligned} \tag{3.24}$$

Then, applying (3.22) to $(\boldsymbol{\rho},\mathbf{z}) = \bar{\mathbf{S}}(\varphi) - \bar{\mathbf{S}}(\psi) = (\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\zeta}}, \bar{\mathbf{u}} - \bar{\mathbf{w}})$, and then employing (3.23), (3.24), and the definitions of $\mathbf{a}_\varphi$, $\mathbf{a}_\psi$, $\mathbf{G}_\varphi$, and $\mathbf{G}_\psi$ (cf. (2.13), (2.14)), we arrive at

$$\begin{aligned} \|\bar{\mathbf{S}}(\varphi) - \bar{\mathbf{S}}(\psi)\| \,&\leqslant\, \bar{C} \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{\mathbf{a}_\varphi(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\zeta}},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\bar{\mathbf{u}} - \bar{\mathbf{w}}) + \mathbf{b}(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\zeta}},\mathbf{v}) - \mathbf{c}(\bar{\mathbf{u}} - \bar{\mathbf{w}},\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|} \\ &=\, \bar{C} \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{\mathbf{a}_\psi(\bar{\boldsymbol{\zeta}},\boldsymbol{\tau}) - \mathbf{a}_\varphi(\bar{\boldsymbol{\zeta}},\boldsymbol{\tau}) + \mathbf{b}(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\zeta}},\mathbf{v}) - \mathbf{c}(\bar{\mathbf{u}} - \bar{\mathbf{w}},\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|} \\ &=\, \bar{C} \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \left(\frac{\mu(\varphi) - \mu(\psi)}{\mu(\psi)\mu(\varphi)} \right) \bar{\boldsymbol{\zeta}}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} - \int_\Omega (\varphi - \psi)\, \boldsymbol{f} \cdot \mathbf{v}}{\|(\boldsymbol{\tau},\mathbf{v})\|}\,, \end{aligned} \tag{3.25}$$

whose right-hand side coincides precisely with the one of [7, eq. (3.44)]. For the rest of the proof we refer to [7, Lemma 3.7], in which the further regularity assumption (3.19) and the continuous injections $\mathrm{i}_4 : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$, $\mathrm{i}_\varepsilon : \mathrm{H}^1(\Omega) \to \mathrm{L}^{n/\varepsilon}(\Omega)$, and $\mathrm{i}_{\tilde{\varepsilon}} : \mathrm{H}^\varepsilon(\Omega) \to \mathrm{L}^{\tilde{\varepsilon}}(\Omega)$, are used. □

Next, for the continuity of the operator $\widetilde{\mathbf{S}}$, and slightly adapting [4, eq. (3.24)], we also require further regularity. More precisely, we assume that $g \in \mathrm{H}^\varepsilon(\Omega)$, for the same $\varepsilon \in (0,1)$ (when $n = 2$) or $\varepsilon \in (\frac{1}{2},1)$ (when $n = 3$) as in (3.19), and that for each $(\varphi,\mathbf{w}) \in \mathrm{H}^1_0(\Omega) \times \mathbf{L}^4(\Omega)$, with $\|\varphi\|_{1,\Omega} + \|\mathbf{w}\|_{0,4;\Omega} \leqslant r$, $r > 0$ given, there holds $\widetilde{\mathbf{S}}(\varphi,\mathbf{w}) := \widetilde{\phi} \in \mathrm{H}^{1+\varepsilon}_0(\Omega)$, and

$$\|\widetilde{\phi}\|_{1+\varepsilon,\Omega} \,\leqslant\, \widetilde{C}_{\widetilde{\mathbf{S}}}(r) \left\{ \gamma_f \, |\Omega|^{1/2} \, |\boldsymbol{k}| \,+\, \|g\|_{\varepsilon,\Omega} \right\}, \tag{3.26}$$

with a constant $\widetilde{C}_{\widetilde{\mathbf{S}}}(r) > 0$, independent of $(\varphi, \mathbf{w})$, but depending on the upper bound r.

At this point we find it important to mention that, unlike [4, Lemma 3.7] and the lemma to be stated next, the proof of the continuity of a similar operator $\widetilde{\mathbf{S}}$ in [3, 7, 8] does not require any extra regularity assumption. The reason is that in those works the diffusivity ϑ depends on the magnitude of the concentration gradient, which allows to handle the whole nonlinear term involved by using tools from the monotone operators theory, whereas in our present context the dependence is only on the concentration, which makes the proof much more intricate.

The continuity of the operator $\widetilde{\mathbf{S}}$ is then given by the following lemma.

Lemma 3.5 *There exists a positive constant $L_{\widetilde{\mathbf{S}}}$, depending on $C_{\widetilde{\mathbf{S}}}$, $\|\mathrm{i}_4\|$, $\|\mathrm{i}_\varepsilon\|$, L_f, and L_ϑ (cf. Lemma 3.2, (2.8), (1.3), (3.26)), such that for all $(\varphi, \mathbf{w})$, $(\psi, \mathbf{z}) \in \mathrm{H}^1_0(\Omega) \times \mathbf{L}^4(\Omega)$ satisfying $\|\mathbf{w}\|_{0,4;\Omega}$, $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\vartheta_1}{2\, c_P\, \|\mathrm{i}_4\|}$, there holds*

$$\begin{aligned}\|\widetilde{\mathbf{S}}(\varphi, \mathbf{w}) - \widetilde{\mathbf{S}}(\psi, \mathbf{z})\|_{1,\Omega} \leqslant L_{\widetilde{\mathbf{S}}} \Big\{ &|\boldsymbol{k}|\, \|\varphi - \psi\|_{0,\Omega} + \|\widetilde{\mathbf{S}}(\psi, \mathbf{z})\|_{1,\Omega}\, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \\ &+ \|\widetilde{\mathbf{S}}(\psi, \mathbf{z})\|_{1+\varepsilon,\Omega}\, \|\varphi - \psi\|_{0,n/\varepsilon;\Omega} \Big\}.\end{aligned} \tag{3.27}$$

Proof. The proof of this lemma is analogous to the one of [4, Lemma 3.7]. It relies on the ellipticity of the bilinear form $A_{\varphi,\mathbf{w}}$ with constant α_A (cf. (3.15)), the Lipschitz-continuity of $\boldsymbol{f}$ and ϑ (cf. (1.3)), the further regularity assumption on the operator $\widetilde{\mathbf{S}}$ (cf. (3.26)), and the Sobolev embeddings of $\mathrm{H}^1(\Omega)$ in $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^{n/\varepsilon}(\Omega)$; the only difference being that the term $\|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}$ is kept as it is instead of bounding it by $\|\mathbf{w} - \mathbf{z}\|_{1,\Omega}$ (and hence (2.2) is not used). □

Having established the continuity of $\bar{\mathbf{S}}$ and $\widetilde{\mathbf{S}}$, we are in position to prove next the continuity of the operator $\mathbf{S}$ and the compactness of $\overline{\mathbf{S}(B)}$, where B is the ball defined by (3.16).

Lemma 3.6 *Assume that the data satisfy the hypotheses of Lemma 3.3, that is (3.17) and (3.18). Then, there exists a positive constant $C_{\mathbf{S}}$, depending only on $L_{\bar{\mathbf{S}}}$ (cf. Lemma 3.4) and $L_{\widetilde{\mathbf{S}}}$ (cf. Lemma 3.5), such that*

$$\begin{aligned}\|\mathbf{S}(\varphi) - \mathbf{S}(\psi)\|_{1,\Omega} \leqslant C_{\mathbf{S}} \Big\{ &|\boldsymbol{k}| \|\varphi - \psi\|_{0,\Omega} + \|\boldsymbol{f}\|_{0,\Omega}\, \|\mathbf{S}(\psi)\|_{1,\Omega}\, \|\varphi - \psi\|_{0,4;\Omega} \\ &+ \Big(\|\mathbf{S}(\psi)\|_{1,\Omega}\, \|\bar{\mathbf{S}}_1(\psi)\|_{\varepsilon,\Omega} + \|\mathbf{S}(\psi)\|_{1+\varepsilon,\Omega} \Big)\, \|\varphi - \psi\|_{0,n/\varepsilon;\Omega} \Big\}\end{aligned} \tag{3.28}$$

for all φ, $\psi \in B$, and hence the operator $\mathbf{S} : B \to B$ is continuous and $\overline{\mathbf{S}(B)}$ is compact.

Proof. The proof follows similarly to those of [7, Lemma 3.9], [4, Lemma 3.9], and [3, Lemma 3.12]. We begin by recalling that the assumptions (3.17) and (3.18) guarantee that $\mathbf{S}(B) \subseteq B$. Then, bearing in mind the definition of $\mathbf{S}$ (cf. (3.7)), straightforward applications of the estimates provided by Lemmas 3.5 (cf. (3.27)) and 3.4 (cf. (3.21)) yield (3.28). In turn, thanks to the Rellich-Kondrachov compactness Theorem (cf. [1, Theorem 6.3], [30, Theorem 1.3.5]) and the ranges for ε specified by the regularity hypotheses (3.19) and (3.26), we know, as already used in the proofs of Lemmas 3.4 and 3.5, that $\mathrm{H}^1(\Omega)$ is compactly embedded in $\mathrm{L}^4(\Omega)$, $\mathrm{L}^2(\Omega)$, and $\mathrm{L}^{n/\varepsilon}(\Omega)$. These compact (and hence continuous) injections together with (3.28) imply the remaining properties of $\mathbf{S}$. □

Now, bounding $\|\mathbf{S}(\varphi)\|_{1,\Omega}$ by r, and $\|\bar{\mathbf{S}}_1(\varphi)\|_{\varepsilon,\Omega}$ and $\|\mathbf{S}(\varphi)\|_{1+\varepsilon,\Omega} = \|\widetilde{\mathbf{S}}\big(\varphi, \bar{\mathbf{S}}_2(\varphi)\big)\|_{1+\varepsilon,\Omega}$ by the estimates provided by (3.19) and (3.26), respectively, and using the aforementioned compact embeddings, we deduce from (3.28) the existence of a positive constant $L_{\mathbf{S}}$, depending on $C_{\mathbf{S}}$, $\bar{C}_{\widetilde{\mathbf{S}}}(r)$, $\widetilde{C}_{\widetilde{\mathbf{S}}}(r)$, r, γ_f, $|\Omega|$, $\|\mathrm{i}_4\|$, and $\|\mathrm{i}_\varepsilon\|$, such that

$$\|\mathbf{S}(\varphi) \,-\, \mathbf{S}(\psi)\|_{1,\Omega} \,\leqslant\, L_{\mathbf{S}} \Big\{ |\boldsymbol{k}| + \|\boldsymbol{f}\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} + \|g\|_{\varepsilon,\Omega} \Big\} \, \|\varphi \,-\, \psi\|_{1,\Omega} \tag{3.29}$$

for all φ, $\psi \in B$.

With the analysis already done, the solvability of (3.8) is addressed by the following theorem.

Theorem 3.7 *Assume that the data satisfy the hypotheses of Lemma 3.3, that is (3.17) and (3.18). Then, the fixed-point equation (3.8) has at least one solution $\phi \in B$, which means, equivalently, that the mixed–primal formulation (2.20) has at least one solution $(\boldsymbol{\sigma}, \mathbf{u}, \phi) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1_0(\Omega)$, with $\phi \in B$. Moreover, there hold*

$$\|\phi\|_{1,\Omega} \,\leqslant\, C_{\widetilde{\mathbf{S}}} \Big\{ \gamma_f \, |\Omega|^{1/2} \, |\boldsymbol{k}| \,+\, \|g\|_{0,\Omega} \Big\} \quad \textit{and} \tag{3.30}$$

$$\|(\boldsymbol{\sigma}, \mathbf{u})\| \,\leqslant\, C_{\bar{\mathbf{S}}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega} \, \|\phi\|_{1,\Omega} \Big\} \,. \tag{3.31}$$

In turn, if the data $\boldsymbol{k}$, $\boldsymbol{f}$, $\mathbf{u}_D$, and g are sufficiently small so that

$$L_{\mathbf{S}} \Big\{ |\boldsymbol{k}| + \|\boldsymbol{f}\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} + \|g\|_{\varepsilon,\Omega} \Big\} \,<\, 1 \,, \tag{3.32}$$

then the above solution of (3.8), and hence that of (2.20), is unique.

Proof. Similarly to the proof of [7, Theorem 3.10] (see, also [3, Theorem 3.13]), and thanks to Lemmas 3.3 and 3.6, we apply the Schauder fixed-point theorem to conclude the existence of solution of (3.8), and hence of (2.20). In addition, the estimates (3.30) and (3.31) follow directly from (3.14) and (3.12), respectively. Furthermore, (3.29) and the assumption (3.32) guarantee that $\mathbf{S}$ is a contraction, so that a straightforward application of Banach's fixed-point theorem completes the proof. □

3.2 The fully–mixed approach

Here we proceed inspired on [8, 28] to analyze the solvability of (2.32) by means also of a fixed-point strategy. In this regard, and given that the main difference between the mixed-primal formulation (2.20) and the fully-mixed formulation (2.32) lies on the way the transport equation is tackled, and hence on the space where the concentration ϕ is sought, the operator associated to the Brinkman flow (analogue of (3.1)) needs only its domain of definition to be modified. More precisely, we now let $\bar{\mathbf{T}} : \mathrm{L}^4(\Omega) \to \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ be the operator defined by

$$\bar{\mathbf{T}}(\varphi) \,=\, (\bar{\mathbf{T}}_1(\varphi), \bar{\mathbf{T}}_2(\varphi)) \,:=\, (\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}) \qquad \forall\, \varphi \in \mathrm{L}^4(\Omega) \,, \tag{3.33}$$

where $(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ is the unique solution (to be confirmed below) of the problem arising from (2.19) when ϕ is replaced by the given φ, that is (3.2). In turn, as the analogue of (3.3), we let $\widetilde{\mathbf{T}} : \mathrm{L}^4(\Omega) \times \mathbf{L}^4(\Omega) \to \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ be the operator defined by

$$\widetilde{\mathbf{T}}(\varphi, \mathbf{w}) \,=\, \big(\widetilde{\mathbf{T}}_1(\varphi, \mathbf{w}), \widetilde{\mathbf{T}}_2(\varphi, \mathbf{w}), \widetilde{\mathbf{T}}_3(\varphi, \mathbf{w})\big) \,:=\, (\widetilde{\phi}, \widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\eta}}) \qquad \forall\, (\varphi, \mathbf{w}) \,\in\, \mathrm{L}^4(\Omega) \times \mathbf{L}^4(\Omega) \,,$$

where $\big(\widetilde{\phi}, \widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\eta}}\big) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ is the unique solution (to be confirmed below) of the problem arising from (2.24) when $\mathcal{A}_{\phi,\mathbf{u}}$ and $\mathcal{F}_\phi$ are replaced by $\mathcal{A}_{\varphi,\mathbf{w}}$ and $\mathcal{F}_\varphi$, respectively, that is

$$\begin{aligned} \mathcal{A}_{\varphi,\mathbf{w}}\big((\widetilde{\phi}, \widetilde{\mathbf{t}}), (\psi, \mathbf{s})\big) \,+\, \mathcal{B}\big((\psi, \mathbf{s}), \widetilde{\boldsymbol{\eta}}\big) \quad &= \quad \mathcal{F}_\varphi\big((\psi, \mathbf{s})\big) \quad \forall\, (\psi, \mathbf{s}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \,, \\ \mathcal{B}\big((\widetilde{\phi}, \widetilde{\mathbf{t}}), \boldsymbol{\chi}\big) \quad &= \quad 0 \qquad\qquad\quad \forall\, \boldsymbol{\chi} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) \,. \end{aligned} \tag{3.34}$$

Then, we set the operator $\mathbf{T} : \mathrm{L}^4(\Omega) \to \mathrm{L}^4(\Omega)$ by

$$\mathbf{T}(\varphi) := \widetilde{\mathbf{T}}_1(\varphi, \bar{\mathbf{T}}_2(\varphi)) \qquad \forall\, \varphi \in \mathrm{L}^4(\Omega)\,, \tag{3.35}$$

and realize that solving (2.32) is equivalent to seeking a fixed point of $\mathbf{T}$, that is, $\phi \in \mathrm{L}^4(\Omega)$ such that

$$\mathbf{T}(\phi) = \phi\,. \tag{3.36}$$

Similarly as in Section 3.1, we now show that the operators $\bar{\mathbf{T}}$ and $\widetilde{\mathbf{T}}$ are well-defined, or equivalently, that the uncoupled problems (3.2) and (3.34) are well-posed. We start by presenting the corresponding result associated with $\bar{\mathbf{T}}$, which, given the already mentioned minor difference with $\bar{\mathbf{S}}$, turns out to be a small modification of Lemma 3.1.

Lemma 3.8 *For each $\varphi \in \mathrm{L}^4(\Omega)$ problem (3.2) has a unique solution $(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$, and hence we can define $\bar{\mathbf{T}}(\varphi) := (\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})$. Moreover, there exists a positive constant $C_{\bar{\mathbf{T}}}$, depending on μ_1, $c(\Omega)$, $\|\mathbf{i}_{4/3}\|$, $\left\|\mathbf{K}^{-1}\right\|_{\infty,\Omega}$, α, and β, and hence independent of φ, such that*

$$\|\bar{\mathbf{T}}(\varphi)\| = \|(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})\| \leqslant C_{\bar{\mathbf{T}}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega}\, \|\varphi\|_{0,4;\Omega} \Big\} \qquad \forall\, \varphi \in \mathrm{L}^4(\Omega)\,. \tag{3.37}$$

Proof. The proof follows almost verbatim from the one of Lemma 3.1, the main difference being the bounding of the functional $\mathbf{G}_\phi$, which in this case is given by (2.31) instead of (2.18). This explains the fact that the constant $C_{\bar{\mathbf{T}}}$ does not depend on $\|\mathrm{i}_4\|$. $\square$

In order to show that (3.34) is well-posed, equivalently, that the operator $\widetilde{\mathbf{T}}$ is well-defined, we apply the generalized Babuška–Brezzi theory (cf. [9, Theorem 2.1, Corollary 2.1]). Indeed, we begin by recalling from (2.29) and (2.30) that $\mathcal{A}_{\varphi,\mathbf{w}}$, $\mathcal{B}$, and F_φ are bounded. Next, proceeding similarly to [20, eq. (3.35)], we readily find that the null space of the bilinear form $\mathcal{B}$ (cf. (2.26)) reduces to

$$\widetilde{\mathbb{V}} = \Big\{ (\psi, \mathbf{s}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) : \quad \nabla\psi = \mathbf{s} \quad \text{in} \quad \Omega \quad \text{and} \quad \psi \in \mathrm{H}^1_0(\Omega) \Big\}\,. \tag{3.38}$$

Having the above, we now prove that for a suitable range of $\mathbf{w} \in \mathbf{L}^4(\Omega)$, $\mathcal{A}_{\varphi,\mathbf{w}}$ becomes $\widetilde{\mathbb{V}}$-elliptic.

Lemma 3.9 *There exists a positive constant $\alpha_{\mathcal{A}}$, depending only on ϑ_1, c_P, and $\|\mathrm{i}_4\|$ (cf. (1.2), (2.1), (2.2)), such that for each $(\varphi, \mathbf{w}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ satisfying $\|\mathbf{w}\|_{0,4;\Omega} \leqslant 2\,\alpha_{\mathcal{A}}$, there holds*

$$\mathcal{A}_{\varphi,\mathbf{w}}\big((\psi,\mathbf{s}),(\psi,\mathbf{s})\big) \geqslant \alpha_{\mathcal{A}}\, \|(\psi,\mathbf{s})\|^2 \qquad \forall\, (\psi,\mathbf{s}) \in \widetilde{\mathbb{V}}\,. \tag{3.39}$$

Proof. Given $(\varphi, \mathbf{w}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ and $(\psi, \mathbf{s}) \in \widetilde{\mathbb{V}}$, we obtain from the definition of $\mathcal{A}_{\varphi,\mathbf{w}}$ (cf. (2.25)), along with the lower bound of ϑ (cf. (1.2)), the Cauchy–Schwarz inequality twice, and the fact that $\|\mathbf{s}\|_{0,\Omega} = |\psi|_{1,\Omega}$ (cf. (3.38)), that

$$\begin{aligned}
\mathcal{A}_{\varphi,\mathbf{w}}\big((\psi,\mathbf{s}),(\psi,\mathbf{s})\big) &= \int_\Omega \vartheta(\varphi)\, |\mathbf{s}|^2 - \int_\Omega \psi\, \mathbf{w}\cdot\mathbf{s} + \varrho\, \|\psi\|^2_{0,\Omega} \\
&\geqslant \vartheta_1\, \|\mathbf{s}\|^2_{0,\Omega} - \|\psi\|_{0,4;\Omega}\, \|\mathbf{w}\|_{0,4;\Omega}\, \|\mathbf{s}\|_{0,\Omega} \\
&= \frac{\vartheta_1}{2}\, \|\mathbf{s}\|^2_{0,\Omega} + \frac{\vartheta_1}{2}\, |\psi|^2_{1,\Omega} - \|\psi\|_{0,4;\Omega}\, \|\mathbf{w}\|_{0,4;\Omega}\, \|\mathbf{s}\|_{0,\Omega}\,.
\end{aligned} \tag{3.40}$$

Next, applying Poincaré's inequality (cf. (2.1)), the continuity of $\mathrm{i}_4 : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ (cf. (2.2)), and Young's inequality, it readily follows from (3.40) that

$$\mathcal{A}_{\varphi,\mathbf{w}}\big((\psi,\mathbf{s}),(\psi,\mathbf{s})\big) \geqslant \frac{1}{2}\Big(\vartheta_1\, \min\big\{1, (c_P\, \|\mathrm{i}_4\|)^{-2}\big\} - \|\mathbf{w}\|_{0,4;\Omega} \Big)\, \|(\psi,\mathbf{s})\|^2\,,$$

from which, defining $\alpha_{\mathcal{A}} := \frac{\vartheta_1 \, \min\left\{1, (c_P \, \|i_4\|)^{-2}\right\}}{4}$, we arrive at (3.39) and conclude the proof. $\square$

In order to apply [9, Theorem 2.1, Corollary 2.1] it only remains to verify that the bilinear form $\mathcal{B}$ satisfies the continuous inf-sup condition. Indeed, this result was already established in [20] and reads as follows.

