Unified numerical analysis for thermoelastic diffusion and thermo-poroelasticity of thin plates

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Abstract

We investigate a coupled hyperbolic-parabolic system modeling thermoelastic diffusion (resp. thermo-poroelasticity) in plates, consisting of a fourth-order hyperbolic partial differential equation for plate deflection and two second-order parabolic partial differential equations for the first moments of temperature and chemical potential (resp. pore pressure). The unique solvability of the system is established via Galerkin approach, and the additional regularity of the solution is obtained under appropriately strengthened data. For numerical approximation, we employ the Newmark method for time discretization of the hyperbolic term and a continuous interior penalty scheme for the spatial discretization of displacement. For the parabolic equations that represent the first moments of temperature and chemical potential (resp. pore pressure), we use the Crank–Nicolson method for time discretization and conforming finite elements for spatial discretization. The convergence of the fully discrete scheme with quasi-optimal rates in space and time is established. The numerical experiments demonstrate the effectiveness of the 2D Kirchhoff–Love plate model in capturing thermoelastic diffusion and thermo-poroelastic behavior in specific materials. We illustrate that as plate thickness decreases, the two-dimensional simulations closely approximate the results of three-dimensional problem. Finally, the numerical experiments also validate the theoretical rates of convergence.

1 Introduction

Scope and presentation of the problem. This study presents a unified analysis of thin plate structures that describe thermoelastic diffusion (TED) and thermo-poroelasticity (TPE). The coupled system comprises of a fourth-order hyperbolic partial differential equation (PDE) governing the plate deflection with two second-order parabolic PDEs describing the first moments of temperature and chemical potential (resp. pressure) in the case of TED (resp. TPE). A combination of the C^0 interior penalty (C^0IP) scheme and conforming finite elements (FEs) is used for spatial discretization. The temporal discretization utilizes the Newmark and Crank–Nicolson schemes for approximating the second and first-order terms, respectively. We establish optimal order theoretical rates of convergence and the numerical experiments validate them.

Thermodiffusion in an elastic solid results from the coupling of strain, temperature, and mass diffusion fields. In the context of TPE, this coupling replaces chemical potential by pore pressure, accounting for the interactions between mechanical deformation, thermal effects, and fluid flow within porous media. The TED phenomena play a critical role in various engineering applications, for example, in satellite, aircraft operations, and in the manufacturing of integrated circuits, integrated resistors, semiconductor substrates, and transistors. Additionally, TED is a key part in the heat and mass transfer processes involved in enhancing oil extraction conditions from deposits. Understanding diffusion properties in thin thermoelastic plates is critical in the study of advance materials predicting stress distribution, material fatigue, and potential failure such as warping or cracking during operation. Also, determination of the flexural motion of fluid-saturated poroelastic plates is an important problem in structural and geotechnical engineering, bioengineering, and geodynamics.

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Constant	*Description*
λ	Lamé's first constant
μ	Shear modulus
ϱ	Measure of the diffusive effect
α_t	Coefficient of thermal expansion
α_c	Coefficient of diffusion expansion
ϖ	Measure of thermodiffusion effect
c_E	Specific heat at constant strain
ρ	Mass density per unit volume
k_1	Coefficient of thermal conductivity
k_2	Coefficient of diffusion conductivity
β^*	Biot-Willis constant
γ^*	Thermal dilation coefficient
ϱ^*	Biot modulus
k_2^*	Permeability

Table 1.1: Physical constants.

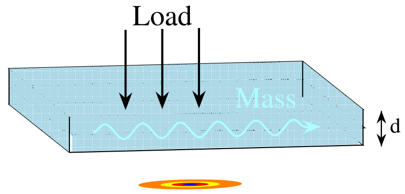

Fig 1.1: 3D plate in reference configuration.

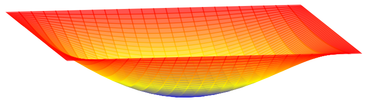
Fig 1.2: Mid-surface at current configuration.

Let $\widehat{\Omega} \subset \mathbb{R}^3$ denote a thin, isotropic, flat plate with a uniform thickness d. Additionally, we define the time interval as $[0, T]$. We denote the mid-surface of the plate as $\Omega \subset \mathbb{R}^2$, which is assumed to lie in alignment with the xy-axis, forming a bounded domain with a Lipschitz continuous boundary Γ. The elastodynamics of the mid-surface of the plate is characterized by the deflection

$$u(\boldsymbol{x}, t) = \frac{1}{d} \int_{-d/2}^{d/2} \hat{u}_3 \, \mathrm{d}z,$$

which represents the transverse displacement $\hat{u}_3(x, y, z, t)$ averaged through the thickness and is a scalar function of $\boldsymbol{x} = (x, y)$ and t only. The first moments of temperature $\hat{\theta}(x, y, z, t)$ and chemical potential (resp. pore pressure) $\hat{p}(x, y, z, t)$ (resp. $\hat{p}^*(x, y, z, t)$) are denoted by

$$\theta(\boldsymbol{x}, t) = \int_{-d/2}^{d/2} z\hat{\theta} \, \mathrm{d}z, \text{ and } p(\boldsymbol{x}, t) = \int_{-d/2}^{d/2} z\hat{p} \, \mathrm{d}z \text{ (resp. } p(\boldsymbol{x}, t) = \int_{-d/2}^{d/2} z\hat{p}^* \, \mathrm{d}z).$$

The authors in [4] formulated a model from the 3D (5.1a)-(5.1c) for TED in thin plates, under the assumption that body forces, external loads, and sources of heat and diffusion are absent. This model is based on the 2D Kirchhoff–Love hypotheses for thin plates, with classical Fourier's law for heat conduction and Fick's law for diffusion. An enhanced *novel* model considered in this article for TED and TPE that include external loads, heat source, and mass diffusion and is presented as follows: the coupled model aims to determine mid-surface deflection u, first moments of temperature θ and chemical potential (resp. pore pressure) p such that

$$u_{tt} - a_0 \Delta u_{tt} + d_0 \Delta^2 u + \alpha \Delta\theta + \beta \Delta p = f(\boldsymbol{x}, t) \qquad \text{in } \Omega \times (0, T], \tag{1.1a}$$
$$a_1 \theta_t - \gamma p_t + b_1 \theta - c_1 \Delta\theta - \alpha \Delta u_t = \phi(\boldsymbol{x}, t) \qquad \text{in } \Omega \times (0, T], \tag{1.1b}$$
$$a_2 p_t - \gamma \theta_t - \kappa \Delta p - \beta \Delta u_t = g(\boldsymbol{x}, t) \qquad \text{in } \Omega \times (0, T], \tag{1.1c}$$
$$u = \partial_{\boldsymbol{n}} u = 0, \ \theta = 0, \ p = 0 \qquad \text{on } \Gamma \times [0, T], \tag{1.1d}$$
$$u|_{t=0} = u^0, \ u_t|_{t=0} = u^{*0}, \ \theta|_{t=0} = \theta^0, \ p|_{t=0} = p^0 \qquad \text{in } \Omega, \tag{1.1e}$$

where $\boldsymbol{n}$ is the outward-pointing unit normal, $\partial_{\boldsymbol{n}} u = \nabla u \cdot \boldsymbol{n}$ is the outer normal derivative of u on $\partial\Omega$, u_t, θ_t, p_t (resp. u_{tt}) denote the first (resp. second)-order derivatives with respect to time. Here, the chemical potential (resp. pore pressure) across the plate is assumed to be linear. The coefficients in the system (1.1) depend on the constants listed in the Table 1.1, with further details deferred to Subsection 5.1.

As mentioned in Subsection 5.1, it is possible to use models of thermoelastic plates with voids or vacuous pores [9, 28, 31, 38]. In this fully coupled system the Kirchhoff–Love equations for the deflection interact with the dynamics of the total amount of fluid, and the thermal energy conservation exhibits a dependence on the plate poromechanics through the thermal stress and thermal dilation contributions. The 3D system of equations

(5.4a)–(5.4c) for TPE closely resembles the 3D TED system (5.1a)–(5.1c), with the primary differences being the physical constants involved and the sign of the coupling constant between the second and third equations. Therefore, by following the dimensional reduction approach and using Darcy's law for fluid flow (in contrast to Fick's law for diffusion) as done in [4, eqns. (9)-(46)], one can derive the 2D TPE model from (5.4a)–(5.4c), which leads to the system (1.1a)–(1.1c).

In this paper, we assume that all coefficients, except for γ, are positive. These assumptions are realistic because, for the TED model (see an explicit representation in Subsection 5.1) and the TPE thin plate model, the coefficients remain positive provided the basic 3D constants listed in the Table 1.1 are positive. Regarding the parameter γ, we allow $\gamma \in \mathbb{R}$. For the TED model the condition $a_1 a_2 - \gamma^2 > 0$ is inherently satisfied due to the material's constitutive properties, as detailed in Table 5.1. This condition is typically assumed for the TPE model to ensure well-posedness and physical realism [14, 53]. This shows that, $|\gamma|/a_1 < a_2/|\gamma|$ and hence there exists some $\gamma_0 > 0$ such that $|\gamma|/a_1 < \gamma_0 < a_2/|\gamma|$, and consequently,

$$a_1 - |\gamma|/\gamma_0 > 0 \qquad \text{and} \qquad a_2 - |\gamma|\gamma_0 > 0. \tag{1.2}$$

Literature overview. The foundational TED theory was initially proposed by Nowacki [44]. Rigorous derivations have been undertaken to establish the linear Kirchhoff–Love thermoelastic plate model, as shown in [36], where the plate is assumed to be homogeneous, as well as elastically and thermally isotropic. Poroelastic models based on the Kirchhoff–Love plate theory and Biot's theory of poroelasticity are discussed in [48] and [37]. In these works, the pressure variation in the longitudinal section is neglected in the former, while a linear pressure distribution across the plate is considered in the latter. The papers [4, 23] discuss hyperbolic problems; the well-posedness of the problem is analyzed using the semigroup theory approach, after transforming the system into an evolution equation by introducing velocity as a new variable with vertical displacement. The model discussed in this paper builds upon the derivations presented in [4], which incorporate diffusion effects in homogeneous and isotropic thermoelastic thin plates. Our analysis uses a Galerkin method and compactness arguments for showing existence and uniqueness of weak and strong solutions [15].

Regarding numerical methods for fully coupled multiphysics system, we mention that [55] employs a mixed element method, the H^1-Galerkin method, and the interior penalty discontinuous Galerkin (dG) method (IP-DG) for spatial discretization of the Kirchhoff–Love thermoelastic system, combined with the backward Euler method for temporal discretization. In [32], a quasi-static poroelastic model is considered, where the pressure moment is discretized using a standard FE approximation, while the biharmonic problem is addressed using a C^0IP method and a two-level scheme with weights for temporal discretization. In the three-dimensional setting, the Biot equations for poromechanics can be coupled with the thermal energy equation leading to a hyperbolic-parabolic system in fully dynamic or elliptic-parabolic system in the quasi-static case. Galerkin methods for this problem are investigated in [54], while mixed FE and dG discretizations are explored in, for example, [3, 14, 15]. Moreover, fully discrete approximations using the conforming P_1 FE method and the implicit Euler scheme are studied for one-dimensional TED problems in porous media [8, 26]. In [40], semi- and fully discrete schemes for solving a one-dimensional TED problem with a moving boundary and quadratic convergence in both time and space are established by employing conforming FEs for spatial discretization and Newmark's time discretization. More recently, in [35], the authors address the steady Biot–Kichhoff–Love problem with centered difference and backward Euler semi-discretization in time, and conforming and non-conforming virtual element methods for spatial discretization. They establish a priori error estimates in the best-approximation form, derive residual-based reliable and efficient a posteriori error estimates in appropriate norms, and demonstrate that these error bounds are robust with respect to the key model parameters.

Main contributions. In this paper, we analyze the unique solvability and numerical approximation for an asymptotic model for TED and TPE plate models consisting of a coupled PDE system of one hyperbolic fourth-order PDE for the plate's vertical deflection, and two second-order parabolic PDEs for the thickness-averaged (first moment) temperature distribution, and chemical potential/pore pressure. The unique solvability of the continuous formulation is based on the classical Galerkin approach (see, for example, [5, 7, 39, 45]). For the spatial discretization, C^0IP method and conforming P_1 elements for temperature and chemical potential (or pore pressure) are employed. In terms of temporal discretization, we adopt Newmark's scheme for the first hyperbolic equation and apply the Crank–Nicolson method for the remaining parabolic equations, ensuring quadratic convergence in time. Following our recent work [43], we utilize a modified Ritz projection for the analysis, based on the companion operator [18]. In conjunction with this, we employ the standard H^1-conforming Ritz projections for temperature and chemical potential/pore pressure to obtain the error estimates.

The key contributions of this work are outlined below:

- The present analysis is robust with respect to the parameter γ. Allowing γ to take values in $\mathbb{R}$ enables a unified analytical framework that accommodates both the thermoelastic diffusion and thermo-poroelastic thin plate models.
- The well-posedness of the fully coupled hyperbolic-parabolic thermoelastic diffusion and thermo-poroelastic systems is demonstrated in Subsection 2.1 under reasonable data regularity conditions. It should be noted that uniqueness for hyperbolic/parabolic coupled problems under the assumptions of Theorem 2.1 is not straightforward. It requires the use of mollified test functions, as explained later in the proof.
- A consistent and stable fully discrete scheme is developed in Section 3. Due to the coupling of second-order terms special care must be taken in the choice of compatible FE spaces that plays a crucial role in the choice of the test functions in the proofs of stability and error estimates. Moreover, similar care is required when approximating coupling terms involving time derivatives of different orders.
- A novel concept of approximating the solution at the initial time step (see (3.7)), while incorporating the approximation properties established in Lemma 4.2, is introduced to the literature, facilitating the development of a fully discrete scheme for general hyperbolic-parabolic coupled systems without any assumptions regarding the solution and its approximation at this time step (see [40]).
- A priori error estimates are derived in the best approximation form in both L^2, H^1 and energy norm for displacement in Section 4. These optimal error rates are also established in L^2 and H^1 norm for temperature and chemical potential/pore pressure. Also, the combination of Newmark–Crank–Nicolson time discretization schemes to approximate the second and first-order time derivatives, respectively, appearing in (1.1) yield quadratic convergence rates.
- The superconvergence of the projected error in the energy norm is established (see Remark 4.3), in turn leading to lower H^s- order estimates with $s = 0, 1$ (resp. $s = 0$) for u (resp. θ and p) as established in Corollary 4.5 (resp. Theorem 4.4). While such superconvergence is expected in uncoupled problems, it is not straightforward in the current coupled problem since the polynomial degrees of the FE spaces V_h and W_h used to approximate first (1.1a) and last two equations (1.1b)-(1.1c) are different. Consequently, the Ritz projection defined in (3.4) lacks orthogonality when the test function is chosen from a FE space V_h.
- Subsection 5.1 demonstrates that the Kirchhoff–Love plate model is effective in capturing TED and TPE behavior in specific materials (such as copper and flat layers of Berea sandstone, respectively). The findings indicate that as the plate thickness decreases, the two-dimensional simulations closely approximate the results from three-dimensional modeling, with a substantial reduction in computational time. This emphasises the efficiency and accuracy of 2D modeling for thin-plate structures.
- Numerical results are provided in Subsections 5.2-5.3 to validate theoretical estimates and illustrate the effective performance of the proposed scheme with different values of γ.

Plan of the paper. This paper is organized as follows. The remainder of this section introduces the common notation used throughout the manuscript. Section 2 provides definitions for solutions in the weak sense, establishes the well-posedness of the system, and discusses regularity for weak solutions. Section 3 details the spatial and temporal discretizations. The fully discrete scheme, its unique solvability, and stability are presented in Section 3.2. The error analysis is discussed in Section 4. In Section 5, we present a few representative numerical examples that confirm the rates of convergence specified by the theoretical analysis. Subsection 5.1 discusses the detailed model description for the thermoelastic diffusion and thermo-poroelastic systems.

Preliminaries. For an open set $O \subset \mathbb{R}^2$, we denote the Sobolev space $W^{m,2}(O)$ by $H^m(O)$ and equip it with the norm $\|w\|_{H^m(O)} = (\sum_{|i|\le m} \|D^i w\|^2_{L^2(O)})^{1/2}$ and semi-norm $|w|^2_{H^m(O)} = (\sum_{|i|=m} \|D^i w\|^2_{L^2(O)})^{1/2}$. For simplicity, we denote L^2 inner product by $(\bullet,\bullet)$ and norm by $\|\bullet\|$. Throughout this paper, $\mathcal{T}$ denotes a shape-regular triangulation of Ω, $H^m(\mathcal{T})$ denotes the Hilbert space $\prod_{K\in\mathcal{T}} H^m(K)$, and $P_r(\mathcal{T})$, the space of globally L^2 functions which are polynomials of degree at most r in each K. The notation ∇ (resp. ∇^2) denotes the gradient (resp. Hessian). The piecewise energy norm is denoted by $|||\bullet|||_{\mathrm{pw}} := |\bullet|_{H^2(\mathcal{T})}$ and D^2_{pw} (resp. Δ_{pw}) stands for the piecewise Hessian (Laplacian).

Let X be a normed space with norm $\|\bullet\|_X$ and $g : (0,T) \to X$ be a measurable function. Then for $1 \le p \le \infty$, we recall that

$$\|g\|_{L^p(0,T;X)} = \|g\|^p_{L^p(X)} := \int_0^T \|g(t)\|^p_X \,\mathrm{d}t,\ 1 \le p < \infty \quad \text{and} \quad \|g\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0\le t\le T} \|g(t)\|_X.$$

Let $L^p(0,T;X) := \{g : (0,T) \to X : \|g\|_{L^p(X)} < \infty\}$. The space $W^{1,p}(0,T;X)$ consists of all functions $u \in L^p(0,T;X)$ such that u_t exists in the weak sense and belongs to $L^p(0,T;X)$. For all non-negative integers k, $C^k([0,T];X)$ denotes all C^k functions $s : [0,T] \to X$ with $\|s\|_{C^k([0,T];X)} = \sum_{0\le i\le k} \max_{0\le t\le T} \|\frac{\partial^i s}{\partial t^i}\| < \infty$.

For real numbers $a > 0$, $b > 0$, and $\epsilon > 0$, we will make repeated use of the Young's inequality $ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$. Finally, as usual, the notation $a \lesssim b$ represents $a \le Cb$, where the generic constant C is independent of both mesh-size and time discretization parameter.

Lemma 1.1 (Gronwall's Lemma [20])**.** *Let g, h, and r be non-negative integrable functions on $[0,T]$ and let g satisfy $g(t) \le h(t) + \int_0^t r(s)g(s)\,\mathrm{d}s$ for all $t \in (0,T)$. Then*

$$g(t) \le h(t) + \int_0^t h(s)r(s)e^{\int_s^t r(\tau)\,d\tau}\,\mathrm{d}s \quad \textit{for all } t \in (0,T).$$

2 Well-posedness and regularity results

In this section, we establish the well-posedness of the problem through the finite Galerkin approach, which follows these steps: (i) construct a sequence of approximate solutions to the continuous problem, (ii) derive a priori bounds on these approximations based on the initial data, (iii) use a compactness argument to show the existence of a limit for a subsequence in the weak topology, and (iv) prove that this limit is the weak solution. After this, we also prove the additional regularity of continuous solution given the extra regularity conditions on given data, which is required in later sections for error analysis.

2.1 Existence and uniqueness of weak solution

Definition 2.1 (Weak solution)**.** *The triplet (u,θ,p) is a weak solution to the problem (1.1) if (1.1e) holds and*

$$u \in C([0,T];H^1(\Omega)) \cap L^\infty(0,T;H^2_0(\Omega)),\ u_t \in C([0,T];L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)), \tag{2.1a}$$

$$\theta, p \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)), \tag{2.1b}$$

satisfy the relations

$$\int_0^T \left[-(u_t, v^t_t) - a_0(\nabla u_t, \nabla v^t_t) + d_0(\nabla^2 u, \nabla^2 v^t) - \alpha(\nabla\theta, \nabla v^t) - \beta(\nabla p, \nabla v^t) \right] \mathrm{d}t$$
$$= \int_0^T (f, v^t)\,\mathrm{d}t + (u^{*0}, v^t(0)) + a_0(\nabla u^{*0}, \nabla v^t(0)), \tag{2.2a}$$

$$\int_0^T \left[-a_1(\theta, \psi^t_t) + \gamma(p, \psi^t_t) + b_1(\theta, \psi^t) + c_1(\nabla\theta, \nabla\psi^t) + \alpha(\nabla u_t, \nabla\psi^t) \right] \mathrm{d}t$$

$$= \int_0^T (\phi, \psi^t)\,\mathrm{d}t + a_1(\theta^0, \psi^t(0)) - \gamma(p^0, \psi^t(0)), \tag{2.2b}$$

$$\int_0^T \left[-a_2(p, q_t^t) + \gamma(\theta, q_t^t) + \kappa(\nabla p, \nabla q^t) + \beta(\nabla u_t, \nabla q^t) \right] \mathrm{d}t$$
$$= \int_0^T (g, q^t)\,\mathrm{d}t + a_2(p^0, q^t(0)) - \gamma(\theta^0, q^t(0)), \tag{2.2c}$$

for any $v^t \in C^2([0,T]; H_0^2(\Omega))$ *and both* $\psi^t, q^t \in C^1([0,T]; H_0^1(\Omega))$.

For any $u \in H_0^2(\Omega)$, $\theta \in H_0^1(\Omega)$, and $p \in H_0^1(\Omega)$, motivated by (2.2) and an appropriate choice of the test functions in (1.2), we define the system energy at any time $0 \le t \le T$ by

$$E(u,\theta,p;t) := \frac{1}{2}\big(\|u_t\|^2 + a_0\|\nabla u_t\|^2 + d_0\|\nabla^2 u\|^2 + (a_1 - |\gamma|/\gamma_0)\|\theta\|^2 + (a_2 - |\gamma|\gamma_0)\|p\|^2\big)$$
$$+ \int_0^t \big(b_1\|\theta\|^2 + c_1\|\nabla\theta\|^2 + \kappa\|\nabla p\|^2\big)\,\mathrm{d}s. \tag{2.3}$$

The following result states existence and uniqueness of solution to (1.1) in the sense of Definition 2.1 and establishes the boundedness of the energy (2.3). The proof is based on the approach outlined in [25, p. 384] (for second-order problems), and extended for coupled fourth- and second-order problems. Details are provided in Appendix A.

Theorem 2.1 (Existence and uniqueness)**.** *Let* $f, \phi, g \in L^2(0,T; L^2(\Omega))$, $u^0 \in H_0^2(\Omega)$, $u^{*0} \in H_0^1(\Omega)$, *and both* $\theta^0, p^0 \in L^2(\Omega)$. *Then, problem* (1.1) *has a unique weak solution* (u,θ,p) *in the sense of Definition 2.1 and the solution satisfies*

$$\operatorname*{ess\,sup}_{t\in[0,T]} E(u,\theta,p;t) \lesssim \|u^{*0}\|^2 + a_0\|u^{*0}\|^2_{H^1(\Omega)} + d_0\|u^0\|^2_{H^2(\Omega)} + (a_1 + |\gamma|/\gamma_0)\|\theta^0\|^2$$
$$+ (a_2 + |\gamma|\gamma_0)\|p^0\|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;L^2(\Omega))} + \|g\|^2_{L^2(0,T;L^2(\Omega))}. \tag{2.4}$$

Next we present an alternate weak formulation under higher regularity assumptions on the initial data. This formulation is utilized later on, to design the fully discrete scheme.

Theorem 2.2 (Alternate weak formulation)**.** *If* $f, \phi, g \in H^1(0,T; L^2(\Omega))$, $u^0 \in H^3(\Omega) \cap H_0^2(\Omega)$, $u^{*0} \in H^2(\Omega) \cap H_0^1(\Omega)$, *and both* $\theta^0, p^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, *then for the weak solution* (u,θ,p), *we have that*

$$\operatorname*{ess\,sup}_{t\in[0,T]} E(u_t,\theta_t,p_t;t) \text{ is bounded.} \tag{2.5}$$

Furthermore, for any $0 \le t \le T$, *the tuple* (u,θ,p) *satisfies*

$$(u_{tt}, v) + a_0(\nabla u_{tt}, \nabla v) + d_0(\nabla^2 u, \nabla^2 v) - \alpha(\nabla\theta, \nabla v) - \beta(\nabla p, \nabla v) = (f, v) \quad \text{for all } v \in H_0^2(\Omega), \tag{2.6a}$$
$$a_1(\theta_t, \psi) - \gamma(p_t, \psi) + b_1(\theta, \psi) + c_1(\nabla\theta, \nabla\psi) + \alpha(\nabla u_t, \nabla\psi) = (\phi, \psi) \quad \text{for all } \psi \in H_0^1(\Omega), \tag{2.6b}$$
$$a_2(p_t, q) - \gamma(\theta_t, q) + \kappa(\nabla p, \nabla q) + \beta(\nabla u_t, \nabla q) = (g, q) \quad \text{for all } q \in H_0^1(\Omega). \tag{2.6c}$$

Proof. Given that $f_t, \phi_t, g_t \in L^2(0,T; L^2(\Omega))$, following Step 1 of the proof of Theorem 2.1 presented in Appendix A, note that, $(\mathrm{d}_m^1(t), \mathrm{d}_m^2(t), \cdots, \mathrm{d}_m^m(t))$ (respectively, $(\eta_m^1(t), \eta_m^2(t), \cdots, \eta_m^m(t))$ and $(\mathrm{l}_m^1(t), \mathrm{l}_m^2(t), \cdots, \mathrm{l}_m^m(t))$), are C^3 (resp. C^2) functions and satisfy (A.2)-(A.3) for $0 \le t \le T$. Next, we differentiate (A.3) with respect to t, and multiply the resulting equations by ${\mathrm{d}_m^k}''(t)$, ${\eta_m^k}'(t)$, and ${\mathrm{l}_m^k}'(t)$, respectively. Summing over $k = 1, 2, \ldots, m$ (for all three equations), readily yields

$$\frac{1}{2}\frac{d}{dt}\big(\|u_{tt}^m\|^2 + a_0\|\nabla u_{tt}^m\|^2 + d_0\|\nabla^2 u_t^m\|^2 + a_1\|\theta_t^m\|^2 + a_2\|p_t^m\|^2\big)$$
$$+ b_1\|\theta_t^m\|^2 + c_1\|\nabla\theta_t^m\|^2 + \kappa\|\nabla p_t^m\|^2 - \gamma\frac{d}{dt}(\theta_t^m, p_t^m) = (f_t, u_{tt}^m) + (\phi_t, \theta_t^m) + (g_t, p_t^m).$$

Then, integrating from 0 to t, and using the Cauchy–Schwarz, Young, and Gronwall's inequalities (similar to the proof of existence in Theorem 2.1 in Appendix A), we arrive at

$$\begin{aligned} E(u_{mt},\theta_{mt},p_{mt};t) \lesssim \|u^m_{tt}(0)\|^2 + a_0\|\nabla u^m_{tt}(0)\|^2 + d_0\|\nabla^2 u^m_t(0)\|^2 + (a_1+|\gamma|/\gamma_0)\|\theta^m_t(0)\|^2 \\ + (a_2+|\gamma|\gamma_0)\|p^m_t(0)\|^2 + \|f_t\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi_t\|^2_{L^2(0,T;L^2(\Omega))} + \|g_t\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{2.7}$$

We now wish to bound the right-hand side of above displayed inequality by known data. Multiply the equations (A.3a), (A.3b), and (A.3c) by ${d^k_m}''(t)$, ${\eta^k_m}'(t)$ and ${1^k_m}'(t)$, respectively. Sum up the resulting equations of the system for $k=1,2,\cdots,m$ and $t=0$, and utilize the definitions in (A.1) to obtain

$$\begin{aligned} (u^m_{tt}(0),u^m_{tt}(0)) + a_0(\nabla u^m_{tt}(0),\nabla u^m_{tt}(0)) - d_0(\nabla\Delta u^m(0),\nabla u^m_{tt}(0)) + \alpha(\Delta\theta^m(0),u^m_{tt}(0)) \\ +\beta(\Delta p^m(0),u^m_{tt}(0)) = (f(0),u^m_{tt}(0)), \\ a_1(\theta^m_t(0),\theta^m_t(0)) - \gamma(p^m_t(0),\theta^m_t(0)) + b_1(\theta^m(0),\theta^m_t(0)) - c_1(\Delta\theta^m(0),\theta^m_t(0)) \\ -\alpha(\Delta u^m_t(0),\theta^m_t(0)) = (\phi(0),\theta^m_t(0)), \\ a_2(p^m_t(0),p^m_t(0)) - \gamma(\theta^m_t(0),p^m_t(0)) - \kappa(\Delta p^m(0),p^m_t(0)) - \beta(\Delta u^m_t(0),p^m_t(0)) = (g(0),p^m_t(0)), \end{aligned}$$

where in the last step we have also used integration by parts and the fact that $u^m_{tt}(0)\in H^2_0(\Omega)$, $\theta^m_t(0)\in H^1_0(\Omega)$ and $p^m_t(0)\in H^1_0(\Omega)$.

Next, we apply once more Cauchy–Schwarz and Young's inequalities together with some elementary manipulation, which gives the following bounds

$$\begin{aligned} \|u^m_{tt}(0)\|^2 + a_0\|\nabla u^m_{tt}(0)\|^2 \le \frac{{d_0}^2}{a_0}\|\nabla\Delta u^m(0)\|^2 + 3\alpha^2\|\Delta\theta^m(0)\|^2 + 3\beta^2\|\Delta p^m(0)\|^2 + 3\|f(0)\|^2, \\ (a_1-\frac{|\gamma|}{\gamma_0})\|\theta^m_t(0)\|^2 + (a_2-|\gamma|\gamma_0)\|p^m_t(0)\|^2 \le \frac{4}{a_1-\frac{|\gamma|}{\gamma_0}}\big(\alpha^2\|\Delta u^m_t(0)\|^2 + b_1^2\|\theta^m(0)\|^2 + c_1^2\|\Delta\theta^m(0)\|^2 + \|\phi(0)\|\big) \\ + \frac{3}{a_2-|\gamma|\gamma_0}\big(\beta^2\|\Delta u^m_t(0)\|^2 + \kappa^2\|\Delta p^m(0)\|^2 + \|g(0)\|^2\big). \end{aligned} \tag{2.8}$$

Then, (2.7)-(2.8) leads to

$$\begin{aligned} E(u_{mt},\theta_{mt},q_m;t) \lesssim d_0\|u^{*0}\|_{H^2(\Omega)} + ({d_0}^2/a_0)\|u^0\|^2_{H^3(\Omega)} + 3\alpha^2\|\theta^0\|^2_{H^2(\Omega)} + 3\beta^2\|p^0\|^2_{H^2(\Omega)} + 3\|f(0)\|^2 \\ + \Big(\frac{a_1+|\gamma|/\gamma_0}{a_1-|\gamma|/\gamma_0} + \frac{a_2+|\gamma|\gamma_0}{a_2-|\gamma|\gamma_0}\Big)\Big[\frac{4}{a_1-|\gamma|/\gamma_0}\big(\alpha^2\|u^{*0}\|^2_{H^2(\Omega)} + b_1^2\|\theta^0\|^2 + c_1^2\|\theta^0\|^2_{H^2(\Omega)} + \|\phi(0)\|\big) \\ + \frac{3}{a_2-|\gamma|\gamma_0}\big(\beta^2\|u^{*0}\|^2_{H^2(\Omega)} + \kappa^2\|p^0\|^2_{H^2(\Omega)} + \|g(0)\|^2\big) \\ + \|f_t\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi_t\|^2_{L^2(0,T;L^2(\Omega))} + \|g_t\|^2_{L^2(0,T;L^2(\Omega))}\Big]. \end{aligned} \tag{2.9}$$

And this, in combination with (A.12a) and (A.12b), readily implies that

$$\begin{aligned} (u^m_t,u^m_{tt},\theta^m_t,p^m_t) \xrightarrow{\text{weak*}} (u_t,u_{tt},\theta_t,p_t) \quad &\text{in } L^\infty\big(0,T;H^2_0(\Omega)\times H^1_0(\Omega)\times(L^2(\Omega))^2\big), \\ (\theta^m_t,p^m_t) \xrightarrow{\text{weak}} (\theta_t,p_t) \quad &\text{in } L^2\big(0,T;(H^1_0(\Omega))^2\big). \end{aligned}$$

Finally, the bounds in (2.5) are established by taking the limit $m\to\infty$ in (2.9). To confirm that (u,θ,p) satisfies (2.6a)-(2.6c), we proceed as in the uniqueness proof of Theorem 2.1, to obtain (A.14a)-(A.14c), but with (f,v), (ϕ,ψ), and (g,q) representing the respective right-hand sides, after which we take the limit as $\varepsilon\to 0$. □

2.2 Additional regularity

The next theorem establishes a priori bounds of the solution and its higher-order time derivatives, provided that the initial and source data are sufficiently smooth. While the specific approach followed here is relatively standard (see, e.g., [42] and the references therein), its adaptation to the present model is novel. A summary of the regularity results is displayed in Table 2.1.

