A mixed-primal finite element method for the coupling of Brinkman–Darcy flow and nonlinear transport

MARIO ALVAREZ

Sección de Matemática, Sede de Occidente, Universidad de Costa Rica,
San Ramón de Alajuela,111-4250 Costa Rica
mario.alvarezguadamuz@ucr.ac.cr

GABRIEL N. GATICA

CI²MA and Departamento de Ingeniería Matemática, Universidad de Concepción,
Casilla 160-C, Concepción, Chile
ggatica@ci2ma.udec.cl

AND

RICARDO RUIZ-BAIER*

Mathematical Institute, University of Oxford, Woodstock Road, OX2 6GG Oxford, UK; and
Universidad Adventista de Chile, Casilla 7-D, Chillán, Chile. Present address: School of
Mathematical Sciences, Monash University, Clayton VIC 3800, Australia
*Corresponding author: ruizbaier@maths.ox.ac.uk

This paper is devoted to the mathematical and numerical analysis of a model describing the interfacial flow-transport interaction in a porous-fluidic domain. The medium consists of a highly permeable material, where the flow of an incompressible viscous fluid is governed by Brinkman equations (written in terms of vorticity, velocity and pressure), and a porous medium where Darcy's law describes fluid motion using filtration velocity and pressure. Gravity and the local fluctuations of a scalar field (representing for instance, the solids volume fraction or the concentration of a contaminant) are the main drivers of the fluid patterns on the whole domain, and the Brinkman-Darcy equations are coupled to a nonlinear transport equation accounting for mass balance of the scalar concentration. We introduce a mixed-primal variational formulation of the problem and establish existence and uniqueness of solution using fixed-point arguments and small-data assumptions. A family of Galerkin discretizations that produce divergence-free discrete velocities is also presented and analysed using similar tools to those employed in the continuous problem. Convergence of the resulting mixed-primal finite element method is proven, and some numerical examples confirming the theoretical error bounds and illustrating the performance of the proposed discrete scheme are reported.

Keywords: nonlinear transport; Brinkman–Darcy coupling; vorticity-based formulation; fixed-point theory; mixed finite elements; error analysis.

1. Introduction

The aim of this paper is to put together an extension of the results from Alvarez *et al.* (2015, 2016a) and Alvarez *et al.* (2016b) dealing with augmented and fully mixed finite element approximations of coupled flow and transport problems, and coupled Brinkman and Darcy flow, respectively. The

coupled system describes the interaction of flow and transport phenomena in two different domains separated by an interface. Such a formalism arises naturally and has been systematically used, in hydrology and biological applications, including for instance subsurface flow, hydraulic fractures, CO_2 sequestration, perfusion of soft living tissues, etc. In obtaining approximate solutions for the problem under consideration one faces marked difficulties. These are related to the coupling of mechanisms that act simultaneously, such as active transport and reaction of the solute and nonlinearities in the diffusion process and in the source term, as well as the heterogeneities, the transmission conditions and the need of preserving physical properties. Even if many numerical solutions are already available (see, e.g., Khaled & Vafai, 2003; Ehrhardt *et al.*, 2009; Joodi *et al.*, 2010; Jena *et al.*, 2013 and the references therein) up to the authors' knowledge the only contributions addressing mathematical and numerical properties of somewhat similar couplings are the recent works of Çeşmelioğlu & Rivière (2012), where existence and stability bounds of weak solutions are established also for the nonlinear Navier–Stokes–Darcy flow coupled with transport; Vassilev & Yotov (2009), where a mixed finite element scheme approximates the Stokes–Darcy system and a local discontinuous Galerkin method is employed to discretize the transport equation; Rui & Zhang (2017) that analyses stabilized velocity–pressure–concentration formulations for a model where viscosity depends on the solute concentration; and the very recent works of Ervin *et al.* (2019) and Zhang *et al.* (2019) focusing on discrete analysis of splitting of time stepping between the conforming discretizations on the two subdomains.

The main difference of these works with respect to our contribution is that we propose a formulation of the problem written in terms of Brinkman vorticity and the transmission conditions we employ are slightly different. In addition, we introduce a mixed-primal finite element method for the Brinkman–Darcy-transport coupling that produces divergence-free discrete velocities. Another clear motivation for using a vorticity-based formulation in the Brinkman domain is that the method is pressure-robust; it computes the vorticity vector directly and avoiding derivative-based postprocessing (this field is of utmost importance in determining properties for a variety of regimes of incompressible fluids, e.g., in subsurface flow or blood flow, or other applications where the patterns of rotational flow are of interest, Speziale, 1987; Pontrelli, 2001; Riaz *et al.*, 2006) and is of relatively low computational cost when compared to other mixed formulations using for instance vorticity tensors. These features are essentially inherited from other schemes such as the methods for Stokes or Brinkman equations advanced in Anaya *et al.* (2013, 2016) and Vassilevski & Villa (2014).

Following our previous work (Alvarez *et al.*, 2016b) the coupling of subdomains is based on a vorticity-based fully mixed formulation for the Brinkman–Darcy problem, whereas a primal formulation for the transport problem is adapted from Alvarez *et al.* (2016a). The solvability of such a coupling will be based on extending the fixed-point strategy introduced in Alvarez *et al.* (2015) and Alvarez *et al.* (2016a) to the present context. In particular, we realize that the primal formulation for the transport problem requires further regularity for the global velocity, initially living in $\mathbf{H}(\mathrm{div}, \Omega)$. In turn, and in contrast with Alvarez *et al.* (2015) and Alvarez *et al.* (2016a), we cannot exploit augmentation techniques to recover $\mathbf{H}^1(\Omega)$ velocities. Instead, a different smoothness assumption is introduced at the level of the continuous analysis of the transport problem and subsequently in the solvability of the Brinkman–Darcy-transport coupling. More precisely, the derivation of existence of weak solutions relies on a strategy combining classical fixed-point arguments, suitable regularity assumptions on the decoupled problems, the Lax–Milgram lemma, preliminary results from Alvarez *et al.* (2016b) and the Sobolev embedding and Rellich–Kondrachov compactness theorems. In addition, sufficiently small data allow us to establish uniqueness of weak solution. On the other hand, the well posedness of the discrete problem is based on the Brouwer fixed-point theorem and analogous arguments to those employed in the continuous analysis. Finally, similar arguments as those utilized in Alvarez *et al.* (2016a,b) allow

us to derive the corresponding Céa estimates for both the Brinkman–Darcy and transport problems, and these lead to natural *a priori* error bounds for the Galerkin scheme.

Outline. This paper has been structured as follows. The remainder of this section presents some notation and preliminary definitions of spaces needed thereafter. The model problem along with boundary data are stated in Section 2. The weak formulation of the problem and its well-posedness analysis in the framework of the Schauder fixed-point theorem are collected in Section 3. The associated Galerkin scheme is then proposed in Section 4 and its solvability is established by the Brouwer fixed-point theorem. Next, we derive in Section 5 some *a priori* error estimates and conclude in Section 6 with a few numerical examples in 2D and 3D, illustrating the good performance of the mixed-primal finite element method and confirming the expected error decay.

Preliminaries. Standard notation will be adopted for Lebesgue and Sobolev spaces. In addition, by $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. For instance, if $\Theta \subseteq \mathrm{R}^n$, $n = 2, 3$ is a domain, $\Lambda \subseteq \mathrm{R}^n$ is a Lipschitz surface and $r \in \mathrm{R}$; we define $\mathbf{H}^r(\Theta) := [\mathrm{H}^r(\Theta)]^n$ and $\mathbf{H}^r(\Lambda) := [\mathrm{H}^r(\Lambda)]^n$. We also recall the definition of the following Hilbert spaces:

$$\mathbf{H}(\mathrm{div}; \Theta) := \left\{\boldsymbol{v} \in \mathbf{L}^2(\Theta) : \quad \mathrm{div}\, \boldsymbol{v} \in \mathrm{L}^2(\Theta)\right\}, \quad \mathbf{H}(\mathbf{curl}; \Theta) := \left\{\boldsymbol{v} \in \mathbf{L}^2(\Theta) : \quad \mathbf{curl}\, \boldsymbol{v} \in \mathrm{L}^2(\Theta)\right\},$$

normed, respectively, with

$$\|\boldsymbol{v}\|_{\mathrm{div};\Theta} := \left\{\|\boldsymbol{v}\|_{0,\Theta}^2 + \|\mathrm{div}\, \boldsymbol{v}\|_{0,\Theta}^2\right\}^{1/2}, \quad \|\boldsymbol{v}\|_{\mathbf{curl};\Theta} := \left\{\|\boldsymbol{v}\|_{0,\Theta}^2 + \|\mathbf{curl}\, \boldsymbol{v}\|_{0,\Theta}^2\right\}^{1/2},$$

where, for any vector field $\boldsymbol{v} := (v_1, \ldots, v_d)^{\mathrm{t}} \in \mathbf{L}^2(\Theta)$,

$$\mathrm{div}\, \boldsymbol{v} := \sum_{i=1}^n \partial_i v_i, \quad \mathbf{curl}\, \boldsymbol{v} := \nabla \times \boldsymbol{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix} \text{ if } n = 3, \text{ and curl } \boldsymbol{v} = \partial_1 v_2 - \partial_2 v_1 \text{ if } n = 2.$$

In addition, we also recall the orthogonal decomposition

$$\mathrm{L}^2(\Theta) = \mathrm{L}_0^2(\Theta) \oplus P_0(\Theta),$$

where $P_0(\Theta)$ is the space of constant functions on Θ and

$$\mathrm{L}_0^2(\Theta) = P_0(\Theta)^\perp := \left\{q \in \mathrm{L}^2(\Theta) : \quad \int_\Theta q = 0\right\}.$$

Equivalently, each $q \in \mathrm{L}^2(\Theta)$ can be uniquely decomposed as $q = q_0 + c$, with

$$q_0 := q - \frac{1}{|\Theta|} \int_\Theta q \in \mathrm{L}_0^2(\Theta) \quad \text{and} \quad c := \frac{1}{|\Theta|} \int_\Theta q \in \mathrm{R},$$

where $L_0^2(\Theta)$ is endowed with the usual norm of $L^2(\Theta)$ and it is easy to see that there holds

$$\|q\|_{0,\Theta}^2 \;=\; \|q_0\|_{0,\Theta}^2 + |\Theta|\,c^2.$$

By **0** we will denote the generic null vector (including the null functional and operator), and we will denote by C and c, with or without subscripts, bars, tildes or hats, generic constants independent of the discretization parameters.

2. Governing equations

Let $\Omega \subset \mathrm{R}^n$, $n = 2, 3$, denote a heterogeneous porous domain composed of two regions: Ω_B, where the viscous flow patterns characterized by velocity $\boldsymbol{u}_B$, vorticity $\boldsymbol{\omega}_B$ and pressure p_B can be governed by the linear Brinkman equations; and Ω_D, where the flow of the immiscible fluid obeys to Darcy's law written in terms of velocity $\boldsymbol{u}_D$ and pressure p_D in the porous domain. These subdomains are separated by an interface Σ, through which exchange of fluid velocities and pressures occurs. We also consider that a given scalar field ϕ (representing, for instance, the concentration of a chemical component, the fluid density, the temperature or the volume fraction or saturation of a solid phase) is advected and diffused on the whole Ω according to the mass conservation principle (or energy conservation if the scalar field stands for, e.g., temperature). The model problem can be summarized as follows:

$$(\text{Brinkman}) \qquad \left.\begin{aligned} \mu\mathbb{K}_B^{-1}\boldsymbol{u}_B + \mu\,\mathbf{curl}\,\boldsymbol{\omega}_B + \nabla p_B &= \phi\boldsymbol{f}_B \\ \boldsymbol{\omega}_B - \mathbf{curl}\,\boldsymbol{u}_B &= \mathbf{0} \\ \operatorname{div}\boldsymbol{u}_B &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_B, \qquad (2.1)$$

coupled with

$$(\text{Darcy}) \qquad \left.\begin{aligned} \mu\mathbb{K}_D^{-1}\boldsymbol{u}_D + \nabla p_D &= \phi\boldsymbol{f}_D \\ \operatorname{div}\boldsymbol{u}_D &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_D, \qquad (2.2)$$

and

$$(\text{Transport}) \qquad \beta\phi - \operatorname{div}(\vartheta(\phi)\nabla\phi - \phi\boldsymbol{u} - f_{bk}(\phi)\boldsymbol{g}) = 0 \quad \text{in} \quad \Omega, \qquad (2.3)$$

where $\mu > 0$ is the constant viscosity of the fluid in the entire domain Ω, and the parameter β is the porosity of the medium (assumed constant inside each subdomain, but possibly discontinuous across Σ). Notice that $\boldsymbol{u}$ in (2.3) refers to the global velocity field defined in both Ω_B and Ω_D that is $\boldsymbol{u} := \mathbf{1}_{\Omega_B}\boldsymbol{u}_B + \mathbf{1}_{\Omega_D}\boldsymbol{u}_D$, where $\mathbf{1}_{\Omega_\star}$ is the characteristic function, $\star \in \{B, D\}$. In addition, $\mathbb{K}_B$ and $\mathbb{K}_D$ are symmetric, bounded and uniformly positive definite tensors $\mathbb{K}_B, \mathbb{K}_D$, which means that there exist $\alpha_{\mathbb{K}_B} > 0$ and $\alpha_{\mathbb{K}_D} > 0$ such that

$$\boldsymbol{v}^{\mathrm{t}}\mathbb{K}_B^{-1}(\boldsymbol{x})\boldsymbol{v} \;\geq\; \alpha_{\mathbb{K}_B}|\boldsymbol{v}|^2 \quad \forall\,\boldsymbol{v} \in \mathrm{R}^n, \forall\,\boldsymbol{x} \in \Omega_B,$$

and

$$\boldsymbol{v}^{\mathrm{t}}\mathbb{K}_D^{-1}(\boldsymbol{x})\boldsymbol{v} \;\geq\; \alpha_{\mathbb{K}_D}|\boldsymbol{v}|^2 \quad \forall\,\boldsymbol{v} \in \mathrm{R}^n, \forall\,\boldsymbol{x} \in \Omega_D.$$

In turn, the tensors $\mathbb{K}_B$ and $\mathbb{K}_D$ characterize the absolute permeability of the Brinkman and Darcy domains, respectively; the function ϑ is a nonlinear diffusivity, and f_{bk} is a nonlinear flux acting on the direction of the gravity acceleration $\boldsymbol{g}$, aligned with the negative x_n−axis. The specific forms of these variable coefficients will be made precise later. In addition, we assume that $\boldsymbol{f}_B \in \mathbf{L}^\infty(\Omega_B)$ and
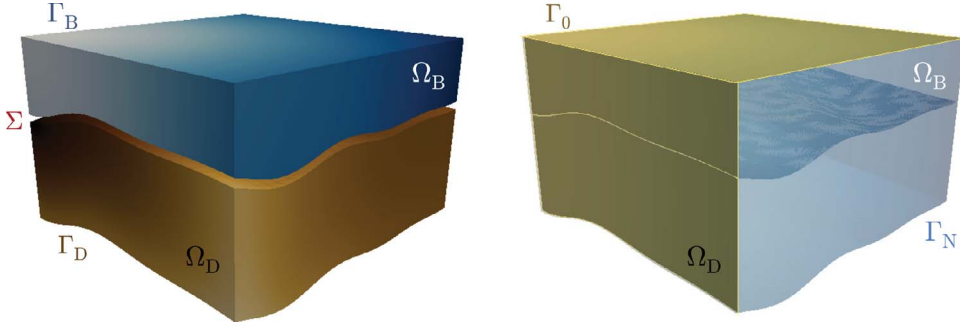
Fig. 1. Sketch of the domains occupied by the incompressible fluid and by the porous medium (Ω_B and Ω_D, respectively), interface Σ and corresponding boundaries.

$f_D \in \mathbf{L}^\infty(\Omega_D)$. We stress that the local fluctuations of ϕ drive the flow patterns only through the external load in the momentum equations. In this sense, the coupling mechanisms considered here are somehow weaker than those studied in Alvarez *et al.* (2015, 2016a) for transport flow in a single domain (where also viscosity was depending on ϕ).

We assume that Ω has a Lipschitz continuous boundary split into two disjoint sub-boundaries with positive measure, according to two criteria: first, $\partial\Omega = \Gamma_B \cup \Gamma_D$, where $\Gamma_B = \partial\Omega_B \setminus \Sigma$ and $\Gamma_D = \partial\Omega_D \setminus \Sigma$ denote pure Brinkman and Darcy borders, respectively; and secondly $\partial\Omega = \Gamma_0 \cup \Gamma_N$, where Γ_0, Γ_N denote the parts of the boundary where homogeneous Dirichlet or Neumann (zero flux) conditions are enforced for ϕ, respectively (see a rough diagram of domains and boundaries in Fig. 1). The considered boundary and transmission conditions are

$$\boldsymbol{u}_D \cdot \boldsymbol{n} = \boldsymbol{u}_B \cdot \boldsymbol{n} \quad \text{and} \quad p_D = p_B \quad \text{on} \quad \Sigma,$$
$$\boldsymbol{\omega}_B \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_B = \Sigma \cup \Gamma_B, \ \boldsymbol{u}_B \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_B \quad \text{and} \quad \boldsymbol{u}_D \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_D, \quad (2.4)$$
$$\phi = 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad (\vartheta(\phi)\nabla\phi - \phi\boldsymbol{u} - f_{\text{bk}}(\phi)\boldsymbol{g}) \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_N,$$

where $\boldsymbol{n}$ denotes the outward normal at Ω_B and Ω_D. Note that interface conditions are not required in the transport equation, as the continuity of ϕ and of the corresponding fluxes is incorporated naturally in the formulation.