Lemma 3.10 *Letting $\beta_{\mathcal{B}} := \frac{1}{2}$, there holds*

$$\sup_{\substack{(\psi,\mathbf{s})\in \mathrm{L}^4(\Omega)\times \mathbf{L}^2(\Omega)\\ (\psi,\mathbf{s})\neq \mathbf{0}}} \frac{\mathcal{B}\big((\psi,\mathbf{s}),\boldsymbol{\chi}\big)}{\|(\psi,\mathbf{s})\|} \,\geqslant\, \beta_{\mathcal{B}}\, \|\boldsymbol{\chi}\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\chi} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,. \tag{3.41}$$

Proof. See [20, Lemma 3.3, ineq. (3.45)]. $\square$

We now establish that the linear problem (3.34) is well-posed, equivalently that $\widetilde{\mathbf{T}}$ is well-defined.

Lemma 3.11 *For each $(\varphi,\mathbf{w}) \in \mathrm{L}^4(\Omega)\times \mathbf{L}^4(\Omega)$ such that $\|\mathbf{w}\|_{0,4;\Omega} \,\leqslant\, 2\,\alpha_{\mathcal{A}}$, problem (3.34) has a unique solution $(\widetilde{\phi},\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\eta}}) \in \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, and hence we can define $\widetilde{\mathbf{T}}(\varphi,\mathbf{w}) := (\widetilde{\phi},\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\eta}})$. Moreover, there exists a positive constant $C_{\widetilde{\mathbf{T}}}$, depending on $\alpha_{\mathcal{A}}$, $\beta_{\mathcal{B}}$, ϑ_2, and ϱ, such that*

$$\|\widetilde{\mathbf{T}}(\varphi,\mathbf{w})\| \;=\; \|(\widetilde{\phi},\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\eta}})\| \,\leqslant\, C_{\widetilde{\mathbf{T}}}\left\{\gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,4/3;\Omega}\right\}. \tag{3.42}$$

Proof. Bearing in mind the aforementioned boundedness of $\mathcal{A}_{\varphi,\mathbf{w}}$, $\mathcal{B}$, and $\mathcal{F}_{\varphi}$, as well as Lemmas 3.9 and 3.10, the proof follows from a straightforward application of the generalized Babuška–Brezzi theory (cf. [9, Theorem 2.1, Corollary 2.1]). In particular, the indicated upper bound of $\mathbf{w}$ and the first inequality of (2.29) yield

$$\|\mathcal{A}_{\varphi,\mathbf{w}}\| \,\leqslant\, \vartheta_2 \,+\, 2\,\alpha_{\mathcal{A}} \,+\, |\Omega|^{1/2}\,\varrho \,=:\, \|\mathcal{A}\|\,, \tag{3.43}$$

which, along with the upper bounds of $\|\mathcal{B}\|$ (cf. second inequality of (2.29)) and $\|\mathcal{F}_{\varphi}\|$ (cf. (2.30)), and employing [9, eqs. (2.15) and (2.16), Corollary 2.1], imply (3.42) and finish the proof. $\square$

We remark that the well-posedness of the uncoupled problems (3.2), with $\varphi \in \mathrm{L}^4(\Omega)$ given, and (3.34), with $(\varphi,\mathbf{w}) \in \mathrm{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ given, confirm the well-definedness of the operators $\bar{\mathbf{T}}$ and $\widetilde{\mathbf{T}}$, respectively, and hence of $\mathbf{T}$ (cf. Section 3.2) as well. Therefore, we now address the solvability analysis of the fixed-point equation (3.36).

We begin by providing conditions under which $\mathbf{T}$ maps a ball into itself. To this end, given $r > 0$, we let W be the closed ball of $\mathrm{L}^4(\Omega)$ with radius r, that is

$$W \,:=\, \left\{\varphi \in \mathrm{L}^4(\Omega): \quad \|\varphi\|_{0,4;\Omega} \,\leqslant\, r\right\}.$$

Then, we have the following result.

Lemma 3.12 *Assume that the data satisfy*

$$C_{\bar{\mathbf{T}}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, r\,\|\boldsymbol{f}\|_{0,\Omega}\right\} \,\leqslant\, 2\,\alpha_{\mathcal{A}}\,, \quad \textit{and} \tag{3.44}$$

$$C_{\widetilde{\mathbf{T}}}\left\{\gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,4/3;\Omega}\right\} \,\leqslant\, r\,. \tag{3.45}$$

Then $\mathbf{T}(W) \subseteq W$.

Proof. Knowing from (3.35) and Lemma 3.11 that $\mathbf{T}(\varphi) := \widetilde{\mathbf{T}}_1(\varphi, \bar{\mathbf{T}}_2(\varphi))$ is well-defined for $\varphi \in \mathrm{L}^4(\Omega)$ if $\|\bar{\mathbf{T}}_2(\varphi)\| \leqslant 2\,\alpha_{\mathcal{A}}$, we deduce, thanks to the bound (3.37) (cf. Lemma 3.8) and the assumption (3.44), that the aforementioned well-definidness is accomplished for $\varphi \in W$. Then, the a priori estimate (3.42) (cf. Lemma 3.11) and the hypothesis (3.45) complete the proof. □

In what follows we aim to prove that the operator $\mathbf{T}$ is continuous, for which, similarly to what was done in Section 3.1, we first show that its constitutive operators $\bar{\mathbf{T}}$ and $\widetilde{\mathbf{T}}$ satisfy that property.

We begin with the corresponding result for $\bar{\mathbf{T}}$ by assuming, as for its counterpart $\bar{\mathbf{S}}$ (cf. Section 3.1), a further regularity assumption. In this regard, and even though these two aforementioned operators only differ in their domain of definition, the fact that the mixed approach (3.34) looks for the concentration in $\mathrm{L}^4(\Omega)$ instead of $\mathrm{H}_0^1(\Omega)$ as in the primal formulation (3.4), leads to a similar, but a bit more restrictive hypothesis. Indeed, we now need to assume that $\mathbf{u}_D \in \mathbf{H}^{1/2+\delta}(\Gamma)$ for some $\delta \in [1/2, 1)$ (when $n = 2$) or $\delta \in [3/4, 1)$ (when $n = 3$), and that for each $\varphi \in \mathrm{L}^4(\Omega)$ with $\|\varphi\|_{0,4;\Omega} \leqslant r$, $r > 0$ given, there holds $\bar{\mathbf{T}}(\varphi) := (\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}) \in \big(\mathbb{H}_0(\mathrm{div}_{4/3};\Omega) \cap \mathbb{H}^\delta(\Omega)\big) \times \mathbf{W}^{\delta,4}(\Omega)$, and

$$\|\bar{\boldsymbol{\sigma}}\|_{\delta,\Omega} + \|\bar{\mathbf{u}}\|_{\delta,4;\Omega} \leqslant \bar{C}_{\bar{\mathbf{T}}}(r) \Big\{ \|\mathbf{u}_D\|_{1/2+\delta,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega} \Big\}, \tag{3.46}$$

with a constant $\bar{C}_{\bar{\mathbf{T}}}(r) > 0$, independent of φ, but depending on the upper bound r of its $\mathrm{L}^4(\Omega)$-norm. As compared with the assumption for $\bar{\mathbf{S}}$ (cf. (3.19)), the only difference lies on the more demanding range for the present regularity index δ (which was denoted ε there). The need of it will become apparent next in the proof of Lemma 3.13, which establishes the continuity of $\bar{\mathbf{T}}$. In fact, the aforementioned range stipulates, equivalently, that $n/\delta \leqslant 4$, thus yielding a continuous injection i_δ of $\mathrm{L}^4(\Omega)$ into $\mathrm{L}^{n/\delta}(\Omega)$.

Lemma 3.13 *There exists a positive constant $L_{\bar{\mathbf{T}}}$, depending on $L_{\bar{\mathbf{S}}}$ (cf. Lemma 3.4) and $\|\mathrm{i}_\delta\|$, such that*

$$\|\bar{\mathbf{T}}(\varphi) - \bar{\mathbf{T}}(\psi)\| \leqslant L_{\bar{\mathbf{T}}} \Big\{ \|\bar{\mathbf{T}}_1(\psi)\|_{\delta,\Omega} + \|\boldsymbol{f}\|_{0,\Omega} \Big\} \|\varphi - \psi\|_{0,4;\Omega} \qquad \forall\, \varphi,\, \psi \in \mathrm{L}^4(\Omega)\,. \tag{3.47}$$

Proof. It is easily seen that the proof of Lemma 3.4 still holds for φ, $\psi \in \mathrm{L}^4(\Omega)$, instead of φ, $\psi \in \mathrm{H}_0^1(\Omega)$, whence we obtain again (3.21) as is, that is, using the present notation,

$$\|\bar{\mathbf{T}}(\varphi) - \bar{\mathbf{T}}(\psi)\| \leqslant L_{\bar{\mathbf{S}}} \Big\{ \|\bar{\mathbf{T}}_1(\psi)\|_{\delta,\Omega} \|\varphi - \psi\|_{0,n/\delta;\Omega} + \|\boldsymbol{f}\|_{0,\Omega} \|\varphi - \psi\|_{0,4;\Omega} \Big\}.$$

Then, bounding $\|\varphi - \psi\|_{0,n/\delta;\Omega}$ by $\|\mathrm{i}_\delta\| \, \|\varphi - \psi\|_{0,4;\Omega}$ in the foregoing inequality, we arrive at (3.47) with $L_{\bar{\mathbf{T}}} := L_{\bar{\mathbf{S}}} \max\big\{1, \|\mathrm{i}_\delta\|\big\}$. □

We find it important to remark here that, differently from $\bar{\mathbf{S}}$ (cf. Lemma 3.4), whose domain $\mathrm{H}_0^1(\Omega)$ is compactly embedded in $\mathrm{L}^4(\Omega)$, and thus in $\mathrm{L}^{n/\delta}(\Omega)$, in the present case of $\bar{\mathbf{T}}$, which acts on $\mathrm{L}^4(\Omega)$, we lack those compactness properties, whence later on we will not be able to apply Schauder theorem, as in Theorem 3.7, but just the classical Banach fixed-point theorem.

Furthermore, similarly to the operator $\widetilde{\mathbf{S}}$ from Section 3.1, we also require further regularity for $\widetilde{\mathbf{T}}$. More precisely, we assume that $g \in \mathrm{W}^{\delta,4/3}(\Omega)$, for the same $\delta \in [1/2, 1)$ (when $n = 2$) or $\delta \in [\frac{3}{4}, 1)$ (when $n = 3$), and that for each $(\varphi, \mathbf{w}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$, with $\|\varphi\|_{0,4;\Omega} + \|\mathbf{w}\|_{0,4;\Omega} \leqslant r$, $r > 0$ given, there holds $\widetilde{\mathbf{T}}(\varphi, \mathbf{w}) := (\widetilde{\phi}, \widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\eta}}) \in \mathrm{L}^4(\Omega) \times \mathbf{H}^\delta(\Omega) \times \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, and

$$\|\widetilde{\varphi}\|_{0,4;\Omega} + \|\widetilde{\mathbf{t}}\|_{\delta,\Omega} + \|\widetilde{\boldsymbol{\eta}}\|_{\mathrm{div}_{4/3};\Omega} \leqslant \widetilde{C}_{\widetilde{\mathbf{T}}}(r) \Big\{ \gamma_f \, |\Omega|^{1/2} \, |\boldsymbol{k}| + \|g\|_{\delta,4/3;\Omega} \Big\}, \tag{3.48}$$

with a constant $\widetilde{C}_{\widetilde{\mathbf{T}}}(r) > 0$, independent of $(\varphi, \mathbf{w})$, but depending on the upper bound r. Note here that, while we could have also considered further regularity for $\widetilde{\phi}$ and $\widetilde{\boldsymbol{\eta}}$, it suffices to assume it only for the second component of the operator $\widetilde{\mathbf{T}}$.

The continuity of the operator $\widetilde{\mathbf{T}}$ is stated as follows.

Lemma 3.14 *There exists a positive constant $L_{\widetilde{\mathbf{T}}}$, depending on $|\Omega|$, $\|\mathrm{i}_\delta\|$, δ, L_f and L_ϑ (cf. (1.3)), $\|\mathcal{A}\|$ (cf. (3.43)), $\alpha_{\mathcal{A}}$, and $\beta_{\mathcal{B}}$, such that for all $(\varphi, \mathbf{w})$, $(\psi, \mathbf{z}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ satisfying $\|\mathbf{w}\|_{0,4;\Omega}$, $\|\mathbf{z}\|_{0,4;\Omega} \leqslant 2\,\alpha_{\mathcal{A}}$, there holds*

$$\begin{aligned} \|\widetilde{\mathbf{T}}(\varphi, \mathbf{w}) \,-\, \widetilde{\mathbf{T}}(\psi, \mathbf{z})\| \,\leqslant\, L_{\widetilde{\mathbf{T}}} \Big\{ \big(|\boldsymbol{k}| + \|\widetilde{\mathbf{T}}_2(\psi, \mathbf{z})\|_{\delta,\Omega}\big) \, \|\varphi \,-\, \psi\|_{0,4;\Omega} \\ +\, \|\widetilde{\mathbf{T}}_1(\psi, \mathbf{z})\|_{0,4;\Omega} \, \|\mathbf{w} \,-\, \mathbf{z}\|_{0,4;\Omega} \Big\} . \end{aligned} \tag{3.49}$$

Proof. The proof follows similarly to that of [8, Lemma 3.9], which, in turn, makes use of some ideas employed in that of [4, Lemma 3.7]. We begin by letting $(\varphi, \mathbf{w})$, $(\psi, \mathbf{z}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ such that $\|\mathbf{w}\|_{0,4;\Omega}$, $\|\mathbf{z}\|_{0,4;\Omega} \leqslant 2\,\alpha_{\mathcal{A}}$, and defining $\widetilde{\mathbf{T}}(\varphi, \mathbf{w}) := (\widetilde{\phi}, \widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\eta}})$ and $\widetilde{\mathbf{T}}(\psi, \mathbf{z}) := (\widetilde{\theta}, \widetilde{\mathbf{r}}, \widetilde{\boldsymbol{\xi}})$. According to their respective problems (3.34), and using in particular from the second equations of them that $(\widetilde{\phi}, \widetilde{\mathbf{t}})$ and $(\widetilde{\theta}, \widetilde{\mathbf{r}})$ belong to $\widetilde{\mathbb{V}}$ (cf. (3.38)), we deduce, thanks to the $\widetilde{\mathbb{V}}$-ellipticity of $\mathcal{A}_{\varphi,\mathbf{w}}$ (cf. (3.39) in Lemma 3.9), and after introducing the null expression $\mathcal{A}_{\psi,\mathbf{z}}\big((\widetilde{\theta}, \widetilde{\mathbf{r}}), (\widetilde{\phi}, \widetilde{\mathbf{t}}) - (\widetilde{\theta}, \widetilde{\mathbf{r}})\big) - \mathcal{F}_\psi\big((\widetilde{\phi}, \widetilde{\mathbf{t}}) - (\widetilde{\theta}, \widetilde{\mathbf{r}})\big)$, that

$$\alpha_{\mathcal{A}} \, \|(\widetilde{\phi}, \widetilde{\mathbf{t}}) - (\widetilde{\theta}, \widetilde{\mathbf{r}})\|^2 \,\leqslant\, \big(\mathcal{F}_\varphi - \mathcal{F}_\psi\big)\big((\widetilde{\phi}, \widetilde{\mathbf{t}}) - (\widetilde{\theta}, \widetilde{\mathbf{r}})\big) - \big(\mathcal{A}_{\varphi,\mathbf{w}} - \mathcal{A}_{\psi,\mathbf{z}}\big)\big((\widetilde{\theta}, \widetilde{\mathbf{r}}), (\widetilde{\phi}, \widetilde{\mathbf{t}}) - (\widetilde{\theta}, \widetilde{\mathbf{r}})\big) . \tag{3.50}$$

Next, bearing in mind the definitions of the functionals and bilinear forms appearing in the foregoing equation (cf. (2.27), (2.25)), and employing the Cauchy–Schwarz and Hölder inequalities, as well as the Lipschitz-continuity of f_{bk} and ϑ (cf. (1.3)), we find that

$$\begin{aligned} \big(\mathcal{F}_\varphi - \mathcal{F}_\psi\big)\big((\widetilde{\phi}, \widetilde{\mathbf{t}}) - (\widetilde{\theta}, \widetilde{\mathbf{r}})\big) &= \left| \int_\Omega \Big\{ f_{\mathrm{bk}}(\varphi) - f_{\mathrm{bk}}(\psi) \Big\} \, \boldsymbol{k} \cdot (\widetilde{\mathbf{t}} - \widetilde{\mathbf{r}}) \right| \\ &\leqslant\, L_f \, |\boldsymbol{k}| \, \|\varphi - \psi\|_{0,\Omega} \, \|\widetilde{\mathbf{t}} - \widetilde{\mathbf{r}}\|_{0,\Omega} \, , \end{aligned} \tag{3.51}$$

and that for each $(\psi, \mathbf{s}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)$ there holds

$$\begin{aligned} \big(\mathcal{A}_{\varphi,\mathbf{w}} - \mathcal{A}_{\psi,\mathbf{z}}\big)\big((\widetilde{\theta}, \widetilde{\mathbf{r}}), (\psi, \mathbf{s})\big) &= \left| \int_\Omega \Big\{ \vartheta(\varphi) - \vartheta(\psi) \Big\} \, \widetilde{\mathbf{r}} \cdot \mathbf{s} - \int_\Omega \widetilde{\theta} \, (\mathbf{w} - \mathbf{z}) \cdot \mathbf{s} \right| \\ &\leqslant\, \Big\{ L_\vartheta \, \|\varphi - \psi\|_{0,2\ell;\Omega} \, \|\widetilde{\mathbf{r}}\|_{0,2j;\Omega} + \|\widetilde{\theta}\|_{0,4;\Omega} \, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \Big\} \, \|\mathbf{s}\|_{0,\Omega} \, , \end{aligned} \tag{3.52}$$

where ℓ, $j \in (1, +\infty)$, conjugate to each other, will be fixed later on. In this way, replacing (3.51), and (3.52) with $(\psi, \mathbf{s}) = (\widetilde{\phi}, \widetilde{\mathbf{t}}) - (\widetilde{\theta}, \widetilde{\mathbf{r}})$, into (3.50), and performing minor algebraic manipulations, we arrive at

$$\|(\widetilde{\phi}, \widetilde{\mathbf{t}}) - (\widetilde{\theta}, \widetilde{\mathbf{r}})\| \,\leqslant\, \frac{1}{\alpha_{\mathcal{A}}} \Big\{ L_f \, |\boldsymbol{k}| \, \|\varphi - \psi\|_{0,\Omega} + L_\vartheta \, \|\varphi - \psi\|_{0,2\ell;\Omega} \, \|\widetilde{\mathbf{r}}\|_{0,2j;\Omega} + \|\widetilde{\theta}\|_{0,4;\Omega} \, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \Big\} . \tag{3.53}$$

Furthermore, relying on the inf-sup condition of $\mathcal{B}$ (cf. (3.41)) and the respective first equations of (3.34), we infer

$$\beta_{\mathcal{B}} \, \|\widetilde{\boldsymbol{\eta}} - \widetilde{\boldsymbol{\xi}}\| \,\leqslant\, \sup_{\substack{(\psi, \mathbf{s}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \\ (\psi, \mathbf{s}) \neq \mathbf{0}}} \frac{\big(\mathcal{F}_\varphi - \mathcal{F}_\psi\big)\big((\psi, \mathbf{s})\big) - \Big\{ \mathcal{A}_{\varphi,\mathbf{w}}\big((\widetilde{\phi}, \widetilde{\mathbf{t}}), (\psi, \mathbf{s})\big) - \mathcal{A}_{\psi,\mathbf{z}}\big((\widetilde{\theta}, \widetilde{\mathbf{r}}), (\psi, \mathbf{s})\big) \Big\}}{\|(\psi, \mathbf{s})\|} .$$

The first term in the numerator can be bound analogously to (3.51), obtaining

$$\big|\big(\mathcal{F}_\varphi - \mathcal{F}_\psi\big)\big((\psi,\mathbf{s})\big)\big| \;\leqslant\; L_f\,|\boldsymbol{k}|\,\|\varphi-\psi\|_{0,\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,, \tag{3.54}$$

whereas the second one is bounded upon adding and substracting $\mathcal{A}_{\varphi,\mathbf{w}}\big((\widetilde{\theta},\widetilde{\mathbf{r}}),(\psi,\mathbf{s})\big)$, and then using the boundedness of $\mathcal{A}_{\varphi,\mathbf{w}}$ with constant $\|\mathcal{A}\|$ (cf. (3.43)), and the estimate (3.52), all of which gives

$$\begin{aligned}
&\big|\mathcal{A}_{\varphi,\mathbf{w}}\big((\widetilde{\phi},\widetilde{\mathbf{t}}),(\psi,\mathbf{s})\big) - \mathcal{A}_{\psi,\mathbf{z}}\big((\widetilde{\theta},\widetilde{\mathbf{r}}),(\psi,\mathbf{s})\big)\big|\\
&\quad\leqslant\; \big|\mathcal{A}_{\varphi,\mathbf{w}}\big((\widetilde{\phi},\widetilde{\mathbf{t}})-(\widetilde{\theta},\widetilde{\mathbf{r}}),(\psi,\mathbf{s})\big)\big| + \big|\big(\mathcal{A}_{\varphi,\mathbf{w}}-\mathcal{A}_{\psi,\mathbf{z}}\big)\big((\widetilde{\theta},\widetilde{\mathbf{r}}),(\psi,\mathbf{s})\big)\big|\\
&\quad\leqslant\; \Big\{\|\mathcal{A}\|\,\|(\widetilde{\phi},\widetilde{\mathbf{t}})-(\widetilde{\theta},\widetilde{\mathbf{r}})\| + L_\vartheta\,\|\varphi-\psi\|_{0,2\ell;\Omega}\,\|\widetilde{\mathbf{r}}\|_{0,2j;\Omega} + \|\widetilde{\theta}\|_{0,4;\Omega}\,\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\Big\}\,\|(\psi,\mathbf{s})\|\,.
\end{aligned} \tag{3.55}$$

