Primal-mixed finite element methods for the coupled Biot and Poisson–Nernst–Planck equations*

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Abstract

We introduce and analyze conservative primal-mixed finite element methods for numerically solving the coupled Biot poroelasticity and Poisson–Nernst–Planck equations (modeling ion transport in deformable porous media). For the poroelasticity, we consider a primal-mixed, four-field formulation in terms of the solid displacement, the fluid pressure, the Darcy flux, and the total pressure. In turn, the Poisson–Nernst–Planck equations are formulated in terms of the electrostatic potential, the electric field, the ionized particle concentrations, their gradients, and the total ionic fluxes. The weak formulation is posed in suitable Banach spaces, and it exhibits the structure of a perturbed block-diagonal operator consisting in turn of perturbed and generalized saddle-point problems for the Biot equations, a generalized saddle-point problem for the Poisson equations, and a perturbed twofold saddle-point problem for the Nernst–Planck equations. The well-posedness analysis hinges on the Banach fixed-point theorem along with small data assumptions, the Babuška–Brezzi theory in Banach spaces, and a slight variant of recent abstract results for perturbed saddle-point problems, again in Banach spaces. The associated Galerkin scheme is addressed similarly, employing the Brouwer and Banach theorems to yield existence and uniqueness of discrete solution. A priori error estimates are derived, and rates of convergence for specific finite element subspaces satisfying the required discrete inf-sup conditions are established. Finally, several numerical examples validating the theoretical error bounds, and illustrating the performance of the proposed family of finite element methods, are presented.

1 Introduction

Scope. We study a mathematical model for the transport of electrolytes through an electrically charged fully saturated and deformable porous medium. The electro-hydrostatics are described by the Nernst–Planck relations (mass balance for the counterions) and a mixed Poisson problem (the Gauss law) while the fluid movement of the electrolyte solution within the pores of the poroelastic structure are modeled with the Biot equations – one of the most common models for coupled fluid flow and mechanical deformations of porous structures – written in mixed form. Homogenization of models of ion transport in poroelastic media can be found in [38, 44] (see also [1] for theory and application in nuclear waste disposal in argillaceous rocks). Other applications of macroscopic models where fixed charges yield Debye layers include polymer

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gels, mechanical actuators for soft robotics, and charged proteoglycans in the solid scaffold of hydrated biological tissues such as articular cartilage [35, 36, 46, 47, 49]. As far as we know, no mixed finite element methods (that is, formulations that include other variables of interest in addition to solid displacement, fluid pressure, electrostatic potential, and ionic concentrations) have been developed – including formulation and theoretical analysis – for this particular problem.

Regarding the two sub-problems separately (Biot and Poisson–Nernst–Planck equations), let us start mentioning that mixed methods for poromechanical equations (and solving not just for the displacement-pressure pair) are abundant in the recent literature (see, for example, the very different formulations in [2, 5, 9, 11, 13, 39, 31, 34, 40, 48, 50] and the references therein). We focus on formulations that maintain robustness with respect to the Lamé parameters of the solid phase and that are mass conservative, which means that they satisfy locally a flux balance. From those works we refer to [33, 10], where, besides displacement and pressure, one uses the total pressure and the relative fluid velocity (Darcy flux) as unknowns.

We also stress that the coupling of Biot equations in mixed form to other physical effects (interface contact, thermal properties, second- and fourth-order transport, etc.) can be substantially more difficult to analyze. Again, focusing on mixed methods, we refer, for example, to [12, 30, 42, 41, 45]. Note that, in some cases, augmented methods allow the recovery of a Hilbertian framework. Nevertheless, such an approach is not feasible for our problem, since it is not possible to readily construct the required Hilbertian norm for the Darcy filtration velocity, which is implied in the advective terms of the Nernst–Planck equations.

The nonlinear coupling structure of the problem we tackle here has similar components as in the aforementioned works, also including the Biot-heat equations recently analyzed in [15]. Such frameworks are based on a Banach spaces approach, which we follow herein. In this regard, we refer as well to similar multiphysics coupled problems addressed with generalizations of the fixed-point and saddle-point abstract framework to Banach spaces [14, 16, 18]. On the other hand, the analysis of fully mixed methods for the Poisson–Nernst–Planck equations coupled with Stokes and Navier–Stokes equations has been recently advanced in [24, 23], respectively, and also using a Banach spaces framework. In contrast with these formulations for the hydro-electro-chemical systems, in our model, the linear momentum balance of the poroelasticity problem involves the gradients of the ionic concentrations, which suggests a different type of mixed formulation for these equations, using in particular the gradient of the ionic concentrations as additional variable, and yielding again a first-order structure of the coupled equations, but now exhibiting a twofold saddle point form. In general, the type of methods we propose here inherits appealing features such as more flexibility in data assumptions and solution regularity, obtaining all variables of interest without postprocessing, and preserving balance equations exactly.

Plan of the paper. We have organized the contents of this paper as follows. Essential notations and fundamental definitions are gathered towards the end of this introductory section. In Section 2, we present the Biot–Poisson–Nernst–Planck equations. In particular, the auxiliary unknowns are introduced here. In Section 3, we establish the primal-mixed variational formulation of the problem by breaking down the analysis according to the three set of equations comprising the coupled model. Appropriate integration by parts formulae, coupled with the Cauchy–Schwarz and Hölder inequalities, play a vital role in determining the appropriate Lebesgue and related spaces to which the unknowns and corresponding test functions must belong. In Section 4, we employ a fixed-point strategy to examine the solvability of the continuous formulation. The Babuška–Brezzi and related theories, such as the one for perturbed saddle-point problems, all in Banach spaces, are applied to investigate the corresponding uncoupled problems, and subsequently, the classical Banach theorem is invoked to establish the existence of a unique solution. The Galerkin scheme is introduced in Section 5 and a fixed-point approach analogous to that of Section 4 is employed to investigate its well-posedness. Under appropriate stability conditions on the finite element subspaces used, the existence and uniqueness of the solution are proven by applying the Brouwer and Banach theorems, along with the discrete versions of the theories employed in the continuous analysis. The error analysis is also conducted there and a corresponding Céa estimate is derived. Next, in Section 6, we introduce specific finite element subspaces that meet the used assumptions. Rates of convergence of the resulting discrete scheme are also established. Finally, several numerical examples confirming these theoretical findings and illustrating the good performance of the method are presented in Section 7.

Notation conventions and preliminaries. Throughout the paper Ω is an open and bounded Lipschitz-continuous domain of R^d, $d \in \{2,3\}$, which satisfies a uniform exterior ball condition, and whose outward unit normal on its boundary Γ is denoted $\boldsymbol{n}$. We remark in advance that the above geometric assumption on Ω is rather of technical character, and will be employed only to prove the continuous and discrete versions of a particular inf-sup condition arising from the analysis (cf. Lemmas 4.2 and 6.1). Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in [1, +\infty)$, and Sobolev spaces $\mathrm{W}^{\ell,t}(\Omega)$, with $\ell \geq 0$, whose corresponding norms and seminorms, either for the scalar, vector, or tensorial version, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{\ell,t;\Omega}$, and $|\cdot|_{\ell,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and that when $t = 2$, we simply write $\mathrm{H}^\ell(\Omega)$ instead of $\mathrm{W}^{\ell,2}(\Omega)$, with its norm and seminorm denoted by $\|\cdot\|_{\ell,\Omega}$ and $|\cdot|_{\ell,\Omega}$, respectively. Now, letting $t,\, t' \in (1, +\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we let $\mathrm{W}^{1/t',t}(\Gamma)$ and $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the trace space of $\mathrm{W}^{1,t}(\Omega)$ and its dual, respectively, and denote the duality pairing between them by $\langle\cdot,\cdot\rangle$. In particular, when $t = t' = 2$, we simply write $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ instead of $\mathrm{W}^{1/2,2}(\Gamma)$ and $\mathrm{W}^{-1/2,2}(\Gamma)$, respectively. Also, given any generic scalar functional space M, we let $\mathbf{M}$ be its vector counterparts. Furthermore, for any vector fields $\boldsymbol{v} = (v_i)_{i=1,d}$ and $\boldsymbol{w} = (w_i)_{i=1,d}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla \boldsymbol{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,d}, \quad \operatorname{div}(\boldsymbol{v}) := \sum_{j=1}^{d} \frac{\partial v_j}{\partial x_j} \quad \text{and} \quad \boldsymbol{v} \otimes \boldsymbol{w} := (v_i w_j)_{i,j=1,d}.$$

In addition, for any tensor $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,d}$, we let $\mathbf{div}(\boldsymbol{\tau})$ denote the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose as $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,d}$.

2 The model problem

We consider a homogeneous porous medium constituted by a mixture of incompressible grains and charged interstitial fluid occupying the domain Ω. In the mixture, we assume the presence of positively and negatively charged ions (for example, binary monovalent completely dissociated electrolytes Na^+ and Cl^-). For a given body force $\boldsymbol{f}$ and mass source g, neglecting convective, gravitational, and inertial terms, the steady-state balance of linear momentum for the mixture and mass balance for the fluid content (using the modified Darcy law) are expressed as

$$-\mathbf{div}(\boldsymbol{\sigma}) = \boldsymbol{f} \quad \text{in} \quad \Omega \quad \text{and} \quad c_0\, p + \alpha \operatorname{div}(\boldsymbol{u}) - \operatorname{div}\left(\frac{\kappa}{\nu}\nabla p\right) = g \quad \text{in} \quad \Omega, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the overall Cauchy stress tensor of the solid-fluid-electrochemical mixture, $\boldsymbol{u}$ is the unknown vector of displacement of the solid particles and p is the reference bulk pressure of the fluid. The remaining parameters are the permeability of the porous solid κ, the constrained specific storage coefficient c_0, the Biot–Willis parameter α, and the viscosity of the pore fluid ν. Following the modified Terzaghi decomposition, the constitutive equation for $\boldsymbol{\sigma}$ is conformed by the effective poroelastic stress through Hooke's law for infinitesimal deformation and Biot's consolidation, plus an active macroscopic stress tensor governing the electrochemical interaction between the electrolyte solution and charged molecules as follows (the dependence on the electric field – known as Maxwell's stress – can be found in, e.g., [1, 38, 44], and that on the ionic concentrations in [47])

$$\boldsymbol{\sigma} \,=\, 2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \,+\, \lambda\,\mathrm{div}(\boldsymbol{u})\,\mathbb{I} \,-\, \alpha\,p\,\mathbb{I} \,+\, \varepsilon\,\nabla\chi\otimes\nabla\chi \,-\, \frac{\varepsilon}{2}|\nabla\chi|^2\,\mathbb{I} \,-\, \delta(\xi_1-\xi_2)\,\mathbb{I} \quad \text{in} \quad \Omega\,, \tag{2.2}$$

where ε is the electric conductivity, δ is an osmotic parameter, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathtt{t}})$ is the tensor of infinitesimal strains, and λ, μ are the Lamé constants of the solid matrix. The fields ξ_1 and ξ_2 are the solute concentrations of positive and negatively charged ions, respectively, and χ is the macroscopic dimensionless electrostatic potential. They satisfy the following system of equations (current conservation and mass balance of the charged species)

$$\begin{array}{rcll} -\,\mathrm{div}(\varepsilon\,\nabla\chi) & = & \xi_1-\xi_2 & \text{in} \quad \Omega\,,\\ \xi_1 \,-\, \dfrac{\kappa}{\nu}\nabla p\cdot\nabla\xi_1 \,-\, \mathrm{div}\big(\kappa_1(\nabla\xi_1 \,+\, q_1\,\xi_1\nabla\chi)\big) & = & f_1 & \text{in} \quad \Omega\,,\\ \xi_2 \,-\, \dfrac{\kappa}{\nu}\nabla p\cdot\nabla\xi_2 \,-\, \mathrm{div}\big(\kappa_2(\nabla\xi_2 \,+\, q_2\,\xi_2\nabla\chi)\big) & = & f_2 & \text{in} \quad \Omega\,, \end{array} \tag{2.3}$$

where $q_1 = 1$, $q_2 = -1$, f_1, f_2 are external charge sources, and κ_1, κ_2 are the diffusivities of the cations and anions, respectively. Here we have assumed that the balance equations are scaled with the porosity (assumed constant) and the scaling is absorbed in the external sources. Note that the second term on the left-hand sides of the second and third rows of (2.3) is the advection using the filtration (Darcy's seepage) flux, which indicates that the ionic particles diffuse in the mixture and are advected in the interstitial fluid.

We emphasize here that a recent study [15] delved into the poroelasticity problem when coupled with the heat equation, which are represented by a Biot and convection-diffusion equations, respectively, with this latter depending on the Darcy seepage velocity and the total stress. In that study, we employed a fully-mixed formulation, meaning that for the Biot equation we utilized a mixed approach to explicitly obtain the solution for the total stress and displacement. Therefore, a natural progression from the findings of that study is to also incorporate a fully-mixed approach for the present Biot–Poisson–Nernst–Planck equations presented in this work.

Now, we follow [33, 10] and, in order to maintain robustness of the formulation in the regime of nearly incompressible solid matrix and to achieve mass conservativity of the Biot system, we adopt a four-field formulation for the poroelasticity system (2.1) introducing the total pressure θ, and the Darcy seepage velocity $\boldsymbol{z}$, as the following additional unknowns

$$\theta \,:=\, -\lambda\,\mathrm{div}(\boldsymbol{u}) \,+\, \alpha\,p \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{z} \,:=\, -\,\frac{\kappa}{\nu}\,\nabla p \quad \text{in} \quad \Omega\,. \tag{2.4}$$

In turn, we notice that for a sufficiently smooth vector function $\boldsymbol{w}$ we have

$$\mathbf{div}(\boldsymbol{w}\otimes\boldsymbol{w}) \,=\, (\mathrm{div}\,\boldsymbol{w})\,\boldsymbol{w} \,+\, (\nabla\boldsymbol{w})\,\boldsymbol{w} \quad \text{and} \quad \nabla(|\boldsymbol{w}|^2) \,=\, 2\,(\nabla\boldsymbol{w})^{\mathtt{t}}\,\boldsymbol{w}\,.$$

Thus, since $\nabla\boldsymbol{w}$ is symmetric for $\boldsymbol{w} = \nabla\chi$, a combination of the first equation of (2.1) with (2.2) and the definition of the total pressure θ allows obtaining

$$-\,\mathbf{div}(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \theta\,\mathbb{I}) - \mathrm{div}(\varepsilon\,\nabla\chi)\nabla\chi \,+\, \delta(\nabla\xi_1 - \nabla\xi_2) \,=\, \boldsymbol{f} \quad \text{in} \quad \Omega\,. \tag{2.5}$$

Next, for the mass balance (cf. second equation of (2.1)) we use the definition of the total pressure θ and of the Darcy flux $\boldsymbol{z}$ (cf. (2.4)) to have

$$\left(c_0 \,+\, \frac{\alpha^2}{\lambda}\right)p - \frac{\alpha}{\lambda}\theta \,+\, \mathrm{div}(\boldsymbol{z}) \,=\, g \quad \text{in} \quad \Omega\,.$$

In addition, in order to maintain the uniqueness of the solution for p, in the limiting cases when $c_0 = 0$ and $\lambda \to \infty$, we impose:

$$\int_\Omega p = 0\,. \tag{2.6}$$

Likewise, with the aim of obtaining a current and mass conservative formulation for the Poisson–Nernst–Planck system (2.3), first we make use of the electric current $\boldsymbol{\varphi}$ defined as

$$\boldsymbol{\varphi} \,:=\, \varepsilon\,\nabla\chi \quad \text{in} \quad \Omega\,,$$

which, jointly with the first row of (2.3), gives

$$-\mathrm{div}(\boldsymbol{\varphi}) \,=\, \xi_1 - \xi_2 \quad \text{in} \quad \Omega\,.$$

In turn, for each $i \in \{1, 2\}$, we define the ionic concentration gradients $\boldsymbol{t}_i$, and total (diffusive plus advective) flux of ionic species $\boldsymbol{\sigma}_i$, which are defined as follows

$$\boldsymbol{t}_i \,:=\, \nabla\xi_i \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{\sigma}_i \,:=\, \kappa_i(\boldsymbol{t}_i \,+\, q_i\varepsilon^{-1}\xi_i\,\boldsymbol{\varphi}) - \xi_i\,\boldsymbol{z} \quad \text{in} \quad \Omega\,.$$

This is a similar approach as in the mixed methods from [24], but the $\boldsymbol{t}_i$ are not used there. Here we need these chemical potentials to manage the last term on the right-hand side of the momentum balance (2.5). Finally, for each $i \in \{1, 2\}$ we use the identity

$$\mathrm{div}(\xi_i\,\boldsymbol{z}) = \boldsymbol{z}\cdot\nabla\xi_i \,+\, \xi_i\,\mathrm{div}(\boldsymbol{z})\,,$$

which, in combination with the second and third rows of (2.3), yields

$$\xi_i - \mathrm{div}(\boldsymbol{\sigma}_i) - \xi_i\,\mathrm{div}(\boldsymbol{z}) \,=\, f_i \quad \text{in} \quad \Omega\,.$$

In summary, these steps lead to the following Biot–Poisson–Nernst–Planck equations in terms of the unknowns $\boldsymbol{u}$, θ, $\boldsymbol{z}$, p, $\boldsymbol{\varphi}$, χ, $\boldsymbol{t}_i$, $\boldsymbol{\sigma}_i$ and ξ_i, $i \in \{1, 2\}$, as

$$-\mathbf{div}(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \theta\,\mathbb{I}) \,+\, \varepsilon^{-1}\,(\xi_1 - \xi_2)\,\boldsymbol{\varphi} \,+\, \delta(\boldsymbol{t}_1 - \boldsymbol{t}_2) \,=\, \boldsymbol{f} \qquad \text{in} \quad \Omega\,, \tag{2.7a}$$

$$\theta - \alpha\,p \,+\, \lambda\,\mathrm{div}(\boldsymbol{u}) \,=\, 0 \qquad \text{in} \quad \Omega\,, \tag{2.7b}$$

$$\frac{\nu}{\kappa}\boldsymbol{z} \,+\, \nabla p \,=\, \mathbf{0} \qquad \text{in} \quad \Omega\,, \tag{2.7c}$$

$$\left(c_0 \,+\, \frac{\alpha^2}{\lambda}\right)p \,-\, \frac{\alpha}{\lambda}\theta \,+\, \mathrm{div}(\boldsymbol{z}) \,=\, g \qquad \text{in} \quad \Omega\,, \tag{2.7d}$$

$$\boldsymbol{\varphi} \,-\, \varepsilon\,\nabla\chi \,=\, \mathbf{0} \qquad \text{in} \quad \Omega\,, \tag{2.7e}$$

$$-\mathrm{div}(\boldsymbol{\varphi}) \,=\, \xi_1 - \xi_2 \qquad \text{in} \quad \Omega\,, \tag{2.7f}$$

$$\boldsymbol{t}_i \,-\, \nabla\xi_i \,=\, \mathbf{0} \qquad \text{in} \quad \Omega\,, \tag{2.7g}$$

$$-\boldsymbol{\sigma}_i \,+\, \kappa_i\boldsymbol{t}_i \,+\, q_i\,\kappa_i\,\varepsilon^{-1}\,\xi_i\,\boldsymbol{\varphi} \,-\, \xi_i\,\boldsymbol{z} \,=\, \mathbf{0} \qquad \text{in} \quad \Omega\,, \tag{2.7h}$$

$$\xi_i \,-\, \mathrm{div}(\boldsymbol{\sigma}_i) \,-\, \xi_i\,\mathrm{div}(\boldsymbol{z}) \,=\, f_i \qquad \text{in} \quad \Omega\,. \tag{2.7i}$$

We endow (2.7a)-(2.7d) with the following boundary conditions

$$\boldsymbol{u} = \mathbf{0} \quad \text{and} \quad \boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma\,, \tag{2.8}$$

and pure Dirichlet boundary conditions with given data $\chi_{\mathtt{D}}$, $\xi_{i,\mathtt{D}}$, $i \in \{1, 2\}$, are considered for (2.7e)-(2.7i):

$$\chi = \chi_{\mathtt{D}} \quad \text{and} \quad \xi_i = \xi_{i,\mathtt{D}} \quad \text{on} \quad \Gamma\,. \tag{2.9}$$

3 The weak formulation

In this section, we derive a primal-mixed formulation of the system (2.7) - (2.9). To this end, we first provide some preliminaries, and then split the analysis according to the respective decoupled problems, namely those given by the poroelasticity, electrostatic potential, and ionized particles concentration equations.