Regularity estimate. It is well known [1,10] that if $\Phi^* \in H^{-r}(\Omega)$ (resp. $F^* \in H^{-s}(\Omega)$) are such that $-\Delta\chi = \Phi^*$ (resp. $\Delta^2 w = F^*$) then

$$\|\chi\|_{H^{2-r}(\Omega)} \leq C_{\mathrm{reg}}(r)\|\Phi^*\|_{H^{-r}(\Omega)} \text{ (resp.} \|w\|_{H^{4-s}(\Omega)} \leq C_{\mathrm{reg}}(s)\|F^*\|_{H^{-s}(\Omega)}), \tag{2.10}$$

for all $1-\sigma^1_{\mathrm{reg}} \leq r \leq 1$ (resp. $2-\sigma^2_{\mathrm{reg}} \leq s \leq 2$), where $\sigma^1_{\mathrm{reg}} > 0$ (resp. $\sigma^2_{\mathrm{reg}} > 0$), is the elliptic regularity index of the Laplace (resp. biharmonic) operator, and the constants $C_{\mathrm{reg}}(r)$ (resp. $C_{\mathrm{reg}}(s)$) depend only on Ω and s (resp. r). Lowest-order FE schemes typically achieve at most linear convergence in energy norm, so it is reasonable to assume throughout the paper that $\sigma = \min\{1, \sigma^1_{\mathrm{reg}}, \sigma^2_{\mathrm{reg}}\}$, whence $0 < \sigma \leq 1$. Note that if Ω is a convex polygon, then $\sigma = 1$, whereas for non-convex polygons we have $1/2 < \sigma < 1$. The elliptic regularity index σ plays an important role in determining the rate of convergence presented in Section 3.2. We are now in a position to state the regularity of the weak solution. From the estimates (2.5), we can write (also using (1.1b)-(1.1c)):

$$\begin{aligned}
-c_1\Delta\theta &= \phi - a_1\theta_t + \gamma p_t - b_1\theta + \alpha\Delta u_t := \Phi^* \in L^2(\Omega), && \text{for all } 0 \leq t \leq T,\\
-\kappa\Delta p &= g - a_2 p_t + \gamma\theta_t + \beta\Delta u_t := G^* \in L^2(\Omega), && \text{for all } 0 \leq t \leq T,\\
d_0\Delta^2 u &= f - u_{tt} + a_0\Delta u_{tt} - \frac{\alpha}{c_1}\Phi^* - \frac{\beta}{\kappa}G^* := F^* \in H^{-1}(\Omega), && \text{for all } 0 \leq t \leq T,
\end{aligned}$$

where in the last equation we have used the fact that $\|\nabla u_{tt}\|$ is bounded (cf. (2.5)), and hence $a_0\Delta u_{tt} \in H^{-1}(\Omega)$, whence $F^* \in H^{-1}(\Omega)$. Then we utilize (2.10) to see that, for all $1/2 < \sigma \leq 1$, there holds

$$u \in L^\infty(0,T; H^{2+\sigma}(\Omega)) \text{ and } \theta, p \in L^\infty(0,T; H^{1+\sigma}(\Omega)). \tag{2.11}$$

The next theorem guarantees higher regularity of weak solution (needed for the error estimates in Section 4). The proof is based on the arguments used in [21, Prop. 2.5.2], and details are postponed to Appendix A.

Theorem 2.3 (Regularity)**.** *(a) Let $f, \phi, g \in H^2(0,T; L^2(\Omega))$, $u^0, u^{*0} \in H^3(\Omega) \cap H^2_0(\Omega), \theta^0 \in H^2(\Omega) \cap H^1_0(\Omega)$, and $p^0 \in H^2(\Omega) \cap H^1_0(\Omega)$. Assume that the compatibility conditions*

$$u_{tt}(0) - a_0\Delta u_{tt}(0) = f(0) - d_0\Delta^2 u^0 - \alpha\Delta\theta^0 - \beta\Delta p^0, \tag{2.12a}$$
$$a_1\theta_t(0) - \gamma p_t(0) = \phi(0) - b_1\theta^0 + c_1\Delta\theta^0 + \alpha\Delta u^{*0}, \tag{2.12b}$$
$$a_2 p_t(0) - \gamma\theta_t(0) = g(0) + \kappa\Delta p^0 + \beta\Delta u^{*0}, \tag{2.12c}$$

hold and $(u_{tt}(0), \theta_t(0), p_t(0))$ belongs to $(H^2(\Omega) \cap H^1_0(\Omega))^3$. Then,

$$\operatorname*{ess\,sup}_{t\in[0,T]}\big(\|u_t\|^2_{H^{2+\sigma}(\Omega)} + \|\theta_t\|^2_{H^{1+\sigma}(\Omega)} + \|p_t\|^2_{H^{1+\sigma}(\Omega)} + E(u_{tt}, \theta_{tt}, p_{tt}; t)\big) \textit{ is bounded.} \tag{2.13}$$

(b) Let $f, \phi, g \in H^3(0,T; L^2(\Omega)$, $u^0, u^{*0}, u_{tt}(0) \in H^3(\Omega) \cap H^2_0(\Omega), \theta^0, \theta_t(0), p^0, p_t(0) \in H^2(\Omega) \cap H^1_0(\Omega)$. Assume that the compatibility conditions

$$u_{ttt}(0) - a_0\Delta u_{ttt}(0) = f_t(0) - d_0\Delta^2 u_t(0) - \alpha\Delta\theta_t(0) - \beta\Delta p_t(0), \tag{2.14a}$$
$$a_1\theta_{tt}(0) - \gamma p_{tt}(0) = \phi_t(0) - b_1\theta_t(0) + c_1\Delta\theta_t(0) + \alpha\Delta u_{tt}(0), \tag{2.14b}$$
$$a_2 p_{tt}(0) - \gamma\theta_{tt}(0) = g_t(0) + \kappa\Delta p_t(0) + \beta\Delta u_{tt}(0). \tag{2.14c}$$

hold and $(u_{ttt}(0), \theta_{tt}(0), p_{tt}(0))$ belongs to $(H^2(\Omega) \cap H^1_0(\Omega))^3$. Then,

$$\operatorname*{ess\,sup}_{t\in[0,T]}\big(\|u_{tt}\|^2_{H^{2+\sigma}(\Omega)} + \|\theta_{tt}\|^2_{H^{1+\sigma}(\Omega)} + \|p_{tt}\|^2_{H^{1+\sigma}(\Omega)} + E(u_{ttt}, \theta_{ttt}, p_{ttt}; t)\big) \textit{ is bounded.}$$

Remark 2.4. *In accordance with the above regularity result, if we define*

$$F(t, \boldsymbol{x}) := f(t, \boldsymbol{x}) - u_{tt} + a_0\Delta u_{tt} - \alpha\Delta\theta - \beta\Delta p, \tag{2.15}$$

then, there exist positive constants C_F and C'_F, such that

$$(i)\ \|F\|_{L^\infty(0,T;L^2(\Omega))} \leq C_F \quad \textit{and} \quad (ii)\ \|F_t\|_{L^2(0,T;L^2(\Omega))} \leq C'_F. \tag{2.16}$$

Description	Assumptions on data	Conclusions
Theorem 2.1 (Existence, Uniqueness, & Energy bound for solution)	$u^0 \in H_0^2(\Omega)$, $u^{*0} \in H_0^1(\Omega)$ θ^0, $p^0 \in L^2(\Omega)$ $f(t)$, $\phi(t)$, $g(t) \in L^2(0,T;L^2(\Omega))$	$u \in L^\infty(0,T;H_0^2(\Omega))$ $u_t \in L^\infty(0,T;H_0^1(\Omega))$ θ, $p \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$
Theorem 2.2 (Weak formulation (2.6) & Energy bound for time derivative of the solution)	$u^0 \in H^3(\Omega) \cap H_0^2(\Omega)$ u^{*0}, θ^0, $p^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ $f(t)$, $\phi(t)$, $g(t) \in H^1(0,T;L^2(\Omega))$	$u \in L^\infty(0,T;H^{2+\sigma}(\Omega) \cap H_0^2(\Omega))$ $u_t \in L^\infty(0,T;H_0^2(\Omega))$, $u_{tt} \in L^\infty(0,T;H_0^1(\Omega))$ θ, $p \in L^\infty(0,T;H^{1+\sigma}(\Omega) \cap H_0^1(\Omega))$ θ_t, $p_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$
Theorem 2.3(a) (Additional regularity of solution & Energy bound for second-order time derivative)	u^0, $u^{*0} \in H^3(\Omega) \cap H_0^2(\Omega)$ $\theta(0)$, $p(0) \in H^2(\Omega) \cap H_0^1(\Omega)$ $u_{tt}(0)$, $\theta_t(0)$, $p_t(0) \in H^2(\Omega) \cap H_0^1(\Omega)$ $f(t)$, $\phi(t)$, $g(t) \in H^2(0,T;L^2(\Omega))$ (2.12) holds	$u_t \in L^\infty(0,T;H^{2+\sigma}(\Omega) \cap H_0^2(\Omega))$ $u_{tt} \in L^\infty(0,T;H_0^2(\Omega))$, $u_{ttt} \in L^\infty(0,T;H_0^1(\Omega))$ θ_t, $p_t \in L^\infty(0,T;H^{1+\sigma}(\Omega) \cap H_0^1(\Omega))$ θ_{tt}, $p_{tt} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$
Theorem 2.3(b) (Energy bound for third-order time derivative. Sufficient conditions for error analysis in Lemma 4.2 & Theorem 4.4)	u^0, u^{*0}, $u_{tt}(0) \in H^3(\Omega) \cap H_0^2(\Omega)$ $u_{ttt}(0)$, θ^0, $\theta_t(0)$, p^0, $p_t(0)$, $\theta_{tt}(0)$, $p_{tt}(0) \in H^2(\Omega) \cap H_0^1(\Omega)$ $f(t)$, $\phi(t)$, $g(t) \in H^3(0,T;L^2(\Omega))$ (2.14) holds	$u_{tt} \in L^\infty(0,T;H^{2+\sigma}(\Omega) \cap H_0^2(\Omega))$ $u_{ttt} \in L^\infty(0,T;H_0^2(\Omega))$, $u_{tttt} \in L^\infty(0,T;H_0^1(\Omega))$ θ_{tt}, $p_{tt} \in L^\infty(0,T;H^{1+\sigma}(\Omega) \cap H_0^1(\Omega))$ θ_{ttt}, $p_{ttt} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$

Table 2.1: Summary of regularity assumptions and corresponding results.

3 Fully discrete scheme and stability

This section develops the numerical framework for the coupled hyperbolic-parabolic system (1.1a)-(1.1c). Subsection 3.1 introduces the FE spaces and projection operators, highlighting the need for a modified Ritz projection for the displacement variable. Subsection 3.2 presents the *first* fully discrete scheme with explicit initial error estimates, contrasting with previous works that begin at the second time step. Subsection 3.3 establishes the unconditional stability of the scheme.

3.1 Space discretization

We now define the FE spaces and projection operators, and highlight their approximation properties. Additionally, we discuss the necessity for a *modified* Ritz projection, specifically for the displacement variable.

Let $K \in \mathcal{T}$ be any triangle in the shape-regular triangulation $\mathcal{T}$ of $\bar{\Omega}$. We denote its diameter by h_K, its area by $|K|$, and use $\boldsymbol{n}_K$ to refer to the outward unit normal vector on ∂K. Define $h := \max_{K\in\mathcal{T}} h_K$. The sets of interior and boundary vertices of $\mathcal{T}$ are denoted by $\mathcal{V}(\Omega)$ and $\mathcal{V}(\partial\Omega)$, respectively, with the combined set represented as $\mathcal{V} = \mathcal{V}(\Omega) \cup \mathcal{V}(\partial\Omega)$. Similarly, we use $\mathcal{E}(\Omega)$ and $\mathcal{E}(\partial\Omega)$ for the sets of interior and boundary edges, and write $\mathcal{E} = \mathcal{E}(\Omega) \cup \mathcal{E}(\partial\Omega)$. For any edge $e \in \mathcal{E}$, the corresponding edge patch $\omega(e)$ is defined as $\mathrm{int}(K_+ \cup K_-)$ if $e = \partial K_+ \cap \partial K_- \in \mathcal{E}(\Omega)$, and $\mathrm{int}(K)$ when $e \in \mathcal{E}(\partial\Omega)$. Consider two neighbouring triangles, K_+ and K_-, with the unit normal vector along e satisfying $\boldsymbol{n}_{K_+}|_e = \boldsymbol{n}|_e = -\boldsymbol{n}_{K_-}|_e$, directed outward from K_+ towards K_-. The jump of a function φ, written as $[\![\varphi]\!]$, is defined by $\varphi|_{K_+} - \varphi|_{K_-}$ if $e = \partial K_+ \cap \partial K_- \in \mathcal{E}(\Omega)$ and $\varphi|_e$ if $e \in \mathcal{E}(\partial\Omega)$.The average $\{\!\{\varphi\}\!\}$ is defined by $\frac{1}{2}(\varphi|_{K_+} + \varphi|_{K_-})$ if $e = \partial K_+ \cap \partial K_- \in \mathcal{E}(\Omega)$ and $\varphi|_e$ if $e \in \mathcal{E}(\partial\Omega)$.

Let $V_h := P_2(\mathcal{T}) \cap H_0^1(\Omega) \subset \mathcal{H}^2(\mathcal{T})$ and $W_h := P_1(\mathcal{T}) \cap H_0^1(\Omega) \subset H_0^1(\Omega)$ be finite-dimensional subspaces and define the bilinear form $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ by

$$a_h(w_h, v_h) := \int_\Omega D^2_{\mathrm{pw}} w_h : D^2_{\mathrm{pw}} v_h \,\mathrm{d}\boldsymbol{x} - \sum_{e\in\mathcal{E}} \int_e [\![\nabla w_h]\!] \cdot \{\!\{D^2_{\mathrm{pw}} v_h\}\!\}\boldsymbol{n} \,\mathrm{d}s \\ - \sum_{e\in\mathcal{E}} \int_e [\![\nabla v_h]\!] \cdot \{\!\{D^2_{\mathrm{pw}} w_h\}\!\}\boldsymbol{n} \,\mathrm{d}s + \sum_{e\in\mathcal{E}} \frac{\sigma_{\mathrm{IP}}}{h_e} \int_e \left[\!\!\left[\frac{\partial w_h}{\partial \boldsymbol{n}}\right]\!\!\right] \left[\!\!\left[\frac{\partial v_h}{\partial \boldsymbol{n}}\right]\!\!\right] \mathrm{d}s,$$

with respect to a mesh-dependent (broken) norm on V_h defined by

$$\|v_h\|_h^2 := \|D^2_{\mathrm{pw}} v_h\|^2 + \sum_{e\in\mathcal{E}} \frac{\sigma_{\mathrm{IP}}}{h_e} \int_e \left[\!\!\left[\frac{\partial v_h}{\partial \boldsymbol{n}}\right]\!\!\right]^2 \mathrm{d}s,$$

where, D^2_{pw} is the piecewise Hessian and the penalty parameter $\sigma_{\mathrm{IP}} > 0$ is chosen sufficiently large [12].

It is well-known that $a_h(\bullet,\bullet)$ is symmetric, continuous, and elliptic, i.e., there exist $C_{\text{Coer}}, C_{\text{Cont}} > 0$ such that for all $w_h, v_h \in V_h$ (see, for e.g., [17])

$$a_h(w_h, v_h) = a_h(v_h, w_h), \quad C_{\text{Coer}}\|w_h\|_h^2 \le a_h(w_h, w_h), \quad a_h(w_h, v_h) \le C_{\text{Cont}}\|w_h\|_h\|v_h\|_h. \tag{3.1}$$

The nonconforming Morley FE space [18] is defined as follows:

$$\begin{aligned}\mathrm{M}(\mathcal{T}) := \{v_{\mathrm{M}} \in P_2(\mathcal{T}) : \ & v_{\mathrm{M}} \text{ is continuous at interior vertices and its normal derivatives}\\ & \text{are continuous at the midpoints of interior edges, } v_{\mathrm{M}} \text{ vanishes at the vertices of } \partial\Omega\\ & \text{and its normal derivatives vanish at the midpoints of boundary edges of } \partial\Omega\}.\end{aligned}$$

Definition 3.1 (Morley interpolation [18])**.** *For all $v_{\mathrm{pw}} \in H^2(\mathcal{T})$, the extended Morley interpolation operator $I_{\mathrm{M}} : H^2(\mathcal{T}) \to \mathrm{M}(\mathcal{T})$ is defined by*

$$(I_{\mathrm{M}}v_{\mathrm{pw}})(z) := |\mathcal{T}(z)|^{-1} \sum_{K\in\mathcal{T}(z)} (v_{\mathrm{pw}}|_K)(z) \quad \textit{and} \quad \fint_e \frac{\partial(I_{\mathrm{M}}v_{\mathrm{pw}})}{\partial \boldsymbol{n}}\,\mathrm{d}s := \fint_e \left\{\!\!\left\{ \frac{\partial v_{\mathrm{pw}}}{\partial \boldsymbol{n}} \right\}\!\!\right\} \mathrm{d}s.$$

In case of an interior vertex z, $\mathcal{T}(z)$ represents the collection of attached triangles, and $|\mathcal{T}(z)|$ indicates the number of such triangles connected to vertex z.

Lemma 3.1 (Companion operator and properties [18, 19])**.** *Let $\mathrm{HCT}(\mathcal{T})$ denote the Hsieh–Clough–Tocher element. There exists a linear mapping $J : \mathrm{M}(\mathcal{T}) \to (\mathrm{HCT}(\mathcal{T}) + P_8(\mathcal{T})) \cap H_0^2(\Omega)$ such that any $w_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$ satisfies*

(i) $Jw_{\mathrm{M}}(z) = w_{\mathrm{M}}(z)$ *for* $z \in \mathcal{V}$,

(ii) $\nabla(Jw_{\mathrm{M}})(z) = |\mathcal{T}(z)|^{-1} \sum_{K\in\mathcal{T}(z)} (\nabla w_{\mathrm{M}}|_K)(z)$ *for* $z \in \mathcal{V}(\Omega)$,

(iii) $\fint_e \frac{\partial Jw_{\mathrm{M}}}{\partial \boldsymbol{n}}\,\mathrm{d}s = \fint_e \frac{\partial w_{\mathrm{M}}}{\partial \boldsymbol{n}}\,\mathrm{d}s$ *for any* $e \in \mathcal{E}$,

(iv) $|\!|\!| w_{\mathrm{M}} - Jw_{\mathrm{M}} |\!|\!|_{\mathrm{pw}} \lesssim \min_{v\in H_0^2(\Omega)} |\!|\!| w_{\mathrm{M}} - v |\!|\!|_{\mathrm{pw}}$,

(v) $\|v_h - Qv_h\|_{H^s(\mathcal{T})} \le C_1 h^{2-s} \min_{v\in H_0^2(\Omega)} \|v - v_h\|_h$ *for* $v_h \in V_h$, $C_1 > 0$, *and* $0 \le s \le 2$.

Here $Q = JI_{\mathrm{M}}$ is a smoother operator defined from V_h to $H_0^2(\Omega)$.

Ritz projection operators

The error control associated with the fully discrete approximation employs Ritz projection operators defined from $H_0^2(\Omega)$ (resp. $H_0^1(\Omega)$) into V_h (resp. W_h) for u (resp. θ and p). It should be noted that since V_h is not a subspace of $H_0^2(\Omega)$, the standard definition

$$a_h(\mathcal{R}_h w, v_h) = (\nabla^2 w, \nabla^2 v_h) \qquad \text{for all } v_h \in V_h,$$

does not hold for $v_h \in V_h \subset H^2(\mathcal{T})$ for the nonstandard C^0IP scheme proposed herein.

Alternative approaches that define Ritz projections for nonstandard methods (see, e.g., [22, 29] for the fourth-order nonlinear parabolic extended Fisher–Kolmogorov equation) often require higher regularity $u \in H^3(\Omega) \cap H_0^2(\Omega)$, which might not hold for non-convex domains. See the discussion in Section 2.2 for non-convex polygons. Our recent work [43] addresses this issue by means of the *modified* Ritz projection (see Definition 3.2 below), which utilizes a smoother operator $Q : V_h \to H_0^2(\Omega)$ defined as JI_{M}, where J (resp. I_{M}) denotes the companion (resp. extended Morley interpolation) operator from Lemma 3.1 (resp. Lemma 3.1). The *modified Ritz projection* $\mathcal{R}_h : H_0^2(\Omega) \to V_h$ for the displacement variable is defined as follows:

$$a_h(\mathcal{R}_h w, v_h) = (\nabla^2 w, \nabla^2 Q v_h) \qquad \text{for all } v_h \in V_h,\ w \in H_0^2(\Omega). \tag{3.2}$$

Lemma 3.2 (Approximation properties for $\mathcal{R}_h$ [41, Appendix])**.** *Let $w \in H_0^2(\Omega) \cap H^{2+\sigma}(\Omega)$, where $\sigma \in (1/2, 1]$, and let $\mathcal{R}_h w$ be its Ritz projection defined in (3.2). Then, there exists a constant $C_2 > 0$ such that*

$$\|w - \mathcal{R}_h w\| + \|\nabla(w - \mathcal{R}_h w)\| + h^{\sigma}\|w - \mathcal{R}_h w\|_h \le C_2 h^{2\sigma}\|w\|_{H^{2+\sigma}(\Omega)}. \tag{3.3}$$

Next, we define the H^1-*conforming elliptic projection* $\Pi_h : H_0^1(\Omega) \to W_h$ [24] for the first moments of temperature and pressure as:

$$(\nabla \Pi_h \chi, \nabla \chi_h) = (\nabla \chi, \nabla \chi_h) \qquad \text{for all } \chi_h \in W_h. \tag{3.4}$$

Lemma 3.3 (Approximation properties for Π_h [24, Theorem 32.15]). *Let $\chi \in H_0^1(\Omega) \cap H^{1+\sigma}(\Omega)$ for some $\sigma \in (1/2, 1]$. Then, there exists a constant $C_3 > 0$ such that*

$$\|\chi - \Pi_h \chi\| + h^\sigma \|\nabla(\chi - \Pi_h \chi)\| \leq C_3 h^{2\sigma} \|\chi\|_{H^{1+\sigma}(\Omega)}. \tag{3.5}$$

3.2 Fully discrete scheme

This subsection discusses a fully discrete scheme for (2.6). To the best of our knowledge, this is the first fully discrete scheme for a hyperbolic-parabolic coupled problem with explicit initial error estimates. In contrast with [40], where the initial error is assumed to be bounded with the required convergence and the formulation begins from the second time step, our approach starts from the initial step and provides a general framework for defining the fully discrete formulation for any coupled hyperbolic-parabolic system.

For a positive integer N, consider the partition $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ of the interval $[0, T]$ with $t_n = n\Delta t$, and $\Delta t = T/N$ being the time step. For any function $v(\boldsymbol{x}, t)$, the following notations are adopted:

$$v^n := v(\boldsymbol{x}, t_n) = v(t_n), \quad v^{n+1/2} := \frac{1}{2}\left(v^{n+1} + v^n\right), \quad v^{n,1/4} := \frac{1}{4}\left(v^{n+1} + 2v^n + v^{n-1}\right) = \frac{1}{2}\left(v^{n+1/2} + v^{n-1/2}\right),$$

$$\bar{\partial}_t v^{n+1/2} := \frac{v^{n+1} - v^n}{\Delta t}, \quad \bar{\partial}_t^2 v^n := \frac{v^{n+1} - 2v^n + v^{n-1}}{(\Delta t)^2}, \quad \delta_t v^n := \frac{v^{n+1} - v^{n-1}}{2\Delta t}.$$

Let $(U^n, \Theta^n, P^n) = (U(t_n), \Theta(t_n), P(t_n))$ denote the approximation of the continuous solution (u, θ, p) at time t_n. Considering the following approximation of the initial solution

$$(U^0, \Theta^0, P^0) = (\mathcal{R}_h u^0, \Pi_h \theta^0, \Pi_h p^0), \tag{3.6}$$

we compute $(U^1, \Theta^1, P^1) \in V_h \times W_h \times W_h$ by solving the following elliptic system for all $(v_h, \psi_h, q_h) \in V_h \times W_h \times W_h$

$$\begin{aligned} &2(\Delta t)^{-1}\left[(\bar{\partial}_t U^{1/2} - u^{*0}, v_h) + a_0(\nabla \bar{\partial}_t U^{1/2} - \nabla u^{*0}, \nabla v_h)\right] \\ &\qquad + d_0 a_h(U^{1/2}, v_h) - \alpha(\nabla \Theta^{1/2}, \nabla v_h) - \beta(\nabla P^{1/2}, \nabla v_h) = (f^{1/2}, v_h), \end{aligned} \tag{3.7a}$$

$$a_1(\bar{\partial}_t \Theta^{1/2}, \psi_h) - \gamma \bar{\partial}_t P^{1/2}, \psi_h) + b_1(\Theta^{1/2}, \psi_h) + c_1(\nabla \Theta^{1/2}, \nabla \psi_h) + \alpha(\nabla \bar{\partial}_t U^{1/2}, \nabla \psi_h) = (\phi^{1/2}, \psi_h), \tag{3.7b}$$

$$a_2(\bar{\partial}_t P^{1/2}, q_h) - \gamma(\bar{\partial}_t \Theta^{1/2}, q_h) + \kappa(\nabla P^{1/2}, \nabla q_h) + \beta(\nabla \bar{\partial}_t U^{1/2}, \nabla q_h) = (g^{1/2}, q_h). \tag{3.7c}$$

The solution is calculated at t_1 using (3.7a)-(3.7c) in order to align the two numerical schemes, since the Newmark scheme (3.8a) requires solutions at t_0 and t_1 to compute the solution at t_2, while the Crank–Nicolson scheme begins its computation from t_1. The idea of the discrete equation (3.7a) is based on the discretisation of biharmonic wave [43] for coupled system, and (3.7b)-(3.7c) are based on the Crank–Nicolson method to determine the solution at t_1. The construction of (3.7) guarantees quadratic convergence in time, implying that also the fully discrete scheme is quadratically convergent.

For $n = 1, 2, \cdots, N-1$, the fully discrete scheme consists in finding $(U^{n+1}, \Theta^{n+1}, P^{n+1}) \in V_h \times W_h \times W_h$ such that for all $(v_h, \psi_h, q_h) \in V_h \times W_h \times W_h$

$$(\bar{\partial}_t^2 U^n, v_h) + a_0(\nabla \bar{\partial}_t^2 U^n, \nabla v_h) + d_0 a_h(U^{n,1/4}, v_h) - \alpha(\nabla \Theta^{n,1/4}, \nabla v_h) - \beta(\nabla P^{n,1/4}, \nabla v_h) = (f^{n,1/4}, v_h), \tag{3.8a}$$

$$\begin{aligned} a_1(\bar{\partial}_t \Theta^{n+1/2}, \psi_h) - \gamma(\bar{\partial}_t P^{n+1/2}, \psi_h) + b_1(\Theta^{n+1/2}, \psi_h) + c_1(\nabla \Theta^{n+1/2}, \nabla \psi_h) \\ + \alpha(\nabla \bar{\partial}_t U^{n+1/2}, \nabla \psi_h) = (\phi^{n+1/2}, \psi_h), \end{aligned} \tag{3.8b}$$

$$a_2(\bar{\partial}_t P^{n+1/2}, q_h) - \gamma(\bar{\partial}_t \Theta^{n+1/2}, q_h) + \kappa(\nabla P^{n+1/2}, \nabla q_h) + \beta(\nabla \bar{\partial}_t U^{n+1/2}, \nabla q_h) = (g^{n+1/2}, q_h). \tag{3.8c}$$

This section and the rest of the paper uses the discrete Gronwall Lemma that is stated below.

Lemma 3.4 (Discrete Gronwall Lemma [30]). *Let $\{v_n\}$, $\{w_n\}$, and $\{y_n\}$ be three non-negative sequences, with $\{y_n\}$ monotone, that satisfy $v_m + w_m \leq y_m + \nu \sum_{n=0}^{m-1} v_n$, $\nu > 0$, $v_0 + w_0 \leq y_0$. Then for $m \geq 0$, it holds that $v_m + w_m \leq y_m e^{m\nu}$.*

Remark 3.5 (Identities)**.** *Before proceeding further, we state the following identities, which lead to telescopic sums and are used in Theorem 3.6 and Theorem 4.4. For any discrete functions $Q^n \in V_h$ and $S^n \in W_h$, $n = 0, 1, 2, \cdots, N$ there hold*

$$2\Delta t(\bar{\partial}_t^2 Q^n, \delta_t Q^n) = \|\bar{\partial}_t Q^{n+1/2}\|^2 - \|\bar{\partial}_t Q^{n-1/2}\|^2, \tag{3.9a}$$

$$2\Delta t(\nabla\bar{\partial}_t^2 Q^n, \nabla\delta_t Q^n) = \|\nabla\bar{\partial}_t Q^{n+1/2}\|^2 - \|\nabla\bar{\partial}_t Q^{n-1/2}\|^2, \tag{3.9b}$$

$$2\Delta t a_h(Q^{n,1/4}, \delta_t Q^n) = a_h(Q^{n+1/2}, Q^{n+1/2}) - a_h(Q^{n-1/2}, Q^{n-1/2}), \tag{3.9c}$$

$$2\Delta t(\bar{\partial}_t S^{n+1/2}, S^{n+1/2}) = \|S^{n+1}\|^2 - \|S^n\|^2. \tag{3.9d}$$

3.3 Stability

Here we demonstrate the stability of the fully discrete scheme in (3.8) and establish a uniform bound of the solution $(U^{m+1}, \Theta^{m+1}, P^{m+1})$ at t_{m+1} for $1 \le m \le N-1$ in terms of the solution at t_0, t_1, and the load/source functions. For any $\chi_h^n, Q_h^n \in W_h; n \in \{1, 2, \cdots, m\}$ with $1 \le m \le N-1$, define

$$\|(\chi_h^m, Q_h^m)\|_H^2 := \Delta t \sum_{n=1}^{m} \Big[b_1\|\chi_h^{n+1/2}\|^2 + c_1\|\nabla\chi_h^{n+1/2}\|^2 + \kappa\|\nabla Q_h^{n+1/2}\|^2 \Big]. \tag{3.10}$$

Also, we define

$$((U^0, U^1, f)) := 6\|\bar{\partial}_t U^{1/2}\|^2 + 4a_0\|\nabla\bar{\partial}_t U^{1/2}\|^2 + 4d_0 C_{\mathrm{Cont}}\|U^{1/2}\|_h^2 + 4T^2\|f\|^2_{L^\infty(0,T;L^2(\Omega))},$$

$$((\Theta^0, \Theta^1, \phi)) := \frac{1}{2}(3a_1 + |\gamma|/\gamma_0)\|\Theta^1\|^2 + c_1\Delta t\|\nabla\Theta^{1/2}\|^2 + \frac{T^2}{a_1 - |\gamma|/\gamma_0}\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))},$$

$$((P^0, P^1, g)) := \frac{1}{2}(3a_2 + |\gamma|\gamma_0)\|P^1\|^2 + \kappa\Delta t\|\nabla P^{1/2}\|^2 + \frac{T^2}{a_2 - |\gamma|\gamma_0}\|g\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

In relation with (2.3), we define the discrete energy of (1.1) at time t_n, for $m = 1, \cdots, N$, as

$$\begin{aligned} E_h(U^{m+1}, \Theta^{m+1}, P^{m+1}) := {} & \|\bar{\partial}_t U^{m+1/2}\|^2 + a_0\|\nabla\bar{\partial}_t U^{m+1/2}\|^2 + d_0 C_{\mathrm{Coer}}\|U^{m+1/2}\|_h^2 \\ & + (a_1 - |\gamma|/\gamma_0)\|\Theta^{m+1}\|^2 + (a_2 - |\gamma|\gamma_0)\|P^{m+1}\|^2 + \|(\Theta^m, P^m)\|_H^2. \end{aligned}$$

Then, as in Theorem 2.2, the next theorem leads to the well-posedness of (3.8).