In this way, employing (3.54) and (3.55) within the preliminary estimate for $\|\widetilde{\boldsymbol{\eta}}-\widetilde{\boldsymbol{\xi}}\|$, we get

$$\begin{aligned}
\beta_{\mathcal{B}}\,\|\widetilde{\boldsymbol{\eta}}-\widetilde{\boldsymbol{\xi}}\| \;\leqslant\; & L_f\,|\boldsymbol{k}|\,\|\varphi-\psi\|_{0,\Omega} + \|\mathcal{A}\|\,\|(\widetilde{\phi},\widetilde{\mathbf{t}})-(\widetilde{\theta},\widetilde{\mathbf{r}})\|\\
&+ L_\vartheta\,\|\varphi-\psi\|_{0,2\ell;\Omega}\,\|\widetilde{\mathbf{r}}\|_{0,2j;\Omega} + \|\widetilde{\theta}\|_{0,4;\Omega}\,\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,.
\end{aligned} \tag{3.56}$$

Next, given the further regularity assumption (3.48), we proceed as in [4, Lemma 3.7] (see, also, proof of Lemma 3.4) by setting, as in (3.20),

$$\widetilde{\delta} \;:=\; \begin{cases} \frac{2}{1-\delta}, & \text{if } n=2\,,\\ \frac{6}{3-2\delta}, & \text{if } n=3\,,\end{cases} \tag{3.57}$$

and recalling the continuous embedding $\mathbf{i}_{\widetilde{\delta}}:\mathbf{H}^\delta(\Omega)\to\mathbf{L}^{\widetilde{\delta}}(\Omega)$. Then, noting that for the range of δ specified in (3.48), that is $\delta\geqslant n/4$, there holds $\widetilde{\delta}\geqslant 4$, we can choose $j=\widetilde{\delta}/2$, thus yielding

$$\|\widetilde{\mathbf{r}}\|_{0,2j;\Omega} \;=\; \|\widetilde{\mathbf{r}}\|_{0,\widetilde{\delta};\Omega} \;\leqslant\; \|\mathbf{i}_{\widetilde{\delta}}\|\,\|\widetilde{\mathbf{r}}\|_{\delta,\Omega} \;=\; \|\mathbf{i}_{\widetilde{\delta}}\|\,\|\widetilde{\mathbf{T}}_2(\psi,\mathbf{z})\|_{\delta,\Omega}\,. \tag{3.58}$$

In turn, it follows that

$$2\ell = \frac{2j}{j-1} = \begin{cases}\frac{2}{\delta}, & \text{if } n=2,\\ \frac{3}{\delta}, & \text{if } n=3\end{cases} = \frac{n}{\delta}\,,$$

so that, using again the continuous injection $\mathrm{i}_\delta:\mathrm{L}^4(\Omega)\to\mathrm{L}^{n/\delta}(\Omega)$, we obtain

$$\|\varphi-\psi\|_{0,2\ell;\Omega} = \|\varphi-\psi\|_{0,n/\delta;\Omega} \;\leqslant\; \|\mathrm{i}_\delta\|\,\|\varphi-\psi\|_{0,4;\Omega}\,. \tag{3.59}$$

Finally, employing the inequalities (3.58) and (3.59) in both (3.53) and (3.56), performing several algebraic manipulations, in particular replacing the resulting estimate from (3.53) into (3.56), and then adding them, recalling that $\widetilde{\theta}=\widetilde{\mathbf{T}}_1(\psi,\mathbf{z})$, and using that $\|\cdot\|_{0,\Omega}\leqslant|\Omega|^{1/4}\,\|\cdot\|_{0,4;\Omega}$, we readily get (3.49) and finish the proof. □

As a consequence of Lemmas 3.13 and 3.14, we now provide the continuity of $\mathbf{T}$.

Lemma 3.15 *Assume that the data satisfy the hypotheses of Lemma 3.12, that is (3.44) and (3.45). Then, there exists a positive constant $C_{\mathbf{T}}$, depending only on $L_{\bar{\mathbf{T}}}$ (cf. Lemma 3.13) and $L_{\widetilde{\mathbf{T}}}$ (cf. Lemma 3.14), such that*

$$\begin{aligned}
\|\mathbf{T}(\varphi)-\mathbf{T}(\psi)\|_{0,4;\Omega} \;\leqslant\; & C_{\mathbf{T}}\Big\{|\boldsymbol{k}| \;+\; \|\widetilde{\mathbf{T}}_2\big(\psi,\bar{\mathbf{T}}_2(\psi)\big)\|_{\delta,\Omega}\\
&+\;\|\mathbf{T}(\psi)\|_{0,4;\Omega}\Big(\|\bar{\mathbf{T}}_1(\psi)\|_{\delta,\Omega} \;+\; \|\boldsymbol{f}\|_{0,\Omega}\Big)\Big\}\,\|\varphi\;-\;\psi\|_{0,4;\Omega}
\end{aligned} \tag{3.60}$$

for all $\varphi,\,\psi\in W$.

Proof. We begin by noting that, given φ, ψ in W, the estimate (3.37) and the assumption (3.44) on the data ensure that $(\varphi, \bar{\mathbf{T}}_2(\varphi))$ and $(\psi, \bar{\mathbf{T}}_2(\psi))$ satisfy the hypotheses of Lemma 3.14, and hence $\mathbf{T}(\varphi)$ and $\mathbf{T}(\psi)$ are well-defined (cf. (3.35)). Needless to say here, we also stress that (3.44) and (3.45) guarantee that $\mathbf{T}(W) \subseteq W$. Having said the above, the deduction of (3.60) follows by straightforward applications of Lemmas 3.14 and 3.13. $\square$

Similarly as for the derivation of (3.29), we now bound $\|\mathbf{T}(\psi)\|_{0,4;\Omega}$ by r, and $\|\bar{\mathbf{T}}_1(\psi)\|_{\delta,\Omega}$ and $\|\widetilde{\mathbf{T}}_2\big(\psi, \bar{\mathbf{T}}_2(\psi)\big)\|_{\delta,\Omega}$ by the estimates given by (3.46) and (3.48), respectively. In this way, we infer from (3.60) the existence of a positive constant $L_{\mathbf{T}}$, depending on $C_{\mathbf{T}}$, $\bar{C}_{\bar{\mathbf{T}}}(r)$, $\widetilde{C}_{\widetilde{\mathbf{T}}}(r)$, r, γ_f, and $|\Omega|$, such that

$$\|\mathbf{T}(\varphi) \,-\, \mathbf{T}(\psi)\|_{0,4;\Omega} \,\leqslant\, L_{\mathbf{T}} \Big\{ |\boldsymbol{k}| + \|\boldsymbol{f}\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2+\delta,\Gamma} + \|g\|_{\delta,4/3;\Omega} \Big\} \, \|\varphi \,-\, \psi\|_{0,4;\Omega}, \tag{3.61}$$

for all φ, $\psi \in W$.

We are now in position to state the unique solvability of (3.36).

Theorem 3.16 *Assume that, in addition to the hypotheses of Lemma 3.12, that is (3.44) and (3.45), the data satisfy*

$$L_{\mathbf{T}} \Big\{ |\boldsymbol{k}| + \|\boldsymbol{f}\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2+\delta,\Gamma} + \|g\|_{\delta,4/3;\Omega} \Big\} \,<\, 1\,. \tag{3.62}$$

Then, the fixed-point equation (3.36) has a unique solution $\phi \in W$, which means, equivalently, that the fully–mixed formulation (2.32) has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ and $(\phi, \mathbf{t}, \boldsymbol{\eta}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, with $\phi \in W$. Moreover, there hold

$$\|(\phi, \mathbf{t}, \boldsymbol{\eta})\| \,\leqslant\, C_{\widetilde{\mathbf{T}}} \Big\{ \gamma_f \, |\Omega|^{1/2} \, |\boldsymbol{k}| \,+\, \|g\|_{0,4/3;\Omega} \Big\} \quad \textit{and} \tag{3.63}$$

$$\|(\boldsymbol{\sigma}, \mathbf{u})\| \,\leqslant\, C_{\bar{\mathbf{T}}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega} \, \|\phi\|_{0,4;\Omega} \Big\}\,. \tag{3.64}$$

Proof. It was already established by Lemma 3.12 that $\mathbf{T}$ maps the ball W into itself. Then, knowing from (3.61) and the assumption (3.62) that $\mathbf{T}$ is a contraction, the unique solvability of (3.36), and hence of (2.32), follows from the Banach fixed-point theorem (see, e.g. [19, Theorem 3.7-1]). In turn, estimates (3.63) and (3.64) arise from (3.42) and (3.37), respectively. $\square$

4 The Galerkin schemes

In this section we introduce Galerkin schemes for the mixed-primal and fully-mixed formulations given by (2.20) and (2.32), respectively, and address their well-posedness by employing discrete analogues of the fixed-point strategies developed in Sections 3.1 and 3.2.

4.1 Preliminaries

Let $\big\{\mathcal{T}_h\big\}_{h>0}$ be a family of regular triangulations of Ω made up of triangles K (resp. tetrahedra K in R^3) of diameter h_K. Note that h stands for both the index of $\mathcal{T}_h$ and its corresponding meshsize $h := \max\Big\{ h_K : \ K \in \mathcal{T}_h \Big\}$. Now, given an integer $\ell \geqslant 0$, for each $K \in \mathcal{T}_h$ we let $\mathrm{P}_\ell(K)$ be the space of polynomial functions on K of degree $\leqslant \ell$, and define the corresponding local Raviart–Thomas space of order ℓ as

$$\mathbf{RT}_\ell(K) \,:=\, \mathbf{P}_\ell(K) \,\oplus\, \mathrm{P}_\ell(K)\,\mathbf{x}\,,$$

where, according to the notations described in Section 1.1, $\mathbf{P}_\ell(K) = [\mathrm{P}_\ell(K)]^n$, and $\mathbf{x}$ is the generic vector in R^n. In addition, we let $\mathbb{RT}_\ell(K)$ be the tensor version of $\mathbf{RT}_\ell(K)$, that is, denoting by $\boldsymbol{\tau}_i$ the i-th row of a tensor $\boldsymbol{\tau}$, we set

$$\mathbb{RT}_\ell(K) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(K) : \quad \boldsymbol{\tau}_i \in \mathbf{RT}_\ell(K) \quad \forall\, i \in \{1, ..., n\} \Big\}.$$

4.2 The mixed–primal method

Given an integer $k \geqslant 0$, we introduce the finite element subspaces:

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \tag{4.1}$$

$$\mathbf{H}_h^{\mathbf{u}} := \Big\{ \mathbf{v}_h \in \mathbf{L}^4(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \quad \text{and} \tag{4.2}$$

$$\mathrm{H}_h^{\phi} := \Big\{ \psi_h \in C(\Omega) \cap \mathrm{H}_0^1(\Omega) : \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \tag{4.3}$$

so that the Galerkin scheme associated with (2.20) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi}$ such that

$$\begin{aligned}
\mathbf{a}_{\phi_h}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \,+\, \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) \,-\, \mathbf{c}(\mathbf{u}_h, \mathbf{v}_h) &= \mathbf{G}_{\phi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}, \\
A_{\mathbf{u}_h}(\phi_h, \psi_h) &= G_{\phi_h}(\psi_h) && \forall\, \psi_h \in \mathrm{H}_h^{\phi}.
\end{aligned} \tag{4.4}$$

We emphasize that the definitions of the bilinear forms $\mathbf{a}_{\phi_h}$, $\mathbf{b}$, $\mathbf{c}$, and $A_{\mathbf{u}_h}$, and the linear functionals $\mathbf{F}$, $\mathbf{G}_{\phi_h}$, and G_{ϕ_h}, are given in (2.13), (2.4), (2.14), and (2.5), respectively, with $\phi = \phi_h$ and $\mathbf{u} = \mathbf{u}_h$.

Next, as previously announced, we adopt the discrete version of the fixed-point strategy used in Section 3.1 to analyze the solvability of (4.4). We first introduce the operator $\bar{\mathbf{S}}_h : \mathrm{H}_h^{\phi} \to \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ defined by

$$\bar{\mathbf{S}}_h(\varphi_h) = (\bar{\mathbf{S}}_{1,h}(\varphi_h), \bar{\mathbf{S}}_{2,h}(\varphi_h)) := (\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}, \tag{4.5}$$

where $(\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h)$ is the unique solution (to be confirmed below) of the first and second rows of (4.4) with the given φ_h, that is

$$\begin{aligned}
\mathbf{a}_{\varphi_h}(\bar{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) \,+\, \mathbf{b}(\boldsymbol{\tau}_h, \bar{\mathbf{u}}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\
\mathbf{b}(\bar{\boldsymbol{\sigma}}_h, \mathbf{v}_h) \,-\, \mathbf{c}(\bar{\mathbf{u}}_h, \mathbf{v}_h) &= \mathbf{G}_{\varphi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}.
\end{aligned} \tag{4.6}$$

In turn, we let $\widetilde{\mathbf{S}}_h : \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{u}} \to \mathrm{H}_h^{\phi}$ be the discrete version of $\widetilde{\mathbf{S}}$ (cf. (3.3)), which is defined by

$$\widetilde{\mathbf{S}}_h(\varphi_h, \mathbf{w}_h) := \widetilde{\phi}_h \qquad \forall\, (\varphi_h, \mathbf{w}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{u}},$$

where $\widetilde{\phi}_h \in \mathrm{H}_h^{\phi}$ is the unique solution (to be confirmed below) of the discrete analogue of (3.4), that is

$$A_{\varphi_h, \mathbf{w}_h}(\widetilde{\phi}_h, \widetilde{\psi}_h) = G_{\varphi_h}(\widetilde{\psi}_h) \qquad \forall\, \widetilde{\psi}_h \in \mathrm{H}_h^{\phi}, \tag{4.7}$$

and the bilinear form $A_{\varphi_h, \mathbf{w}_h}$ and the linear functional G_{φ_h} are given by (3.5) and (3.6), respectively, with $\varphi = \varphi_h$ and $\mathbf{w} = \mathbf{w}_h$. Then, we define the operator $\mathbf{S}_h : \mathrm{H}_h^{\phi} \to \mathrm{H}_h^{\phi}$ by

$$\mathbf{S}_h(\varphi_h) := \widetilde{\mathbf{S}}_h(\varphi_h, \bar{\mathbf{S}}_{2,h}(\varphi_h)) \qquad \forall\, \varphi_h \in \mathrm{H}_h^{\phi}, \tag{4.8}$$

and realize that solving (4.4) is equivalent to seeking a fixed point of $\mathbf{S}_h$, that is $\phi_h \in \mathrm{H}_h^\phi$ such that

$$\mathbf{S}_h(\phi_h) \;=\; \phi_h\,. \tag{4.9}$$

Analogously to the continuous case, the well-definedness of the discrete operators $\bar{\mathbf{S}}_h$ and $\widetilde{\mathbf{S}}_h$, and hence of $\mathbf{S}_h$, hinges on the discrete problems (4.6) and (4.7) being well-posed, which we address in what follows. We begin with (4.6) by resorting to [23, Theorem 3.5], discrete analogue of [23, Theorem 3.4], which was applied to derive the well-posedness of (3.2). Indeed, we recall again from (2.15) and (2.16) that the bilinear forms $\mathbf{a}_{\varphi_h}$, $\mathbf{b}$, and $\mathbf{c}$ are all bounded, and that $\mathbf{a}_{\varphi_h}$ and $\mathbf{c}$ are both symmetric and positive semi-definite, whence assumption i) of [23, Theorem 3.5] is accomplished. Next, the discrete kernel of $\mathbf{b}$ is given by

$$\mathbb{V}_h \;:=\; \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\Big\}\,,$$

which, using from (4.1) and (4.2) that $\mathbf{div}(\mathbb{H}_h^{\boldsymbol{\sigma}}) \subseteq \mathbf{H}_h^{\mathbf{u}}$, reduces to

$$\mathbb{V}_h \;=\; \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega\Big\}\,,$$

thus showing that $\mathbb{V}_h \subseteq \mathbb{V}$ (cf. (3.9)). It follows from (3.10) that

$$\mathbf{a}_{\varphi_h}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \;\geqslant\; \alpha_{\mathtt{d}}\, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h\,, \tag{4.10}$$

with $\alpha_{\mathtt{d}} := \alpha$, which proves assumption ii) of [23, Theorem 3.5]. In turn, we recall from [20, Lemma 5.5, Section 5.4] that there exists a positive constant $\beta_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \;\geqslant\; \beta_{\mathtt{d}}\, \|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,, \tag{4.11}$$

which verifies assumption iii) of [23, Theorem 3.5]. Hereafter, we use the subscript "$\mathtt{d}$" to identify constants that arise in the discrete analyses and that are independent of the mesh size h.

According to the previous discussion, we are now able to prove the well-definedness of $\bar{\mathbf{S}}_h$, which constitutes the discrete analogue of Lemma 3.1.

Lemma 4.1 *For each $\varphi_h \in \mathrm{H}_h^\phi$ problem (4.6) has a unique solution $(\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$, and hence we can define $\bar{\mathbf{S}}_h(\varphi_h) := (\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h)$. Moreover, there exists a positive constant $C_{\bar{\mathbf{S}},\mathtt{d}}$, depending on μ_1, $\|\mathrm{i}_4\|$, $c(\Omega)$, $\|\mathbf{i}_{4/3}\|$, $\|\mathbf{K}^{-1}\|_{\infty,\Omega}$, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, and hence independent of φ_h, such that*

$$\|\bar{\mathbf{S}}_h(\varphi_h)\| \;=\; \|(\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h)\| \;\leqslant\; C_{\bar{\mathbf{S}},\mathtt{d}} \Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} \;+\; \|\boldsymbol{f}\|_{0,\Omega}\, \|\varphi_h\|_{1,\Omega}\Big\} \qquad \forall\, \varphi_h \in \mathrm{H}_h^\phi\,. \tag{4.12}$$

Proof. The proof, being analogous to the one of Lemma 3.1, follows from the previous analysis and a straightforward application of [23, Theorem 3.5]. In particular, the boundedness of the linear functionals $\mathbf{F}$ and $\mathbf{G}_{\varphi_h}$, as stated in (2.16) and (2.18), along with the a priori estimates provided by [23, Theorem 3.5, eq. (3.67)], imply (4.12). Further details are omitted. $\square$

The following result, taken from [4, Lemma 4.2], states that the operator $\widetilde{\mathbf{S}}_h$ is well-defined, thus yielding the discrete analogue of Lemma 3.2.

Lemma 4.2 *Let $\varphi_h \in \mathrm{H}_h^\phi$ and $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{w}_h\|_{0,4;\Omega} < \frac{\vartheta_1}{2\,c_P\,\|\mathrm{i}_4\|}$ (cf. (1.2), (2.1), (2.8)). Then, problem (4.7) has a unique solution $\widetilde{\phi}_h \in \mathrm{H}_h^\phi$, whence we can define $\widetilde{\mathbf{S}}_h(\varphi_h, \mathbf{w}_h) := \widetilde{\phi}_h$. Moreover, letting $C_{\widetilde{\mathbf{S}},\mathtt{d}} = C_{\widetilde{\mathbf{S}}} := \frac{2\,c_P^2}{\vartheta_1}$ (cf. Lemma 3.2), which is independent of $(\varphi_h, \mathbf{w}_h)$, there holds*

$$\|\widetilde{\mathbf{S}}_h(\varphi_h, \mathbf{w}_h)\|_{1,\Omega} \,=\, \|\widetilde{\phi}_h\|_{1,\Omega} \,\leqslant\, C_{\widetilde{\mathbf{S}},\mathtt{d}} \Big\{ \gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,\Omega} \Big\} . \tag{4.13}$$

In what follows we address the well-posedness of the discrete mixed–primal formulation (4.4) by using the Brouwer fixed-point theorem (cf. [18, cf. Theorem 9.9-2]) to study the solvability of the equivalent fixed-point equation (4.9). To this end, we now introduce the ball

$$B_h \,:=\, \Big\{ \varphi_h \in \mathrm{H}_h^\phi : \quad \|\varphi_h\|_{1,\Omega} \,\leqslant\, r \Big\} ,$$

and establish next the discrete version of Lemma 3.3.

Lemma 4.3 *Assume that the data satisfy the discrete analogues of (3.17) and (3.18) (cf. Lemma 3.3), that is*

$$C_{\bar{\mathbf{S}},\mathtt{d}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, r\,\|\boldsymbol{f}\|_{0,\Omega} \Big\} \,\leqslant\, \frac{\vartheta_1}{2\,c_P\,\|\mathrm{i}_4\|} , \quad \text{and} \tag{4.14}$$

$$C_{\widetilde{\mathbf{S}},\mathtt{d}} \Big\{ \gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,\Omega} \Big\} \,\leqslant\, r . \tag{4.15}$$

Then $\mathbf{S}_h(B_h) \subseteq B_h$.

Proof. It is a direct consequence of Lemmas 4.1 and 4.2. Needless to say, note that actually (3.18) and (4.15) coincide since $C_{\widetilde{\mathbf{S}}} = C_{\widetilde{\mathbf{S}},\mathtt{d}}$. □

We now present the discrete analogue of Lemma 3.4, for which, knowing in advance that no regularity assumption could be applied in this case, we simply resort to a $\mathrm{L}^4 - \mathrm{L}^4 - \mathrm{L}^2$ argument in the corresponding bounding process.