3.1 Preliminaries

We start by considering, for each $t \in [1, +\infty)$, the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega) \Big\},$$
$$\mathbf{H}^t(\mathrm{div};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^2(\Omega) \Big\},$$
$$\mathbf{H}^t(\mathrm{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega) \Big\},$$

which are endowed with the natural norms:

$$\begin{aligned}
\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} &:= \|\boldsymbol{\tau}\|_{0;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega),\\
\|\boldsymbol{\tau}\|_{t,\mathrm{div};\Omega} &:= \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div};\Omega),\\
\|\boldsymbol{\tau}\|_{t,\mathrm{div}_t;\Omega} &:= \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega).
\end{aligned}$$

We recall that, proceeding as in [26, eqn. (1.43), Section 1.3.4] (see also [21, Section 3.1]), one can prove that for each $t \in \begin{cases} (1, +\infty) & \text{in } \mathrm{R}^2, \\ [6/5, +\infty) & \text{in } \mathrm{R}^3, \end{cases}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{n}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_t;\Omega) \times \mathrm{H}^1(\Omega), \tag{3.3}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. In turn, given $t,\, t' \in (1, +\infty)$ conjugate to each other, there also holds (cf. [25, Corollary B.57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{n}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}^t(\mathrm{div}_t;\Omega) \times \mathrm{W}^{1,t'}(\Omega), \tag{3.4}$$

where $\langle \cdot,\cdot \rangle$ denotes in (3.4) the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$.

Now, we notice that there are at least four key expressions in (2.7a)-(2.7i) that need to be looked at carefully before confirming adequate Sobolev and Lebesgue exponents that will specify the trial and test spaces. These are $(\xi_1 - \xi_2)\,\boldsymbol{\varphi}$, $\xi_i\,\boldsymbol{\varphi}$, $\xi_i\,\boldsymbol{z}$, and $\xi_i\,\mathrm{div}(\boldsymbol{z})$. Given test functions $\boldsymbol{v}$, $\boldsymbol{s}_i$ and η_i associated with $\boldsymbol{u}$, $\boldsymbol{t}_i$ and ξ_i, respectively, a straightforward application of the Cauchy–Schwarz and Hölder inequalities yield

$$\left| \int_\Omega (\xi_1 - \xi_2)\,\boldsymbol{\varphi}\cdot\boldsymbol{v} \right| \le \|\xi_1 - \xi_2\|_{0,2l;\Omega}\, \|\boldsymbol{\varphi}\|_{0,2j;\Omega}\, \|\boldsymbol{v}\|_{0,\Omega}, \tag{3.5a}$$
$$\left| \int_\Omega \xi_i\,\boldsymbol{\varphi}\cdot\boldsymbol{s}_i \right| \le \|\xi_i\|_{0,2l;\Omega}\, \|\boldsymbol{\varphi}\|_{0,2j;\Omega}\, \|\boldsymbol{s}_i\|_{0,\Omega}, \tag{3.5b}$$
$$\left| \int_\Omega \xi_i\,\boldsymbol{z}\cdot\boldsymbol{s}_i \right| \le \|\xi_i\|_{0,2l;\Omega}\, \|\boldsymbol{z}\|_{0,2j;\Omega}\, \|\boldsymbol{s}_i\|_{0,\Omega}, \tag{3.5c}$$
$$\left| \int_\Omega \xi_i\,\mathrm{div}(\boldsymbol{z})\,\eta_i \right| \le \|\xi_i\|_{0,2l;\Omega}\, \|\mathrm{div}(\boldsymbol{z})\|_{0,\Omega}\, \|\eta_i\|_{0,2j;\Omega}, \tag{3.5d}$$

where $l, j \in (1, +\infty)$ are conjugate to each other. In this way, denoting

$$r := 2j, \quad s := \frac{2j}{2j-1}\ (\text{conjugate of } r), \quad \rho := 2l, \quad \varrho := \frac{2l}{2l-1}\ (\text{conjugate of } \rho), \tag{3.6}$$

it follows that the above expressions are integrable for $\xi_i \in \mathrm{L}^\rho(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, $\boldsymbol{z} \in \mathbf{H}^r(\mathrm{div};\Omega)$, $\boldsymbol{v} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{s}_i \in \mathbf{L}^2(\Omega)$ and, assuming that $\rho > r$ (a condition that will be satisfied below in (3.7)), we can consider

$\eta_i \in \mathrm{L}^\rho(\Omega)$. Moreover, since we are aiming to apply (3.3) to $\boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$ and $\xi_i \in \mathrm{L}^\rho(\Omega)$, we need that $\mathrm{H}^1(\Omega)$ is continuously embedded in $\mathrm{L}^\rho(\Omega)$. The latter is guaranteed for $\rho \in [1,+\infty)$ when $n=2$, and $\rho \in [1,6]$ when $n=3$.

On the other hand, in the forthcoming analysis we require a result on the $\mathrm{W}^{1,r}(\Omega)$-solvability of an auxiliary Poisson equation (in showing a continuous inf-sup condition). For this we need that $4/3 \le r \le 4$ when $n=2$, and $3/2 \le r \le 3$ when $n=3$. Thus, since $r = \frac{\rho}{l-1}$, intersecting this with the previous restrictions on ρ, we find the following feasible ranges for r, s, ρ and ϱ:

$$\begin{cases} r\in(2,4] \quad \text{and} \quad s\in[4/3,2) & \text{if } n=2\,, \\ \quad r=3 \quad \text{and} \quad s=3/2 & \text{if } n=3\,, \end{cases} \qquad \begin{cases} \rho\in[4,+\infty) \quad \text{and} \quad \varrho\in(1,4/3] & \text{if } n=2\,, \\ \quad \rho=6 \quad \text{and} \quad \varrho=6/5 & \text{if } n=3\,. \end{cases} \tag{3.7}$$

In turn, in view of the essential boundary conditions for displacement and Darcy flux in (2.8), we consider the following closed subspaces of Hilbert and Banach spaces

$$\mathbf{H}^1_0(\Omega) := \Big\{ \boldsymbol{v} \in \mathbf{H}^1(\Omega) : \quad \boldsymbol{v}|_\Gamma = \mathbf{0} \Big\}, \tag{3.8a}$$

$$\mathbf{H}^s_0(\mathrm{div}_s;\Omega) := \Big\{ \boldsymbol{w} \in \mathbf{H}^s(\mathrm{div}_s;\Omega) : \quad (\boldsymbol{w}\cdot\boldsymbol{n})|_\Gamma = 0 \Big\}, \tag{3.8b}$$

$$\mathbf{H}^r_0(\mathrm{div};\Omega) := \Big\{ \boldsymbol{w} \in \mathbf{H}^r(\mathrm{div};\Omega) : \quad (\boldsymbol{w}\cdot\boldsymbol{n})|_\Gamma = 0 \Big\}. \tag{3.8c}$$

Here the boundary specification is to be understood in the sense of traces. In addition, for $t \in [1,+\infty)$ we define

$$\mathrm{L}^t_0(\Omega) := \Big\{ q \in \mathrm{L}^t(\Omega) : \quad \int_\Omega q = 0 \Big\}. \tag{3.9}$$

As announced at the beginning of the section, in what follows we rewrite each variational formulation of Biot, Poisson and Nernst–Planck equations independently, ending up with three systems whose coupling is carried out via a fixed-point iteration. We also provide preliminary properties of the bilinear forms involved in each sub-problem.

3.2 Primal-mixed formulation of the poroelasticity equations

In this section, we follow very closely [33, Section 2] to derive the variational formulation of the poroelasticity equations (2.7a)-(2.7d) and (2.8), which, given ξ_1, ξ_2, $\boldsymbol{t}_1$, and $\boldsymbol{t}_2$, consist of finding $\boldsymbol{u}$, θ, $\boldsymbol{z}$, and p, all the above in suitable spaces, such that

$$-\mathbf{div}(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \theta\,\mathbb{I}) + \varepsilon^{-1}(\xi_1-\xi_2)\,\boldsymbol{\varphi} + \delta(\boldsymbol{t}_1-\boldsymbol{t}_2) = \boldsymbol{f} \qquad \text{in} \quad \Omega\,, \tag{3.10a}$$

$$\theta - \alpha\,p + \lambda\,\mathrm{div}(\boldsymbol{u}) = 0 \qquad \text{in} \quad \Omega\,, \tag{3.10b}$$

$$\frac{\nu}{\kappa}\boldsymbol{z} + \nabla p = \mathbf{0} \qquad \text{in} \quad \Omega\,, \tag{3.10c}$$

$$\left(c_0 + \frac{\alpha^2}{\lambda}\right)p - \frac{\alpha}{\lambda}\theta + \mathrm{div}(\boldsymbol{z}) = g \qquad \text{in} \quad \Omega\,, \tag{3.10d}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{and} \quad \boldsymbol{z}\cdot\boldsymbol{n} = 0 \qquad \text{on} \quad \Gamma\,. \tag{3.10e}$$

We begin by testing (3.10a) against $\boldsymbol{v} \in \mathbf{H}^1_0(\Omega)$ (cf. (3.8a)), which satisfies the bound given by (3.5a). In this way, applying (3.3) with $t=2$, and employing the first boundary condition in (3.10e), we obtain

$$2\mu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}) - \int_\Omega \theta\,\mathrm{div}(\boldsymbol{v}) = \int_\Omega \Big(\boldsymbol{f} - \varepsilon^{-1}(\xi_1-\xi_2)\,\boldsymbol{\varphi} - \delta(\boldsymbol{t}_1-\boldsymbol{t}_2)\Big)\cdot\boldsymbol{v} \qquad \forall\,\boldsymbol{v}\in\mathbf{H}^1_0(\Omega)\,. \tag{3.11}$$

Note that, thanks to the Cauchy–Schwarz's inequality and (3.5a), each term in (3.11) makes sense for $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$, $\theta \in \mathrm{L}^2(\Omega)$, $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, $\xi_i \in \mathrm{L}^\rho(\Omega)$, and $\boldsymbol{t}_i \in \mathbf{L}^2(\Omega)$, $i \in \{1,2\}$. Next, we test (3.10b) against $\vartheta \in \mathrm{L}^2(\Omega)$, which gives

$$-\int_\Omega \vartheta \operatorname{div}(\boldsymbol{u}) - \frac{1}{\lambda}\int_\Omega \theta\,\vartheta + \frac{\alpha}{\lambda}\int_\Omega p\,\vartheta = 0 \qquad \forall\,\vartheta \in \mathrm{L}^2(\Omega)\,. \tag{3.12}$$

On the other hand, recalling from (3.5c) and (3.5d) that $\boldsymbol{z} \in \mathbf{H}^r(\operatorname{div};\Omega)$, and bearing in mind the second boundary condition in (3.10e), we deduce that $\boldsymbol{z}$ must be sought in $\mathbf{H}_0^r(\operatorname{div};\Omega)$ (cf. (3.8b)), whence (3.10c) suggests to look originally for $p \in \mathrm{W}^{1,r}(\Omega)$. In this way, testing (3.10c) against $\boldsymbol{w} \in \mathbf{H}_0^s(\operatorname{div}_s;\Omega)$ (cf. (3.8c)), and employing (3.4), we formally get

$$\frac{\nu}{\kappa}\int_\Omega \boldsymbol{z}\cdot\boldsymbol{w} - \int_\Omega p \operatorname{div}(\boldsymbol{w}) \;=\; 0 \qquad \forall\,\boldsymbol{w} \in \mathbf{H}_0^s(\operatorname{div}_s;\Omega)\,, \tag{3.13}$$

from whose second term and (2.6), we notice that it suffices to look for the pressure p in the space $\mathrm{L}_0^r(\Omega)$ (cf. (3.9)). In turn, since $\operatorname{div}(\boldsymbol{z})$ belongs to $\mathrm{L}^2(\Omega)$, we test (3.10d) against $q \in \mathrm{L}_0^2(\Omega)$ obtaining

$$\frac{\alpha}{\lambda}\int_\Omega \theta\, q - \int_\Omega q \operatorname{div}(\boldsymbol{z}) - \left(c_0 + \frac{\alpha^2}{\lambda}\right)\int_\Omega p\, q = -\int_\Omega g\, q \qquad \forall\, q \in \mathrm{L}_0^2(\Omega)\,, \tag{3.14}$$

which requires assuming that $g \in \mathrm{L}^2(\Omega)$. In addition, knowing that $\vartheta \in \mathrm{L}^2(\Omega)$, $p \in \mathrm{L}_0^r(\Omega)$, and $q \in \mathrm{L}_0^2(\Omega)$, and recalling from (3.7) that $r > 2$, which certainly yields $\mathrm{L}^r(\Omega) \subset \mathrm{L}^2(\Omega)$, we realize that the third terms of (3.12) and (3.14) make sense as well. According to the foregoing discussion, and aiming to conveniently rewrite the system of equations (3.11) - (3.14), we now introduce the spaces

$$\begin{gathered}\mathbf{X} := \mathbf{H}_0^1(\Omega)\,, \quad \mathbf{X}_2 \;:=\; \mathbf{H}_0^r(\operatorname{div};\Omega)\,, \quad \mathbf{X}_1 \;:=\; \mathbf{H}_0^s(\operatorname{div}_s;\Omega)\,,\\ \mathbf{Q} = \mathrm{L}^2(\Omega)\,, \quad \mathbf{Q}_1 \;:=\; \mathrm{L}_0^r(\Omega)\,, \quad \text{and} \quad \mathbf{Q}_2 \;:=\; \mathrm{L}_0^2(\Omega)\,,\end{gathered}$$

which are endowed, respectively, with the norms

$$\begin{gathered}\|\boldsymbol{v}\|_{\mathbf{X}} \;:=\; \|\boldsymbol{v}\|_{1,\Omega}\,, \quad \|\boldsymbol{z}\|_{\mathbf{X}_2} \;:=\; \|\boldsymbol{z}\|_{r,\operatorname{div};\Omega}\,, \quad \|\boldsymbol{w}\|_{\mathbf{X}_1} \;:=\; \|\boldsymbol{w}\|_{s,\operatorname{div}_s;\Omega}\,,\\ \|\vartheta\|_{\mathbf{Q}} \;:=\; \|\vartheta\|_{0,\Omega}\,, \quad \|p\|_{\mathbf{Q}_1} \;:=\; \|p\|_{0,r;\Omega}\,, \quad \text{and} \quad \|q\|_{\mathbf{Q}_2} \;:=\; \|q\|_{0,\Omega}\,.\end{gathered}$$

In this way, given $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, $\boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathrm{L}^\rho(\Omega)\times\mathrm{L}^\rho(\Omega)$, $\boldsymbol{t} = (\boldsymbol{t}_1,\boldsymbol{t}_2) \in \mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$, and $p \in \mathrm{L}_0^r(\Omega)$, (3.11) and (3.12) can be reformulated as: Find $(\boldsymbol{u},\theta) \in \mathbf{X}\times\mathbf{Q}$ such that

$$\begin{array}{lcll}\mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) \;+\; \mathbf{b}_s(\boldsymbol{v},\theta) & = & \mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}(\boldsymbol{v}) & \forall\,\boldsymbol{v} \in \mathbf{X}\,,\\ \mathbf{b}_s(\boldsymbol{u},\vartheta) \;-\; \mathbf{c}_s(\theta,\vartheta) \;+\mathbf{e}_s(p,\vartheta) & = & 0 & \forall\,\vartheta \in \mathbf{Q}\,,\end{array} \tag{3.15}$$

where the bilinear forms $\mathbf{a}_s : \mathbf{X}\times\mathbf{X} \to \mathrm{R}$, $\mathbf{b}_s : \mathbf{X}\times\mathbf{Q} \to \mathrm{R}$, $\mathbf{c}_s : \mathbf{Q}\times\mathbf{Q} \to \mathrm{R}$, and $\mathbf{e}_s : \mathbf{Q}_1\times\mathbf{Q} \to \mathrm{R}$, and the functional $\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}} : \mathbf{X} \to \mathrm{R}$, are defined, respectively, as

$$\begin{array}{lcll}\mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) & := & 2\mu\displaystyle\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}) & \forall\,(\boldsymbol{u},\boldsymbol{v}) \in \mathbf{X}\times\mathbf{Q}\,,\\ \mathbf{b}_s(\boldsymbol{v},\vartheta) & := & -\displaystyle\int_\Omega \vartheta\operatorname{div}(\boldsymbol{v}) & \forall\,(\boldsymbol{v},\vartheta) \in \mathbf{X}\times\mathbf{Q}\,,\\ \mathbf{c}_s(\theta,\vartheta) & := & \displaystyle\int_\Omega \theta\,\vartheta & \forall\,\theta,\vartheta \in \mathbf{Q}\,,\\ \mathbf{e}_s(p,\vartheta) & := & \displaystyle\frac{\alpha}{\lambda}\int_\Omega p\,\vartheta & \forall\,(p,\vartheta) \in \mathbf{Q}_2\times\mathbf{Q}\,, \quad \text{and}\end{array} \tag{3.16}$$

$$\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}(\boldsymbol{v}) := \int_\Omega \Big(\boldsymbol{f} \;-\; \varepsilon^{-1}(\xi_1-\xi_2)\boldsymbol{\varphi} \;-\; \delta(\boldsymbol{t}_1-\boldsymbol{t}_2)\Big)\cdot\boldsymbol{v} \qquad \forall\,\boldsymbol{v} \in \mathbf{X}\,. \tag{3.17}$$

Similarly, given $\theta \in \mathrm{L}^2(\Omega)$, (3.13) and (3.14) can be reformulated as: Find $(\boldsymbol{z}, p) \in \mathbf{X}_2 \times \mathbf{Q}_1$ such that

$$\begin{array}{llll} \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) \,+\, \mathbf{d}_1(\boldsymbol{w},p) & & = \; 0 & \forall\, \boldsymbol{w} \in \mathbf{X}_1\,, \\ \mathbf{d}_2(\boldsymbol{z},q) & +\,\mathbf{e}_f((\theta,p),q) & = \; \mathbf{G}(q) & \forall\, q \in \mathbf{Q}_2\,, \end{array} \tag{3.18}$$

where the bilinear forms $\mathbf{a}_f : \mathbf{X}_2 \times \mathbf{X}_1 \to \mathrm{R}$, $\mathbf{d}_i : \mathbf{X}_i \times \mathbf{Q}_i \to \mathrm{R}$, $i \in \{1,2\}$, and $\mathbf{e}_f : (\mathbf{Q} \times \mathbf{Q}_1) \times \mathbf{Q}_2 \to \mathrm{R}$, and the functional $\mathbf{G} : \mathbf{Q}_2 \to \mathrm{R}$, are given, respectively, by

$$\begin{array}{rcll} \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) & := & \dfrac{\nu}{\kappa}\displaystyle\int_\Omega \boldsymbol{z}\cdot\boldsymbol{w} & \forall\,(\boldsymbol{z},\boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{X}_1\,, \\ \mathbf{d}_i(\boldsymbol{w},q) & := & -\displaystyle\int_\Omega q\,\mathrm{div}(\boldsymbol{w}) & \forall\,(\boldsymbol{w},q) \in \mathbf{X}_i \times \mathbf{Q}_i\,, \\ \mathbf{e}_f((\theta,p),q) & := & \dfrac{\alpha}{\lambda}\displaystyle\int_\Omega \theta\,q - \left(c_0 \,+\, \dfrac{\alpha^2}{\lambda}\right)\displaystyle\int_\Omega p\,q & \forall\,((\theta,p),q) \in (\mathbf{Q}\times\mathbf{Q}_1)\times\mathbf{Q}_2\,, \quad \text{and} \\ \mathbf{G}(q) & := & \displaystyle\int_\Omega g\,q & \forall\, q \in \mathbf{Q}_2\,. \end{array} \tag{3.19}$$

Summarizing, given $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, $\boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathrm{L}^\rho(\Omega)\times\mathrm{L}^\rho(\Omega)$, and $\boldsymbol{t} = (\boldsymbol{t}_1,\boldsymbol{t}_2) \in \mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$, the primal-mixed formulation for the poroelasticity equations (cf. (3.10)) reduces to gathering (3.15) and (3.18), that is: Find $((\boldsymbol{u},\theta),(\boldsymbol{z},p)) \in (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$ such that

$$\begin{array}{lllll} \mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) \,+\, \mathbf{b}_s(\boldsymbol{v},\theta) & & & = \; \mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}(\boldsymbol{v}) & \forall\,\boldsymbol{v} \in \mathbf{X}\,, \\ \mathbf{b}_s(\boldsymbol{u},\vartheta) \,-\, \mathbf{c}_s(\theta,\vartheta) & & +\,\mathbf{e}_s(p,\vartheta) & = \; 0 & \forall\,\vartheta \in \mathbf{Q}\,, \\ & \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) \,+\, \mathbf{d}_1(\boldsymbol{w},p) & & = \; 0 & \forall\,\boldsymbol{w} \in \mathbf{X}_1\,, \\ & \mathbf{d}_2(\boldsymbol{z},q) & +\,\mathbf{e}_f((\theta,p),q) & = \; \mathbf{G}(q) & \forall\, q \in \mathbf{Q}_2\,. \end{array} \tag{3.20}$$

It is important to stress here that, ignoring the bilinear forms $\mathbf{e}_s$ and $\mathbf{e}_f$, the left-hand side of (3.20) shows a block-diagonal structure with perturbed and generalized saddle-point problems, respectively, as the first and second block. We take advantage of this fact later on in Section 4.2.