Theorem 3.6 (Stability)**.** *Let $f, \phi, g \in L^\infty(0,T; L^2(\Omega))$, $u^0 \in H_0^2(\Omega)$, $u^{*0} \in H_0^1(\Omega)$, and both $\theta^0, p^0 \in H_0^1(\Omega)$. Then, the scheme (3.8) is unconditionally stable. Moreover, for $1 \le m \le N-1$, the following bound holds:*

$$E_h(U^{m+1}, \Theta^{m+1}, P^{m+1}) \lesssim ((U^0, U^1, f)) + ((\Theta^0, \Theta^1, \phi)) + ((P^0, P^1, g)).$$

The constant absorbed in "$\lesssim$" above depends on T and on the model coefficients $a_0, c_1, \alpha, \beta, \kappa$.

Proof. The proof follows in six steps as outlined below.

Step 1 (Key inequality). We multiply (3.8a) by $8\Delta t$, then choose $v_h = \delta_t U^n$ in (3.8a), and utilize (3.9a)-(3.9c) to show that

$$\begin{aligned} & 4\big[\|\bar{\partial}_t U^{n+1/2}\|^2 - \|\bar{\partial}_t U^{n-1/2}\|^2 + a_0\|\nabla\bar{\partial}_t U^{n+1/2}\|^2 - a_0\|\nabla\bar{\partial}_t U^{n-1/2}\|^2\big] + 4d_0 a_h(U^{n+1/2}, U^{n+1/2}) \\ & \quad - 4d_0 a_h(U^{n-1/2}, U^{n-1/2}) = 8\Delta t(\alpha\nabla\Theta^{n,1/4} + \beta\nabla P^{n,1/4}, \nabla\delta_t U^n) + 8\Delta t(f^{n,1/4}, \delta_t U^n) \\ & \quad = 4\Delta t(\alpha\nabla\Theta^{n,1/4} + \beta\nabla P^{n,1/4}, \nabla(\bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2})) + 4\Delta t(f^{n,1/4}, \bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}), \end{aligned} \tag{3.11}$$

with the identity $2\delta_t U^n = \bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}$ in the last equality. Next we choose $\psi_h = 2\Delta t\Theta^{n+1/2}$ in (3.8b), $q_h = 2\Delta t P^{n+1/2}$ in (3.8c), employ the identity (3.9d) and add the two resulting equations to obtain

$$\begin{aligned} & a_1\Delta t\bar{\partial}_t\|\Theta^{n+1/2}\|^2 + 2\Delta t\big[b_1\|\Theta^{n+1/2}\|^2 + c_1\|\nabla\Theta^{n+1/2}\|^2 + \kappa\|\nabla P^{n+1/2}\|^2\big] \\ & \qquad + 2\gamma\big[(P^n, \Theta^n) - (P^{n+1}, \Theta^{n+1})\big] + a_2\Delta t\bar{\partial}_t\|P^{n+1/2}\|^2 \end{aligned}$$

$$= -2\Delta t\big[(\alpha\nabla\Theta^{n+1/2} + \beta\nabla P^{n+1/2}, \nabla\bar{\partial}_t U^{n+1/2})\big] + 2\Delta t\big[(\phi^{n+1/2}, \Theta^{n+1/2}) + (g^{n+1/2}, P^{n+1/2})\big]. \tag{3.12}$$

We also combine the coupling terms on the right-hand sides of (3.11)-(3.12), and utilize the term $\Theta^{n,1/4} := \frac{1}{2}\left(\Theta^{n+1/2} + \Theta^{n-1/2}\right)$ twice (analogously for $P^{n,1/4}$). Elementary manipulations lead to the cancellation of some terms, and we eventually arrive at

$$\begin{aligned} &4(\alpha\nabla\Theta^{n,1/4} + \beta\nabla P^{n,1/4}, \nabla(\bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2})) - 2(\alpha\nabla\Theta^{n+1/2} + \beta\nabla P^{n+1/2}, \nabla\bar{\partial}_t U^{n+1/2}) \\ &= \big[4(\alpha\nabla\Theta^{n,1/4} + \beta\nabla P^{n,1/4}, \nabla\bar{\partial}_t U^{n-1/2})\big] + \big[2(\alpha\nabla\Theta^{n-1/2} + \beta\nabla P^{n-1/2}, \nabla\bar{\partial}_t U^{n+1/2})\big] := A^n + B^n. \end{aligned} \tag{3.13}$$

Then we add (3.11)-(3.12), utilize (3.13) and then sum the resulting equation for $n = 1, 2, \cdots, m$, for any $m = 1, \cdots, N-1$, and in turn use (3.1) and (3.10) to arrive at the *key inequality*

$$\begin{aligned} &4\Big[\|\bar{\partial}_t U^{m+1/2}\|^2 + a_0\|\nabla\bar{\partial}_t U^{m+1/2}\|^2 + d_0 C_{\text{Coer}}\|U^{m+1/2}\|_h^2\Big] + a_1\|\Theta^{m+1}\|^2 + a_2\|P^{m+1}\|^2 + 2\|(\Theta^m, P^m)\|_H^2 \\ &\le 4\Big[\|\bar{\partial}_t U^{1/2}\|^2 + a_0\|\nabla\bar{\partial}_t U^{1/2}\|^2 + d_0 C_{\text{Cont}}\|U^{1/2}\|_h^2\Big] + a_1\|\Theta^1\|^2 + a_2\|P^1\|^2 + 2\gamma\big[(P^{m+1}, \Theta^{m+1}) - (P^1, \Theta^1)\big] \\ &\quad + \Delta t\sum_{n=1}^{m}\Big[A^n + B^n + 4(f^{n,1/4}, \bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}) + (\phi^{n+1/2}, \Theta^{n+1} + \Theta^n) + (g^{n+1/2}, P^{n+1} + P^n)\Big]. \end{aligned} \tag{3.14}$$

Step 2 (Bound for $\Delta t\sum_{n=1}^m A^n$). Using the definition $\Theta^{n,1/4} := \frac{1}{2}\left(\Theta^{n+1/2} + \Theta^{n-1/2}\right)$ (and an analogous expression for $P^{n,1/4}$) yields

$$\begin{aligned} \sum_{n=1}^m A^n &= \sum_{n=1}^m 4(\alpha\nabla\Theta^{n,1/4} + \beta\nabla P^{n,1/4}, \nabla\bar{\partial}_t U^{n-1/2}) \\ &= 2\alpha\sum_{n=1}^m(\nabla\Theta^{n+1/2} + \nabla\Theta^{n-1/2}, \nabla\bar{\partial}_t U^{n-1/2}) + 2\beta\sum_{n=1}^m(\nabla P^{n+1/2} + \nabla P^{n-1/2}, \nabla\bar{\partial}_t U^{n-1/2}). \end{aligned} \tag{3.15}$$

Next we can apply Cauchy–Schwarz inequality and Young's inequality ($ab \le a^2/2\epsilon + b^2\epsilon/2$) with $\epsilon = 4/c_1$ to bound the first term on the right-hand side of (3.15) by

$$\begin{aligned} 2\alpha\sum_{n=1}^m(\nabla\Theta^{n+1/2} + \nabla\Theta^{n-1/2}, \nabla\bar{\partial}_t U^{n-1/2}) &\le \frac{c_1}{2}\sum_{n=1}^m\left(\|\nabla\Theta^{n+1/2}\|^2 + \|\nabla\Theta^{n-1/2}\|^2\right) + \sum_{n=1}^m\frac{4\alpha^2}{c_1}\|\nabla\bar{\partial}_t U^{n-1/2}\|^2 \\ &\le c_1\sum_{n=1}^m\|\nabla\Theta^{n+1/2}\|^2 + \frac{c_1}{2}\|\nabla\Theta^{1/2}\|^2 + \sum_{n=1}^m\frac{4\alpha^2}{c_1}\|\nabla\bar{\partial}_t U^{n-1/2}\|^2 \end{aligned}$$

with elementary manipulations and addition of $\frac{4\alpha^2}{c_1}\|\nabla\Theta^{m+1/2}\|^2$ in the last step. Similar arguments bounds the second term on the right-hand side of (3.15). A combination of all this in (3.15) (after multiplying by Δt) shows

$$\begin{aligned} \Delta t\sum_{n=1}^m A^n \le \Delta t\Big(c_1\sum_{n=1}^m\|\nabla\Theta^{n+1/2}\|^2 + \kappa\sum_{n=1}^m\|\nabla P^{n+1/2}\|^2\Big) + \frac{\Delta t}{2}\Big(c_1\|\nabla\Theta^{1/2}\|^2 + \kappa\|\nabla P^{1/2}\|^2\Big) \\ + 4\Delta t\Big(\frac{\alpha^2}{c_1} + \frac{\beta^2}{\kappa}\Big)\sum_{n=1}^m\|\nabla\bar{\partial}_t U^{n-1/2}\|^2. \end{aligned} \tag{3.16}$$

Step 3 (Bound for $\Delta t\sum_{n=1}^m B^n$). First, we rewrite $\Delta t\sum_{n=1}^m B^n$ as

$$\Delta t\sum_{n=1}^{m-1}(2\alpha\nabla\Theta^{n-1/2} + 2\beta\nabla P^{n-1/2}, \nabla\bar{\partial}_t U^{n+1/2}) + 2\Delta t(\alpha\nabla\Theta^{m-1/2} + \beta\nabla P^{m-1/2}, \nabla\bar{\partial}_t U^{m+1/2}).$$

Then, it suffices to apply Cauchy–Schwarz inequality and Young's inequality with $\epsilon = 2/c_1$ (resp. $\epsilon = 2/\kappa$) to the first (resp. second) term in the summation on the right-hand side above, to obtain

$$\sum_{n=1}^{m-1}2(\alpha\nabla\Theta^{n-1/2}, \nabla\bar{\partial}_t U^{n+1/2}) \le \sum_{n=1}^{m-1}\left(\frac{c_1}{2}\|\nabla\Theta^{n-1/2}\|^2 + \frac{2\alpha^2}{c_1}\|\nabla\bar{\partial}_t U^{n+1/2}\|^2\right)$$

$$\leq \sum_{n=1}^{m-1} \frac{c_1}{2}\|\nabla\Theta^{n+1/2}\|^2 + \frac{c_1}{2}\|\nabla\Theta^{1/2}\|^2 + \frac{2\alpha^2}{c_1}\sum_{n=1}^{m-1}\|\nabla\bar{\partial}_t U^{n+1/2}\|^2$$

$$\left(\text{resp. } \sum_{n=1}^{m-1} 2(\beta\nabla P^{n-1/2}, \nabla\bar{\partial}_t U^{n+1/2}) \leq \sum_{n=1}^{m-1}\frac{\kappa}{2}\|\nabla P^{n+1/2}\|^2 + \frac{\kappa}{2}\|\nabla P^{1/2}\|^2 + \frac{2\beta^2}{\kappa}\sum_{n=1}^{m-1}\|\nabla\bar{\partial}_t U^{n+1/2}\|^2\right),$$

with an addition of a non-negative term $\frac{c_1}{2}\|\nabla\Theta^{m-1/2}\|^2$ (resp. $\frac{\kappa}{2}\|\nabla P^{m-1/2}\|^2$) on the right-hand side. An analogous simplification (with $\epsilon = a_0$) in the Young's inequality leads to

$$2\Delta t(\alpha\nabla\Theta^{m-1/2} + \beta\nabla P^{m-1/2}, \nabla\bar{\partial}_t U^{m+1/2}) \leq a_0^{-1}(\Delta t)^2\left(\alpha^2\|\nabla\Theta^{m-1/2}\|^2 + \beta^2\|\nabla P^{m-1/2}\|^2\right) + 2a_0\|\nabla\bar{\partial}_t U^{m+1/2}\|^2$$
$$\leq a_0^{-1}(\Delta t)^2\sum_{n=1}^{m-1}\left(\alpha^2\|\nabla\Theta^{n+1/2}\|^2 + \beta^2\|\nabla P^{n+1/2}\|^2\right) + 2a_0\|\nabla\bar{\partial}_t U^{m+1/2}\|^2,$$

where there is an over bound by $a_0^{-1}(\Delta t)^2\sum_{n=1}^{m-2}\left(\alpha^2\|\nabla\Theta^{n+1/2}\|^2 + \beta^2\|\nabla P^{n+1/2}\|^2\right)$ in the last step. A combination of all this yields

$$\Delta t\sum_{n=1}^{m} B^n \leq 2a_0\|\nabla\bar{\partial}_t U^{m+1/2}\|^2 + \frac{\Delta t}{2}\left(c_1\|\nabla\Theta^{1/2}\|^2 + \kappa\|\nabla P^{1/2}\|^2\right) + \frac{\Delta t}{2}\sum_{n=1}^{m-1}\left(c_1\|\nabla\Theta^{n+1/2}\|^2 + \kappa\|\nabla P^{n+1/2}\|^2\right)$$
$$+ \frac{(\Delta t)^2}{a_0}\sum_{n=1}^{m-1}\left(\alpha^2\|\nabla\Theta^{n+1/2}\|^2 + \beta^2\|\nabla P^{n+1/2}\|^2\right) + 2\Delta t\left(\frac{\alpha^2}{c_1} + \frac{\beta^2}{\kappa}\right)\sum_{n=1}^{m-1}\|\nabla\bar{\partial}_t U^{n+1/2}\|^2. \quad (3.17)$$

Step 4 (Bounds for load and source terms). One more application of Cauchy–Schwarz inequality and Young's inequality with $\epsilon = 1/2T$, results in the following bound

$$4\Delta t\sum_{n=1}^{m}(f^{n,1/4}, \bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}) \leq 4T\Delta t\sum_{n=1}^{m}\|f^{n,1/4}\|^2 + \frac{\Delta t}{T}\sum_{n=1}^{m}\|\bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}\|^2.$$

Note that $\Delta t\sum_{n=1}^{m}\|f^{n,1/4}\|^2 \leq m\Delta t\|f\|^2_{L^\infty(0,T;L^2(\Omega))} \leq T\|f\|^2_{L^\infty(0,T;L^2(\Omega))}$. Moreover, $\|\bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}\|^2 \leq 2\|\bar{\partial}_t U^{n+1/2} + \|^2 + 2\|\bar{\partial}_t U^{n-1/2}\|^2$ shows

$$\frac{\Delta t}{T}\sum_{n=1}^{m}\|\bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}\|^2 \leq 2\frac{\Delta t}{T}\|\bar{\partial}_t U^{1/2}\|^2 + 2\frac{\Delta t}{T}\|\bar{\partial}_t U^{m+1/2}\|^2 + 4\frac{\Delta t}{T}\sum_{n=1}^{m-1}\|\bar{\partial}_t U^{n+1/2}\|^2.$$

A combination all this with $\frac{\Delta t}{T} \leq 1$ yields

$$4\Delta t\sum_{n=1}^{m}(f^{n,1/4}, \bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}) \leq 4T^2\|f\|^2_{L^\infty(0,T;L^2(\Omega))} + 2\|\bar{\partial}_t U^{1/2}\|^2$$
$$+ 2\|\bar{\partial}_t U^{m+1/2}\|^2 + 4\frac{\Delta t}{T}\sum_{n=1}^{m}\|\bar{\partial}_t U^{n-1/2}\|^2.$$

Moreover, the same arguments, with $\epsilon = \frac{1}{2T}(a_1 - |\gamma|/\gamma_0)$ (resp. $\epsilon = \frac{1}{2T}(a_2 - |\gamma|\gamma_0)$) used in Young's inequality, also lead to the following bounds

$$\Delta t\sum_{n=1}^{m}(\phi^{n+1/2}, \Theta^{n+1} + \Theta^n) \leq \frac{T^2}{a_1 - |\gamma|/\gamma_0}\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{a_1 - |\gamma|/\gamma_0}{2}\|\Theta^1\|^2$$
$$+ \frac{a_1 - |\gamma|/\gamma_0}{2}\|\Theta^{m+1}\|^2 + (a_1 - |\gamma|/\gamma_0)\frac{\Delta t}{T}\sum_{n=1}^{m}\|\Theta^n\|^2. \quad (3.18a)$$

$$\Bigg(\text{resp. } \Delta t\sum_{n=1}^{m}(g^{n+1/2}, P^{n+1} + P^n) \leq \frac{T^2}{a_2 - |\gamma|\gamma_0}\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{a_1 - |\gamma|\gamma_0}{2}\|P^1\|^2$$

$$+\frac{a_1-|\gamma|\gamma_0}{2}\|P^{m+1}\|^2+(a_1-|\gamma|\gamma_0)\frac{\Delta t}{T}\sum_{n=1}^{m}\|P^n\|^2.\Big) \tag{3.18b}$$

Step 5 (bound for $2\gamma(P^{m+1},\Theta^{m+1})-2\gamma(P^1,\Theta^1)$). A triangle inequality plus Cauchy–Schwarz and Young's inequalities with $\epsilon=1/\gamma_0$ lead to

$$2\gamma(P^{m+1},\Theta^{m+1})-2\gamma(P^1,\Theta^1)\le|\gamma|\gamma_0\|P^{m+1}\|^2+|\gamma|/\gamma_0\|\Theta^{m+1}\|^2+|\gamma|\gamma_0\|P^1\|^2+|\gamma|/\gamma_0\|\Theta^1\|^2.$$

Step 6 (Consolidation). A combination of (3.16)-(3.18) and (3.14) together with elementary manipulations (adding the non-negative term $2\Delta t\Big(\frac{\alpha^2}{c_1}+\frac{\beta^2}{\kappa}\Big)\|\nabla\bar{\partial}_t U^{1/2}\|^2$ on the right-hand side), yields the bound:

$$\begin{aligned}
&2\|\bar{\partial}_t U^{m+1/2}\|^2+2a_0\|\nabla\bar{\partial}_t U^{m+1/2}\|^2+4d_0C_{\text{Coer}}\|U^{m+1/2}\|_h^2+\frac{1}{2}(a_1-|\gamma|/\gamma_0)\|\Theta^{m+1}\|^2+\frac{1}{2}(a_2-|\gamma|\gamma_0)\|P^{m+1}\|^2\\
&\quad+2\|(\Theta^m,P^m)\|_H^2-\Delta t\Big(c_1\|\nabla\Theta^{m+1/2}\|^2+\kappa\|\nabla P^{m+1/2}\|^2\Big)-\frac{3\Delta t}{2}\sum_{n=1}^{m-1}\Big(c_1\|\nabla\Theta^{n+1/2}\|^2+\kappa\|\nabla P^{n+1/2}\|^2\Big)\\
&\le((U^0,U^1,f))+((\Theta^0,\Theta^1,\phi))+((P^0,P^1,g))\\
&\quad+2\frac{\Delta t}{T}\Big(\sum_{n=1}^{m}2\|\bar{\partial}_t U^{n-1/2}\|^2+\frac{1}{2}(a_1-|\gamma|/\gamma_0)\sum_{n=1}^{m}\|\Theta^n\|^2+\frac{1}{2}(a_2-|\gamma|\gamma_0)\sum_{n=1}^{m}\|P^n\|^2\Big)\\
&\quad+6\Delta t\Big(\frac{\alpha^2}{c_1}+\frac{\beta^2}{\kappa}\Big)\sum_{n=0}^{m-1}\|\nabla\bar{\partial}_t U^{n+1/2}\|^2+\frac{(\Delta t)^2}{a_0}\sum_{n=1}^{m-1}\Big(\alpha^2\|\nabla\Theta^{n+1/2}\|^2+\beta^2\|\nabla P^{n+1/2}\|^2\Big).
\end{aligned}\tag{3.19}$$

Utilize the definition (3.10) to obtain

$$\begin{aligned}
&2\|(\Theta^m,P^m)\|_H^2-\Delta t\Big(c_1\|\nabla\Theta^{m+1/2}\|^2+\kappa\|\nabla P^{m+1/2}\|^2\Big)-\frac{3\Delta t}{2}\sum_{n=1}^{m-1}\Big(c_1\|\nabla\Theta^{n+1/2}\|^2+\kappa\|\nabla P^{n+1/2}\|^2\Big)\\
&=\frac{\Delta t}{2}\Big(c_1\|\nabla\Theta^{m+1/2}\|^2+\kappa\|\nabla P^{m+1/2}\|^2+\sum_{n=1}^{m}\big(4b_1\|\Theta^{n+1/2}\|^2+c_1\|\nabla\Theta^{n+1/2}\|^2+\kappa\|\nabla P^{n+1/2}\|^2\big)\Big]\\
&\ge\frac{\Delta t}{2}\Big(c_1\|\nabla\Theta^{m+1/2}\|^2+\kappa\|\nabla P^{m+1/2}\|^2\Big)+\frac{1}{2}\|(\Theta^m,P^m)\|_H^2,
\end{aligned}$$

and the elementary manipulations

$$6\Delta t\Big(\frac{\alpha^2}{c_1}+\frac{\beta^2}{\kappa}\Big)\sum_{n=0}^{m-1}\|\nabla\bar{\partial}_t U^{n+1/2}\|^2=\frac{\Delta t}{T}\Big(\frac{3T\alpha^2}{a_0c_1}+\frac{3T\beta^2}{a_0\kappa}\Big)\sum_{n=0}^{m-1}2a_0\|\nabla\bar{\partial}_t U^{n+1/2}\|^2,$$

$$\frac{(\Delta t)^2}{a_0}\sum_{n=1}^{m-1}\Big(\alpha^2\|\nabla\Theta^{n+1/2}\|^2+\beta^2\|\nabla P^{n+1/2}\|^2\Big)\le\frac{\Delta t}{T}\Big(\frac{2T\alpha^2}{a_0c_1}+\frac{2T\beta^2}{a_0\kappa}\Big)\frac{\Delta t}{2}\sum_{n=1}^{m-1}\Big(c_1\|\nabla\Theta^{n+1/2}\|^2+\kappa\|\nabla P^{n+1/2}\|^2\Big)$$

in (3.19) to show that

$$\begin{aligned}
&2\|\bar{\partial}_t U^{m+1/2}\|^2+2a_0\|\nabla\bar{\partial}_t U^{m+1/2}\|^2+\frac{1}{2}(a_1-|\gamma|/\gamma_0)\|\Theta^{m+1}\|^2+\frac{1}{2}(a_2-|\gamma|\gamma_0)\|P^{m+1}\|^2\\
&\qquad+\frac{\Delta t}{2}\Big(c_1\|\nabla\Theta^{m+1/2}\|^2+\kappa\|\nabla P^{m+1/2}\|^2\Big)+\frac{1}{2}\|(\Theta^m,P^m)\|_H^2+4d_0C_{\text{Coer}}\|U^{m+1/2}\|_h^2\\
&\le((U^0,U^1,f))+((\Theta^0,\Theta^1,\phi))+((P^0,P^1,g))+C\frac{\Delta t}{T}\sum_{n=0}^{m-1}\Big[2\|\bar{\partial}_t U^{n+1/2}\|^2+2a_0\|\bar{\partial}_t\nabla U^{n+1/2}\|^2\\
&\qquad+\frac{1}{2}(a_1-|\gamma|/\gamma_0)\|\Theta^{n+1}\|^2+\frac{1}{2}(a_2-|\gamma|\gamma_0)\|P^{n+1}\|^2+\frac{\Delta t}{2}\Big(c_1\|\nabla\Theta^{n+1/2}\|^2+\kappa\|\nabla P^{n+1/2}\|^2\Big)\Big],
\end{aligned}$$

where $C=\max\{2,\frac{3T\alpha^2}{a_0c_1}+\frac{3T\beta^2}{a_0\kappa}\}$. Then we invoke Lemma 1.1 and (3.10) to arrive at

$$2\|\bar{\partial}_t U^{m+1/2}\|^2+2a_0\|\nabla\bar{\partial}_t U^{m+1/2}\|^2+\frac{1}{2}(a_1-|\gamma|/\gamma_0)\|\Theta^{m+1}\|^2+\frac{1}{2}(a_2-|\gamma|\gamma_0)\|P^{m+1}\|^2$$

$$+\frac{\Delta t}{2}\Big(c_1\|\nabla\Theta^{m+1/2}\|^2+\kappa\|\nabla P^{m+1/2}\|^2\Big)+\frac{1}{2}\|(\Theta^m,P^m)\|_H^2+4d_0C_{\mathrm{Coer}}\|U^{m+1/2}\|_h^2$$
$$\le e^{m\frac{C\Delta t}{T}}\Big(((U^0,U^1,f))+((\Theta^0,\Theta^1,\phi))+((P^0,P^1,g))\Big)\le e^C\Big(((U^0,U^1,f))+((\Theta^0,\Theta^1,\phi))+((P^0,P^1,g))\Big),$$

with $m\Delta t\le T$ in the last step. Then, one can ignore the non-negative term $\frac{1}{2}\Delta t[c_1\|\nabla\Theta^{m+1/2}\|^2+\kappa\|\nabla P^{m+1/2}\|^2]$ on the left-hand side to conclude the proof. □

Remark 3.7. *For existence of unique solution, it suffices to show that* $(0,0,0)$ *is the only solution of the fully discrete problem* (3.6)-(3.8) *with homogeneous initial conditions and load/source functions. From* (3.6)*, it is evident that if* $u^0=\theta^0=p^0=0$*, then* $U^0=\Theta^0=P^0=0$*, which, together with* $u^{*0}=0$*, leads to* $U^1=\Theta^1=P^1=0$ *from* (3.7)*. Then, we can utilize* $U^0=\Theta^0=P^0=U^1=\Theta^1=P^1=0$ *in Theorem 3.6 to show that* $U^{m+1}=\Theta^{m+1}=P^{m+1}=0$ *for all* $1\le m\le N-1$*.*

4 Error Analysis

This section establishes the error estimates for the fully discrete scheme presented in the previous section. In Subsection 4.1, we prove the error estimates at the initial time steps t_0 and t_1 for the scheme (3.6)-(3.7). The subsequent subsection provides error estimates for the scheme (3.8) in different norms. Let us consider the following decomposition of errors at time t_n, for $n=1,\dots,N$

$$u(t_n)-U^n=\big(u(t_n)-\mathcal{R}_hu(t_n)\big)+\big(\mathcal{R}_hu(t_n)-U^n\big):=\rho^n+\zeta^n, \tag{4.1a}$$
$$\theta(t_n)-\Theta^n=\big(\theta(t_n)-\Pi_h\theta(t_n)\big)+\big(\Pi_h\theta(t_n)-\Theta^n\big):=\eta^n+\Psi^n, \tag{4.1b}$$
$$p(t_n)-P^n=\big(p(t_n)-\Pi_hp(t_n)\big)+\big(\Pi_hp(t_n)-P^n\big):=\varrho^n+\xi^n, \tag{4.1c}$$

where $\mathcal{R}_h$ and Π_h are the projections defined in (3.2) and (3.4), respectively.