For the sake of our analysis, the variable coefficients need to satisfy the following requirements: there exist positive constants $\vartheta_1, \vartheta_2, \gamma_1, \gamma_2, L_\vartheta$ and $L_{f_{\text{bk}}}$, such that

$$\vartheta_1 \le \vartheta(s) \le \vartheta_2 \quad \text{and} \quad \gamma_1 \le f_{\text{bk}}(s) \le \gamma_2 \qquad \forall s \in \mathrm{R}, \quad (2.5)$$

$$|\vartheta(s) - \vartheta(t)| \le L_\vartheta |s - t| \qquad \forall s, t \in \mathrm{R} \quad (2.6)$$

and

$$|f_{\text{bk}}(s) - f_{\text{bk}}(t)| \le L_{f_{\text{bk}}} |s - t| \qquad \forall s, t \in \mathrm{R}. \quad (2.7)$$

In view of deriving a weak form of (2.1)–(2.3) and according to the boundary data (2.4) we introduce the following functional spaces:

$$\mathbf{H}_{\mathrm{B}}(\mathrm{div}; \Omega_{\mathrm{B}}) := \left\{ v_{\mathrm{B}} \in \mathbf{H}(\mathrm{div}; \Omega_{\mathrm{B}}) : \quad v_{\mathrm{B}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{B}} \right\},$$

$$\mathbf{H}_{0}(\mathbf{curl}; \Omega_{\mathrm{B}}) := \left\{ z_{\mathrm{B}} \in \mathbf{H}(\mathbf{curl}; \Omega_{\mathrm{B}}) : \quad z_{\mathrm{B}} \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega_{\mathrm{B}} \right\},$$

$$\mathbf{H}_{\mathrm{D}}(\mathrm{div}; \Omega_{\mathrm{D}}) := \left\{ v_{\mathrm{D}} \in \mathbf{H}(\mathrm{div}; \Omega_{\mathrm{D}}) : \quad v_{\mathrm{D}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \right\}$$

and

$$\mathrm{H}_{\Gamma_0}^1(\Omega) := \{ \psi \in \mathrm{H}^1(\Omega) : \psi|_{\Gamma_0} = 0 \},$$

for which, thanks to the generalized Poincaré inequality, there exists $c_p > 0$, depending only on Ω and Γ_0, such that

$$\|\psi\|_{1,\Omega} \leq c_p \, |\psi|_{1,\Omega}, \qquad \forall \, \psi \in \mathrm{H}_{\Gamma_0}^1(\Omega). \tag{2.8}$$

3. Weak formulation and its solvability analysis

In this section we proceed similarly as in Alvarez *et al.* (2015) and Alvarez *et al.* (2016a), to derive a suitable variational formulation of (2.1)–(2.4) and analyse its solvability by means of a fixed-point strategy.

3.1 *Continuous mixed-primal formulation*

The continuity of pressure across the interface allows us to define its trace

$$\lambda := p_{\mathrm{D}}|_{\Sigma} = p_{\mathrm{B}}|_{\Sigma} \in \mathrm{H}^{1/2}(\Sigma). \tag{3.1}$$

Then, after testing the Brinkman momentum equation in (2.1) against $v_{\mathrm{B}} \in \mathbf{H}_{\mathrm{B}}(\mathrm{div}; \Omega_{\mathrm{B}})$ and integrating by parts, we get

$$\mu \int_{\Omega_{\mathrm{B}}} \mathbb{K}_{\mathrm{B}}^{-1} u_{\mathrm{B}} \cdot v_{\mathrm{B}} + \mu \int_{\Omega_{\mathrm{B}}} v_{\mathrm{B}} \cdot \mathbf{curl}\, \omega_{\mathrm{B}} - \int_{\Omega_{\mathrm{B}}} p_{\mathrm{B}} \, \mathrm{div}\, v_{\mathrm{B}} + \langle v_{\mathrm{B}} \cdot \mathbf{n}, \lambda \rangle_{\Sigma} = \int_{\Omega_{\mathrm{B}}} \phi f_{\mathrm{B}} \cdot v_{\mathrm{B}} \quad \forall v_{\mathrm{B}} \in \mathbf{H}_{\mathrm{B}}(\mathrm{div}; \Omega_{\mathrm{B}}).$$

Next, testing the constitutive equation in (2.1) against $z_{\mathrm{B}} \in \mathbf{H}_{0}(\mathbf{curl}; \Omega_{\mathrm{B}})$ and integrating by parts, we obtain

$$\int_{\Omega_{\mathrm{B}}} \omega_{\mathrm{B}} \cdot z_{\mathrm{B}} - \int_{\Omega_{\mathrm{B}}} u_{\mathrm{B}} \cdot \mathbf{curl}\, z_{\mathrm{B}} = 0 \qquad \forall z_{\mathrm{B}} \in \mathbf{H}_{0}(\mathbf{curl}; \Omega_{\mathrm{B}}).$$

In turn, the incompressibility constraint in (2.1) in weak form reads

$$\int_{\Omega_{\mathrm{B}}} q_{\mathrm{B}} \, \mathrm{div}\, u_{\mathrm{B}} = 0 \qquad \forall q_{\mathrm{B}} \in \mathrm{L}^2(\Omega_{\mathrm{B}}).$$

On the other hand, testing the first equation of (2.2) with functions in $\mathbf{H}_D(\mathrm{div};\Omega_D)$, integrating by parts, using the boundary conditions and employing (3.1), we get

$$\mu \int_{\Omega_D} \mathbb{K}_D^{-1}\boldsymbol{u}_D \cdot \boldsymbol{v}_D - \int_{\Omega_D} p_D \, \mathrm{div}\,\boldsymbol{v}_D - \langle \boldsymbol{v}_D \cdot \boldsymbol{n}, \lambda\rangle_\Sigma = \int_{\Omega_D} \phi \boldsymbol{f}_D \cdot \boldsymbol{v}_D \qquad \forall \boldsymbol{v}_D \in \mathbf{H}_D(\mathrm{div};\Omega_D).$$

In addition, the second equation in (2.2) is tested as

$$\int_{\Omega_D} q_D \, \mathrm{div}\,\boldsymbol{v}_D = 0 \qquad \forall q_D \in \mathrm{L}^2(\Omega_D).$$

Finally, the continuity of normal velocities across Σ (*cf*. first equation in (2.4)) is imposed weakly as follows

$$\langle \boldsymbol{u}_B \cdot \boldsymbol{n} - \boldsymbol{u}_D \cdot \boldsymbol{n}, \xi\rangle_\Sigma = 0 \qquad \forall \xi \in \mathrm{H}^{1/2}(\Sigma).$$

Therefore, given $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, we arrive at the following mixed formulation for the Brinkman–Darcy coupling: find $\vec{\boldsymbol{u}} := (\boldsymbol{u}_B, \boldsymbol{\omega}_B, \boldsymbol{u}_D) \in \mathbf{H}$ and $\vec{p} := (p_B, p_D, \lambda) \in \mathbf{Q}$, such that

$$\begin{aligned} \mathscr{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathscr{B}(\vec{\boldsymbol{v}},\vec{p}) &= \mathscr{F}_\phi(\vec{\boldsymbol{v}}) \quad \forall \vec{\boldsymbol{v}} := (\boldsymbol{v}_B, \boldsymbol{z}_B, \boldsymbol{v}_D) \in \mathbf{H}, \\ \mathscr{B}(\vec{\boldsymbol{u}},\vec{q}) &= 0 \qquad \forall \vec{q} := (q_B, q_D, \lambda) \in \mathbf{Q}, \end{aligned} \tag{3.2}$$

where the product spaces are

$$\mathbf{H} := \mathbf{H}_B(\mathrm{div};\Omega_B) \times \mathbf{H}_0(\mathbf{curl};\Omega_B) \times \mathbf{H}_D(\mathrm{div};\Omega_D), \quad \mathbf{Q} := \mathrm{L}^2(\Omega_B) \times \mathrm{L}^2(\Omega_D) \times \mathrm{H}^{1/2}(\Sigma),$$

the bilinear forms $\mathscr{A} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$ and $\mathscr{B} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$ are defined by

$$\begin{aligned} \mathscr{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) &:= \mu \int_{\Omega_B} \mathbb{K}_B^{-1}\boldsymbol{u}_B \cdot \boldsymbol{v}_B + \mu \int_{\Omega_B} \boldsymbol{\omega}_B \cdot \boldsymbol{z}_B + \mu \int_{\Omega_B} \boldsymbol{v}_B \cdot \mathbf{curl}\,\boldsymbol{\omega}_B \\ &\quad - \mu \int_{\Omega_B} \boldsymbol{u}_B \cdot \mathbf{curl}\,\boldsymbol{z}_B + \mu \int_{\Omega_D} \mathbb{K}_D^{-1}\boldsymbol{u}_D \cdot \boldsymbol{v}_D, \\ \mathscr{B}(\vec{\boldsymbol{v}},\vec{q}) &:= -\int_{\Omega_B} q_B \, \mathrm{div}\,\boldsymbol{v}_B - \int_{\Omega_D} q_D \, \mathrm{div}\,\boldsymbol{v}_D + \langle \boldsymbol{v}_B \cdot \boldsymbol{n} - \boldsymbol{v}_D \cdot \boldsymbol{n}, \xi\rangle_\Sigma, \end{aligned}$$

for all $\vec{\boldsymbol{v}} \in \mathbf{H}, \vec{q} \in \mathbf{Q}$, and $\mathscr{F} \in \mathbf{H}'$ is the functional defined by

$$\mathscr{F}_\psi(\vec{\boldsymbol{v}}) := \int_{\Omega_B} \psi \boldsymbol{f}_B \cdot \boldsymbol{v}_B + \int_{\Omega_D} \psi \boldsymbol{f}_D \cdot \boldsymbol{v}_D \quad \forall \vec{\boldsymbol{v}} \in \mathbf{H}. \tag{3.3}$$

Next, we observe that the solution for (3.2) is not unique. Indeed, it suffices to consider $\vec{p} := (c,c,c)$, with $c \in \mathrm{R}$ to see that $(\mathbf{0},\vec{p})$ is also solution to the homogeneous system (see Alvarez *et al.*, 2016b, Theorem 3.1). In order to amend this, the formulation can be modified as follows. Find $(\vec{\boldsymbol{u}},\vec{p}) \in \mathbf{H} \times \mathbf{Q}_0$, such that

$$\begin{aligned} \mathscr{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathscr{B}(\vec{\boldsymbol{v}},\vec{p}) &= \mathscr{F}_\phi(\vec{\boldsymbol{v}}) \quad \forall \vec{\boldsymbol{v}} \in \mathbf{H}, \\ \mathscr{B}(\vec{\boldsymbol{u}},\vec{q}) &= 0 \qquad \forall \vec{q} \in \mathbf{Q}_0, \end{aligned} \tag{3.4}$$

where $\mathbf{Q}_0 := \mathrm{L}^2_0(\Omega_B) \times \mathrm{L}^2(\Omega_D) \times \mathrm{H}^{1/2}(\Sigma)$.

On the other hand, given $\boldsymbol{u}$ in a suitable space (to be made precise in Lemma 3.2, below), testing with functions in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, integrating by parts and using the boundary data, we deduce the following primal formulation for the transport problem: find $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ such that

$$\mathscr{C}_{\boldsymbol{u}}(\phi, \psi) = \int_\Omega f_{\mathrm{bk}}(\phi)\,\boldsymbol{g} \cdot \nabla \psi \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega), \tag{3.5}$$

where the form $\mathscr{C}_{\boldsymbol{u}}$ is defined by

$$\mathscr{C}_{\boldsymbol{u}}(\phi, \psi) := \int_\Omega \vartheta(\phi)\,\nabla\phi \cdot \nabla\psi - \int_\Omega \phi\,\boldsymbol{u} \cdot \nabla\psi + \int_\Omega \beta\,\phi\,\psi \qquad \forall\, \phi,\,\psi \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{3.6}$$

The mixed-primal formulation of our original coupled problem (2.1)–(2.3) and (2.4), reduces then to (3.4) and (3.5), that is: find $(\vec{\boldsymbol{u}}, \vec{p}, \phi) \in \mathbf{H} \times \mathbf{Q}_0 \times \mathrm{H}^1_{\Gamma_0}(\Omega)$ such that

$$\begin{aligned}
\mathscr{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) + \mathscr{B}(\vec{\boldsymbol{v}}, \vec{p}) &= \mathscr{F}_\phi(\vec{\boldsymbol{v}}) && \forall\, \vec{\boldsymbol{v}} \in \mathbf{H}, \\
\mathscr{B}(\vec{\boldsymbol{u}}, \vec{q}) &= 0 && \forall\, \vec{q} \in \mathbf{Q}_0, \\
\mathscr{C}_{\boldsymbol{u}}(\phi, \psi) &= \int_\Omega f_{\mathrm{bk}}(\phi)\,\boldsymbol{g} \cdot \nabla\psi && \forall\, \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega).
\end{aligned} \tag{3.7}$$

The well posedness of (3.7) will be addressed in Sections 3.3 and 3.4. We anticipate now that the analysis of (3.5) (in particular, the estimation of the second term defining $\mathscr{C}_{\boldsymbol{u}}$) requires further regularity of the global velocity.

3.2 *Fixed-point strategy*

We describe a fixed-point framework for (3.7). We commence by introducing the operator $\mathbf{S}^{\mathrm{flow}}$: $\mathrm{H}^1_{\Gamma_0}(\Omega) \longrightarrow \mathbf{H}(\mathrm{div}; \Omega)$ defined as

$$\mathbf{S}^{\mathrm{flow}}(\phi) := \boldsymbol{u} := \begin{cases} \boldsymbol{u}_{\mathrm{B}} & \text{in} \quad \Omega_{\mathrm{B}} \\ \boldsymbol{u}_{\mathrm{D}} & \text{in} \quad \Omega_{\mathrm{D}} \end{cases} \qquad \forall\, \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$$

that gives the flow velocity, where the Brinkman and Darcy velocities $\boldsymbol{u}_{\mathrm{B}}$ and $\boldsymbol{u}_{\mathrm{D}}$ are part of the unique solution, to be confirmed below, of the interfacial flow problem (3.4) for the given concentration ϕ.

In turn, we also introduce the operator $\mathbf{S}^{\mathrm{adv}} : \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathbf{H}(\mathrm{div}; \Omega) \longrightarrow \mathrm{H}^1_{\Gamma_0}(\Omega)$ defined as

$$\mathbf{S}^{\mathrm{adv}}(\phi, \boldsymbol{u}) := \widetilde{\phi} \qquad \forall\, (\phi, \boldsymbol{u}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathbf{H}(\mathrm{div}; \Omega),$$

where, given $(\phi, \boldsymbol{u}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathbf{H}(\mathrm{div}; \Omega)$, $\widetilde{\phi}$ is the unique solution (to be confirmed below) of the linear advection problem arising from (3.5) after performing the following actions: replacing the nonlinear expression $\vartheta(\phi)\,\nabla\phi$ appearing in the first term of $\mathscr{C}_{\boldsymbol{u}}$ (cf. (3.6)) by the linear one $\vartheta(\phi)\,\nabla\widetilde{\phi}$; incorporating $\boldsymbol{u}$ into the second term defining $\mathscr{C}_{\boldsymbol{u}}$ (cf. (3.6)), thus making it bilinear; and realizing

that the expression on the right-hand side of (3.5) becomes a linear functional of ψ. In this way, the aforementioned linear problem reduces to find $\widetilde{\phi} \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ such that

$$\mathscr{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi}) = \mathscr{H}_\phi(\widetilde{\psi}) \qquad \forall \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega), \tag{3.8}$$

where, as previously described, the new bilinear form $\mathscr{C}_{\phi,\boldsymbol{u}}$ and linear functional $\mathscr{H}_\phi$ adopt the definitions

$$\mathscr{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi}) := \int_\Omega \vartheta(\phi)\, \nabla\widetilde{\phi} \cdot \nabla\widetilde{\psi} - \int_\Omega \widetilde{\phi}\, \boldsymbol{u} \cdot \nabla\widetilde{\psi} + \int_\Omega \beta\, \widetilde{\phi}\, \widetilde{\psi} \qquad \forall \widetilde{\phi}, \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega) \tag{3.9}$$

and

$$\mathscr{H}_\phi(\widetilde{\psi}) := \int_\Omega f_{\mathrm{bk}}(\phi)\, \boldsymbol{g} \cdot \nabla\widetilde{\psi} \qquad \forall \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{3.10}$$

Here, we stress in advance that $\mathbf{S}^{\mathrm{adv}}$ will be well defined, not in the whole space $\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathbf{H}(\mathrm{div}; \Omega)$, but only in a subspace of it (see Lemma 3.2 below).

Finally, we define the operator $\mathbf{T} : \mathrm{H}^1_{\Gamma_0}(\Omega) \longrightarrow \mathrm{H}^1_{\Gamma_0}(\Omega)$ as

$$\mathbf{T}(\phi) := \mathbf{S}^{\mathrm{adv}}(\phi, \mathbf{S}^{\mathrm{flow}}(\phi)) \qquad \forall \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega) \tag{3.11}$$

and realize that solving (3.7) is equivalent to finding $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ such that

$$\mathbf{T}(\phi) = \phi. \tag{3.12}$$

Indeed, $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ is solution of (3.12) if and only if $\phi = \mathbf{S}^{\mathrm{adv}}(\phi, \mathbf{S}^{\mathrm{flow}}(\phi))$, which means that, once $\boldsymbol{u} = \mathbf{S}^{\mathrm{flow}}(\phi)$ is obtained as part of the solution of (3.4) with the given ϕ and then employed jointly with ϕ to solve (3.8), the resulting solution of the latter is the same initial ϕ. Equivalently, this says that

$$\mathscr{C}_{\phi,\boldsymbol{u}}(\phi, \widetilde{\psi}) = \mathscr{H}_\phi(\widetilde{\psi}) \qquad \forall \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega),$$

which, bearing in mind the definitions of $\mathscr{C}_{\boldsymbol{u}}$ (cf. (3.6)) and $\mathscr{C}_{\phi,\boldsymbol{u}}$ (cf. (3.9)) and redenoting $\widetilde{\psi}$ as simply ψ, can be rewritten as

$$\mathscr{C}_{\boldsymbol{u}}(\phi, \psi) = \int_\Omega f_{\mathrm{bk}}(\phi)\, \boldsymbol{g} \cdot \nabla\psi \qquad \forall \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega),$$

which confirms that ϕ is the solution of (3.5). Hence, $(\vec{\boldsymbol{u}}, \vec{p}, \phi) \in \mathbf{H} \times \mathbf{Q}_0 \times \mathrm{H}^1_{\Gamma_0}(\Omega)$, with $(\vec{\boldsymbol{u}}, \vec{p})$ solving (3.4) for the given ϕ, is solution of the original coupled problem (3.7).

