A finite volume element method for simulating secondary settling tanks

Raimund Bürger[1,∗], **Ricardo Ruiz-Baier**[2], and **Héctor Torres**[3]

[1] CI²MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

[2] CMCS-MATHICSE-SB, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland

[3] Departamento de Matemáticas, Facultad de Ciencias, Universidad de La Serena, Av. Cisternas 1200, La Serena, Chile

This contribution summarizes a new finite volume element (FVE) method as a unified scheme for coupled sedimentation-flow problems in secondary settling tanks in wastewater treatment. The model is two-dimensional in an axisymmetric setting. A new numerical example is presented.

1 Model problem

Models for the simulation of secondary settling tanks in wastewater treatment are usually of macroscopic type, where the governing PDEs are represent the mass and linear momentum balance equations for the solid and liquid phases. A well-known challenge for 2D or 3D numerical simulation is the strong coupling between the field of solids concentration ϕ and the flow field of the mixture, which is defined by the velocity $\boldsymbol{u}$ and the pressure p. (Additional equations of motion for $\boldsymbol{u}$ and p do not arise in 1D models [1].) The governing equations can be written as follows, where ϕ, $\boldsymbol{u}$ and p are sought:

$$\partial_t \phi + \nabla \cdot \boldsymbol{F}(\phi, \boldsymbol{u}) = \Delta A(\phi), \quad -\nabla \cdot \big(\mu(\phi)\varepsilon(\boldsymbol{u}) - \lambda p \boldsymbol{I}\big) = \boldsymbol{G}(\phi), \quad \lambda \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \tilde{\Omega}, t > 0. \tag{1}$$

Here $\boldsymbol{F}(\phi, \boldsymbol{u}) = \phi \boldsymbol{u} + f_{\mathrm{bk}}(\phi)\boldsymbol{k}$ is a flux vector, where f_{bk} is the material specific Kynch batch flux density function and $\boldsymbol{k}$ is the upward-pointing unit vector, the diffusion term $\Delta A(\phi)$ models sediment compressibility (where A is assumed to be a strictly increasing function of ϕ), and $\mu(\phi)\varepsilon(\boldsymbol{u}) - \lambda p \boldsymbol{I}$ denotes the Cauchy stress tensor, where $\varepsilon(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})$ and μ is concentration-dependent viscosity function. In the forcing term $\boldsymbol{G} = \lambda \phi g \boldsymbol{k}$, the constant g is the acceleration of gravity and λ is another constant. We here choose $f_{\mathrm{bk}}(\phi) = u_\infty \phi(1 - \phi/\phi_{\max})^2$ and $A(\phi) = \int_0^\phi a(s)\,\mathrm{d}s$, where $a(\phi) := f_{\mathrm{bk}}(\phi)\sigma_{\mathrm{e}}'(\phi)/[(\varrho_{\mathrm{s}} - \varrho_{\mathrm{f}})g\phi]$ and ϱ_{s} and ϱ_{f} are the solid and fluid mass densities. Moreover, σ_{e} is the so-called effective solid stress function that is assumed to satisfy $\sigma_{\mathrm{e}}'(0) = 0$ and $\sigma_{\mathrm{e}}'(\phi) > 0$ for $\phi > 0$. Thus, $(1)_1$ is a two-point degenerate parabolic PDE. The viscosity function is chosen as $\mu(\phi) = (1 - \phi/\tilde{\phi}_{\max})^{-\beta}$ with $\tilde{\phi}_{\max} > \phi_{\max}$.

The initial and boundary conditions are as follows (see Figure 1). We consider $t \in [0, T]$, and the spatial domain $(r, z) \in \Omega \subset \mathbb{R}^2$ corresponding to an axisymmetric setting. Initial data are specified for ϕ and $\boldsymbol{u}$. The unit is fed through the portion of the boundary $\Gamma_{\mathrm{in}} \subset \partial\Omega$, where $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{in}}$ and $\phi = \phi_{\mathrm{in}}$ are given. The boundary parts Γ_{out} and Γ_{c} correspond to discharge and overflow outlets, where $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{out}}$ and $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{c}}$ are specified. On the remainder of $\partial\Omega$ we impose no-slip conditions ($\boldsymbol{u} = \boldsymbol{0}$) and zero-flux conditions for ϕ.