4.1 Initial error bounds

Since our discrete formulation is split into two parts, solutions at t_0 and t_1 are determined using (3.6)-(3.7), whereas the solutions at $t_2,t_3,\dots,t_n$ are computed using (3.8)—it is thus necessary to estimate the initial error at time levels t_0 and t_1 before we proceed to derive the error estimates. To do so, we take the average of the equations in system (2.6) at t_0 and t_1 as

$$(u_{tt}^{1/2},v)+a_0(\nabla u_{tt}^{1/2},\nabla v)+d_0(\nabla^2u^{1/2},\nabla^2v)-\alpha(\nabla\theta^{1/2},\nabla v)-\beta(\nabla p^{1/2},\nabla v)=(f^{1/2},v),$$
$$a_1(\theta_t^{1/2},\psi)-\gamma(p_t^{1/2},\psi)+b_1(\theta^{1/2},\psi)+c_1(\nabla\theta^{1/2},\nabla\psi)+\alpha(\nabla u_t^{1/2},\nabla\psi)=(\phi^{1/2},\psi),$$
$$a_2(p_t^{1/2},q)-\gamma(\theta_t^{1/2},q)+\kappa(\nabla p^{1/2},\nabla q)+\beta(\nabla u_t^{1/2},\nabla q)=(g^{1/2},q),$$

for all $v\in H_0^2(\Omega)$ and both $\psi,q\in H_0^1(\Omega)$. Let us observe that for the smoother Q defined in Section 3 there holds Range $(Q)\subset H_0^2(\Omega)$, and, readily from the definitions, we have that $W_h\subset H_0^1(\Omega)$. Then, for any $v_h\in V_h$ and $\psi_h,q_h\in W_h$, we can choose $Qv_h\in H_0^2(\Omega)$ and $\psi_h,q_h\in W_h\subset H_0^1(\Omega)$ as test functions in the last system of equations and employ the definitions of the projections $\mathcal{R}_h$ and Π_h from (3.2) and (3.4), respectively to arrive at

$$(u_{tt}^{1/2},Qv_h)+a_0(\nabla u_{tt}^{1/2},\nabla Qv_h)+d_0a_h(\mathcal{R}_hu^{1/2},v_h)-\alpha(\nabla\theta^{1/2},\nabla Qv_h)$$
$$+\beta(\nabla p^{1/2},\nabla Qv_h)=(f^{1/2},Qv_h), \tag{4.2a}$$
$$a_1(\theta_t^{1/2},\psi_h)-\gamma(p_t^{1/2},\psi_h)+b_1(\theta^{1/2},\psi_h)+c_1(\nabla\Pi_h\theta^{1/2},\nabla\psi_h)+\alpha(\nabla u_t^{1/2},\nabla\psi_h)=(\phi^{1/2},\psi_h), \tag{4.2b}$$
$$a_2(p_t^{1/2},q_h)-\gamma(\theta_t^{1/2},q_h)+\kappa(\nabla\Pi_hp^{1/2},\nabla q_h)+\beta(\nabla u_t^{1/2},\nabla q_h)=(g^{1/2},q_h). \tag{4.2c}$$

Since all the terms $u_{tt}^{1/2},\theta^{1/2},p^{1/2},$ and $(Q-I)v_h$ belong to $H_0^1(\Omega)$, an integration by parts leads to

$$(a_0\nabla u_{tt}^{1/2}-\beta\nabla p^{1/2}-\alpha\nabla\theta^{1/2},\nabla(Q-I)v_h)=(-a_0\Delta u_{tt}^{1/2}+\beta\Delta p^{1/2}+\alpha\Delta\theta^{1/2},(Q-I)v_h). \tag{4.3}$$

Next we recall $F := F(t, \boldsymbol{x}) = f(t, \boldsymbol{x}) - u_{tt} + a_0 \Delta u_{tt} - \alpha \Delta \theta - \beta \Delta p$ from (2.15). This and (4.3) in (4.2a) with some basic manipulations yields

$$\begin{aligned}
&2(\Delta t)^{-1}(\bar{\partial}_t u^{1/2}, v_h) + 2a_0(\Delta t)^{-1}(\nabla \bar{\partial}_t u^{1/2}, \nabla v_h) + d_0 a_h(\mathcal{R}_h u^{1/2}, v_h) - \alpha(\nabla \theta^{1/2}, \nabla v_h) - \beta(\nabla p^{1/2}, \nabla v_h) \\
&= (F^{1/2}, (Q - I)v_h) + 2(\Delta t)^{-1}(\bar{\partial}_t u^{1/2}, v_h) - (u_{tt}^{1/2}, v_h) + 2a_0(\Delta t)^{-1}(\nabla \bar{\partial}_t u^{1/2}, \nabla v_h) - a_0(\nabla u_{tt}^{1/2}, \nabla v_h). \quad (4.4)
\end{aligned}$$

Let us now define the initial truncation terms R^0, r_0, τ_0, and s_0 as follows:

$$R^0 := 2(\Delta t)^{-1}(\bar{\partial}_t u^{1/2} - u^{*0}) - u_{tt}^{1/2}, \quad r_0 := \bar{\partial}_t u^{1/2} - u_t^{1/2}, \tau_0 := \bar{\partial}_t \theta^{1/2} - \theta_t^{1/2}, \quad s_0 := \bar{\partial}_t p^{1/2} - p_t^{1/2}. \quad (4.5)$$

Utilizing these definitions and subtracting the equation (3.7a) form (4.4), (3.7b) from (4.2b), and (3.7c) from (4.2c), we can obtain the following system

$$\begin{aligned}
&2(\Delta t)^{-1}(\bar{\partial}_t(u^{1/2} - U^{1/2}), v_h) + 2(\Delta t)^{-1} a_0(\nabla \bar{\partial}_t(u^{1/2} - U^{1/2}), \nabla v_h) + d_0 a_h(\mathcal{R}_h u^{1/2} - U^{1/2}, v_h) \\
&\quad - \alpha(\nabla(\theta^{1/2} - \Theta^{1/2}), \nabla v_h) - \beta(\nabla(p^{1/2} - P^{1/2}), \nabla v_h) = (F^{1/2}, (Q - I)v_h) + (R_0, v_h) + a_0(\nabla R_0, \nabla v_h), \\
&a_1(\bar{\partial}_t(\theta^{1/2} - \Theta^{1/2}), \psi_h) - \gamma(\bar{\partial}_t(p^{1/2} - P^{1/2}), \psi_h) + b_1(\theta^{1/2} - \Theta^{1/2}, \psi_h) + c_1(\nabla(\Pi_h \theta^{1/2} - \Theta^{1/2}), \nabla \psi_h) \\
&\quad + \alpha(\nabla \bar{\partial}_t(u^{1/2} - U^{1/2}), \nabla \psi_h) = a_1(\tau_0, \psi_h) - \gamma(s_0, \psi_h) + \alpha(\nabla r_0, \nabla \psi_h), \\
&a_2(\bar{\partial}_t(p^{1/2} - P^{1/2}), q_h) - \gamma(\bar{\partial}_t(\theta^{1/2} - \Theta^{1/2}), q_h) + \kappa(\nabla(\Pi_h p^{1/2} - P^{1/2}), \nabla q_h) \\
&\quad + \beta(\nabla \bar{\partial}_t(u^{1/2} - U^{1/2}), \nabla q_h) = a_2(s_0, q_h) - \gamma(\tau_0, p_h) + \beta(\nabla r_0, \nabla q_h).
\end{aligned}$$

In turn, the error decomposition described in (4.1) leads to

$$\begin{aligned}
&2(\Delta t)^{-1}(\bar{\partial}_t \zeta^{1/2}, v_h) + 2(\Delta t)^{-1} a_0(\nabla \bar{\partial}_t \zeta^{1/2}, \nabla v_h) + d_0 a_h(\zeta^{1/2}, v_h) - \alpha(\nabla \Psi^{1/2}, \nabla v_h) - \beta(\nabla \xi^{1/2}, \nabla v_h) \\
&\quad = -2(\Delta t)^{-1}(\bar{\partial}_t \rho^{1/2}, v_h) - 2(\Delta t)^{-1} a_0(\nabla \bar{\partial}_t \rho^{1/2}, \nabla v_h) + \alpha(\nabla \eta^{1/2}, \nabla v_h) + \beta(\nabla \varrho^{1/2}, \nabla v_h) \\
&\qquad + (F^{1/2}, (Q - I)v_h) + (R_0, v_h) + a_0(\nabla R_0, \nabla v_h) \qquad \text{for all } v_h \in V_h, \qquad (4.6a) \\
&a_1(\bar{\partial}_t \Psi^{1/2}, \psi_h) - \gamma(\bar{\partial}_t \xi^{1/2}, \psi_h) + b_1(\Psi^{1/2}, \psi_h) + c_1(\nabla \Psi^{1/2}, \nabla \psi_h) + \alpha(\nabla \bar{\partial}_t \zeta^{1/2}, \nabla \psi_h) \\
&\quad = -a_1(\bar{\partial}_t \eta^{1/2}, \psi_h) + \gamma(\bar{\partial}_t \varrho^{1/2}, \psi_h) - b_1(\eta^{1/2}, \psi_h) - \alpha(\nabla \bar{\partial}_t \rho^{1/2}, \nabla \psi_h) \\
&\qquad + a_1(\tau_0, \psi_h) - \gamma(s_0, \psi_h) + \alpha(\nabla r_0, \nabla \psi_h) \qquad \text{for all } \psi_h \in W_h, \qquad (4.6b) \\
&a_2(\bar{\partial}_t \xi^{1/2}, q_h) - \gamma(\bar{\partial}_t \Psi^{1/2}, q_h) + \kappa(\nabla \xi^{1/2}, \nabla q_h) + \beta(\nabla \bar{\partial}_t \zeta^{1/2}, \nabla q_h) = -a_2(\bar{\partial}_t \varrho^{1/2}, q_h) \\
&\quad + \gamma(\bar{\partial}_t \eta^{1/2}, q_h) - \beta(\nabla \bar{\partial}_t \rho^{1/2}, \nabla q_h) + a_2(s_0, q_h) - \gamma(\tau_0, q_h) + \beta(\nabla r_0, \nabla q_h) \quad \text{for all } q_h \in W_h. \qquad (4.6c)
\end{aligned}$$

Note that $\|\bar{\partial}_t \rho^{1/2}\| = (1/\Delta t)\|\int_0^{t_1} \rho_t(t)\, \mathrm{d}s\|$ and $\|\nabla \bar{\partial}_t \rho^{1/2}\| = (1/\Delta t)\|\int_0^{t_1} \nabla \rho_t(t)\, \mathrm{d}s\|$. Then, definition (4.1a) and the approximation property in (3.3) yield the bounds

$$\|\bar{\partial}_t \rho^{1/2}\| + \|\nabla \bar{\partial}_t \rho^{1/2}\| \le C_2 h^{2\sigma} \|u_t\|_{L^\infty(0,t_1;H^{2+\sigma}(\Omega))}, \qquad (4.7a)$$

$$\sqrt{\Delta t}\big(\|\bar{\partial}_t \rho^{1/2}\| + \|\nabla \bar{\partial}_t \rho^{1/2}\|\big) \le C_2 h^{2\sigma} \|u_t\|_{L^2(0,t_1;H^{2+\sigma}(\Omega))}. \qquad (4.7b)$$

Further, the definitions (4.1b)–(4.1c) and the approximation property in (3.5) lead to

$$\|\eta^1 - \eta^0\| + \|\eta^{1/2}\| + h^\sigma \|\nabla \eta^{1/2}\| \le 3C_3 h^{2\sigma} \|\theta\|_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))}, \qquad (4.8a)$$

and $$\|\varrho^1 - \varrho^0\| + \|\varrho^{1/2}\| + h^\sigma \|\nabla \varrho^{1/2}\| \le 3C_3 h^{2\sigma} \|p\|_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))}. \qquad (4.8b)$$

The following lemma, whose proof involves Taylor series expansion and the Cauchy–Schwarz inequality, provides truncation error estimates that will be utilized later in this section.

Lemma 4.1 (Truncation error bounds [30,33]). *For $\varphi \in H^4(0, T; L^2(\Omega))$, the following inequalities hold*

$$(a) \quad \|2\Delta t^{-1}(\bar{\partial}_t \varphi^{1/2} - \varphi_t(0)) - \varphi_{tt}^{1/2}\| \le \Delta t \|\varphi_{ttt}\|_{L^\infty(0,t_1;L^2(\Omega))}, \qquad (4.9a)$$

$$(b) \quad \|\bar{\partial}_t \varphi^{n+1/2} - \varphi_t^{n+1/2}\| \lesssim (\Delta t)^{3/2} \|\varphi_{ttt}\|_{L^2(t_n,t_{n+1};L^2(\Omega))} \text{ for } n = 0, 1, \dots, N-1, \qquad (4.9b)$$

$$(c) \quad \sum_{n=1}^{N-1} \|\bar{\partial}_t^2 \varphi^n - \varphi_{tt}^{n,1/4}\|^2 \lesssim (\Delta t)^3 \|\varphi_{tttt}\|^2_{L^2(0,T;L^2(\Omega))}. \qquad (4.9c)$$

Next, we present the initial error estimates for (3.7), which will be used to prove the next theorem. Before proceeding we define the following quantities, all bounded thanks to Theorems 2.2 and 2.3

$$
\begin{aligned}
L_{(u,\theta,p,t_1)} &:= \|u_t\|^2_{L^2(0,t_1;H^{2+\sigma}(\Omega))} + \|u_t\|^2_{L^\infty(0,t_1;H^{2+\sigma}(\Omega))} + \|\theta\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))} + \|p\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))},\\
M_{(u,\theta,p,t_1)} &:= \|u_{ttt}\|^2_{L^\infty(0,t_1;H^1(\Omega))} + \|\theta_{ttt}\|^2_{L^2(0,t_1;L^2(\Omega))} + \|p_{ttt}\|^2_{L^2(0,t_1;L^2(\Omega))}.
\end{aligned}
$$

Lemma 4.2 (Initial error bounds). *Under the regularity assumptions on given data as stated in Theorems 2.1-2.3, the following estimates are satisfied:*

$$
\begin{aligned}
&\|\bar{\partial}_t\zeta^{1/2}\|^2 + a_0\|\nabla\bar{\partial}_t\zeta^{1/2}\|^2 + d_0C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2 + (a_1 - |\gamma|/\gamma_0)\|\Psi^{1/2}\|^2 + (a_2 - |\gamma|\gamma_0)\|\xi^{1/2}\|^2\\
&\quad + \Delta t\big[b_1\|\Psi^{1/2}\|^2 + c_1\|\nabla\Psi^{1/2}\|^2 + \kappa\|\nabla\xi^{1/2}\|^2\big] \lesssim h^{4\sigma} + L_{(u,\theta,p,t_1)}h^{4\sigma} + M_{(u,\theta,p,t_1)}(\Delta t)^4,
\end{aligned}
$$

where the absorbed constant in "$\lesssim$" depends on C_F, C_{Coer}, C_{Cont}, C_1,C_2, C_3, T, and the model coefficients.

Proof. The proof follows in five steps below.
Step 1 (Key inequality). From the choice of test function $v_h = \zeta^{1/2} \in V_h$ in (4.6a) and the identity $\zeta^{1/2} = \frac{\zeta^1+\zeta^0}{2} = \frac{\zeta^1-\zeta^0}{2} = \frac{\Delta t}{2}\bar{\partial}_t\zeta^{1/2}$ that follows from $\zeta^0 = 0$ (see (3.6)), we obtain

$$
\begin{aligned}
&\|\bar{\partial}_t\zeta^{1/2}\|^2 + a_0\|\nabla\bar{\partial}_t\zeta^{1/2}\|^2 + d_0a_h(\zeta^{1/2},\zeta^{1/2}) - (\nabla(\alpha\Psi^{1/2} + \beta\xi^{1/2}), \nabla\zeta^{1/2})\\
&\quad = -(\bar{\partial}_t\rho^{1/2}, \bar{\partial}_t\zeta^{1/2}) - a_0(\nabla\bar{\partial}_t\rho^{1/2}, \nabla\bar{\partial}_t\zeta^{1/2}) + (\nabla(\alpha\eta^{1/2} + \beta\varrho^{1/2}), \nabla\zeta^{1/2})\\
&\quad\quad + (F^{1/2}, (Q-I)\zeta^{1/2}) + \frac{1}{2}\Delta t(R_0, \bar{\partial}_t\zeta^{1/2}) + \frac{a_0}{2}\Delta t(\nabla R_0, \nabla\bar{\partial}_t\zeta^{1/2}).
\end{aligned}
\tag{4.10}
$$

We can then proceed to multiply (4.6b) by $\Delta t/2$ and choose $\psi_h = \Psi^{1/2} \in W_h$ as test function, and utilize

$$
\zeta^{1/2} = \frac{\Delta t}{2}\bar{\partial}_t\zeta^{1/2}, \quad \Psi^{1/2} = \frac{\Delta t}{2}\bar{\partial}_t\Psi^{1/2}, \quad \xi^{1/2} = \frac{\Delta t}{2}\bar{\partial}_t\xi^{1/2},
$$

(from (3.6)) to get

$$
\begin{aligned}
&a_1\|\Psi^{1/2}\|^2 - \gamma(\xi^{1/2}, \Psi^{1/2}) + \frac{b_1}{2}\Delta t\|\Psi^{1/2}\|^2 + \frac{c_1}{2}\Delta t\|\nabla\Psi^{1/2}\|^2 + \alpha(\nabla\zeta^{1/2}, \nabla\Psi^{1/2})\\
&= -\frac{a_1}{2}(\eta^1 - \eta^0, \Psi^{1/2}) + \frac{\gamma}{2}(\varrho^1 - \varrho^0, \Psi^{1/2}) - \frac{b_1}{2}\Delta t(\eta^{1/2}, \Psi^{1/2}) - \frac{\alpha}{2}\Delta t(\nabla\bar{\partial}_t\rho^{1/2}, \nabla\Psi^{1/2})\\
&\quad + \frac{a_1}{2}\Delta t(\tau_0, \Psi^{1/2}) - \frac{\gamma}{2}\Delta t(s_0, \Psi^{1/2}) + \frac{\alpha}{2}\Delta t(\nabla r_0, \nabla\Psi^{1/2}).
\end{aligned}
\tag{4.11}
$$

Similarly, we multiply (4.6c) by $\Delta t/2$, choose $q_h = \xi^{1/2} \in W_h$, and use $\Psi^{1/2} = \frac{\Delta t}{2}\bar{\partial}_t\Psi^{1/2}$, $\xi^{1/2} = \frac{\Delta t}{2}\bar{\partial}_t\xi^{1/2}$ to arrive at

$$
\begin{aligned}
&a_2\|\xi^{1/2}\|^2 - \gamma(\Psi^{1/2}, \xi^{1/2}) + \frac{\kappa}{2}\Delta t\|\nabla\xi^{1/2}\|^2 + \beta(\nabla\zeta^{1/2}, \nabla\xi^{1/2}) = -\frac{a_2}{2}(\varrho^1 - \varrho^0, \xi^{1/2})\\
&\quad + \frac{\gamma}{2}(\eta^1 - \eta^0, \xi^{1/2}) - \frac{\beta}{2}\Delta t(\nabla\bar{\partial}_t\rho^{1/2}, \nabla\xi^{1/2}) + \frac{\Delta t}{2}(a_2s_0 - \gamma\tau_0, \xi^{1/2}) + \frac{\beta}{2}\Delta t(\nabla r_0, \nabla\xi^{1/2}).
\end{aligned}
\tag{4.12}
$$

A summation of (4.10)–(4.12) leads to the cancellation of the term $(\nabla(\alpha\Psi^{1/2}+\beta\xi^{1/2}), \nabla\zeta^{1/2})$. This, the coercivity of $a_h(\cdot,\cdot)$ from (3.1), and an appropriate regrouping of the terms lead to

$$
\begin{aligned}
&\|\bar{\partial}_t\zeta^{1/2}\|^2 + a_0\|\nabla\bar{\partial}_t\zeta^{1/2}\|^2 + d_0C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2 + a_1\|\Psi^{1/2}\|^2 + a_2\|\xi^{1/2}\|^2\\
&\quad + \frac{\Delta t}{2}\big[b_1\|\Psi^{1/2}\|^2 + c_1\|\nabla\Psi^{1/2}\|^2 + \kappa\|\nabla\xi^{1/2}\|^2\big] \le (F^{1/2}, (Q-I)\zeta^{1/2}) + (\nabla(\alpha\eta^{1/2} + \beta\varrho^{1/2}), \nabla\zeta^{1/2})\\
&+ \big[-(\bar{\partial}_t\rho^{1/2}, \bar{\partial}_t\zeta^{1/2}) - a_0(\nabla(\bar{\partial}_t\rho^{1/2}, \nabla\bar{\partial}_t\zeta^{1/2}) + \frac{1}{2}\Delta t(R_0, \bar{\partial}_t\zeta^{1/2}) + \frac{1}{2}\Delta t(R_0, \nabla\bar{\partial}_t\zeta^{1/2})\big]\\
&+ \frac{\Delta t}{2}\big[-\alpha(\nabla\bar{\partial}_t\rho^{1/2}, \nabla\Psi^{1/2}) - \beta(\nabla\bar{\partial}_t\rho^{1/2}, \nabla\xi^{1/2}) + \alpha(\nabla r_0, \nabla\Psi^{1/2}) + \beta(\nabla r_0, \nabla\xi^{1/2})\big]\\
&+ \frac{1}{2}\big[-a_1(\eta^1 - \eta^0, \Psi^{1/2}) + a_1\Delta t(\tau_0, \Psi^{1/2}) + \gamma(\varrho^1 - \varrho^0, \Psi^{1/2}) - \gamma\Delta t(s_0, \Psi^{1/2}) - b_1\Delta t(\eta^{1/2}, \Psi^{1/2})\big]
\end{aligned}
$$

$$+\frac{1}{2}\big[-a_2(\varrho^1-\varrho^0,\xi^{1/2})+\gamma(\eta^1-\eta^0,\xi^{1/2})+a_2\Delta t(s_0,\xi^{1/2})-\gamma\Delta t(\tau_0,\xi^{1/2})\big]+2\gamma(\Psi^{1/2},\xi^{1/2})$$
$$:=T_1+T_2+T_3+T_4+T_5+T_6+T_7. \tag{4.13}$$

Step 2 (Bound for T_1). An application of Cauchy–Schwarz inequality and the bounds from Lemma 3.1(v) yields

$$T_1:=(F^{1/2},(Q-I)\zeta^{1/2})\le\|F^{1/2}\|\|(Q-I)\zeta^{1/2}\|\le C_1h^2\|F^{1/2}\|\|\zeta^{1/2}\|_h.$$

Then we can utilize Young's inequality with $\epsilon=2(d_0C_{\mathrm{Coer}})^{-1}$ to show that

$$T_1\le C_1^2(d_0C_{\mathrm{Coer}})^{-1}h^4\|F^{1/2}\|^2+\frac{d_0}{4}C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2\le C_1^2C_F^2(d_0C_{\mathrm{Coer}})^{-1}h^4+\frac{d_0}{4}C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2,$$

with the bound $\|F^{1/2}\|\le C_F$ from the regularity result (2.16) in the last step.

Step 3 (Bound for T_2). Note that $\eta^{1/2},\varrho^{1/2}\in H_0^1(\Omega)$, and $Q\zeta^{1/2}\in H_0^2(\Omega)$. Some elementary manipulations and an integration by parts show

$$\begin{aligned}T_2&:=\alpha(\nabla\eta^{1/2},\nabla\zeta^{1/2})+\beta(\nabla\varrho^{1/2},\nabla\zeta^{1/2})\\&=\alpha(\nabla\eta^{1/2},\nabla(I-Q)\zeta^{1/2})+\beta(\nabla\varrho^{1/2},\nabla(I-Q)\zeta^{1/2})-\alpha(\eta^{1/2},\Delta(Q\zeta^{1/2}))-\beta(\varrho^{1/2},\Delta(Q\zeta^{1/2})).\end{aligned} \tag{4.14}$$

Using Cauchy–Schwarz's inequality, $\|\nabla(I-Q)\zeta^{1/2}\|\le C_1h\|\zeta^{1/2}\|_h$ from Lemma 3.1(v) (with $s=1$ and $v=0$), and Young's inequality (applied twice with $\epsilon=8(d_0C_{\mathrm{Coer}})^{-1}$), we can readily bound the first two terms on the right-hand side of (4.14) as

$$\begin{aligned}&\alpha(\nabla\eta^{1/2},\nabla(I-Q)\zeta^{1/2})+\beta(\nabla\varrho^{1/2},\nabla(I-Q)\zeta^{1/2})\\&\quad\le C_1h\big(\alpha\|\nabla\eta^{1/2}\|\|\zeta^{1/2}\|_h+\beta\|\nabla\varrho^{1/2}\|\|\zeta^{1/2}\|_h\big)\\&\quad\le 4C_1^2h^2(d_0C_{\mathrm{Coer}})^{-1}\big(\alpha^2\|\nabla\eta^{1/2}\|^2+\beta^2\|\nabla\varrho^{1/2}\|^2\big)+\frac{d_0}{8}C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2\\&\quad\le Ch^{2+2\sigma}\big(\|\theta\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))}+\|p\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))}\big)+\frac{d_0}{8}C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2,\end{aligned} \tag{4.15}$$

where we have utilized (4.8) in the last inequality.

Note that $\|\Delta(Q\zeta^{1/2})\|\le\|Q\zeta^{1/2}\|_h$ and a triangle inequality with Lemma 3.1(v) shows $\|\Delta(Q\zeta^{1/2})\|\le\Lambda\|\zeta^{1/2}\|_h$ for $\Lambda>0$. Therefore, combining this with the Cauchy–Schwarz and Young's inequality (as in the last step) lead to

$$\begin{aligned}-\alpha(\eta^{1/2},\Delta(Q\zeta^{1/2}))-\beta(\varrho^{1/2},\Delta(Q\zeta^{1/2}))&\le\alpha\Lambda\|\eta^{1/2}\|\|\zeta^{1/2}\|_h+\beta\Lambda\|\varrho^{1/2}\|\|\zeta^{1/2}\|_h\\&\le 4\Lambda^2(d_0C_{\mathrm{Coer}})^{-1}\big(\alpha^2\|\eta^{1/2}\|^2+\beta^2\|\varrho^{1/2}\|^2\big)+\frac{d_0}{8}C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2\\&\le Ch^{4\sigma}\big(\|\theta\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))}+\|p\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))}\big)+\frac{d_0}{8}C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2,\end{aligned} \tag{4.16}$$

with estimates in the last step from (4.8). In addition, a combination of (4.15)-(4.16) in (4.14) yields

$$T_2\le C(h^{2+2\sigma}+h^{4\sigma})\big(\|\theta\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))}+\|p\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))}\big)+\frac{d_0}{4}C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2. \tag{4.17}$$

Step 4 (Bounds for T_3–T_7). The estimates for T_3–T_7 follow a similar approach. We apply the Cauchy–Schwarz and Young's inequalities (with $\epsilon=2,2,1,1$ for the four terms, respectively, in T_3) to reveal the bound

$$\begin{aligned}T_3&:=-(\bar\partial_t\rho^{1/2},\bar\partial_t\zeta^{1/2})-a_0(\nabla\bar\partial_t\rho^{1/2},\nabla\bar\partial_t\zeta^{1/2})+\frac{1}{2}\Delta t(R_0,\bar\partial_t\zeta^{1/2})+\frac{a_0}{2}\Delta t(\nabla R_0,\nabla\bar\partial_t\zeta^{1/2})\\&\le C_{T_3}\big(\|\bar\partial_t\rho^{1/2}\|^2+\|\nabla\bar\partial_t\rho^{1/2}\|^2+(\Delta t)^2\|R_0\|^2+(\Delta t)^2\|\nabla R_0\|^2\big)+\frac{1}{2}\|\bar\partial_t\zeta^{1/2}\|^2+\frac{a_0}{2}\|\nabla\bar\partial_t\zeta^{1/2}\|^2,\end{aligned}$$

where $C_{T_3}=\max\{1,a_0\}$. We then employ (4.7a) to bound the first two terms and (4.5) and Lemma 4.1 to bound the third and fourth terms on the right-hand side above to obtain

$$T_3\le C\big(h^{4\sigma}\|u_t\|^2_{L^\infty(0,t_1;H^{2+\sigma}(\Omega))}+(\Delta t)^4\|u_{ttt}\|^2_{L^\infty(0,t_1;H^1(\Omega))}\big)+\frac{1}{2}\|\bar\partial_t\zeta^{1/2}\|^2+\frac{a_0}{2}\|\nabla\bar\partial_t\zeta^{1/2}\|^2.$$

(Here C denotes a generic constant independent of the discretization parameters). Now, similar arguments (with $\epsilon = 2/c_1, 2/\kappa, 2/c_1, 2/\kappa$ in the Young's inequalities for the terms in T_4) lead to

$$\begin{aligned}T_4 &:= \frac{\Delta t}{2}\big[-\alpha(\nabla\bar{\partial}_t\rho^{1/2}, \nabla\Psi^{1/2}) - \beta(\nabla\bar{\partial}_t\rho^{1/2}, \nabla\xi^{1/2}) + \alpha(\nabla r_0, \nabla\Psi^{1/2}) + \beta(\nabla r_0, \nabla\xi^{1/2})\big] \\ &\leq C_{T_4}\Big(\Delta t\|\nabla\bar{\partial}_t\rho^{1/2}\|^2 + \Delta t\|\nabla r_0\|^2\Big) + \frac{c_1}{4}\Delta t\|\nabla\Psi^{1/2}\|^2 + \frac{\kappa}{4}\Delta t\|\nabla\xi^{1/2}\|^2,\end{aligned}$$

with $C_{T_4} = \max\{\frac{1}{2}\alpha^2 c_1^{-1}, \frac{1}{2}\beta^2\kappa^{-1}\}$. An application of (4.7b) leads to

$$T_4 \leq C\Big(h^{4\sigma}\|u_t\|^2_{L^2(0,t_1;H^{2+\sigma}(\Omega))} + (\Delta t)^4\|u_{ttt}\|^2_{L^\infty(0,t_1;H^1(\Omega))}\Big) + \frac{c_1}{4}\Delta t\|\nabla\Psi^{1/2}\|^2 + \frac{\kappa}{4}\Delta t\|\nabla\xi^{1/2}\|^2.$$

A further application of Cauchy–Schwarz and Young's inequalities (details on the choice of ϵ in the rest of the proof are skipped for brevity) leads to

$$\begin{aligned}T_5 &:= \frac{1}{2}\big[-a_1(\eta^1 - \eta^0, \Psi^{1/2}) + a_1\Delta t(\tau_0, \Psi^{1/2}) + \gamma(\varrho^1 - \varrho^0, \Psi^{1/2}) - \gamma\Delta t(s_0, \Psi^{1/2}) - b_1\Delta t(\eta^{1/2}, \Psi^{1/2})\big] \\ &\leq C\big(\|\eta^1 - \eta^0\|^2 + \|\eta^{1/2}\|^2 + \|\varrho^1 - \varrho^0\|^2 + (\Delta t)^2\|\tau_0\|^2 + (\Delta t)^2\|s_0\|^2\big) + \frac{a_1 - |\gamma|/\gamma_0}{2}\|\Psi^{1/2}\|^2 + \frac{b_1}{4}\Delta t\|\Psi^{1/2}\|^2 \\ &\leq C\big(h^{4\sigma}\|\theta\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))} + h^{4\sigma}\|p\|^2_{L^\infty(0,t_1;H^{1+\sigma}(\Omega))} + (\Delta t)^4\|\theta_{ttt}\|^2_{L^2(0,t_1;L^2(\Omega))} + (\Delta t)^4\|p_{ttt}\|^2_{L^2(0,t_1;L^2(\Omega))}\big) \\ &\quad + \frac{a_1 - |\gamma|/\gamma_0}{2}\|\Psi^{1/2}\|^2 + \frac{b_1}{4}\Delta t\|\Psi^{1/2}\|^2.\end{aligned}$$

In the last inequality, the bounds for the first three terms on the right-hand side are obtained from (4.8), and the last two from Lemma 4.1, respectively. Then, using similar arguments as above also show that

$$\begin{aligned}T_6 &:= \frac{1}{2}\big[-a_2(\varrho^1 - \varrho^0, \xi^{1/2}) + \gamma(\eta^1 - \eta^0, \xi^{1/2}) + a_2\Delta t(s_0, \xi^{1/2}) - \gamma\Delta t(\tau_0, \xi^{1/2})\big] \\ &\leq C\Big(h^{4\sigma}\|\theta\|^2_{L^\infty(0,t_1;H^2(\Omega))} + h^{4\sigma}\|p\|^2_{L^\infty(0,t_1;H^2(\Omega))} + (\Delta t)^4\|\theta_{ttt}\|^2_{L^2(0,t_1;L^2(\Omega))} + (\Delta t)^4\|p_{ttt}\|^2_{L^\infty(0,t_1;L^2(\Omega))}\Big) \\ &\qquad + \frac{a_2 - |\gamma|\gamma_0}{2}\|\xi^{1/2}\|^2.\end{aligned}$$

The Cauchy–Schwarz and Young's inequalities lead to $T_7 := 2\gamma(\Psi^{1/2}, \xi^{1/2}) \leq \frac{|\gamma|}{\gamma_0}\|\Psi^{1/2}\|^2 + |\gamma|\gamma_0\|\xi^{1/2}\|^2$.

Step 5 (Conclusion). It suffices to put together the bounds for T_1-T_7 in (4.13) and the fact that $h^2 \leq |\Omega|^{1-\sigma}h^{2\sigma} \lesssim h^{2\sigma}$, to finish the proof. □

Remark 4.3. *The decomposition* (4.14) *and the analysis in Step 3 of Lemma* 4.2 *are aimed at proving the superconvergence ($h^{4\sigma}$ rates) of the projected errors $\zeta^{1/2}$ and $\Psi^{1/2}$, $\xi^{1/2}$ in the norms $\|\zeta^{1/2}\|_h^2$ and $\|(\Psi^1, \xi^1)\|^2_H$, respectively. This is achieved using the approximation properties of the smoother Q from Lemma* 3.1*(v). The same arguments are also applied in Step 2 of Theorem* 4.4 *to obtain the superconvergence of the projected errors $\zeta^{m+1/2}$ and $\Psi^{m+1/2}$, $\xi^{m+1/2}$ for all $1 \leq m \leq N-1$ in $\|\zeta^{m+1/2}\|_h^2$ and $\|(\Psi^m, \xi^m)\|^2_H$, respectively; and is achieved in* (4.37)*. This superconvergence also yields more elegant lower H^s- order estimates with $s = 0, 1$ (resp. $s = 0$) for u (resp. θ and p) established in Corollary* 4.5 *(resp. Theorem* 4.4*).*

4.2 Error Estimates

We present the error estimates for (3.8). To do this, we first derive the error equations, which will subsequently be used in Theorem 4.4, with appropriate choices of test functions, to establish the estimates.