We end this section by remarking that direct applications of the Hölder and Cauchy–Schwarz inequalities allow us to conclude that the above bilinear forms and the functional $\mathbf{G}$ are bounded with positive constants given by

$$\begin{gathered} \|\mathbf{a}_s\| \,:=\, 2\mu\,, \qquad \|\mathbf{b}_s\|\,,\|\mathbf{c}_s\| \,:=\, 1\,, \quad \|\mathbf{e}_s\| \,:=\, C_r(\Omega)\,\frac{\alpha}{\lambda}\,, \quad \|\mathbf{a}_f\| \,:=\, \frac{\nu}{\kappa}\,, \\ \|\mathbf{d}_1\|\,,\|\mathbf{d}_2\| := 1\,, \quad \|\mathbf{e}_f\| \,:=\, \max\left\{\frac{\alpha}{\lambda}\,,\, C_r(\Omega)\,\left(c_0 + \frac{\alpha^2}{\lambda}\right)\right\}\,, \quad \text{and} \quad \|\mathbf{G}\| = \|g\|_{0,\Omega}\,, \end{gathered} \tag{3.21}$$

where $C_r(\Omega) \,:=\, |\Omega|^{\frac{r-2}{2r}}$. In addition, for each $\boldsymbol{v} \in \mathbf{X}_1$ there holds

$$\begin{gathered} |\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}(\boldsymbol{v})| \,\le\, \|\mathbf{F}\|\Big\{\|\boldsymbol{f}\|_{0,\Omega} \,+\, \|\boldsymbol{\varphi}\|_{0,r;\Omega}\,\|\xi_1-\xi_2\|_{0,\rho;\Omega} \,+\, \|\boldsymbol{t}_1-\boldsymbol{t}_2\|_{0,\Omega}\Big\}\|\boldsymbol{v}\|_{\mathbb{X}_1}\,, \quad \text{with} \\ \|\mathbf{F}\| \,:=\, \max\big\{1,\varepsilon^{-1},\delta\big\}\,. \end{gathered} \tag{3.22}$$

3.3 Mixed formulation of the electrostatic potential equations.

We first recall that the electrostatic potential equations are given by (2.7e) - (2.7f), and the Dirichlet boundary condition for χ in (2.9), that is

$$\boldsymbol{\varphi} - \varepsilon\,\nabla\chi \,=\, \mathbf{0} \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\boldsymbol{\varphi}) \,=\, \xi_1 - \xi_2 \quad \text{in} \quad \Omega\,, \quad \chi \,=\, \chi_{\mathtt{D}} \quad \text{on} \quad \Gamma\,. \tag{3.23}$$

Then, following [24, Section 3.3], we set the trial and test spaces

$$\mathrm{X}_1 := \mathbf{H}^s(\mathrm{div}_s;\Omega)\,,\quad \mathrm{X}_2 := \mathbf{H}^r(\mathrm{div}_r;\Omega)\,,\quad \mathrm{M}_1 := \mathrm{L}^r(\Omega) \quad\text{and}\quad \mathrm{M}_2 := \mathrm{L}^s(\Omega)\,,$$

which are provided with the norms

$$\|\boldsymbol{\psi}\|_{\mathrm{X}_1} := \|\boldsymbol{\psi}\|_{s,\mathrm{div}_s;\Omega}\,,\quad \|\boldsymbol{\varphi}\|_{\mathrm{X}_2} := \|\boldsymbol{\varphi}\|_{r,\mathrm{div}_r;\Omega}\,,\quad \|\chi\|_{\mathrm{M}_1} := \|\chi\|_{0,r;\Omega} \quad\text{and}\quad \|\gamma\|_{\mathrm{M}_2} := \|\gamma\|_{0,s;\Omega}\,,$$

and deduce that, given $\boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathrm{L}^\rho(\Omega)\times\mathrm{L}^\rho(\Omega)$, the weak formulation of (3.23) reduces to the generalized saddle-point problem: Find $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ such that

$$\begin{aligned} a(\boldsymbol{\varphi},\boldsymbol{\psi}) + b_1(\boldsymbol{\psi},\chi) &= G(\boldsymbol{\psi}) \qquad \forall\,\boldsymbol{\psi} \in \mathrm{X}_1\,,\\ b_2(\boldsymbol{\varphi},\gamma) &= F_{\boldsymbol{\xi}}(\gamma) \qquad \forall\,\gamma \in \mathrm{M}_2\,, \end{aligned} \tag{3.24}$$

where the bilinear forms $a : \mathrm{X}_2 \times \mathrm{X}_1 \to \mathrm{R}$, and $b_i : \mathrm{X}_i \times \mathrm{M}_i \to \mathrm{R}$, with $i \in \{1,2\}$, and the linear functionals $G : \mathrm{X}_1 \to \mathrm{R}$ and $F_{\boldsymbol{\xi}} : \mathrm{M}_2 \to \mathrm{R}$, are given, respectively, by

$$\begin{aligned} a(\boldsymbol{\varphi},\boldsymbol{\psi}) &:= \int_\Omega \varepsilon^{-1}\,\boldsymbol{\varphi}\cdot\boldsymbol{\psi} \qquad \forall\,(\boldsymbol{\varphi},\boldsymbol{\psi}) \in \mathrm{X}_2 \times \mathrm{X}_1\,,\\ b_i(\boldsymbol{\psi},\gamma) &:= \int_\Omega \gamma\,\mathrm{div}(\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\psi},\gamma) \in \mathrm{X}_i \times \mathrm{M}_i\,, i \in \{1,2\}\,,\\ G(\boldsymbol{\psi}) &:= \langle \boldsymbol{\psi}\cdot\boldsymbol{n}, \chi_{\mathtt{D}}\rangle \qquad \forall\,\boldsymbol{\psi} \in \mathrm{X}_1\,,\\ F_{\boldsymbol{\xi}}(\gamma) &:= -\int_\Omega (\xi_1 - \xi_2)\,\gamma \qquad \forall\,\gamma \in \mathrm{M}_2\,. \end{aligned}$$

Straightforward applications of Hölder's inequality allows us to conclude that a and b_i, with $i \in \{1,2\}$, are bounded with constants given by

$$\|a\| := \varepsilon^{-1} \quad\text{and}\quad \|b_1\|, \|b_2\| := 1\,. \tag{3.25}$$

By similar arguments there holds

$$|F_{\boldsymbol{\xi}}(\gamma)| \le \|F\|\,\|\xi_1 - \xi_2\|_{0,\rho;\Omega}\,\|\gamma\|_{\mathrm{M}_2} \quad \forall\,\gamma \in \mathrm{M}_2\,,\quad\text{with}\quad \|F\| := |\Omega|^{\frac{\rho-r}{\rho r}}\,. \tag{3.26}$$

In turn, regarding the boundedness of G, we invoke [25, Lemma A.36] and the surjectivity of the trace operator mapping $\mathrm{W}^{1,r}(\Omega)$ onto $\mathrm{W}^{1/s,r}(\Gamma)$, which imply the existence of a constant c_r, such that for the given $\chi_{\mathtt{D}} \in \mathrm{W}^{1/s,r}(\Gamma)$, there exists $v_{\mathtt{D}} \in \mathrm{W}^{1,r}(\Omega)$ satisfying $v_{\mathtt{D}}|_\Gamma = \chi_{\mathtt{D}}$ and the estimate $\|v_{\mathtt{D}}\|_{1,r;\Omega} \le c_r\,\|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma}$, which, thanks to (3.4), yields

$$|G(\boldsymbol{\psi})| \le \|G\|\,\|\boldsymbol{\psi}\|_{\mathrm{X}_1} \quad \forall\,\boldsymbol{\psi} \in \mathrm{X}_1\,,\quad\text{with}\quad \|G\| := c_r\,\|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma}\,. \tag{3.27}$$

3.4 Mixed formulation of the ionized particles concentration equations

In what follows we deduce the weak formulation of the Nernst–Planck equations (2.7g) - (2.7i), and the Dirichlet boundary condition for ξ_i in (2.9), for $i \in \{1,2\}$, which, given $\boldsymbol{\varphi} \in \mathbf{H}^r(\mathrm{div}_r;\Omega)$ and $\boldsymbol{z} \in \mathbf{H}^r(\mathrm{div};\Omega)$, consist in finding $\boldsymbol{t}_i \in \mathbf{L}^2(\Omega)$, $\xi_i \in \mathrm{L}^\rho(\Omega)$, and $\boldsymbol{\sigma}_i$ in a suitable space to be made precise, such that

$$\boldsymbol{t}_i - \nabla\xi_i = \mathbf{0} \qquad \text{in}\quad \Omega\,, \tag{3.28a}$$

$$-\boldsymbol{\sigma}_i + \kappa_i\,\boldsymbol{t}_i + q_i\,\kappa_i\,\varepsilon^{-1}\,\xi_i\,\boldsymbol{\varphi} - \xi_i\,\boldsymbol{z} = \mathbf{0} \qquad \text{in}\quad \Omega\,, \tag{3.28b}$$

$$\xi_i - \mathrm{div}(\boldsymbol{\sigma}_i) - \xi_i\,\mathrm{div}(\boldsymbol{z}) = f_i \qquad \text{in}\quad \Omega\,. \tag{3.28c}$$

$$\xi_i = \xi_{i,\mathtt{D}} \qquad \text{on}\quad \Gamma\,. \tag{3.28d}$$

Note that the spaces to which $\boldsymbol{t}_i$ and ξ_i are indicated to belong, for $i \in \{1,2\}$, were derived in Section 3.2 after analyzing the validity of (3.11). These belongings are confirmed next, but we need to suppose momentarily that $\xi_i \in \mathrm{H}^1(\Omega)$, which implies assuming as well that $\xi_{i,\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$. Indeed, we begin by testing (3.28a) against $\boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$, so that applying (3.3) with $t = \varrho$ to the aforementioned $\boldsymbol{\tau}_i$ and $\xi_i \in \mathrm{H}^1(\Omega)$, and using the Dirichlet boundary condition for ξ_i (cf. (3.28d)), we get

$$\int_\Omega \boldsymbol{t}_i \cdot \boldsymbol{\tau}_i \,+\, \int_\Omega \xi_i \,\mathrm{div}(\boldsymbol{\tau}_i) \,=\, \langle \boldsymbol{\tau}_i \cdot \boldsymbol{n}, \xi_{i,\mathrm{D}} \rangle \qquad \forall\, \boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)\,, \tag{3.29}$$

from which it suffices to look for ξ_i in $\mathrm{L}^\rho(\Omega)$, as previously announced. In turn, bearing in mind (3.5b) and (3.5c), we test (3.28b) against $\boldsymbol{s}_i \in \mathbf{L}^2(\Omega)$, thus arriving at

$$\kappa_i \int_\Omega \boldsymbol{t}_i \cdot \boldsymbol{s}_i - \int_\Omega \boldsymbol{\sigma}_i \cdot \boldsymbol{s}_i \,+\, q_i\, \varepsilon^{-1} \kappa_i \int_\Omega \xi_i\, \boldsymbol{\varphi} \cdot \boldsymbol{s}_i - \int_\Omega \xi_i\, \boldsymbol{z} \cdot \boldsymbol{s}_i = 0 \qquad \forall\, \boldsymbol{s}_i \in \mathbf{L}^2(\Omega)\,, \tag{3.30}$$

from where it only remains to observe that the second term on the left-hand side makes sense for $\boldsymbol{\sigma}_i \in \mathbf{L}^2(\Omega)$. Furthermore, assuming that f_i belongs to $\mathrm{L}^\varrho(\Omega)$, we test (3.28c) against $\eta_i \in \mathrm{L}^\rho(\Omega)$ and obtain

$$\int_\Omega \eta_i \,\mathrm{div}(\boldsymbol{\sigma}_i) - \int_\Omega \xi_i\, \eta_i \,+\, \int_\Omega \xi_i \,\mathrm{div}(\boldsymbol{z})\, \eta_i \,=\, -\int_\Omega f_i\, \eta_i \qquad \forall\, \eta_i \in \mathrm{L}^\rho(\Omega)\,, \tag{3.31}$$

whose first term on the left-hand side is well-defined if $\mathrm{div}(\boldsymbol{\sigma}_i)$ belongs to $\mathrm{L}^\varrho(\Omega)$, whence we now look for $\boldsymbol{\sigma}$ in $\mathbf{H}(\mathrm{div}_\varrho;\Omega)$. In addition, being $\rho \geq r > 2$ (cf. (3.7)), it is easily seen, thanks to the Cauchy–Schwarz and Hölder inequalities, that the second and third term makes sense as well. Consequently, we now introduce the spaces

$$\mathcal{H}_1 \,:=\, \mathbf{L}^2(\Omega)\,, \quad \mathcal{H}_2 \,:=\, \mathbf{H}(\mathrm{div}_\varrho;\Omega)\,, \quad \mathcal{M} \,:=\, \mathrm{L}^\rho(\Omega)\,,$$

which are endowed, respectively, with the norms

$$\|\boldsymbol{s}\|_{\mathcal{H}_1} \,:=\, \|\boldsymbol{s}\|_{0,\Omega} \quad \forall\, \boldsymbol{s} \in \mathcal{H}_1\,, \quad \|\boldsymbol{\tau}\|_{\mathcal{H}_2} \,:=\, \|\boldsymbol{\tau}\|_{\mathrm{div}_\varrho;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathcal{H}_2\,, \quad \|\eta\|_{\mathcal{M}} \,:=\, \|\eta\|_{0,\rho;\Omega} \quad \forall\, \eta \in \mathcal{M}\,,$$

define

$$\boldsymbol{\mathcal{H}} \,:=\, \mathcal{H}_1 \times \mathcal{H}_2 \quad \text{with product norm} \quad \|\vec{\boldsymbol{s}}\|_{\boldsymbol{\mathcal{H}}} := \|\boldsymbol{s}\|_{\mathcal{H}_1} + \|\boldsymbol{\tau}\|_{\mathcal{H}_2} \quad \forall\, \vec{\boldsymbol{s}} := (\boldsymbol{s},\boldsymbol{\tau}) \in \boldsymbol{\mathcal{H}}\,,$$

and set the notations

$$\vec{\boldsymbol{t}}_i := (\boldsymbol{t}_i, \boldsymbol{\sigma}_i)\,, \quad \vec{\boldsymbol{r}}_i \,:=\, (\boldsymbol{r}_i, \boldsymbol{\zeta}_i)\,, \quad \vec{\boldsymbol{s}}_i := (\boldsymbol{s}_i, \boldsymbol{\tau}_i) \in \boldsymbol{\mathcal{H}}\,.$$

Then, adding (3.29) and (3.30), and gathering the result with (3.31), we conclude that, given $(\boldsymbol{z},\boldsymbol{\varphi}) \in \mathbf{X}_2 \times \mathrm{X}_2$, the mixed formulation of (3.28a) - (3.28d) reduces to: Find $(\vec{\boldsymbol{t}}_i, \xi_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}$ such that

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{t}}_i, \vec{\boldsymbol{s}}_i) \,+\, \mathcal{B}(\vec{\boldsymbol{s}}_i, \xi_i) \quad + \mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i, \xi_i) \quad &= \quad \mathcal{G}(\vec{\boldsymbol{s}}_i) \qquad \forall\, \vec{\boldsymbol{s}}_i \in \boldsymbol{\mathcal{H}}\,, \\ \mathcal{B}(\vec{\boldsymbol{t}}_i, \eta_i) \,-\, \mathcal{C}(\xi_i, \eta_i) \quad + \mathcal{D}_{\boldsymbol{z}}(\xi_i, \eta_i) \quad &= \quad \mathcal{F}(\eta_i) \qquad \forall\, \eta_i \in \mathcal{M}\,, \end{aligned} \tag{3.32}$$

where the bilinear forms $\mathcal{A} : \boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{H}} \to \mathrm{R}$, $\mathcal{B} : \boldsymbol{\mathcal{H}} \times \mathcal{M} \to \mathrm{R}$, $\mathcal{C} : \mathcal{M} \times \mathcal{M}$, $\mathcal{D}_{\boldsymbol{z}} : \mathcal{M} \times \mathcal{M} \to \mathrm{R}$, and $\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}} : \boldsymbol{\mathcal{H}} \times \mathcal{M} \to \mathrm{R}$, are defined, respectively, as

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{t}}_i, \vec{\boldsymbol{s}}_i) \quad &:= \quad \kappa_i \int_\Omega \boldsymbol{t}_i \cdot \boldsymbol{s}_i \,-\, \int_\Omega \boldsymbol{\sigma}_i \cdot \boldsymbol{s}_i \,+\, \int_\Omega \boldsymbol{\tau}_i \cdot \boldsymbol{t}_i \qquad &&\forall\, \vec{\boldsymbol{t}}_i, \vec{\boldsymbol{s}}_i \in \boldsymbol{\mathcal{H}}\,, \\ \mathcal{B}(\vec{\boldsymbol{s}}_i, \eta_i) \quad &:= \quad \int_\Omega \eta_i \,\mathrm{div}(\boldsymbol{\tau}_i) &&\forall\, (\vec{\boldsymbol{s}}_i, \eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,, \\ \mathcal{C}(\xi_i, \eta_i) \quad &:= \quad \int_\Omega \xi_i\, \eta_i &&\forall\, \xi_i, \eta_i \in \mathcal{M}\,, \\ \mathcal{D}_{\boldsymbol{z}}(\xi_i, \eta_i) \quad &:= \quad \int_\Omega \xi_i \,\mathrm{div}(\boldsymbol{z})\, \eta_i &&\forall\, \xi_i,\, \eta_i \in \mathcal{M}\,, \quad \text{and} \\ \mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i, \eta_i) \quad &:= \quad -\int_\Omega \eta_i\, \boldsymbol{z} \cdot \boldsymbol{s}_i + q_i\, \varepsilon^{-1} \kappa_i \int_\Omega \eta_i\, \boldsymbol{\varphi} \cdot \boldsymbol{s}_i &&\forall\, (\vec{\boldsymbol{s}}_i, \eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,, \end{aligned} \tag{3.33}$$

whereas the functionals $\mathcal{G} : \boldsymbol{\mathcal{H}} \to \mathrm{R}$ and $\mathcal{F} : \mathcal{M} \to \mathrm{R}$ are given, respectively, by

$$\mathcal{G}(\vec{\boldsymbol{s}}_i) := \langle \boldsymbol{\tau}_i \cdot \boldsymbol{n}, \xi_{i,\mathtt{D}} \rangle \quad \text{and} \quad \mathcal{F}(\eta_i) := -\int_\Omega f_i \, \eta_i \,.$$

We remark here that, ignoring the bilinear forms $\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}$ and $\mathcal{D}_{\boldsymbol{z}}$, the structure of the left-hand side of (3.32) corresponds to that of a perturbed saddle-point problem.