Lemma 4.4 *There exists a positive constant $L_{\bar{\mathbf{S}},\mathtt{d}}$, depending on μ_1, L_μ, $\|\mathbf{i}_{4/3}\|$, $\|\boldsymbol{K}^{-1}\|_{\infty,\Omega}$, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, such that for all φ_h, $\psi_h \in \mathrm{H}_h^\phi$ there holds*

$$\|\bar{\mathbf{S}}_h(\varphi_h) - \bar{\mathbf{S}}_h(\psi_h)\| \,\leqslant\, L_{\bar{\mathbf{S}},\mathtt{d}} \Big\{ \|\bar{\mathbf{S}}_{1,h}(\psi_h)\|_{0,4;\Omega} \,+\, \|\boldsymbol{f}\|_{0,\Omega} \Big\} \, \|\varphi_h - \psi_h\|_{0,4;\Omega} . \tag{4.16}$$

Proof. It follows analogously to the proof of Lemma 3.4. In fact, by means of the corresponding discrete global inf-sup condition satisfied by the operator $\bar{\mathbf{S}}_h$ (cf. [23, eq. (3.42)]), which holds with a constant $\bar{C}_{\mathtt{d}}$ depending only on $\|\mathbf{a}_{\varphi_h}\| \,=\, \frac{1}{\mu_1}$, $\|\mathbf{c}\| \,=\, \|\mathbf{i}_{4/3}\|\,\|\boldsymbol{K}^{-1}\|_{\infty,\Omega}$, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, we obtain the discrete analogue of (3.25), from which we continue as in the proof of [7, Lemma 4.6]. More precisely, employing the Cauchy–Schwarz inequality twice in the resulting first term of that analogue, which constitutes the aforementioned $\mathrm{L}^4 - \mathrm{L}^4 - \mathrm{L}^2$ argument, we arrive, similarly as [7, eq. (4.17)], to

$$\|\bar{\mathbf{S}}_h(\varphi_h) - \bar{\mathbf{S}}_h(\psi_h)\| \,\leqslant\, \bar{C}_{\mathtt{d}} \Big\{ L_\mu\,\mu_1^{-2}\,\|\bar{\mathbf{S}}_{1,h}(\psi_h)\|_{0,4;\Omega} \,+\, \|\boldsymbol{f}\|_{0,\Omega} \Big\} \, \|\varphi_h - \psi_h\|_{0,4;\Omega} \qquad \forall\, \varphi_h,\, \psi_h \in \mathrm{H}_h^\phi ,$$

which yields (4.16) and finishes the proof. □

For the continuity of $\widetilde{\mathbf{S}}_h$ we state the discrete analogue of Lemma 3.5 as follows.

Lemma 4.5 *There exists a positive constant $L_{\widetilde{\mathbf{S}},\mathbf{d}}$, depending on $C_{\widetilde{\mathbf{S}}}$ $(= C_{\widetilde{\mathbf{S}},\mathbf{d}})$, $\|\mathrm{i}_4\|$, L_f, and L_ϑ, such that for all $(\varphi_h, \mathbf{w}_h)$, $(\psi_h, \mathbf{z}_h) \in \mathrm{H}_h^\phi \times \mathbf{H}_h^{\mathbf{u}}$ satisfying $\|\mathbf{w}_h\|_{0,4;\Omega}$, $\|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \frac{\vartheta_1}{2\,\|\mathrm{i}_4\|\, c_P}$, there holds*

$$\|\widetilde{\mathbf{S}}_h(\varphi_h, \mathbf{w}_h) - \widetilde{\mathbf{S}}_h(\psi_h, \mathbf{z}_h)\|_{1,\Omega} \leqslant L_{\widetilde{\mathbf{S}},\mathbf{d}} \Big\{ |\boldsymbol{k}|\, \|\varphi_h - \psi_h\|_{0,\Omega} + \|\widetilde{\mathbf{S}}_h(\psi_h, \mathbf{z}_h)\|_{1,\Omega}\, \|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega} + \|\nabla \widetilde{\mathbf{S}}_h(\psi_h, \mathbf{z}_h)\|_{0,4;\Omega}\, \|\varphi_h - \psi_h\|_{0,4;\Omega} \Big\}.$$

Proof. The proof follows almost identically to that of [4, Lemma 4.5], except that, as in the proof of Lemma 3.5, we do not bound $\|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}$ by $\|\mathbf{w} - \mathbf{z}\|_{1,\Omega}$. Note that, being $\nabla \widetilde{\mathbf{S}}_h(\psi_h, \mathbf{z}_h)$ piecewise polynomial, $\|\nabla \widetilde{\mathbf{S}}_h(\psi_h, \mathbf{z}_h)\|_{0,4;\Omega}$ is finite and hence well-defined. $\square$

We are now in position to state the continuity of the discrete fixed-point operator $\mathbf{S}_h$.

Lemma 4.6 *Assume that the data satisfy the hypotheses of Lemma 4.3, that is (4.14) and (4.15). Then, there exists a positive constant $L_{\mathbf{S},\mathbf{d}}$, depending only on $|\Omega|$, $L_{\bar{\mathbf{S}},\mathbf{d}}$ (cf. Lemma 4.4), and $L_{\widetilde{\mathbf{S}},\mathbf{d}}$ (cf. Lemma 4.5), such that*

$$\|\mathbf{S}_h(\varphi_h) - \mathbf{S}_h(\psi_h)\|_{1,\Omega} \leqslant L_{\mathbf{S},\mathbf{d}} \Big\{ |\boldsymbol{k}| + \|\mathbf{S}_h(\psi_h)\|_{1,\Omega} \big(\|\bar{\mathbf{S}}_{1,h}(\psi_h)\|_{0,4;\Omega} + \|\boldsymbol{f}\|_{0,\Omega} \big) + \|\nabla\, \mathbf{S}_h(\psi_h)\|_{0,4;\Omega} \Big\} \|\varphi_h - \psi_h\|_{0,4;\Omega} \tag{4.17}$$

for all φ_h, $\psi_h \in B_h$, and hence the operator $\mathbf{S}_h : B_h \to B_h$ is continuous.

Proof. We first recall from Lemma 4.3 that (4.14) and (4.15) guarantee that $\mathbf{S}_h$ maps B_h into itself. Then, bearing in mind the definition of $\mathbf{S}_h$ (cf. (4.8)), inequality (4.17) follows after applying Lemmas 4.5 and 4.4, and taking into account the continuous injection of $\mathrm{L}^4(\Omega)$ into $\mathrm{L}^2(\Omega)$ (with boundedness constant $|\Omega|^{1/4}$). Thus, the continuity of $\mathbf{S}_h : B_h \to B_h$ is a consequence of (4.17) and the embedding i_4 of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^4(\Omega)$ (cf. (2.2)). $\square$

Consequently, thanks to Brouwer's fixed-point theorem (cf. [18, cf. Theorem 9.9-2]) and Lemmas 4.3, 4.6, 4.1, and 4.2, we now establish the main result of this section.

Theorem 4.7 *Assume that the data satisfy the hypotheses of Lemma 4.3, that is (4.14) and (4.15). Then, the fixed-point equation (4.9) has at least one solution $\phi_h \in B_h$, which means, equivalently, that the Galerkin scheme (4.4) has at least one solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^\phi$, with $\phi_h \in B_h$. Moreover, there hold*

$$\|\phi_h\|_{1,\Omega} \leqslant C_{\widetilde{\mathbf{S}},\mathbf{d}} \Big\{ \gamma_f\, |\Omega|^{1/2}\, |\boldsymbol{k}| + \|g\|_{0,\Omega} \Big\} \quad \text{and} \tag{4.18}$$

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leqslant C_{\bar{\mathbf{S}},\mathbf{d}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega}\, \|\phi_h\|_{1,\Omega} \Big\}. \tag{4.19}$$

We remark that the lack of an appropriate, uniform in h, upper bound for $\|\nabla\, \mathbf{S}_h(\varphi_h)\|_{0,4;\Omega}$ prevents us from using (4.17) to derive a contraction estimate that would let the Banach fixed-point theorem to ensure uniqueness of the discrete solution for small enough data.

4.3 The fully–mixed method

Given an integer $k \geqslant 0$, we now introduce the finite element subspaces:

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \tag{4.20}$$

$$\mathbf{H}_h^{\mathbf{u}} := \Big\{ \mathbf{v}_h \in \mathbf{L}^4(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \tag{4.21}$$

$$\mathrm{H}_h^{\phi} := \Big\{ \psi_h \in \mathrm{L}^4(\Omega) : \quad \psi_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \tag{4.22}$$

$$\mathbf{H}_h^{\mathbf{t}} := \Big\{ \mathbf{s}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{s}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \quad \text{and} \tag{4.23}$$

$$\mathbf{H}_h^{\boldsymbol{\eta}} := \Big\{ \boldsymbol{\chi}_h \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) : \quad \boldsymbol{\chi}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \tag{4.24}$$

so that the Galerkin scheme associated with (2.32) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\phi_h, \mathbf{t}_h, \boldsymbol{\eta}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{t}} \times \mathbf{H}_h^{\boldsymbol{\eta}}$ such that

$$\begin{array}{rcll} \mathbf{a}_{\phi_h}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) & = & \mathbf{F}(\boldsymbol{\tau}_h) & \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) - \mathbf{c}(\mathbf{u}_h, \mathbf{v}_h) & = & \mathbf{G}_{\phi_h}(\mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}, \\ \mathcal{A}_{\phi_h,\mathbf{u}_h}\big((\phi_h, \mathbf{t}_h), (\psi_h, \mathbf{s}_h)\big) + \mathcal{B}\big((\psi_h, \mathbf{s}_h), \boldsymbol{\eta}_h\big) & = & \mathcal{F}_{\phi_h}\big((\psi_h, \mathbf{s}_h)\big) & \forall\, (\psi_h, \mathbf{s}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{t}}, \\ \mathcal{B}\big((\phi_h, \mathbf{t}_h), \boldsymbol{\chi}_h\big) & = & 0 & \forall\, \boldsymbol{\chi}_h \in \mathbf{H}_h^{\boldsymbol{\eta}}. \end{array} \tag{4.25}$$

Note that the bilinear forms $\mathbf{a}_{\phi_h}$, $\mathbf{b}$, $\mathbf{c}$, $\mathcal{A}_{\phi_h,\mathbf{u}_h}$, and $\mathcal{B}$, and the linear functionals $\mathbf{F}$, $\mathbf{G}_{\phi_h}$, and $\mathcal{F}_{\phi_h}$, are defined in (2.13), (2.25), (2.26), (2.14), and (2.27), respectively, with $\phi = \phi_h$ and $\mathbf{u} = \mathbf{u}_h$.

We find it important to stress here that (4.25) yields momentum conservation properties in an approximate sense. In order to explain this, we first let $\mathcal{P}_h^k : \mathrm{L}^1(\Omega) \to \mathrm{H}_h^{\phi}$ be the projector defined for each $v \in \mathrm{L}^1(\Omega)$ as the unique element $\mathcal{P}_h^k(v) \in \mathrm{H}_h^{\phi}$ such that

$$\int_\Omega \mathcal{P}_h^k(v)\, \psi_h = \int_\Omega v\, \psi_h \qquad \forall\, \psi_h \in \mathrm{H}_h^{\phi}.$$

Analogously, we set the projector $\boldsymbol{\mathcal{P}}_h^k : \mathbf{L}^1(\Omega) \to \mathbf{H}_h^{\mathbf{u}}$, or simply say that $\boldsymbol{\mathcal{P}}_h^k$ is the vector version of $\mathcal{P}_h^k$. Then, according to the definitions of $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{G}_{\phi_h}$ (cf. (2.13), (2.14)), the second equation of (4.25) can be rewritten as

$$\int_\Omega \big(\phi_h\, \boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h) - \boldsymbol{K}^{-1}\, \mathbf{u}_h\big) \cdot \mathbf{v}_h = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}},$$

which says, equivalently, that

$$\boldsymbol{\mathcal{P}}_h^k\big(\phi_h\, \boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h) - \boldsymbol{K}^{-1}\, \mathbf{u}_h\big) = \mathbf{0} \quad \text{in} \quad \Omega. \tag{4.26}$$

In turn, bearing in mind the definitions of $\mathcal{A}_{\phi_h,\mathbf{u}_h}$, $\mathcal{B}$, and $\mathcal{F}_{\phi_h}$ (cf. (2.25), (2.26), (2.27)), and taking $\mathbf{s}_h = 0$ in the third row of (4.25), we obtain

$$\int_\Omega \big(g + \mathrm{div}(\boldsymbol{\eta}_h) - \varrho\, \phi_h\big)\, \psi_h = 0 \qquad \forall\, \psi_h \in \mathrm{H}_h^{\phi},$$

that is

$$\mathcal{P}_h^k\big(g + \mathrm{div}(\boldsymbol{\eta}_h) - \varrho\, \phi_h\big) = 0 \quad \text{in} \quad \Omega. \tag{4.27}$$

The identities (4.26) and (4.27) constitute approximate verification of the continuous momentum equations given by (cf. (1.7)) $\boldsymbol{K}^{-1}\, \mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) = \phi\, \boldsymbol{f}$ and $\varrho\, \phi - \mathrm{div}(\boldsymbol{\eta}) = g$, respectively.

The solvability of (4.25) is addressed in what follows by applying the discrete version of the fixed-point strategy employed in Section 3.2. To this end, we first let $\bar{\mathbf{T}}_h : \mathrm{H}_h^{\phi} \to \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ be the operator defined by

$$\bar{\mathbf{T}}_h(\varphi_h) = (\bar{\mathbf{T}}_{1,h}(\varphi_h), \bar{\mathbf{T}}_{2,h}(\varphi_h)) := (\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h) \qquad \forall\, \varphi_h \in \mathrm{H}_h^{\phi}, \tag{4.28}$$

where $(\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ is the unique solution (to be confirmed below) of the system formed by the first and second rows of (4.25) with the given φ_h, that is

$$\begin{aligned} \mathbf{a}_{\varphi_h}(\bar{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) \,+\, \mathbf{b}(\boldsymbol{\tau}_h, \bar{\mathbf{u}}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \\ \mathbf{b}(\bar{\boldsymbol{\sigma}}_h, \mathbf{v}_h) \,-\, \mathbf{c}(\bar{\mathbf{u}}_h, \mathbf{v}_h) &= \mathbf{G}_{\varphi_h}(\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,. \end{aligned} \tag{4.29}$$

Analogously to $\bar{\mathbf{S}}$ (cf. (3.1)) and $\bar{\mathbf{T}}$ (cf. (3.33)), we stress here that $\bar{\mathbf{S}}_h$ (cf. (4.5)) and $\bar{\mathbf{T}}_h$ (cf. (4.28)) differ only in their domains of definition, both denoted H_h^{ϕ}, which are given by subspaces of $\mathrm{H}_0^1(\Omega)$ and $\mathrm{L}^4(\Omega)$, respectively. In this way, and as noticed below, the corresponding well-definedness results practically coincide. In turn, we let $\widetilde{\mathbf{T}}_h : \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{u}} \to \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{t}} \times \mathbf{H}_h^{\boldsymbol{\eta}}$ be the operator given by

$$\widetilde{\mathbf{T}}_h(\varphi_h, \mathbf{w}_h) \,=\, (\widetilde{\mathbf{T}}_{1,h}(\varphi_h, \mathbf{w}_h), \widetilde{\mathbf{T}}_{2,h}(\varphi_h, \mathbf{w}_h), \widetilde{\mathbf{T}}_{3,h}(\varphi_h, \mathbf{w}_h)) \,:=\, (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h, \widetilde{\boldsymbol{\eta}}_h) \tag{4.30}$$

for all $(\varphi_h, \mathbf{w}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{u}}$, where $(\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h, \widetilde{\boldsymbol{\eta}}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{t}} \times \mathbf{H}_h^{\boldsymbol{\eta}}$ is the unique solution of the system formed by the third and fourth rows of (4.25) with the given $(\varphi_h, \mathbf{w}_h)$, that is

$$\begin{aligned} \mathcal{A}_{\varphi_h, \mathbf{w}_h}\big((\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h), (\psi_h, \mathbf{s}_h)\big) \,+\, \mathcal{B}\big((\psi_h, \mathbf{s}_h), \widetilde{\boldsymbol{\eta}}_h\big) &= \mathcal{F}_{\varphi_h}\big((\psi_h, \mathbf{s}_h)\big) \qquad \forall\, (\psi_h, \mathbf{s}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{t}}\,, \\ \mathcal{B}\big((\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h), \boldsymbol{\chi}_h\big) &= 0 \qquad \forall\, \boldsymbol{\chi}_h \in \mathbf{H}_h^{\boldsymbol{\eta}}\,. \end{aligned} \tag{4.31}$$

Then, we define the operator $\mathbf{T}_h : \mathrm{H}_h^{\phi} \to \mathrm{H}_h^{\phi}$ by

$$\mathbf{T}_h(\varphi_h) \,:=\, \widetilde{\mathbf{T}}_{1,h}(\varphi_h, \bar{\mathbf{T}}_{2,h}(\varphi_h)) \qquad \forall\, \varphi_h \in \mathrm{H}_h^{\phi}\,, \tag{4.32}$$

and realize that solving (4.25) is equivalent to seeking a fixed point of $\mathbf{T}_h$, that is $\phi_h \in \mathrm{H}_h^{\phi}$ such that

$$\mathbf{T}_h(\phi_h) \,=\, \phi_h\,. \tag{4.33}$$

In order to ensure that all of the above makes sense, we now show that operators $\bar{\mathbf{T}}_h$ and $\widetilde{\mathbf{T}}_h$ are well-defined, which reduces to establishing the discrete analogues of Lemmas 3.8 and 3.11. The first of them reads as follows.

Lemma 4.8 *For each $\varphi_h \in \mathrm{H}_h^{\phi}$ problem (4.29) has a unique solution $(\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$, and hence we can define $\bar{\mathbf{T}}_h(\varphi_h) := (\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h)$. Moreover, there exists a positive constant $C_{\bar{\mathbf{T}},\mathtt{d}}$, depending on μ_1, $c(\Omega)$, $\|\mathbf{i}_{4/3}\|$, $\|\mathbf{K}^{-1}\|_{\infty,\Omega}$, $\alpha_{\mathtt{d}}$ (cf. (4.10)), and $\beta_{\mathtt{d}}$ (cf. (4.11)), and hence independent of φ_h, such that*

$$\|\bar{\mathbf{T}}_h(\varphi_h)\| = \|(\bar{\boldsymbol{\sigma}}_h, \bar{\mathbf{u}}_h)\| \,\leqslant\, C_{\bar{\mathbf{T}},\mathtt{d}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\, \|\varphi_h\|_{0,4;\Omega} \Big\} \qquad \forall\, \varphi_h \in \mathrm{H}_h^{\phi}\,. \tag{4.34}$$

Proof. The proof is essentially identical to that of Lemma 4.1, the only difference being that the bound for $\mathbf{G}_{\varphi_h}$ in the present context is given by (2.31) and not by (2.18), whence $C_{\bar{\mathbf{T}},\mathtt{d}}$ does not depend on $\|\mathrm{i}_4\|$. $\square$

Next, we turn to prove that $\widetilde{\mathbf{T}}_h$ is well-defined, equivalently that (4.31) is well-posed, for which we now apply the generalized discrete Babuška–Brezzi theory (cf. [9, Corollary 2.2]). In this regard, we first recall that, given $(\varphi_h, \mathbf{w}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{u}}$, the bilinear forms $\mathcal{A}_{\varphi_h, \mathbf{w}_h}$ and $\mathcal{B}$, and the functional F_{φ}, are bounded (cf. (2.29), (2.30)). Then, we let $\widetilde{\mathbb{V}}_h$ be the discrete null space of the bilinear form $\mathcal{B}$, that is

$$\widetilde{\mathbb{V}}_h \,=\, \Big\{ (\psi_h, \mathbf{s}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{t}} : \quad \mathcal{B}\big((\psi_h, \mathbf{s}_h), \boldsymbol{\chi}_h\big) = 0 \quad \forall\, \boldsymbol{\chi}_h \in \mathbf{H}_h^{\boldsymbol{\eta}} \Big\}\,. \tag{4.35}$$

In order to verify the hypotheses of [9, Corollary 2.2], we first resort to a result proved in [8], which makes use of the abstract equivalence provided by [20, Lemma 5.1] as well as of the inequalities given in [20, eqs. (5.64) and (5.65)]. More precisely, we have the following lemma (cf. [8, Lemma 4.2]).

Lemma 4.9 *There exist positive constants $\beta_{\mathcal{B},\mathtt{d}}$ and $C_{\mathcal{A},\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{(\psi_h,\mathbf{s}_h)\in \mathrm{H}_h^{\phi}\times\mathbf{H}_h^{\mathbf{t}}\\(\psi_h,\mathbf{s}_h)\neq\mathbf{0}}} \frac{\mathcal{B}\big((\psi_h,\mathbf{s}_h),\boldsymbol{\chi}_h\big)}{\|(\psi_h,\mathbf{s}_h)\|} \,\geqslant\, \beta_{\mathcal{B},\mathtt{d}}\,\|\boldsymbol{\chi}_h\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall\,\boldsymbol{\chi}_h\in\mathbf{H}_h^{\boldsymbol{\eta}}\,, \tag{4.36}$$

and

$$\|\mathbf{s}_h\|_{0,\Omega} \,\geqslant\, C_{\mathcal{A},\mathtt{d}}\,\|\psi_h\|_{0,4;\Omega} \qquad \forall\,(\psi_h,\mathbf{s}_h)\in\widetilde{\mathbb{V}}_h\,. \tag{4.37}$$

Having (4.36) provided the discrete inf-sup condition for $\mathcal{B}$, thus yielding the discrete analogue of Lemma 3.10, we now employ (4.37) to prove next the $\widetilde{\mathbb{V}}_h$-ellipticity of $\mathcal{A}_{\varphi_h,\mathbf{w}_h}$. Indeed, we notice in advance that, not being able to deduce from (4.35) that $\widetilde{\mathbb{V}}_h$ is contained in $\widetilde{\mathbb{V}}$ (cf. (3.38)), the aforementioned property for $\mathcal{A}_{\varphi_h,\mathbf{w}_h}$ does not follow from the one provided by Lemma 3.9, but from a suitable modification of its proof, which makes use of (4.37) instead of Poincaré's inequality (cf. (2.1)) along with the continuity of i_4 (cf. (2.2)).