Error equations

A linear combination of the equations in the system (2.6), evaluated at $t = t_{n-1}$, $t = t_n$, and $t = t_{n+1}$, for $n = 1, 2, \cdots, N-1$, yields

$$\begin{aligned}&(u_{tt}^{n,1/4}, Qv_h) + a_0(\nabla u_{tt}^{n,1/4}, \nabla Qv_h) \\ &\qquad + d_0(\nabla^2 u^{n,1/4}, \nabla^2 Qv_h) - \alpha(\nabla\theta^{n,1/4}, \nabla Qv_h) - \beta(\nabla p^{n,1/4}, \nabla Qv_h) = (f^{n,1/4}, Qv_h), \qquad (4.18a)\end{aligned}$$

$$
\begin{aligned}
a_1(\theta_t^{n+1/2}, \psi_h) - \gamma(p_t^{n+1/2}, \psi_h) + b_1(\theta^{n+1/2}, \psi_h) \\
+ c_1(\nabla\theta^{n+1/2}, \nabla\psi_h) + \alpha(\nabla u_t^{n+1/2}, \nabla\psi_h) = (\phi^{n+1/2}, \psi_h),
\end{aligned} \tag{4.18b}
$$

$$
a_2(p_t^{n+1/2}, q_h) - \gamma(\theta_t^{n+1/2}, q_h) + \kappa(\nabla p^{n+1/2}, \nabla q_h) + \beta(\nabla u_t^{n+1/2}, \nabla q_h) = (g^{n+1/2}, q_h), \tag{4.18c}
$$

for all $v_h \in V_h$ and $\psi_h, q_h \in W_h$, where as earlier we have used Range$(Q) \subset H_0^2(\Omega)$ and $W_h \subset H_0^1(\Omega)$. Next we recall the definition of $F = f(t, \boldsymbol{x}) - u_{tt} + a_0\Delta u_{tt} - \alpha\Delta\theta - \beta\Delta p$ from (2.15) and define the truncation terms as follows:

$$
R^n := \bar{\partial}_t^2 u^n - u_{tt}^{n,1/4}, \;\; r^n := \bar{\partial}_t u^{n+1/2} - u_t^{n+1/2}, \;\; \tau^n := \bar{\partial}_t \theta^{n+1/2} - \theta_t^{n+1/2}, \text{ and } s^n := \bar{\partial}_t p^{n+1/2} - p_t^{n+1/2}. \tag{4.19}
$$

Subtracting (3.8a) from (4.18a) and employ $a_h(\mathcal{R}_h u^{n,1/4}, v_h) = (\nabla^2 u^{n,1/4}, \nabla^2 Q v_h)$ from (3.2), we readily obtain

$$
\begin{aligned}
&(\bar{\partial}_t^2 u^n - \bar{\partial}_t^2 U^n, v_h) + a_0(\nabla(\bar{\partial}_t^2 u^n - \bar{\partial}_t^2 U^n), \nabla v_h) + d_0 a_h(\mathcal{R}_h u^{n,1/4} - U^{n,1/4}, v_h) \\
&- \alpha(\nabla(\theta^{n,1/4} - \Theta^{n,1/4}), \nabla v_h) - \beta(\nabla(p^{n,1/4} - P^{n,1/4}), \nabla v_h) = (F^{n,1/4}, (Q - I)v_h) + (R^n, v_h) + a_0(\nabla R^n, \nabla v_h).
\end{aligned}
$$

An appeal to the splitting in (4.1) leads to the first relation of the error equation of the system as

$$
\begin{aligned}
&(\bar{\partial}_t^2 \zeta^n, v_h) + a_0(\nabla\bar{\partial}_t^2 \zeta^n, \nabla v_h) + d_0 a_h(\zeta^{n,1/4}, v_h) - \alpha(\nabla\Psi^{n,1/4}, \nabla v_h) \\
&- \beta(\nabla\xi^{n,1/4}, \nabla v_h) = -(\bar{\partial}_t^2 \rho^n, v_h) - a_0(\nabla\bar{\partial}_t^2 \rho^n, \nabla v_h) + \alpha(\nabla\eta^{n,1/4}, \nabla v_h) \\
&+ \beta(\nabla\varrho^{n,1/4}, \nabla v_h) + (F^{n,1/4}, (Q - I)v_h) + (R^n, v_h) + a_0(\nabla R^n, \nabla v_h).
\end{aligned} \tag{4.20}
$$

We can then subtract (3.8b) from (4.18b) and utilize the definition of Π_h from (3.4) to obtain

$$
\begin{aligned}
&a_1(\bar{\partial}_t(\theta^{n+1/2} - \Theta^{n+1/2}), \psi_h) - \gamma(\bar{\partial}_t(p^{n+1/2} - P^{n+1/2}), \psi_h) + b_1(\theta^{n+1/2} - \Theta^{n+1/2}, \psi_h) \\
&+ c_1(\nabla(\Pi_h\theta^{n+1/2} - \Theta^{n+1/2}), \nabla\psi_h) + \alpha(\nabla\bar{\partial}_t(u^{n+1/2} - U^{n+1/2}), \nabla\psi_h) = a_1(\tau^n, \psi_h) - \gamma(s^n, \psi_h) + \alpha(\nabla r^n, \nabla\psi_h).
\end{aligned}
$$

The splitting from (4.1) reveals the second relation in the error equation as

$$
\begin{aligned}
a_1(\bar{\partial}_t\Psi^{n+1/2}, \psi_h) - \gamma(\bar{\partial}_t\xi^{n+1/2}, \psi_h) + b_1(\Psi^{n+1/2}, \psi_h) + c_1(\nabla\Psi^{n+1/2}, \nabla\psi_h) + \alpha(\nabla\bar{\partial}_t\zeta^{n+1/2}, \nabla\psi_h) \\
= -a_1(\bar{\partial}_t\eta^{n+1/2}, \psi_h) + \gamma(\bar{\partial}_t\varrho^{n+1/2}, \psi_h) - b_1(\eta^{n+1/2}, \psi_h) \\
- \alpha(\nabla\bar{\partial}_t\rho^{n+1/2}, \nabla\psi_h) + a_1(\tau^n, \psi_h) - \gamma(s^n, \psi_h) + \alpha(\nabla r^n, \nabla\psi_h).
\end{aligned} \tag{4.21}
$$

Furthermore, we subtract (3.8c) from (4.18c), and utilize the same arguments as above along with (4.1) to obtain the third error relation of the error equation as

$$
\begin{aligned}
&a_2(\bar{\partial}_t\xi^{n+1/2}, q_h) - \gamma(\bar{\partial}_t\Psi^{n+1/2}, q_h) + \kappa(\nabla\xi^{n+1/2}, \nabla q_h) + \beta(\nabla\bar{\partial}_t\zeta^{n+1/2}, \nabla q_h) \\
&= -a_2(\bar{\partial}_t\varrho^{n+1/2}, q_h) + \gamma(\bar{\partial}_t\eta^{n+1/2}, q_h) - \beta(\nabla\bar{\partial}_t\rho^{n+1/2}, \nabla q_h) + a_2(s^n, q_h) - \gamma(\tau^n, q_h) + \beta(\nabla r^n, \nabla q_h).
\end{aligned} \tag{4.22}
$$

Some useful bounds

Next we present some bounds that will be useful in the proof of Theorem 4.4. The estimates from (3.3) for $\rho(t)$, (3.5) for $\eta(t)$ and $\varrho(t)$ are used to demonstrate the bounds

$$
\|\eta^{m,1/4}\| + h^\sigma\|\nabla\eta^{m,1/4}\| \leq C_3 h^{2\sigma}\|\theta\|_{L^\infty(t_{m-1}, t_{m+1}; H^{1+\sigma}(\Omega))}, \text{ for any } 1 \leq m \leq N, \tag{4.23a}
$$

$$
\|\varrho^{m,1/4}\| + h^\sigma\|\nabla\varrho^{m,1/4}\| \leq C_3 h^{2\sigma}\|p\|_{L^\infty(t_{m-1}, t_{m+1}; H^{1+\sigma}(\Omega))}, \text{ for any } 1 \leq m \leq N, \tag{4.23b}
$$

$$
\begin{aligned}
&\Big(\Delta t \sum_{n=1}^m \|\nabla\bar{\partial}_t\rho^{n+1/2}\|^2\Big)^{1/2} + \Big(\Delta t \sum_{n=1}^m \|\eta^{n+1/2}\|^2\Big)^{1/2} + \Big(\Delta t \sum_{n=1}^m \|\bar{\partial}_t\eta^{n+1/2}\|^2\Big)^{1/2} + \Big(\Delta t \sum_{n=1}^m \|\bar{\partial}_t\varrho^{n+1/2}\|^2\Big)^{1/2} \\
&\lesssim T^{1/2} h^{2\sigma}\big[\|u_t\|_{L^\infty(0,T;H^{2+\sigma}(\Omega))} + \|\theta\|_{L^\infty(0,T;H^{1+\sigma}(\Omega))} + \|\theta_t\|_{L^\infty(0,T;H^{1+\sigma}(\Omega))} + \|p_t\|_{L^\infty(0,T;H^{1+\sigma}(\Omega))}\big],
\end{aligned} \tag{4.23c}
$$

where in last inequality we have used $m\Delta t \leq T$. The Taylor series estimate $\|\bar{\partial}_t^2\rho^n\|^2 \leq \frac{2}{3}(\Delta t)^{-1}\int_{t_{n-1}}^{t_{n+1}}\|\rho_{tt}(t)\|^2\,\mathrm{d}t$ (resp. $\|\nabla\bar{\partial}_t^2\rho^n\|^2 \leq \frac{2}{3}(\Delta t)^{-1}\int_{t_{n-1}}^{t_{n+1}}\|\nabla\rho_{tt}(t)\|^2\,\mathrm{d}t$) along with (3.3) reveals that

$$
\Big(\Delta t \sum_{n=1}^m \big(\|\bar{\partial}_t^2\rho^n\|^2 + a_0\|\nabla\bar{\partial}_t^2\rho^n\|^2\big)\Big)^{1/2} \lesssim h^{2\sigma}\|u_{tt}\|_{L^2(0,T;H^{2+\sigma}(\Omega))}. \tag{4.24}
$$

Also, the definition (4.19), and the truncation estimates from Lemma 4.1 are given by

$$\Big(\Delta t\sum_{n=1}^{m}\| R^n\|^2\Big)^{1/2}\lesssim (\Delta t)^2\|u_{tttt}\|_{L^2(0,T;L^2(\Omega))},\ \text{and}\ \Big(\Delta t\sum_{n=1}^{m}\|\nabla R^n\|^2\Big)^{1/2}\lesssim (\Delta t)^2\|\nabla u_{tttt}\|_{L^2(0,T;L^2(\Omega))},\tag{4.25a}$$

$$\Big(\Delta t\sum_{n=1}^{m}\|\nabla r^n\|^2\Big)^{1/2}+\Big(\Delta t\sum_{n=1}^{m}\|\tau^n\|^2\Big)^{1/2}+\Big(\Delta t\sum_{n=1}^{m}\|s^n\|^2\Big)^{1/2}$$
$$\lesssim (\Delta t)^2\Big[\|u_{ttt}\|_{L^2(0,T;H^1(\Omega))}+\|\theta_{ttt}\|_{L^2(0,T;L^2(\Omega))}+\|p_{ttt}\|_{L^2(0,T;L^2(\Omega))}\Big].\tag{4.25b}$$

Note that $\sum_{k=1}^{\ell}\|b^{k-1}+b^k\|^2\leq 2\sum_{k=1}^{\ell}\|b^{k-1}\|^2+2\sum_{k=1}^{\ell}\|b^k\|^2\leq 4\sum_{k=1}^{\ell}\|b^{k-1}\|^2+2\|b^\ell\|^2$, with addition of $2\|b^0\|^2$ on the right-hand side in the last expression. This and an application of Cauchy–Schwarz and Young's inequalities lead to

$$\pm\sum_{k=1}^{\ell}(a^k,b^{k-1}+b^k)\leq\frac{\epsilon}{2}\sum_{k=1}^{\ell}\|a^k\|^2+\frac{1}{2\epsilon}\sum_{k=1}^{\ell}\|b^{k-1}+b^k\|^2\leq\frac{\epsilon}{2}\sum_{k=1}^{\ell}\|a^k\|^2+\frac{2}{\epsilon}\sum_{k=1}^{\ell}\|b^{k-1}\|^2+\frac{1}{\epsilon}\|b^\ell\|^2.\tag{4.26}$$

Main result

Before proceeding to establish the error estimates at $t=t_2,t_3,\cdots,t_N$, we first note that the following quantities are bounded, thanks to Table 2.1

$$L_{(u,\theta,p,T)}:=\|u\|^2_{L^\infty(0,T;H^{2+\sigma}(\Omega))}+\|u_t\|^2_{L^\infty(0,T;H^{2+\sigma}(\Omega))}+\|u_{tt}\|^2_{L^2(0,T;H^{2+\sigma}(\Omega))}+\|\theta\|^2_{L^\infty(0,T;H^{1+\sigma}(\Omega))}$$
$$+\|\theta_t\|^2_{L^\infty(0,T;H^{1+\sigma}(\Omega))}+\|p\|^2_{L^\infty(0,T;H^{1+\sigma}(\Omega))}+\|p_t\|^2_{L^\infty(0,T;H^{1+\sigma}(\Omega))},\tag{4.27a}$$
$$M_{(u,\theta,p,T)}:=\|u_{ttt}\|^2_{L^2(0,T;H^1(\Omega))}+\|u_{tttt}\|^2_{L^2(0,T;H^1(\Omega))}+\|\theta_{ttt}\|^2_{L^2(0,T;L^2(\Omega))}+\|p_{ttt}\|^2_{L^2(0,T;L^2(\Omega))}.\tag{4.27b}$$

Theorem 4.4 (Error estimates)**.** *Under the regularity assumptions on given data as stated in Theorem 2.3, for* $1\leq m\leq N-1$, *the following estimates are satisfied:*

$$\|\bar\partial_t(u^{m+1/2}-U^{m+1/2})\|^2+a_0\|\nabla\bar\partial_t(u^{m+1/2}-U^{m+1/2})\|^2+(a_1-|\gamma|/\gamma_0)\|\theta^{m+1}-\Theta^{m+1}\|^2$$
$$+(a_2-|\gamma|\gamma_0)\|p^{m+1}-P^{m+1}\|^2+d_0h^{2\sigma}\|u^{m+1/2}-U^{m+1/2}\|_h^2+h^{2\sigma}\|(\theta^m-\Theta^m,p^m-P^m)\|^2_H$$
$$\lesssim h^{4\sigma}+\big[L_{(u,\theta,p,t_1)}+L_{(u,\theta,p,T)}\big]h^{4\sigma}+M_{(u,\theta,p,T)}(\Delta t)^4,$$

where the absorbed constant in "$\lesssim$" depends on $C_{\rm Coer}$, $C_{\rm Cont}$, C_1,C_2, C_3, T, and the model coefficients.

Proof. The proof is divided into six steps-the first step derives a key inequality, this is followed by bounds for the terms in the key inequality in Steps 2-5, and Step 6 consolidates the proof.

Step 1 (Key inequality). Let us multiply (4.20) by $2\Delta t$, choose $v_h=\delta_t\zeta^n$ and utilize the identities (3.9a)-(3.9c). This yields

$$\|\bar\partial_t\zeta^{n+1/2}\|^2-\|\bar\partial_t\zeta^{n-1/2}\|^2+a_0\|\nabla\bar\partial_t\zeta^{n+1/2}\|^2-a_0\|\nabla\bar\partial_t\zeta^{n-1/2}\|^2+d_0a_h(\zeta^{n+1/2},\zeta^{n+1/2})-d_0a_h(\zeta^{n-1/2},\zeta^{n-1/2})$$
$$=2\Delta t\Big[(F^{n,1/4},(Q-I)\delta_t\zeta^n)+(\alpha\nabla\eta^{n,1/4}+\beta\nabla\varrho^{n,1/4},\nabla\delta_t\zeta^n)+(\alpha\nabla\Psi^{n,1/4}+\beta\nabla\xi^{n,1/4},\nabla\delta_t\zeta^n)$$
$$-(\bar\partial_t^2\rho^n,\delta_t\zeta^n)-a_0(\nabla\bar\partial_t^2\rho^n,\nabla\delta_t\zeta^n)+(R^n,\delta_t\zeta^n)+a_0(\nabla R^n,\nabla\delta_t\zeta^n)\Big]$$
$$=2(F^{n,1/4},(Q-I)(\zeta^{n+1/2}-\zeta^{n-1/2}))+2(\alpha\nabla\eta^{n,1/4}+\beta\nabla\varrho^{n,1/4},\nabla(\zeta^{n+1/2}-\zeta^{n-1/2}))$$
$$+\Delta t(\alpha\nabla\Psi^{n,1/4}+\beta\nabla\xi^{n,1/4},\bar\partial_t(\zeta^{n+1/2}+\zeta^{n-1/2}))-\Delta t(\bar\partial_t^2\rho^n-R^n,\bar\partial_t(\zeta^{n+1/2}+\zeta^{n-1/2}))$$
$$-\Delta t a_0(\nabla(\bar\partial_t^2\rho^n-R^n),\nabla(\bar\partial_t(\zeta^{n+1/2}+\zeta^{n-1/2})))$$

with $\Delta t\delta_t\zeta^n=\zeta^{n+1/2}-\zeta^{n-1/2}$ for the first two terms on the right-hand side and $\delta_t\zeta^n=\frac{1}{2}(\bar\partial_t\zeta^{n+1/2}+\bar\partial_t\zeta^{n-1/2})$ for the remaining terms. Next, we again multiply the equation (4.21) by $2\Delta t$ and choose $\psi_h=\Psi^{n+1/2}$ as the test function. Then utilize (3.9d)(i) to obtain

$$a_1\|\Psi^{n+1}\|^2-a_1\|\Psi^n\|^2+2\Delta t\Big[b_1\|\Psi^{n+1/2}\|^2+c_1\|\nabla\Psi^{n+1/2}\|^2+\alpha(\nabla\bar\partial_t\zeta^{n+1/2},\nabla\Psi^{n+1/2})\Big]$$

$$= 2\Delta t\Big[- a_1(\bar\partial_t\eta^{n+1/2}, \Psi^{n+1/2}) + \gamma(\bar\partial_t\xi^{n+1/2}, \Psi^{n+1/2}) + \gamma(\bar\partial_t\varrho^{n+1/2}, \Psi^{n+1/2}) - b_1(\eta^{n+1/2}, \Psi^{n+1/2})$$
$$- \alpha(\nabla\bar\partial_t\rho^{n+1/2}, \nabla\Psi^{n+1/2}) + a_1(\tau^n, \Psi^{n+1/2}) - \gamma(s^n, \Psi^{n+1/2}) + \alpha(\nabla r^n, \nabla\Psi^{n+1/2})\Big].$$

Similarly, we multiply (4.22) by $2\Delta t$, select the test function $q_h = \xi^{n+1/2}$, and employ (3.9d)(ii) to get

$$a_2\|\xi^{n+1}\|^2 - a_2\|\xi^n\|^2 + 2\Delta t\Big[\kappa\|\nabla\xi^{n+1/2}\|^2 + \beta(\nabla\bar\partial_t\zeta^{n+1/2}, \nabla\xi^{n+1/2})\Big]$$
$$= 2\Delta t\Big[- a_2(\bar\partial_t\varrho^{n+1/2}, \xi^{n+1/2}) + \gamma(\bar\partial_t\Psi^{n+1/2}, \xi^{n+1/2}) + \gamma(\bar\partial_t\eta^{n+1/2}, \xi^{n+1/2})$$
$$- \beta(\nabla\bar\partial_t\rho^{n+1/2}, \nabla\xi^{n+1/2}) + a_2(s^n, \xi^{n+1/2}) - \gamma(\tau^n, \xi^{n+1/2}) + \beta(\nabla r^n, \nabla\xi^{n+1/2})\Big].$$

After adding the last three displayed equations (after multiplying the first equation by 4) and summing for $n = 1, 2, \cdots, m$, where $1 \le m \le N-1$, we can then use (3.1) and (3.10) to produce

$$4\|\bar\partial_t\zeta^{m+1/2}\|^2 + 4a_0\|\nabla\bar\partial_t\zeta^{m+1/2}\|^2 + 4d_0C_{\mathrm{Coer}}\|\zeta^{m+1/2}\|_h^2 + a_1\|\Psi^{m+1}\|^2 + a_2\|\xi^{m+1}\|^2 + 2\|(\Psi^m, \xi^m)\|_H^2$$
$$\le 8\sum_{n=1}^m \Big[(F^{n,1/4}, (Q-I)(\zeta^{n+1/2} - \zeta^{n-1/2})) + (\nabla(\alpha\eta^{n,1/4} + \beta\varrho^{n,1/4}), \nabla(\zeta^{n+1/2} - \zeta^{n-1/2}))\Big]$$
$$+ \Delta t\sum_{n=1}^m \Big[4(\alpha\nabla\Psi^{n,1/4} + \beta\nabla\xi^{n,1/4}, \nabla(\bar\partial_t\zeta^{n+1/2} + \bar\partial_t\zeta^{n-1/2}) - 2(\alpha\nabla\Psi^{n+1/2} + \beta\nabla\xi^{n+1/2}, \nabla\bar\partial_t\zeta^{n+1/2})\Big]$$
$$- 4\Delta t\sum_{n=1}^m \Big((\bar\partial_t^2\rho^n - R^n, \bar\partial_t(\zeta^{n+1/2} + \zeta^{n-1/2})) + a_0(\nabla(\bar\partial_t^2\rho^n - R^n), \nabla(\bar\partial_t(\zeta^{n+1/2} + \zeta^{n-1/2})))\Big]$$
$$+ 2\Delta t\sum_{n=1}^m \Big[- a_1(\bar\partial_t\eta^{n+1/2}, \Psi^{n+1/2}) + \gamma(\bar\partial_t\varrho^{n+1/2}, \Psi^{n+1/2}) - b_1(\eta^{n+1/2}, \Psi^{n+1/2})$$
$$- \alpha(\nabla\bar\partial_t\rho^{n+1/2}, \nabla\Psi^{n+1/2}) + a_1(\tau^n, \Psi^{n+1/2}) - \gamma(s^n, \Psi^{n+1/2}) + \alpha(\nabla r^n, \nabla\Psi^{n+1/2})\Big]$$
$$+ 2\Delta t\sum_{n=1}^m \Big[- a_2(\bar\partial_t\varrho^{n+1/2}, \xi^{n+1/2}) + \gamma(\bar\partial_t\eta^{n+1/2}, \xi^{n+1/2}) - \beta(\nabla\bar\partial_t\rho^{n+1/2}, \nabla\xi^{n+1/2})$$
$$+ a_2(s^n, \xi^{n+1/2}) - \gamma(\tau^n, \xi^{n+1/2}) + \beta(\nabla r^n, \nabla\xi^{n+1/2})\Big] + 2\gamma\sum_{n=1}^m \Big[(\Psi^{n+1}, \xi^{n+1}) - (\Psi^n, \xi^n)\Big]$$
$$+ \Big[4\|\bar\partial_t\zeta^{1/2}\|^2 + 4a_0\|\nabla\bar\partial_t\zeta^{1/2}\|^2 + 4d_0C_{\mathrm{Cont}}\|\zeta^{1/2}\|_h^2 + a_1\|\Psi^1\|^2 + a_2\|\xi^1\|^2\Big]$$
$$=: T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7. \tag{4.28}$$

Step 2 (Bound for T_1). The summation by parts formula $\sum_{n=1}^m g^{n,1/4}(h^{n+1/2} - h^{n-1/2}) = g^{m,1/4}h^{m+1/2} - g^{1,1/4}h^{1/2} - \sum_{n=1}^{m-1}(g^{n+1,1/4} - g^{n,1/4})h^{n+1/2}$, together with the observation $g^{n+1,1/4} - g^{n,1/4} = 1/4\int_{t_{n-1}}^{t_{n+2}} g_t\,\mathrm{d}t + 1/4\int_{t_n}^{t_{n+1}} g_t\,\mathrm{d}t$ reveals

$$T_1 = 8(F^{m,1/4}, (Q-I)\zeta^{m+1/2}) - 8(F^{1,1/4}, (Q-I)\zeta^{1/2})$$
$$- 2\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+2}} (F_t(t), (Q-I)\zeta^{n+1/2})\,\mathrm{d}t - 2\sum_{n=1}^{m-1}\int_{t_n}^{t_{n+1}} (F_t(t), (Q-I)\zeta^{n+1/2})\,\mathrm{d}t$$
$$+ 8(\nabla(\alpha\eta^{m,1/4} + \beta\varrho^{m,1/4}), \nabla\zeta^{m+1/2}) - 8(\nabla(\alpha\eta^{1,1/4} + \beta\varrho^{1,1/4}), \nabla\zeta^{1/2})$$
$$- 2\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+2}} (\nabla(\alpha\eta_t(t) + \beta\varrho_t(t)), \nabla\zeta^{n+1/2})\,\mathrm{d}t - 2\sum_{n=1}^{m-1}\int_{t_n}^{t_{n+1}} (\nabla(\alpha\eta_t(t) + \beta\varrho_t(t)), \nabla\zeta^{n+1/2})\,\mathrm{d}t. \tag{4.29}$$

An application of Cauchy–Schwarz inequality, the bounds form Lemma 3.1(v), the Young inequality (with $\epsilon = 4(d_0C_{\mathrm{Coer}})^{-1}, 4(d_0C_{\mathrm{Coer}})^{-1}$, respectively for the first two terms), and (2.16)(i) result in

$$8(F^{m,1/4}, (Q-I)\zeta^{m+1/2}) \le 8C_1h^2\|F^{m,1/4}\|\|\zeta^{m+1/2}\|_h \le 16C_1^2C_F^2(d_0C_{\mathrm{Coer}})^{-1}h^4 + d_0C_{\mathrm{Coer}}\|\zeta^{m+1/2}\|_h^2,$$

$$8(F^{1,1/4},(I-Q)\zeta^{1/2}) \leq 8C_1h^2\|F^{1,1/4}\|\|\zeta^{1/2}\|_h \leq 16C_1^2C_F^2(d_0C_{\mathrm{Coer}})^{-1}h^4 + d_0C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2.$$

Analogous arguments with $\epsilon = 6T(d_0C_{\mathrm{Coer}})^{-1}, 2T(d_0C_{\mathrm{Coer}})^{-1}$, respectively for the third and fourth terms give

$$2\int_{t_{n-1}}^{t_{n+2}}(F_t(t),(I-Q)\zeta^{n+1/2})\,\mathrm{d}t \leq 6TC_1^2(d_0C_{\mathrm{Coer}})^{-1}h^4\|F_t\|^2_{L^2(t_{n-1},t_{n+2};L^2(\Omega))} + \frac{d_0}{2T}C_{\mathrm{Coer}}\Delta t\|\zeta^{n+1/2}\|_h^2,$$
$$2\int_{t_n}^{t_{n+1}}(F_t(t),(I-Q)\zeta^{n+1/2})\,\mathrm{d}t \leq 2TC_1^2(d_0C_{\mathrm{Coer}})^{-1}h^4\|F_t\|^2_{L^2(t_n,t_{n+1};L^2(\Omega))} + \frac{d_0}{2T}C_{\mathrm{Coer}}\Delta t\|\zeta^{n+1/2}\|_h^2.$$

In the last two displayed inequalities, we have used $\int_{t_{n-1}}^{t_{n+2}}\|\zeta^{n+1/2}\|_h^2\,\mathrm{d}t = 3\Delta t\|\zeta^{n+1/2}\|_h^2$ and $\int_{t_n}^{t_{n+1}}\|\zeta^{n+1/2}\|_h^2\,\mathrm{d}t = \Delta t\|\zeta^{n+1/2}\|_h^2$, respectively. Hence, it follows from (2.16)(ii) that

$$2\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+2}}(F_t(t),(I-Q)\zeta^{n+1/2})\,\mathrm{d}t \leq 6TC_1^2(C_F')^2(d_0C_{\mathrm{Coer}})^{-1}h^4 + \frac{d_0}{2T}C_{\mathrm{Coer}}\Delta t\sum_{n=1}^{m-1}\|\zeta^{n+1/2}\|_h^2,$$
$$2\sum_{n=1}^{m-1}\int_{t_n}^{t_{n+1}}(F_t(t),(I-Q)\zeta^{n+1/2})\,\mathrm{d}t \leq 2TC_1^2(C_F')^2(d_0C_{\mathrm{Coer}})^{-1}h^4 + \frac{d_0}{2T}C_{\mathrm{Coer}}\Delta t\sum_{n=1}^{m-1}\|\zeta^{n+1/2}\|_h^2.$$

Elementary manipulations analogous to (4.14) show

$$\begin{aligned}8(\nabla(\alpha\eta^{m,1/4}+\beta\varrho^{m,1/4}),\nabla\zeta^{m+1/2}) &= 8(\nabla(\alpha\eta^{m,1/4}+\beta\varrho^{m,1/4}),\nabla(I-Q)\zeta^{m+1/2})\\ &\quad - 8(\alpha\eta^{m,1/4}+\beta\varrho^{m,1/4},\Delta(Q\zeta^{m+1/2})).\end{aligned} \tag{4.30}$$

Then, similar to (4.15), utilize Cauchy–Schwarz's inequality, $\|\nabla(Q-I)\zeta^{m+1/2}\| \leq C_1h\|\zeta^{m+1/2}\|_h$ from Lemma 3.1(v) (with $s=1$ and $v=0$), and Young's inequality for first two terms on the right-hand side of (4.30) as

$$\begin{aligned}8(\nabla(\alpha\eta^{m,1/4}+\beta\varrho^{m,1/4}),\nabla(I-Q)\zeta^{m+1/2}) &\leq 8C_1h\big(\alpha\|\nabla\eta^{m,1/4}\|\|\zeta^{m+1/2}\|_h + \beta\|\nabla\varrho^{m,1/4}\|\|\zeta^{m+1/2}\|_h\big)\\ &\leq Ch^2\big(\|\nabla\eta^{m,1/4}\|^2 + \|\nabla\varrho^{m,1/4}\|^2\big) + \frac{d_0}{2}C_{\mathrm{Coer}}\|\zeta^{m+1/2}\|_h^2.\end{aligned} \tag{4.31}$$

For third and fourth terms of right-hand side of (4.30), we note that $\|\Delta(Q\zeta^{m+1/2})\| \leq \|Q\zeta^{m+1/2}\|_h$ and a triangle inequality with Lemma 3.1(v) shows $\|\Delta(Q\zeta^{m+1/2})\| \leq \Lambda\|\zeta^{m+1/2}\|_h$ for $\Lambda > 0$. Therefore, combining this with the Cauchy–Schwarz and Young's inequality (as in (4.16)) lead to

$$\begin{aligned}-8(\alpha\eta^{m,1/4}+\beta\varrho^{m,1/4},\Delta(Q\zeta^{m+1/2})) &\leq 8\alpha\Lambda\|\eta^{m,1/4}\|\|\zeta^{m+1/2}\|_h + 8\beta\Lambda\|\varrho^{m,1/4}\|\|\zeta^{m+1/2}\|_h\\ &\leq 64\Lambda^2(d_0C_{\mathrm{Coer}})^{-1}\big(\alpha^2\|\eta^{m,1/4}\|^2 + \beta^2\|\varrho^{m,1/4}\|^2\big) + \frac{d_0}{2}C_{\mathrm{Coer}}\|\zeta^{m+1/2}\|_h^2.\end{aligned} \tag{4.32}$$

A combination of (4.31)-(4.32) in (4.30) and bounds from (4.23a)-(4.23b) yields

$$\begin{aligned}&8(\nabla(\alpha\eta^{m,1/4}+\beta\varrho^{m,1/4}),\nabla\zeta^{m+1/2})\\ &\qquad\leq C(h^{2+2\sigma}+h^{4\sigma})\big(\|\theta\|^2_{L^\infty(t_{m-1},t_{m+1};H^{1+\sigma}(\Omega))} + \|p\|^2_{L^\infty(t_{m-1},t_{m+1};H^{1+\sigma}(\Omega))}\big) + d_0C_{\mathrm{Coer}}\|\zeta^{m+1/2}\|_h^2.\end{aligned}$$

Analogous arguments lead to

$$\begin{aligned}&-8(\nabla(\alpha\eta^{1,1/4}+\beta\varrho^{1,1/4}),\nabla\zeta^{1/2}) \leq C(h^{2+2\sigma}+h^{4\sigma})\big(\|\theta\|^2_{L^\infty(0,t_2;H^{1+\sigma}(\Omega))} + \|p\|^2_{L^\infty(0,t_2;H^{1+\sigma}(\Omega))}\big)\\ &\qquad + d_0C_{\mathrm{Coer}}\|\zeta^{1/2}\|_h^2,\\ &-\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+2}}\Big[(\nabla(\alpha\eta_t(t)+\beta\varrho_t(t)),\nabla\zeta^{n+1/2})\Big]\,\mathrm{d}t - \sum_{n=1}^{m-1}\int_{t_n}^{t_{n+1}}\Big[(\nabla(\alpha\eta_t(t)+\beta\varrho_t(t)),\nabla\zeta^{n+1/2})\Big]\,\mathrm{d}t\\ &\qquad\leq C(h^{2+2\sigma}+h^{4\sigma})\big(\|\theta_t\|^2_{L^2(0,T;H^{1+\sigma}(\Omega))} + \|p_t\|^2_{L^2(0,T;H^{1+\sigma}(\Omega))}\big) + \frac{d_0}{T}C_{\mathrm{Coer}}\Delta t\sum_{n=1}^{m-1}\|\zeta^{n+1/2}\|_h^2.\end{aligned}$$

A combination of all this in (4.29) establishes

$$T_1 \leq C\Big(h^4 + (h^{2+2\sigma}+h^{4\sigma})\big(\|\theta\|^2_{L^\infty(t_0,t_2;H^{1+\sigma}(\Omega))} + \|p\|^2_{L^\infty(t_0,t_2;H^{1+\sigma}(\Omega))}$$

$$+ \|\theta\|^2_{L^\infty(t_{m-1},t_{m+1};H^{1+\sigma}(\Omega))} + \|p\|^2_{L^\infty(t_{m-1},t_{m+1};H^{1+\sigma}(\Omega))} + \|\theta_t\|^2_{L^2(0,T;H^{1+\sigma}(\Omega))} + \|p_t\|^2_{L^2(0,T;H^{1+\sigma}(\Omega))}\Big)\Big)$$
$$+ 2d_0 C_{\mathrm{Coer}}\|\zeta^{1/2}\|^2_h + 2d_0 C_{\mathrm{Coer}}\|\zeta^{m+1/2}\|^2_h + \frac{2d_0}{T} C_{\mathrm{Coer}}\Delta t \sum_{n=1}^{m-1}\|\zeta^{n+1/2}\|^2_h, \tag{4.33}$$

where the generic constant C depends on C_1, C_F, C_F', d_0^{-1}, and C_{coer}^{-1}.