## 2 Finite Volume Element (FVE) method and numerical example

The FVE method can be summarized as follows [2]. We discretize $(1)_1$ in time by a semi-implicit method. Starting from the appropriate weak formulation and employing standard stabilization techniques, we discretize (1) in space by a finite element (FE) scheme on a triangular "primal" mesh $\mathcal{T}_h$. Since ϕ may be discontinuous ($(1)_1$ may be strongly degenerate), we choose piecewise linear, possibly discontinuous functions for ϕ, piecewise bilinear, continuous functions for $\boldsymbol{u}$ and piecewise linear, continuous functions for p. By standard discretization techniques we then obtain a Galerkin FE formulation for the coupled problem (1) including boundary conditions. Now, FVE methods are intermediate between FV and FE methods: FVE schemes are locally conservative, and at the same time permit L^2 estimates in a natural way. The basic idea is to reformulate the FE scheme as a FV scheme on a dual mesh (see Fig. 2). Precisely, we introduce a dual partition $\mathcal{T}_h^\star$ and diamond mesh $\mathcal{T}_h'$ such that the FE method can then be expressed as a FV method where test functions are constant on the elements of $\mathcal{T}_h^\star$ and $\mathcal{T}_h'$. We define the following FE spaces for the approximation of ϕ, $\boldsymbol{u}$ and p, respectively:

$$S_h := \{s \in L_1^2(\Omega) : s|_K \in \mathbb{P}_1(K) \,\forall K \in \mathcal{T}\}, \quad \boldsymbol{\mathcal{V}}_h^0 := \{\boldsymbol{v} \in H_{1,0}^1(\Omega) \cap \boldsymbol{C}^0(\bar{\Omega}) : \boldsymbol{v}|_K \in \mathbb{P}_1(K)^2 \,\forall K \in \mathcal{T}\},$$

$$Q_h := \{q \in L_{1,0}^2(\Omega) \cap C^0(\bar{\Omega}) : q|_K \in \mathbb{P}_1(K) \,\forall K \in \mathcal{T}\}, \quad \boldsymbol{\mathcal{V}}_h := \{\boldsymbol{v} \in \boldsymbol{\mathcal{V}} \cap \boldsymbol{C}^0(\bar{\Omega}) : \boldsymbol{v}|_K \in \mathbb{P}_1(K)^2 \,\forall K\},$$

$$\boldsymbol{\mathcal{V}}_h^\star := \{\boldsymbol{v} \in \boldsymbol{L}_1^2(\Omega) : \boldsymbol{v}|_{K_j^\star} \in \mathbb{P}_0(K_j^\star)^2 \,\forall K_j^\star \in \mathcal{T}^\star, \boldsymbol{v}|_{K_j^\star} = \boldsymbol{u}_{\mathrm{D}} \text{ if } K_j^\star \text{ is a boundary volume}\},$$

$$S_h' := \big\{s \in L_1^2(\Omega) : s|_{K_F} \in \mathbb{P}_0(K_F) \,\forall K_F \in \mathcal{T}'\big\}.$$