3.3 *Well posedness of the uncoupled problem*

In this section we show that the uncoupled problems (3.4) and (3.8) are in fact well posed. We begin the solvability analysis with the following result, whose proof is a direct consequence of Alvarez *et al.* (2016b, Theorem 3.2). Let us remark that similar vorticity-based formulations for Brinkman–Darcy equations can be analysed using a different approach, as done recently in Anaya *et al.* (2019).

LEMMA 3.1   For each $\phi \in H^1_{\Gamma_0}(\Omega)$ problem (3.4) has a unique solution $(\vec{u}, \vec{p}) \in \mathbf{H} \times \mathbf{Q}_0$. Moreover, there exists $C_{\mathbf{S}^{\text{flow}}} > 0$, independent of ϕ, such that

$$\|\mathbf{S}^{\text{flow}}(\phi)\|_{\text{div},\Omega} \leq \|(\vec{u}, \vec{p})\|_{\mathbf{H} \times \mathbf{Q}_0} \leq C_{\mathbf{S}^{\text{flow}}} \|\phi\|_{0,\Omega} \left\{ \|f_B\|_{\infty,\Omega_B} + \|f_D\|_{\infty,\Omega_D} \right\}, \qquad \forall \phi \in H^1_{\Gamma_0}(\Omega).$$
(3.13)

For the purpose of the next result, which provides the solvability of the uncoupled problem (3.8), we require that the global velocity $\boldsymbol{u}$ belong to $\mathbf{H}(\text{div}; \Omega) \cap \mathbf{H}^\delta(\Omega)$ for some $\delta \in (0, 1)$ (when $n = 2$) or $\delta \in (1/2, 1)$ (when $n = 3$). In turn, according to the aforementioned range for δ, we recall that the Sobolev embedding theorem (*cf.* Quarteroni & Valli, 1994, Theorem 1.3.4; Adams & Fournier, 2003, Theorem 4.12) establishes the continuous injection $\boldsymbol{i}_\delta : \mathbf{H}^\delta(\Omega) \longrightarrow \mathbf{L}^{\delta^*}(\Omega)$ with boundedness constant $\mathbf{C}^*_\delta$, where

$$\delta^* := \begin{cases} \frac{2}{1-\delta} & \text{if} \quad n = 2, \\ \frac{6}{3-2\delta} & \text{if} \quad n = 3, \end{cases}$$
(3.14)

and it also guarantees that the injection i : $H^1(\Omega) \to L^{n/\delta}(\Omega)$ is compact, and hence continuous, with constant C_δ. In addition, we set $r_0 := \frac{\vartheta_1}{2c_p C_\delta C^*_\delta}$, where ϑ_1 and c_p are the constants given in (2.5) and (2.8), respectively.

LEMMA 3.2   Let $\phi \in H^1_{\Gamma_0}(\Omega)$, and $\boldsymbol{u} \in \mathbf{H}(\text{div}; \Omega) \cap \mathbf{H}^\delta(\Omega)$ for some $\delta \in (0, 1)$ (when $n = 2$) or $\delta \in (1/2, 1)$ (when $n = 3$), such that $\|\boldsymbol{u}\|_{\delta,\Omega} < r_0$. Then, the problem (3.8) has a unique solution $\mathbf{S}^{\text{adv}}(\phi, \boldsymbol{u}) := \widetilde{\phi} \in H^1_{\Gamma_0}(\Omega)$. Moreover, there exists $C_{\mathbf{S}^{\text{adv}}} > 0$, independent of $(\phi, \boldsymbol{u})$, such that

$$\|\mathbf{S}^{\text{adv}}(\phi, \boldsymbol{u})\|_{1,\Omega} \leq C_{\mathbf{S}^{\text{adv}}} \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}|.$$
(3.15)

Proof.   We first notice that $\mathscr{C}_{\phi,\boldsymbol{u}}$ (*cf.* (3.9)) is clearly a bilinear form. In turn, employing the upper bound of ϑ (*cf.* (2.5)), Cauchy–Schwarz's inequality and Hölder's inequality, it readily follows from (3.9) that

$$|\mathscr{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi})| \leq \vartheta_2 |\widetilde{\phi}|_{1,\Omega} |\widetilde{\psi}|_{1,\Omega} + \|\widetilde{\phi}\|_{L^{2q}(\Omega)} \|\boldsymbol{u}\|_{\mathbf{L}^{2p}(\Omega)} |\widetilde{\psi}|_{1,\Omega} + \beta \|\widetilde{\phi}\|_{0,\Omega} \|\widetilde{\psi}\|_{0,\Omega},$$
(3.16)

where $p, q \in [1, +\infty)$ are such that $1/p + 1/q = 1$. Next, choosing p such that $2p = \delta^*$ (*cf.* (3.14)), it readily follows that

$$2q := \frac{2p}{p-1} = \frac{n}{\delta}.$$
(3.17)

The range of δ implies that we can apply the continuous injections $\boldsymbol{i}_\delta : \mathbf{H}^\delta(\Omega) \longrightarrow \mathbf{L}^{\delta^*}(\Omega)$ and i : $H^1(\Omega) \to L^{n/\delta}(\Omega)$ in combination with the bound for $\|\boldsymbol{u}\|_{\delta,\Omega}$ to deduce from (3.16) the existence of a positive constant $\|\mathscr{C}\|$ (depending on $\vartheta_1, \vartheta_2, \beta, \|\boldsymbol{i}_\delta\|, \|\text{i}\|$ and c_p) such that

$$|\mathscr{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi})| \leq \|\mathscr{C}\| \|\widetilde{\phi}\|_{1,\Omega} \|\widetilde{\psi}\|_{1,\Omega} \quad \forall \widetilde{\phi}, \widetilde{\psi} \in H^1_{\Gamma_0}(\Omega).$$

This proves that $\mathscr{C}_{\phi,\boldsymbol{u}}$ is bounded independently of ϕ and $\boldsymbol{u}$. On the other hand, applying the same argument used for the derivation of second term on the right-hand side of (3.16) and using (3.14), (3.17)

and (2.8), we find that for each $\widetilde{\phi} \in H^1_{\Gamma_D}(\Omega)$ there holds

$$
\mathscr{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi},\widetilde{\phi}) = \int_\Omega \vartheta(\phi)\,|\nabla\widetilde{\phi}|^2 - \int_\Omega \widetilde{\phi}\boldsymbol{u}\cdot\nabla\widetilde{\phi} + \beta\,\|\widetilde{\phi}\|_{0,\Omega}^2
$$

$$
\geq\ \vartheta_1\,|\widetilde{\phi}|_{1,\Omega}^2 - \mathbf{C}_\delta^*\,\|\widetilde{\phi}\|_{L^{n/\delta}(\Omega)}\,\|\boldsymbol{u}\|_{\delta,\Omega}\,|\widetilde{\phi}|_{1,\Omega} + \beta\,\|\widetilde{\phi}\|_{0,\Omega}^2
$$

$$
\geq\ (\vartheta_1 - c_p\mathbf{C}_\delta\mathbf{C}_\delta^*\,\|\boldsymbol{u}\|_{\delta,\Omega})\,|\widetilde{\phi}|_{1,\Omega}^2
$$

$$
\geq\ \frac{\vartheta_1}{2}\,|\widetilde{\phi}|_{1,\Omega}^2\ \geq\ \frac{\vartheta_1}{2c_p^2}\,\|\widetilde{\phi}\|_{1,\Omega}^2, \tag{3.18}
$$

which proves that $\mathscr{C}_{\phi,\boldsymbol{u}}$ is $H^1_{\Gamma_0}(\Omega)$-elliptic with constant $\widetilde{\alpha} := \frac{\vartheta_1}{2c_p^2}$, also independently of both ϕ and $\boldsymbol{u}$. Next, applying Cauchy–Schwarz inequality and the upper bound for f_{bk} given in (2.5), we easily deduce that

$$
|\mathscr{H}_\phi(\widetilde{\psi})|\ \leq\ \gamma_2\,|\Omega|^{1/2}\,|\boldsymbol{g}|\,\|\widetilde{\psi}\|_{1,\Omega}\qquad \forall\,\widetilde{\psi}\in H^1_{\Gamma_D}(\Omega),
$$

which says that $\mathscr{H}_\phi \in H^1_{\Gamma_0}(\Omega)'$ and $\|\mathscr{H}_\phi\| \leq \gamma_2\,|\Omega|^{1/2}\,|\boldsymbol{g}|$. Consequently, a direct application of the Lax–Milgram lemma implies the existence of a unique solution $\widetilde{\phi} := \mathbf{S}^{\mathrm{adv}}(\phi,\boldsymbol{u}) \in H^1_{\Gamma_0}(\Omega)$ of (3.8), and the corresponding continuous dependence result becomes (3.15) with $C_{\mathbf{S}^{\mathrm{adv}}} = \frac{1}{\alpha} = \frac{2c_p^2}{\vartheta_1}$. $\qquad\square$

REMARK 3.3 The bound for $\|\boldsymbol{u}\|_{\delta,\Omega}$ used in Lemma 3.2 could also have been taken as

$$
\|\boldsymbol{u}\|_{\delta,\Omega}\ <\ \varepsilon\,\frac{\vartheta_1}{c_p\mathbf{C}_\delta\mathbf{C}_\delta^*},
$$

with any $\varepsilon \in (0,1)$, however choosing simply $\varepsilon = \frac{1}{2}$ yields a joint maximization of the ellipticity constant of $\mathscr{C}_{\phi,\boldsymbol{u}}$. In addition, dropping the term $\beta\,\|\widetilde{\phi}\|_{0,\Omega}^2$ in (3.18) we have assumed that β is small and then utilized the Poincaré inequality (2.8). When β is instead large, say $\beta \geq \vartheta_1$, then the aforementioned expression is maintained throughout the derivation of (3.18), implying that the Poincaré inequality (2.8) is not required.

We end this section by introducing adequate regularity hypotheses on $\mathbf{S}^{\mathrm{flow}}$, which will be employed to guarantee that the operator $\mathbf{T}$ is well defined. In addition, we also assume sufficient regularity of $\mathbf{S}^{\mathrm{adv}}$ in order to establish its Lipschitz continuity and also that of $\mathbf{T}$. In fact, for the remainder of this paper we follow Alvarez *et al.* (2016a), Eq. (3.23) and Eq. (3.24)) and consider the following two hypotheses.

REGULARITY Hypothesis 3.4 For $\boldsymbol{f}_{\mathrm{B}} \in \mathbf{L}^\infty(\Omega_{\mathrm{B}})$, $\boldsymbol{f}_{\mathrm{D}} \in \mathbf{L}^\infty(\Omega_{\mathrm{D}})$ and for each $\phi \in H^1_{\Gamma_0}(\Omega)$ with $\|\phi\|_{1,\Omega} \leq r$, $r > 0$ given, there holds $\mathbf{S}^{\mathrm{flow}}(\phi) \in \mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{H}^\delta(\Omega)$, for some $\delta \in (0,1)$ (when $n=2$) or $\delta \in (1/2,1)$ (when $n=3$), with

$$
\|\mathbf{S}^{\mathrm{flow}}(\phi)\|_{\delta,\Omega}\ \leq\ \widehat{C}_{\mathbf{S}^{\mathrm{flow}}}(r)\,\|\phi\|_{0,\Omega}\,\Big\{\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}}\Big\}, \tag{3.19}
$$

where $\widehat{C}_{\mathbf{S}^{\mathrm{flow}}}(r)$ is a positive constant independent of ϕ, but depending on the upper bound r of its norm.

REGULARITY Hypothesis 3.5 For each $(\varphi,\boldsymbol{w}) \in H^1_{\Gamma_0}(\Omega) \times (\mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{H}^\delta(\Omega))$, with $\delta \in (0,1)$ (when $n=2$) or $\delta \in (1/2,1)$ (when $n=3$), and $\|\varphi\|_{1,\Omega} + \|\boldsymbol{w}\|_{\mathrm{div},\Omega} + \|\boldsymbol{w}\|_\delta \leq r$, $r > 0$ given, there holds

$\mathbf{S}^{\mathrm{adv}}(\varphi, \boldsymbol{w}) \in \mathrm{H}_{\Gamma_0}^{1+\delta}(\Omega)$, with

$$\|\mathbf{S}^{\mathrm{adv}}(\varphi, \boldsymbol{w})\|_{1+\delta, \Omega} \leq \widehat{C}_{\mathbf{S}^{\mathrm{adv}}}(r) \, C(\Omega) \, |\boldsymbol{g}|, \tag{3.20}$$

where $C(\Omega)$ is a positive constant depending only on the domain Ω and $\widehat{C}_{\mathbf{S}^{\mathrm{adv}}}(r)$ is a positive constant independent of $(\varphi, \boldsymbol{w})$, but depending on the upper bound r of its norm.

Hypothesis 3.4 will be needed in the proof of Lemma 3.6. More precisely, it is required to invoke Lemma 3.2, which is in turn crucial to prove that $\mathbf{T}$ is well defined. Subsequently, the estimate (3.19) is employed in Lemma 3.8 to bound an expression of the form $\|\mathbf{S}^{\mathrm{flow}}(\phi - \varphi)\|_{\mathbf{L}^{2p}(\Omega)}$ in terms of $\|\mathbf{S}^{\mathrm{flow}}(\phi - \varphi)\|_{\delta, \Omega}$, and hence depending on the right-hand side of (3.19). In turn, the further regularity from Hypothesis 3.5 is used in the proof of Lemma 3.7 to bound an expression of the form $\|\nabla \mathbf{S}^{\mathrm{adv}}(\varphi, \boldsymbol{w})\|_{\mathbb{L}^{2p}(\Omega)}$ in terms of $\|\mathbf{S}^{\mathrm{adv}}(\varphi, \boldsymbol{w})\|_{1+\delta, \Omega}$, which in the proof of Theorem 3.10, can be bounded by the data appearing in the right-hand side of (3.20).

3.4 *Solvability of the fixed-point equation*

The well posedness of the uncoupled problems (3.4) and (3.8) confirms that the operators $\mathbf{S}^{\mathrm{flow}}$, $\mathbf{S}^{\mathrm{adv}}$ and $\mathbf{T}$ are well defined. In order to address the solvability of the fixed-point equation (3.12), we verify the hypotheses of the Schauder fixed-point theorem (see, e.g., Ciarlet, 2013, Theorem 9.12-1(b)).

We first define the following set, for any $r > 0$

$$\mathrm{W}^r := \left\{ \phi \in \mathrm{H}_{\Gamma_0}^1(\Omega) : \quad \|\phi\|_{1, \Omega} \leq r \right\}, \tag{3.21}$$

and we have the following result.

LEMMA 3.6 Given $r > 0$ and (3.21) assume that

$$\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty, \Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty, \Omega_{\mathrm{D}}} < \frac{r_0}{r \widehat{C}_{\mathbf{S}^{\mathrm{flow}}}(r)} \qquad \text{and} \qquad C_{\mathbf{S}^{\mathrm{adv}}} \, \gamma_2 \, |\Omega|^{1/2} \, |\boldsymbol{g}| \leq r, \tag{3.22}$$

where r_0 is the constant specified right before Lemma 3.2. Then $\mathbf{T}(\mathrm{W}^r) \subseteq \mathrm{W}^r$.

Proof. Let us consider a given $\phi \in \mathrm{W}^r$. We then combine the arguments from Lemma 3.1, the estimate (3.19), the first condition in (3.22) and Lemma 3.2 to conclude that $\mathbf{S}^{\mathrm{adv}}(\phi, \mathbf{S}^{\mathrm{flow}}(\phi)) := \mathbf{T}(\phi)$ is well defined. Next, according to the definition of the operator $\mathbf{T}$ (*cf.* (3.11)) and the continuous dependence estimate (3.15) it readily follows that

$$\|\mathbf{T}(\phi)\|_{1, \Omega} = \|\mathbf{S}^{\mathrm{adv}}(\phi, \mathbf{S}^{\mathrm{flow}}(\phi))\|_{1, \Omega} \leq C_{\mathbf{S}^{\mathrm{adv}}} \, \gamma_2 \, |\Omega|^{1/2} \, |\boldsymbol{g}|,$$

which, due to the second inequality in (3.22), proves that $\mathbf{T}(\phi) \in \mathrm{W}^r$, thus finishing the proof. $\qquad \square$

Our next goal is to establish the continuity and compactness of $\mathbf{T}$, which is addressed in the following two lemmas.