Applying once again the Cauchy–Schwarz and Hölder inequalities, and using the continuous injection $i_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$, we readily show that the bilinear forms $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and the functionals $\mathcal{G}$ and $\mathcal{F}$, are all bounded with respective constants given by

$$\begin{gathered} \|\mathcal{A}\| := \max\big\{\kappa_i, 1\big\}\,, \quad \|\mathcal{B}\| := 1\,, \quad \|\mathcal{C}\| := |\Omega|^{\frac{\rho-2}{\rho}}\,, \\ \|\mathcal{G}\| := \big(1 + \|i_\rho\|\big)\, \|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma}\,, \quad \text{and} \quad \|\mathcal{F}\| := \|f_i\|_{0,\varrho;\Omega}\,. \end{gathered} \tag{3.34}$$

Likewise, there hold

$$\begin{aligned} |\mathcal{D}_{\boldsymbol{z}}(\xi_i,\eta_i)| &\le \|\mathcal{D}\|\, \|\boldsymbol{z}\|_{\mathbf{X}_2}\, \|\xi_i\|_{\mathcal{M}}\, \|\eta_i\|_{\mathcal{M}} && \forall\, \xi_i,\eta_i \in \mathcal{M}\,, \\ |\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i,\eta_i)| &\le \|\mathcal{E}\|\, \|(\boldsymbol{z},\boldsymbol{\varphi})\|_{\mathbf{X}_2\times \mathrm{X}_2}\, \|\vec{\boldsymbol{s}}_i\|_{\boldsymbol{\mathcal{H}}}\, \|\eta_i\|_{\mathcal{M}} && \forall\, (\vec{\boldsymbol{s}}_i,\eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,, \end{aligned} \tag{3.35}$$

with

$$\|\mathcal{D}\| := 1\,, \quad \text{and} \quad \|\mathcal{E}\| := \max\big\{\varepsilon^{-1}\,\kappa_i, 1\big\}\,. \tag{3.36}$$

3.5 Weak formulation of the full coupled problem

According to the analysis in Sections 3.2, 3.3, and 3.4, we conclude that, under the assumption that $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $g \in \mathrm{L}^2(\Omega)$, $\chi_{\mathtt{D}} \in \mathrm{W}^{1/s,r}(\Gamma)$, $\xi_{i,\mathtt{D}} \in \mathrm{H}^{1/2}(\Gamma)$, and $f_i \in \mathrm{L}^\varrho(\Omega)$, $i \in \{1,2\}$, the primal-mixed formulation of the Biot–Poisson–Nernst–Planck problem (2.7a) - (2.9) is obtained by gathering (3.20), (3.24), and (3.32), so that it becomes: Find $(\boldsymbol{u},\theta) \in \mathbf{X}\times\mathbf{Q}$, $(\boldsymbol{z},p) \in \mathbf{X}_2 \times \mathbf{Q}_1$, $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ and $(\vec{\boldsymbol{t}}_i,\xi_i) \in \boldsymbol{\mathcal{H}}\times\mathcal{M}$ such that

$$\begin{array}{llllll} \mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) + \mathbf{b}_s(\boldsymbol{v},\theta) & & & = & \mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}(\boldsymbol{v}) & \forall\, \boldsymbol{v} \in \mathbf{X}\,, \\ \mathbf{b}_s(\boldsymbol{u},\vartheta) - \mathbf{c}_s(\theta,\vartheta) & & + \mathbf{e}_s(p,\vartheta) & = & 0 & \forall\, \vartheta \in \mathbf{Q}\,, \\ & \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) + \mathbf{d}_1(\boldsymbol{w},p) & & = & 0 & \forall\, \boldsymbol{w} \in \mathbf{X}_1\,, \\ & \mathbf{d}_2(\boldsymbol{z},q) & + \mathbf{e}_f((\theta,p),q) & = & \mathbf{G}(q) & \forall\, q \in \mathbf{Q}_2\,, \\ a(\boldsymbol{\varphi},\boldsymbol{\psi}) + b_1(\boldsymbol{\psi},\chi) & & & = & G(\boldsymbol{\psi}) & \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \\ b_2(\boldsymbol{\varphi},\gamma) & & & = & F_{\boldsymbol{\xi}}(\gamma) & \forall\, \gamma \in \mathrm{M}_2\,, \\ \mathcal{A}(\vec{\boldsymbol{t}}_i,\vec{\boldsymbol{s}}_i) + \mathcal{B}(\vec{\boldsymbol{s}}_i,\xi_i) & & + \mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i,\xi_i) & = & \mathcal{G}(\vec{\boldsymbol{s}}_i) & \forall\, \vec{\boldsymbol{s}}_i \in \boldsymbol{\mathcal{H}}\,, \\ \mathcal{B}(\vec{\boldsymbol{t}}_i,\eta_i) - \mathcal{C}(\xi_i,\eta_i) & & + \mathcal{D}_{\boldsymbol{z}}(\xi_i,\eta_i) & = & \mathcal{F}(\eta_i) & \forall\, \eta_i \in \mathcal{M}\,. \end{array} \tag{3.37}$$

4 Continuous solvability analysis

In this section, we proceed similarly as in [24] (see also [18, 28]), and adopt a fixed-point strategy to study the solvability of (3.37). To this end, we define operators solving the decoupled problems, and in terms of them we set the fixed-point equation that is equivalent to (3.37). Then, we analyze the well-posedness of the aforementioned problems and equation.

4.1 Fixed-point approach

We begin by defining the spaces

$$\boldsymbol{\mathcal{H}}_1 := \mathcal{H}_1 \times \mathcal{H}_1 \quad \text{and} \quad \boldsymbol{\mathcal{M}} := \mathcal{M} \times \mathcal{M}\,,$$

which are endowed with the product norms

$$\begin{aligned} \|\boldsymbol{r}\|_{\boldsymbol{\mathcal{H}}_1} &:= \|\boldsymbol{r}_1\|_{\mathcal{H}_1} + \|\boldsymbol{r}_2\|_{\mathcal{H}_1} \quad \forall\, \boldsymbol{r} := (\boldsymbol{r}_1, \boldsymbol{r}_2) \in \boldsymbol{\mathcal{H}}_1 \quad \text{and} \\ \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{M}}} &:= \|\eta_1\|_{\mathcal{M}} + \|\eta_2\|_{\mathcal{M}} \quad \forall\, \boldsymbol{\eta} := (\eta_1, \eta_2) \in \boldsymbol{\mathcal{M}}\,, \end{aligned}$$

and additionally set the notations

$$\boldsymbol{t} := (\boldsymbol{t}_1, \boldsymbol{t}_2) \in \boldsymbol{\mathcal{H}}_1 \qquad \text{and} \qquad \boldsymbol{\xi} := (\xi_1, \xi_2) \in \boldsymbol{\mathcal{M}}\,.$$

Now, let $\mathbf{S} : \mathrm{X}_2 \times \boldsymbol{\mathcal{M}} \times \boldsymbol{\mathcal{H}}_1 \to \mathbf{X}_2$ be the operator defined for each $(\boldsymbol{\phi}, \boldsymbol{\eta}, \boldsymbol{r}) \in \mathrm{X}_2 \times \boldsymbol{\mathcal{M}} \times \boldsymbol{\mathcal{H}}_1$ by

$$\mathbf{S}(\boldsymbol{\phi}, \boldsymbol{\eta}, \boldsymbol{r}) := \boldsymbol{z}, \tag{4.1}$$

where $((\boldsymbol{u}, \theta), (\boldsymbol{z}, p)) \in (\mathbf{X} \times \mathbf{Q}) \times (\mathbf{X}_2 \times \mathbf{Q}_1)$ is the unique solution (to be confirmed below) of problem (3.20) when $\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}$ is replaced by $\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}$, that is

$$\begin{array}{llcll} \mathbf{a}_s(\boldsymbol{u}, \boldsymbol{v}) + \mathbf{b}_s(\boldsymbol{v}, \theta) & & = & \mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}(\boldsymbol{v}) & \forall\, \boldsymbol{v} \in \mathbf{X}\,, \\ \mathbf{b}_s(\boldsymbol{u}, \vartheta) - \mathbf{c}_s(\theta, \vartheta) & + \mathbf{e}_s(p, \vartheta) & = & 0 & \forall\, \vartheta \in \mathbf{Q}\,, \\ \qquad \mathbf{a}_f(\boldsymbol{z}, \boldsymbol{w}) \,+\, \mathbf{d}_1(\boldsymbol{w}, p) & & = & 0 & \forall\, \boldsymbol{w} \in \mathbf{X}_1\,, \\ \qquad \mathbf{d}_2(\boldsymbol{z}, q) & + \mathbf{e}_f((\theta, p), q) & = & \mathbf{G}(q) & \forall\, q \in \mathbf{Q}_2\,. \end{array} \tag{4.2}$$

In turn, let $\widetilde{\mathbf{S}} : \boldsymbol{\mathcal{M}} \to \mathrm{X}_2$ be the operator defined for each $\boldsymbol{\eta} \in \boldsymbol{\mathcal{M}}$ by

$$\widetilde{\mathbf{S}}(\boldsymbol{\eta}) := \boldsymbol{\varphi}\,,$$

where $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ is the unique solution (to be confirmed below) of problem (3.24) with $F_{\boldsymbol{\eta}}$ instead of $F_{\boldsymbol{\xi}}$, that is

$$\begin{array}{lcll} a(\boldsymbol{\varphi}, \boldsymbol{\psi}) \,+\, b_1(\boldsymbol{\psi}, \chi) & = & G(\boldsymbol{\psi}) & \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \\ b_2(\boldsymbol{\varphi}, \gamma) & = & F_{\boldsymbol{\eta}}(\gamma) & \forall\, \gamma \in \mathrm{M}_2\,. \end{array} \tag{4.3}$$

Furthermore, we let $\mathrm{T}_i : \mathbf{X}_2 \times \mathrm{X}_2 \to \mathcal{H}_1$ and $\Xi_i : \mathbf{X}_2 \times \mathrm{X}_2 \to \mathcal{M}$, $i \in \{1, 2\}$, be the operators defined for each $(\boldsymbol{w}, \boldsymbol{\phi}) \in \mathbf{X}_2 \times \mathrm{X}_2$ by

$$\mathrm{T}_i(\boldsymbol{w}, \boldsymbol{\phi}) := \boldsymbol{t}_i \quad \text{and} \quad \Xi_i(\boldsymbol{w}, \boldsymbol{\phi}) := \xi_i\,,$$

where $(\vec{\boldsymbol{t}}_i, \xi_i) = ((\boldsymbol{t}_i, \boldsymbol{\sigma}_i), \xi_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}$ is the unique solution (to be confirmed below) of problem (3.32) when $\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}$ and $\mathcal{D}_{\boldsymbol{z}}$ are replaced by $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{D}_{\boldsymbol{w}}$, respectively, that is

$$\begin{array}{lllcll} \mathcal{A}(\vec{\boldsymbol{t}}_i, \vec{\boldsymbol{s}}_i) & +\; \mathcal{B}(\vec{\boldsymbol{s}}_i, \xi_i) & +\; \mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}(\vec{\boldsymbol{s}}_i, \xi_i) & = & \mathcal{G}(\vec{\boldsymbol{s}}_i) & \forall\, \vec{\boldsymbol{s}}_i \in \boldsymbol{\mathcal{H}}\,, \\ \mathcal{B}(\vec{\boldsymbol{t}}_i, \eta_i) & -\; \mathcal{C}(\xi_i, \eta_i) & +\; \mathcal{D}_{\boldsymbol{w}}(\xi_i, \eta_i) & = & \mathcal{F}(\eta_i) & \forall\, \eta_i \in \mathcal{M}\,. \end{array} \tag{4.4}$$

As a consequence, we can set the operators $\boldsymbol{\Xi} : \mathbf{X}_2 \times \mathrm{X}_2 \to \boldsymbol{\mathcal{M}}$ and $\mathbf{T} : \mathbf{X}_2 \times \mathrm{X}_2 \to \boldsymbol{\mathcal{H}}_1$ as

$$\boldsymbol{\Xi}(\boldsymbol{w}, \boldsymbol{\phi}) := (\Xi_1(\boldsymbol{w}, \boldsymbol{\phi}), \Xi_2(\boldsymbol{w}, \boldsymbol{\phi})) = \boldsymbol{\xi} \quad \text{and} \quad \mathbf{T}(\boldsymbol{w}, \boldsymbol{\phi}) := (\mathrm{T}_1(\boldsymbol{w}, \boldsymbol{\phi}), \mathrm{T}_2(\boldsymbol{w}, \boldsymbol{\phi})) = \boldsymbol{t},$$

for all $(\boldsymbol{w}, \boldsymbol{\phi}) \in \mathbf{X}_2 \times \mathrm{X}_2$. Finally, we introduce the operator $\boldsymbol{\Pi} : \mathbf{X}_2 \times \mathrm{X}_2 \to \mathbf{X}_2 \times \mathrm{X}_2$ defined for each $(\boldsymbol{w}, \boldsymbol{\phi}) \in \mathbf{X}_2 \times \mathrm{X}_2$ by

$$\boldsymbol{\Pi}(\boldsymbol{w}, \boldsymbol{\phi}) := \big(\mathbf{S}(\boldsymbol{\phi}, \boldsymbol{\Xi}(\boldsymbol{w}, \boldsymbol{\phi}), \mathbf{T}(\boldsymbol{w}, \boldsymbol{\phi})), \widetilde{\mathbf{S}}(\boldsymbol{\Xi}(\boldsymbol{w}, \boldsymbol{\phi}))\big)\,, \tag{4.5}$$

and realize that solving (3.37) is equivalent to finding a fixed point of $\boldsymbol{\Pi}$, that is, $(\boldsymbol{z}, \boldsymbol{\varphi}) \in \mathbf{X}_2 \times \mathrm{X}_2$ such that

$$\boldsymbol{\Pi}(\boldsymbol{z}, \boldsymbol{\varphi}) = (\boldsymbol{z}, \boldsymbol{\varphi})\,. \tag{4.6}$$

4.2 Well-definedness of the operator S

We first apply an abstract result on perturbed saddle-point problems in Hilbert spaces (cf. [8, Theorem 4.3.1]) and the generalized Babuška–Brezzi theory (cf. [6, Theorem 2.1, Corollary 2.1, Section 2.1]) to the bilinear form arising from (4.2) when $\mathbf{e}_s$ and $\mathbf{e}_f$ are dropped, and then employ the Banach–Nečas–Babuška theorem (cf. [25, Theorem 2.6]) to conclude that the whole problem (4.2) is well-posed, which is equivalent to stating that $\mathbf{S}$ (cf. (4.1)) is well-defined. For this purpose, we now introduce the spaces

$$\mathbb{X} := \mathbf{X} \times \mathbf{Q} \times \mathbf{X}_2 \times \mathbf{Q}_1 \quad \text{and} \quad \mathbb{Q} := \mathbf{X} \times \mathbf{Q} \times \mathbf{X}_1 \times \mathbf{Q}_2\,,$$

which are endowed with the norms

$$\begin{aligned} \|\vec{\boldsymbol{u}}\|_{\mathbb{X}} &:= \|\boldsymbol{u}\|_{\mathbf{X}} + \|\theta\|_{\mathbf{Q}} + \|\boldsymbol{z}\|_{\mathbf{X}_2} + \|p\|_{\mathbf{Q}_1} \quad \forall\, \vec{\boldsymbol{u}} := (\boldsymbol{u}, \theta, \boldsymbol{z}, p) \in \mathbb{X}\,, \quad \text{and} \\ \|\vec{\boldsymbol{v}}\|_{\mathbb{Q}} &:= \|\boldsymbol{v}\|_{\mathbf{X}} + \|\vartheta\|_{\mathbf{Q}} + \|\boldsymbol{w}\|_{\mathbf{X}_1} + \|q\|_{\mathbf{Q}_2} \quad \forall\, \vec{\boldsymbol{v}} := (\boldsymbol{v}, \vartheta, \boldsymbol{w}, q) \in \mathbb{Q}\,. \end{aligned}$$

Then, as announced, we let $\mathbf{A} : \mathbb{X} \times \mathbb{Q} \to \mathrm{R}$ be the bounded bilinear form arising from (4.2) after adding the left-hand sides of its equations, but without including $\mathbf{e}_s$ and $\mathbf{e}_f$, that is

$$\begin{aligned} \mathbf{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) &:= \mathbf{a}_s(\boldsymbol{u}, \boldsymbol{v}) + \mathbf{b}_s(\boldsymbol{v}, \theta) + \mathbf{b}_s(\boldsymbol{u}, \vartheta) - \mathbf{c}_s(\theta, \vartheta) \\ &\quad + \mathbf{a}_f(\boldsymbol{z}, \boldsymbol{w}) + \mathbf{d}_1(\boldsymbol{w}, p) + \mathbf{d}_2(\boldsymbol{z}, q) \end{aligned} \tag{4.7}$$

for all $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in \mathbb{X} \times \mathbb{Q}$. Note that the boundedness $\mathbf{A}$ follows from those of $\mathbf{a}_s$, $\mathbf{b}_s$, $\mathbf{b}_s$, $\mathbf{c}_s$, $\mathbf{a}_f$, $\mathbf{d}_1$, and $\mathbf{d}_2$ (cf. (3.21)). In addition, as noticed in advance in Section 3.2, we now stress that $\mathbf{A}$ shows the matrix representation

$$\left(\begin{array}{cc|cc} \mathbf{a}_s & \mathbf{b}_s & & \\ \mathbf{b}_s & -\mathbf{c}_s & & \\ \hline & & \mathbf{a}_f & \mathbf{d}_1 \\ & & \mathbf{d}_2 & \end{array}\right), \tag{4.8}$$

whose block-diagonal structure, composed by the perturbed and generalized saddle-point matrix operators given, respectively, by $\begin{pmatrix} \mathbf{a}_s & \mathbf{b}_s \\ \mathbf{b}_s & -\mathbf{c}_s \end{pmatrix}$ and $\begin{pmatrix} \mathbf{a}_f & \mathbf{d}_1 \\ \mathbf{d}_2 & \end{pmatrix}$, is evident.