∗ Corresponding author: e-mail rburger@ing-mat.udec.cl, phone +56 41 2203126, fax +56 41 2522055
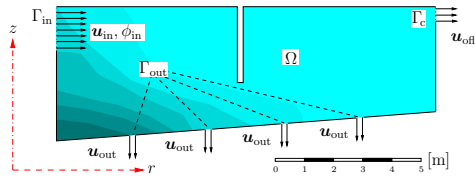
Fig. 1: Secondary settling tank [3, 4] with a center feed, peripheral and radial effluent overflows, a skirt baffle, and four suction lifts for sludge removal, of width 0.25 m each.
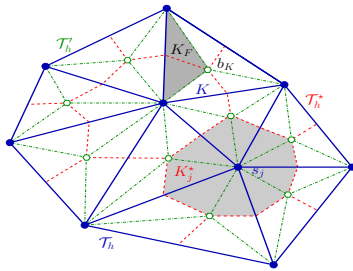
Fig. 2: Primal mesh $\mathcal{T}_h$, interior node-centered control volumes of the dual mesh $\mathcal{T}^\star$ (dashed), and triangular elements of the diamond mesh $\mathcal{T}'$ (dash-dotted).
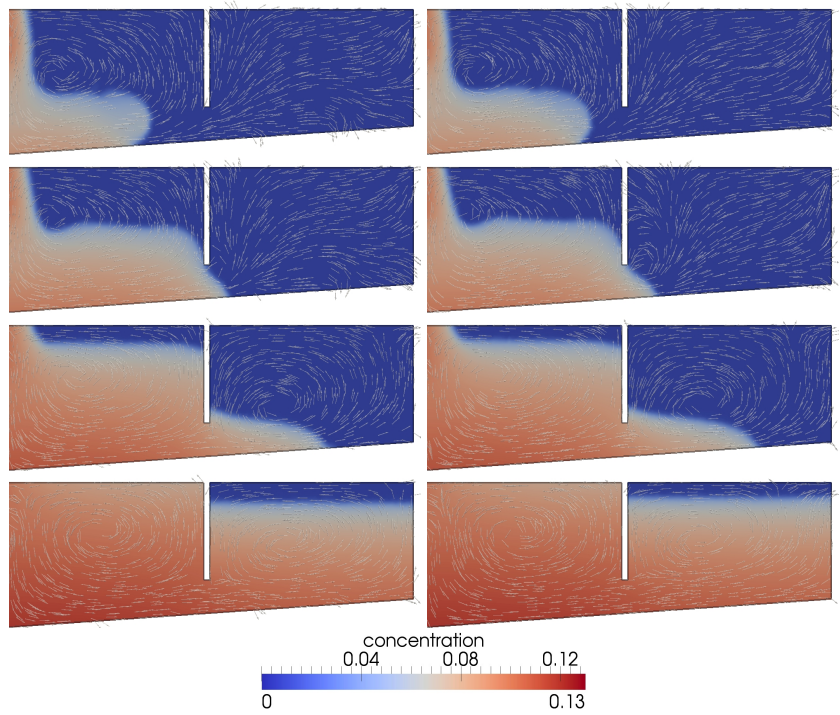
Fig. 3: Settling of the mixture at times $t = 500,\ 1000,\ 2000,\ 7500$ [s], for $\nu = 0.8$ (left panels) and $\nu = 0.01$ (right panels).

This procedure involves so-called transfer operators $\mathcal{P}_h : \mathcal{V}_h \to \mathcal{V}_h^\star$ and $\mathcal{R}_h : S_h \to S'_h$ that lead to a semi-discrete, stabilized Galerkin FE formulation. One eventually obtains the following space-time discrete scheme:

1. Given ϕ_h^n, compute $\boldsymbol{u}_h^n$ and p_h^n by solving the Stokes problem

$$\big(\mu(\phi_h^n)\boldsymbol{\varepsilon_a}(\boldsymbol{u}_h^n), \boldsymbol{\varepsilon_a}(\boldsymbol{v}_h)\big)_{1,\Omega} + \big(\mu(\phi_h^n)u_{h,r}^n, v_{h,r}\big)_{-1,\Omega} - \lambda\big(p_h^n, \nabla_{\mathbf{a}} \cdot \boldsymbol{v}_h\big)_{1,\Omega} - \big(\boldsymbol{G}(\phi_h^n), \mathcal{P}_h\boldsymbol{v}_h\big)_{1,\Omega}$$
$$+ \sum_{F\in\mathcal{E}_h^{\mathrm{int}}} \tfrac{h_F}{\kappa_F^{\boldsymbol{u}}}\big(\llbracket\mu(\phi_h^n)\boldsymbol{\varepsilon_a}(\boldsymbol{u}_h^n)\cdot\boldsymbol{n}_F\rrbracket_F, \llbracket\mu(\phi_h^n)\boldsymbol{\varepsilon_a}(\boldsymbol{v}_h)\cdot\boldsymbol{n}_F\rrbracket_F\big)_{1,F} = 0 \quad \forall\boldsymbol{v}_h\in\mathcal{V}_h^0,$$
$$\lambda\big(q_h, \nabla_{\mathbf{a}}\cdot\boldsymbol{u}_h^n\big)_{1,\Omega} + \sum_{K\in\mathcal{T}}\tfrac{h_K^2}{\kappa_K^p}\big(\lambda\nabla_{\mathbf{a}}p_h^n - \boldsymbol{G}(\phi_h^n), \nabla_{\mathbf{a}}q_h\big)_{1,K} = 0 \quad \forall q_h\in Q_h.$$