Step 3 (Bound for T_2). Utilize the arguments similar to (3.13) (with Θ^n, U^n, P^n replaced by Ψ^n, ζ^n, ξ^n, respectively) to obtain

$$T_2 := \Delta t \sum_{n=1}^{m} \Big[4(\alpha\nabla\Psi^{n,1/4} + \beta\nabla\xi^{n,1/4}, \nabla(\bar\partial_t\zeta^{n+1/2} + \bar\partial_t\zeta^{n-1/2})) - 2(\alpha\nabla\Psi^{n+1/2} + \beta\nabla\xi^{n+1/2}, \nabla\bar\partial_t\zeta^{n+1/2})\Big]$$
$$= 4\Delta t \sum_{n=1}^{m}(\alpha\nabla\Psi^{n,1/4} + \beta\nabla\xi^{n,1/4}, \nabla\bar\partial_t\zeta^{n-1/2}) + 2\Delta t\sum_{n=1}^{m}(\alpha\nabla\Psi^{n-1/2} + \beta\nabla\xi^{n-1/2}, \nabla\bar\partial_t\zeta^{n+1/2}).$$

Follow the approach used in Steps 2-3 of Theorem 3.6 (more precisely see the bounds (3.16)-(3.17)) to show

$$T_2 \le c_1\Delta t\|\nabla\Psi^{1/2}\|^2 + \kappa\Delta t\|\nabla\xi^{1/2}\|^2 + \Delta t\sum_{n=1}^{m}\Big[c_1\|\nabla\Psi^{n+1/2}\|^2 + \kappa\|\nabla\xi^{n+1/2}\|^2\Big]$$
$$+ \frac{\Delta t}{2}\sum_{n=1}^{m-1}\Big[c_1\|\nabla\Psi^{n+1/2}\|^2 + \kappa\|\nabla\xi^{n+1/2}\|^2\Big] + \frac{(\Delta t)^2}{a_0}\sum_{n=1}^{m-1}\Big[\alpha^2\|\nabla\Psi^{n-1/2}\|^2 + \beta^2\|\nabla\xi^{n-1/2}\|^2\Big]$$
$$+ 2a_0\|\nabla\bar\partial_t\zeta^{m+1/2}\|^2 + 4\Delta t\left(\frac{\alpha^2}{c_1} + \frac{\beta^2}{\kappa}\right)\sum_{n=1}^{m}\|\nabla\bar\partial_t\zeta^{n-1/2}\|^2 + 2\Delta t\left(\frac{\alpha^2}{c_1} + \frac{\beta^2}{\kappa}\right)\sum_{n=1}^{m-1}\|\nabla\bar\partial_t\zeta^{n+1/2}\|^2. \tag{4.34}$$

Step 4 (Bound for $T_3 - T_5$). A repeated application of (4.26) with $\epsilon = 8T$ yields

$$T_3 := -4\Delta t\sum_{n=1}^{m}\Big((\bar\partial_t^2\rho^n - R^n, \bar\partial_t(\zeta^{n+1/2} + \zeta^{n-1/2})) + a_0(\nabla(\bar\partial_t^2\rho^n - R^n), \nabla(\bar\partial_t(\zeta^{n+1/2} + \zeta^{n-1/2})))\Big]$$
$$\le 16T\Delta t\sum_{n=1}^{m}\Big[\|\bar\partial_t^2\rho^n\|^2 + a_0\|\nabla\bar\partial_t^2\rho^n\|^2 + \|R^n\|^2 + a_0\|\nabla R^n\|^2\Big]$$
$$+ 2\frac{\Delta t}{T}\sum_{n=1}^{m}\Big[\|\bar\partial_t\zeta^{n-1/2}\|^2 + a_0\|\nabla\bar\partial_t\zeta^{n-1/2}\|^2\Big] + \frac{\Delta t}{T}\Big[\|\bar\partial_t\zeta^{m+1/2}\|^2 + a_0\|\nabla\bar\partial_t\zeta^{m+1/2}\|^2\Big]$$
$$\le C\left(h^{4\sigma}\|u_{tt}\|^2_{L^2(0,T;H^{2+\sigma}(\Omega))} + (\Delta t)^4\|\nabla u_{tttt}\|^2_{L^2(0,T;L^2(\Omega))}\right)$$
$$+ 2\frac{\Delta t}{T}\sum_{n=1}^{m}\Big[\|\bar\partial_t\zeta^{n-1/2}\|^2 + a_0\|\nabla\bar\partial_t\zeta^{n-1/2}\|^2\Big] + \Big[\|\bar\partial_t\zeta^{m+1/2}\|^2 + a_0\|\nabla\bar\partial_t\zeta^{m+1/2}\|^2\Big]$$

with (4.24),(4.25a) and $\Delta t/T \le 1$ applied for the last term (in the last line). Another repeated application of (4.26) with $\epsilon = 8T(a_1 - |\gamma|/\gamma_0)^{-1}$ and $\Psi^{n+1/2} = \frac{1}{2}(\Psi^n + \Psi^{n+1})$ leads to a bound for four terms in T_4 as

$$2\Delta t\sum_{n=1}^{m}\big[-a_1(\bar\partial_t\eta^{n+1/2}, \Psi^{n+1/2}) + \gamma(\bar\partial_t\varrho^{n+1/2}, \Psi^{n+1/2}) + a_1(\tau^n, \Psi^{n+1/2}) - \gamma(s^n, \Psi^{n+1/2})\big]$$
$$\le 4\Delta t T(a_1 - |\gamma|/\gamma_0)^{-1}\sum_{n=1}^{m}\big[a_1^2\|\bar\partial_t\eta^{n+1/2}\|^2 + \gamma^2\|\bar\partial_t\varrho^{n+1/2}\|^2 + a_1^2\|\tau^n\|^2 + \gamma^2\|s^n\|^2\big]$$
$$+ \frac{\Delta t}{T}(a_1 - |\gamma|/\gamma_0)\sum_{n=1}^{m}\|\Psi^n\|^2 + \frac{\Delta t}{2T}(a_1 - |\gamma|/\gamma_0)\|\Psi^{m+1}\|^2.$$

On the other hand, Cauchy–Schwarz and Young's inequalities (with $\epsilon = 1/4, c_1/8, c_1/8$) bound the remaining terms of T_4 as

$$2\Delta t\sum_{n=1}^{m}\big[-b_1(\eta^{n+1/2}, \Psi^{n+1/2}) - \alpha(\nabla\bar\partial_t\rho^{n+1/2}, \nabla\Psi^{n+1/2}) + \alpha(\nabla r^n, \nabla\Psi^{n+1/2})\big]$$

$$\leq \frac{\Delta t}{4}\sum_{n=1}^{m}\left[b_1\|\Psi^{n+1/2}\|^2 + c_1\|\nabla\Psi^{n+1/2}\|^2\right] + 4\Delta t\sum_{n=1}^{m}\left[b_1\|\eta^{n+1/2}\|^2 + 2\alpha^2 c_1^{-1}\|\nabla\bar{\partial}_t\rho^{n+1/2}\|^2 + 2\alpha^2 c_1^{-1}\|\nabla r^n\|^2\right].$$

A combination of last two inequalities leads to the bound

$$\begin{aligned}
T_4 &:= 2\Delta t\sum_{n=1}^{m}\Big[-a_1(\bar{\partial}_t\eta^{n+1/2},\Psi^{n+1/2}) + \gamma(\bar{\partial}_t\varrho^{n+1/2},\Psi^{n+1/2}) - b_1(\eta^{n+1/2},\Psi^{n+1/2})\\
&\qquad - \alpha(\nabla\bar{\partial}_t\rho^{n+1/2},\nabla\Psi^{n+1/2}) + a_1(\tau^n,\Psi^{n+1/2}) - \gamma(s^n,\Psi^{n+1/2}) + \alpha(\nabla r^n,\nabla\Psi^{n+1/2})\Big]\\
&\leq \Delta t\sum_{n=1}^{m}\left[\frac{(a_1-|\gamma|/\gamma_0)}{T}\|\Psi^n\|^2 + \frac{b_1}{4}\|\Psi^{n+1/2}\|^2 + \frac{c_1}{4}\|\nabla\Psi^{n+1/2}\|^2\right] + \frac{\Delta t}{2T}(a_1-|\gamma|/\gamma_0)\|\Psi^{m+1}\|^2\\
&\quad + C\Delta t\sum_{n=1}^{m}\left[\|\bar{\partial}_t\eta^{n+1/2}\|^2 + \|\bar{\partial}_t\varrho^{n+1/2}\|^2 + \|\eta^{n+1/2}\|^2 + \|\nabla\bar{\partial}_t\rho^{n+1/2}\|^2 + \|\tau^n\|^2 + \|s^n\|^2 + \|\nabla r^n\|^2\right],
\end{aligned}$$

with a generic constant that depends on the material parameters. Analogous steps are now employed to bound $T_5 := \Delta t\sum_{n=1}^{m}\left[(-a_2\bar{\partial}_t\varrho^{n+1/2}+\gamma\bar{\partial}_t\eta^{n+1/2},\xi^{n+1/2}) - \beta(\nabla\bar{\partial}_t\rho^{n+1/2},\nabla\xi^{n+1/2}) + (a_2 s^n - \gamma\tau^n,\xi^{n+1/2}) + \beta(\nabla r^n,\nabla\xi^{n+1/2})\right]$ as

$$\begin{aligned}
T_5 &\leq \Delta t\sum_{n=1}^{m}\left[\frac{(a_2-|\gamma|\gamma_0)}{T}\|\xi^n\|^2 + \frac{\kappa}{4}\|\nabla\xi^{n+1/2}\|^2\right] + \frac{\Delta t}{2T}(a_2-|\gamma|\gamma_0)\|\xi^{m+1}\|^2\\
&\quad + C\Delta t\sum_{n=1}^{m}\left[\|\bar{\partial}_t\varrho^{n+1/2}\|^2 + \|\bar{\partial}_t\eta^{n+1/2}\|^2 + \|\nabla\bar{\partial}_t\rho^{n+1/2}\|^2 + \|\tau^n\|^2 + \|s^n\|^2 + \|\nabla r^n\|^2\right].
\end{aligned}$$

Next we put together the bounds for $T_3 - T_5$, use (4.23c) and (4.25b) for controlling the terms in the last lines of T_4 and T_5, recall the definitions (3.10) and (4.27) to arrive at

$$\begin{aligned}
T_3 + T_4 + T_5 &\leq C\big(L_{(u,\theta,p,T)}h^{4\sigma} + M_{(u,\theta,p,T)}(\Delta t)^4\big) + \|\bar{\partial}_t\zeta^{m+1/2}\|^2 + a_0\|\nabla\bar{\partial}_t\zeta^{m+1/2}\|^2 + \frac{1}{4}\|(\Psi^m,\xi^m)\|_H^2\\
&\quad + \frac{1}{2}(a_1-|\gamma|/\gamma_0)\|\Psi^{m+1}\|^2 + \frac{1}{2}(a_2-|\gamma|\gamma_0)\|\xi^{m+1}\|^2 + 2\frac{\Delta t}{T}\sum_{n=1}^{m}\left[\|\bar{\partial}_t\zeta^{n-1/2}\|^2 + a_0\|\nabla\bar{\partial}_t\zeta^{n-1/2}\|^2\right]\\
&\quad + \frac{\Delta t}{T}\sum_{n=1}^{m}\left[(a_1-|\gamma|/\gamma_0)\|\Psi^n\|^2 + (a_2-|\gamma|\gamma_0)\|\xi^n\|^2\right],
\end{aligned}\tag{4.35}$$

with $\Delta t/T \leq 1$ utilized in two terms above that involve $\|\Psi^{m+1}\|^2$ and $\|\xi^{m+1}\|^2$.

Step 5 (Bounds for T_6 and T_7). Elementary manipulations show

$$T_6 := 2\gamma\sum_{n=1}^{m}\Big[(\Psi^{n+1},\xi^{n+1}) - (\Psi^n,\xi^n)\Big] \leq \frac{|\gamma|}{\gamma_0}\|\Psi^{m+1}\|^2 + |\gamma|\gamma_0\|\xi^{m+1}\|^2 + |\gamma|\Big(\|\Psi^1\|^2 + \|\xi^1\|^2\Big).$$

This, $\Psi^0 = 0$ and $\xi^0 = 0$, and the definition $T_7 := 4\|\bar{\partial}_t\zeta^{1/2}\|^2 + 4a_0\|\nabla\bar{\partial}_t\zeta^{1/2}\|^2 + 4d_0 C_{\mathrm{Cont}}\|\zeta^{1/2}\|_h^2 + a_1\|\Psi^1\|^2 + a_2\|\xi^1\|^2$ lead to

$$\begin{aligned}
T_6 + T_7 &\leq 4\|\bar{\partial}_t\zeta^{1/2}\|^2 + 4a_0\|\nabla\bar{\partial}_t\zeta^{1/2}\|^2 + 4d_0 C_{\mathrm{Cont}}\|\zeta^{1/2}\|_h^2\\
&\quad + 2\Big(a_1 + |\gamma|\Big)\|\Psi^{1/2}\|^2 + 2\Big(a_2 + |\gamma|\Big)\|\xi^{1/2}\|^2 + \frac{|\gamma|}{\gamma_0}\|\Psi^{m+1}\|^2 + |\gamma|\gamma_0\|\xi^{m+1}\|^2\\
&\leq C\big(h^{4\sigma} + L_{(u,\theta,p,t_1)}h^{4\sigma} + M_{(u,\theta,p,t_1)}(\Delta t)^4\big) + \frac{|\gamma|}{\gamma_0}\|\Psi^{m+1}\|^2 + |\gamma|\gamma_0\|\xi^{m+1}\|^2,
\end{aligned}\tag{4.36}$$

where elementary manipulations and Lemma 4.2 were used in the last step.

Step 6 (Consolidation). First note that, by definition (3.10) and some basic manipulations, we can assert that

$$\frac{7}{4}\|(\Psi^m,\xi^m)\|_H^2 - \Delta t\sum_{n=1}^{m}\left[c_1\|\nabla\Psi^{n+1/2}\|^2 + \kappa\|\nabla\xi^{n+1/2}\|^2\right] - \frac{\Delta t}{2}\sum_{n=1}^{m-1}\left[c_1\|\nabla\Psi^{n+1/2}\|^2 + \kappa\|\nabla\xi^{n+1/2}\|^2\right]$$

$$= \frac{\Delta t}{4}\sum_{n=1}^{m}\Big[7b_1\|\Psi^{n+1/2}\|^2 + c_1\|\nabla\Psi^{n+1/2}\|^2 + \kappa\|\nabla\xi^{n+1/2}\|^2\Big] + \frac{\Delta t}{2}\Big[c_1\|\nabla\Psi^{m+1/2}\|^2 + \kappa\|\nabla\xi^{m+1/2}\|^2\Big]$$
$$\geq \frac{\Delta t}{2}\Big[c_1\|\nabla\Psi^{m+1/2}\|^2 + \kappa\|\nabla\xi^{m+1/2}\|^2\Big] + \frac{1}{4}\|(\Psi^m, \xi^m)\|_H^2.$$

This, a combination of (4.33)-(4.36) in (4.28) with $2d_0C_{\text{Coer}}\|\zeta^{1/2}\|_h^2 + c_1\Delta t\|\nabla\Psi^{1/2}\|^2 + \kappa\Delta t\|\nabla\xi^{1/2}\|^2 \lesssim h^{4\sigma} + L_{(u,\theta,p,t_1)}h^{4\sigma} + M_{(u,\theta,p,t_1)}(\Delta t)^4$ from Lemma 4.2 to bound the terms that involve the initial bounds in T_1 and T_2, and some elementary manipulations of the constants yield

$$\begin{aligned}
&3\|\bar\partial_t\zeta^{m+1/2}\|^2 + a_0\|\nabla\bar\partial_t\zeta^{m+1/2}\|^2 + 2d_0C_{\text{Coer}}\|\zeta^{m+1/2}\|_h^2 + \frac{1}{2}(a_1 - |\gamma|/\gamma_0)\|\Psi^{m+1}\|^2\\
&\quad + \frac{1}{2}(a_2 - |\gamma|\gamma_0)\|\xi^{m+1}\|^2 + \frac{\Delta t}{2}c_1\|\nabla\Psi^{m+1/2}\|^2 + \frac{\Delta t}{2}\kappa\|\nabla\xi^{m+1/2}\|^2 + \frac{1}{4}\|(\Psi^m, \xi^m)\|_H^2\\
&\lesssim h^{4\sigma} + \big[L_{(u,\theta,p,t_1)} + L_{(u,\theta,p,T)}\big]h^{4\sigma} + \big[M_{(u,\theta,p,t_1)} + M_{(u,\theta,p,T)}\big](\Delta t)^4\\
&\quad + \frac{\Delta t}{T}\nu\sum_{n=0}^{m-1}\Big[3\|\bar\partial_t\zeta^{n+1/2}\|^2 + a_0\|\nabla\bar\partial_t\zeta^{n-1/2}\|^2 + 2d_0C_{\text{Coer}}\|\zeta^{n+1/2}\|_h^2 + \frac{1}{2}(a_1 - |\gamma|/\gamma_0)\|\Psi^{n+1}\|^2\\
&\quad + \frac{1}{2}(a_2 - |\gamma|\gamma_0)\|\xi^{n+1}\|^2 + \frac{\Delta t}{2}c_1\|\nabla\Psi^{n+1/2}\|^2 + \frac{\Delta t}{2}\kappa\Delta t\|\nabla\xi^{n+1/2}\|^2\big)\Big].
\end{aligned}$$

The constant ν in the right-hand side of the above expression is manipulated for an easy application of Gronwall's Lemma 1.1. Now, we apply Lemma 3.4 to arrive at

$$\begin{aligned}
&\|\bar\partial_t\zeta^{m+1/2}\|^2 + a_0\|\nabla\bar\partial_t\zeta^{m+1/2}\|^2 + d_0\|\zeta^{m+1/2}\|_h^2 + (a_1 - |\gamma|/\gamma_0)\|\Psi^{m+1}\|^2\\
&\qquad + (a_2 - |\gamma|\gamma_0)\|\xi^{m+1}\|^2 + \|(\Psi^m, \xi^m)\|_H^2 + \frac{\Delta t}{2}\Big[c_1\|\nabla\Psi^{m+1/2}\|^2 + \kappa\|\nabla\xi^{m+1/2}\|^2\Big]\\
&\qquad \lesssim h^{4\sigma} + \big[L_{(u,\theta,p,t_1)} + L_{(u,\theta,p,T)}\big]h^{4\sigma} + \big[M_{(u,\theta,p,t_1)} + M_{(u,\theta,p,T)}\big](\Delta t)^4.
\end{aligned} \tag{4.37}$$

We ignore the non-negative term $\frac{\Delta t}{2}\left[c_1\|\nabla\Psi^{m+1/2}\|^2 + \kappa\|\nabla\xi^{m+1/2}\|^2\right]$ from the left-hand side and apply the definitions (4.1a)-(4.1c), triangle inequality and the projections estimates from (3.3)-(3.5), which eventually lead to the desired estimates. □

The L^2-estimates for θ and p have already been derived in the above theorem, while for u, we present the following result.

Corollary 4.5 (L^2 and H^1-estimates for deflection)**.** *Suppose that (u, θ, p) and (U^n, Θ^n, P^n) solve (2.6a)-(2.6c) and (3.8a)-(3.8c), respectively. Then, under the assumptions of Theorem 4.4, for $1 \leq m \leq N-1$, the following error estimate holds*

$$\|u^{m+1} - U^{m+1}\|^2 + a_0\|\nabla(u^{m+1} - U^{m+1})\|^2 \lesssim h^{4\sigma} + (\Delta t)^4.$$

Proof. Ignoring the last four non-negative terms on the left-hand side of (4.37), we can obtain

$$\|\bar\partial_t\zeta^{m+1/2}\|^2 + a_0\|\nabla\bar\partial_t\zeta^{m+1/2}\|^2 + \|\zeta^{m+1/2}\|_h^2 \lesssim h^{4\sigma} + \Delta t^4, \qquad \text{for } 1 \leq m \leq N-1.$$

Note that $\zeta^{m+1} = \zeta^{m+1/2} + \frac{1}{2}\Delta t\bar\partial_t\zeta^{m+1/2}$ (resp. $\nabla\zeta^{m+1} = \nabla\zeta^{m+1/2} + \frac{1}{2}\Delta t\nabla\bar\partial_t\zeta^{m+1/2}$). Then, by the discrete Poincaré inequality we have $\|\zeta^{m+1/2}\| \leq \|\zeta^{m+1/2}\|_h$ (resp.$\|\nabla\zeta^{m+1/2}\| \leq \|\zeta^{m+1/2}\|_h$), and hence

$$\|\zeta^{m+1}\|^2 + a_0\|\nabla\zeta^{m+1}\| \lesssim \|\bar\partial_t\zeta^{m+1/2}\|^2 + \|\zeta^{m+1/2}\|_h^2 + a_0\|\nabla\bar\partial_t^2\zeta^{m+1/2}\| + a_0\|\zeta^{m+1/2}\|_h \lesssim (1 + a_0)(h^{4\sigma} + (\Delta t)^4).$$

Therefore, simply using triangle inequality we can obtain

$$\|u^{m+1} - U^{m+1}\|^2 + a_0\|\nabla(u^{m+1} - U^{m+1})\|^2 \lesssim \|\rho^{m+1}\|^2 + \|\zeta^{m+1}\|^2 + a_0\|\nabla\rho^{m+1}\|^2 + a_0\|\nabla\zeta^{m+1}\|^2.$$

A combination of last three inequalities and (3.3) for $\|\rho^{m+1}\|^2 + a_0\|\nabla\rho^{m+1}\|^2$ lead to the desired result. □

5 Numerical results

In this section, we investigate the application of the Kirchhoff–Love plate model in Subsection 5.1 to capture TED in copper and TPE in flat Berea sandstone. Subsections 5.2-5.3 provide numerical results that validate theoretical estimates and illustrate the effective performance of the proposed scheme across different values of the parameter γ. The penalty parameter σ_{IP} is chosen according to [13]. All simulations were conducted with the finite element library FEniCS [2], and executed on a desktop machine equipped with an Intel® Core™ i5-7500 CPU (Kaby Lake architecture), featuring 4 cores and 4 threads, operating at a base frequency of 3.4 GHz.

5.1 Example 1: Verification of Kirchhoff's model: 2D vs 3D TED and TPE plate models

In this subsection, we illustrate Kirchhoff's hypothesis by comparing the solution of the three-dimensional (3D) model in (5.1) (resp. (5.4)) for TED (resp. TPE) with two-dimensional (2D) model in (1.1), while systematically varying the plate thickness d. The plots in Figures 5.4-5.6 demonstrate that as the plate becomes thinner, the solution curves (plotted against time) from the 3D model approximates those for the 2D model.

Building upon classical theory as described in [44] (see also [4, Eq. (9)]), given a space-time dependent loading $(\hat{f}_1(t), \hat{f}_2(t), \hat{f}_3(t)) = \boldsymbol{f}(t) : \widehat{\Omega} \to \mathbb{R}^3$ ($\widehat{\Omega} \subset \mathbb{R}^3$), prescribed heat source $\hat{\phi}(t) : \widehat{\Omega} \to \mathbb{R}$, and total amount of mass source/sink $\hat{g}(t) : \widehat{\Omega} \to \mathbb{R}$, the 3D TED model seeks the displacement vector $\boldsymbol{u} = (\hat{u}_1(t), \hat{u}_2(t), \hat{u}_3(t))$, the small temperature increment $\hat{\theta} = T_{\mathrm{abs}} - T_0$ (with T_{abs}, T_0 as the absolute and reference temperature, respectively), and the chemical potential $\hat{p}$ such that

$$\rho \boldsymbol{u}_{tt} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \rho \boldsymbol{f} \qquad \text{in } \widehat{\Omega} \times [0, T], \tag{5.1a}$$

$$\Big(\frac{\rho c_E}{T_0} + \frac{\varpi^2}{\varrho}\Big)\hat{\theta}_t + \frac{\varpi}{\varrho}\hat{p}_t + \nabla \cdot \boldsymbol{q} + \gamma_1 \nabla \cdot \boldsymbol{u}_t = \hat{\phi} \qquad \text{in } \widehat{\Omega} \times [0, T], \tag{5.1b}$$

$$\frac{1}{\varrho}\hat{p}_t + \frac{\varpi}{\varrho}\hat{\theta}_t + \nabla \cdot \boldsymbol{p} + \gamma_2 \nabla \cdot \boldsymbol{u}_t = \hat{g} \qquad \text{in } \widehat{\Omega} \times [0, T]. \tag{5.1c}$$

The $\boldsymbol{\sigma} = 2\mu \nabla_{\mathrm{sym}} \boldsymbol{u} + [\lambda_0 \nabla \cdot \boldsymbol{u} - \gamma_1 \hat{\theta} - \gamma_2 \hat{p}]\mathbf{I}$ above is the total Cauchy stress tensor, $\boldsymbol{q} = -k_1 \nabla \hat{\theta}$ is the heat flux (Fourier's law) and $\boldsymbol{p} = -k_2 \nabla \hat{p}$ the diffusive flux (Fick's law). The constants involved in the definition of $\boldsymbol{\sigma}$ are given by

$$\lambda_0 = \lambda - \frac{(3\lambda + 2\mu)^2 \alpha_c^2}{\varrho}, \quad \gamma_1 = (3\lambda + 2\mu)\big(\alpha_t + \frac{\varpi}{\varrho}\alpha_c\big), \quad \gamma_2 = \frac{(3\lambda + 2\mu)\alpha_c}{\varrho},$$

with the basic parameters from Table 1.1. The surfaces at $z = -d/2, d/2$ are subject to traction-free and zero-flux boundary conditions, while the remaining boundaries are governed by homogeneous Dirichlet conditions.

A dimensional reduction analysis in [4, Eqs. (9)-(46)] derives a 2D model (1.1) from the 3D model (5.1) which seeks transverse displacement, first moments of temperature and chemical potential

$$u = \frac{1}{d}\int_{-d/2}^{d/2} \hat{u}_3 \,\mathrm{d}z, \ \theta = \int_{-d/2}^{d/2} z\hat{\theta}\,\mathrm{d}z, \ \text{and} \ \ p = \int_{-d/2}^{d/2} z\hat{p}\,\mathrm{d}z. \tag{5.2}$$

The transformation of the model coefficients in this process is given in Table 5.1 and that of moments of the right-hand side functions (forces and sources) by

$$f = \frac{1}{d}\int_{-d/2}^{d/2} \hat{f}_3 \,\mathrm{d}z, \ \ \phi = \frac{12}{\rho d^4}\int_{-d/2}^{d/2} z\hat{\phi}\,\mathrm{d}z, \ \text{and} \ \ g = \frac{12}{\rho d^4}\int_{-d/2}^{d/2} z\hat{g}\,\mathrm{d}z. \tag{5.3}$$

It is very important to note that the the constant λ_0 is assumed to satisfy $\lambda_0 + \mu > 0$ [46] and this condition makes all the coefficients ecxept γ in the 2D model (1.1) positive, see Table 5.1.

Another example of diffusion in porous media is the phenomenon of TPE. Consider now that the domain $\widehat{\Omega} \subset \mathbb{R}^3$ is fully saturated with a viscous fluid. The flow occurs also in the xy plane and the poroelastic material is subject to thermal energy effects. Given the mechanical load $(\hat{f}_1(t), \hat{f}_2(t), \hat{f}_3(t)) = \boldsymbol{f}(t) : \widehat{\Omega} \to \mathbb{R}^3$ ($\widehat{\Omega} \subset \mathbb{R}^3$), prescribed heat source $\hat{\phi}(t) : \widehat{\Omega} \to \mathbb{R}$, and fluid mass source $\hat{g}^*(t) : \widehat{\Omega} \to \mathbb{R}$, the three-dimensional TPE equations [14, 16, 53] seeks displacement $\boldsymbol{u}$, the small temperature increment $\hat{\theta}$, and pore pressure $\hat{p}^*$ such that

$$\rho \boldsymbol{u}_{tt} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \rho \boldsymbol{f} \qquad \text{in } \widehat{\Omega} \times [0, T], \tag{5.4a}$$

$$\frac{\rho c_E}{T_0}\hat{\theta}_t - 3\gamma^*\hat{p}_t + \nabla\cdot\boldsymbol{q} + \gamma_1^*\nabla\cdot\boldsymbol{u}_t = \hat{\phi} \qquad \text{in } \hat{\Omega}\times[0,T], \tag{5.4b}$$

$$\frac{1}{\varrho^*}\hat{p}_t^* - 3\gamma^*\hat{\theta}_t + \nabla\cdot\boldsymbol{p}^* + \gamma_2^*\nabla\cdot\boldsymbol{u}_t = \hat{g}^* \qquad \text{in } \hat{\Omega}\times[0,T], \tag{5.4c}$$

The surfaces at $z = -d/2, d/2$ are subject to traction-free and zero-flux boundary conditions, while the remaining boundaries are governed by homogeneous Dirichlet conditions. Here, $\boldsymbol{\sigma} = 2\mu\nabla_{\text{sym}}\boldsymbol{u} + [\lambda\nabla\cdot\boldsymbol{u} - \gamma_1^*\hat{\theta} - \gamma_2^*\hat{p}^*]\mathbf{I}$, $\boldsymbol{q} = -k_1\nabla\hat{\theta}$ (Fourier's law) and $\boldsymbol{p} = -k_2^*\nabla\hat{p}$ (Darcy's law) represent the total stress tensor, heat and fluid flux respectively. Also $\gamma_1^* = \alpha_t(3\lambda + 2\mu)$, $\gamma_2^* = \beta^*$, and the other constants are defined in Table 1.1.