Lemma 4.10 *There exists a positive constant $\alpha_{\mathcal{A},\mathtt{d}}$, depending only on ϑ_1 and $C_{\mathcal{A},\mathtt{d}}$ (cf. (1.2), (4.37)), such that for each $(\varphi_h,\mathbf{w}_h)\in\mathrm{H}_h^{\phi}\times\mathbf{H}_h^{\mathbf{u}}$ satisfying $\|\mathbf{w}_h\|_{0,4;\Omega}\leqslant 2\,\alpha_{\mathcal{A},\mathtt{d}}$, there holds*

$$\mathcal{A}_{\varphi_h,\mathbf{w}_h}\big((\psi_h,\mathbf{s}_h),(\psi_h,\mathbf{s}_h)\big) \,\geqslant\, \alpha_{\mathcal{A},\mathtt{d}}\,\|(\psi_h,\mathbf{s}_h)\|^2 \qquad \forall\,(\psi_h,\mathbf{s}_h)\in\widetilde{\mathbb{V}}_h\,. \tag{4.38}$$

Proof. Given $(\varphi_h,\mathbf{w}_h)\in\mathrm{H}_h^{\phi}\times\mathbf{H}_h^{\mathbf{u}}$ and $(\psi_h,\mathbf{s}_h)\in\widetilde{\mathbb{V}}_h$, and similarly as for the derivation of (3.40) (cf. proof of Lemma 3.9), we now employ the lower bound of ϑ (cf. (1.2)), the Cauchy–Schwarz inequality twice, and the estimate (4.37), to deduce that

$$\begin{aligned}
\mathcal{A}_{\varphi_h,\mathbf{w}_h}\big((\psi_h,\mathbf{s}_h),(\psi_h,\mathbf{s}_h)\big) \,&=\, \int_\Omega \vartheta(\varphi_h)\,|\mathbf{s}_h|^2 \,-\, \int_\Omega \psi_h\,\mathbf{w}_h\cdot\mathbf{s}_h \,+\, \varrho\,\|\psi_h\|_{0,\Omega}^2\\
&\geqslant\, \vartheta_1\,\|\mathbf{s}_h\|_{0,\Omega}^2 \,-\, \|\psi_h\|_{0,4;\Omega}\,\|\mathbf{w}_h\|_{0,4;\Omega}\,\|\mathbf{s}_h\|_{0,\Omega}\\
&\geqslant\, \frac{\vartheta_1}{2}\,\min\big\{1,C_{\mathcal{A},\mathtt{d}}^2\big\}\,\|(\psi_h,\mathbf{s}_h)\|^2 \,-\, \|\psi_h\|_{0,4;\Omega}\,\|\mathbf{w}_h\|_{0,4;\Omega}\,\|\mathbf{s}_h\|_{0,\Omega}\,.
\end{aligned} \tag{4.39}$$

Then, applying Young's inequality to the last term of (4.39), it readily follows that

$$\mathcal{A}_{\varphi_h,\mathbf{w}_h}\big((\psi_h,\mathbf{s}_h),(\psi_h,\mathbf{s}_h)\big) \,\geqslant\, \frac{1}{2}\Big(\vartheta_1\,\min\big\{1,C_{\mathcal{A},\mathtt{d}}^2\big\} \,-\, \|\mathbf{w}\|_{0,4;\Omega}\Big)\,\|(\psi,\mathbf{s})\|^2\,,$$

from which, defining $\alpha_{\mathcal{A},\mathtt{d}} := \frac{\vartheta_1\,\min\{1,C_{\mathcal{A},\mathtt{d}}^2\}}{4}$, we arrive at (4.38) and conclude the proof. □

We are now in position to establish that problem (4.31) is well-posed, equivalently that $\widetilde{\mathbf{T}}_h$ is well-defined. In other words, the discrete analogue of Lemma 3.11 reads as follows.

Lemma 4.11 *For each $(\varphi_h,\mathbf{w}_h)\in\mathrm{H}_h^{\phi}\times\mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{w}_h\|_{0,4;\Omega}\leqslant 2\,\alpha_{\mathcal{A},\mathtt{d}}$, problem (4.31) has a unique solution $(\widetilde{\phi}_h,\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\eta}}_h)\in\mathrm{H}_h^{\phi}\times\mathbf{H}_h^{\mathbf{t}}\times\mathbf{H}_h^{\boldsymbol{\eta}}$, and hence we can define $\widetilde{\mathbf{T}}_h(\varphi_h,\mathbf{w}_h) := (\widetilde{\phi}_h,\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\eta}}_h)$. Moreover, there exists a positive constant $C_{\widetilde{\mathbf{T}},\mathtt{d}}$, depending on $\alpha_{\mathcal{A},\mathtt{d}}$, $\beta_{\mathcal{B},\mathtt{d}}$, ϑ_2, and ϱ, such that*

$$\|\widetilde{\mathbf{T}}_h(\varphi_h,\mathbf{w}_h)\| \,=\, \|(\widetilde{\phi}_h,\widetilde{\mathbf{t}}_h,\widetilde{\boldsymbol{\eta}}_h)\| \,\leqslant\, C_{\widetilde{\mathbf{T}},\mathtt{d}}\Big\{\gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{0,4/3;\Omega}\Big\}\,. \tag{4.40}$$

Proof. Let $(\varphi_h, \mathbf{w}_h) \in \mathrm{H}_h^\phi \times \mathbf{H}_h^{\mathbf{u}}$ be as stated. Bearing in mind the boundedness of the bilinear forms and the functional involved in (4.31), and thanks to Lemmas 4.9 and 4.10, the proof follows, as previously announced, from a direct application of the generalized discrete Babuška–Brezzi theory (cf. [9, Corollary 2.2]). In particular, and analogously to the derivation of (3.43), the first inequality of (2.29) yields

$$\|\mathcal{A}_{\varphi_h,\mathbf{w}_h}\| \leqslant \vartheta_2 + 2\,\alpha_{\mathcal{A},\mathtt{d}} + |\Omega|^{1/2}\,\varrho := \|\mathcal{A}\|_{\mathtt{d}}\,, \tag{4.41}$$

so that, employing now [9, Corollary 2.2, eqs. (2.24) and (2.25)] along with (4.41) and the upper bound of $\|\mathcal{F}_{\varphi_h}\|$ (cf. (2.30)), we arrive at (4.40) and conclude the proof $\square$

Similarly as done for the discrete mixed-primal scheme (cf. Section 4.2), we now aim to employ Brouwer's fixed-point theorem (cf. [18, cf. Theorem 9.9-2]) to address the well-posedness of (4.25) by means of the solvability analysis of the equivalent fixed-point equation (4.33). For this purpose, we introduce the ball

$$W_h := \Big\{\varphi_h \in \mathrm{H}_h^\phi : \quad \|\varphi_h\|_{0,4;\Omega} \leqslant r\Big\}\,,$$

and prove next the discrete version of Lemma 3.12.

Lemma 4.12 *Assume that the data satisfy the discrete analogues of* (3.44) *and* (3.45) *(cf. Lemma* 3.12*), that is*

$$C_{\bar{\mathbf{T}},\mathtt{d}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + r\,\|\boldsymbol{f}\|_{0,\Omega}\Big\} \leqslant 2\,\alpha_{\mathcal{A},\mathtt{d}}\,, \quad \textit{and} \tag{4.42}$$

$$C_{\widetilde{\mathbf{T}},\mathtt{d}}\Big\{\gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| + \|g\|_{0,4/3;\Omega}\Big\} \leqslant r\,. \tag{4.43}$$

Then $\mathbf{T}_h(W_h) \subseteq W_h$.

Proof. It follows analogously to the proof of Lemma 3.12, by recalling now the definition of $\mathbf{T}_h$ (cf. (4.32)), and bearing in mind the assumptions required, as well as the a priori estimates provided, by Lemmas 4.8 and 4.11. Further details are omitted. $\square$

The continuity of $\bar{\mathbf{T}}_h$ and $\widetilde{\mathbf{T}}_h$, and hence of $\mathbf{T}_h$, is our following goal. We begin with $\bar{\mathbf{T}}_h$ by establishing next the discrete analogue of Lemma 3.13.

Lemma 4.13 *There exists a positive constant $L_{\bar{\mathbf{T}},\mathtt{d}}$, depending on μ_1, L_μ, $\|\mathbf{i}_{4/3}\|$, $\|\boldsymbol{K}^{-1}\|_{\infty,\Omega}$, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, such that for all φ_h, $\psi_h \in \mathrm{H}_h^\phi$ there holds*

$$\|\bar{\mathbf{T}}_h(\varphi_h) - \bar{\mathbf{T}}_h(\psi_h)\| \leqslant L_{\bar{\mathbf{T}},\mathtt{d}}\Big\{\|\bar{\mathbf{T}}_{1,h}(\psi_h)\|_{0,4;\Omega} + \|\boldsymbol{f}\|_{0,\Omega}\Big\}\,\|\varphi_h - \psi_h\|_{0,4;\Omega}\,. \tag{4.44}$$

Proof. Since $\bar{\mathbf{S}}_h$ and $\bar{\mathbf{T}}_h$ differ only in their domains of definitions, which are given by suitable subspaces of $\mathrm{H}_0^1(\Omega)$ and $\mathrm{L}^4(\Omega)$, respectively, the present proof is basically the same of Lemma 4.4. We omit further details and just stress that the respective constants, namely $L_{\bar{\mathbf{S}},\mathtt{d}}$ and $L_{\bar{\mathbf{T}},\mathtt{d}}$, coincide. $\square$

In turn, the continuity of $\widetilde{\mathbf{T}}_h$, that is the discrete analogue of Lemma 3.14, is stated below.

Lemma 4.14 *There exists a positive constant $L_{\widetilde{\mathbf{T}},\mathtt{d}}$, depending on $|\Omega|$, L_f and L_ϑ (cf.* (1.3)*), $\|\mathcal{A}\|_{\mathtt{d}}$ (cf.* (4.41)*), $\alpha_{\mathcal{A},\mathtt{d}}$, and $\beta_{\mathcal{B},\mathtt{d}}$, such that for all $(\varphi_h, \mathbf{w}_h)$, $(\psi_h, \mathbf{z}_h) \in \mathrm{H}_h^\phi \times \mathbf{H}_h^{\mathbf{u}}$ satisfying $\|\mathbf{w}_h\|_{0,4;\Omega}$, $\|\mathbf{z}_h\|_{0,4;\Omega} \leqslant 2\,\alpha_{\mathcal{A},\mathtt{d}}$, there holds*

$$\begin{aligned}\|\widetilde{\mathbf{T}}_h(\varphi_h, \mathbf{w}_h) - \widetilde{\mathbf{T}}_h(\psi_h, \mathbf{z}_h)\| \leqslant L_{\widetilde{\mathbf{T}},\mathtt{d}}\Big\{&\big(|\boldsymbol{k}| + \|\widetilde{\mathbf{T}}_{2,h}(\psi_h, \mathbf{z}_h)\|_{0,4;\Omega}\big)\,\|\varphi_h - \psi_h\|_{0,4;\Omega} \\ &+ \|\widetilde{\mathbf{T}}_{1,h}(\psi_h, \mathbf{z}_h)\|_{0,4;\Omega}\,\|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega}\Big\}\,.\end{aligned} \tag{4.45}$$

Proof. Being the proof analogous to that of Lemma 3.14, we provide in what follows the main steps of it. Indeed, given $(\varphi_h, \mathbf{w}_h)$, $(\psi_h, \mathbf{z}_h) \in \mathrm{H}_h^\phi \times \mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{w}_h\|_{0,4;\Omega}$, $\|\mathbf{z}_h\|_{0,4;\Omega} \leqslant 2\,\alpha_{\mathcal{A},\mathtt{d}}$, we first define $\widetilde{\mathbf{T}}_h(\varphi_h, \mathbf{w}_h) := (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h, \widetilde{\boldsymbol{\eta}}_h)$ and $\widetilde{\mathbf{T}}_h(\psi_h, \mathbf{z}_h) := (\widetilde{\theta}_h, \widetilde{\mathbf{r}}_h, \widetilde{\boldsymbol{\xi}}_h)$. Then, employing the $\widetilde{\mathbb{V}}_h$-ellipticity of $\mathcal{A}_{\varphi_h,\mathbf{w}_h}$, we obtain the discrete analogue of (3.53) (with $\ell = j = 2$), namely

$$\begin{aligned} \|(\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h) - (\widetilde{\theta}_h, \widetilde{\mathbf{r}}_h)\| \leqslant \frac{1}{\alpha_{\mathcal{A},\mathtt{d}}} \Big\{ & L_f\,|\boldsymbol{k}|\,\|\varphi_h - \psi_h\|_{0,\Omega} + L_\vartheta\,\|\varphi_h - \psi_h\|_{0,4;\Omega}\,\|\widetilde{\mathbf{r}}_h\|_{0,4;\Omega} \\ & + \|\widetilde{\theta}_h\|_{0,4;\Omega}\,\|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega} \Big\}. \end{aligned} \tag{4.46}$$

Similarly, by means of the discrete inf-sup condition of $\mathcal{B}$ (cf. (4.36)) and the boundedness of $\mathcal{A}_{\varphi_h,\mathbf{w}_h}$ with constant $\|\mathcal{A}\|_{\mathtt{d}}$ (cf. inequality (4.41) in proof of Lemma 4.11), we obtain the discrete analogue of (3.56) (with $\ell = j = 2$), that is

$$\begin{aligned} \beta_{\mathcal{B},\mathtt{d}}\,\|\widetilde{\boldsymbol{\eta}}_h - \widetilde{\boldsymbol{\xi}}_h\| \leqslant\ & L_f\,|\boldsymbol{k}|\,\|\varphi_h - \psi_h\|_{0,\Omega} + \|\mathcal{A}\|_{\mathtt{d}}\,\|(\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h) - (\widetilde{\theta}_h, \widetilde{\mathbf{r}}_h)\| \\ & + L_\vartheta\,\|\varphi_h - \psi_h\|_{0,4;\Omega}\,\|\widetilde{\mathbf{r}}_h\|_{0,4;\Omega} + \|\widetilde{\theta}_h\|_{0,4;\Omega}\,\|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega}\,. \end{aligned} \tag{4.47}$$

Thus, using the continuous injection of $\mathrm{L}^4(\Omega)$ into $\mathrm{L}^2(\Omega)$ (with boundedness constant $|\Omega|^{1/4}$), and combining (4.46) and (4.47), we arrive at the desired estimate (4.45), thus ending the proof. $\square$

The continuity of the discrete fixed-point operator $\mathbf{T}_h$ is given by the following lemma.

Lemma 4.15 *Assume that the data satisfy the hypotheses of Lemma 4.12, that is (4.42) and (4.43). Then, there exists a positive constant $C_{\mathbf{T},\mathtt{d}}$, depending only on $L_{\bar{\mathbf{T}},\mathtt{d}}$ (cf. Lemma 4.13) and $L_{\widetilde{\mathbf{T}},\mathtt{d}}$ (cf. Lemma 4.14), such that*

$$\begin{aligned} \|\mathbf{T}_h(\varphi_h) - \mathbf{T}_h(\psi_h)\|_{0,4;\Omega} \leqslant C_{\mathbf{T},\mathtt{d}} \Big\{ & |\boldsymbol{k}| + \|\widetilde{\mathbf{T}}_{2,h}\big(\psi_h, \bar{\mathbf{T}}_{2,h}(\psi_h)\big)\|_{0,4;\Omega} \\ & + \|\mathbf{T}_h(\psi_h)\|_{0,4;\Omega} \Big(\|\bar{\mathbf{T}}_{1,h}(\psi_h)\|_{0,4;\Omega} + \|\boldsymbol{f}\|_{0,\Omega}\Big)\Big\}\,\|\varphi_h - \psi_h\|_{0,4;\Omega} \end{aligned} \tag{4.48}$$

for all φ_h, $\psi_h \in W_h$.

Proof. It proceeds analogously to the proof of Lemma 3.15. In fact, we first observe that, given φ_h, ψ_h in W_h, the a priori bound given by (4.34) and the assumption (4.42) guarantee that both $(\varphi_h, \bar{\mathbf{T}}_{2,h}(\varphi_h))$ and $(\psi_h, \bar{\mathbf{T}}_{2,h}(\psi_h))$ accomplish the hypotheses of Lemma 4.14, thus yielding $\mathbf{T}_h(\varphi_h)$ and $\mathbf{T}_h(\psi_h)$ to be well-defined (cf. (4.32)). In this regard, we also notice that (4.42) and (4.43) ensure that $\mathbf{T}_h(W_h) \subseteq W_h$. Finally, it is easily seen that (4.48) follows from the continuity estimates provided by Lemmas 4.14 and 4.13. $\square$

Similarly as observed for the operator $\mathbf{S}_h$ in Lemma 4.6 (cf. Section 4.2), we stress here that the lack of appropriate, uniform in h, upper bounds for $\|\widetilde{\mathbf{T}}_{2,h}\big(\psi_h, \bar{\mathbf{T}}_{2,h}(\psi_h)\big)\|_{0,4;\Omega}$ and $\|\bar{\mathbf{T}}_{1,h}(\psi_h)\|_{0,4;\Omega}$, stops us of concluding from (4.48) that $\mathbf{T}_h$ is a contraction, and hence applying the Banach fixed-point theorem becomes unfeasible. The above considerations suggest to employ, instead, the Brouwer fixed-point theorem (cf. [18, cf. Theorem 9.9-2]). More precisely, thanks to this abstract result and Lemmas 4.12, 4.15, 4.8, and 4.11, we are able to establish the following result.

Theorem 4.16 *Assume that the data satisfy the hypotheses of Lemma 4.12, that is (4.42) and (4.43). Then, the fixed-point equation (4.33) has at least one solution $\phi_h \in W_h$, which means, equivalently, that the Galerkin scheme (4.25) has at least one solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\phi_h, \mathbf{t}_h, \boldsymbol{\eta}_h) \in \mathrm{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}} \times \mathbf{H}_h^{\boldsymbol{\eta}}$, with $\phi_h \in W_h$. Moreover, there hold*

$$\|(\phi_h, \mathbf{t}_h, \boldsymbol{\eta}_h)\| \leqslant C_{\widetilde{\mathbf{T}},\mathtt{d}} \Big\{\gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| + \|g\|_{0,4/3;\Omega}\Big\}, \quad \textit{and} \tag{4.49}$$

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leqslant C_{\bar{\mathbf{T}},\mathtt{d}} \Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega}\,\|\phi_h\|_{0,4;\Omega}\Big\}. \tag{4.50}$$

5 A priori error analysis

In this section we derive a priori error estimates for the discrete mixed–primal and fully–mixed schemes given by (4.4) and (4.25), respectively, and then use the approximation properties of the finite element subspaces involved to derive the corresponding rates of convergence. In what follows, given a subspace Z_h of an arbitrary Banach space $(\mathrm{Z}, \|\cdot\|_{\mathrm{Z}})$, we set

$$\operatorname{dist}(z, \mathrm{Z}_h) := \inf_{z_h \in \mathrm{Z}_h} \|z - z_h\|_{\mathrm{Z}} \qquad \forall\, z \in \mathrm{Z}\,.$$

5.1 The mixed–primal method

Let $(\boldsymbol{\sigma}, \mathbf{u}, \phi) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}_0^1(\Omega)$, with $\phi \in B$, be the unique solution of (2.20), which is guaranteed by the second part of Theorem 3.7, and let $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi}$, with $\phi_h \in B_h$, a solution of (4.4), whose existence was established by Theorem 4.7. We are interested in deriving the Céa estimate for the global error

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \phi) - (\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h)\|\,.$$

To this end, and similarly as in the proof of Lemma 3.4, we let $\mathbf{X} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ with discrete counterpart $\mathbf{X}_h := \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$, and introduce the bilinear form arising from the adding of the two equations forming (2.19), that is, given $\varphi \in \mathrm{H}_0^1(\Omega)$,

$$\mathbf{A}_\varphi\big((\boldsymbol{\rho}, \mathbf{z}), (\boldsymbol{\tau}, \mathbf{v})\big) := \mathbf{a}_\varphi(\boldsymbol{\rho}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{z}) + \mathbf{b}(\boldsymbol{\rho}, \mathbf{v}) - \mathbf{c}(\mathbf{u}, \mathbf{v}) \qquad \forall\, (\boldsymbol{\rho}, \mathbf{z}),\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}\,. \tag{5.1}$$

It follows that the first two rows of (2.20) and (4.4) can be rewritten, respectively, as

$$\mathbf{A}_\phi\big((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})\big) = \mathbf{F}(\boldsymbol{\tau}) + \mathbf{G}_\phi(\mathbf{v}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathrm{H}\,, \tag{5.2}$$

and

$$\mathbf{A}_{\phi_h}\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\big) = \mathbf{F}(\boldsymbol{\tau}_h) + \mathbf{G}_{\phi_h}(\mathbf{v}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{X}_h\,. \tag{5.3}$$

It is clear from (2.15) that for each $\varphi \in \mathrm{H}_0^1(\Omega)$, $\mathbf{A}_\varphi$ is bounded with respective constant, denoted $\|\mathbf{A}\|$, depending only on $\|\mathbf{a}_\varphi\| = \frac{1}{\mu_1}$, $\|\mathbf{b}\| = 1$, and $\|\mathbf{c}\| = \|\mathbf{i}_{4/3}\|\,\|\mathbf{K}^{-1}\|_{\infty,\Omega}$ (cf. (2.16)), and hence independent of φ. Furthermore, we recall from (3.22) (cf. proof of Lemma 3.4) that $\mathbf{A}_\varphi$ satisfies the continuous inf-sup condition

$$\alpha_{\mathbf{A}}\,\|(\boldsymbol{\rho}, \mathbf{z})\| \leqslant \sup_{\substack{(\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X} \\ (\boldsymbol{\tau}, \mathbf{v}) \neq \mathbf{0}}} \frac{\mathbf{A}_\varphi\big((\boldsymbol{\rho}, \mathbf{z}), (\boldsymbol{\tau}, \mathbf{v})\big)}{\|(\boldsymbol{\tau}, \mathbf{v})\|} \qquad \forall\, (\boldsymbol{\rho}, \mathbf{z}) \in \mathbf{X}\,, \tag{5.4}$$

where $\alpha_{\mathbf{A}} := 1/\bar{C}$ depends only on $\|\mathbf{a}_\varphi\| = \frac{1}{\mu_1}$, $\|\mathbf{c}\| = \|\mathbf{i}_{4/3}\|\,\|\boldsymbol{K}^{-1}\|_{\infty,\Omega}$, α, and β. In turn, as stated in the proof of Lemma 4.4, we also have the discrete analogue of (5.4), which means that for each $\varphi_h \in \mathrm{H}_h^{\phi}$ there holds

$$\alpha_{\mathbf{A},\mathtt{d}}\,\|(\boldsymbol{\rho}_h, \mathbf{z}_h)\| \leqslant \sup_{\substack{(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{X}_h \\ (\boldsymbol{\tau}_h, \mathbf{v}_h) \neq \mathbf{0}}} \frac{\mathbf{A}_{\varphi_h}\big((\boldsymbol{\rho}_h, \mathbf{z}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\big)}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|} \qquad \forall\, (\boldsymbol{\rho}_h, \mathbf{z}_h) \in \mathbf{X}_h\,, \tag{5.5}$$

where $\alpha_{\mathbf{A},\mathtt{d}} := 1/\bar{C}_{\mathtt{d}}$ depends only on $\|\mathbf{a}_{\varphi_h}\| = \frac{1}{\mu_1}$, $\|\mathbf{c}\| = \|\mathbf{i}_{4/3}\|\,\|\boldsymbol{K}^{-1}\|_{\infty,\Omega}$, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$.