2. Given $\boldsymbol{u}_h^n$ and p_h^n computed in Step 1, obtain ϕ_h^{n+1} from

$$\big((\phi_h^{n+1} - \phi_h^n)/\Delta t^n, \mathcal{R}_h s_h\big)_{1,\Omega} - \big(\boldsymbol{F}(\phi_h^{n+1}, \boldsymbol{u}_h^n) - a(\phi_h^n)\nabla_{\mathbf{a}}\phi_h^{n+1}, \nabla_{\mathbf{a}}s_h\big)_{1,\Omega}$$
$$+ \sum_{F\in\mathcal{E}_h\cap\Gamma_{\mathrm{in}}}\big(\tfrac{\kappa_F^\phi}{h_F^2}\phi_h^{n+1}, \mathcal{R}_h s_h\big)_{1,F} + \big(\tfrac{\kappa_F^\phi}{h_F^2}\phi_{\mathrm{in}}, \mathcal{R}_h s_h\big)_{1,F}$$
$$+ \sum_{F\in\mathcal{E}_h^{\mathrm{int}}}\big(\widehat{\boldsymbol{F}}(\phi_h^{n+1}, \boldsymbol{u}_h^n)\cdot\boldsymbol{n}_F - \big\{a(\phi_h^n)\nabla_{\mathbf{a}}\phi_h^{n+1}\cdot\boldsymbol{n}_F\big\}_F + \tfrac{\kappa_F^\phi}{h_F^2}\llbracket\mathcal{R}_h\phi_h^{n+1}\rrbracket_F, \llbracket\mathcal{R}_h s_h\rrbracket_F\big)_{1,F} = 0 \quad \forall s_h\in S_h.$$

We choose the parameters $\varrho_{\mathrm{s}} - \varrho_{\mathrm{f}} = 1562\,\mathrm{kg/m^3}$, $\phi_{\max} = 0.9$, $u_\infty = 2.2\times10^{-3}\,\mathrm{m/s}$, $\phi_{\mathrm{in}} = 0.1$, $g = 9.81\,\mathrm{m/s^2}$, $u_{z,\mathrm{out}} = \nu u_{z,\mathrm{in}}$, $u_{r,\mathrm{ofl}} = \frac{9-\nu}{52}u_{z,\mathrm{in}}$, $u_{r,\mathrm{in}} = 0.019\,\mathrm{m/s}$, $\alpha = 5$, $\beta = 2.5$, $\lambda = 1$, $\sigma_0 = 0.05$, $h = 0.0961\,\mathrm{m}$, $\Delta t = 5\,\mathrm{s}$, $\kappa_F^\phi = \frac{1}{12}$, $\kappa_F^{\boldsymbol{u}} = 100$, $\kappa_F^p = 500$. The primal mesh $\mathcal{T}$ is composed by 7410 elements and 4206 interior nodes. The boundary conditions for velocity at the suction lifts are given by $\boldsymbol{u} = (0, -u_{z,\mathrm{out}}/4)$, where $u_{z,\mathrm{out}} = \nu u_{r,\mathrm{in}}$, with $\nu = 0.8$ and $\nu = 0.01$ corresponding to operating conditions that differ from those studied in [2]. See Figure 3 for numerical results.

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