The above property yields an advantageous feature when showing below the corresponding global inf-sup conditions. More precisely, introducing

$$\mathcal{S}_1(\vec{\boldsymbol{u}}) := \sup_{\substack{\vec{\boldsymbol{v}} \in \mathbb{Q} \\ \vec{\boldsymbol{v}} \neq \mathbf{0}}} \frac{\mathbf{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})}{\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}}} \quad \forall\, \vec{\boldsymbol{u}} \in \mathbb{X} \qquad \text{and} \qquad \mathcal{S}_2(\vec{\boldsymbol{v}}) := \sup_{\substack{\vec{\boldsymbol{u}} \in \mathbb{X} \\ \vec{\boldsymbol{u}} \neq \mathbf{0}}} \frac{\mathbf{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})}{\|\vec{\boldsymbol{u}}\|_{\mathbb{X}}} \quad \forall\, \vec{\boldsymbol{v}} \in \mathbb{Q}\,,$$

we aim to prove next the existence of a positive constant $\boldsymbol{\alpha}_{\mathbf{A}}$ such that

$$\mathcal{S}_1(\vec{\boldsymbol{u}}) \geq \boldsymbol{\alpha}_{\mathbf{A}} \|\vec{\boldsymbol{u}}\|_{\mathbb{X}} \qquad \forall\, \vec{\boldsymbol{u}} \in \mathbb{X}\,, \quad \text{and,} \tag{4.9a}$$

$$\mathcal{S}_2(\vec{\boldsymbol{v}}) \geq \boldsymbol{\alpha}_{\mathbf{A}} \|\vec{\boldsymbol{v}}\|_{\mathbb{Q}} \qquad \forall\, \vec{\boldsymbol{v}} \in \mathbb{Q}\,. \tag{4.9b}$$

To this end, and according to (4.8), we decompose $\mathbf{A}$ as

$$\mathbf{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) := \mathbf{A}_s((\boldsymbol{u}, \theta), (\boldsymbol{v}, \vartheta)) + \mathbf{A}_f((\boldsymbol{z}, p), (\boldsymbol{w}, q)) \qquad \forall\, (\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \in \mathbb{X} \times \mathbb{Q}\,, \tag{4.10}$$

where $\mathbf{A}_s : (\mathbf{X} \times \mathbf{Q}) \times (\mathbf{X} \times \mathbf{Q}) \to \mathrm{R}$ and $\mathbf{A}_f : (\mathbf{X}_2 \times \mathbf{Q}_1) \times (\mathbf{X}_1 \times \mathbf{Q}_2) \to \mathrm{R}$ are the bilinear forms defined, respectively, by

$$\mathbf{A}_s\big((\boldsymbol{u}, \theta), (\boldsymbol{v}, \vartheta)\big) := \mathbf{a}_s(\boldsymbol{u}, \boldsymbol{v}) + \mathbf{b}_s(\boldsymbol{v}, \theta) + \mathbf{b}_s(\boldsymbol{u}, \vartheta) - \mathbf{c}_s(\theta, \vartheta),$$

for all $(\boldsymbol{u}, \theta)$, $(\boldsymbol{v}, \vartheta) \in \mathbf{X} \times \mathbf{Q}$, and

$$\mathbf{A}_f\big((\boldsymbol{z}, p), (\boldsymbol{w}, q)\big) := \mathbf{a}_f(\boldsymbol{z}, \boldsymbol{w}) + \mathbf{d}_1(\boldsymbol{w}, p) + \mathbf{d}_2(\boldsymbol{z}, q),$$

for all $\big((\boldsymbol{z},p),\,(\boldsymbol{w},q)\big) \in (\mathbf{X}_2 \times \mathbf{Q}_1) \times (\mathbf{X}_1 \times \mathbf{Q}_2$. Thus, thanks to (4.10), it is straightforward to see that there holds

$$\mathcal{S}_1(\vec{\boldsymbol{u}}) \,\geq\, \frac{1}{2} \left\{ \sup_{\substack{(\boldsymbol{v},\vartheta)\in \mathbf{X}\times\mathbf{Q} \\ (\boldsymbol{v},\vartheta)\neq\mathbf{0}}} \frac{\mathbf{A}_s((\boldsymbol{u},\theta),(\boldsymbol{v},\vartheta))}{\|(\boldsymbol{v},\vartheta)\|_{\mathbf{X}\times\mathbf{Q}}} + \sup_{\substack{(\boldsymbol{w},q)\in \mathbf{X}_1\times\mathbf{Q}_2 \\ (\boldsymbol{w},q)\neq\mathbf{0}}} \frac{\mathbf{A}_f((\boldsymbol{z},p),(\boldsymbol{w},q))}{\|(\boldsymbol{w},q)\|_{\mathbf{X}_1\times\mathbf{Q}_2}} \right\} \quad \forall\, \vec{\boldsymbol{u}} \in \mathbb{X}\,, \tag{4.11}$$

whence, in order to prove (4.9a), it suffices to show that there exist positive constants $\boldsymbol{\alpha}_s$ and $\boldsymbol{\alpha}_f$ such that

$$\sup_{\substack{(\boldsymbol{v},\vartheta)\in \mathbf{X}\times\mathbf{Q} \\ (\boldsymbol{v},\vartheta)\neq\mathbf{0}}} \frac{\mathbf{A}_s((\boldsymbol{u},\theta),(\boldsymbol{v},\vartheta))}{\|(\boldsymbol{v},\vartheta)\|_{\mathbf{X}\times\mathbf{Q}}} \,\geq\, \boldsymbol{\alpha}_s\, \|(\boldsymbol{u},\theta)\|_{\mathbf{X}\times\mathbf{Q}} \qquad \forall\,(\boldsymbol{u},\theta)\in\mathbf{X}\times\mathbf{Q} \quad \text{and} \tag{4.12a}$$

$$\sup_{\substack{(\boldsymbol{w},q)\in \mathbf{X}_1\times\mathbf{Q}_2 \\ (\boldsymbol{w},q)\neq\mathbf{0}}} \frac{\mathbf{A}_f((\boldsymbol{z},p),(\boldsymbol{w},q))}{\|(\boldsymbol{w},q)\|_{\mathbf{X}_1\times\mathbf{Q}_2}} \,\geq\, \boldsymbol{\alpha}_f\, \|(\boldsymbol{z},p)\|_{\mathbf{X}_2\times\mathbf{Q}_1} \qquad \forall\,(\boldsymbol{z},p)\in\mathbf{X}_2\times\mathbf{Q}_1\,. \tag{4.12b}$$

In this regard, and because of the matrix representation of $\mathbf{A}_s$ (cf. upper block in (4.8)), we find that establishing (4.12a) is equivalent to proving that the bilinear forms $\mathbf{a}_s$, $\mathbf{b}_s$ and $\mathbf{c}_s$ satisfy the hypotheses of the abstract result in Hilbert spaces provided by [8, Theorem 4.3.1]. Indeed, we first notice from (3.16) that $\mathbf{a}_s$ and $\mathbf{c}_s$ are clearly symmetric and positive semi-definite. In addition, applying the Körn and Poincaré inequalities, which say, respectively, that $\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2 \,\geq\, \frac{1}{2}\,|\boldsymbol{v}|_{1,\Omega}^2$ and $|\boldsymbol{v}|_{1,\Omega}^2 \,\geq\, C_P\,\|\boldsymbol{v}\|_{1,\Omega}^2$ for all $\boldsymbol{v}\in\mathbf{H}_0^1(\Omega)$, where C_P is a fixed positive constant, we readily deduce that

$$\mathbf{a}_s(\boldsymbol{v},\boldsymbol{v}) \,=\, 2\mu\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2 \,\geq\, \alpha_s \|\boldsymbol{v}\|_{\mathbf{X}}^2 \quad \forall\,\boldsymbol{v}\in\mathbf{X}\,, \tag{4.13}$$

with the constant $\alpha_s \,=\, \mu\, C_P$, thus proving that $\mathbf{a}_s$ is $\mathbf{X}$-elliptic. Furthermore, we know from [29, Chapter I, eqn. (5.14)] that there exists a positive constant β_s such that

$$\sup_{\substack{\boldsymbol{v}\in\mathbf{X} \\ \boldsymbol{v}\neq\mathbf{0}}} \frac{\mathbf{b}_s(\boldsymbol{v},\vartheta)}{\|\boldsymbol{v}\|_{\mathbf{X}}} \,\geq\, \beta_s\,\|\vartheta\|_{\mathbf{Q}} \qquad \forall\,\vartheta\in\mathbf{Q}\,. \tag{4.14}$$

Therefore, under the hypotheses of [8, Theorem 4.3.1], the a priori estimates given by [8, Proposition 2.11, eqn. (4.3.21)] imply that there exists a positive constant $\boldsymbol{\alpha}_s$, depending on $\|\mathbf{a}_s\|$, $\|\mathbf{c}_s\|$, α_s, and β_s, such that (4.12a) holds.

In turn, due to the matrix representation of $\mathbf{A}_f$ (cf. lower block in (4.8)), we realize that proving (4.12b) is equivalent to verifying that the bilinear forms $\mathbf{a}_f$, $\mathbf{d}_1$, and $\mathbf{d}_2$ satisfy the hypotheses of the generalized Babuška-Brezzi theory (cf. [6, Theorem 2.1, Section 2.1]). In fact, we first observe that the kernels of the bilinear forms $\mathbf{d}_i$ (cf. (3.19)), $i\in\{1,2\}$, are given, respectively, by

$$\mathbf{K}_1 \,:=\, \Big\{\boldsymbol{w}\in\mathbf{H}_0^s(\mathrm{div}_s;\Omega):\quad \mathrm{div}(\boldsymbol{w}) = 0 \quad\text{in}\quad \Omega\Big\} \quad \text{and}$$

$$\mathbf{K}_2 \,:=\, \Big\{\boldsymbol{w}\in\mathbf{H}_0^r(\mathrm{div};\Omega):\quad \mathrm{div}(\boldsymbol{w}) = 0 \quad\text{in}\quad \Omega\Big\}\,.$$

Thus, resorting to [27], we have the required continuous inf-sup conditions for $\mathbf{a}_f$.

Lemma 4.1. *There exists a positive constant α_f such that*

$$\sup_{\substack{\boldsymbol{w}\in\mathbf{K}_1 \\ \boldsymbol{w}\neq\mathbf{0}}} \frac{\mathbf{a}_f(\boldsymbol{z},\boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathbf{X}_1}} \,\geq\, \alpha_f\,\|\boldsymbol{z}\|_{\mathbf{X}_2} \qquad \forall\,\boldsymbol{z}\in\mathbf{K}_2\,, \quad \textit{and} \tag{4.15a}$$

$$\sup_{\boldsymbol{z}\in\mathbf{K}_2} \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) \,>\, 0 \qquad \forall\,\boldsymbol{w}\in\mathbf{K}_1\,,\,\boldsymbol{w}\neq\mathbf{0}\,. \tag{4.15b}$$

Proof. It reduces to a minor modification of the proof of [27, Lemma 2.6], which first yields (4.15a) with $\alpha_f := \frac{\nu}{\kappa\|D_s\|}$, where D_s is the bounded linear operator defined in [27, Lemma 2.3]. In turn, proceeding similarly there holds

$$\sup_{\boldsymbol{z}\in\mathbf{K}_2} \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) \,\geq\, \frac{\nu}{\kappa}\,\|\boldsymbol{w}\|_{0,s;\Omega}^s \quad \forall\,\boldsymbol{w}\in\mathbf{K}_1\,,$$

which proves (4.15b). □

Furthermore, regarding the continuous inf-sup conditions to be satisfied by the bilinear forms $\mathbf{d}_i$, $i\in\{1,2\}$, we stress that the one for $\mathbf{d}_1$ can be found in [27, Lemma 2.7], whereas the one for $\mathbf{d}_2$, to be provided next, makes use of the fact that Ω has been assumed to satisfy a uniform exterior ball condition (cf. Notation conventions and preliminaries in Section 1). Indeed, Lemma 4.2 below, and later on Lemma 6.1, are the only places where this hypothesis is employed.

Lemma 4.2. *There exists a constant $\beta_2 > 0$ such that*

$$\sup_{\substack{\boldsymbol{w}\in\mathbf{X}_2\\ \boldsymbol{w}\neq\mathbf{0}}} \frac{\mathbf{d}_2(\boldsymbol{w},q)}{\|\boldsymbol{w}\|_{\mathbf{X}_2}} \,\geq\, \beta_2\,\|q\|_{\mathbf{Q}_2} \qquad \forall\,q\in\mathbf{Q}_2\,. \tag{4.16}$$

Proof. Thanks to the aforementioned geometric assumption on Ω, we can apply [32, Theorem 1.1] to deduce that, given $q\in\mathbf{Q}_2 := \mathrm{L}^2_0(\Omega)$, there exists a unique $u\in\mathrm{H}^2(\Omega)$ such that

$$\Delta u \,=\, q \quad\text{in}\quad \Omega\,,\quad \nabla u\cdot\nu \,=\, 0 \quad\text{on}\quad \Gamma\,,\quad \int_\Omega u \,=\, 0\,,$$

and

$$\|u\|_{2,\Omega} \,\leq\, C\,\|q\|_{0,\Omega},$$

with a constant $C>0$ depending only on Ω. Thus, letting $\widetilde{\boldsymbol{w}} := \nabla u\in\mathrm{H}^1(\Omega)$, it follows that $\mathrm{div}(\widetilde{\boldsymbol{w}}) = q$ in Ω and $\widetilde{\boldsymbol{w}}\cdot\nu = 0$ on Γ. In addition, due to the continuous embedding $i_r : \mathrm{H}^1(\Omega)\to\mathrm{L}^r(\Omega)$, which is valid for the range of r specified in (3.7), we get

$$\|\widetilde{\boldsymbol{w}}\|_{0,r;\Omega} \,\leq\, \|i_r\|\,\|\widetilde{\boldsymbol{w}}\|_{1,\Omega} \,\leq\, \|i_r\|\,\|u\|_{2,\Omega} \,\leq\, C\,\|i_r\|\,\|q\|_{0,\Omega}\,,$$

so that we readily conclude that $\widetilde{\boldsymbol{w}}\in\mathbf{X}_2 \,=\, \mathbf{H}^r_0(\mathrm{div};\Omega)$ and

$$\|\widetilde{\boldsymbol{w}}\|_{\mathbf{X}_2} \,=\, \|\widetilde{\boldsymbol{w}}\|_{0,r;\Omega} \,+\, \|\mathrm{div}(\widetilde{\boldsymbol{w}})\|_{0,\Omega} \,\leq\, \big(1 \,+\, C\,\|i_r\|\big)\,\|q\|_{0,\Omega}\,. \tag{4.17}$$

Finally, bounding the supremum by below with $\widetilde{\boldsymbol{w}}$, and using (4.17), we obtain

$$\sup_{\substack{\boldsymbol{w}\in\mathbf{X}_2\\ \boldsymbol{w}\neq\mathbf{0}}} \frac{\mathbf{d}_2(\boldsymbol{w},q)}{\|\boldsymbol{w}\|_{\mathbf{X}_2}} \,\geq\, \frac{\mathbf{d}_2(\widetilde{\boldsymbol{w}},q)}{\|\widetilde{\boldsymbol{w}}\|_{\mathbf{X}_2}} \,\geq\, \frac{1}{\big(1 \,+\, C\,\|i_r\|\big)}\,\|q\|_{0,\Omega}\,,$$

which proves (4.16) with $\beta_2 := \big(1 \,+\, C\,\|i_r\|\big)^{-1}$. □

Consequently, thanks to the previous discussion, the required hypotheses of [6, Theorem 2.1, Section 2.1] are satisfied, and hence the a priori estimates provided by [6, Corollary 2.1, Section 2.1] imply that there exists a positive constant $\boldsymbol{\alpha}_f$, depending on $\|\mathbf{a}_f\|$, α_f, β_1 (the constant of the continuous inf-sup condition for $\mathbf{d}_1$ in [27, Lemma 2.7]), and β_2, such that (4.12b) holds.

Thus, having proved (4.12a) and (4.12b), the required inf-sup condition (4.9a) follows straightforwardly from (4.11), which gives the constant $\boldsymbol{\alpha}_{\mathbf{A}} := \frac{1}{2}\min\big\{\boldsymbol{\alpha}_s,\boldsymbol{\alpha}_f\big\}$. Similarly, using that $\mathbf{A}_s$ is a symmetric bilinear form, and that the transpose of $\mathbf{A}_f$, defined as $\mathbf{A}_f^{\mathtt{t}}\big((\boldsymbol{w},q),(\boldsymbol{z},p)\big) := \mathbf{A}_f\big((\boldsymbol{z},p),(\boldsymbol{w},q)\big)$, also satisfies the hypotheses of the generalized Babuška-Brezzi theory, we are able to prove (4.9b) by using analogue arguments to those yielding (4.9a). In particular, note that the matrix representation of $\mathbf{A}_f^{\mathtt{t}}$ arises from the one of $\mathbf{A}_f$ after exchanging $\mathbf{d}_1$ and $\mathbf{d}_2$, that is $\begin{pmatrix} \mathbf{a}_f & \mathbf{d}_2 \\ \mathbf{d}_1 & \end{pmatrix}$, and hence the hypotheses of [6, Theorem 2.1, Section 2.1] are clearly attained.

Now, we set the product spaces $\mathbb{X} := (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$ and $\mathbb{Q} := (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_1\times\mathbf{Q}_2)$, so that, given $(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}) \in \mathrm{X}_2\times\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}_1$, (4.2) is equivalent to finding $\vec{\boldsymbol{u}} = ((\boldsymbol{u},\theta),(\boldsymbol{z},p)) \in \mathbb{X}$ such that

$$\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathbf{e}_s(p,\vartheta) + \mathbf{e}_f((\theta,p),q) = \mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}(\boldsymbol{v}) + \mathbf{G}(q) \qquad \forall\,\vec{\boldsymbol{v}} = ((\boldsymbol{v},\vartheta),(\boldsymbol{w},q)) \in \mathbb{Q}\,. \tag{4.18}$$

Hence, employing (4.9a) and the boundedness of $\|\mathbf{e}_s\|$ and $\|\mathbf{e}_f\|$ (cf. (3.21)), we find that

$$\sup_{\substack{\vec{\boldsymbol{v}}\in\mathbb{Q}\\ \vec{\boldsymbol{v}}\neq\mathbf{0}}} \frac{\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathbf{e}_s(p,\vartheta) + \mathbf{e}_f((\theta,p),q)}{\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}}} \geq \Big\{\alpha_{\mathbf{A}} - \max\big\{\|\mathbf{e}_s\|,\|\mathbf{e}_f\|\big\}\Big\}\|\vec{\boldsymbol{u}}\|_{\mathbb{X}} \qquad \forall\,\vec{\boldsymbol{u}}\in\mathbb{X}\,,$$

from which, under the assumption that

$$\max\big\{\|\mathbf{e}_s\|,\|\mathbf{e}_f\|\big\} := C_r(\Omega)\,\max\left\{c_0 + \frac{\alpha^2}{\lambda},\frac{\alpha}{\lambda}\right\} \leq \frac{\boldsymbol{\alpha}_{\mathbf{A}}}{2}\,, \tag{4.19}$$

we deduce that

$$\sup_{\substack{\vec{\boldsymbol{v}}\in\mathbb{Q}\\ \vec{\boldsymbol{v}}\neq\mathbf{0}}} \frac{\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathbf{e}_s(p,\vartheta) + \mathbf{e}_f((\theta,p),q)}{\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}}} \geq \frac{\boldsymbol{\alpha}_{\mathbf{A}}}{2}\|\vec{\boldsymbol{u}}\|_{\mathbb{X}} \qquad \forall\,\vec{\boldsymbol{u}}\in\mathbb{X}\,. \tag{4.20}$$

Analogously, but employing (4.9b) instead of (4.9a), and assuming again (4.19), we obtain

$$\sup_{\substack{\vec{\boldsymbol{u}}\in\mathbb{X}\\ \vec{\boldsymbol{u}}\neq\mathbf{0}}} \frac{\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathbf{e}_s(p,\vartheta) + \mathbf{e}_f((\theta,p),q)}{\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}}} \geq \frac{\boldsymbol{\alpha}_{\mathbf{A}}}{2}\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}} \qquad \forall\,\vec{\boldsymbol{v}}\in\mathbb{Q}\,. \tag{4.21}$$

Note that (4.19) becomes feasible for sufficiently small c_0 and for the quasi-incompressible case ($\lambda \to +\infty$).