LEMMA 3.7 There exists a positive constant $\widetilde{C} > 0$, depending on $L_{f_{\mathrm{bk}}}, L_{\vartheta}, \mathbf{C}_\delta^*$ and $\widetilde{\alpha}$ (*cf.* (2.7), (2.6) and Lemma 3.2), and the boundedness constant C_δ^* of the injection $\mathrm{i}_\delta : \mathrm{H}^\delta(\Omega) \to \mathrm{L}^{\delta^*}(\Omega)$, such that for all

$(\phi, \boldsymbol{u}), (\varphi, \boldsymbol{w}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times (\mathbf{H}(\mathrm{div}; \Omega) \cap \mathbf{H}^\delta(\Omega))$, with $\|\boldsymbol{u}\|_{\delta, \Omega}, \|\boldsymbol{w}\|_{\delta, \Omega} < r_0$ (cf. Lemma 3.2), there holds

$$\|\mathbf{S}^{\mathrm{adv}}(\phi, \boldsymbol{u}) - \mathbf{S}^{\mathrm{adv}}(\varphi, \boldsymbol{w})\|_{1, \Omega} \leq \widetilde{C} \left\{ |\boldsymbol{g}| \, \|\phi - \varphi\|_{0, \Omega} + \|\mathbf{S}^{\mathrm{adv}}(\varphi, \boldsymbol{w})\|_{\mathrm{L}^{n/\delta}(\Omega)} \|\boldsymbol{u} - \boldsymbol{w}\|_{\delta, \Omega} \right.$$
$$\left. + \|\mathbf{S}^{\mathrm{adv}}(\varphi, \boldsymbol{w})\|_{1+\delta, \Omega} \|\phi - \varphi\|_{\mathrm{L}^{n/\delta}(\Omega)} \right\}. \tag{3.23}$$

Proof. Given $(\phi, \boldsymbol{u}), (\varphi, \boldsymbol{w})$ as stated we let $\widetilde{\phi} := \mathbf{S}^{\mathrm{adv}}(\phi, \boldsymbol{u})$ and $\widetilde{\varphi} := \mathbf{S}^{\mathrm{adv}}(\varphi, \boldsymbol{w})$ that is (cf. (3.8))

$$\mathscr{C}_{\phi, \boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi}) = \mathscr{H}_\phi(\widetilde{\psi}) \quad \text{and} \quad \mathscr{C}_{\varphi, \boldsymbol{w}}(\widetilde{\varphi}, \widetilde{\psi}) = \mathscr{H}_\varphi(\widetilde{\psi}) \qquad \forall \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega).$$

According to the ellipticity of $\mathscr{C}_{\phi, \boldsymbol{u}}$ with constant $\widetilde{\alpha}$ we can subtract and add $\mathscr{H}_\varphi(\widetilde{\phi} - \widetilde{\varphi}) = \mathscr{C}_{\varphi, \boldsymbol{w}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})$ to the previous expression, leading to

$$\widetilde{\alpha} \, \|\widetilde{\phi} - \widetilde{\varphi}\|^2_{1, \Omega} \leq \mathscr{C}_{\phi, \boldsymbol{u}}(\widetilde{\phi}, \widetilde{\phi} - \widetilde{\varphi}) - \mathscr{C}_{\phi, \boldsymbol{u}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})$$
$$= \mathscr{H}_\phi(\widetilde{\phi} - \widetilde{\varphi}) - \mathscr{H}_\varphi(\widetilde{\phi} - \widetilde{\varphi}) + \mathscr{C}_{\varphi, \boldsymbol{w}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi}) - \mathscr{C}_{\phi, \boldsymbol{u}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})$$
$$= \int_\Omega \left(f_{\mathrm{bk}}(\phi) - f_{\mathrm{bk}}(\varphi) \right) \boldsymbol{g} \cdot \nabla(\widetilde{\phi} - \widetilde{\varphi}) + \int_\Omega \widetilde{\varphi} \, (\boldsymbol{u} - \boldsymbol{w}) \cdot \nabla(\widetilde{\phi} - \widetilde{\varphi})$$
$$+ \int_\Omega \left(\vartheta(\varphi) - \vartheta(\phi) \right) \nabla \widetilde{\varphi} \cdot \nabla(\widetilde{\phi} - \widetilde{\varphi}), \tag{3.24}$$

where for the last equality we have employed definitions (3.9) and (3.10). Then we can apply Cauchy–Schwarz's inequality, Hölder's inequality, Hypothesis 3.5, the Lipschitzcontinuity (2.6) and (2.7), which allows us to treat the last two terms in (3.24), much as in (3.18). Eventually, these steps imply the estimate

$$\widetilde{\alpha} \, \|\widetilde{\phi} - \widetilde{\varphi}\|^2_{1, \Omega} \leq \left\{ L_{f_{\mathrm{bk}}} |\boldsymbol{g}| \, \|\phi - \varphi\|_{0, \Omega} + \mathrm{C}^*_\delta \, \|\widetilde{\varphi}\|_{\mathrm{L}^{n/\delta}(\Omega)} \|\boldsymbol{u} - \boldsymbol{w}\|_{\delta, \Omega} \right\} |\widetilde{\phi} - \widetilde{\varphi}|_{1, \Omega}$$
$$+ L_\vartheta \, \mathrm{C}^*_\delta \, \|\phi - \varphi\|_{\mathrm{L}^{n/\delta}(\Omega)} \|\nabla \widetilde{\varphi}\|_{\delta, \Omega} \, |\widetilde{\phi} - \widetilde{\varphi}|_{1, \Omega}, \tag{3.25}$$

and combining inequalities (3.24) and (3.25) readily gives (3.23), thus finishing the proof. $\square$

The following result is a straightforward consequence of Lemma 3.7.

LEMMA 3.8 Given $r > 0$ and the definition of W^r from (3.21) let us assume the condition (3.22). Then, for all $\phi, \varphi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, there holds

$$\|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{1, \Omega} \leq \left\{ \widetilde{C} |\boldsymbol{g}| + r \, \widetilde{C} \, \widehat{C}_{\mathbf{S}^{\mathrm{flow}}}(r) (\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty, \Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty, \Omega_{\mathrm{D}}}) \, \|\mathbf{T}(\varphi)\|_{\mathrm{L}^{n/\delta}(\Omega)} \right\} \|\phi - \varphi\|_{0, \Omega}$$
$$+ \widetilde{C} \, \|\mathbf{T}(\varphi)\|_{1+\delta, \Omega} \|\phi - \varphi\|_{\mathrm{L}^{n/\delta}(\Omega)}, \tag{3.26}$$

where $\widetilde{C}$ and $\widehat{C}_{\mathbf{S}^{\mathrm{flow}}}(r)$ are the constants given in Lemma 3.7 and estimate (3.19), respectively.

Proof. It suffices to recall from Section 3.2 that $\mathbf{T}(\phi) = \mathbf{S}^{\mathrm{adv}}(\phi, \mathbf{S}^{\mathrm{flow}}(\phi))$ $\forall \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ and then directly apply Lemmas 3.6, 3.7, the linearity of $\mathbf{S}^{\mathrm{flow}}$ and the estimate (3.19). $\square$

The announced properties of $\mathbf{T}$ are proved now.

LEMMA 3.9   The map $\mathbf{T} : W^r \rightarrow W^r$ is continuous and $\overline{\mathbf{T}(W^r)}$ is compact.

Proof.   It follows almost verbatim as in Alvarez *et al.* (2015, Lemma 3.12). Essentially, the proof is a consequence of the Rellich–Kondrachov compactness theorem (Adams & Fournier, 2003, Theorem 6.3), the specified range of the constant δ involved in Hypotheses 3.4 and 3.5, and the well-known fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence.   □

Finally, the main result of this section is stated as follows.

THEOREM 3.10   Assume that the hypotheses of the Lemmas 3.6– 3.9 are met. Then, the mixed-primal problem (3.7) has at least one solution $(\vec{u}, \vec{p}, \phi) \in \mathbf{H} \times \mathbf{Q}_0 \times \mathrm{H}^1_{\Gamma_0}(\Omega)$ with $\phi \in W^r$, satisfying the continuous dependences

$$\|(\vec{u}, \vec{p})\|_{\mathbf{H} \times \mathbf{Q}_0} \leq r \, C_{\mathbf{S}^{\mathrm{flow}}} \left\{ \|\mathbf{f}_{\mathrm{B}}\|_{\infty, \Omega_{\mathrm{B}}} + \|\mathbf{f}_{\mathrm{D}}\|_{\infty, \Omega_{\mathrm{D}}} \right\}, \tag{3.27}$$

$$\|\phi\|_{1,\Omega} \leq C_{\mathbf{S}^{\mathrm{adv}}} \, \gamma_2 |\Omega|^{1/2} |\mathbf{g}|, \tag{3.28}$$

where $C_{\mathbf{S}^{\mathrm{flow}}}$ and $C_{\mathbf{S}^{\mathrm{adv}}}$ are the constants specified in Lemmas 3.1 and 3.2, respectively. Moreover, if the data $\mathbf{f}_{\mathrm{B}}, \mathbf{f}_{\mathrm{D}}$ and $\mathbf{g}$ are sufficiently small so that

$$\widetilde{C}(1 + \widehat{C}_{\mathbf{S}^{\mathrm{adv}}}(r) \, C_\delta C(\Omega)) \, |\mathbf{g}| + r^2 \, \widetilde{C} C_\delta \, \widehat{C}_{\mathbf{S}^{\mathrm{flow}}}(r) \left(\|\mathbf{f}_{\mathrm{B}}\|_{\infty, \Omega_{\mathrm{B}}} + \|\mathbf{f}_{\mathrm{D}}\|_{\infty, \Omega_{\mathrm{D}}} \right) < 1, \tag{3.29}$$

where $\widetilde{C}$, $\widehat{C}_{\mathbf{S}^{\mathrm{flow}}}(r)$, $\widehat{C}_{\mathbf{S}^{\mathrm{adv}}}(r)$ and $C(\Omega)$ are the constants from Lemma 3.7, and estimates (3.19) and (3.20), respectively, and C_δ is the boundedness constant of the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{n/\delta}(\Omega)$, then the solution ϕ is unique in W^r.

Proof.   According to the equivalence between (3.7) and the fixed-point equation (3.12), and thanks to Lemmas 3.6 and 3.9, the existence of solution is a direct consequence of the Schauder fixed-point theorem (Ciarlet, 2013, Theorem 9.12-1(b)). In turn, the estimates (3.27) and (3.28) follow from (3.13) and (3.15), respectively. Finally, given another solution $\varphi \in W^r$ of (3.12), the estimates (3.26), together with the bounds

$$\|\mathbf{T}(\varphi)\|_{1,\Omega} = \|\varphi\|_{1,\Omega} \leq r, \qquad \|\varphi\|_{1+\delta,\Omega} \leq \widehat{C}_{\mathbf{S}^{\mathrm{adv}}}(r) \, C(\Omega) \, |\mathbf{g}| \quad (cf. \, (3.20)),$$

and

$$\|\varphi\|_{\mathrm{L}^{n/\delta}(\Omega)} \leq C_\delta \, \|\varphi\|_{1,\Omega} \quad \forall \, \varphi \in \mathrm{H}^1(\Omega),$$

confirm (3.29) as a sufficient condition for uniqueness.   □

Alternatively to the fixed-point analysis developed in this section, one could appeal to Banach's fixed-point theory starting from the same setting, that is, exactly the same fixed-point equation of Section 3.2. It is also possible to define a different Picard problem, where the global velocity is the fixed-point or a completely different structure by appealing to monolithic approaches that do not separate flow from transport equations. Another option is to perform a linearization such that also the linear parts of the global flux of the nonlinear advection equation are now part of the bilinear form. That is, moving terms from (3.10) to (3.9). These (or many other possible) choices can be explored case by case, especially in terms of the structure of the solvability analysis, but also regarding the suitability in light of the specific

application. For instance, the strength of the coupling mechanisms can vary substantially from one flow regime to another. The strategy we have adopted here is adequate at least for the regime of laminar flow and transport dominated by nonlinear diffusion.

4. Galerkin scheme

Let $\mathscr{T}_h$ be a regular family of triangulations of $\bar{\Omega}_{\mathrm{B}} \cup \bar{\Omega}_{\mathrm{D}}$ by tetrahedra K of diameter h_K with meshsize $h := \max\{h_K :\ K \in \mathscr{T}_h\}$, such that $\mathscr{T}_h(\Omega_\star) := \left\{K \in \mathscr{T}_h :\ K \subseteq \bar{\Omega}_\star\right\}$ is a triangulation of $\Omega_\star$ for each $\star \in \{\mathrm{B}, \mathrm{D}\}$. We further denote by $\mathscr{T}_h(\Sigma)$ the triangulation on Σ induced by $\mathscr{T}_h$ (either from Ω_{B} or Ω_{D}). Also, we introduce an independent partition $\mathscr{T}_{\tilde{h}}(\Sigma)$ of Σ by triangles $\widetilde{T}$ of diameter $h_{\widetilde{T}}$ and define $\tilde{h} := \max\left\{h_{\widetilde{T}} :\ \widetilde{T} \in \mathscr{T}_{\tilde{h}}(\Sigma)\right\}$.

Let us now consider finite dimensional subspaces of the test and trial spaces appearing in Section 3:

$$\mathbf{H}_h^{\mathrm{B}} \subseteq \mathbf{H}_{\mathrm{B}}(\mathrm{div}; \Omega_{\mathrm{B}}), \qquad \mathbf{H}_{0,h}^{\mathrm{B}} \subseteq \mathbf{H}_0(\mathbf{curl}; \Omega_{\mathrm{B}}), \qquad \mathbf{H}_h^{\mathrm{D}} \subseteq \mathbf{H}_{\mathrm{D}}(\mathrm{div}; \Omega_{\mathrm{D}}), \qquad \mathbf{X}_h \subset \mathbf{H}(\mathrm{div}; \Omega),$$

$$\mathrm{Q}_h^{\mathrm{B}} \subseteq \mathrm{L}^2(\Omega_{\mathrm{B}}), \qquad \mathrm{Q}_h^{\mathrm{D}} \subseteq \mathrm{L}^2(\Omega_{\mathrm{D}}), \qquad \mathrm{Q}_h^{\Sigma} \subseteq \mathrm{H}^{1/2}(\Sigma), \qquad \mathrm{H}_h^{\phi} \subseteq \mathrm{H}_{\Gamma_0}^1(\Omega). \tag{4.1}$$

Hence, setting the global spaces

$$\mathbf{H}_h := \mathbf{H}_h^{\mathrm{B}} \times \mathbf{H}_{0,h}^{\mathrm{B}} \times \mathbf{H}_h^{\mathrm{D}} \quad \text{and} \quad \mathbf{Q}_{0,h} := \mathrm{Q}_{h,0}^{\mathrm{B}} \times \mathrm{Q}_h^{\mathrm{D}} \times \mathrm{Q}_h^{\Sigma},$$

the Galerkin scheme for (3.7) becomes the following: find $(\vec{\boldsymbol{u}}_h, \vec{p}_h, \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h} \times \mathrm{H}_h^{\phi}$ such that

$$
\begin{aligned}
\mathscr{A}(\vec{\boldsymbol{u}}_h, \vec{\boldsymbol{v}}_h) + \mathscr{B}(\vec{\boldsymbol{v}}_h, \vec{p}_h) &= \mathscr{F}_{\phi_h}(\vec{\boldsymbol{v}}_h) & \forall \vec{\boldsymbol{v}}_h \in \mathbf{H}_h, \\
\mathscr{B}(\vec{\boldsymbol{u}}_h, \vec{q}_h) &= 0 & \forall \vec{q}_h \in \mathbf{Q}_{0,h}, \\
\mathscr{C}_{\boldsymbol{u}_h}(\phi_h, \psi_h) &= \mathscr{H}_{\phi_h}(\psi_h) & \forall \psi_h \in \mathrm{H}_h^{\phi}.
\end{aligned}
\tag{4.2}
$$

In order to guarantee the well posedness of the Galerkin scheme (4.2), the subspaces introduced in (4.1) can be chosen as follows (see Alvarez *et al.*, 2016b, Section 4.1)

$$
\begin{aligned}
\mathbf{H}_h^{\star} &:= \left\{\boldsymbol{v}_h^{\star} \in \mathbf{H}_{\star}(\mathrm{div}; \Omega_{\star}) :\quad \boldsymbol{v}_h^{\star}|_K \in \mathbb{RT}_0(K) \quad \forall K \in \mathscr{T}_h(\Omega_{\star})\right\}, \\
\mathrm{Q}_h^{\star} &:= \left\{q_h \in \mathrm{L}^2(\Omega_{\star}) :\quad q_h|_K \in P_0(K) \quad \forall K \in \mathscr{T}_h(\Omega_{\star})\right\}, \\
\mathrm{Q}_{h,0}^{\star} &:= \mathrm{Q}_h^{\star} \cap \mathrm{L}_0^2(\Omega_{\star}), \\
\mathrm{H}_h^{\phi} &:= \left\{\psi_h \in C(\Omega) \cap \mathrm{H}_{\Gamma_0}^1(\Omega) :\quad \psi_h|_K \in \mathrm{P}_1(K) \quad \forall K \in \mathscr{T}_h\right\},
\end{aligned}
$$

with $\star \in \{\mathrm{B}, \mathrm{D}\}$, and where for any $K \in \mathscr{T}_h(\Omega_{\star})$, we denote

$$\mathbb{RT}_0(K) := \mathbf{P}_0(K) \oplus P_0(K)\,\boldsymbol{x},$$

as the local Raviart–Thomas space of lowest order. In addition, we set

$$\mathbf{H}_{0,h}^{\mathrm{B}} := \left\{ z_h^{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl}; \Omega_{\mathrm{B}}) : \quad z_h^{\mathrm{B}}|_K \in \mathbb{ND}_1(K) \quad \forall K \in \mathscr{T}_h(\Omega_{\mathrm{B}}) \right\},$$

where for any $K \in \mathscr{T}_h(\Omega_{\mathrm{B}})$, the local edge space of Nédélec type is

$$\mathbb{ND}_1(K) := \mathbf{P}_0(K) \oplus \mathbf{P}_0(K) \times \boldsymbol{x} = \left\{ w : K \to \mathbb{C}^3 : \quad w(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x} \quad \forall \boldsymbol{x} \in K, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3 \right\}.$$

In turn, we set $\mathbf{X}_h := \mathbf{1}_{\Omega_{\mathrm{B}}} \mathbf{H}_h^{\mathrm{B}} + \mathbf{1}_{\Omega_{\mathrm{D}}} \mathbf{H}_h^{\mathrm{D}}$, whereas for the interfacial pressure, we consider the finite element space

$$\mathrm{Q}_{\tilde{h}}^{\Sigma} := \left\{ \lambda_{\tilde{h}} \in \mathscr{C}^0(\Sigma) : \quad \lambda_{\tilde{h}}|_{\tilde{T}} \in P_1(\tilde{T}) \quad \forall \tilde{T} \in \mathscr{T}_{\tilde{h}}(\Sigma) \right\}.$$

4.1 *Discrete fixed-point strategy*

A first important remark is that the hypotheses of additional regularity that we employed to prove Lemmas 3.2 and 3.7 are neither needed nor applicable in the discrete case. In consequence, it is not possible to reuse the same fixed-point strategy from Section 3.2. Instead, we require that the operator associated with the discrete version of problem (3.4) is uniformly bounded. For simplicity of the presentation the fixed-point scheme and its analysis will be restricted to the 2D case. Given $r > 0$ we proceed to define

$$\mathbf{Z}_h^r := \left\{ \boldsymbol{u}_h \in \mathbf{X}_h : \quad ||\boldsymbol{u}_h||_{\star} \leq r \right\}, \tag{4.3}$$

where $|| \cdot ||_{\star} := \| \cdot \|_{\mathrm{div}, \Omega} + \| \cdot \|_{\mathbf{L}^s(\Omega)}$, with $s > 2$. We anticipate that the stipulated range for s will allow us to invoke certain Sobolev embeddings in the subsequent analysis, more precisely, in the proof of Lemma 4.2 and Theorem 4.9, below.