Having established the above, we now apply a slight variant of the first Strang's Lemma (cf. [25, Lemma 2.27]) to the context given by the continuous and discrete schemes (5.2) and (5.3), respectively, thus obtaining the existence of a positive constant $C_{\mathbf{A}}$, depending only on $\|\mathbf{A}\|$ and $\alpha_{\mathbf{A},\mathtt{d}}$, such that

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leqslant C_{\mathbf{A}}\Big\{\operatorname{dist}\big((\boldsymbol{\sigma}, \mathbf{u}), \mathbf{X}_h\big) + \|\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\|_{(\mathbf{H}_h^{\mathbf{u}})'} + \|\big(\mathbf{A}_\phi - \mathbf{A}_{\phi_h}\big)\big((\boldsymbol{\sigma}, \mathbf{u}), \cdot\big)\|_{\mathbf{X}_h'}\Big\}\,, \tag{5.6}$$

where the consistency terms are given by

$$\|\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\|_{(\mathbf{H}_h^{\mathbf{u}})'} \,:=\, \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{\big(\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\big)(\mathbf{v}_h)}{\|\mathbf{v}_h\|_{0,4;\Omega}}\,, \quad \text{and} \tag{5.7}$$

$$\|\big(\mathbf{A}_\phi - \mathbf{A}_{\phi_h}\big)\big((\boldsymbol{\sigma},\mathbf{u}),\cdot\big)\|_{\mathbf{X}_h'} \,=\, \|(\mathbf{a}_\phi - \mathbf{a}_{\phi_h})(\boldsymbol{\sigma},\cdot)\|_{\left(\mathbb{H}_h^{\boldsymbol{\sigma}}\right)'} \,:=\, \sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{(\mathbf{a}_\phi - \mathbf{a}_{\phi_h})(\boldsymbol{\sigma},\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}}\,. \tag{5.8}$$

We stress here that the aforementioned variant arises when the first component of the discrete bilinear form can be evaluated in the exact solution. In this case, and after subtracting and adding the latter in the first component of both forms, the consistency term regarding them (cf. last expression in [25, Lemma 2.27, eq. (2.21)]) becomes separated from the respective infimum, and hence can be handled independently from it. This is precisely the situation with $\mathbf{A}_\phi$ and $\mathbf{A}_{\phi_h}$, which explains the way (5.6) has been derived.

Now, according to the definition of $\mathbf{G}_\phi$ (cf. (2.14)), and proceding as for the boundedness of this functional (cf. second inequality in (2.17)), we readily find that

$$\big(\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\big)(\mathbf{v}_h) = \int_\Omega \big(\phi_h - \phi\big)\,\boldsymbol{f}\cdot\mathbf{v}_h \,\leqslant\, \|\mathrm{i}_4\|\,\|\boldsymbol{f}\|_{0,\Omega}\,\|\phi-\phi_h\|_{1,\Omega}\,\|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\,\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}\,,$$

which yields

$$\|\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\|_{(\mathbf{H}_h^{\mathbf{u}})'} \,\leqslant\, \|\mathrm{i}_4\|\,\|\boldsymbol{f}\|_{0,\Omega}\,\|\phi-\phi_h\|_{1,\Omega}\,. \tag{5.9}$$

In turn, recalling the definition of $\mathbf{a}_\phi$ (cf. (2.13)), we get

$$(\mathbf{a}_\phi - \mathbf{a}_{\phi_h})(\boldsymbol{\sigma},\boldsymbol{\tau}_h) \,=\, \int_\Omega \left(\frac{\mu(\phi_h)-\mu(\phi)}{\mu(\phi)\mu(\phi_h)}\right)\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}_h^{\mathrm{d}} \qquad \forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\,, \tag{5.10}$$

so that, using the lower bound and the Lipschitz-continuity of μ (cf. (1.2), (1.3)), the Cauchy–Schwarz and Hölder inequalities, the regularity assumption (3.19), and the continuous injections given by $\mathrm{i}_{\tilde{\varepsilon}} : \mathrm{H}^\varepsilon(\Omega)\to\mathrm{L}^{\tilde{\varepsilon}}(\Omega)$ and $\mathrm{i}_\varepsilon : \mathrm{H}^1(\Omega)\to\mathrm{L}^{n/\varepsilon}(\Omega)$, exactly as in the proof of [7, Lemma 3.7] (see, also, the last part of the proof of Lemma 3.4), we find from (5.10) that

$$(\mathbf{a}_\phi - \mathbf{a}_{\phi_h})(\boldsymbol{\sigma},\boldsymbol{\tau}_h) \,\leqslant\, L_\mu\,\mu_1^{-2}\,\|\mathrm{i}_{\tilde{\varepsilon}}\|\,\|\mathrm{i}_\varepsilon\|\,\|\boldsymbol{\sigma}\|_{\varepsilon,\Omega}\,\|\phi-\phi_h\|_{1,\Omega}\,\|\boldsymbol{\tau}_h\|_{0,\Omega} \qquad \forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\,, \tag{5.11}$$

from which, along with the upper bound of $\|\boldsymbol{\sigma}\|_{\varepsilon,\Omega}$ provided by (3.19), we arrive at

$$\|(\mathbf{a}_\phi - \mathbf{a}_{\phi_h})(\boldsymbol{\sigma},\cdot)\|_{\left(\mathbb{H}_h^{\boldsymbol{\sigma}}\right)'} \,\leqslant\, L_\mu\,\mu_1^{-2}\,\|\mathrm{i}_{\tilde{\varepsilon}}\|\,\|\mathrm{i}_\varepsilon\|\,\bar{C}_{\mathbf{S}}(r)\left\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\right\}\|\phi-\phi_h\|_{1,\Omega}\,. \tag{5.12}$$

In this way, replacing (5.9) and (5.12) back into (5.6), we deduce the existence of a positive constant $\bar{C}_{\mathbf{A}}$, depending only on $C_{\mathbf{A}}$, L_μ, μ_1, $\|\mathrm{i}_4\|$, $\|\mathrm{i}_{\tilde{\varepsilon}}\|$, $\|\mathrm{i}_\varepsilon\|$, and $\bar{C}_{\mathbf{S}}(r)$, such that

$$\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,\leqslant\, C_{\mathbf{A}}\,\mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{X}_h\big) \,+\, \bar{C}_{\mathbf{A}}\left\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\right\}\|\phi-\phi_h\|_{1,\Omega}\,. \tag{5.13}$$

On the other hand, regarding the third rows of (2.20) and (4.4), which read

$$\begin{aligned} A_{\mathbf{u}}(\phi,\psi) &= G_\phi(\psi) \qquad \forall\,\psi\in\mathrm{H}_0^1(\Omega)\,,\\ A_{\mathbf{u}_h}(\phi_h,\psi_h) &= G_{\phi_h}(\psi_h) \qquad \forall\,\psi_h\in\mathrm{H}_h^\phi\,, \end{aligned} \tag{5.14}$$

we resort to the closely related result given by [4, Lemma 5.3]. More precisely, proceeding almost verbatim to its proof, we are able to show that there exist positive constants D_A (depending on $C_{\bar{\mathbf{S}}}$, $C_{\widetilde{\mathbf{S}}}$, ϑ_2, ϱ, $\|\mathrm{i}_4\|$, and r), $\bar{D}_A$ (depending on $C_{\widetilde{\mathbf{S}}}$, $\widetilde{C}_{\widetilde{\mathbf{S}}}(r)$, L_f, L_ϑ, γ_f, $\|\mathrm{i}_\varepsilon\|$, $\|\mathrm{i}_{\widehat{\varepsilon}}\|$, and $|\Omega|$), and $\widetilde{D}_A$ (depending on $C_{\widetilde{\mathbf{S}}}$, $\|\mathrm{i}_4\|$, and r), such that

$$
\begin{aligned}
\|\phi-\phi_h\|_{1,\Omega} \,\leqslant\, & D_A\Big\{1+\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\boldsymbol{f}\|_{0,\Omega}\Big\}\,\mathrm{dist}\big(\phi,\mathrm{H}_h^\phi\big)\\
+ \quad & \bar{D}_A\Big\{|\boldsymbol{k}|+\|g\|_{\varepsilon,\Omega}\Big\}\,\|\phi-\phi_h\|_{1,\Omega} \,+\, \widetilde{D}_A\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.
\end{aligned}
\tag{5.15}
$$

Then, assuming that there holds

$$
\bar{D}_A\Big\{|\boldsymbol{k}|+\|g\|_{\varepsilon,\Omega}\Big\} \,\leqslant\, \frac{1}{2}\,,
\tag{5.16}
$$

we obtain from (5.15) that

$$
\|\phi-\phi_h\|_{1,\Omega} \,\leqslant\, 2\,D_A\Big\{1+\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\boldsymbol{f}\|_{0,\Omega}\Big\}\,\mathrm{dist}\big(\phi,\mathrm{H}_h^\phi\big) \,+\, 2\,\widetilde{D}_A\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.
\tag{5.17}
$$

Next, replacing (5.17) in (5.13), we deduce the existence of positive constants $E_{\mathbf{A}}$ $(= C_{\mathbf{A}})$, $\bar{E}_{\mathbf{A}}$ (depending on $\bar{C}_{\mathbf{A}}$, D_A, $\|\mathbf{u}_D\|_{1/2,\Gamma}$, $\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma}$, and $\|\boldsymbol{f}\|_{0,\Omega}$), and $\widetilde{E}_{\mathbf{A}}$ $(= 2\,\bar{C}_{\mathbf{A}}\,\widetilde{D}_A)$, such that

$$
\begin{aligned}
\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,\leqslant\, & E_{\mathbf{A}}\,\mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{X}_h\big) \,+\, \bar{E}_{\mathbf{A}}\,\mathrm{dist}\big(\phi,\mathrm{H}_h^\phi\big)\\
+ \quad & \widetilde{E}_{\mathbf{A}}\,\Big\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\Big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.
\end{aligned}
\tag{5.18}
$$

We are now in position to establish the final a priori error estimate for the mixed-primal method.

Theorem 5.1 *In addition to the hypotheses required by Theorems 3.7 and 4.7, assume that the data are sufficiently small so that (5.16) and*

$$
\widetilde{E}_{\mathbf{A}}\,\Big\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\Big\} \,\leqslant\, \frac{1}{2}\,,
\tag{5.19}
$$

are satisfied. Then, there exists a constant $C>0$, independent of h, but depending on data, such that

$$
\|(\boldsymbol{\sigma},\mathbf{u},\phi)-(\boldsymbol{\sigma}_h,\mathbf{u}_h,\phi_h)\| \,\leqslant\, C\,\Big\{\mathrm{dist}\big(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}\big) \,+\, \mathrm{dist}\big(\mathbf{u},\mathbf{H}_h^{\mathbf{u}}\big) \,+\, \mathrm{dist}\big(\phi,\mathrm{H}_h^\phi\big)\Big\}\,.
\tag{5.20}
$$

Proof. It follows employing (5.19) in (5.18), combining the resulting estimate with (5.17), and finally noticing that $\mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{X}_h\big) \,=\, \mathrm{dist}\big(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}\big) \,+\, \mathrm{dist}\big(\mathbf{u},\mathbf{H}_h^{\mathbf{u}}\big)$. $\square$

5.2 The fully–mixed method

Let $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)$ and $(\phi,\mathbf{t},\boldsymbol{\eta}) \in \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, with $\phi\in W$, be the unique solution of (2.32), which is guaranteed by Theorem 3.16, and let $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}$ and $(\phi_h,\mathbf{t}_h,\boldsymbol{\eta}_h)\in\mathrm{H}_h^\phi\times\mathbf{H}_h^{\mathbf{t}}\times\mathbf{H}_h^{\boldsymbol{\eta}}$, with $\phi_h\in W_h$, be a solution of (4.25), whose existence was established by Theorem 4.16. We are interested now in deriving the Céa estimate for the global error

$$
\|(\boldsymbol{\sigma},\mathbf{u},\phi,\mathbf{t},\boldsymbol{\eta}) \,-\, (\boldsymbol{\sigma}_h,\mathbf{u}_h,\phi_h,\mathbf{t}_h,\boldsymbol{\eta}_h)\|\,.
$$

We begin with the estimate regarding $\|(\boldsymbol{\sigma},\mathbf{u}) \,-\, (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|$, whose derivation is, except some minor differences, basically the same one provided in the previous section. The reason for it is that the first two rows of (2.20) (resp. (4.4)) and (2.32) (resp. (4.25)) differ only on the space where ϕ (resp. ϕ_h) is taken, which is $\mathrm{H}_0^1(\Omega)$ (resp. a subspace of it) for the former, and $\mathrm{L}^4(\Omega)$ (resp. a subspace of it) for the latter. As a consequence, instead of (5.9) we obtain

$$\|\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\|_{(\mathbf{H}_h^{\mathbf{u}})'} \leqslant \|\boldsymbol{f}\|_{0,\Omega}\,\|\phi - \phi_h\|_{0,4;\Omega}\,, \tag{5.21}$$

whereas, using once again the lower bound and the Lipschitz-continuity of μ (cf. (1.2), (1.3)), the Cauchy–Schwarz and Hölder inequalities, the regularity assumption (3.46), and the continuous injections given by $\mathrm{i}_{\tilde{\delta}} : \mathrm{H}^\delta(\Omega) \to \mathrm{L}^{\tilde{\delta}}(\Omega)$ and $\mathrm{i}_\delta : \mathrm{L}^4(\Omega) \to \mathrm{L}^{n/\delta}(\Omega)$, similarly as for the derivation of (5.11), we find now from (5.10) that

$$(\mathbf{a}_\phi - \mathbf{a}_{\phi_h})(\boldsymbol{\sigma}, \boldsymbol{\tau}_h) \leqslant L_\mu\,\mu_1^{-2}\,\|\mathrm{i}_{\tilde{\delta}}\|\,\|\mathrm{i}_\delta\|\,\|\boldsymbol{\sigma}\|_{\delta,\Omega}\,\|\phi - \phi_h\|_{0,4;\Omega}\,\|\boldsymbol{\tau}_h\|_{0,\Omega} \qquad \forall\,\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,.$$

In this way, employing the upper bound of $\|\boldsymbol{\sigma}\|_{\delta,\Omega}$ provided by (3.46), we arrive at

$$\|(\mathbf{a}_\phi - \mathbf{a}_{\phi_h})(\boldsymbol{\sigma}, \cdot)\|_{\left(\mathbb{H}_h^{\boldsymbol{\sigma}}\right)'} \leqslant L_\mu\,\mu_1^{-2}\,\|\mathrm{i}_{\tilde{\delta}}\|\,\|\mathrm{i}_\delta\|\,\bar{C}_{\bar{\mathbf{T}}}(r)\left\{\|\mathbf{u}_D\|_{1/2+\delta,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega}\right\}\|\phi - \phi_h\|_{0,4;\Omega}\,, \tag{5.22}$$

and hence, replacing (5.21) and (5.22) back into (5.6), being this latter inequality still valid here, we deduce the existence of a positive constant $\bar{C}_{\mathbf{A}}$, depending only on $C_{\mathbf{A}}$, L_μ, μ_1, $\|\mathrm{i}_{\tilde{\delta}}\|$, $\|\mathrm{i}_\delta\|$, and $\bar{C}_{\bar{\mathbf{T}}}(r)$, such that

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leqslant C_{\mathbf{A}}\,\mathrm{dist}\big((\boldsymbol{\sigma}, \mathbf{u}), \mathbf{X}_h\big) + \bar{C}_{\mathbf{A}}\left\{\|\mathbf{u}_D\|_{1/2+\delta,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega}\right\}\|\phi - \phi_h\|_{0,4;\Omega}\,. \tag{5.23}$$

Note that, while the present constant $\bar{C}_{\mathbf{A}}$ does not necessarily coincide with the one from (5.13) (cf. Section 4.2), we use the same notation just for simplicity.

On the other hand, regarding $\|(\phi, \mathbf{t}, \boldsymbol{\eta}) - (\phi_h, \mathbf{t}_h, \boldsymbol{\eta}_h)\|$, which has to do with the third and fourth rows of (2.32) and (4.25), we simply apply the general Strang estimate provided in [26, Theorem 2.2, eqs. (2.26) and (2.27)]. Thus, denoting $\mathbf{Y}_h := \mathrm{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}}$, we find that there exists a positive constant $D_{\mathcal{A}}$, depending only on $\alpha_{\mathcal{A},\mathtt{d}}$ (cf. Lemma 4.10), $\beta_{\mathcal{B},\mathtt{d}}$ (cf. Lemma 4.9), and $\|\mathcal{A}\|_{\mathtt{d}}$ (cf. (4.41)), which in turn, depends on ϑ_2, $\alpha_{\mathcal{A},\mathtt{d}}$, $|\Omega|$, and ϱ, such that

$$\begin{aligned} \|(\phi, \mathbf{t}, \boldsymbol{\eta}) - (\phi_h, \mathbf{t}_h, \boldsymbol{\eta}_h)\| \leqslant D_{\mathcal{A}}\Big\{&\mathrm{dist}\big((\phi, \mathbf{t}), \mathbf{Y}_h\big) + \mathrm{dist}\big(\boldsymbol{\eta}, \mathbf{H}_h^{\boldsymbol{\eta}}\big) \\ &+ \|\mathcal{F}_\phi - \mathcal{F}_{\phi_h}\|_{\mathbf{Y}_h'} + \big\|\big(\mathcal{A}_{\phi,\mathbf{u}} - \mathcal{A}_{\phi_h,\mathbf{u}_h}\big)\big((\phi, \mathbf{t}), \cdot\big)\big\|_{\mathbf{Y}_h'}\Big\}\,, \end{aligned} \tag{5.24}$$

where the consistency terms are given by

$$\|\mathcal{F}_\phi - \mathcal{F}_{\phi_h}\|_{\mathbf{Y}_h'} := \sup_{\substack{(\psi_h,\mathbf{s}_h)\in\mathbf{Y}_h \\ (\psi_h,\mathbf{s}_h)\neq\mathbf{0}}} \frac{\big(\mathcal{F}_\phi - \mathcal{F}_{\phi_h}\big)\big((\psi_h, \mathbf{s}_h)\big)}{\|(\psi_h, \mathbf{s}_h)\|}\,, \quad \text{and} \tag{5.25}$$

$$\big\|\big(\mathcal{A}_{\phi,\mathbf{u}} - \mathcal{A}_{\phi_h,\mathbf{u}_h}\big)\big((\phi, \mathbf{t}), \cdot\big)\big\|_{\mathbf{Y}_h'} := \sup_{\substack{(\psi_h,\mathbf{s}_h)\in\mathbf{Y}_h \\ (\psi_h,\mathbf{s}_h)\neq\mathbf{0}}} \frac{\big(\mathcal{A}_{\phi,\mathbf{u}} - \mathcal{A}_{\phi_h,\mathbf{u}_h}\big)\big((\phi, \mathbf{t}), (\psi_h, \mathbf{s}_h)\big)}{\|(\psi_h, \mathbf{s}_h)\|}\,. \tag{5.26}$$

Hence, proceeding as we did in (3.51), and using that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{1/4}\,\|\cdot\|_{0,4;\Omega}$, we readily obtain

$$\|\mathcal{F}_\phi - \mathcal{F}_{\phi_h}\|_{\mathbf{Y}_h'} \leqslant L_f\,|\Omega|^{1/4}\,|\boldsymbol{k}|\,\|\phi - \phi_h\|_{0,4;\Omega}\,. \tag{5.27}$$