We are now in a position to establish the well-definedness of $\mathbf{S}$.

Lemma 4.3. *Assume that the data satisfy (4.19). Then, for each $(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}) \in \mathrm{X}_2\times\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}_1$, there exists a unique $((\boldsymbol{u},\theta),(\boldsymbol{z},p)) \in (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$ solution to (4.2), and hence we can define $\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}) := \boldsymbol{z} \in \mathbf{X}_2$. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending on $\alpha_{\mathbf{A}}$, ε, and δ, such that*

$$\begin{aligned} \|\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\|_{\mathbf{X}_2} &= \|\boldsymbol{z}\|_{\mathbf{X}_2} \leq \|(\boldsymbol{u},\theta)\|_{\mathbf{X}\times\mathbf{Q}} + \|(\boldsymbol{z},p)\|_{\mathbf{X}_2\times\mathbf{Q}_1} \\ &\leq C_{\mathbf{S}}\Big\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{M}}}\,\|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\boldsymbol{r}\|_{\boldsymbol{\mathcal{H}}_1}\Big\}\,. \end{aligned} \tag{4.22}$$

Proof. Thanks to the boundedness of $\mathbf{A}$, $\mathbf{e}_s$, and $\mathbf{e}_f$, and the global inf-sup conditions (4.20) and (4.21), a direct application of the Banach–Nečas–Babuška theorem (cf. [25, Theorem 2.6]) provides the existence of a unique solution $((\boldsymbol{u},\theta),(\boldsymbol{z},p)) \in (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$ to (4.2). Moreover, the a priori estimate (4.22) follows from [25, eqn. (2.5)] along with the boundedness of $\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}$ (cf. (3.22)) and $\mathbf{G}$ (cf. (3.21)). $\square$

4.3 Well-definedness of the operator $\widetilde{\mathrm{S}}$

We now prove that (4.3) is well-posed (equivalently, that $\widetilde{\mathbf{S}}$ is well-defined), by resorting to the analysis from [24, Section 4.2.2], where the same bilinear forms involved here arose. Indeed, the continuous inf-sup conditions for a, b_1, and b_2 that are required by the Babuška–Brezzi theory (cf. [6, Theorem 2.1, Corollary 2.1, Section 2.1]) for the unique solvability of (4.3), were established in [24, Lemmas 4.3 and 4.4] with constants that here we denote $\widetilde{\alpha}$, $\widetilde{\beta}_1$, and $\widetilde{\beta}_2$, respectively. In particular, recall that those regarding a involve the kernels K_i of the bilinear forms b_i, $i \in \{1,2\}$. Thus, a simple application of the aforementioned theory implies the following result, which, up to minor differences, coincides with [24, Theorem 4.5].

Lemma 4.4. *For each $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \boldsymbol{\mathcal{M}}$, there exists a unique $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ solution to (4.3), and hence one can define $\widetilde{\mathbf{S}}(\boldsymbol{\eta}) := \boldsymbol{\varphi} \in \mathrm{X}_2$. Moreover, there exist positive constants $C_{\widetilde{\mathbf{S}}}$ and $\widetilde{C}_{\widetilde{\mathbf{S}}}$, which depend on ε, c_r (cf. (3.27)), $|\Omega|$, ρ, r, $\widetilde{\alpha}$, $\widetilde{\beta}_1$, and $\widetilde{\beta}_2$, such that*

$$\|\widetilde{\mathbf{S}}(\boldsymbol{\eta})\|_{\mathrm{X}_2} = \|\boldsymbol{\varphi}\|_{\mathrm{X}_2} \,\leq\, C_{\widetilde{\mathbf{S}}} \Big\{ \|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} + \|\boldsymbol{\eta}\|_{0,\rho;\Omega} \Big\}, \quad \textit{and} \tag{4.23a}$$

$$\|\chi\|_{\mathrm{M}_1} \leq \widetilde{C}_{\widetilde{\mathbf{S}}} \Big\{ \|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} + \|\boldsymbol{\eta}\|_{0,\rho;\Omega} \Big\} . \tag{4.23b}$$

Proof. We omit further details and just mention that the derivations of (4.23a) and (4.23b) make use of the boundedness of $F_{\boldsymbol{\eta}}$ (cf. (3.26)) and G (cf. (3.27)). $\square$

4.4 Well-definedness of the operators $\mathbf{T}$ and $\boldsymbol{\Xi}$

In this section, we follow the approaches from [18, Section 3.2.2] and [28, Section 3.3] to prove that the operators $\mathbf{T}$ and $\boldsymbol{\Xi}$ are well-defined. More precisely, we first apply [6, Theorem 2.1] and [28, Theorem 3.2] to the formulation arising from (4.4) when $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{D}_{\boldsymbol{w}}$ are dropped, and then employ the Banach–Nečas–Babuška theorem (cf. [25, Theorem 2.6]) to conclude that the full system (4.4) is well-posed for each $i \in \{1,2\}$. To this end, as announced above, and similarly as in Section 4.2, we let $\boldsymbol{\mathcal{A}} : \big(\boldsymbol{\mathcal{H}} \times \mathcal{M}\big) \times \big(\boldsymbol{\mathcal{H}} \times \mathcal{M}\big) \to \mathrm{R}$ be the bounded bilinear defined by the sum of the left-hand sides of (4.4), excluding $\mathcal{D}_{\boldsymbol{w}}$ and $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$, that is

$$\boldsymbol{\mathcal{A}}((\vec{\boldsymbol{r}}_i, \xi_i), (\vec{\boldsymbol{s}}_i, \eta_i)) \,:=\, \mathcal{A}(\vec{\boldsymbol{r}}_i, \vec{\boldsymbol{s}}_i) \,+\, \mathcal{B}(\vec{\boldsymbol{s}}_i, \xi_i) \,+\, \mathcal{B}(\vec{\boldsymbol{r}}_i, \eta_i) \,-\, \mathcal{C}(\xi_i, \eta_i) \tag{4.24}$$

for all $(\vec{\boldsymbol{r}}_i, \xi_i)$, $(\vec{\boldsymbol{s}}_i, \eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}$, and proceed to show next that $\boldsymbol{\mathcal{A}}$ satisfies continuous inf-sup conditions with respect to its first and second components. Needless to say, the boundedness of $\boldsymbol{\mathcal{A}}$ follows from those of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ (cf. (3.34)).

It follows from (4.24) that the aforementioned property for $\boldsymbol{\mathcal{A}}$ is equivalent to proving that the bilinear forms $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, satisfy the hypotheses of [28, Theorem 3.2], which is actually a slight improvement of the original result for perturbed saddle-point problems provided by [22, Theorem 3.4]. In this regard, we first notice from (3.33) that $\mathcal{A}$ and $\mathcal{C}$ are positive semi-definite, that is

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{s}}_i, \vec{\boldsymbol{s}}_i) \,&\geq\, \kappa_i \|\boldsymbol{s}_i\|^2_{0,\Omega} \,\geq\, 0 \qquad \forall\, \vec{\boldsymbol{s}}_i \in \boldsymbol{\mathcal{H}}\,, \quad \text{and} \\ \mathcal{C}(\eta_i, \eta_i) \,&=\, \|\eta_i\|^2_{0,\Omega} \,\geq\, 0 \qquad \forall\, \eta_i \in \mathcal{M}\,. \end{aligned}$$

In turn, it is readily seen that $\mathcal{C}$ is symmetric, and that the null space V of $\mathcal{B}$ is given by

$$V := \mathcal{H}_1 \times V_0\,, \quad \text{where} \quad V_0 \,:=\, \Big\{ \boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_{\varrho}; \Omega) : \quad \mathrm{div}(\boldsymbol{\tau}_i) = 0 \quad \text{in} \quad \Omega \Big\} . \tag{4.25}$$

In addition, $\mathcal{A}$ shows the matrix representation $\begin{pmatrix} A & B_1 \\ B_2 & \end{pmatrix}$, where $A : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathrm{R}$, $B_1 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathrm{R}$, and $B_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathrm{R}$ are the bilinear forms defined as

$$\begin{aligned} A(\boldsymbol{t}_i, \boldsymbol{s}_i) \quad &:= \quad \kappa_i \int_\Omega \boldsymbol{t}_i \cdot \boldsymbol{s}_i \qquad && \forall\, \boldsymbol{t}_i, \boldsymbol{s}_i \in \mathcal{H}_1 \\ B_1(\boldsymbol{s}_i, \boldsymbol{\tau}_i) \quad &:= \quad -\int_\Omega \boldsymbol{\tau}_i \cdot \boldsymbol{s}_i \qquad && \forall\, (\boldsymbol{s}_i, \boldsymbol{\tau}_i) \in \mathcal{H}_1 \times \mathcal{H}_2\,, \\ B_2(\boldsymbol{s}_i, \boldsymbol{\tau}_i) \quad &:= \quad \int_\Omega \boldsymbol{\tau}_i \cdot \boldsymbol{s}_i \qquad && \forall\, (\boldsymbol{s}_i, \boldsymbol{\tau}_i) \in \mathcal{H}_1 \times \mathcal{H}_2\,. \end{aligned} \tag{4.26}$$

According to the above, and similarly as in Section 4.2, we deduce that in order for $\mathcal{A}$ to satisfy the inf-sup conditions specified in [28, eqns. (3.31) and (3.32), Theorem 3.2], we just need to prove that A, B_1, and B_2 verify the hypothesis of [6, Theorem 2.1]. In particular, it is easily seen that A is $\mathcal{H}_1$-elliptic since

$$A(\boldsymbol{s}_i, \boldsymbol{s}_i) \,=\, \kappa_i \, \|\boldsymbol{s}_i\|^2_{0,\Omega} \quad \forall\, \boldsymbol{s}_i \in \mathcal{H}_1\,, \tag{4.27}$$

and hence A satisfies the assumptions of [6, Theorem 2.1, eqns. (2.8) and (2.9)]. Note that this holds irrespective of the conditions defining the kernels $\mathcal{K}_j$ of $B_j|_{\mathbf{L}^2(\Omega)\times V_0}$, $j \in \{1,2\}$, which, due to the fact that $B_1 \,=\, -B_2$, are given by

$$\mathcal{K}_1 \,=\, \mathcal{K}_2 \,=\, \mathcal{K} \,:=\, \left\{ \boldsymbol{s}_i \in \mathcal{H}_1 : \quad \int_\Omega \boldsymbol{s}_i \cdot \boldsymbol{\tau}_i = 0 \quad \forall\, \boldsymbol{\tau}_i \in V_0 \right\}. \tag{4.28}$$

Indeed, all what is needed is that $\mathcal{K}$ be contained in $\mathcal{H}_1$. Furthermore, regarding the bilinear forms B_1 and B_2, we now consider $\boldsymbol{\tau}_i \in V_0$ (cf. (4.25)), that is $\boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$ such that $\mathrm{div}(\boldsymbol{\tau}_i) \,=\, 0$ in Ω, and observe that, bounding by below with $\boldsymbol{s}_i \,=\, -\boldsymbol{\tau}_i$ (for B_1) and $\boldsymbol{s}_i \,=\, \boldsymbol{\tau}_i$ (for B_2), there holds for each $j \in \{1,2\}$

$$\sup_{\substack{\boldsymbol{s}_i \in \mathcal{H}_1 \\ \boldsymbol{s}_i \neq \mathbf{0}}} \frac{B_j(\boldsymbol{s}_i, \boldsymbol{\tau}_i)}{\|\boldsymbol{s}_i\|_{\mathcal{H}_1}} \,=\, \sup_{\substack{\boldsymbol{s}_i \in \mathbf{L}^2(\Omega) \\ \boldsymbol{s}_i \neq \mathbf{0}}} \frac{B_j(\boldsymbol{s}_i, \boldsymbol{\tau}_i)}{\|\boldsymbol{s}_i\|_{0,\Omega}} \,\geq\, \|\boldsymbol{\tau}_i\|_{0,\Omega} \,=\, \|\boldsymbol{\tau}_i\|_{\mathcal{H}_2} \qquad \forall\, \boldsymbol{\tau}_i \in V_0\,. \tag{4.29}$$

Hence, thanks to (4.27) and (4.29), we can apply [6, Theorem 2.1] to conclude that there exists a positive $\widehat{\alpha}$, depending only on κ_i, such that the whole bilinear form $\mathcal{A}$ satisfies

$$\sup_{\substack{\vec{\boldsymbol{s}}_i \in V \\ \vec{\boldsymbol{s}}_i \neq \mathbf{0}}} \frac{\mathcal{A}(\vec{\boldsymbol{r}}_i, \vec{\boldsymbol{s}}_i)}{\|\vec{\boldsymbol{s}}_i\|_{\boldsymbol{\mathcal{H}}}} \,\geq\, \widehat{\alpha}\, \|\vec{\boldsymbol{r}}_i\|_{\boldsymbol{\mathcal{H}}} \qquad \forall\, \vec{\boldsymbol{r}}_i \in V\,. \tag{4.30}$$

Moreover, exchanging the roles of B_1 and B_2, and applying again [6, Theorem 2.1], we conclude that

$$\sup_{\substack{\vec{\boldsymbol{r}}_i \in V \\ \vec{\boldsymbol{r}}_i \neq \mathbf{0}}} \frac{\mathcal{A}(\vec{\boldsymbol{r}}_i, \vec{\boldsymbol{s}}_i)}{\|\vec{\boldsymbol{r}}_i\|_{\boldsymbol{\mathcal{H}}}} \,\geq\, \widehat{\alpha}\, \|\vec{\boldsymbol{s}}_i\|_{\boldsymbol{\mathcal{H}}} \qquad \forall\, \vec{\boldsymbol{s}}_i \in V\,.$$

On the other hand, we know from [27, Lemma 2.9] (see also [18, eqn. (3.23)]) that $\mathcal{B}$ (cf. (3.33)) satisfies the required continuous inf-sup condition, which means that there exists a positive constant $\beta_{\mathcal{B}}$ such that

$$\sup_{\substack{\vec{\boldsymbol{s}}_i \in \boldsymbol{\mathcal{H}} \\ (\boldsymbol{s}_i \times \boldsymbol{\tau}_i) \neq \mathbf{0}}} \frac{\mathcal{B}(\vec{\boldsymbol{s}}_i, \eta_i)}{\|\vec{\boldsymbol{s}}_i\|_{\boldsymbol{\mathcal{H}}}} \,\geq\, \beta_{\mathcal{B}}\, \|\eta_i\|_{0,\rho,\Omega} \qquad \forall\, \eta_i \in \mathcal{M}\,.$$

Thus, having $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfied the hypotheses of [28, Theorem 3.2], we deduce the existence of a positive constant $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}$, depending only on $\widehat{\alpha}$, $\beta_{\mathcal{B}}$, $\|\mathcal{A}\|$, and $\|\mathcal{C}\|$, such that

$$\sup_{\substack{(\vec{\boldsymbol{s}}_i,\eta_i) \in \boldsymbol{\mathcal{H}}\times\mathcal{M} \\ (\vec{\boldsymbol{s}}_i,\eta_i) \neq \mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}((\vec{\boldsymbol{r}}_i, \xi_i), (\vec{\boldsymbol{s}}_i, \eta_i))}{\|(\vec{\boldsymbol{s}}_i, \eta_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}} \,\geq\, \boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}\, \|(\vec{\boldsymbol{r}}_i, \xi_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} \qquad \forall\, (\vec{\boldsymbol{r}}_i, \xi_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,, \quad \text{and} \tag{4.31a}$$

$$\sup_{\substack{(\vec{\boldsymbol{r}}_i,\widetilde{\xi}_i) \in \boldsymbol{\mathcal{H}}\times\mathcal{M} \\ (\vec{\boldsymbol{r}}_i,\xi_i) \neq \mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}((\vec{\boldsymbol{r}}_i, \xi_i), (\vec{\boldsymbol{s}}_i, \eta_i))}{\|(\vec{\boldsymbol{r}}_i, \xi_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}} \,\geq\, \boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}\, \|(\vec{\boldsymbol{s}}_i, \eta_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} \qquad \forall\, (\vec{\boldsymbol{s}}_i, \eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,. \tag{4.31b}$$

Going back to (4.4) with the given $(\boldsymbol{w}, \boldsymbol{\phi}) \in \mathbf{X}_2 \times \mathrm{X}_2$, we let $\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}} : (\boldsymbol{\mathcal{H}} \times \mathcal{M}) \times (\boldsymbol{\mathcal{H}} \times \mathcal{M}) \to \mathrm{R}$ be the bounded bilinear form arising after adding its left-hand sides, that is (cf. (4.24))

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{\boldsymbol{r}}_i, \xi_i), (\vec{\boldsymbol{s}}_i, \eta_i)) \,:=\, \boldsymbol{\mathcal{A}}((\vec{\boldsymbol{r}}_i, \xi_i), (\vec{\boldsymbol{s}}_i, \eta_i)) \,+\, \mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}(\vec{\boldsymbol{s}}_i, \xi_i) \,+\, \mathcal{D}_{\boldsymbol{w}}(\xi_i, \eta_i), \tag{4.32}$$

for all $(\vec{\boldsymbol{r}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}$. In this way, (4.4) can be rewritten equivalently, as: Find $(\vec{\boldsymbol{t}}_i,\xi_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}$ such that

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{\boldsymbol{t}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i)) \,=\, \mathcal{G}(\vec{\boldsymbol{s}}_i)\,+\,\mathcal{F}(\eta_i) \qquad \forall\,(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}\,. \tag{4.33}$$

Note that the boundedness of $\boldsymbol{\mathcal{A}}$, $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$, and $\mathcal{D}_{\boldsymbol{w}}$ (cf. (3.34), (3.35), and (3.36)) guarantees that $\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}$ is bounded as well. In turn, bearing in mind (4.32), (4.31a), and again the boundedness of $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{D}_{\boldsymbol{w}}$, we readily show that for each $(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2$ there holds

$$\sup_{\substack{(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}\\(\vec{\boldsymbol{s}}_i,\eta_i)\neq\mathbf{0}}}\frac{\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{\boldsymbol{r}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i))}{\|(\vec{\boldsymbol{s}}_i,\eta_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}} \,\geq\, \Big\{\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}} - \max\big\{\|\mathcal{D}\|,\|\mathcal{E}\|\big\}\,\|(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2}\Big\}\,\|(\vec{\boldsymbol{r}}_i,\xi_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}\,,$$

for all $(\vec{\boldsymbol{r}}_i,\xi_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}$, from which, assuming that

$$\|(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2} \,\leq\, \mathtt{R} \,:=\, \frac{\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}}{2\,\max\big\{\|\mathcal{D}\|,\|\mathcal{E}\|\big\}}\,, \tag{4.34}$$

we conclude that

$$\sup_{\substack{(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}\\(\vec{\boldsymbol{s}}_i,\eta_i)\neq\mathbf{0}}}\frac{\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{\boldsymbol{r}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i))}{\|(\vec{\boldsymbol{s}}_i,\eta_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}} \,\geq\, \frac{\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}}{2}\|(\vec{\boldsymbol{r}}_i,\xi_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} \qquad \forall\,(\vec{\boldsymbol{r}}_i,\xi_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}\,. \tag{4.35}$$

Proceeding similarly as above, but now employing (4.31b) instead of (4.31a), and under the same assumption (4.34), we arrive at

$$\sup_{\substack{(\vec{\boldsymbol{r}}_i,\xi_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}\\(\vec{\boldsymbol{r}}_i,\xi_i)\neq\mathbf{0}}}\frac{\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{\boldsymbol{r}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i))}{\|(\vec{\boldsymbol{r}}_i,\widetilde{\xi}_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}} \,\geq\, \frac{\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}}{2}\|(\vec{\boldsymbol{s}}_i,\eta_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} \qquad \forall\,(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}\,. \tag{4.36}$$

The well-definedness of the components of $\mathbf{T}$ and $\boldsymbol{\Xi}$, and hence of themselves, can be stated now.