We now set $Y_h := \mathrm{H}_h^{\phi} \times \mathbf{Z}_h^r$ and introduce the operator $\mathbf{S}_h^{\mathrm{adv}} : Y_h \to \mathrm{H}_h^{\phi}$ defined by

$$\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h) := \varphi_h \qquad \forall (\phi_h, \boldsymbol{u}_h) \in Y_h,$$

where φ_h is the unique solution (to be established below) of the linear problem: find $\varphi_h \in \mathrm{H}_h^{\phi}$ such that

$$\mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\varphi_h, \psi_h) = \mathscr{H}_{\phi_h}(\psi_h) \qquad \forall \psi_h \in \mathrm{H}_h^{\phi}, \tag{4.4}$$

for given $(\phi_h, \boldsymbol{u}_h)$, where

$$\mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\varphi_h, \psi_h) := \int_{\Omega} \vartheta(\phi_h) \nabla \varphi_h \cdot \nabla \psi_h - \int_{\Omega} \varphi_h \boldsymbol{u}_h \cdot \nabla \psi_h + \int_{\Omega} \beta \varphi_h \psi_h \qquad \forall \varphi_h, \psi_h \in \mathrm{H}_h^{\phi},$$

and

$$\mathscr{H}_{\phi_h}(\psi_h) := \int_{\Omega} f_{\mathrm{bk}}(\phi_h) \boldsymbol{g} \cdot \nabla \psi_h \qquad \forall \psi_h \in \mathrm{H}_h^{\phi}.$$

In turn, we define the operator $\mathbf{S}_h^{\mathrm{flow}} : \mathrm{H}_h^{\phi} \to \mathbf{X}_h$ as

$$\mathbf{S}_h^{\mathrm{flow}}(\varphi_h) := \boldsymbol{w}_h \qquad \forall \varphi_h \in \mathrm{H}_h^{\phi}, \tag{4.5}$$

where $w_h|_{\Omega_B} = w_h^B$ and $w_h|_{\Omega_D} = w_h^D$ are the first and third components of $\vec{w}_h \in \mathbf{H}_h$, which in turn is the first component of the unique solution (to be confirmed below) of the discrete problem associated with (3.4): find $(\vec{w}_h, \vec{r}_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h}$, such that

$$
\begin{aligned}
\mathscr{A}(\vec{w}_h, \vec{v}_h) + \mathscr{B}(\vec{v}_h, \vec{r}_h) &= \mathscr{F}_{\varphi_h}(\vec{v}_h) \quad &\forall \vec{v}_h \in \mathbf{H}_h, \\
\mathscr{B}(\vec{w}_h, \vec{q}_h) &= 0 \quad &\forall \vec{q}_h \in \mathbf{Q}_{0,h},
\end{aligned}
\tag{4.6}
$$

with φ_h given. Therefore, by introducing the operator $\mathbf{T}_h : Y_h \to \mathrm{H}_h^\phi \times \mathbf{X}_h$ as

$$
\mathbf{T}_h(\phi_h, \boldsymbol{u}_h) := (\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h), \mathbf{S}_h^{\mathrm{flow}}(\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h))) \qquad \forall (\phi_h, \boldsymbol{u}_h) \in Y_h,
\tag{4.7}
$$

we see that solving (4.2) is equivalent to finding a fixed point of $\mathbf{T}_h$ that is the following: find $(\phi_h, \boldsymbol{u}_h) \in Y_h$ such that

$$
\mathbf{T}_h(\phi_h, \boldsymbol{u}_h) = (\phi_h, \boldsymbol{u}_h).
\tag{4.8}
$$

Certainly, all the above makes sense if we guarantee that the discrete problems (4.4) and (4.6) are well posed. This is precisely the purpose of the next section.

4.2 *Well posedness of the uncoupled problem*

In this section, we establish the well posedness of both (4.6) and (4.4), thus confirming that $\mathbf{S}_h^{\mathrm{flow}}$, $\mathbf{S}_h^{\mathrm{adv}}$ and hence $\mathbf{T}_h$ are well defined.

LEMMA 4.1 For each $\phi_h \in \mathrm{H}_h^\phi$ the problem (4.6) has a unique solution $(\vec{\boldsymbol{u}}_h, \vec{p}_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h}$. Moreover, there exists a positive constant C_2, independent of h, such that

$$
\|\mathbf{S}_h^{\mathrm{flow}}(\phi_h)\|_{\mathrm{div},\Omega} \leq \|(\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H} \times \mathbf{Q}_0} \leq C_2 \|\phi_h\|_{0,\Omega} \left\{ \|\boldsymbol{f}_B\|_{\infty,\Omega_B} + \|\boldsymbol{f}_D\|_{\infty,\Omega_D} \right\} \qquad \forall \phi_h \in \mathrm{H}_h^\phi.
\tag{4.9}
$$

Proof. It follows directly from Alvarez *et al.* (2016b, Theorem 4.1). □

LEMMA 4.2 Assume that $r \in \left(0, \frac{\vartheta_1}{2C_s c_p}\right)$, where C_s is the boundedness constant of the injection $i_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^{\frac{2s}{s-2}}(\Omega)$, with $s > 2$. Then, for each $(\phi_h, \boldsymbol{u}_h) \in Y_h$, we see that problem (4.4) has a unique solution $\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h) := \varphi_h \in \mathrm{H}_h^\phi$. Moreover, by denoting $C_1 := \frac{1}{\tilde{\alpha}}$, with $\tilde{\alpha}$ as in the proof of Lemma 3.2, there holds

$$
\|\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h)\|_{1,\Omega} \leq C_1 \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}|, \qquad \forall (\phi_h, \boldsymbol{u}_h) \in Y_h.
\tag{4.10}
$$

Proof. It proceeds as in the proof of Lemma 3.2. Indeed, employing the same arguments used to derive (3.16), we deduce that

$$
|\mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\tilde{\phi}_h, \tilde{\psi}_h)| \leq \vartheta_2 |\tilde{\phi}_h|_{1,\Omega} |\tilde{\psi}_h|_{1,\Omega} + \|\tilde{\phi}_h\|_{\mathrm{L}^{2\tilde{q}}(\Omega)} \|\boldsymbol{u}_h\|_{\mathbf{L}^{2\tilde{p}}(\Omega)} |\tilde{\psi}_h|_{1,\Omega} + \beta \|\tilde{\phi}_h\|_{0,\Omega} \|\tilde{\psi}_h\|_{0,\Omega},
\tag{4.11}
$$

where $\tilde{p}, \tilde{q} \in [1, +\infty)$ are such that $1/\tilde{p} + 1/\tilde{q} = 1$. Thus, choosing $\tilde{p}$ such that $2\tilde{p} = s$, with $s > 2$, it readily follows that

$$
2\tilde{q} := \frac{2s}{s-2} > 1.
\tag{4.12}
$$

We can then use (4.3) and the continuity of the injection $i_s : H^1(\Omega) \to L^{\frac{2s}{s-2}}(\Omega)$ with constant C_s (since for 2D this injection is compact), to deduce the existence of a positive constant $\|\mathscr{C}\|$ depending on ϑ_2, β, r and C_s, such that

$$|\mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\widetilde{\phi}_h, \widetilde{\psi}_h)| \leq \|\mathscr{C}\| \, \|\widetilde{\phi}_h\|_{1,\Omega} \, \|\widetilde{\psi}_h\|_{1,\Omega} \quad \forall \, \widetilde{\phi}_h, \, \widetilde{\psi}_h \in H^1_{\Gamma_0}(\Omega), \tag{4.13}$$

which proves that $\mathscr{C}_{\phi_h, \boldsymbol{u}_h}$ is bounded independently of ϕ_h and $\boldsymbol{u}_h$. On the other hand, we can apply the same argument to handle the trilinear term on the right-hand side of (4.11) in the derivation of (4.13). Moreover, employing estimates (2.5) and (2.8), we find that for each $\widetilde{\phi}_h \in H^1_{\Gamma_0}(\Omega)$, there holds

$$\mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\widetilde{\phi}_h, \widetilde{\phi}_h) \geq \vartheta_1 |\widetilde{\phi}_h|^2_{1,\Omega} - \|\widetilde{\phi}_h\|_{L^{2s/s-2}(\Omega)} \|\boldsymbol{u}_h\|_{\mathbf{L}^s(\Omega)} |\widetilde{\phi}_h|_{1,\Omega} + \beta \|\widetilde{\phi}_h\|^2_{0,\Omega}$$

$$\geq (\vartheta_1 - C_s \, c_p r) |\widetilde{\phi}_h|^2_{1,\Omega} \geq \frac{\vartheta_1}{2} |\widetilde{\phi}_h|^2_{1,\Omega} \geq \frac{\vartheta_1}{2c_p^2} \|\widetilde{\phi}_h\|^2_{1,\Omega},$$

which proves that $\mathscr{C}_{\phi_h, \boldsymbol{u}_h}$ is elliptic on $H^\phi_h \times H^\phi_h$, with the same constant $\widetilde{\alpha}$ from Lemma 3.2. In addition, the fact that $\|\mathscr{H}_\phi\|$ is bounded independently of ϕ (*cf*. Proof of Lemma 3.2) confirms the same upper bound for $\|\mathscr{H}_{\phi_h}\|_{(H^\phi_h)'}$. The rest of the proof is a straightforward application of Lax–Milgram's lemma. □

REMARK 4.3 The manipulation of the term $\beta \|\widetilde{\phi}_h\|^2_{0,\Omega}$ (in the derivation of the ellipticity of $\mathscr{C}_{\phi_h, \boldsymbol{u}_h}$) is done analogously as described at the end of the proof of Lemma 3.2.

We end this section with a hypothesis of uniform boundedness on the operator $\mathbf{S}^{\text{flow}}_h$, which will be required to guarantee that the operator $\mathbf{T}_h$ is well defined and continuous within a certain ball.

HYPOTHESIS 4.4 There exists $s > 2$ such that the operator $\mathbf{S}^{\text{flow}}_h : (H^\phi_h, \| \cdot \|_{1,\Omega}) \to (\mathbf{X}_h, \| \cdot \|_{s,\Omega})$ (*cf*. (4.5)) is uniformly bounded, that is

$$\|\mathbf{S}^{\text{flow}}_h(\phi_h)\|_{\mathbf{L}^s(\Omega)} \leq \widetilde{C}_s \, \|\phi_h\|_{1,\Omega} \quad \forall \, \phi_h \in H^\phi_h, \tag{4.14}$$

where $\widetilde{C}_s$ is a positive constant independent of h.

Estimate (4.14) will be needed in the proof of Lemma 4.5, below. More specifically, it is required to bound an expression of the form $\|\mathbf{S}^{\text{flow}}_h(\mathbf{S}^{\text{adv}}_h(\,\cdot\,))\|_{\mathbf{L}^s(\Omega)}$ in terms of $\|\mathbf{S}^{\text{adv}}_h(\,\cdot\,)\|_{1,\Omega}$, which in turn is bounded by data (*cf*. (4.10)). The hypothesis will also be required to handle the expression $\|\mathbf{S}^{\text{flow}}_h(\mathbf{S}^{\text{adv}}_h(\phi_h)) - \mathbf{S}^{\text{flow}}_h(\mathbf{S}^{\text{adv}}_h(\varphi_h))\|_{\mathbf{L}^s(\Omega)}$ when deriving the Lipschitz continuity of $\mathbf{T}_h$. Even if a rigorous proof for (4.14) can be quite difficult to obtain in Section 6, we will provide numerical evidence that such $\widetilde{C}_s$ exists and that it is independent of h.

4.3 *Solvability of the fixed-point equation*

We now aim to show the solvability of (4.2) by analysing the equivalent fixed-point equation (4.8). To this end, we proceed to verify the hypotheses of the Brouwer fixed-point theorem (Ciarlet, 2013, Theorem 9.9-2).

We start by defining the following set:

$$\mathbf{W}_h := \{(\phi_h, \boldsymbol{u}_h) \in H^\phi_h \times \mathbf{X}_h : |||(\phi_h, \boldsymbol{u}_h)||| \leq r\}, \tag{4.15}$$

where r was previously fixed in (4.3) and

$$|||(\phi_h, \boldsymbol{u}_h)||| := \|\phi_h\|_{1,\Omega} + \|\boldsymbol{u}_h\|_\star = \|\phi_h\|_{1,\Omega} + \|\boldsymbol{u}_h\|_{\mathrm{div},\Omega} + \|\boldsymbol{u}_h\|_{\mathbf{L}^s(\Omega)}.$$

LEMMA 4.5 Let $\mathbf{W}_h$ be as in (4.15) and assume that the data $\boldsymbol{g}, \boldsymbol{f}_B$ and $\boldsymbol{f}_D$ are sufficiently small so that

$$(\widetilde{C}_s + C_1)\gamma_2|\Omega|^{1/2}|\boldsymbol{g}| + C_1 C_2 \gamma_2|\Omega|^{1/2}|\boldsymbol{g}| \left\{ \|\boldsymbol{f}_B\|_{\infty,\Omega_B} + \|\boldsymbol{f}_D\|_{\infty,\Omega_D} \right\} \leq r. \tag{4.16}$$

Then $\mathbf{T}_h(\mathbf{W}_h) \subseteq \mathbf{W}_h$.

Proof. Given $(\phi_h, \boldsymbol{u}_h) \in \mathbf{W}_h$ we get from (4.7) and the estimates (4.14), (4.9) and (4.10) that

$$
\begin{aligned}
|||\mathbf{T}_h(\phi_h, \boldsymbol{u}_h)||| &= |||(\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h), \mathbf{S}_h^{\mathrm{flow}}(\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h)))||| \\
&= \|\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h)\|_{1,\Omega} + \|\mathbf{S}_h^{\mathrm{flow}}(\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h))\|_{\mathrm{div},\Omega} + \|\mathbf{S}_h^{\mathrm{flow}}(\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h))\|_{\mathbf{L}^s(\Omega)} \\
&\leq (\widetilde{C}_s + C_1)\gamma_2|\Omega|^{1/2}|\boldsymbol{g}| + C_1 C_2 \gamma_2|\Omega|^{1/2}|\boldsymbol{g}| \{\|\boldsymbol{f}_B\|_{\infty,\Omega_B} + \|\boldsymbol{f}_D\|_{\infty,\Omega_D}\},
\end{aligned}
$$

and hence, employing the condition (4.16), we conclude that $\mathbf{T}_h(\phi_h, \boldsymbol{u}_h) \in \mathbf{W}_h$. □

The continuity of $\mathbf{T}_h$ will be proved in the following two lemmas, where we derive Lipschitz-type estimates for $\mathbf{S}_h^{\mathrm{flow}}$ and $\mathbf{S}_h^{\mathrm{adv}}$.

LEMMA 4.6 Let C_2 be the constant given in Lemma 4.1. Then, there holds

$$\|\mathbf{S}_h^{\mathrm{flow}}(\phi_h) - \mathbf{S}_h^{\mathrm{flow}}(\varphi_h)\|_{\mathrm{div},\Omega} \leq C_2 \|\phi_h - \varphi_h\|_{0,\Omega} \left\{ \|\boldsymbol{f}_B\|_{\infty,\Omega_B} + \|\boldsymbol{f}_D\|_{\infty,\Omega_D} \right\} \qquad \forall \phi_h, \varphi_h \in \mathrm{H}_h^\phi. \tag{4.17}$$

Proof. Given $\phi_h, \varphi_h \in \mathrm{H}_h^\phi$ we let $\vec{u}_h := (\boldsymbol{u}_h^B, \omega_h^B, \boldsymbol{u}_h^D) \in \mathbf{H}_h$, $\vec{p}_h := (p_h^B, p_h^D, \lambda_h) \in \mathbf{Q}_{0,h}$ and $\vec{w}_h = (\boldsymbol{w}_h^B, \boldsymbol{x}_h^B, \boldsymbol{w}_h^D) \in \mathbf{H}_h$, $\vec{r}_h := (r_h^B, r_h^D, \chi_h) \in \mathbf{Q}_{0,h}$ be the solution of (4.6) so that $\boldsymbol{u}_h = \boldsymbol{u}_h^B + \boldsymbol{u}_h^D =: \mathbf{S}_h^{\mathrm{flow}}(\phi_h)$ and $\boldsymbol{w}_h = \boldsymbol{w}_h^B + \boldsymbol{w}_h^D =: \mathbf{S}_h^{\mathrm{flow}}(\varphi_h)$. Then, employing the linearity of the forms $\mathscr{A}$ and $\mathscr{B}$ we deduce from (4.6) that

$$
\begin{aligned}
\mathscr{A}(\vec{u}_h - \vec{w}_h, \vec{v}_h) + \mathscr{B}(\vec{v}_h, \vec{p}_h - \vec{r}_h) &= \mathscr{F}_{\phi_h - \varphi_h}(\vec{v}_h) \quad \forall \vec{v}_h := (\boldsymbol{v}_h^B, z_h^B, \boldsymbol{v}_h^D) \in \mathbf{H}, \\
\mathscr{B}(\vec{u}_h - \vec{w}_h, \vec{q}_h) &= 0 \qquad\qquad \forall \vec{q}_h := (q_h^B, q_h^D, \xi_h) \in \mathbf{Q}_{0,h}.
\end{aligned}
$$

Moreover, since $\mathbf{S}_h^{\mathrm{flow}}(\phi_h - \varphi_h) = \mathbf{S}_h^{\mathrm{flow}}(\phi_h) - \mathbf{S}_h^{\mathrm{flow}}(\varphi_h)$ the bound (4.17) follows directly from estimate (4.9). □

LEMMA 4.7 Let $L_{f_{\mathrm{bk}}}, L_\vartheta$ and $\widetilde{\alpha}$ be the constants given in (2.7), (2.6) and Lemma 3.2, respectively. Then,

$$
\begin{aligned}
\|\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h) - \mathbf{S}_h^{\mathrm{adv}}(\varphi_h, \boldsymbol{w}_h)\|_{1,\Omega} \leq \frac{1}{\widetilde{\alpha}} \Big\{ &L_{f_{\mathrm{bk}}}|\boldsymbol{g}|\|\phi_h - \varphi_h\|_{0,\Omega} + L_\vartheta \|\nabla \mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h)\|_{\mathbf{L}^4(\Omega)} \|\phi_h - \varphi_h\|_{\mathbf{L}^4(\Omega)} \\
&+ \|\mathbf{S}_h^{\mathrm{adv}}(\phi_h, \boldsymbol{u}_h)\|_{\mathbf{L}^{2\widetilde{q}}(\Omega)} \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{\mathbf{L}^{2\widetilde{p}}(\Omega)} \Big\},
\end{aligned}
\tag{4.18}
$$

$\forall (\phi_h, \boldsymbol{u}_h), (\varphi_h, \boldsymbol{w}_h) \in \mathrm{H}_h^\phi \times \mathbf{Z}_h^r$, and where $\widetilde{p}, \widetilde{q} \in [1, +\infty)$ are such that $1/\widetilde{p} + 1/\widetilde{q} = 1$.