The three-dimensional TPE model (5.4) exhibits structural similarities to the three-dimensional TED model (5.1). By employing a dimensional reduction approach analogous to that in [4], we derive a two-dimensional TPE model (1.1) which seeks transverse displacement, first moments of temperature and pore pressure

$$u = \frac{1}{d}\int_{-d/2}^{d/2}\hat{u}_3\,\mathrm{d}z,\;\; \theta = \int_{-d/2}^{d/2} z\hat{\theta}\,\mathrm{d}z,\;\; \text{and}\;\; p = \int_{-d/2}^{d/2} z\hat{p}^*\,\mathrm{d}z. \tag{5.5}$$

The moments of the right-hand side functions (forces and sources) of this 2D plate model are given as

$$f = \frac{1}{d}\int_{-d/2}^{d/2}\hat{f}_3\,\mathrm{d}z,\;\;\; \phi = \frac{12}{\rho d^4}\int_{-d/2}^{d/2} z\hat{\phi}\,\mathrm{d}z,\;\; \text{and}\;\; g = \frac{12}{\rho d^4}\int_{-d/2}^{d/2} z\hat{g}^*\,\mathrm{d}z, \tag{5.6}$$

and the parametrization of the coefficients given in Table 5.1.

Coefficient	*2D-TED Model*	*2D-TPE Model*	*Coefficient*	*2D-TED Model*	*2D-TPE Model*
a_0	$\frac{d^2}{12}$	$\frac{d^2}{12}$	γ	$-\frac{12}{\rho d^4}\left(\frac{\varpi}{\varrho} + \frac{\gamma_1\gamma_2}{\lambda_0+2\mu}\right)$	$\frac{12}{\rho d^4}\left(3\gamma^* - \frac{\gamma_1\gamma_2}{\lambda+2\mu}\right)$
d_0	$\frac{4\mu d^2(\lambda_0+\mu)}{12\rho(\lambda_0+2\mu)}$	$\frac{4\mu d^2(\lambda+\mu)}{12\rho(\lambda+2\mu)}$	b_1	$\frac{12k_1}{\rho d^3}$	$\frac{12k_1}{\rho d^3}$
α	$\frac{2\mu\gamma_1}{\rho d(\lambda_0+2\mu)}$	$\frac{2\mu\gamma_1}{\rho d(\lambda+2\mu)}$	c_1	$\frac{12k_1}{\rho d^4}$	$\frac{12k_1}{\rho d^4}$
β	$\frac{2\mu\gamma_2}{\rho d(\lambda_0+2\mu)}$	$\frac{2\mu\gamma_2}{\rho d(\lambda+2\mu)}$	a_2	$\frac{12}{\rho d^4}\left(\frac{1}{\varrho} + \frac{\gamma_2^2}{\lambda_0+2\mu}\right)$	$\frac{12}{\rho d^4}\left(\frac{1}{\varrho^*} + \frac{\gamma_2^2}{\lambda+2\mu}\right)$
a_1	$\frac{12}{\rho d^4}\left(\frac{\rho c_E}{T_0} + \frac{\varpi^2}{\varrho} + \frac{\gamma_1^2}{\lambda_0+2\mu}\right)$	$\frac{12}{\rho d^4}\left(\frac{\rho c_E}{T_0} + \frac{\gamma_1^2}{\lambda+2\mu}\right)$	κ	$\frac{12k_2}{\rho d^4}$	$\frac{12k_2^*}{\rho d^4}$

Table 5.1: Coefficients in the 2D model (1.1a)-(1.1c) for thermoelastic diffusion and thermo-poroelastic cases.

Table 5.2: Constants in 3D-TED model [49].

Constant	**Value**	**SI Unit**
λ	7.76×10^{10}	$\mathrm{kgm^{-1}s^{-2}}$
μ	3.36×10^{10}	$\mathrm{kgm^{-1}s^{-2}}$
ϱ	9.0×10^{5}	$\mathrm{m^5kg^{-1}s^{-2}}$
α_t	1.78×10^{-5}	$\mathrm{K^{-1}}$
α_c	1.98×10^{-4}	$\mathrm{m^4kg^{-1}}$
ϖ	1.2×10^{4}	$\mathrm{m^2s^{-2}K^{-1}}$
ρ	8954	$\mathrm{kgm^{-3}}$
c_E	383.1	$\mathrm{J\,kg^{-1}\,K^{-1}}$
T_0	293	K
k_1	386	$\mathrm{W\,m^{-1}\,K^{-1}}$
k_2	8.5×10^{-9}	$\mathrm{kgsm^{-3}}$

Table 5.3: Constants in 3D-TPE model [11,47,52].

Constant	**Value**	**SI Unit**
λ	10.22×10^{9} [52]	$\mathrm{kg\,m^{-1}\,s^{-2}}$
μ	4.09×10^{9} [52]	$\mathrm{kg\,m^{-1}\,s^{-2}}$
α_t	3×10^{-5} [11]	$\mathrm{K^{-1}}$
ϱ^*	12×10^{9} [47]	$\mathrm{kg\,m^{-1}\,s^{-2}}$
β^*	0.79 [47]	–
ρ	2280 [52]	$\mathrm{kgm^{-3}}$
c_E	800 [11]	$\mathrm{J\,kg^{-1}\,K^{-1}}$
T_0	293 [11]	K
γ^*	5×10^{-5}	$\mathrm{K^{-1}}$
k_1	1×10^{-6} [11]	$\mathrm{W\,m^{-1}\,K^{-1}}$
k_2	1.9×10^{-13} [11]	$\mathrm{m^2}$

Example 1. 3D model coefficients for copper (left) and Berea sandstone (right) plate.

Thermoelastic diffusion plate model verification: Our objective is to illustrate that the 2D TED model (1.1), effectively approximates the 3D TED model described by (5.1) in the sense that if (U^n, Θ^n, P^n) is the approximation of the solution (u, θ, p) of the 2D model (1.1) at time $t = t_n$ computed with the discrete formulation

(3.8) and $(\hat{\mathbf{U}}^n, \hat{\Theta}^n, \hat{P}^n)$ is the discrete solution of (5.1) at $t = t_n$, then (U^n, Θ^n, P^n) approximates the triplet $(\int_{-d/2}^{d/2} \hat{U}_3^n \,\mathrm{d}z, \int_{-d/2}^{d/2} z\hat{\Theta}^n \,\mathrm{d}z, \int_{-d/2}^{d/2} z\hat{P}^n \,\mathrm{d}z)$ with $\hat{\mathbf{U}}^n = (\hat{U}_1^n, \hat{U}_2^n, \hat{U}_3^n)$, as motivated by (5.2).

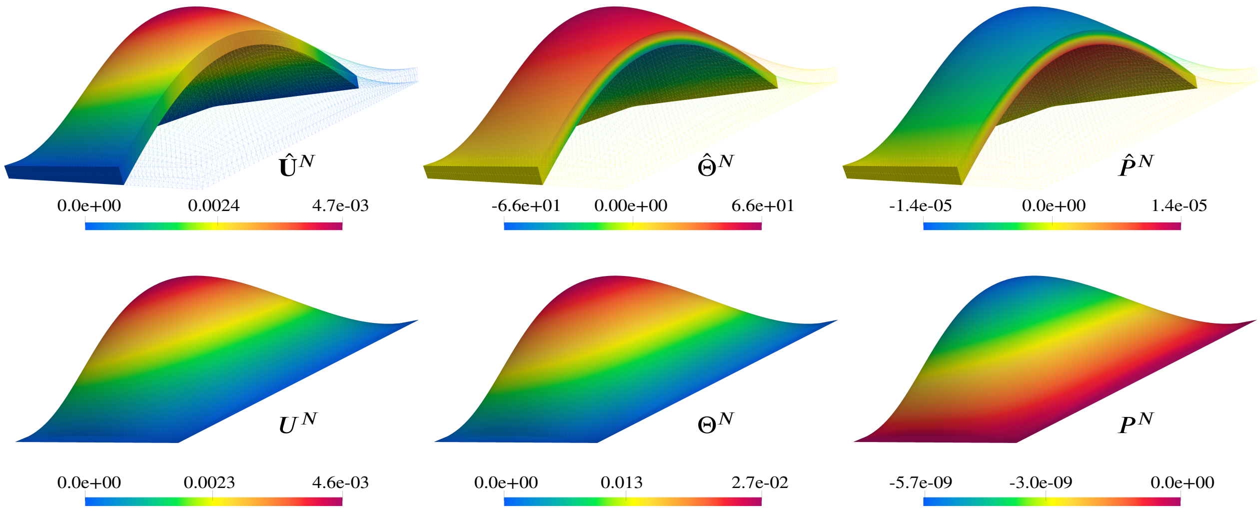

Fig 5.2: Example 1. 3D (upper) and 2D displacement (lower), temperature, and chemical potential at final time T for TED model.

To achieve this, we solve the 3D system (5.1) using continuous FE spaces: $(\mathcal{P}^1(\mathcal{T}))^3$ for displacement $\boldsymbol{u}$, and $\mathcal{P}^1(\mathcal{T})$ for temperature $\hat{\theta}$ and pressure $\hat{p}$, with $\hat{\Omega} = [0,1] \times [0,1] \times [-d/2, d/2]$. The temporal discretization is handled by the Newmark scheme for (5.1a) and by Crank–Nicolson scheme for (5.1b)-(5.1c). Homogeneous Dirichlet boundary conditions are set on all the sides except the surfaces $z = -d/2, d/2$ where the plate is assumed traction free and subject to zero heat/diffusion flux (in line with the theoretical discussion in [4]). In the 3D setting, the load, heat, and mass sources are defined as

$$\mathbf{f} = \left(0, 0, t^2 \sin(\pi x)\sin(\pi y)\right), \ \hat{\phi} = txy(x-1)(y-1), \text{ and } \hat{g} = t\sin(\pi x)\sin(\pi y), \tag{5.7}$$

whereas for 2D we use equation (5.3). Initial conditions are set to zero in both 2D and 3D cases. The parameters used in the 3D model (5.1) assume typical values for copper plate [49]. See Table 5.2.

Let $T = 10$, $\Delta t = 1/8$, and consider the cells $\hat{\Omega}_c = [5/64, 6/64] \times [5/64, 6/64] \times [-d/2, d/2]$, and $\Omega_c = [5/64, 6/64] \times [5/64, 6/64]$. At time $t = t_n$, we will use the following output quantities

$$U_{3\mathrm{D}}^n := \frac{1}{|\hat{\Omega}_c|}\int_{\hat{\Omega}_c} \hat{U}_3 d\hat{\boldsymbol{x}}, \ \Theta_{3\mathrm{D}}^n := \frac{1}{|\hat{\Omega}_c|}\int_{\hat{\Omega}_c} z\hat{\Theta}^n \, d\hat{\boldsymbol{x}}, \text{ and } P_{3\mathrm{D}}^n := \frac{1}{|\hat{\Omega}_c|}\int_{\hat{\Omega}_c} z\hat{P}^n \, d\hat{\boldsymbol{x}} \ \text{ for } \hat{\boldsymbol{x}} = (x, y, z),$$
$$U_{2\mathrm{D}}^n := \frac{1}{|\Omega_c|}\int_{\Omega_c} U^n d\boldsymbol{x}, \ \Theta_{2\mathrm{D}}^n := \frac{1}{|\Omega_c|}\int_{\Omega_c} \Theta^n \, \boldsymbol{x}, \ P_{2\mathrm{D}}^n := \frac{1}{|\Omega_c|}\int_{\Omega_c} zP^n \, \boldsymbol{x} \ \text{ for } \boldsymbol{x} = (x, y).$$

The simulations in Figure 5.4 reveal that as the plate thickness d decreases (from upper to lower), the results of the 2D model approximate those of the 3D model and the solutions at the final time are plotted in Figure 5.2. Furthermore, as expected the computational efficiency is significantly improved: the 2D model requires approximately 138 (resp. 131) seconds, whereas the 3D model takes about 567 seconds (resp. 564) seconds for a plate width $d = 0.5$ (resp $d = 0.005$).

Thermo-poroelastic plate model verification: Motivated by [50], in this experiment we choose a flat Berea sandstone with material parameters given in Table 5.3, and repeat the last experiment. In 3D we consider (5.4) with the same load/source functions as in (5.7), and the transformation of source functions from 3D to 2D is given in (5.6). The transformation of 3D to 2D parameters is given in Table 5.1. The quantities $(U_{3\mathrm{D}}^n, \Theta_{3\mathrm{D}}^n, P_{3\mathrm{D}}^n)$ and $(U_{2\mathrm{D}}^n, \Theta_{2\mathrm{D}}^n, P_{2\mathrm{D}}^n)$ are defined similarly as in the last experiment. Moreover we also consider $T = 100$, $\Delta t = 10/8$, and the cells $\hat{\Omega}_c = [5/64, 6/64] \times [5/64, 6/64] \times [-d/2, d/2]$, and $\Omega_c = [5/64, 6/64] \times [5/64, 6/64]$. The simulations in Figure 5.6 reveal that as the plate thickness d decreases (from upper to bottom), the 2D model's results converge to those of the 3D model. Furthermore, the computational efficiency is significantly improved: the 2D model requires approximately 138 (resp. 85) seconds, whereas the 3D model takes about 566 (resp. 537) seconds for plate width $d = 0.5$ (resp. $d = 0.005$).

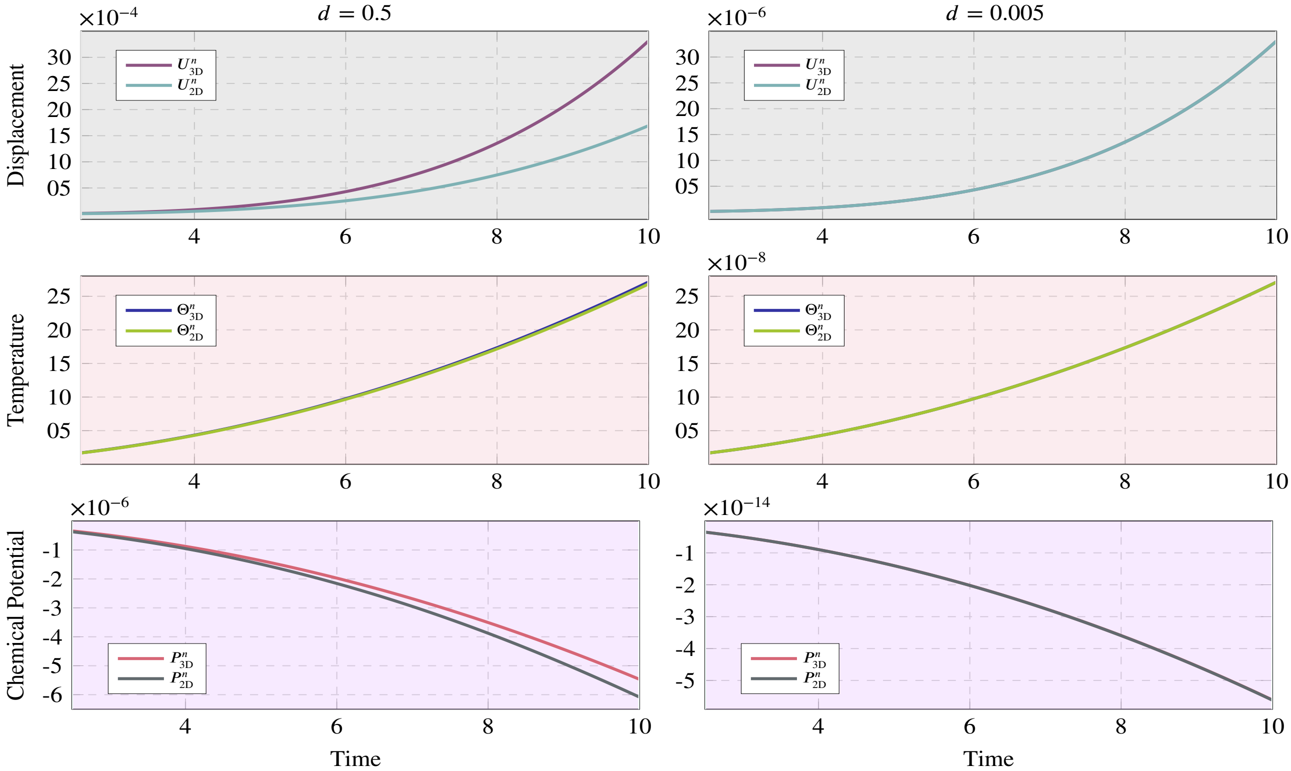

Fig 5.4: Example 1. 2D $(U^n_{2D}, \Theta^n_{2D}, P^n_{2D})$ and 3D $(U^n_{3D}, \Theta^n_{3D}, P^n_{3D})$ solution vs time t_n with plate thickness $d = 0.5$ (left) and $d = 0.005$ (right) for the TED model.

5.2 Example 2: Convergence against smooth solutions

The theoretical results of Section 3.2 are validated in this section by choosing a smooth manufactured solution of (1.1a)-(1.1a). We consider the spatial domain $\Omega = (0,1)^2$ and time interval $[0,1]$. All model parameters are set to 1, except for $a_1 = 35, a_2 = 40$, and $\gamma = 1$ (and $\gamma = -1$) which are selected so that the condition (1.2) is satisfied and the result are robust with respect to γ. The transverse load f, heat source ϕ, and a total amount of mass source g and the initial data u^0, u^{*0}, θ^0 and p^0 are chosen such that the exact solution of (1.1) is given by

$$u(\boldsymbol{x},t) = \exp(5t)(x(x-1)y(y-1))^2,$$
$$\theta(\boldsymbol{x},t) = \exp(-t)\sin(\pi x)\sin(\pi y), \qquad p(\boldsymbol{x},t) = \cos(t)\sin(\pi x)\sin(\pi y),$$

and hence our theoretical regularity results with $\sigma = 1$ (as well as the clamped boundary conditions) are satisfied.

We construct a sequence of successively refined uniform triangular meshes $\mathcal{T}^i$ of Ω of size h_i and split the time domain using the refined time step $\Delta t = 2^{-3/2}h_i$. For each mesh refinement, we calculate errors as

$$\|e_u\|^{\ell^\infty} := \max_{0\le n\le N} \|u^n - U^n\|, \tag{5.8a}$$
$$\|\nabla e_u\|^{\ell^\infty} := \max_{0\le n\le N} \|\nabla(u^n - U^n)\|, \qquad \|\hat{e}_u\|_h^{\ell^\infty} := \max_{0\le n\le N-1} \|u^{n+1/2} - U^{n+1/2}\|_h, \tag{5.8b}$$
$$\|e_\theta\|^{\ell^\infty} := \max_{0\le n\le N} \|\theta^n - \Theta^n\|, \qquad \|\nabla \hat{e}_\theta\|^{\ell^2} := \Big(\Delta t \sum_{n=0}^{N-1} \|\nabla(\theta^{n+1/2} - \Theta^{n+1/2})\|^2\Big)^{1/2}, \tag{5.8c}$$
$$\|e_p\|^{\ell^\infty} := \max_{0\le n\le N} \|p^n - P^n\|, \qquad \|\nabla \hat{e}_p\|^{\ell^2} := \Big(\Delta t \sum_{n=0}^{N-1} \|\nabla(p^{n+1/2} - P^{n+1/2})\|^2\Big)^{1/2}. \tag{5.8d}$$

The experimental rates of convergence in space are computed as $\texttt{Rate} = \log(e_{i+1}/e_i)[\log(h_{i+1}/h_i)]^{-1}$, where e_i denotes a norm of the error on the mesh $\mathcal{T}^i$. Then by Theorem 4.4 and Corollary 4.5, the expected convergence rates are of order $\mathcal{O}(h^\sigma)$ for $\|\hat{e}_u\|_h^{\ell^\infty}, \|\nabla\hat{e}_\theta\|^{\ell^2}, \|\nabla\hat{e}_p\|^{\ell^2}$ and $\mathcal{O}(h^{2\sigma})$ for $\|e_u\|^{\ell^\infty}, \|\nabla e_u\|^{\ell^\infty}, \|e_\theta\|^{\ell^\infty}, \|e_p\|^{\ell^\infty}$ norms defined in (5.8). Table 5.4 shows the error history and convergence results for u, θ and p and the numerical

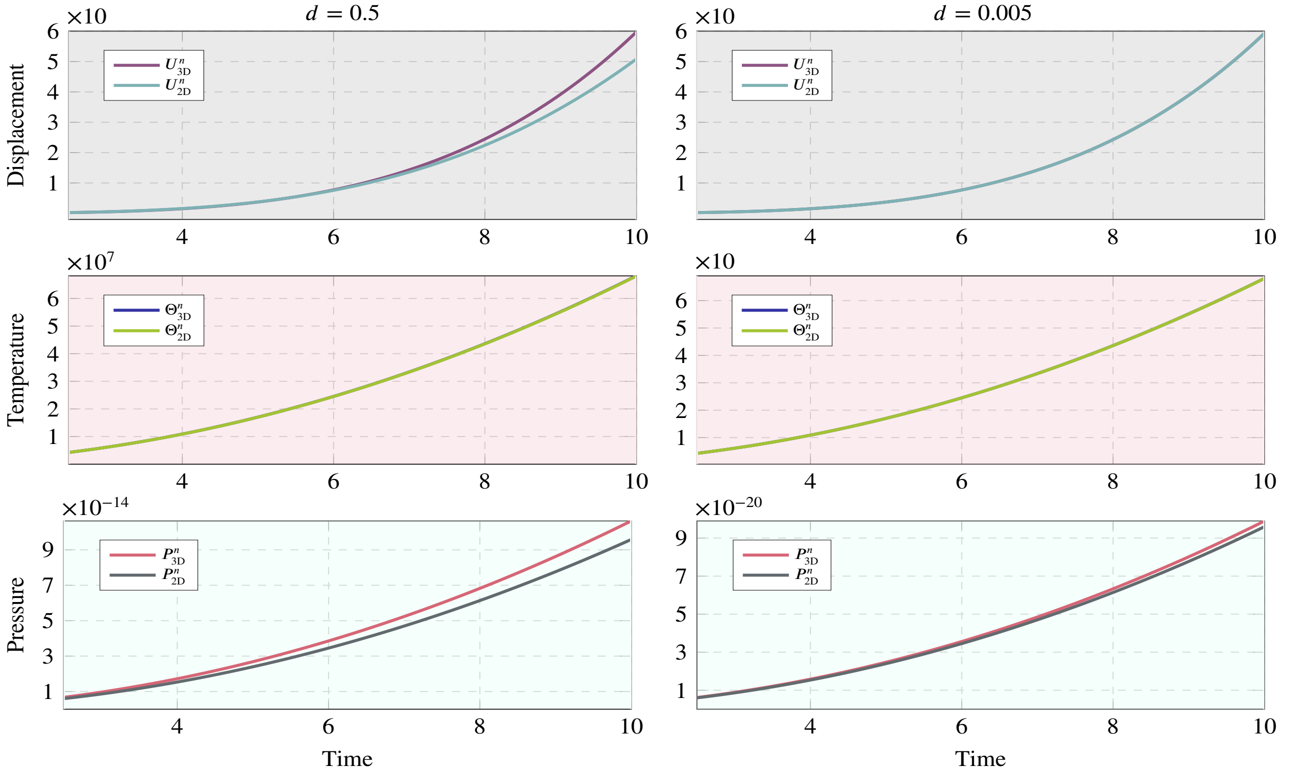

Fig 5.6: Example 1. 2D ($U^n_{2D}, \Theta^n_{2D}, P^n_{2D}$) and 3D ($U^n_{3D}, \Theta^n_{3D}, P^n_{3D}$) solution vs time t_n with plate thickness $d = 0.5$ (left) and $d = 0.005$ (right panels) for the TPE model.

solution at final time given in Figure 5.8 for $\gamma = -1$. In all cases, the numerical results are consistent with the expected theoretical results.

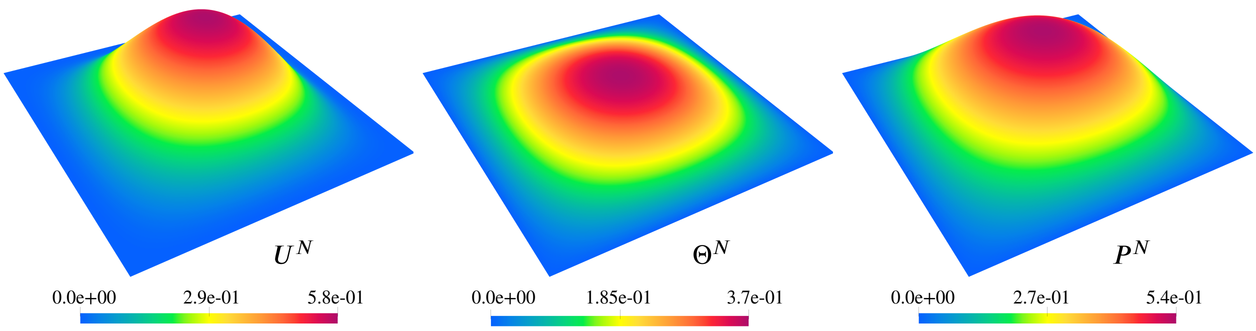

Fig 5.8: Example 2. Numerical solution shown at final time T for $\gamma = -1$.

5.3 Example 3: Convergence for a non-convex domain

This example illustrates the convergence of the proposed method even when the domain Ω is non-convex, constituting a case where $\sigma < 1$. Consider $\Omega = [-1, 1]^2 \setminus [-1, 0]^2$, $T = 1$, and choose the load and source functions such that the triplet (u, θ, p) in polar coordinates is given by

$$u(r, \varphi, t) = t^2(r^2 \sin^2(\varphi) - 1)^2(r^2 \cos^2(\varphi)^2 - 1)r^{1+\upsilon}G(r, \varphi + \pi/2),$$
$$\theta(r, \varphi, t) = p(r, \varphi, t) = 2t(r^2 \sin^2(\varphi) - 1)(r^2 \cos^2(\varphi) - 1)r^{2/3} \sin\big(2/3(\varphi + \pi/2)\big),$$

where

$$G(r, \varphi) = \Big(\frac{1}{\upsilon - 1} \sin\big((\upsilon - 1)\frac{3\pi}{2}\big) - \frac{1}{\upsilon + 1} \sin\big((\upsilon + 1)\frac{3\pi}{2}\big)\Big)\Big(\cos\big((\upsilon - 1)\varphi\big) - \cos((\upsilon + 1)\varphi)\Big)$$

h	$\|e_u\|^{\ell^\infty}$	Rate	$\|\nabla e_u\|^{\ell^\infty}$	Rate	$\|\hat{e}_u\|_h^{\ell^\infty}$	Rate	$\|e_\theta\|^{\ell^\infty}$	Rate	$\|\nabla \hat{e}_\theta\|^{\ell^2}$	Rate	$\|e_p\|^{\ell^\infty}$	Rate	$\|\nabla \hat{e}_p\|^{\ell^2}$	Rate
$\gamma = -1$ (TED)														
0.3536	8.93e-02	$\star$	4.39e-01	$\star$	5.30e+00	$\star$	7.91e-02	$\star$	5.57e-01	$\star$	7.91e-02	$\star$	7.15e-01	$\star$
0.1768	2.99e-02	1.5801	1.48e-01	1.5628	3.36e+00	0.6564	2.11e-02	1.9038	2.86e-01	0.9630	2.11e-02	1.9038	3.69e-01	0.9549
0.0884	8.14e-03	1.8757	4.13e-02	1.8461	1.84e+00	0.8684	5.38e-03	1.9745	1.43e-01	0.9941	5.38e-03	1.9745	1.86e-01	0.9903
0.0442	2.07e-03	1.9753	1.06e-02	1.9588	9.54e-01	0.9496	1.35e-03	1.9935	7.17e-02	0.9995	1.35e-03	1.9935	9.30e-02	0.9981
0.0221	5.12e-04	2.0159	2.64e-03	2.0060	4.84e-01	0.9802	3.38e-04	1.9984	3.58e-02	1.0000	3.38e-04	1.9984	4.65e-02	0.9996
0.0110	1.07e-04	2.2608	5.77e-04	2.1952	2.43e-01	0.9913	8.45e-05	1.9996	1.79e-02	1.0000	8.45e-05	1.9996	2.32e-02	0.9999
$\gamma = 1$ (TPE)														
0.3536	8.92e-02	$\star$	4.38e-01	$\star$	5.30e+00	$\star$	7.91e-02	$\star$	5.57e-01	$\star$	7.91e-02	$\star$	5.57e-01	$\star$
0.1768	2.98e-02	1.5806	1.48e-01	1.5633	3.36e+00	0.6564	2.11e-02	1.9038	2.86e-01	0.9624	2.11e-02	1.9038	2.85e-01	0.9649
0.0884	8.13e-03	1.8760	4.12e-02	1.8463	1.84e+00	0.8684	5.38e-03	1.9745	1.43e-01	0.9943	5.38e-03	1.9745	1.43e-01	0.9940
0.0442	2.07e-03	1.9754	1.06e-02	1.9589	9.54e-01	0.9496	1.35e-03	1.9935	7.17e-02	0.9996	1.35e-03	1.9935	7.17e-02	0.9993
0.0221	5.11e-04	2.0160	2.64e-03	2.0061	4.84e-01	0.9802	3.38e-04	1.9984	3.59e-02	1.0000	3.38e-04	1.9984	3.58e-02	0.9999
0.0110	1.07e-04	2.2609	5.77e-04	2.1952	2.43e-01	0.9913	8.45e-05	1.9996	1.79e-02	1.0000	8.45e-05	1.9996	1.79e-02	1.0000

Table 5.4: Example 2. Error decay with respect to mesh refinement, and convergence rates in the norms (5.8b)-(5.8d) with smooth exact solution. Errors and rates for displacement, temperature, and chemical potential (resp. pore pressure) are represented by black, red, and violet (resp. aquamarine) colors in the background.

$$-\Big(\frac{1}{\upsilon-1}\sin\big((\upsilon-1)\varphi\big)-\frac{1}{\upsilon+1}\sin\big((\upsilon+1)\varphi\big)\Big)\Big(\cos\big((\upsilon-1)3\pi/2\big)-\cos\big((\upsilon+1)3\pi/2\big)\Big).$$

It is easy to check that $u \in C^\infty([0,T]; H^{2+\sigma}(\Omega)\cap H^2_0(\Omega))$, $\theta, p \in C^\infty([0,T]; H^{1+\sigma}(\Omega)\cap H^1_0(\Omega))$ with $\sigma = \upsilon = 0.5444837$ [13, 27]. Then by Theorem 4.4 and Corollary 4.5, the expected convergence rates are of order $\mathcal{O}(h^\sigma)$ for $\|\hat{e}_u\|_h^{\ell^\infty}, \|\nabla\hat{e}_\theta\|^{\ell^2}, \|\nabla\hat{e}_p\|^{\ell^2}$ and $\mathcal{O}(h^{2\sigma})$ for $\|e_u\|^{\ell^\infty}, \|\nabla e_u\|^{\ell^\infty}, \|e_\theta\|^{\ell^\infty}, \|e_p\|^{\ell^\infty}$ norms defined in (5.8). In this case, the model coefficients as well as the used norms are as in Example 2. Furthermore, we take $\Delta t = 1/4$ and the experimental convergence rates are reported in Table 5.5, exhibiting the anticipated behavior.

h	$\|e_u\|^{\ell^\infty}$	Rate	$\|\nabla e_u\|^{\ell^\infty}$	Rate	$\|\hat{e}_u\|_h^{\ell^\infty}$	Rate	$\|e_\theta\|^{\ell^\infty}$	Rate	$\|\nabla \hat{e}_\theta\|^{\ell^2}$	Rate	$\|e_p\|^{\ell^\infty}$	Rate	$\|\nabla \hat{e}_p\|^{\ell^2}$	Rate
$\gamma = -1$ (TED)														
0.7071	3.38e-01	$\star$	1.25e+00	$\star$	9.63e+00	0.0729	1.76e-01	$\star$	1.04e+00	$\star$	1.74e-01	$\star$	1.04e+00	$\star$
0.3536	1.61e-01	1.0685	5.92e-01	1.0788	6.38e+00	0.5932	6.05e-02	1.5427	5.04e-01	1.0466	5.78e-02	1.5927	5.02e-01	1.0509
0.1768	4.56e-02	1.8225	1.68e-01	1.8217	3.27e+00	0.9633	1.74e-02	1.7988	2.56e-01	0.9799	1.65e-02	1.8069	2.55e-01	0.9751
0.0884	1.35e-02	1.7593	4.93e-02	1.7664	1.72e+00	0.9302	5.27e-03	1.7226	1.38e-01	0.8914	5.02e-03	1.7184	1.38e-01	0.8902
0.0442	4.40e-03	1.6138	1.65e-02	1.5811	9.45e-01	0.8629	1.76e-03	1.5774	7.72e-02	0.8357	1.69e-03	1.5736	7.72e-02	0.8354
0.0221	1.61e-03	1.4491	6.35e-03	1.3734	5.49e-01	0.7835	6.65e-04	1.4069	4.45e-02	0.7933	6.34e-04	1.4118	4.45e-02	0.7933
0.0110	6.08e-04	1.4062	2.61e-03	1.2853	3.36e-01	0.7078	3.03e-04	1.1340	2.63e-02	0.7581	2.83e-04	1.1615	2.63e-02	0.7581
$\gamma = 1$ (TPE)														
0.7071	3.38e-01	$\star$	1.25e+00	$\star$	9.63e+00	$\star$	1.77e-01	$\star$	1.04e+00	$\star$	1.75e-01	$\star$	1.04e+00	$\star$
0.3536	1.61e-01	1.0688	5.92e-01	1.0791	6.38e+00	0.5932	6.16e-02	1.5218	5.05e-01	1.0447	5.89e-02	1.5715	5.03e-01	1.0491
0.1768	4.56e-02	1.8228	1.67e-01	1.8219	3.27e+00	0.9634	1.78e-02	1.7958	2.56e-01	0.9820	1.69e-02	1.8036	2.55e-01	0.9771
0.0884	1.35e-02	1.7594	4.92e-02	1.7665	1.72e+00	0.9302	5.37e-03	1.7243	1.38e-01	0.8920	5.12e-03	1.7202	1.38e-01	0.8907
0.0442	4.40e-03	1.6138	1.65e-02	1.5811	9.45e-01	0.8629	1.80e-03	1.5790	7.72e-02	0.8358	1.72e-03	1.5752	7.72e-02	0.8355
0.0221	1.61e-03	1.4491	6.35e-03	1.3734	5.49e-01	0.7835	6.79e-04	1.4049	4.45e-02	0.7933	6.47e-04	1.4097	4.45e-02	0.7933
0.0110	6.08e-04	1.4065	2.61e-03	1.2854	3.36e-01	0.7078	3.12e-04	1.1234	2.63e-02	0.7581	2.92e-04	1.1496	2.63e-02	0.7581

Table 5.5: Example 3. Error history in the norms from (5.8b)–(5.8d) for an L-shaped domain. Errors and rates for displacement, temperature, and chemical potential (resp. pore pressure) are represented by black, red, and violet (resp. aquamarine) colors in the background.