Lemma 4.5. *For each $i\in\{1,2\}$, and for each $(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2$ satisfying (4.34), there exists a unique $(\vec{\boldsymbol{t}}_i,\xi_i)=\big((\boldsymbol{t}_i,\boldsymbol{\sigma}_i),\xi_i\big)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}$ solution of (4.4), and hence we can define $\mathrm{T}_i(\boldsymbol{w},\boldsymbol{\phi}):=\boldsymbol{t}_i\in\mathcal{H}_1$ and $\Xi_i(\boldsymbol{w},\boldsymbol{\phi}):=\xi_i\in\mathcal{M}$. Moreover, there exists a positive constant $C_{\mathbf{T}}$, independent of $(\boldsymbol{w},\boldsymbol{\phi})$, such that*

$$\begin{aligned}\|\mathrm{T}_i(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathcal{H}_1} \,+\, \|\Xi_i(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathcal{M}} &\,=\, \|\boldsymbol{t}_i\|_{\mathcal{H}_1} \,+\, \|\xi_i\|_{\mathcal{M}}\\ &\,\leq\, \|(\vec{\boldsymbol{t}}_i,\xi_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} \,\leq\, C_{\mathbf{T}}\Big\{\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\}\,.\end{aligned} \tag{4.37}$$

Proof. Thanks to (4.35) and (4.36), the proof reduces to a direct application of [25, Theorem 2.6], where the derivation of the a priori estimate (4.37) makes use of the expressions for $\|\mathcal{G}\|$ and $\|\mathcal{F}\|$ given by (3.34). □

4.5 Solvability analysis of the fixed-point equation

Knowing that the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$, $\mathbf{T}$, $\boldsymbol{\Xi}$, and hence $\boldsymbol{\Pi}$ as well, are well-defined, we now address the solvability of the fixed-point equation (4.5) by means of the Banach fixed-point theorem. We begin by setting the ball

$$\mathrm{W}(\mathtt{R}) \,:=\, \Big\{(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2 : \quad \|(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2} \,\leq\, \mathtt{R}\Big\}\,, \tag{4.38}$$

where $\mathtt{R}>0$ is defined in (4.34), and provide next a condition on the data ensuring that $\boldsymbol{\Pi}$ maps $\mathrm{W}(\mathtt{R})$ into itself. In fact, bearing in mind the definition of $\boldsymbol{\Pi}$ (cf. (4.5)), and employing the a priori estimates for $\mathbf{S}$,

$\widetilde{\mathbf{S}}$, $\mathbf{T}$, and $\boldsymbol{\Xi}$ (cf. (4.22), (4.23a), and (4.37), we deduce the existence of a positive constant $C(\mathtt{R})$, depending only on $C_{\mathbf{S}}$, $C_{\widetilde{\mathbf{S}}}$, $C_{\mathbf{T}}$, and $\mathtt{R}$, such that for each $(\boldsymbol{w},\boldsymbol{\phi})\in \mathrm{W}(\mathtt{R})$ there holds

$$\|\boldsymbol{\Pi}(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2} \,\le\, C(\mathtt{R})\left\{\|\boldsymbol{f}\|_{0,\Omega} \,+\, \|g\|_{0,\Omega} \,+\, \|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} \,+\, \sum_{i=1}^{2}\Big(\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big)\right\}. \tag{4.39}$$

A straightforward consequence of (4.39) implies the following result.

Lemma 4.6. *Assume that the data are sufficiently small so that*

$$C(\mathtt{R})\left\{\|\boldsymbol{f}\|_{0,\Omega} \,+\, \|g\|_{0,\Omega} \,+\, \|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} \,+\, \sum_{i=1}^{2}\Big(\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big)\right\} \,\le\, \mathtt{R}\,. \tag{4.40}$$

Then, $\boldsymbol{\Pi}\big(\mathrm{W}(\mathtt{R})\big) \,\subseteq\, \mathrm{W}(\mathtt{R})$.

Our following goal is to show that $\boldsymbol{\Pi}$ is Lipschitz-continuous, for which it suffices to show that $\mathbf{S}$, $\widetilde{\mathbf{S}}$, $\boldsymbol{\Xi}$, and $\mathbf{T}$ satisfy suitable continuity properties. We begin with the corresponding result for $\mathbf{S}$.

Lemma 4.7. *There exists a positive constant $L_{\mathbf{S}}$, depending on ε, δ, and $\boldsymbol{\alpha}_{\mathbf{A}}$, such that*

$$\|\mathbf{S}(\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t})-\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\|_{\mathbf{X}_2} \,\le\, L_{\mathbf{S}}\left\{\|\boldsymbol{\xi}\|_{\boldsymbol{\mathcal{M}}}\,\|\boldsymbol{\varphi}-\boldsymbol{\phi}\|_{\mathrm{X}_2} \,+\, \|\boldsymbol{\phi}\|_{\mathrm{X}_2}\,\|\boldsymbol{\xi}-\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{M}}} \,+\, \|\boldsymbol{t}-\boldsymbol{r}\|_{\boldsymbol{\mathcal{H}}_1}\right\}, \tag{4.41}$$

for all $(\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t})$, $(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\in \mathrm{X}_2\times\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}_1$.

Proof. Given $(\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t})$, $(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\in \mathrm{X}_2\times\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}_1$, we let $\mathbf{S}(\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}) := \boldsymbol{z}\in\mathbf{X}_2$ and $\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}) := \boldsymbol{z}_0\in\mathbf{X}_2$, where $\vec{\boldsymbol{u}} = ((\boldsymbol{u},\theta),(\boldsymbol{z},p))\in\mathbb{X}$ and $\vec{\boldsymbol{u}}_0 = ((\boldsymbol{u}_0,\theta_0),(\boldsymbol{z}_0,p_0))\in\mathbb{X}$ are the respective solutions of (4.2), equivalently (4.18), that is

$$\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \,+\, \mathbf{e}_s(p,\vartheta) \,+\, \mathbf{e}_f((\theta,p),q) \,=\, \mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}(\boldsymbol{v}) \,+\, \mathbf{G}(q) \qquad \forall\,\vec{\boldsymbol{v}} := ((\boldsymbol{v},\vartheta),(\boldsymbol{w},q))\in\mathbb{Q}\,,$$

and

$$\mathbf{A}(\vec{\boldsymbol{u}}_0,\vec{\boldsymbol{v}}) \,+\, \mathbf{e}_s(p_0,\vartheta) \,+\, \mathbf{e}_f((\theta_0,p_0),q) \,=\, \mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}(\boldsymbol{v}) \,+\, \mathbf{G}(q) \qquad \forall\,\vec{\boldsymbol{v}} := ((\boldsymbol{v},\vartheta),(\boldsymbol{w},q))\in\mathbb{Q}\,.$$

It follows from the foregoing identities and the bilinearity of $\mathbf{A}$, $\mathbf{e}_s$, and $\mathbf{e}_f$, that

$$\mathbf{A}(\vec{\boldsymbol{u}}-\vec{\boldsymbol{u}}_0,\vec{\boldsymbol{v}}) \,+\, \mathbf{e}_s(p-p_0,\vartheta) \,+\, \mathbf{e}_f((\theta-\theta_0,p-p_0),q) = (\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}-\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}})(\boldsymbol{v}) \qquad \forall\,\vec{\boldsymbol{v}}\in\mathbb{Q}\,, \tag{4.42}$$

so that, applying the global inf-sup condition (4.20) to $\vec{\boldsymbol{u}}-\vec{\boldsymbol{u}}_0$, and using (4.42), we find that

$$\begin{aligned}\|\mathbf{S}(\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t})-\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\|_{\mathbf{X}_2} &= \|\boldsymbol{z}-\boldsymbol{z}_0\|_{r,\mathrm{div};\Omega} \,\le\, \|\vec{\boldsymbol{u}}-\vec{\boldsymbol{u}}_0\|_{\mathbb{X}}\\ &\le\, \frac{2}{\boldsymbol{\alpha}_{\mathbf{A}}}\sup_{\substack{\vec{\boldsymbol{v}}\in\mathbb{Q}\\ \vec{\boldsymbol{v}}\neq\mathbf{0}}}\frac{(\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}-\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}})(\boldsymbol{v})}{\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}}}\,,\end{aligned} \tag{4.43}$$

where, for each $\vec{\boldsymbol{v}} := ((\boldsymbol{v},\vartheta),(\boldsymbol{w},q))\in\mathbb{Q}$ (cf. (3.17))

$$(\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}-\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}})(\boldsymbol{v}) \,=\, \int_\Omega\Big(-\,\varepsilon^{-1}\big((\xi_1-\xi_2)\boldsymbol{\varphi}-(\eta_1-\eta_2)\boldsymbol{\phi}\big) \,-\, \delta\big((\boldsymbol{t}_1-\boldsymbol{t}_2)-(\boldsymbol{r}_1-\boldsymbol{r}_2)\big)\Big)\cdot\boldsymbol{v}\,. \tag{4.44}$$

Then, adding and subtracting the expression $(\xi_1-\xi_2)\boldsymbol{\phi}\cdot\boldsymbol{v}$, we get

$$\int_\Omega \varepsilon^{-1}\big((\xi_1-\xi_2)\boldsymbol{\varphi}-(\eta_1-\eta_2)\boldsymbol{\phi}\big)\cdot\boldsymbol{v} \,=\, \int_\Omega \varepsilon^{-1}\Big\{(\xi_1-\xi_2)(\boldsymbol{\varphi}-\boldsymbol{\phi})-\big((\eta_1-\xi_1)-(\eta_2-\xi_2)\big)\boldsymbol{\phi}\Big\}\cdot\boldsymbol{v}\,,$$

from which, employing the Cauchy–Schwarz and Hölder inequalities, we find that

$$\int_\Omega \varepsilon^{-1} \big((\xi_1-\xi_2)\boldsymbol{\varphi}-(\eta_1-\eta_2)\boldsymbol{\phi}\big)\cdot\boldsymbol{v} \,\le\, \varepsilon^{-1}\Big\{\|\boldsymbol{\xi}\|_{0,\rho;\Omega}\,\|\boldsymbol{\varphi}-\boldsymbol{\phi}\|_{r,\mathrm{div}_r;\Omega} + \|\boldsymbol{\phi}\|_{r,\mathrm{div}_r;\Omega}\,\|\boldsymbol{\eta}-\boldsymbol{\xi}\|_{0,\rho;\Omega}\Big\}\,\|\boldsymbol{v}\|_{0,\Omega}\,. \tag{4.45}$$

In turn, proceeding similarly, we readily obtain

$$\int_\Omega \delta\big((\boldsymbol{t}_1-\boldsymbol{t}_2)-(\boldsymbol{r}_1-\boldsymbol{r}_2)\big)\cdot\boldsymbol{v} \,=\, \int_\Omega \delta\big((\boldsymbol{t}_1-\boldsymbol{r}_1)\,+\,(\boldsymbol{r}_2-\boldsymbol{t}_2)\big)\cdot\boldsymbol{v} \,\le\, \delta\,\|\boldsymbol{t}-\boldsymbol{r}\|_{0,\Omega}\,\|\boldsymbol{v}\|_{0,\Omega}\,. \tag{4.46}$$

In this way, replacing (4.45) and (4.46) back into (4.44), and then the resulting estimate in (4.43), we arrive at (4.41) with $L_{\mathbf{S}} := \frac{2}{\alpha_{\mathbf{A}}}\,\max\big\{\varepsilon^{-1},\delta\big\}$. $\square$

Next, we resort to a result from [24] to establish the continuity of $\widetilde{\mathbf{S}}$.

Lemma 4.8. *There exists a positive constant $L_{\widetilde{\mathbf{S}}}$, depending only on Ω, the inf-sup constants $\widetilde{\alpha}$ and $\widetilde{\beta}_2$ (cf. Section 4.3), and $\|a\|$ (cf. (3.25)), such that*

$$\|\widetilde{\mathbf{S}}(\boldsymbol{\xi})-\widetilde{\mathbf{S}}(\boldsymbol{\eta})\|_{\mathrm{X}_2} \,\le\, L_{\widetilde{\mathbf{S}}}\,\|\boldsymbol{\xi}-\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{M}}} \qquad \forall\,\boldsymbol{\xi},\,\boldsymbol{\eta}\in\boldsymbol{\mathcal{M}}\,. \tag{4.47}$$

Proof. It reduces to the same proof of [24, Lemma 4.9]. $\square$

Recalling that $\mathrm{W}(\mathtt{R})$ is the closed ball defined by (4.38), we now prove the continuity of $\mathbf{T}$ and $\boldsymbol{\Xi}$.

Lemma 4.9. *There exists a positive constant $L_{\mathbf{T}}$, depending only on $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}$, $C_{\mathbf{T}}$, ε, and κ_i, $i\in\{1,2\}$, such that*

$$\begin{aligned}&\|\mathbf{T}(\boldsymbol{z},\boldsymbol{\varphi})-\mathbf{T}(\boldsymbol{w},\boldsymbol{\phi})\|_{\boldsymbol{\mathcal{H}}_1} \,+\, \|\boldsymbol{\Xi}(\boldsymbol{z},\boldsymbol{\varphi})-\boldsymbol{\Xi}(\boldsymbol{w},\boldsymbol{\phi})\|_{\boldsymbol{\mathcal{M}}}\\ &\qquad\le\, L_{\mathbf{T}}\,\sum_{i=1}^{2}\Big\{\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\}\,\|(\boldsymbol{z},\boldsymbol{\varphi})-(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2}\,.\end{aligned} \tag{4.48}$$

for all $(\boldsymbol{z},\boldsymbol{\varphi})$, $(\boldsymbol{w},\boldsymbol{\phi})\in\mathrm{W}(\mathtt{R})$.

Proof. Given $(\boldsymbol{z},\boldsymbol{\varphi})$, $(\boldsymbol{w},\boldsymbol{\phi})\in\mathrm{W}(\mathtt{R})$, we let for each $i\in\{1,2\}$

$$\mathrm{T}_i(\boldsymbol{z},\boldsymbol{\varphi}) \,:=\, \boldsymbol{t}_i\in\mathcal{H}_1\,,\quad \Xi_i(\boldsymbol{z},\boldsymbol{\varphi}) \,:=\, \xi_i\in\mathcal{M}\,,\quad \mathrm{T}_i(\boldsymbol{w},\boldsymbol{\phi}) \,:=\, \boldsymbol{r}_i\in\mathcal{H}_1\,,\quad \text{and}\quad \Xi_i(\boldsymbol{w},\boldsymbol{\phi}) \,:=\, \varkappa_i\in\mathcal{M}\,,$$

where $(\vec{\boldsymbol{t}}_i,\xi_i)=\big((\boldsymbol{t}_i,\boldsymbol{\sigma}_i),\xi_i\big)$, $(\vec{\boldsymbol{r}}_i,\varkappa_i)=\big((\boldsymbol{r}_i,\boldsymbol{\zeta}_i),\varkappa_i\big)\,\in\,\boldsymbol{\mathcal{H}}\times\mathcal{M}$ are the respective solutions of (4.33), that is

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{z},\boldsymbol{\varphi}}\big((\vec{\boldsymbol{t}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big) \,=\, \mathcal{G}(\vec{\boldsymbol{s}}_i) \,+\, \mathcal{F}(\eta_i) \quad\text{and}\quad \boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)) \,=\, \mathcal{G}(\vec{\boldsymbol{s}}_i) \,+\, \mathcal{F}(\eta_i)\,,$$

for all $(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}$. It follows from the foregoing identities and the definitions of the bilinear forms $\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}$ (cf. (4.24), (4.32)), and $\mathcal{D}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{E}_{\boldsymbol{w}}$ (cf. (3.33)), that

$$\begin{aligned}\boldsymbol{\mathcal{A}}_{\boldsymbol{z},\boldsymbol{\varphi}}\big((\vec{\boldsymbol{t}}_i,\xi_i)-(\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big) \,&=\, \boldsymbol{\mathcal{A}}_{\boldsymbol{z},\boldsymbol{\varphi}}\big((\vec{\boldsymbol{t}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big) - \boldsymbol{\mathcal{A}}_{\boldsymbol{z},\boldsymbol{\varphi}}\big((\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big)\\ &=\, \boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}\big((\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big) - \boldsymbol{\mathcal{A}}_{\boldsymbol{z},\boldsymbol{\varphi}}\big((\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big)\\ &=\, \mathcal{E}_{\boldsymbol{w}-\boldsymbol{z},\boldsymbol{\phi}-\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i,\varkappa_i) \,+\, \mathcal{D}_{\boldsymbol{w}-\boldsymbol{z}}(\varkappa_i,\eta_i)\,.\end{aligned} \tag{4.49}$$

Hence, applying the global inf-sup condition (4.35) to the bilinear form $\boldsymbol{\mathcal{A}}_{\boldsymbol{z},\boldsymbol{\varphi}}$ and the vector $(\vec{\boldsymbol{t}}_i,\xi_i)-(\vec{\boldsymbol{r}}_i,\varkappa_i)$, and employing (4.49) and the boundedness of $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{D}_{\boldsymbol{w}}$ (cf. (3.35)), we find that

$$\begin{aligned}\|(\vec{\boldsymbol{t}}_i,\xi_i)-(\vec{\boldsymbol{r}}_i,\varkappa_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} \,&\le\, \frac{2}{\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}}\,\sup_{\substack{(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}\\ (\vec{\boldsymbol{s}}_i,\eta_i)\neq\mathbf{0}}}\frac{\mathcal{E}_{\boldsymbol{w}-\boldsymbol{z},\boldsymbol{\phi}-\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i,\varkappa_i) \,+\, \mathcal{D}_{\boldsymbol{w}-\boldsymbol{z}}(\varkappa_i,\eta_i)}{\|(\vec{\boldsymbol{s}}_i,\eta_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}}\,,\\ &\le\, \frac{2\,\max\big\{\|\mathcal{E}\|,\|\mathcal{D}\|\big\}}{\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}}\,\|\varkappa_i\|_{0,\rho,\Omega}\,\|(\boldsymbol{z},\boldsymbol{\varphi})-(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2}\,,\end{aligned}$$

from which, along with the a priori estimate (4.37) for $\|\varkappa_i\|_{0,\varrho,\Omega}$, $i \in \{1,2\}$, and the expressions for $\|\mathcal{E}\|$ and $\|\mathcal{D}\|$ (cf. (3.36)), we conclude (4.48) with $L_{\mathbf{T}}$ as indicated. $\square$

Having derived the continuity properties of the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$, $\mathbf{T}$, and $\boldsymbol{\Xi}$, we now look at the one of the fixed-point operator $\boldsymbol{\Pi}$. Indeed, given $(\boldsymbol{z},\boldsymbol{\varphi})$, $(\boldsymbol{w},\boldsymbol{\phi}) \in \mathrm{W}(\mathtt{R})$, we first observe from the definition of $\boldsymbol{\Pi}$ (cf. (4.5)) that

$$\begin{aligned} &\|\boldsymbol{\Pi}(\boldsymbol{z},\boldsymbol{\varphi}) - \boldsymbol{\Pi}(\boldsymbol{w},\boldsymbol{\phi})\| \\ &\quad = \|\mathbf{S}\big(\boldsymbol{\varphi},\boldsymbol{\Xi}(\boldsymbol{z},\boldsymbol{\varphi}),\mathbf{T}(\boldsymbol{z},\boldsymbol{\varphi})\big) - \mathbf{S}\big(\boldsymbol{\phi},\boldsymbol{\Xi}(\boldsymbol{w},\boldsymbol{\phi}),\mathbf{T}(\boldsymbol{w},\boldsymbol{\phi})\big)\| + \|\widetilde{\mathbf{S}}\big(\boldsymbol{\Xi}(\boldsymbol{z},\boldsymbol{\varphi})\big) - \widetilde{\mathbf{S}}\big(\boldsymbol{\Xi}(\boldsymbol{w},\boldsymbol{\phi})\big)\|\,. \end{aligned} \tag{4.50}$$

Note that here and below, for simplicity, we omit the subscripts of the norms involved. Then, employing the respective properties of $\mathbf{S}$ (cf. (4.41)) and $\widetilde{\mathbf{S}}$ (cf. (4.47)), and performing some algebraic manipulations, we deduce from (4.50) that

$$\begin{aligned} &\|\boldsymbol{\Pi}(\boldsymbol{z},\boldsymbol{\varphi}) - \boldsymbol{\Pi}(\boldsymbol{w},\boldsymbol{\phi})\| \\ &\quad \le L_{\mathbf{S}}\,\|\boldsymbol{\Xi}(\boldsymbol{z},\boldsymbol{\varphi})\|\,\|\boldsymbol{\varphi}-\boldsymbol{\phi}\| + \big(L_{\widetilde{\mathbf{S}}} + L_{\mathbf{S}}\,\|\boldsymbol{\phi}\|\big)\,\|\boldsymbol{\Xi}(\boldsymbol{z},\boldsymbol{\varphi}) - \boldsymbol{\Xi}(\boldsymbol{w},\boldsymbol{\phi})\| + L_{\mathbf{S}}\,\|\mathbf{T}(\boldsymbol{z},\boldsymbol{\varphi}) - \mathbf{T}(\boldsymbol{w},\boldsymbol{\phi})\|\,. \end{aligned}$$

In this way, the foregoing inequality along with the a priori estimate for $\|\boldsymbol{\Xi}(\boldsymbol{z},\boldsymbol{\varphi})\|$ (cf. (4.37)), the fact that $\|\boldsymbol{\phi}\| \le \mathtt{R}$, and the Lipschitz-continuity of $\mathbf{T}$ and $\boldsymbol{\Xi}$ (cf. (4.48)), yield the existence of a positive constant $L_{\boldsymbol{\Pi}}$, depending only on $L_{\mathbf{S}}$, $C_{\mathbf{T}}$, $L_{\widetilde{\mathbf{S}}}$, $L_{\mathbf{T}}$, and $\mathtt{R}$, such that

$$\|\boldsymbol{\Pi}(\boldsymbol{z},\boldsymbol{\varphi}) - \boldsymbol{\Pi}(\boldsymbol{w},\boldsymbol{\phi})\| \,\le\, L_{\boldsymbol{\Pi}} \sum_{i=1}^{2} \Big\{\|\xi_{i,\mathtt{D}}\|_{1/2;\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\}\,\|(\boldsymbol{z},\boldsymbol{\varphi}) - (\boldsymbol{w},\boldsymbol{\phi})\|\,. \tag{4.51}$$

As a consequence of (4.51), we state next the main result of this section.