Proof. Given $(\phi_h, \boldsymbol{u}_h), (\varphi_h, \boldsymbol{w}_h)$ as stated we let $\widetilde{\phi}_h := \mathbf{S}_h^{\text{adv}}(\phi_h, \boldsymbol{u}_h)$ and $\widetilde{\varphi}_h := \mathbf{S}_h^{\text{adv}}(\varphi_h, \boldsymbol{w}_h)$, that is

$$\mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\widetilde{\phi}_h, \widetilde{\psi}_h) = \mathscr{H}_{\phi_h}(\widetilde{\psi}_h) \quad \text{and} \quad \mathscr{C}_{\varphi_h, \boldsymbol{w}_h}(\widetilde{\varphi}_h, \widetilde{\psi}_h) = \mathscr{H}_{\varphi_h}(\widetilde{\psi}_h) \qquad \forall \widetilde{\psi}_h \in \mathrm{H}_h^{\phi}.$$

Next we proceed analogously as in the proof of Lemma 3.7. We apply Cauchy–Schwarz's inequality, the Lipschitz continuity estimates (2.6) and (2.7), and Hölder's inequality to the second term on the right-hand side for the discrete version of (3.24). For the last term we use an $\mathrm{L}^4 - \mathrm{L}^4 - \mathrm{L}^2$ argument. This gives

$$\widetilde{\alpha} \|\widetilde{\phi}_h - \widetilde{\varphi}_h\|_{1,\Omega}^2 \leq \left\{ L_{f_{\text{bk}}} |\boldsymbol{g}| \|\phi_h - \varphi_h\|_{0,\Omega} + \|\widetilde{\varphi}_h\|_{\mathrm{L}^{2\widetilde{q}}(\Omega)} \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{\mathbf{L}^{2\widetilde{p}}(\Omega)} \right\} |\widetilde{\phi}_h - \widetilde{\varphi}_h|_{1,\Omega}$$
$$+ L_\vartheta \|\phi_h - \varphi_h\|_{\mathrm{L}^4(\Omega)} \|\nabla \widetilde{\varphi}_h\|_{\mathrm{L}^4(\Omega)} |\widetilde{\phi}_h - \widetilde{\varphi}_h|_{1,\Omega}.$$

Then, since the elements of H_h^{ϕ} are piecewise polynomials it follows that $\|\nabla \widetilde{\varphi}_h\|_{\mathrm{L}^4(\Omega)} < +\infty$, and hence the foregoing equation yields (4.18). $\qquad\square$

We now can establish a Lipschitz continuity-type estimate for $\mathbf{T}_h$.

LEMMA 4.8   Given $r > 0$ and $\mathbf{W}_h$ as in (4.15) assume the condition (4.16). Then, there exist constants $C_3, C_4, C_5 > 0$, depending only upon $L_{f_{\text{bk}}}, L_\vartheta, \widetilde{\alpha}, C_2, \widetilde{C}_s$ (*cf.* (2.7), (2.6), Lemma 3.2, Lemma 4.6 and (4.14)) and the data, such that, for all $(\phi_h, \boldsymbol{u}_h), (\varphi_h, \boldsymbol{w}_h) \in \mathbf{W}_h$, there holds

$$|||\mathbf{T}_h(\phi_h, \boldsymbol{u}_h) - \mathbf{T}_h(\varphi_h, \boldsymbol{w}_h)||| \leq C_3 \|\phi_h - \varphi_h\|_{0,\Omega} + C_4 \|\nabla \mathbf{S}_h^{\text{adv}}(\phi_h, \boldsymbol{u}_h)\|_{\mathrm{L}^4(\Omega)} \|\phi_h - \varphi_h\|_{\mathrm{L}^4(\Omega)}$$
$$+ C_5 \|\mathbf{S}_h^{\text{adv}}(\phi_h, \boldsymbol{u}_h)\|_{\mathrm{L}^{2\widetilde{q}}(\Omega)} \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{\mathbf{L}^{2\widetilde{p}}(\Omega)},$$

where $\widetilde{p}, \widetilde{q} \in [1, +\infty)$ are such that $1/\widetilde{p} + 1/\widetilde{q} = 1$.

Proof. It suffices to recall from Section 4.1 that

$$\mathbf{T}_h(\psi_h, \boldsymbol{v}_h) := (\mathbf{S}_h^{\text{adv}}(\psi_h, \boldsymbol{v}_h), \mathbf{S}_h^{\text{flow}}(\mathbf{S}_h^{\text{adv}}(\psi_h, \boldsymbol{v}_h))) \qquad \forall (\psi_h, \boldsymbol{v}_h) \in Y_h,$$

and then apply estimate (4.14) and Lemmas 4.6 and 4.7. $\qquad\square$

Consequently, from the foregoing Lemma, choosing $2\widetilde{p}$ and $2\widetilde{q}$ as in the proof of Lemma 4.2 that is $2\widetilde{p} = s$ and hence $2\widetilde{q} := \frac{2s}{s-2} > 1$, and employing the continuous injection $\mathrm{i} : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$, the continuity of $\mathbf{T}_h$ is derived. Thanks to Brouwer's fixed-point theorem (Ciarlet, 2013, Theorem 9.9-2) and Lemmas 4.5 and 4.8, we establish the main result of this section.

THEOREM 4.9   Under the assumptions of Lemma 4.5 the Galerkin scheme (4.2) has at least one solution $(\vec{\boldsymbol{u}}_h, \vec{p}_h, \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h} \times \mathrm{H}_h^{\phi}$ with $(\phi_h, \boldsymbol{u}_h) \in \mathbf{W}_h$. Moreover,

$$\|\phi_h\|_{1,\Omega} \leq C_1 \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}|,$$
$$\|(\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H} \times \mathbf{Q}_0} \leq C_1 C_2 \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}| \left\{ \|\boldsymbol{f}_{\mathrm{B}}\|_{\infty, \Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty, \Omega_{\mathrm{D}}} \right\},$$

where C_1, C_2 and γ_2 are the constants provided by Lemmas 4.1 and 4.2, as well as (2.5), respectively.

REMARK 4.10 The extension to 3D case of our discrete fixed-point analysis is based on a new range for the parameter s in (4.3), Lemma 4.2 and Hypothesis 4.14. More precisely, one needs now to take $s > 3$ in (4.3) to guarantee the compactness (and hence the continuity) of the injection $\mathrm{i}_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^{\frac{3s}{s-3}}(\Omega)$. This is crucial in the proof of Lemma 4.2.

5. *A priori* error estimate

Given $(\vec{u}, \vec{p}, \phi) \in \mathbf{H} \times \mathbf{Q}_0 \times \mathrm{H}^1_{\Gamma_0}(\Omega)$ with $\phi \in \mathrm{W}$ and $(\vec{u}_h, \vec{p}_h, \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h} \times \mathrm{H}^\phi_h$ with $(\phi_h, \boldsymbol{u}_h) \in \mathbf{W}_h$ we aim to derive suitable *a priori* error estimates. To this end, we first observe from (3.7) and (4.2) that the above problems can be rewritten as follows:

$$(BD) \begin{cases} \mathscr{A}(\vec{u}, \vec{v}) + \mathscr{B}(\vec{v}, \vec{p}) = \mathscr{F}_\phi(\vec{v}) & \forall \vec{v} \in \mathbf{H}, \\ \mathscr{B}(\vec{u}, \vec{q}) = 0 & \forall \vec{q} \in \mathbf{Q}_0, \end{cases}$$

$$(BD_h) \begin{cases} \mathscr{A}(\vec{u}_h, \vec{v}_h) + \mathscr{B}(\vec{v}_h, \vec{p}_h) = \mathscr{F}_{\phi_h}(\vec{v}_h) & \forall \vec{v}_h \in \mathbf{H}_h, \\ \mathscr{B}(\vec{u}_h, \vec{q}_h) = 0 & \forall \vec{q}_h \in \mathbf{Q}_{0,h}, \end{cases}$$

and

$$(T) \qquad \mathscr{C}_{\boldsymbol{u}}(\phi, \psi) = \mathscr{H}_\phi(\psi) \qquad \forall \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega),$$

$$(T_h) \quad \mathscr{C}_{\boldsymbol{u}_h}(\phi_h, \psi_h) = \mathscr{H}_{\phi_h}(\psi_h) \qquad \forall \psi_h \in \mathrm{H}^\phi_h.$$

LEMMA 5.1 There exists $C_{\mathrm{B,D}} > 0$, independent of h, such that

$$\|(\vec{u}, \vec{p}) - (\vec{u}_h, \vec{p}_h)\|_{\mathbf{H} \times \mathbf{Q}_0}$$

$$\leq C_{\mathrm{B,D}} \left\{ \mathrm{dist}(\vec{u}, \mathbf{H}_h) + \mathrm{dist}(\vec{p}, \mathbf{Q}_{0,h}) + \|\phi - \phi_h\|_{1,\Omega} \left(\|\boldsymbol{f}_\mathrm{B}\|_{\infty,\Omega_\mathrm{B}} + \|\boldsymbol{f}_\mathrm{D}\|_{\infty,\Omega_\mathrm{D}} \right) \right\}. \tag{5.1}$$

Proof. We derive a Strang-type error estimate for (BD) and (BD_h). For this we proceed similarly as in Gatica *et al.* (2016, Section 4) and deduce the existence of a positive constant $C_{\mathrm{B,D}}$, independent of h, such that

$$\|(\vec{u}, \vec{p}) - (\vec{u}_h, \vec{p}_h)\|_{\mathbf{H} \times \mathbf{Q}_0} \leq C_{\mathrm{B,D}} \left\{ \mathrm{dist}(\vec{u}, \mathbf{H}_h) + \mathrm{dist}(\vec{p}, \mathbf{Q}_{0,h}) + \|\mathscr{F}_\phi - \mathscr{F}_{\phi_h}\|_{\mathbf{H}'_h} \right\}. \tag{5.2}$$

Next, according to the definition of $\mathscr{F}_\phi$ and $\mathscr{F}_{\phi_h}$ (*cf.* (3.3)), and applying Cauchy–Schwarz's inequality, we easily deduce that

$$\|\mathscr{F}_\phi - \mathscr{F}_{\phi_h}\|_{\mathbf{H}'_h} \leq \|\phi - \phi_h\|_{0,\Omega} \left(\|\boldsymbol{f}_\mathrm{B}\|_{\infty,\Omega_\mathrm{B}} + \|\boldsymbol{f}_\mathrm{D}\|_{\infty,\Omega_\mathrm{D}} \right). \tag{5.3}$$

The proof is completed after replacing (5.3) into (5.2). $\qquad\square$

We now derive a Céa estimate for the term $\|\phi - \phi_h\|_{1,\Omega}$, focusing in the 2D case. For notational convenience we define the following constants, all independent of the data $\boldsymbol{g}, \boldsymbol{f}_\mathrm{B}$ and $\boldsymbol{f}_\mathrm{D}$,

$$K_1 := C_{\mathbf{S}^{\mathrm{adv}}} \left\{ L_{f_{\mathrm{bk}}} + L_\vartheta \, C_\delta \, C^*_\delta \, \widehat{C}_{\mathbf{S}^{\mathrm{adv}}}(r) \, \gamma_2 \, |\Omega|^{1/2} \right\}, \quad K_2 := C_{\mathbf{S}^{\mathrm{adv}}} \, (r C_s + \beta + \vartheta_2) + 1 \quad \text{and} \quad K_3 = C_{\mathbf{S}^{\mathrm{adv}}}.$$

Here $\widehat{C}_{\mathbf{S}^{\mathrm{adv}}}(r)$ and $C_{\mathbf{S}^{\mathrm{adv}}}$ are given in (3.20) and (3.15), and C_s, C_δ, C_δ^* are the boundedness constants of the continuous injections

$$i_s : H^1(\Omega) \longrightarrow L^{\frac{2s}{s-2}}(\Omega), \quad i : H^1(\Omega) \longrightarrow L^{2/\delta}(\Omega), \quad i_\delta : H^\delta(\Omega) \longrightarrow L^{\delta^*}(\Omega), \tag{5.4}$$

respectively, where $s > 2$, $\delta \in (0, 1)$ and $\delta^* := 2/(1 - \delta)$. In addition, in order to handle a term in the derivation of the Céa estimate for $\|\phi - \phi_h\|_{1,\Omega}$, we will further assume that $\phi \in L^\infty(\Omega)$.

LEMMA 5.2   Assume that $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$ and that the data $\boldsymbol{g}$ satisfy

$$K_1 |\boldsymbol{g}| \leq \frac{1}{2}. \tag{5.5}$$

Then, there holds

$$\|\phi - \phi_h\|_{1,\Omega} \leq 2K_2 \operatorname{dist}(\phi, H_h^\phi) + 2K_3 \|\phi\|_{\infty,\Omega} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}. \tag{5.6}$$

Proof.   Triangle inequality immediately gives

$$\|\phi - \phi_h\|_{1,\Omega} \leq \|\phi - \varphi_h\|_{1,\Omega} + \|\phi_h - \varphi_h\|_{1,\Omega} \qquad \forall \, \varphi_h \in H_h^\phi. \tag{5.7}$$

Then we employ the ellipticity of the bilinear form $\mathscr{C}_{\phi_h, \boldsymbol{u}_h}$ with constant $\widetilde{\alpha}$, and subsequently add and subtract one term, we deduce that

$$\begin{aligned}
\widetilde{\alpha} \|\phi_h - \varphi_h\|_{1,\Omega}^2 &\leq \mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\phi_h - \varphi_h, \phi_h - \varphi_h) \\
&\leq \left| \mathscr{H}_{\phi_h}(\phi_h - \varphi_h) - \mathscr{H}_\phi(\phi_h - \varphi_h) \right| + \left| \mathscr{C}_{\phi, \boldsymbol{u}}(\phi, \phi_h - \varphi_h) - \mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\varphi_h, \phi_h - \varphi_h) \right|.
\end{aligned} \tag{5.8}$$

Next, according to (3.10) and applying Cauchy–Schwarz's inequality, we get

$$\left| \mathscr{H}_{\phi_h}(\phi_h - \varphi_h) - \mathscr{H}_\phi(\phi_h - \varphi_h) \right| \leq L_{f_{\mathrm{bk}}} |\boldsymbol{g}| \|\phi - \phi_h\|_{0,\Omega} |\phi_h - \varphi_h|_{1,\Omega}. \tag{5.9}$$