Similarly, following the same steps yielding (3.52), and then employing the continuous injections $\mathrm{i}_\delta : \mathrm{L}^4(\Omega) \to \mathrm{L}^{n/\delta}(\Omega)$ and $\mathbf{i}_{\widetilde{\delta}} : \mathbf{H}^\delta(\Omega) \to \mathbf{L}^{\widetilde{\delta}}(\Omega)$, we find that

$$\begin{aligned}&\big(\mathcal{A}_{\phi,\mathbf{u}} - \mathcal{A}_{\phi_h,\mathbf{u}_h}\big)\big((\phi,\mathbf{t}),(\psi_h,\mathbf{s}_h)\big)\\ &\quad \leqslant \ \Big\{ L_\vartheta\,\|\mathrm{i}_\delta\|\,\|\mathbf{i}_{\widetilde{\delta}}\|\,\|\phi-\phi_h\|_{0,4;\Omega}\,\|\mathbf{t}\|_{\delta,\Omega} + \|\phi\|_{0,4;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,\|\mathbf{s}_h\|_{0,\Omega}\,,\end{aligned}$$

from which, invoking the upper bound of $\|\mathbf{t}\|_{\delta,\Omega}$ provided by (3.48), using that $\|\phi\|_{0,4;\Omega} \leqslant r$, and replacing the resulting estimate in (5.26), we arrive at

$$\begin{aligned}&\big\|\big(\mathcal{A}_{\phi,\mathbf{u}} - \mathcal{A}_{\phi_h,\mathbf{u}_h}\big)\big((\phi,\mathbf{t}),\cdot\big)\big\|_{\mathbf{Y}_h'}\\ &\quad \leqslant \ L_\vartheta\,\|\mathrm{i}_\delta\|\,\|\mathbf{i}_{\widetilde{\delta}}\|\,\widetilde{C}_{\widetilde{\mathbf{T}}}(r)\,\Big\{\gamma_f\,|\Omega|^{1/2}\,|\boldsymbol{k}| \,+\, \|g\|_{\delta,4/3;\Omega}\Big\}\,\|\phi-\phi_h\|_{0,4;\Omega} + r\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.\end{aligned} \tag{5.28}$$

Thus, replacing (5.27) and (5.28) in (5.24), we deduce the existence of positive constants $\bar{D}_{\mathcal{A}}$ (depending on $D_{\mathcal{A}}$, L_f, L_ϑ, γ_f, $\|\mathrm{i}_\delta\|$, $\|\mathbf{i}_{\widetilde{\delta}}\|$, $\widetilde{C}_{\widetilde{\mathbf{T}}}(r)$, and $|\Omega|$), and $\widetilde{D}_{\mathcal{A}}$ $(= D_{\mathcal{A}}\,r)$, such that

$$\begin{aligned}\|(\phi,\mathbf{t},\boldsymbol{\eta}) \,-\, (\phi_h,\mathbf{t}_h,\boldsymbol{\eta}_h)\| \,&\leqslant\, D_{\mathcal{A}}\Big\{\mathrm{dist}\big((\phi,\mathbf{t}),\mathbf{Y}_h\big) + \mathrm{dist}\big(\boldsymbol{\eta},\mathbf{H}_h^{\boldsymbol{\eta}}\big)\Big\}\\ &+\ \ \bar{D}_{\mathcal{A}}\Big\{|\boldsymbol{k}| \,+\, \|g\|_{\delta,4/3;\Omega}\Big\}\,\|\phi-\phi_h\|_{0,4;\Omega} \,+\, \widetilde{D}_{\mathcal{A}}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.\end{aligned} \tag{5.29}$$

Then, assuming now that

$$\bar{D}_{\mathcal{A}}\Big\{|\boldsymbol{k}| \,+\, \|g\|_{\delta,4/3;\Omega}\Big\} \,\leqslant\, \frac{1}{2}\,, \tag{5.30}$$

we get from (5.29)

$$\|(\phi,\mathbf{t},\boldsymbol{\eta}) \,-\, (\phi_h,\mathbf{t}_h,\boldsymbol{\eta}_h)\| \,\leqslant\, 2\,D_{\mathcal{A}}\Big\{\mathrm{dist}\big((\phi,\mathbf{t}),\mathbf{Y}_h\big) + \mathrm{dist}\big(\boldsymbol{\eta},\mathbf{H}_h^{\boldsymbol{\eta}}\big)\Big\} \,+\, 2\,\widetilde{D}_{\mathcal{A}}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,. \tag{5.31}$$

Next, bounding $\|\phi-\phi_h\|_{0,4;\Omega}$ in (5.23) by the right hand side of (5.31), we find positive constants $E_{\mathcal{A}}$ $(= C_{\mathbf{A}})$, $\bar{E}_{\mathcal{A}}$ (depending on $\bar{C}_{\mathbf{A}}$, $D_{\mathcal{A}}$, $\|\mathbf{u}_D\|_{1/2+\delta,\Gamma}$, and $\|\boldsymbol{f}\|_{0,\Omega}$), and $\widetilde{E}_{\mathcal{A}}$ $(= 2\,\bar{C}_{\mathbf{A}}\,\widetilde{D}_{\mathcal{A}})$, such that

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,&\leqslant\, E_{\mathcal{A}}\,\mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{X}_h\big) \,+\, \bar{E}_{\mathcal{A}}\Big\{\mathrm{dist}\big((\phi,\mathbf{t}),\mathbf{Y}_h\big) + \mathrm{dist}\big(\boldsymbol{\eta},\mathbf{H}_h^{\boldsymbol{\eta}}\big)\Big\}\\ &+\ \ \widetilde{E}_{\mathcal{A}}\Big\{\|\mathbf{u}_D\|_{1/2+\delta,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\Big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.\end{aligned} \tag{5.32}$$

The final a priori error estimate for the fully-mixed method is then stated as follows.

Theorem 5.2 *In addition to the hypotheses required by Theorems 3.16 and 4.16, assume that the data are sufficiently small so that (5.30) and*

$$\widetilde{E}_{\mathcal{A}}\Big\{\|\mathbf{u}_D\|_{1/2+\delta,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\Big\} \,\leqslant\, \frac{1}{2}\,, \tag{5.33}$$

are satisfied. Then, there exists a constant $C > 0$, independent of h, but depending on data, such that

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u},\phi,\mathbf{t},\boldsymbol{\eta}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h,\phi_h,\mathbf{t}_h,\boldsymbol{\eta}_h)\| \,&\leqslant\, C\,\Big\{\mathrm{dist}\big(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}\big) \,+\, \mathrm{dist}\big(\mathbf{u},\mathbf{H}_h^{\mathbf{u}}\big)\\ &+\ \ \mathrm{dist}\big(\phi,\mathrm{H}_h^{\phi}\big) \,+\, \mathrm{dist}\big(\mathbf{t},\mathbf{H}_h^{\mathbf{t}}\big) \,+\, \mathrm{dist}\big(\boldsymbol{\eta},\mathbf{H}_h^{\boldsymbol{\eta}}\big)\Big\}\,.\end{aligned} \tag{5.34}$$

Proof. Similarly as for the proof of Theorem 5.1, it suffices to employ (5.33) in (5.32), combine the resulting estimate with (5.31), and then decompose $\mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{X}_h\big)$ and $\mathrm{dist}\big((\phi,\mathbf{t}),\mathbf{Y}_h\big)$ in terms of their respective components. $\square$

5.3 The rates of convergence

In this section we establish the rates of convergence of (4.4) and (4.25). We begin with the former by recalling the approximation properties of the respective finite elements subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and H_h^{ϕ}. As usual, those properties are derived by suitable projection and interpolation operators, along with interpolation estimates in Sobolev spaces (see e.g. [10, 11, 20, 27]). The ones for (4.4) read as follows:

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists $C > 0$, independent of h, such that for each $l \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}) := \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \leqslant C\,h^l \left\{ \|\boldsymbol{\tau}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega} \right\},$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) := \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \leqslant C\,h^l\,\|\mathbf{v}\|_{l,4;\Omega},$$

and

$(\mathbf{AP}_h^{\phi})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathrm{H}^{l+1}(\Omega)$, there holds

$$\operatorname{dist}(\psi, \mathrm{H}_h^{\phi}) := \inf_{\psi_h \in \mathrm{H}_h^{\phi}} \|\psi - \psi_h\|_{1,\Omega} \leqslant C\,h^l\,\|\psi\|_{l+1,\Omega}.$$

Then, the rates of convergence of the Galerkin scheme (4.4) are given by the following theorem.

Theorem 5.3 *In addition to the hypotheses of Theorems 3.7, 4.7, and 5.1, assume that there exists $l \in (0, k+1]$ such that $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$, and $\phi \in \mathrm{H}^{l+1}(\Omega)$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|\phi - \phi_h\|_{1,\Omega} \\ &\qquad \leqslant C\,h^l \left\{ \|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} + \|\mathbf{u}\|_{l,4;\Omega} + \|\phi\|_{l+1,\Omega} \right\}. \end{aligned}$$

Proof. It follows from Theorem 5.1 along with the above properties $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, $(\mathbf{AP}_h^{\mathbf{u}})$, and $(\mathbf{AP}_h^{\phi})$. □

We now add the a approximation properties of the remaining finite element subspaces, besides $\mathbb{H}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$, employed in (4.25), namely H_h^{ϕ}, $\mathbf{H}_h^{\mathbf{t}}$, and $\mathbf{H}_h^{\boldsymbol{\eta}}$:

$(\mathbf{AP}_h^{\phi})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathrm{W}^{l,4}(\Omega)$, there holds

$$\operatorname{dist}(\psi, \mathrm{H}_h^{\phi}) := \inf_{\psi_h \in \mathrm{H}_h^{\phi}} \|\psi - \psi_h\|_{0,4;\Omega} \leqslant C\,h^l\,\|\psi\|_{l,4;\Omega},$$

$(\mathbf{AP}_h^{\mathbf{t}})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{s} \in \mathbf{H}^l(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{s}, \mathbf{H}_h^{\mathbf{t}}) := \inf_{\mathbf{s}_h \in \mathbf{H}_h^{\mathbf{t}}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \leqslant C\,h^l\,\|\mathbf{s}\|_{l,\Omega},$$

and

$\left(\mathbf{AP}_h^{\boldsymbol{\eta}}\right)$ there exists $C > 0$, independent of h, such that for each $l \in (0, k+1]$, and for each $\boldsymbol{\chi} \in \mathbf{H}^l(\Omega) \cap \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, with $\mathrm{div}(\boldsymbol{\chi}) \in \mathrm{W}^{l,4/3}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\chi}, \mathbf{H}_h^{\boldsymbol{\eta}}) := \inf_{\boldsymbol{\chi}_h \in \mathbf{H}_h^{\boldsymbol{\eta}}} \|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{\mathrm{div}_{4/3};\Omega} \leqslant C\, h^l \left\{ \|\boldsymbol{\chi}\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\chi})\|_{l,4/3;\Omega} \right\}.$$

The rates of convergence of the Galerkin scheme (4.25) are then stated in the following theorem.

Theorem 5.4 *In addition to the hypotheses of Theorems 3.16, 4.16, and 5.2, assume that there exists* $l \in (0, k+1]$ *such that* $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$, $\phi \in \mathrm{W}^{l,4}(\Omega)$, $\mathbf{t} \in \mathbf{H}^l(\Omega)$, $\boldsymbol{\eta} \in \mathbf{H}^l(\Omega) \cap \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, *and* $\mathrm{div}(\boldsymbol{\eta}) \in \mathrm{W}^{l,4/3}(\Omega)$. *Then, there exists a positive constant* C*, independent of* h*, such that*

$$\begin{aligned} &\|(\boldsymbol{\sigma}, \mathbf{u}, \phi, \mathbf{t}, \boldsymbol{\eta}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h, \mathbf{t}_h, \boldsymbol{\eta}_h)\| \\ &\quad \leqslant C\, h^l \left\{ \|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} + \|\mathbf{u}\|_{l,4;\Omega} + \|\phi\|_{l,4;\Omega} + \|\mathbf{t}\|_{l,\Omega} + \|\boldsymbol{\eta}\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\eta})\|_{l,4/3;\Omega} \right\}. \end{aligned}$$

Proof. It follows from Theorem 5.2 along with $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, $(\mathbf{AP}_h^{\mathbf{u}})$, and the three foregoing approximation properties. □

6 Numerical tests

In this section we consider four examples to illustrate the performance of our mixed finite element methods on sets of quasi-uniform triangulations of their domains. As previously indicated, we use the finite element subspaces given by (4.1), (4.2), and (4.3), for the mixed–primal scheme (4.4), whereas those defined by (4.20), (4.21), (4.22), (4.23), and (4.24), are considered for the fully–mixed one (4.25). In what follows, we refer to the corresponding sets of finite element subpaces generated by $k = 0$ and $k = 1$, simply as $\mathbb{RT}_k - \mathbf{P}_k - \mathrm{P}_{k+1}$ for the mixed–primal case, and $\mathbb{RT}_k - \mathbf{P}_k - \mathrm{P}_k - \mathbf{P}_k - \mathbf{RT}_k$ for the fully–mixed one. The set of computational tests collected in this section have been implemented using the open source finite element library FEniCS [2]. The nonlinear algebraic systems arising from the discrete schemes are solved via Newton's method with a residual tolerance of 10^{-6}, and the linear systems of the respective iterations are solved with the UMFPACK solver [24]. The zero-mean condition for the trace of the pseudostress is enforced using a real Lagrange multiplier.

We now introduce some additional notation. The individual errors are denoted by:

$$\begin{gathered} \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}, \quad \mathsf{e}(p) := \|p - p_h\|_{0,\Omega}, \\ \mathsf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}, \quad \mathsf{e}(\boldsymbol{\eta}) := \|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{\mathrm{div}_{4/3};\Omega}, \quad \text{and} \end{gathered}$$

$$\mathsf{e}(\phi) := \begin{cases} \|\phi - \phi_h\|_{1,\Omega} & \text{for the mixed–primal approach}, \\ \|\phi - \phi_h\|_{0,4;\Omega} & \text{for the fully–mixed approach}. \end{cases}$$

We stress that p_h corresponds to the post-processed pressure p_h suggested by (1.5), that is

$$p_h = -\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma}_h),$$

whose error $\mathsf{e}(p)$ is certainly of the same order of $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$, and hence is controlled by $\mathsf{e}(\boldsymbol{\sigma})$. In addition, while we do not include it in the numerical experiments to be reported in this section, we

DoF	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}\ (\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}\ (\mathbf{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	it
58	0.707	9.42e+01	-	7.03e-01	-	1.33e+00	-	8.40e-01	-	4
202	0.354	6.82e+01	0.47	4.26e-01	0.72	7.99e-01	0.73	8.74e-01	-0.06	4
754	0.177	3.71e+01	0.88	2.28e-01	0.90	4.34e-01	0.88	5.40e-01	0.70	4
2914	0.088	1.78e+01	1.06	1.16e-01	0.97	2.24e-01	0.96	1.94e-01	1.48	4
11458	0.044	8.34e+00	1.09	5.84e-02	0.99	1.13e-01	0.99	5.13e-02	1.92	4
45442	0.022	3.96e+00	1.07	2.92e-02	1.00	5.65e-02	1.00	1.27e-02	2.02	4
180994	0.011	1.91e+00	1.05	1.46e-02	1.00	2.82e-02	1.00	3.21e-03	1.98	4

Table 6.1: Example 1, number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the mixed–primal $\mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_1$.

highlight that the first equation in (1.6) (or (1.7)) suggests a post-processed approximation as well for the velocity gradient, namely

$$(\nabla \mathbf{u})_h := \frac{1}{\mu(\phi_h)}\, \boldsymbol{\sigma}_h^{\mathrm{d}} .$$

Next, for each $\star \in \{\boldsymbol{\sigma}, \mathbf{u}, \phi, p, \mathbf{t}, \boldsymbol{\eta}\}$, the convergence rates $r(\star)$ are computed as

$$\mathtt{r}(\star) = \frac{\log(\mathtt{e}(\star)/\widehat{\mathtt{e}}(\star))}{\log(h/\widehat{h})} ,$$

where $\mathtt{e}$ and $\widehat{\mathtt{e}}$ denote errors produced on two consecutive meshes associated with mesh sizes h and $\widehat{h}$, respectively. In turn, we refer to DoF as the number of degrees of freedom and `it` as the number of Newton iterations.

6.1 Example 1: 2D smooth exact solution for the mixed–primal scheme

In the first computational experiment, we aim to demonstrate the precision of the mixed–primal scheme in two dimensions. To achieve this, we use a manufactured exact solution defined within the unit square $\Omega := (0,1)^2$. We let

$$\mu(\phi) = (1 - 0.5\,\phi)^{-2} , \quad \vartheta(\phi) = \exp(-\phi^2) , \quad f_{\mathrm{bk}}(\phi) = 0.5\,\phi\,(1 - 0.5\,\phi)^2 ,$$
$$\boldsymbol{K} = 0.01\,\mathbb{I} , \quad \varrho = 10 , \quad \text{and} \quad \boldsymbol{k} = (0, -1)^{\mathrm{t}} ,$$

and adjust the source terms $\boldsymbol{f}$ and g in (1.6) to ensure that $\boldsymbol{\sigma}$, $\mathbf{u}$, and ϕ are given by the smooth functions

$$\boldsymbol{\sigma} = \mu(\phi)\,\nabla\mathbf{u} - (x_1^2 - x_2^2)\,\mathbb{I} , \quad \mathbf{u}(x_1, x_2) = \begin{pmatrix} \sin(2\pi x_1)\,\cos(2\pi x_2) \\ -\cos(2\pi x_1)\,\sin(2\pi x_2) \end{pmatrix} ,$$
$$\text{and} \quad \phi(x_1, x_2) = 15 - 15\,\exp(-x_1\,(x_1 - 1)\,x_2\,(x_2 - 1)) .$$

Note that ϕ vanishes in Γ and $\mathbf{u}_D$ is imposed according to the exact solution. Tables 6.1–6.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations for the approximations. The experiments confirm the theoretical convergences rates $\mathcal{O}(h^{k+1})$ for $k = 0, 1$, provided by Theorem 5.3. The Newton method converges in four iterations for all cases, the convergence being therefore independent of the mesh size.

DoF	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}\ (\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}\ (\mathbf{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	it
170	0.707	5.23e+01	-	3.09e-01	-	3.65e-01	-	1.28e+00	-	4
626	0.354	1.87e+01	1.48	1.41e-01	1.14	1.10e-01	1.74	3.51e-01	1.87	4
2402	0.177	4.94e+00	1.92	3.56e-02	1.98	2.96e-02	1.89	1.05e-01	1.74	4
9410	0.088	1.26e+00	1.97	9.06e-03	1.97	7.66e-03	1.95	2.87e-02	1.88	4
37250	0.044	3.16e-01	2.00	2.27e-03	1.99	1.95e-03	1.98	7.36e-03	1.96	4
148226	0.022	7.88e-02	2.00	5.69e-04	2.00	4.91e-04	1.99	1.85e-03	1.99	4

Table 6.2: Example 1, number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the mixed–primal $\mathbb{RT}_1 - \mathbf{P}_1 - \mathrm{P}_2$.

6.2 Example 2: 2D smooth exact solution for the fully–mixed scheme

In this second example, we demonstrate the accuracy of the fully-mixed scheme in two dimensions by examining a manufactured exact solution defined on $\Omega := (0,1)^2$. We set μ, ϑ, f_{bk}, $\boldsymbol{K}$, ϱ, and $\boldsymbol{k}$ as in Example 1. Then, the source terms $\boldsymbol{f}$ and g in (1.7) are adjusted in such a manner that the resulting smooth functions $\boldsymbol{\sigma}$, $\mathbf{u}$, ϕ, $\mathbf{t}$, and $\boldsymbol{\eta}$ are given by

$$\boldsymbol{\sigma} \,=\, \mu(\phi)\,\nabla\mathbf{u} \,-\, (x_1^2 \,-\, x_2^2)\,\mathbb{I}\,, \quad \mathbf{u}(x_1,x_2) \,=\, \begin{pmatrix} \sin(2\pi x_1)\,\cos(2\pi x_2) \\ -\cos(2\pi x_1)\,\sin(2\pi x_2) \end{pmatrix},$$

$$\phi(x_1,x_2) \,=\, 15 \,-\, 15\,\exp(-x_1\,(x_1-1)\,x_2\,(x_2-1))\,,$$

$$\mathbf{t} \,=\, \nabla\phi\,, \quad \text{and} \quad \boldsymbol{\eta} \,=\, \vartheta(\phi)\,\mathbf{t} \,-\, \phi\,\mathbf{u} \,-\, f_{\mathrm{bk}}(\phi)\,\boldsymbol{k}\,.$$

It should be noted that ϕ vanishes at Γ, and $\mathbf{u}_D$ is imposed in accordance with the exact solution. Tables 6.3–6.4 display the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations required for each approximation. The experimental results confirm the theoretical convergence rates of $\mathcal{O}(h^{k+1})$ with $k = 0,1$, as provided by Theorem 5.4. Notably, the Newton method converges in four iterations for all cases, indicating that the convergence was independent of the mesh size. In Figure 6.1, we present the solution obtained with the fully–mixed $\mathbb{RT}_1 - \mathbf{P}_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ approximation, utilizing a mesh size $h = 0.022$ and $32,768$ triangle elements (equivalent to $312,065$ DoF). Furthermore, we confirm that the Galerkin scheme associated with the fully–mixed formulation provides conservation of momentum in the approximate sense established by (4.26) and (4.27). This fact is illustrated in Table 6.5, which displays the computed l^∞-norm for both $\boldsymbol{\mathcal{P}}_h^0\big(\phi_h\,\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h) - \boldsymbol{K}^{-1}\,\mathbf{u}_h\big)$ and $\mathcal{P}_h^0\big(g + \mathrm{div}(\boldsymbol{\eta}_h) - \varrho\,\phi_h\big)$.