Acknowledgments

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A Appendix

Proof of Theorem 2.1. The proof of existence is presented in Steps 1-4 and that of uniqueness using energy arguments in Step 5.

Step 1 (Construction of a sequence of approximate solutions). It is well known that there exists an orthogonal basis $\{w^1, w^2, \cdots\}$ (resp. $\{y^1, y^2, \cdots\}$) of $H_0^2(\Omega)$ (resp. $H_0^1(\Omega)$) and this also forms an orthonormal basis of $L^2(\Omega)$ [6, 34]. For a fixed integer m, we proceed to write

$$u^m(t) := \sum_{k=1}^{m} \mathrm{d}_m^k(t) w^k, \;\; \theta^m(t) := \sum_{k=1}^{m} \eta_m^k(t) y^k \text{ and } p^m(t) := \sum_{k=1}^{m} \mathrm{l}_m^k(t) y^k, \tag{A.1}$$

where the coefficients $\mathrm{d}_m^k(t)$, $\eta_m^k(t)$ and $\mathrm{l}_m^k(t)$ are selected such that

$$\mathrm{d}_m^k(0) = (u^0, w^k), \qquad \mathrm{d}_m^{k\,\prime}(0) + a_0\|\nabla w^k\|^2 \mathrm{d}_m^{k\,\prime}(0) = (u^{*0}, w^k) + a_0(\nabla u^{*0}, \nabla w^k), \tag{A.2a}$$

$$\eta_m^k(0) = (\theta^0, y^k), \qquad \mathrm{l}_m^k(0) = (p^0, y^k) \tag{A.2b}$$

and

$$(u_{tt}^m, w^k) + a_0(\nabla u_{tt}^m, \nabla w^k) + d_0(\nabla^2 u^m, \nabla^2 w^k) - \alpha(\nabla\theta^m, \nabla w^k) - \beta(\nabla p^m, \nabla w^k) = (f, w^k), \tag{A.3a}$$

$$a_1(\theta_t^m, y^k) - \gamma(p_t^m, y^k) + b_1(\theta^m, y^k) + c_1(\nabla\theta^m, \nabla y^k) + \alpha(\nabla u_t^m, \nabla y^k) = (\phi, y^k), \tag{A.3b}$$

$$a_2(p_t^m, y^k) - \gamma(\theta_t^m, y^k) + \kappa(\nabla p^m, \nabla y^k) + \beta(\nabla u_t^m, \nabla y^k) = (g, y^k), \tag{A.3c}$$

hold for all $0 < t \le T$ and $k = 1, 2\cdots, m$. (Since (A.3) forms a linear ODE system with initial conditions (A.2), standard ODE theory [25], guarantees the existence of unique C^2 (resp. C^1) functions $(\mathrm{d}_m^1(t), \mathrm{d}_m^2(t), \cdots, \mathrm{d}_m^m(t))$ (resp. $(\eta_m^1(t), \eta_m^2(t), \cdots, \eta_m^m(t))$ and $(\mathrm{l}_m^1(t), \mathrm{l}_m^2(t), \cdots, \mathrm{l}_m^m(t))$), that satisfy (A.2)-(A.3) for $0 \le t \le T$.)

Step 2 (Derivation of a priori bounds for approximate solutions). We aim to take the limit $m \to \infty$ and hence shall derive estimates that are uniform with respect to m. Multiply the equations (A.3a), (A.3b), and (A.3c) by $\mathrm{d}_m^{k\,\prime}(t)$, $\eta_m^k(t)$, and $\mathrm{l}_m^k(t)$, respectively and sum up the result for $k = 1, 2, \ldots, m$. Then definitions in (A.1) lead to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|u_t^m\|^2 + a_0\|\nabla u_t^m\|^2 + d_0\|\nabla^2 u^m\|^2 + a_1\|\theta^m\|^2 + a_2\|p^m\|^2\big)\\ &\quad + b_1\|\theta^m\|^2 + c_1\|\nabla\theta^m\|^2 + \kappa\|\nabla p^m\|^2 - \gamma\frac{d}{dt}(\theta^m, p^m) = (f, u_t^m) + (\phi, \theta^m) + (g, p^m).\end{aligned}$$

An integration from 0 to t and simple manipulations show

$$\begin{aligned}&\frac{1}{2}\big(\|u_t^m\|^2 + a_0\|\nabla u_t^m\|^2 + d_0\|\nabla^2 u^m\|^2 + a_1\|\theta^m\|^2 + a_2\|p^m\|^2\big) + \int_0^t \big(b_1\|\theta^m\|^2 + c_1\|\nabla\theta^m\|^2 + \kappa\|\nabla p^m\|^2\big)\,\mathrm{d}s\\ &\quad = \frac{1}{2}\big(\|u_t^m(0)\|^2 + a_0\|\nabla u_t^m(0)\|^2 + d_0\|\nabla^2 u^m(0)\|^2 + a_1\|\theta^m(0)\|^2 + a_2\|p^m(0)\|^2\big)\\ &\qquad + \gamma(\theta^m, p^m) - \gamma(\theta^m(0), p^m(0)) + \int_0^t \big((f, u_t^m) + (\phi, \theta^m) + (g, p^m)\big)\,\mathrm{d}s.\end{aligned} \tag{A.4}$$

An application of Cauchy–Schwarz and Young's inequalities to the last three terms in (A.4) yields

$$\begin{aligned}\gamma(\theta^m, p^m) - \gamma(\theta^m(0), p^m(0)) &+ \int_0^t \big((f, u_t^m) + (\phi, \theta^m) + (g, p^m)\big)\, \mathrm{d}s \\ &\le \frac{|\gamma|}{2\gamma_0}\big(\|\theta^m\|^2 + \|\theta^m(0)\|^2\big) + \frac{|\gamma|\gamma_0}{2}\big(\|p^m\|^2 + \|p^m(0)\|^2\big) \\ &\quad + \frac{1}{2}\int_0^t \big(\|f\|^2 + \|\phi\|^2 + \|g\|^2 + \|u_t^m\|^2 + \|\theta^m\|^2 + \|p^m\|^2\big)\, \mathrm{d}s,\end{aligned} \tag{A.5}$$

where γ_0 is defined in (1.2). Next, substitute (A.5) in (A.4), to obtain

$$\begin{aligned}&\|u_t^m\|^2 + a_0\|\nabla u_t^m\|^2 + d_0\|\nabla^2 u^m\|^2 + (a_1 - |\gamma|/\gamma_0)\|\theta^m\|^2 + (a_2 - |\gamma|\gamma_0)\|p^m\|^2 \\ &\qquad + 2\int_0^t \big(b_1\|\theta^m\|^2 + c_1\|\nabla\theta^m\|^2 + \kappa\|\nabla p^m\|^2\big)\, \mathrm{d}s \\ &\le \|u_t^m(0)\|^2 + a_0\|\nabla u_t^m(0)\|^2 + d_0\|\nabla^2 u^m(0)\|^2 + (a_1 + |\gamma|/\gamma_0)\|\theta^m(0)\|^2 + (a_2 + |\gamma|\gamma_0)\|p^m(0)\|^2 \\ &\qquad + \|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;L^2(\Omega))} + \|g\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^t \big(\|u_t^m\|^2 + \|\theta^m\|^2 + \|p^m\|^2\big)\, \mathrm{d}s.\end{aligned} \tag{A.6}$$

We next utilize (A.2a)-(A.2b) to show

$$\begin{aligned}&\|u_t^m(0)\|^2 + a_0\|\nabla u_t^m(0)\|^2 + d_0\|\nabla^2 u^m(0)\|^2 + (a_1 + |\gamma|/\gamma_0)\|\theta^m(0)\|^2 + (a_2 + |\gamma|\gamma_0)\|p^m(0)\|^2 \\ &\quad \lesssim \|u^{*0}\|^2 + a_0\|u^{*0}\|^2_{H^1(\Omega)} + d_0\|u^0\|^2_{H^2(\Omega)} + (a_1 + |\gamma|/\gamma_0)\|\theta^0\|^2 + (a_2 + |\gamma|\gamma_0)\|p^0\|^2.\end{aligned} \tag{A.7}$$

A combination of (A.6)-(A.7) and an application of Lemma 1.1 lead to the bound

$$\begin{aligned}2E(u^m, \theta^m, p^m; t) &= \|u_t^m\|^2 + a_0\|\nabla u_t^m\|^2 + d_0\|\nabla^2 u^m\|^2 + (a_1 - |\gamma|/\gamma_0)\|\theta^m\|^2 + (a_2 - |\gamma|\gamma_0)\|p^m\|^2 \\ &\quad + 2\int_0^t \big(b_1\|\theta^m\|^2 + c_1\|\nabla\theta^m\|^2 + \kappa\|\nabla p^m\|^2\big)\, \mathrm{d}s \\ &\lesssim \|u^{*0}\|^2 + a_0\|u^{*0}\|^2_{H^1(\Omega)} + d_0\|u^0\|^2_{H^2(\Omega)} + (a_1 + |\gamma|/\gamma_0)\|\theta^0\|^2 + (a_2 + |\gamma|\gamma_0)\|p^0\|^2 \\ &\qquad + \|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;L^2(\Omega))} + \|g\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{A.8}$$

Now, fix any $v \in H_0^2(\Omega)$ and $\psi, q \in H_0^1(\Omega)$ with $\|v\|_{H_0^2(\Omega)} \le 1$, $\|\psi\|_{H_0^1(\Omega)} \le 1$, and $\|q\|_{H_0^1(\Omega)} \le 1$. Write $v = v_1 + v_2$, $\psi = \psi_1 + \psi_2$ and $q = q_1 + q_2$, where $v_1 \in \text{span}\,\{w^k\}_{k=1}^m$, and both $\psi_1, q_1 \in \text{span}\,\{y^k\}_{k=1}^m$ with $(v_2, w^k) = (\nabla v_2, \nabla w^k) = (\psi_2, y^k) = (q_2, y^k) = 0$ $(k = 1, 2, \cdots, m)$. Note $\|v_1\|_{H_0^2(\Omega)} \le 1$, $\|\psi_1\|_{H_0^1(\Omega)} \le 1$, and $\|q_1\|_{H_0^1(\Omega)} \le 1$. Then (A.1) and (A.3a) imply that

$$\begin{aligned}\langle u_{tt}^m, v\rangle + a_0\langle\nabla u_{tt}^m, \nabla v\rangle &= (u_{tt}^m, v) + a_0(\nabla u_{tt}^m, \nabla v) = (u_{tt}^m, v_1) + a_0(\nabla u_{tt}^m, \nabla v_1) \\ &= (f, v_1) - d_0(\nabla^2 u^m, \nabla^2 v_1) + \alpha(\nabla\theta^m, \nabla v_1) + \beta(\nabla p^m, \nabla v_1).\end{aligned} \tag{A.9}$$

This and a Cauchy–Schwarz inequality reveals

$$\|u_{tt}^m\|_{H^{-1}(\Omega)} \lesssim |\langle u_{tt}^m, v\rangle + \langle\nabla u_{tt}^m, \nabla v\rangle| \lesssim |\langle u_{tt}^m, v\rangle + a_0\langle\nabla u_{tt}^m, \nabla v\rangle| \lesssim \|f\| + d_0\|\nabla^2 u^m\| + \alpha\|\nabla\theta^m\| + \beta\|\nabla p^m\|.$$

An integration from 0 to T and the bounds from (A.8) allow us to assert that

$$\begin{aligned}\int_0^T \|u_{tt}^m\|_{H^{-1}(\Omega)}\, \mathrm{d}t &\lesssim \|u^{*0}\|^2 + a_0\|u^{*0}\|^2_{H^1(\Omega)} + d_0\|u^0\|^2_{H^2(\Omega)} + (a_1 + |\gamma|/\gamma_0)\|\theta^0\|^2 + (a_2 + |\gamma|\gamma_0)\|p^0\|^2 \\ &\quad + \|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;L^2(\Omega))} + \|g\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{A.10}$$

On the other hand, the combination of (A.1) and (A.3b)–(A.3c) with similar arguments as in (A.9), yields

$$\begin{aligned}&a_1\langle\theta_t^m, \psi\rangle - \gamma\langle p_t^m, \psi\rangle = a_1(\theta_t^m, \psi_1) - \gamma(p_t^m, \psi_1) = (\phi, \psi_1) - b_1(\theta^m, \psi_1) - c_1(\nabla\theta^m, \nabla\psi_1) - \alpha(\nabla u_t^m, \nabla y^k), \\ &a_2\langle p_t^m, q\rangle - \gamma\langle\theta_t^m, q\rangle = a_2(p_t^m, q_1) - \gamma(\theta_t^m, q_1) = (g, q_1) - \kappa(\nabla p^m, \nabla q_1) - \beta(\nabla u_t^m, \nabla q_1).\end{aligned}$$

Applying Cauchy–Schwarz inequality again (shifting the second terms from left to right-hand side in both inequalities and using $|\gamma\langle p_t^m, \psi\rangle| \le |\gamma| \|p_t^m\|_{H^{-1}(\Omega)}$ and $|\gamma\langle \theta_t^m, q\rangle| \le |\gamma| \|\theta_t^m\|_{H^{-1}(\Omega)}$, we readily get

$$
\begin{aligned}
a_1 \|\theta_t^m\|_{H^{-1}(\Omega)} &\le |\gamma| \|p_t^m\|_{H^{-1}(\Omega)} + C\Big(\|\phi\| + b_1 \|\theta^m\| + c_1 \|\nabla\theta^m\| + \alpha \|\nabla u_t^m\| \Big),\\
a_2 \|p_t^m\|_{H^{-1}(\Omega)} &\le |\gamma| \|\theta_t^m\|_{H^{-1}(\Omega)} + C\Big(\|g\| + \kappa \|\nabla p^m\| + \beta \|\nabla u_t^m\| \Big).
\end{aligned}
$$

Next, we multiply the first equation above by ${\gamma_0}^{1/2}$, the second by ${\gamma_0}^{-1/2}$, and add the two inequalities. Again, applying integration from 0 to T and the bounds from (A.8), lead to

$$
\begin{aligned}
&{\gamma_0}^{1/2}(a_1 - |\gamma|/\gamma_0) \int_0^T \|\theta_t^m\|_{H^{-1}(\Omega)} \, \mathrm{d}t + \gamma_0^{-1/2}(a_2 - |\gamma|\gamma_0) \int_0^T \|p_t^m\|_{H^{-1}(\Omega)} \, \mathrm{d}t\\
&\lesssim \|u^{*0}\|^2 + a_0 \|u^{*0}\|^2_{H^1(\Omega)} + d_0 \|u^0\|^2_{H^2(\Omega)} + (a_1 + |\gamma|/\gamma_0)\|\theta^0\|^2 + (a_2 + |\gamma|\gamma_0)\|p^0\|^2\\
&\qquad + \|f\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;L^2(\Omega))} + \|g\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned} \tag{A.11}
$$

Step 3 (Existence of a limit for a subsequences). The estimates in (A.8) indicate that $\{u^m\}_{m=1}^\infty$ and $\{u_t^m\}_{m=1}^\infty$ are bounded in the spaces $L^\infty(0,T; H_0^2(\Omega))$ and $L^\infty([0,T]; H_0^1(\Omega))$, respectively, and both $\{\theta^m\}_{m=1}^\infty$ and $\{p^m\}_{m=1}^\infty$ are bounded in $L^\infty(0,T; L^2(\Omega))$ as well as in $L^2(0,T; H_0^1(\Omega))$. Moreover, the estimates in (A.10) and (A.11) reveal that $\{u_{tt}^m\}_{m=1}^\infty$, $\{\theta_t^m\}_{m=1}^\infty$ and $\{p_t^m\}_{m=1}^\infty$ are bounded in $L^2(0,T; H^{-1}(\Omega))$. Consequently, there exist subsequences $\{u^m\}_{m=1}^\infty$, $\{\theta^m\}_{m=1}^\infty$, and $\{p^m\}_{m=1}^\infty$ (where re-labelling is used), and some $u \in L^\infty(0,T; H_0^2(\Omega))$ with $u_t \in L^\infty(0,T; H_0^1(\Omega))$, and $u_{tt} \in L^2(0,T; H^{-1}(\Omega))$, $\theta, p \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, and $\theta_t, p_t \in L^2(0,T; H^{-1}(\Omega))$ such that

$$
(u^m, u_t^m, \theta^m, p^m) \xrightarrow{\text{weak*}} (u, u_t, \theta, p) \quad \text{in } L^\infty\big(0,T; H_0^2(\Omega) \times H_0^1(\Omega) \times (L^2(\Omega))^2\big), \tag{A.12a}
$$

$$
(\theta^m, p^m) \xrightarrow{\text{weak}} (\theta, p) \quad \text{in } L^2\big(0,T; (H_0^1(\Omega))^2\big), \tag{A.12b}
$$

$$
(u_{tt}^m, \theta_t^m, p_t^m) \xrightarrow{\text{weak}} (u_{tt}, \theta_t, p_t) \quad \text{in } L^2\big(0,T; (H^{-1}(\Omega))^3\big). \tag{A.12c}
$$

Step 4 (Limit is a weak solution). Now we show that (u, θ, p) satisfies (2.2a)-(2.2c). For this, we introduce $\widehat{\mathrm{d}}_{j_0}^k(t) \in C^2[0,T]$, $\widehat{\eta}_{j_0}^k(t)$ and $\widehat{\mathrm{l}}_{j_0}^k(t) \in C^1[0,T]$ such that $\widehat{\mathrm{d}}_{j_0}^k(T) = \widehat{\mathrm{d}}_{j_0 t}^k(T) = \widehat{\eta}_{j_0}^k(T) = \widehat{\mathrm{l}}_{j_0}^k(T) = 0$, and define

$$
\widehat{u}^{j_0} := \sum_k^{j_0} \widehat{\mathrm{d}}_{j_0}^k(t) w^k, \ \widehat{\theta}^{j_0} := \sum_k^{j_0} \widehat{\eta}_{j_0}^k(t) y^k, \text{ and } \widehat{p}^{j_0} := \sum_k^{j_0} \widehat{\mathrm{l}}_{j_0}^k(t) y^k. \tag{A.13}
$$

Multiply (A.3a) by $\widehat{\mathrm{d}}_{j_0}^k(t)$, (A.3b) by $\widehat{\eta}_{j_0}^k(t)$, and (A.3c) by $\widehat{\mathrm{l}}_{j_0}^k(t)$, add the resulting equations for $k = 1, 2, \dots, j_0$, and integrate by parts in t from 0 to T, to obtain

$$
\begin{aligned}
&-\int_0^T (u_t^m, \widehat{u}_t^{j_0}) \, \mathrm{d}t - a_0 \int_0^T (\nabla u_t^m, \nabla \widehat{u}_t^{j_0}) \, \mathrm{d}t + d_0 \int_0^T (\nabla^2 u^m, \nabla^2 \widehat{u}^{j_0}) \, \mathrm{d}t - \alpha \int_0^T (\nabla\theta^m, \nabla\widehat{u}^{j_0}) \, \mathrm{d}t - \beta \int_0^T (\nabla p^m, \nabla \widehat{u}^{j_0}) \, \mathrm{d}t\\
&\quad = \int_0^T (f, \widehat{u}^{j_0}) \, \mathrm{d}t + (u_t^m(0), \widehat{u}^{j_0}(0)) + a_0(\nabla u_t^m(0), \nabla\widehat{u}^{j_0}(0)),\\
&- a_1 \int_0^T (\theta^m, \widehat{\theta}_t^{j_0}) \, \mathrm{d}t + \gamma \int_0^T (p^m, \widehat{\theta}_t^{j_0}) \, \mathrm{d}t + b_1 \int_0^T (\theta^m, \widehat{\theta}^{j_0}) \, \mathrm{d}t + c_1 \int_0^T (\nabla\theta^m, \nabla\widehat{\theta}^{j_0}) \, \mathrm{d}t + \alpha \int_0^T (\nabla u_t^m, \nabla\widehat{\theta}^{j_0}) \, \mathrm{d}t\\
&\quad = \int_0^T (\phi, \widehat{\theta}^{j_0}) \, \mathrm{d}t + a_1(\theta^m(0), \widehat{\theta}_t^{j_0}(0)) - \gamma(p^m(0), \widehat{\theta}^{j_0}(0)),\\
&- a_2 \int_0^T (p^m, \widehat{p}_t^{j_0}) \, \mathrm{d}t + \gamma \int_0^T (\theta^m, \widehat{p}_t^{j_0}) \, \mathrm{d}t + \kappa \int_0^T (\nabla p^m, \nabla\widehat{p}^{j_0}) \, \mathrm{d}t + \beta \int_0^T (\nabla u_t^m, \nabla\widehat{p}^{j_0}) \, \mathrm{d}t\\
&\quad = \int_0^T (g, \widehat{p}^{j_0}) \, \mathrm{d}t + a_2(p^m(0), \widehat{p}_t^{j_0}(0)) - \gamma(\theta^m(0), \widehat{p}^{j_0}(0)),
\end{aligned}
$$

where we have utilized that $\widehat{\mathrm{d}}_{j_0}^k(t)$, $\widehat{\eta}_{j_0}^k(t)$ and $\widehat{\mathrm{l}}_{j_0}^k(t)$ are such that $\widehat{u}^{j_0}(T) = 0$, $\widehat{\theta}^{j_0}(T) = 0$ and $\widehat{p}^{j_0}(T) = 0$.

Then we invoke (A.12a) and (A.12b) to pass to the limit as $m \to \infty$ in the final system of equations. Also, since the functions in (A.13) are dense in $C^2([0,T]; H_0^2(\Omega))$, $C^1([0,T]; H_0^1(\Omega))$, and $C^1([0,T]; H_0^1(\Omega))$, respectively; we can observe that u, θ, p satisfy (2.2a)-(2.2c). Moreover, the regularity stated in (2.1) is guaranteed by (A.12) and [25, Theorem 3, p. 287]. To ensure that (1.1e) holds we can follow verbatim [25, p. 384], and omit further details. We therefore establish the existence of weak solution to (1.1a)-(1.1c), and the bounds (2.4) are a consequence of passing to the limit as m tends to infinity in (A.8), and utilizing (A.12).

Step 5 (Uniqueness). The uniqueness of solution to the coupled system (2.2a)-(2.2c) (under the data regularity provided in the first part of the proof above) was still an open problem as in [5,51], and it is not trivial. However, the uniqueness of solution to the uncoupled system – under the same data assumptions – can be proved using [25, Section 7.1.2-Theorem 4, Section 7.2.1-Theorem 4]. To this end, we follow the approach in [42] to construct mollified test functions that possess sufficient regularity and are compactly supported in the time interval $[0,T]$.

Let us define $\rho_\varepsilon(s) = \varepsilon^{-1}\rho(\varepsilon^{-1}s)$ for $\varepsilon > 0$, where $\rho(t)$ is a function in $C_0^\infty(\mathbb{R})$ satisfying

$$\rho \geq 0, \ \operatorname{supp}\rho \subset [-2,-1], \ \text{and} \ \int_{-\infty}^{\infty} \rho(s)\,\mathrm{d}s = 1.$$

Let us take $v \in H_0^2(\Omega)$, $\psi \in H_0^1(\Omega)$, and $q \in H_0^1(\Omega)$ and further denote

$$\tilde{v}^t(s,\boldsymbol{x}) = \rho_\varepsilon(t-s)v(\boldsymbol{x}), \ \tilde{\psi}^t(s,\boldsymbol{x}) = \rho_\varepsilon(t-s)\psi(\boldsymbol{x}), \ \tilde{q}^t(s,\boldsymbol{x}) = \rho_\varepsilon(t-s)q(\boldsymbol{x}) \text{ for } t \in [0,T].$$

Clearly, $\tilde{v}^t \in C^\infty([0,T]; H_0^2(\Omega))$, $\tilde{\psi}^t \in C^\infty([0,T]; H_0^1(\Omega))$, $\tilde{q}^t \in C^\infty([0,T]; H_0^1(\Omega))$ for $0 \leq t \leq T$. Substituting $\tilde{v}^t$, $\tilde{\psi}^t$, and $\tilde{\psi}^t$ in (2.2a)-(2.2c), and noting that $\rho_\varepsilon(t) = (d/dt)\rho_\varepsilon(t) = 0$ for $0 \leq t \leq T$, yields

$$(u_{\varepsilon tt}(t,\cdot), v) + a_0(\nabla u_{\varepsilon tt}(t,\cdot), \nabla v) + d_0(\nabla^2 u_\varepsilon(t,\cdot), \nabla^2 v) - \alpha(\nabla\theta_\varepsilon(t,\cdot), \nabla v) - \beta(\nabla p_\varepsilon(t,\cdot), \nabla v) = 0, \tag{A.14a}$$

$$a_1(\theta_{\varepsilon t}(t,\cdot), \psi) - \gamma(p_{\varepsilon t}(t,\cdot), \psi) + b_1(\theta_\varepsilon(t,\cdot), \psi) + c_1(\nabla\theta_\varepsilon(t,\cdot), \nabla\psi) + \alpha(\nabla u_{\varepsilon t}(t,\cdot), \nabla\psi) = 0, \tag{A.14b}$$

$$a_2(p_{\varepsilon t}(t,\cdot), q) - \gamma(\theta_{\varepsilon t}(t,\cdot), q) + \kappa(\nabla p_\varepsilon(t,\cdot), \nabla q) + \beta(\nabla u_{\varepsilon t}(t,\cdot), \nabla q) = 0, \tag{A.14c}$$

for any $0 \leq t \leq T$, $v \in H_0^2(\Omega)$, $\psi \in H_0^1(\Omega)$, and $q \in H_0^1(\Omega)$, where

$$u_\varepsilon(t,\boldsymbol{x}) = \int_{-\infty}^{\infty} \rho_\varepsilon(t-s)u(s,\boldsymbol{x})\,\mathrm{d}s, \ \theta_\varepsilon(t,\boldsymbol{x}) = \int_{-\infty}^{\infty} \rho_\varepsilon(t-s)\theta(s,\boldsymbol{x})\,\mathrm{d}s, \ \text{and } p_\varepsilon(t,\boldsymbol{x}) = \int_{-\infty}^{\infty} \rho_\varepsilon(t-s)p(s,\boldsymbol{x})\,\mathrm{d}s.$$

From (A.12a)-(A.12b), it follows that

$$u_{\varepsilon t} \in C^\infty([0,T]; H_0^2(\Omega)), \ \theta_\varepsilon \in C^\infty([0,T]; H_0^1(\Omega)), \ \text{and } p_\varepsilon \in C^\infty([0,T]; H_0^1(\Omega)).$$

For all $t \in [0,T]$, we choose the test functions in (A.14a)-(A.14c) as $v = u_{\varepsilon t}(t,\cdot)$, $\psi = \theta_\varepsilon(t,\cdot)$, and $p = p_\varepsilon(t,\cdot)$, and integrate the resulting equations with respect to t, and follow arguments analogous to Step 2 to show

$$0 \leq E(u_\varepsilon, \theta_\varepsilon, p_\varepsilon; t) \leq 0.$$

(The system (A.14a)-(A.14c) is similar to (A.3a)-(A.3c) with $f = \phi = g = 0$ and zero initial conditions).
Now take the limit as $\varepsilon \to 0$ to obtain

$$E(u, \theta, p; t) = 0,$$

which shows, directly from (2.3), that $u = 0$, $\theta = 0$, and $p = 0$. This completes the proof. □

Proof of Theorem 2.3.
Let $(\tilde{u}, \tilde{\theta}, \tilde{p})$ be the solution to (1.1a)-(1.1c) (its existence is guaranteed by Theorem 2.2) satisfying (2.6a)-(2.6c), and consider the initial conditions

$$\tilde{u}(0,\boldsymbol{x}) = u^{*0}(\boldsymbol{x}) \in H^3(\Omega) \cap H_0^2(\Omega), \ \tilde{u}_t(0,\boldsymbol{x}) = u_{tt}(0) \in H^2(\Omega) \cap H_0^1(\Omega),$$
$$\tilde{\theta}(0,\boldsymbol{x}) = \theta_t(0) \in H^2(\Omega) \cap H_0^1(\Omega), \ \tilde{p}(0,\boldsymbol{x}) = p_t(0) \in H^2(\Omega) \cap H_0^1(\Omega).$$

We then write

$$u(t,\boldsymbol{x}) = u^0 + \int_0^t \tilde{u}(s,\boldsymbol{x})\,\mathrm{d}s, \ \theta(t,\boldsymbol{x}) = \theta^0 + \int_0^t \tilde{\theta}(s,\boldsymbol{x})\,\mathrm{d}s, \ \text{and } p(t,\boldsymbol{x}) = p^0 + \int_0^t \tilde{p}(s,\boldsymbol{x})\,\mathrm{d}s,$$

and can readily employ similar arguments as in Theorem 2.2 and (2.10)-(2.11) to obtain the bounds (2.13). Part (b) follows by analogous arguments.