In turn, adding and subtracting suitable expressions and then applying Hölder's inequality, the upper bound of ϑ (*cf.* (2.5)) and its Lipschitz continuity (*cf.* (2.6)) and using $\phi \in L^\infty(\Omega)$, we find that

$$\begin{aligned}
&\left| \mathscr{C}_{\phi, \boldsymbol{u}}(\phi, \phi_h - \varphi_h) - \mathscr{C}_{\phi_h, \boldsymbol{u}_h}(\varphi_h, \phi_h - \varphi_h) \right| \\
&\quad \leq \quad L_\vartheta \|\phi - \phi_h\|_{L^{2q}(\Omega)} \|\nabla\phi\|_{L^{2p}(\Omega)} |\phi_h - \varphi_h|_{1,\Omega} + \vartheta_2 |\phi - \varphi_h|_{1,\Omega} |\phi_h - \varphi_h|_{1,\Omega} \\
&\quad + \quad \|\phi\|_{\infty,\Omega} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} |\phi_h - \varphi_h|_{1,\Omega} + \|\phi - \varphi_h\|_{L^{2\widetilde{q}}(\Omega)} \|\boldsymbol{u}_h\|_{L^{2\widetilde{p}}(\Omega)} |\phi_h - \varphi_h|_{1,\Omega} \\
&\quad + \quad \beta \|\phi - \varphi_h\|_{0,\Omega} \|\phi_h - \varphi_h\|_{0,\Omega},
\end{aligned} \tag{5.10}$$

where $p, q, \widetilde{p}, \widetilde{q} \in [1, +\infty)$ are such that $1/p + 1/q = 1$ and $1/\widetilde{p} + 1/\widetilde{q} = 1$. In this way choosing $2p$ and $2q$ as in (3.17), $2\widetilde{p}$ and $2\widetilde{q}$ as in (4.12), and applying the continuous embeddings i, i_δ, i_s, estimate (3.20)

and the fact that $\|\boldsymbol{u}_h\|_{\mathbf{L}^s(\Omega)} \leq r$, it follows from (5.10) that

$$
\begin{aligned}
\Big| &\mathscr{C}_{\phi,\boldsymbol{u}}(\phi, \phi_h - \varphi_h) - \mathscr{C}_{\phi_h,\boldsymbol{u}_h}(\varphi_h, \phi_h - \varphi_h) \Big| \\
&\leq \quad L_\vartheta \, C_\delta \, C_\delta^* \, \widehat{C}_{\mathbf{S}^{\mathrm{adv}}}(r) \, \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}| \, \|\phi - \phi_h\|_{1,\Omega} \, \|\phi_h - \varphi_h\|_{1,\Omega} \\
&+ \quad \vartheta_2 \, \|\phi - \varphi_h\|_{1,\Omega} \, \|\phi_h - \varphi_h\|_{1,\Omega} + r \, C_s \, \|\phi - \varphi_h\|_{1,\Omega} \, \|\phi_h - \varphi_h\|_{1,\Omega} \\
&+ \quad \|\phi\|_{\infty,\Omega} \, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \, \|\phi_h - \varphi_h\|_{1,\Omega} + \beta \, \|\phi - \varphi_h\|_{1,\Omega} \, \|\phi_h - \varphi_h\|_{1,\Omega}.
\end{aligned}
\tag{5.11}
$$

Finally, we replace (5.9) and (5.11) into (5.8). The resulting estimate is combined with (5.7), and after recalling (from the proof of Lemma 3.2) that $\widetilde{\alpha} = C_{\mathbf{S}^{\mathrm{adv}}}^{-1}$ and applying several algebraic manipulations, we find that

$$
\|\phi - \phi_h\|_{1,\Omega} \leq K_1 \, |\boldsymbol{g}| \, \|\phi - \phi_h\|_{1,\Omega} + K_2 \, \|\phi - \varphi_h\|_{1,\Omega} + \|\phi\|_{\infty,\Omega} \, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \qquad \forall \, \varphi_h \in \mathrm{H}_h^\phi.
$$

Then, we use the data constraint (5.5) and take the infimum $\varphi_h \in \mathrm{H}_h^\phi$, yielding (5.2). $\qquad\square$

REMARK 5.3 Extending the proof of Lemma 5.2 to the 3D case requires to modify the choice of parameters δ and δ^*, and hence $2p$ and $2q$, similarly as in (3.17). In turn, as discussed in Remark 4.10, one needs to take $s > 3$ and then choose $2\widetilde{p}$ and $2\widetilde{q}$ analogously as in (4.12). This permits to invoke the continuous injection $\mathrm{i}_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^{\frac{3s}{s-3}}(\Omega)$.

We now combine Lemmas 5.1 and 5.2 to derive the Céa estimate for the total error $\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} + \|\phi - \phi_h\|_{1,\Omega}$. Replacing the estimate for $\|\phi - \phi_h\|_{1,\Omega}$ given by (5.6) into the second term on the right-hand side of (5.1), we find that

$$
\begin{aligned}
\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} \leq\ &C_{\mathrm{B,D}} \Big\{ \mathrm{dist}(\vec{\boldsymbol{u}}, \mathbf{H}_h) + \mathrm{dist}(\vec{p}, \mathbf{Q}_{0,h}) \Big\} + \widetilde{K} \, \mathrm{dist}(\phi, \mathrm{H}_h^\phi) \\
&+ 2 C_{\mathrm{B,D}} \|\phi\|_{\infty,\Omega} \Big(\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \Big) \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega},
\end{aligned}
$$

where

$$
\widetilde{K} := 2 K_2 C_{\mathrm{B,D}} \left(\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \right).
$$

Assuming now that the data $\boldsymbol{f}_{\mathrm{B}}$ and $\boldsymbol{f}_{\mathrm{D}}$ satisfy

$$
C_{\mathrm{B,D}} \|\phi\|_{\infty,\Omega} \|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + C_{\mathrm{B,D}} \|\phi\|_{\infty,\Omega} \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \leq \frac{1}{4}
\tag{5.12}
$$

we can conclude that

$$
\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} \leq 2 C_{\mathrm{B,D}} \Big\{ \mathrm{dist}(\vec{\boldsymbol{u}}, \mathbf{H}_h) + \mathrm{dist}(\vec{p}, \mathbf{Q}_{0,h}) \Big\} + 2 \widetilde{K} \, \mathrm{dist}(\phi, \mathrm{H}_h^\phi).
\tag{5.13}
$$

THEOREM 5.4 Assume that $\phi \in \mathrm{H}^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and that the data $\boldsymbol{f}_{\mathrm{B}}, \boldsymbol{f}_{\mathrm{D}}$ and $\boldsymbol{g}$ are sufficiently small so that $K_1 |\boldsymbol{g}| \leq \frac{1}{2}$ and (5.12) hold. Then, there exists a positive constant C independent of h, such that

$$
\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} + \|\phi - \phi_h\|_{1,\Omega} \leq C \left\{ \mathrm{dist}(\vec{\boldsymbol{u}}, \mathbf{H}_h) + \mathrm{dist}(\vec{p}, \mathbf{Q}_{0,h}) + \mathrm{dist}(\phi, \mathrm{H}_h^\phi) \right\}.
\tag{5.14}
$$

Proof. It follows straightforwardly from the Céa estimates (5.13) and (5.6). □

The following theorem provides the rate of convergence of our Galerkin scheme (4.2).

THEOREM 5.5 Let $\mathbf{H}_h := \mathbf{H}_h^{\mathrm{B}} \times \mathbf{H}_{0,h}^{\mathrm{B}} \times \mathbf{H}_h^{\mathrm{D}}, \mathbf{Q}_{h,0} := \mathrm{Q}_{h,0}^{\mathrm{B}} \times \mathrm{Q}_h^{\mathrm{D}} \times \mathrm{Q}_{\tilde{h}}^{\Sigma}$ and H_h^{ϕ}, be the subspaces specified in Section 4. In addition, let $(\vec{\boldsymbol{u}}, \vec{p}, \phi) := ((\boldsymbol{u}_{\mathrm{B}}, \boldsymbol{\omega}_{\mathrm{B}}, \boldsymbol{u}_{\mathrm{D}}), (p_{\mathrm{B}}, p_{\mathrm{D}}, \lambda), \phi) \in \mathbf{H} \times \mathbf{Q}_0 \times \mathrm{H}_{\Gamma_0}^1(\Omega)$ and $(\vec{\boldsymbol{u}}_h, \vec{p}_h, \phi_h)$ $:= ((\boldsymbol{u}_h^{\mathrm{B}}, \boldsymbol{\omega}_h^{\mathrm{B}}, \boldsymbol{u}_h^{\mathrm{D}}), (p_h^{\mathrm{B}}, p_h^{\mathrm{D}}, \lambda_{\tilde{h}}), \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h} \times \mathrm{H}_h^{\phi}$ be the solutions of the continuous and discrete problems (3.7) and (4.2), respectively. Assume that $\boldsymbol{u}_{\star} \in \mathbf{H}^l(\Omega_{\star}), \operatorname{div} \boldsymbol{u}_{\star} \in \mathrm{H}^l(\Omega_{\star}), p_{\star} \in \mathrm{H}^l(\Omega_{\star})$ where $\star \in \{\mathrm{B}, \mathrm{D}\}, \boldsymbol{\omega}_{\mathrm{B}} \in \mathbf{H}^l(\mathbf{curl}; \Omega_{\mathrm{B}}), \lambda \in \mathrm{H}^{1/2+l}(\Sigma)$ and $\phi \in \mathrm{H}_{\Gamma_0}^{1+l}(\Omega)$, for some $l \in (0, 1)$ (when $n = 2$) or $l \in (1/2, 1)$ (when $n = 3$). Then, there exists $C > 0$ and $\widetilde{C} > 0$ independent of h and $\tilde{h}$ such that

$$
\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H} \times \mathbf{Q}_0} + \|\phi - \phi_h\|_{1,\Omega}
$$

$$
\leq Ch^{\min\{l,k+1\}} \Big\{ \|\boldsymbol{u}_{\mathrm{B}}\|_{l,\Omega_{\mathrm{B}}} + \|\operatorname{div}(\boldsymbol{u}_{\mathrm{B}})\|_{l,\Omega_{\mathrm{B}}} + \|\boldsymbol{\omega}_{\mathrm{B}}\|_{\mathbf{H}^l(\mathbf{curl};\Omega_{\mathrm{B}})} + \|\boldsymbol{u}_{\mathrm{D}}\|_{l,\Omega_{\mathrm{D}}}
$$

$$
+ \|\operatorname{div}(\boldsymbol{u}_{\mathrm{D}})\|_{l,\Omega_{\mathrm{D}}} + \|p_{\mathrm{B}}\|_{l,\Omega_{\mathrm{B}}} + \|p_{\mathrm{D}}\|_{l,\Omega_{\mathrm{D}}} + \|\phi\|_{1+l,\Omega} \Big\} + \widetilde{C}\tilde{h}^l \|\lambda\|_{l+1/2,\Sigma}.
$$

Proof. It follows directly from the Céa estimate (5.14) and the approximation properties of the finite element spaces (see Alvarez *et al.*, 2016b, Section 4.2.2 and Ciarlet, 1978). □

6. Numerical examples

Test 1. We begin this section with an accuracy test, where we construct smooth solutions satisfying (2.1)–(2.3) on $\Omega = (0, 2) \times (0, 1)$. The Brinkman and Darcy domains are on the left and right parts of Ω, respectively, and are separated by the curved interface Σ defined by the parameterization

$$
(0, 1) \ni t \mapsto (x_1, x_2) = (1 + 0.15[1/2 - |t - 1/2|]\cos(6\pi t - 3\pi), t),
$$

on which the normal vector is considered pointing to Ω_{B}. The proposed exact solutions are given by

$$
\boldsymbol{u} = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad \boldsymbol{u}_{\mathrm{B}} = \boldsymbol{u}|_{\Omega_{\mathrm{B}}}, \quad \boldsymbol{u}_{\mathrm{D}} = \boldsymbol{u}|_{\Omega_{\mathrm{D}}}, \quad \boldsymbol{\omega}_{\mathrm{B}} = 2\pi \sin(\pi x_1)\sin(\pi x_2),
$$

$$
p = (x_1 - 1/2)(x_2 - 1/2), \quad p_{\mathrm{B}} = p|_{\Omega_{\mathrm{B}}}, \quad p_{\mathrm{D}} = p|_{\Omega_{\mathrm{D}}}, \quad \phi = \frac{5}{2}x_1^2(2 - x_1)x_2(1 - x_2),
$$

from which the necessary forcing, boundary and source terms are generated. The concentration-dependent functions are $\vartheta(\phi) = \phi + (1 - c\phi)^2$ and $f_{\mathrm{bk}}(\phi) = c\phi(1 - c\phi)^2$, and the remaining physical parameters assume the values $c = 1/2, \boldsymbol{f}_{\mathrm{B}} = (1, 0)^2, \boldsymbol{f}_{\mathrm{D}} = (0.1, 0)^{\mathrm{t}}, \mathbb{K}_{\mathrm{B}} = 0.05, \mathbb{K}_{\mathrm{D}} = 0.01, \beta = 0.4\chi|_{\Omega_{\mathrm{B}}} + 0.1\chi|_{\Omega_{\mathrm{D}}}, \boldsymbol{k} = (0, -1)^{\mathrm{t}}$. We recall that two different splittings of the domain boundary $\partial\Omega$ are assumed. First, the distribution of the Brinkman and Darcy boundaries follows the sketch presented in the left panel of Fig. 2. According to (2.4), on Γ_{B} we set slip velocities $\boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} = 0$ and zero tangential vorticity (in this 2D case, it translates to fix the scalar vorticity to zero), but on Σ we prescribe the vorticity by its exact solution. Normal Darcy velocities are fixed on Γ_{D}: $\boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{n} = 0$. Secondly, by construction, the concentration normal flux is zero on the left side of Γ_{B}, which constitutes the Neumann boundary Γ_{N}. The remainder of $\partial\Omega$ conforms the Dirichlet boundary Γ_0, where we impose $\phi = 0$ (see Fig. 2, right). Both domains are rendered with a small gap on the interface, for visualization purposes.
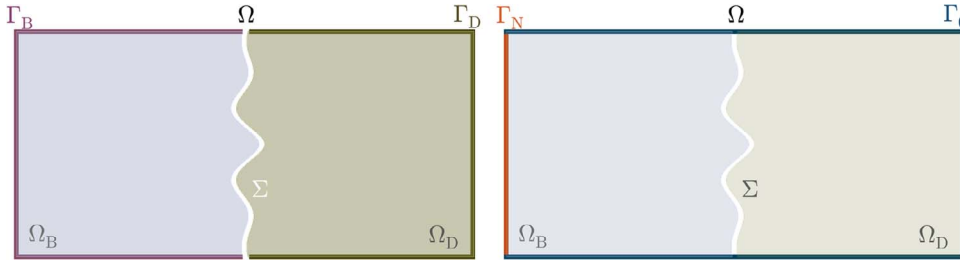
FIG. 2. Test 1. Domain and boundary configuration for the Brinkman–Darcy problem, where $\partial\Omega = \Gamma_{\mathrm{B}} \cup \Gamma_{\mathrm{D}}$ (left panel), and for the transport equation where $\partial\Omega = \Gamma_{\mathrm{N}} \cup \Gamma_0$ (right). The unit normal vector on the interface points towards Ω_{B}.
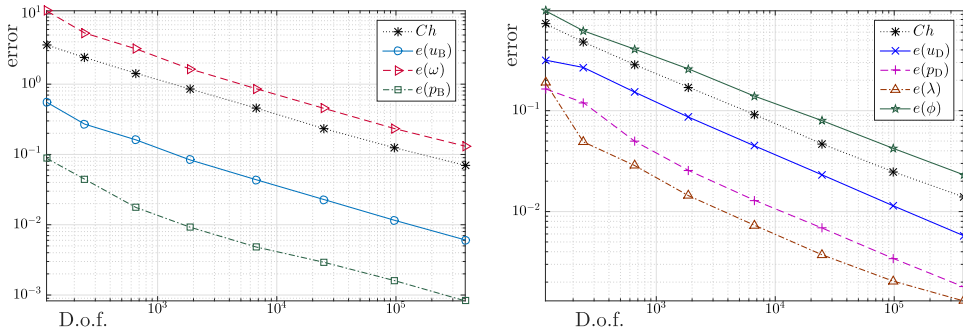

FIG. 3. Test 1. Convergence history for the lowest-order approximation of the coupled Brinkman–Darcy-transport problem.

As usual, to determine the convergence of the method we generate a sequence of successively refined triangulations of Ω (and conforming partitions for Ω_{B}, Ω_{D} and Σ) and proceed to compute errors, and they decay rates according to

$$e(\boldsymbol{u}_{\mathrm{B}}) = \|\boldsymbol{u}_{\mathrm{B}} - \boldsymbol{u}_{\mathrm{B}h}\|_{\mathrm{div},\Omega_{\mathrm{B}}}, \; e(\boldsymbol{\omega}_{\mathrm{B}}) = \|\boldsymbol{\omega}_{\mathrm{B}} - \boldsymbol{\omega}_{\mathrm{B}h}\|_{\mathbf{curl},\Omega_{\mathrm{B}}}, \; e(\boldsymbol{u}_{\mathrm{D}}) = \|\boldsymbol{u}_{\mathrm{D}} - \boldsymbol{u}_{\mathrm{D}h}\|_{\mathrm{div},\Omega_{\mathrm{D}}},$$

$$e(p_{\mathrm{B}}) = \|p_{\mathrm{B}} - p_{\mathrm{B}h}\|_{0,\Omega_{\mathrm{B}}}, \; e(p_{\mathrm{D}}) = \|p_{\mathrm{D}} - p_{\mathrm{D}h}\|_{0,\Omega_{\mathrm{D}}}, \; e(\lambda) = \|\lambda - \lambda_h\|_{0,\Sigma}\|\lambda - \lambda_h\|_{1,\Sigma},$$

$$e(\phi) = \|\phi - \phi_h\|_{1,\Omega}, \; r(\cdot) = -2\log(e(\cdot)/\hat{e}(\cdot))[\log(N/\hat{N})]^{-1},$$

where e and $\hat{e}$ denote errors produced on two consecutive meshes associated with schemes with N and $\hat{N}$ degrees of freedom (D.o.f.), respectively. The results are collected in Fig. 3, where we plot the decaying of individual errors with the meshsize, for the lowest-order scheme. All panels indicate an $O(h)$ convergence, as anticipated by Theorem 5.5. We point out that an average of seven Picard steps (accounting for the coupling between the Brinkman–Darcy and transport problems) are required to reach the stopping tolerance of $1e - 6$, whereas an average of three Newton steps are sufficient to achieve convergence of the inner linearization step (with a tolerance of $1e - 7$) for the nonlinear advection–diffusion problem. We also portray the approximate solutions obtained with the proposed method on a fine mesh (see Fig. 4).