6.3 Example 3: Three dimensional smooth exact solution

In this third example, we examine the cube domain $\Omega = (0,1)^3$, with the same functions μ, ϑ, and f_{bk} from Example 1. In addition, we define

$$\boldsymbol{K} \,=\, 0.01\,\mathbb{I}\,, \quad \varrho \,=\, 10\,, \quad \text{and} \quad \boldsymbol{k} \,=\, (0,-1,-1)^{\mathrm{t}}\,,$$

and adjust the source terms on the right-hand side to obtain exact solutions given by

$$\boldsymbol{\sigma} \,=\, \mu(\phi)\,\nabla\mathbf{u} \,-\, p\,\mathbb{I}\,, \quad \mathbf{u}(x_1,x_2,x_3) \,=\, \begin{pmatrix} \sin(\pi x_1)\,\cos(\pi x_2)\,\cos(\pi x_3) \\ -2\,\cos(\pi x_1)\,\sin(\pi x_2)\,\cos(\pi x_3) \\ \cos(\pi x_1)\,\cos(\pi x_2)\,\sin(\pi x_3) \end{pmatrix},$$

$$\phi(x_1,x_2,x_3) \,=\, -\,\sin(x_1+x_2+x_3)\,, \quad \mathbf{t} \,=\, \nabla\phi\,,$$

$$\text{and} \quad p(x_1,x_2,x_3) \,=\, x_1^4 \,-\, x_2^4 \,-\, x_3^4\,.$$

DoF	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}\ (\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}\ (\mathbf{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	it
89	0.707	1.02e+02	-	9.71e-01	-	2.74e-01	-	4
329	0.354	8.96e+01	0.19	4.67e-01	1.06	1.66e-01	0.73	4
1265	0.177	5.63e+01	0.67	2.46e-01	0.92	8.66e-02	0.94	4
4961	0.088	3.30e+01	0.77	1.22e-01	1.02	4.38e-02	0.98	4
19649	0.044	1.91e+01	0.79	5.95e-02	1.03	2.20e-02	1.00	4
78209	0.022	1.10e+01	0.80	2.94e-02	1.02	1.10e-02	1.00	4
312065	0.011	6.33e+00	0.80	1.46e-02	1.01	5.50e-03	1.00	4
DoF	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\boldsymbol{\eta})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	it
89	0.707	1.23e+00	-	3.30e+00	-	5.34e-01	-	4
329	0.354	7.38e-01	0.74	2.10e+00	0.65	4.83e+00	-3.18	4
1265	0.177	4.10e-01	0.85	1.25e+00	0.75	2.79e+00	0.79	4
4961	0.088	2.10e-01	0.97	6.41e-01	0.96	1.09e+00	1.36	4
19649	0.044	1.06e-01	0.99	3.23e-01	0.99	3.65e-01	1.58	4
78209	0.022	5.29e-02	1.00	1.62e-01	1.00	1.25e-01	1.55	4
312065	0.011	2.64e-02	1.00	8.09e-02	1.00	4.83e-02	1.37	4

Table 6.3: Example 2, number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully–mixed $\mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$.

DoF	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}\ (\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}\ (\mathbf{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	it
265	0.707	5.70e+01	-	3.16e-01	-	1.03e-01	-	4
1009	0.354	2.32e+01	1.30	1.41e-01	1.17	2.66e-02	1.96	4
3937	0.177	6.77e+00	1.78	3.57e-02	1.98	6.81e-03	1.97	4
15553	0.088	1.96e+00	1.79	9.07e-03	1.98	1.71e-03	1.99	4
61825	0.044	5.60e-01	1.81	2.28e-03	1.99	4.29e-04	2.00	4
246529	0.022	1.59e-01	1.82	5.69e-04	2.00	1.07e-04	2.00	4
DoF	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\eta})$	$\mathtt{r}(\boldsymbol{\eta})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	it
265	0.707	4.74e-01	-	1.76e+00	-	2.11e+00	-	4
1009	0.354	1.51e-01	1.65	7.75e-01	1.18	6.36e-01	1.73	4
3937	0.177	4.19e-02	1.85	2.29e-01	1.76	1.13e-01	2.49	4
15553	0.088	1.10e-02	1.93	5.84e-02	1.97	2.44e-02	2.21	4
61825	0.044	2.79e-03	1.98	1.47e-02	1.99	5.81e-03	2.07	4
246529	0.022	7.01e-04	1.99	3.68e-03	1.99	1.43e-03	2.02	4

Table 6.4: Example 2, number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully–mixed $\mathbb{RT}_1 - \mathbf{P}_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ method.

h	0.354	0.177	0.088	0.044	0.022	0.011
$\lVert\boldsymbol{\mathcal{P}}_h^0\big(\phi_h\,\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h) - \boldsymbol{K}^{-1}\,\mathbf{u}_h\big)\rVert_{l^\infty}$	5.7e-14	6.3e-14	1.4e-13	2.4e-10	2.1e-10	1.1e-09
$\lVert\mathcal{P}_h^0\big(g + \mathrm{div}(\boldsymbol{\eta}_h) - \varrho\,\phi_h\big)\rVert_{l^\infty}$	4.6e-15	7.2e-15	1.4e-13	6.9e-11	3.1e-11	5.7e-11

Table 6.5: Example 2, conservation of momentum for the fully–mixed $\mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$ approximation.

The numerical approximation for the fully–mixed $\mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$ method is illustrated in Figure 6.2, employing a mesh size of $h = 0.108$ and $12,288$ tetrahedral elements (totaling 374,785 DoF). In turn, Table 6.6 presents the convergence behavior for a series of quasi-uniform mesh refine-

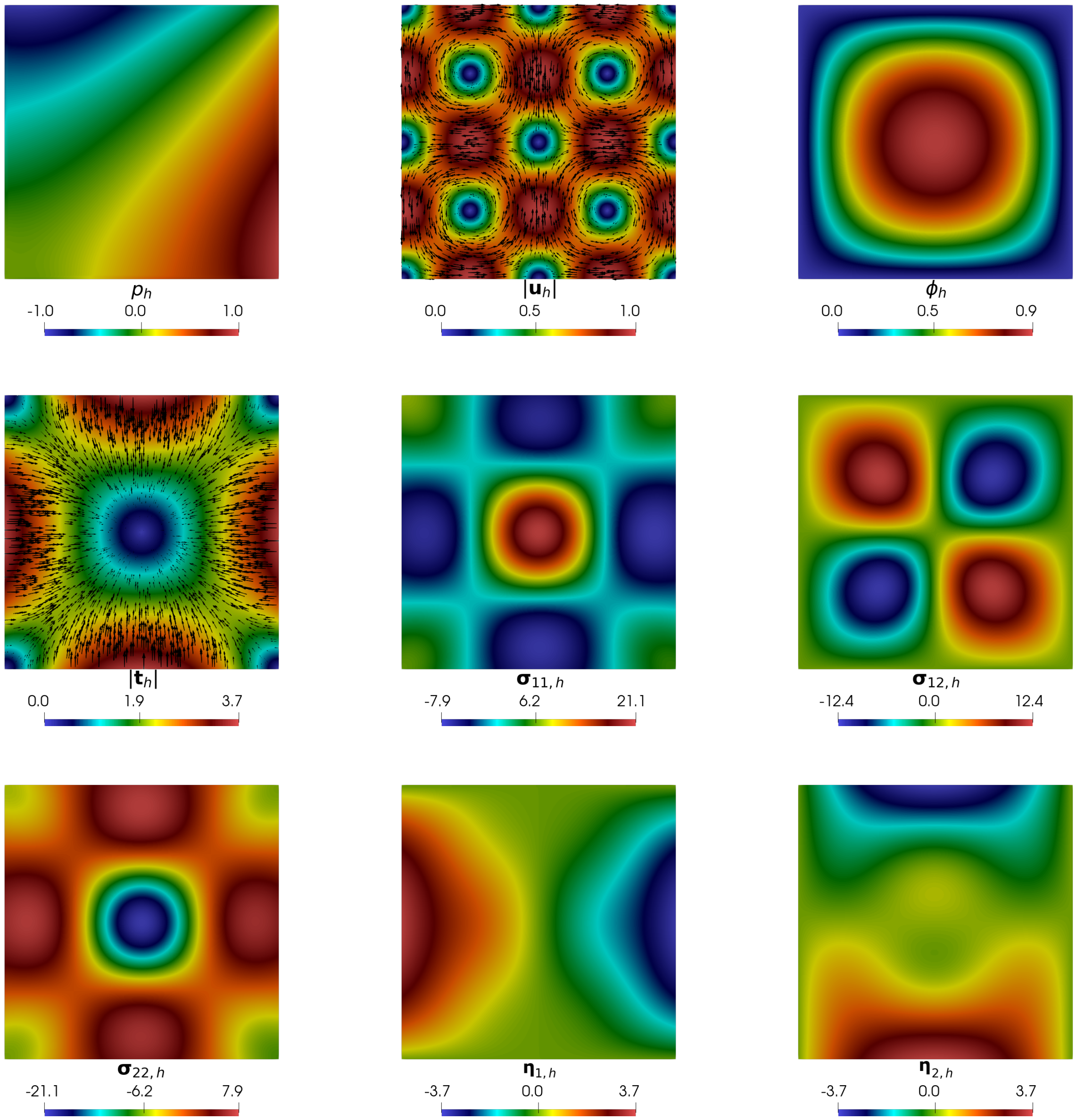

Figure 6.1: Example 2, $\mathbb{RT}_1 - \mathbf{P}_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ approximation of pressure field, magnitude of the velocity, and concentration field (first row); magnitude of the concentration gradient and Cauchy stress components (second row); Cauchy stress component and total flux components (third row).

ments using $k = 0$, which confirms that in this 3D example the fully–mixed finite element method also attains the optimal convergence rate of $\mathcal{O}(h)$ guaranteed by Theorem 5.4.

6.4 Example 4: Settling in a vessel with downward facing inclined walls

We close this section with an application of the proposed numerical schemes in the simulation of sedimentation-compression of a suspension of particles within a porous medium with relatively high but heterogeneous permeability. The problem configuration is adapted from [6] and [31]. The do-

DoF	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}\,(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}\,(\mathbf{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	it
817	0.866	1.43e+01	-	5.66e-01	-	1.57e-01	-	4
6145	0.433	8.53e+00	0.75	3.02e-01	0.91	8.52e-02	0.88	4
47617	0.217	4.48e+00	0.93	1.55e-01	0.96	4.34e-02	0.97	4
374785	0.108	2.27e+00	0.98	7.80e-02	0.99	2.18e-02	0.99	4

DoF	h	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\eta})$	$\mathtt{r}(\boldsymbol{\eta})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	it
817	0.866	4.27e-01	-	1.25e+01	-	6.77e-01	-	4
6145	0.433	2.19e-01	0.97	7.69e+00	0.70	4.12e-01	0.72	4
47617	0.217	1.10e-01	0.99	4.05e+00	0.92	2.33e-01	0.82	4
374785	0.108	5.51e-02	1.00	2.05e+00	0.98	1.21e-01	0.95	4

Table 6.6: Example 3, number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully–mixed $\mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$ scheme.

main consists of an isosceles trapezoid of height 3 [m], base of 2.82 [m] and basal angles of 80°. An unstructured triangular mesh with 27728 elements is used to discretize the domain, resulting in a formulation with 264059 degrees of freedom (in this subsection the numerical tests are run with the fully–mixed formulation and taking the lowest-order polynomial degree). This test considers two main modifications with respect to (1.6) and (1.7): 1) we include the time derivative of the volume fraction ϕ in the transport equation (second equation in (1.6)), which is discretized using the backward Euler scheme with fixed time step of $\Delta t = 0.025$ [s] and the system is evolved until $t = t_{\mathrm{end}} = 10$ [s]; and 2) we use flux-based boundary conditions for the sedimentation equation, corresponding to the case of batch settling. On the whole boundary (inclined walls plus the top and bottom segments) we impose (naturally in the formulation) no-slip boundary conditions $\mathbf{u} = \mathbf{0}$, while we set a no-flux condition using the total flux $\boldsymbol{\eta}\cdot\boldsymbol{\nu} = 0$ (essentially imposed). In this way, the analysis developed in the previous sections can be easily extended, up to minor modifications, to the present case. The initial volume fraction is prescribed as high near the top of the domain (0.75) and a uniform random perturbation of amplitude 0.05 around the value 0.15. The volume fraction-dependent functions (fluid effective viscosity, Kynch batch flux density, and sediment compressibility) and dimensional parameters (taken from [6, 31]) assume the following specifications

$$\mu(\phi) = \mu_0\left(1 - \frac{\phi}{\phi_{\max}}\right)^{-5/2}, \quad f_{\mathrm{bk}}(\phi) = u_\infty\phi\left(1 - \frac{\phi}{\phi_{\max}}\right)^2, \quad \vartheta(\phi) = \frac{\sigma_0\alpha_0 f_{\mathrm{bk}}(\phi)\phi^{\alpha_0-2}}{\phi_c^{\alpha_0}\Delta\rho\bar{g}} + u_\infty,$$
$$\boldsymbol{k} = (0,-1)^{\mathrm{t}}, \quad \boldsymbol{f} = (0,-\bar{g})^{\mathrm{t}}, \quad \alpha_0 = 5, \quad \bar{g} = 9.8\,[\mathrm{m/s^2}], \quad \mu_0 = 2\cdot 10^{-4}\,[\mathrm{Pa\cdot s}],$$
$$u_\infty = 2.2\cdot 10^{-3}\,[\mathrm{m/s}], \quad \phi_{\max} = 0.95, \quad \phi_c = 0.07, \quad \Delta\rho = 1562\,[\mathrm{Kg/m^3}], \quad \sigma_0 = 5.5\times 10^{-2}\,[\mathrm{Pa}].$$

The permeability is considered isotropic but heterogeneous: $\boldsymbol{K} = K(\mathbf{x})\,\mathbb{I}$, where

$$K(\mathbf{x}) = \frac{10K^{\min}}{K^{\min} + K^{\max}\max\{\tilde{K}(\mathbf{x}),0\}}\,[\mathrm{m^2}], \quad \tilde{K}(\mathbf{x}) = \sum_{i=1}^{25}\exp\big(-\frac{1}{r}[(x-q_{i,x})^2 + (y-q_{i,y})^2]\big),$$

and where $(q_{i,x}, q_{i,y})$ are 25 randomly located points in Ω, and $r = 0.0015$ [m].

The top panels of Figure 6.3 present snapshots of the numerically computed volume fraction at different times (and we recall that this is a P_0 field). These plots show the expected behavior in batch settling of particles (higher concentration zones start to accumulate at the bottom of the enclosure). One can clearly notice that the high contrast in permeability induces that part of the solid particles stick for a longer time to the zones of low permeability before settling to the bottom of the vessel. The second row shows fewer snapshots of the total flux (we plot line integral convolutions that indicate the

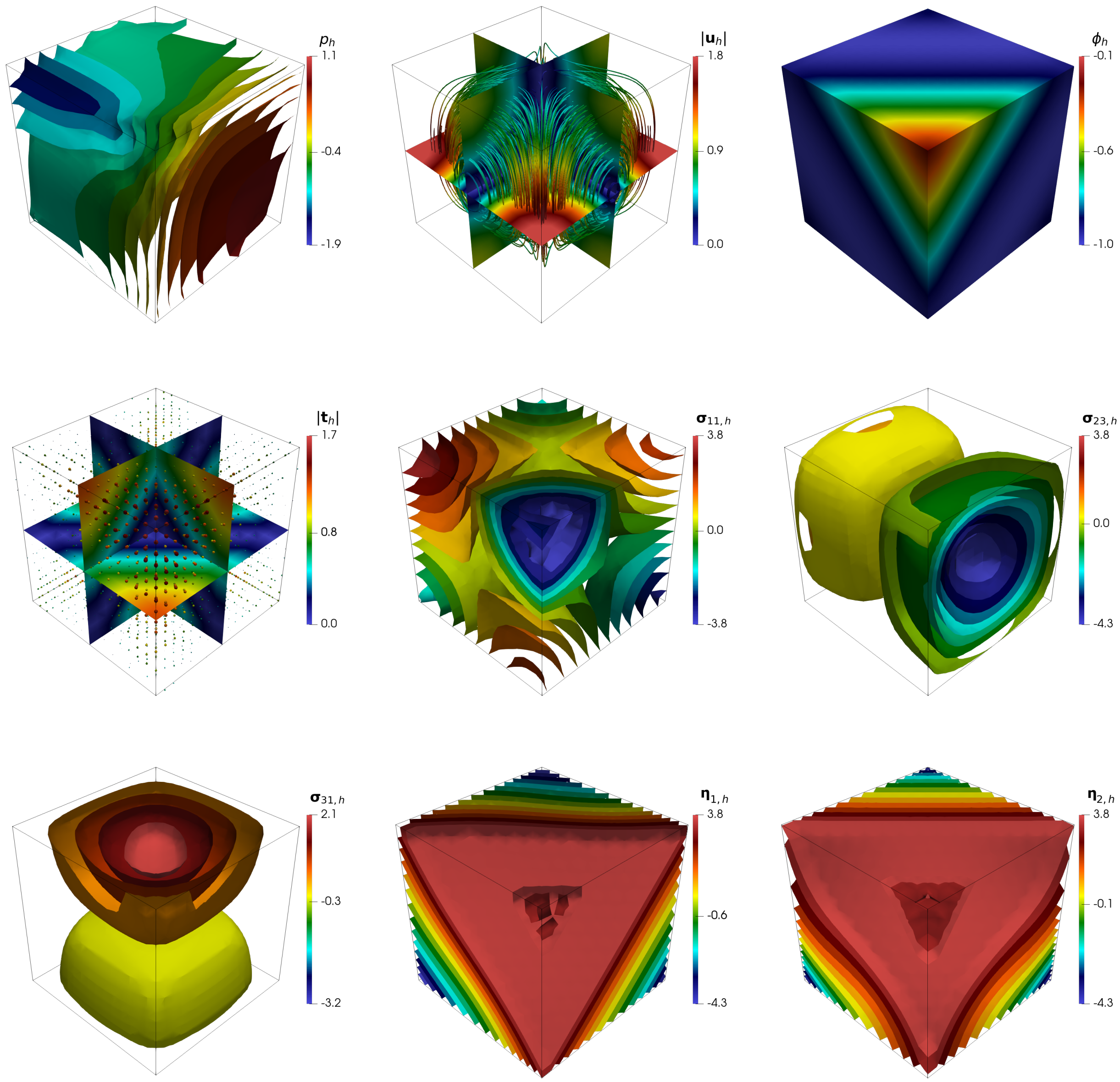

Figure 6.2: Example 3, $\mathbf{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$ approximation of pressure field, magnitude of the velocity, and concentration field (first row); magnitude of the concentration gradient, and Cauchy stress components (second row); Cauchy stress component and total flux components (third row).

directions and magnitude of the vector field). In the velocity plots (third row of Figure 6.3) we can observe how the fluid prefers to flow in the zones of higher permeability, and we also see a boundary layer of higher magnitude forming on the donward facing walls as a result of recirculation effects.

References

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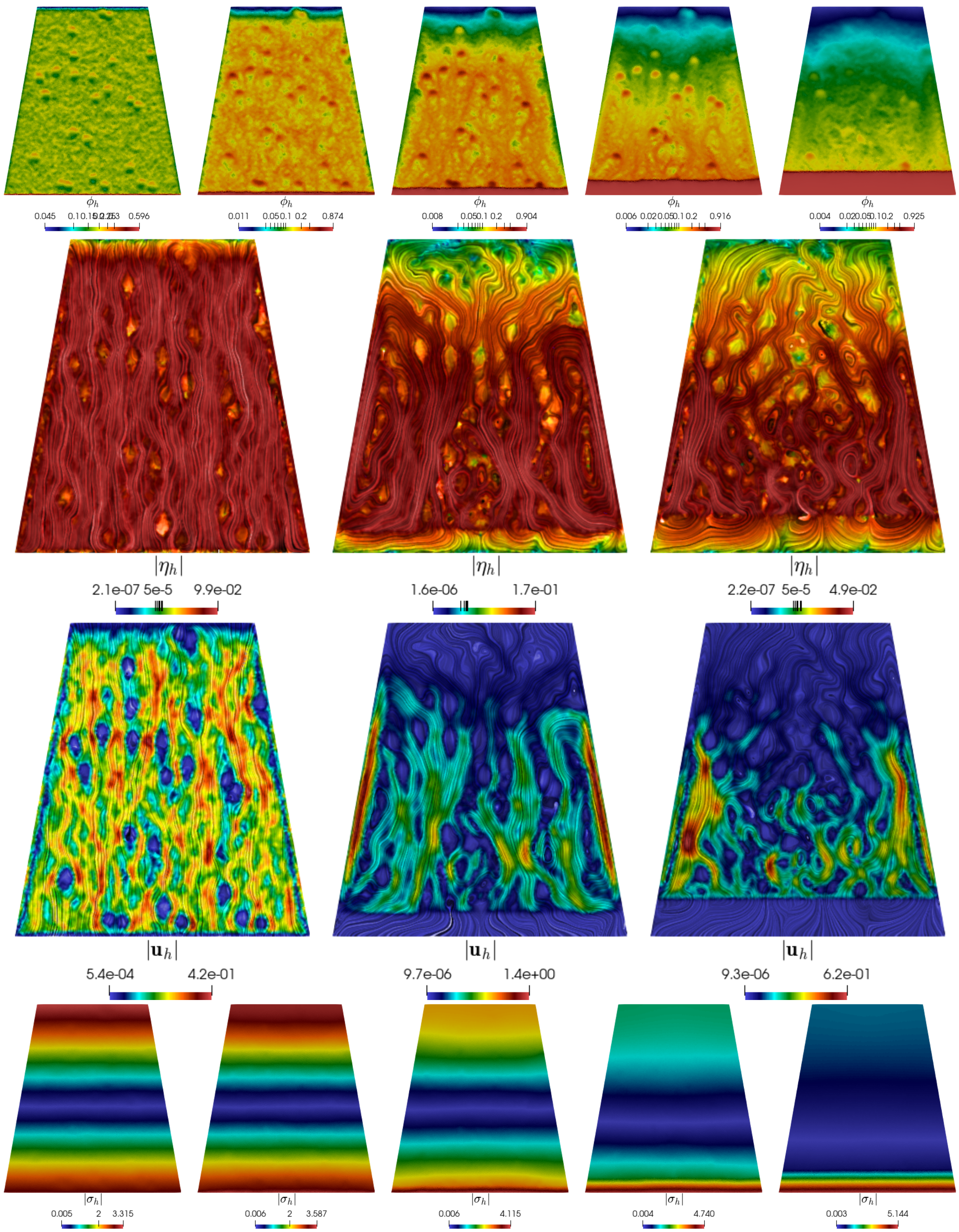

Figure 6.3: Example 4, snapshots of volume fraction profiles and pseudo-stress magnitude (in log-scale) at times $t = 0.25, 1, 2.5, 5, 10$ [s] (top and bottom rows, respectively). The centre rows show magnitude of the total flux (also in log-scale) and velocity magnitude at times $t = 1, 5, 10$ [s]. Solutions computed with the fully–mixed $\mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$ scheme.

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