An important assumption in the solvability analysis was an additional regularity for the discrete velocity, as stated in Hypothesis 4.4. Even if proving this assumption can be very difficult we can at least provide numerical evidence of its validity for the finite element spaces we employ here. For instance, taking the regularity index as $s = 2.5$ and obtaining approximate solutions in the same
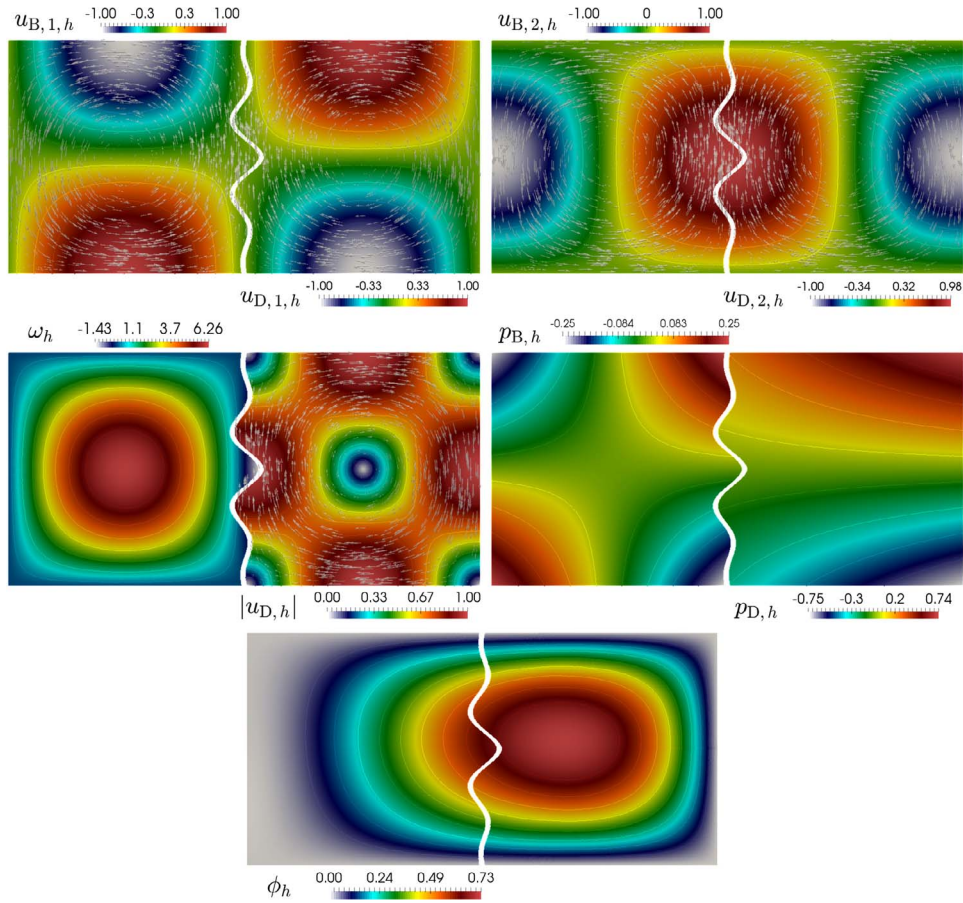
Fig. 4. Test 1. Approximate velocity components, vorticity, pressure and concentration, rendered on both Brinkman and Darcy domains.

refinement levels as mentioned above (whose error history is depicted in Fig. 3), the values reported in Table 1 are produced. The ratios between the $\mathbf{L}^s$−norm of the discrete velocity and the H^1−norm of the concentration are tabulated in the last column and they suggest that the constant $\widetilde{C}_s$ is indeed uniform. These values correspond to the approximate solutions after convergence of the Picard algorithm. Taking the last mesh refinement as an example we can also see in Table 2 how the constant $\widetilde{C}_s$ evolves throughout the nine fixed-point steps.

Test 2. Our second example addresses the applicability of the formulation and the associated numerical scheme in the simulation of groundwater flow, where we have followed the setup adopted in Test 4.1 of Ceşmelioğlu *et al.* (see also Alvarez *et al.*, 2016b). The computational domain now corresponds to the rectangle $\Omega = (0, 12) \times (0, 6)$ (in square metres), where the Brinkman domain (with a maximum height of 4 m) is on the top and the Darcy subdomain (with a maximum height of 2.25 m) on the bottom. The subdomains are separated by a step-shaped polygonal interface (see a sketch in the top-left panel of Fig. 5, where we also depict sample triangular meshes). We consider $\mu = 1\,\mathrm{Pa{\cdot}s}$, $\beta = 0.7\chi|_{\Omega_\mathrm{B}} + 0.07\chi|_{\Omega_\mathrm{D}}$, and the permeabilities are again isotropic and assume the values $\mathbb{K}_\mathrm{B} = 1e-6$

TABLE 1   *Test 1. Convergence history for velocity and concentration, and illustration of Hypothesis* 4.4 *for the lowest-order scheme. Here we have used* $s = 2.5$

D.o.f.	h	$e(\boldsymbol{u}_{\mathrm{B}})$	$r(\boldsymbol{u}_{\mathrm{B}})$	$e(\boldsymbol{u}_{\mathrm{D}})$	$r(\boldsymbol{u}_{\mathrm{D}})$	$e(\phi)$	$r(\phi)$	$\|\mathbf{S}^{\mathrm{flow}}(\phi_h)\|_{\mathbf{L}^s(\Omega)}$	$\|\phi_h\|_{1,\Omega}$	$\widetilde{C}_s$
111	0.723	0.562	–	0.371	–	0.823	–	0.644	1.791	0.5219
237	0.479	0.269	1.786	0.267	0.791	0.622	0.678	0.920	1.825	0.5700
664	0.285	0.160	0.997	0.155	1.045	0.407	0.819	1.022	1.890	0.5695
1920	0.170	0.082	1.308	0.086	1.149	0.264	0.837	0.903	1.894	0.5689
6625	0.099	0.043	1.160	0.044	1.117	0.161	0.975	1.082	1.907	0.5670
24907	0.048	0.022	0.931	0.022	0.950	0.083	1.011	1.080	1.911	0.5656
97907	0.023	0.011	0.956	0.011	0.962	0.022	1.041	1.079	1.911	0.5643
382115	0.014	0.006	1.066	0.006	1.161	0.012	1.008	1.078	1.911	0.5643

TABLE 2   *Test 1. Illustration of Hypothesis* 4.4. *Variation of the constant* $\widetilde{C}_s$ *for the lowest-order scheme on the last refinement level*

0.55012	0.56600	0.56450	0.56435	0.56435	0.56435	0.56435	0.56435	0.56435

m^2, $\mathbb{K}_{\mathrm{D}} = 1e - 7$ m^2. Normal velocities are imposed everywhere on $\partial\Omega$. On the top segment of Γ_{B} and in all Γ_{D} these are simply zero, whereas on the left and right sides of the Brinkman domain we prescribe the parabolic profiles

$$\boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} = \frac{1}{4}(y-4)(y-8), \quad \text{and} \quad \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} = \frac{3}{16}(y-4)(8-y),$$

respectively, as well as the compatible vorticity $\omega_{\mathrm{B}} = \frac{1}{2}(y-6)$ and $\omega_{\mathrm{B}} = \frac{3}{8}(y-6)$, respectively. Regarding the transport equation on the left side of the Brinkman domain (denoted by Γ_0) we impose a maximum solute concentration $\phi = \phi_{\max} = 0.99$, whereas zero total flux is considered on $\Gamma_{\mathrm{N}} = \partial\Omega \setminus \Gamma_0$. The nonlinear diffusion assumes the form $\vartheta(\phi, \boldsymbol{u}) = \exp(-\frac{1}{4}\phi) + 0.01|\boldsymbol{u}| + \beta$, and the flux is simply linear $f_{\mathrm{bk}}(\phi) = 0.001\phi$. We take $\boldsymbol{k} = (0,-1)^{\mathrm{t}}$ and assume that an external source modulates the Brinkman flow (for instance due to rain, to wind or to the leakage of contaminants through the top portion of the boundary that represents the surface) so we use $\boldsymbol{f}_{\mathrm{B}} = (0.01, 0.001)^{\mathrm{t}}$. We employ a second-order scheme resulting on a linear system of 1,517,352 unknowns for the Brinkman–Darcy subproblem and 327,707 D.o.f. for the transport equation. Seven fixed-point iterations were needed to reach the desired tolerance and only two Newton steps were required for the convergence of the inner linearization (probably due to the fact that the nonlinear diffusion is in this case milder than that used in Tests 1 and 3). The results are collected in Fig. 5, which shows flow patterns as well as the solute entering the domain and starting to propagate towards the right.

Test 3. We finalize this section presenting a 3D computation that illustrates the use of our method in the numerical simulation of filtering devices. Flow-rate conditions are taken similarly to those employed in Ervin *et al.* (2009) and Anaya *et al.* (2015), namely a cylindrical geometry aligned with the $y-$axis, with varying cross section, where the Brinkman domain is the region with largest radius ($r = 4$ cm and length $L = 6$ cm), and the Darcy domain constitutes the two other sections of the device (of radii $r = 2$ cm and $r = 3$ cm, and lengths of $L = 3$ cm and $L = 5.1$ cm, respectively). We assume that there is an inlet boundary belonging to Γ_{B} and an outlet disk at the end of the cylinder, on Γ_{D}.
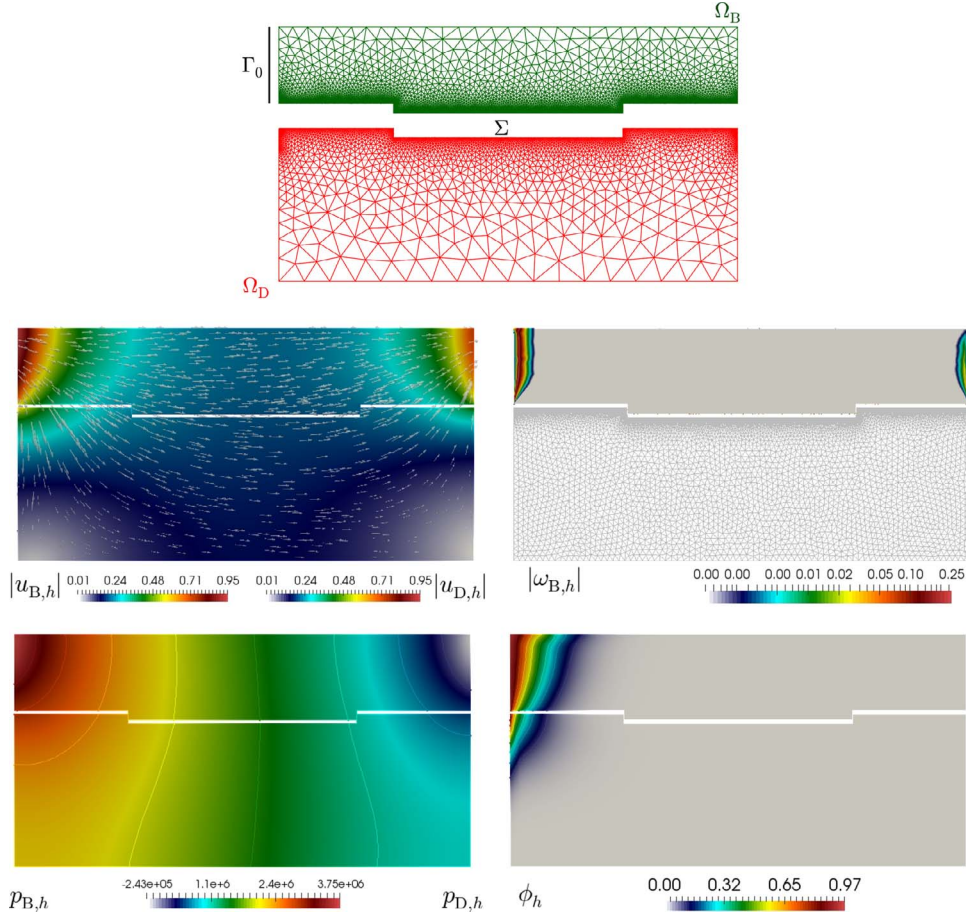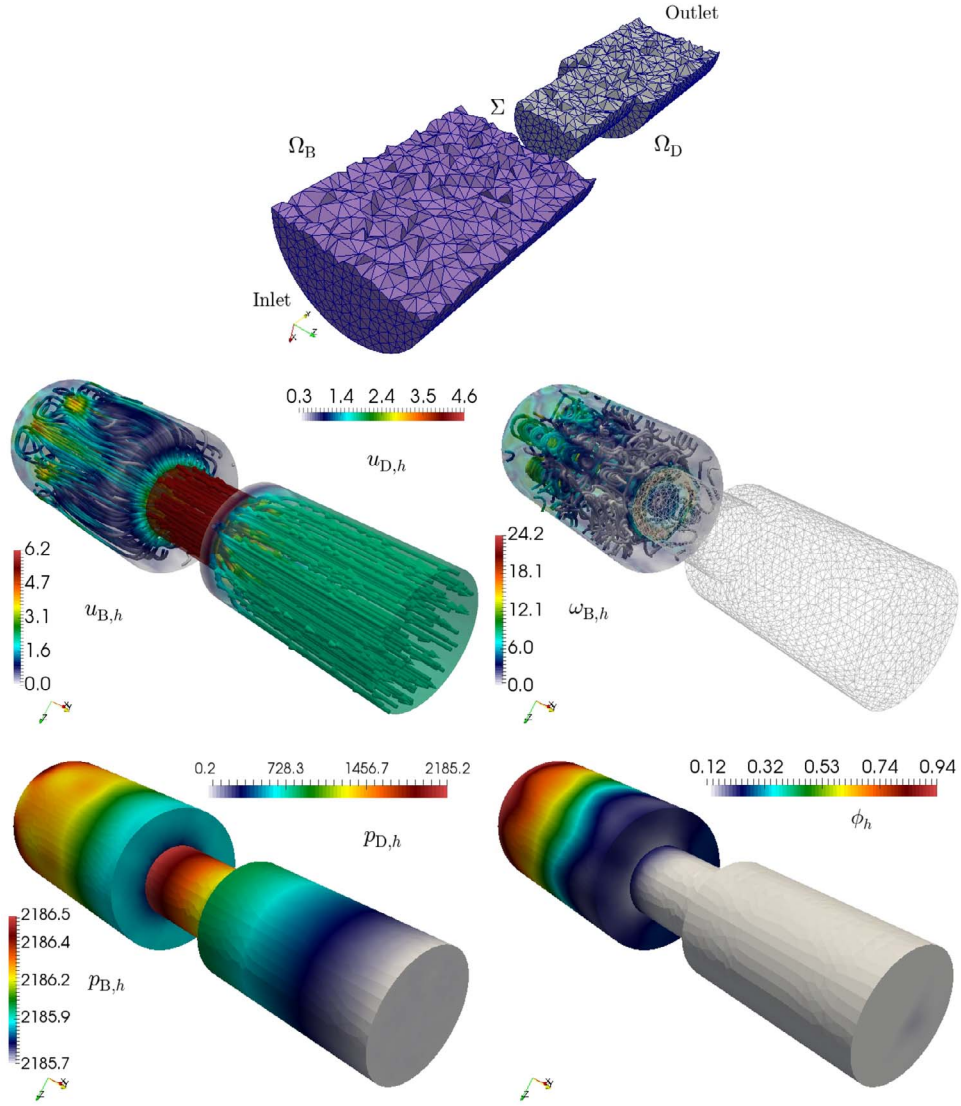
FIG. 5. Test 2. Sample mesh and domain configuration (top), and approximate velocity, vorticity magnitude, pressure and solute concentration produced with a second-order scheme.

The flow is driven essentially by injection of fluid. A Poiseuille Brinkman velocity is prescribed at the inlet, as well as compatible vorticity

$$\boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} = 2\left(1 - \frac{1}{4}(x^2 + z^2)\right), \quad \text{and} \quad \boldsymbol{\omega}_{\mathrm{B}} \times \boldsymbol{n} = \left(-2x\left(1 - \frac{1}{4}(x^2 + z^2)\right), 0, -2x\left(1 - \frac{1}{4}(x^2 + z^2)\right)\right)^{\mathrm{t}},$$

whereas on the outlet boundary we impose a constant Darcy pressure $p_{\mathrm{D}} = p_0 = 0.1$. On the remainder of the domain boundary we set slip conditions for velocity (and zero tangential vorticity on the curved Brinkman boundary). For the transport equation we impose a constant concentration on the inlet and assume zero total flux everywhere else; therefore, the inlet (the disk of radius 4 m and centred at the origin) is the boundary Γ_0 and the remainder of the boundary is Γ_{N}. The interface conditions correspond to the ones stated in (2.4), and a depiction of the domain and boundary setup is presented in the first panel of Fig. 6. The constitutive equations specifying the nonlinear diffusion ϑ and the unidirectional

FIG. 6. Test 3. Sample (coarse) mesh and domain configuration (top), and approximate velocity, vorticity, pressure and solute concentration produced with a first-order scheme.

flux f_{bk} are simply taken as in Test 1 above, with $c = 0.4$. Other model parameters are chosen as

$$\mu = 0.01, \quad \mathbb{K}_B = 0.01, \quad \mathbb{K}_D = 0.00001, \quad \beta = \begin{cases} \frac{1}{2} & \text{in } \Omega_B, \\ 10 & \text{in } \Omega_D \end{cases}, \quad f_D = (0, 1, 0)^t,$$

$$f_B = (\exp(-xy) + x \exp(-x^2), \cos(\pi y) - y \exp(-y^2), xyz - z \exp(-z^2))^t, \quad k = (0, 1, 0)^t,$$

where we note that the hydraulic conductivity is discontinuous across the interface. The domain has been discretized with an unstructured tetrahedral mesh of 74,108 elements, and we have employed a first-order scheme. The approximate solutions are shown in the remaining panels of Fig. 6.5. The first observation from the velocity streamlines is that the nonsymmetric external force f_B rapidly disrupts the Poiseuille profile as the flow moves away from the inlet. We can also see that the Lagrange multiplier enforces correctly the continuity of pressure across the interface, but that there exists a very large Brinkman pressure and a large pressure drop is then seen in the Darcy domain. Also, the tangential components of vorticity slowly decrease when approaching the the interface. As the flow patterns stabilize due to the interfacial conditions the propagation of concentration also becomes very uniform.

For this problem the nonlinear coupling between flow and transport seems to be stronger than before, as the convergence of the Picard algorithm occurred after nine iterations and the inner Newton iterations for the transport problem converged after four steps.

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