Theorem 4.10. *Besides (4.19) and (4.40), assume that the data satisfy*

$$L_{\boldsymbol{\Pi}} \sum_{i=1}^{2} \Big\{\|\xi_{i,\mathtt{D}}\|_{1/2;\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\} \,<\, 1\,. \tag{4.52}$$

Then, the fixed-point equation (4.6) has a unique solution $(\boldsymbol{z},\boldsymbol{\varphi}) \in \mathrm{W}(\mathtt{R})$. Equivalently, the coupled problem (3.37) has a unique solution $\big((\boldsymbol{u},\theta),(\boldsymbol{z},p)\big) \in (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$, $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2\times\mathrm{M}_1$, and $(\vec{\boldsymbol{t}}_i,\xi_i) \in \boldsymbol{\mathcal{H}}\times\mathcal{M}, i \in \{1,2\}$. Moreover, the following a priori estimates hold true

$$\begin{aligned} \|(\boldsymbol{u},\theta)\|_{\mathbf{X}\times\mathbf{Q}} \,+\, \|(\boldsymbol{z},p)\|_{\mathbf{X}_2\times\mathbf{Q}_1} \quad &\le \quad \widetilde{C}_{\mathbf{S}}\,\Big\{\|\boldsymbol{f}\|_{0,\Omega} \,+\, \|g\|_{0,\Omega} \,+\, \sum_{i=1}^{2}\big(\|\xi_{i,\mathtt{D}}\|_{1/2;\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\big)\Big\}\,, \\ \|(\boldsymbol{\varphi},\chi)\|_{\mathrm{X}_2\times\mathrm{M}_1} \quad &\le \quad \widetilde{C}_{\widetilde{\mathbf{S}}}\,\Big\{\|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} \,+\, \sum_{i=1}^{2}\big(\|\xi_{i,\mathtt{D}}\|_{1/2;\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\big)\Big\}\,, \\ \|(\vec{\boldsymbol{t}}_i,\xi_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} \quad &\le \quad C_{\mathbf{T}}\,\Big\{\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\}\,, \quad i \in \{1,2\}\,. \end{aligned}$$

where $\widetilde{C}_{\mathbf{S}}$ and $\widetilde{C}_{\widetilde{\mathbf{S}}}$ are positive constants depending only on $C_{\mathbf{S}}$, $C_{\widetilde{\mathbf{S}}}$, $C_{\mathbf{T}}$, and $\mathtt{R}$.

Proof. Lemma 4.6 guarantees that $\boldsymbol{\Pi}$ maps $\mathrm{W}(\mathtt{R})$ into itself. Hence, in virtue of the equivalence between (3.37) and (4.6), and bearing in mind the Lipschitz-continuity of $\boldsymbol{\Pi}$ (cf. (4.51)) and the hypothesis (4.52), a straightforward application of the Banach fixed-point theorem implies the existence of a unique solution $(\boldsymbol{z},\boldsymbol{\varphi}) \in \mathrm{W}(\mathtt{R})$ of (4.6), and thus a unique solution $((\boldsymbol{u},\theta),(\boldsymbol{z},p)) \in (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$, $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2\times\mathrm{M}_1$, and $(\vec{\boldsymbol{t}}_i,\xi_i) \in \boldsymbol{\mathcal{H}}\times\mathcal{M}$, $i \in \{1,2\}$, of (3.37). In addition, the a priori estimates follow straightforwardly from (4.22), (4.23a), (4.23b), (4.37), and bounding $\|\boldsymbol{\varphi}\|_{0,r;\Omega}$, which appears in the original version of the first estimate above (cf. (4.22)), by $\mathtt{R}$. $\square$

5 A Galerkin scheme

In this section, we introduce a Galerkin scheme for (3.37), and analyze its well-posedness by means of the discrete analogue of the fixed-point approach developed in Section 4. In particular, for the solvability analysis of the Galerkin schemes associated with the decoupled problems studied in Sections 4.2, 4.3, and 4.4, we employ [8, Theorem 4.3.1], and the discrete versions of [25, Theorem 2.6], [6, Theorem 2.1, Corollary 2.1, Section 2.1], and [22, Theorem 3.4], which are given by [25, Theorem 2.22], [6, Corollary 2.2, Section 2.2], and [22, Theorem 3.5], respectively.

5.1 Preliminaries

We begin by considering arbitrary finite element subspaces of the continuous spaces indicated as follows

$$\begin{aligned}
&\mathbf{X}_h \subseteq \mathbf{H}_0^1(\Omega)\,, \quad \mathbf{Q}_h \subseteq \mathrm{L}^2(\Omega)\,, \quad \mathbf{X}_{2,h} \subseteq \mathbf{H}_0^r(\mathrm{div};\Omega)\,,\\
&\mathbf{X}_{1,h} \subseteq \mathbf{H}_0^s(\mathrm{div}_s;\Omega)\,, \quad \mathbf{Q}_{1,h} \subseteq \mathrm{L}_0^r(\Omega)\,, \quad \mathbf{Q}_{2,h} \subseteq \mathrm{L}_0^2(\Omega)\,,\\
&\mathrm{X}_{2,h} \subseteq \mathrm{X}_2\,, \quad \mathrm{X}_{1,h} \subseteq \mathrm{X}_1\,, \quad \mathrm{M}_{1,h} \subseteq \mathrm{M}_1\,, \quad \mathrm{M}_{2,h} \subseteq \mathrm{M}_2\,,\\
&\mathcal{H}_{1,h} \subseteq \mathcal{H}_1\,, \quad \mathcal{H}_{2,h} \subseteq \mathcal{H}_2\,, \quad \text{and} \quad \mathcal{M}_h \subseteq \mathcal{M}\,.
\end{aligned}$$

Hereafter, h stands for both the sub-index of each foregoing subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, that is $h := \max\{h_K : \quad K \in \mathcal{T}_h\}$. Specific finite element subspaces satisfying the stability conditions to be introduced along the analysis will be provided later on in Section 6. Then, setting the notation

$$\vec{\boldsymbol{t}}_{i,h} := (\boldsymbol{t}_{i,h}, \boldsymbol{\sigma}_{i,h})\,, \quad \vec{\boldsymbol{r}}_{i,h} := (\boldsymbol{r}_{i,h}, \boldsymbol{\zeta}_{i,h})\,, \quad \text{and} \quad \vec{\boldsymbol{s}}_{i,h} := (\boldsymbol{s}_{i,h}, \boldsymbol{\tau}_{i,h}) \in \boldsymbol{\mathcal{H}}_h := \mathcal{H}_{1,h} \times \mathcal{H}_{2,h}\,,$$

the Galerkin scheme associated with (3.37) reads: Find $(\boldsymbol{u}_h, \theta_h) \in \mathbf{X}_h \times \mathbf{Q}_h$, $(\boldsymbol{z}_h, p_h) \in \mathbf{X}_{2,h} \times \mathbf{Q}_{1,h}$, $(\boldsymbol{\varphi}_h, \chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $(\vec{\boldsymbol{t}}_{i,h}, \xi_{i,h}) \in \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h$, $i \in \{1,2\}$, such that

$$\begin{array}{llllll}
\mathbf{a}_s(\boldsymbol{u}_h, \boldsymbol{v}_h) & +\ \mathbf{b}_s(\boldsymbol{v}_h, \theta_h) & & & = & \mathbf{F}_{\boldsymbol{\varphi}_h, \boldsymbol{\xi}_h, \boldsymbol{t}_h}(\boldsymbol{v}_h)\,,\\
\mathbf{b}_s(\boldsymbol{u}_h, \vartheta_h) & -\ \mathbf{c}_s(\theta_h, \vartheta_h) & & +\,\mathbf{e}_s(p_h, \vartheta_h) & = & 0\,,\\
& & \mathbf{a}_f(\boldsymbol{z}_h, \boldsymbol{w}_h) + \mathbf{d}_1(\boldsymbol{w}_h, p_h) & & = & 0\,,\\
& & \mathbf{d}_2(\boldsymbol{z}_h, q_h) & +\,\mathbf{e}_f((\theta_h, p_h), q_h) & = & \mathbf{G}(q_h)\,,\\
a(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) & +\ b_1(\boldsymbol{\psi}_h, \chi_h) & & & = & G(\boldsymbol{\psi}_h)\,,\\
b_2(\boldsymbol{\varphi}_h, \gamma_h) & & & & = & F_{\boldsymbol{\xi}_h}(\gamma_h)\,,\\
\mathcal{A}(\vec{\boldsymbol{t}}_{i,h}, \vec{\boldsymbol{s}}_{i,h}) & +\ \mathcal{B}(\vec{\boldsymbol{s}}_{i,h}, \xi_{i,h}) & & +\ \mathcal{E}_{\boldsymbol{z}_h, \boldsymbol{\varphi}_h}(\vec{\boldsymbol{s}}_{i,h}, \xi_{i,h}) & = & \mathcal{G}(\vec{\boldsymbol{s}}_{i,h})\,,\\
\mathcal{B}(\vec{\boldsymbol{t}}_{i,h}, \eta_{i,h}) & -\ \mathcal{C}(\xi_{i,h}, \eta_{i,h}) & & +\ \mathcal{D}_{\boldsymbol{z}_h}(\xi_{i,h}, \eta_{i,h}) & = & \mathcal{F}(\eta_{i,h})\,,
\end{array} \tag{5.1}$$

for all $(\boldsymbol{v}_h, \vartheta_h) \in \mathbf{X}_h \times \mathbf{Q}_h$, $(\boldsymbol{w}_h, q_h) \in \mathbf{X}_{1,h} \times \mathbf{Q}_{2,h}$, $(\boldsymbol{\psi}_h, \gamma_h) \in \mathrm{X}_{1,h} \times \mathrm{M}_{2,h}$, and $(\vec{\boldsymbol{s}}_{i,h}, \eta_{i,h}) \in \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h$.

5.2 Discrete fixed-point approach

In order to analyze the solvability of (5.1), we introduce next the discrete version of the strategy employed in Section 4.1. We begin by adopting the notation

$$\begin{aligned}
\boldsymbol{t}_h &:= (\boldsymbol{t}_{1,h}, \boldsymbol{t}_{2,h})\,, \quad \boldsymbol{r}_h := (\boldsymbol{r}_{1,h}, \boldsymbol{r}_{2,h}) \in \boldsymbol{\mathcal{H}}_{1,h} := \mathcal{H}_{1,h} \times \mathcal{H}_{1,h}\,,\\
\boldsymbol{\xi}_h &:= (\xi_{1,h}, \xi_{2,h})\,, \quad \boldsymbol{\eta}_h := (\eta_{1,h}, \eta_{2,h}) \in \boldsymbol{\mathcal{M}}_h := \mathcal{M}_h \times \mathcal{M}_h\,,
\end{aligned}$$

and by letting $\mathbf{S}_h : \mathrm{X}_{2,h} \times \boldsymbol{\mathcal{M}}_h \times \boldsymbol{\mathcal{H}}_{1,h} \to \mathbf{X}_{2,h}$ be the operator defined by

$$\mathbf{S}_h(\boldsymbol{\phi}_h, \boldsymbol{\eta}_h, \boldsymbol{r}_h) := \boldsymbol{z}_h \qquad \forall\, (\boldsymbol{\phi}_h, \boldsymbol{\eta}_h, \boldsymbol{r}_h) \in \mathrm{X}_{2,h} \times \boldsymbol{\mathcal{M}}_h \times \boldsymbol{\mathcal{H}}_{1,h}\,,$$

where $(\boldsymbol{u}_h, \theta_h) \in \mathbf{X}_h \times \mathbf{Q}_h$ and $(\boldsymbol{z}_h, p_h) \in \mathbf{X}_{2,h} \times \mathbf{Q}_{1,h}$ constitute the unique solution (to be confirmed) of the first four rows of (5.1) with $\mathbf{F}_{\boldsymbol{\phi}_h,\boldsymbol{\eta}_h,\boldsymbol{r}_h}$ instead of $\mathbf{F}_{\boldsymbol{\varphi}_h,\boldsymbol{\xi}_h,\boldsymbol{t}_h}$. Similarly, we define $\widetilde{\mathbf{S}}_h : \boldsymbol{\mathcal{M}}_h \to \mathrm{X}_{2,h}$ as

$$\widetilde{\mathbf{S}}_h(\boldsymbol{\eta}_h) := \boldsymbol{\varphi}_h \quad \forall\, \boldsymbol{\eta}_h \in \boldsymbol{\mathcal{M}}_h\,,$$

where $(\boldsymbol{\varphi}_h, \chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ is the unique solution (to be confirmed) of the fifth and sixth rows of (5.1) with $F_{\boldsymbol{\eta}_h}$ instead of $F_{\boldsymbol{\xi}_h}$. Furthermore, we let $\mathrm{T}_{i,h} : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \mathcal{H}_{1,h}$ and $\Xi_{i,h} : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \mathcal{M}_h$, $i \in \{1,2\}$, be the operators given for each $(\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ by

$$\mathrm{T}_{i,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := \boldsymbol{t}_{i,h} \quad \text{and} \quad \Xi_{i,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := \boldsymbol{\xi}_{i,h}\,,$$

where $(\vec{\boldsymbol{t}}_{i,h}, \xi_{i,h}) = \big((\boldsymbol{t}_{i,h}, \boldsymbol{\sigma}_{i,h}), \xi_{i,h}\big) \in \boldsymbol{\mathcal{H}}_h \times \mathrm{M}_h$ is the unique solution (to be confirmed) of the last two rows of (5.1) with $\mathcal{E}_{\boldsymbol{w}_h,\boldsymbol{\phi}_h}$ and $\mathcal{D}_{\boldsymbol{w}_h}$ instead of $\mathcal{E}_{\boldsymbol{z}_h,\boldsymbol{\varphi}_h}$ and $\mathcal{D}_{\boldsymbol{z}_h}$, respectively. Hence, we can set the operators $\mathbf{T}_h : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \boldsymbol{\mathcal{H}}_{1,h}$ and $\boldsymbol{\Xi}_h : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \boldsymbol{\mathcal{M}}_h$ as

$$\mathbf{T}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := (\mathrm{T}_{1,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h), \mathrm{T}_{2,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h)) = \boldsymbol{t}_h\,, \quad \text{and}$$

$$\boldsymbol{\Xi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := (\Xi_{1,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h), \Xi_{2,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h)) = \boldsymbol{\xi}_h\,,$$

for all $(\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$. Finally, introducing the operator $\boldsymbol{\Pi}_h : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ defined as

$$\boldsymbol{\Pi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := \big(\mathbf{S}_h(\boldsymbol{\phi}_h, \boldsymbol{\Xi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h), \mathbf{T}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h)), \widetilde{\mathbf{S}}_h(\boldsymbol{\Xi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h))\big) \quad \forall\, (\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}\,,$$

we see that solving (5.1) is equivalent to finding a fixed-point of $\boldsymbol{\Pi}_h$, that is, $(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ such that

$$\boldsymbol{\Pi}_h(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) = (\boldsymbol{z}_h, \boldsymbol{\varphi}_h)\,. \tag{5.2}$$

5.3 Well-definedness of the operator $\mathbf{S}_h$

In what follows we proceed as in Section 4.2. In fact, we first observe that the properties of the bilinear forms $\mathbf{a}_s$ and $\mathbf{c}_s$, namely symmetry, positive semi-definiteness, and ellipticity, remain valid as such in the present discrete context. In particular, $\mathbf{a}_s$ is certainly $\mathbf{X}_h$-elliptic with the same constant $\alpha_s := \mu\, C_P$ (cf. (4.13)). Next, in order to continue the analysis, we need to assume the discrete version of (4.14), that is:

(H.1) there exists a positive constant $\beta_{s,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{v}_h \in \mathbf{X}_h \\ \boldsymbol{v}_h \neq \mathbf{0}}} \frac{\mathbf{b}_s(\boldsymbol{v}_h, \vartheta_h)}{\|\boldsymbol{v}_h\|_{\mathbf{Q}}} \geq \beta_{s,\mathrm{d}}\, \|\vartheta_h\|_{\mathbf{Q}} \qquad \forall\, \vartheta_h \in \mathbf{Q}_h\,.$$

Thanks to the above discussion and hypothesis **(H.1)**, we can apply again [8, Theorem 4.3.1] to deduce, similarly to (4.12a), its discrete analogue, that is the existence of a positive constant $\boldsymbol{\alpha}_{s,\mathrm{d}}$, depending on $\|\mathbf{a}_s\|$, $\|\mathbf{c}_s\|$, α_s, and $\beta_{s,\mathrm{d}}$, such that

$$\sup_{\substack{(\boldsymbol{v}_h,\vartheta_h) \in \mathbf{X}_h \times \mathbf{Q}_h \\ (\boldsymbol{v}_h,\vartheta_h) \neq \mathbf{0}}} \frac{\mathbf{A}_s((\boldsymbol{u}_h, \theta_h), (\boldsymbol{v}_h, \vartheta_h))}{\|(\boldsymbol{v}_h, \vartheta_h)\|_{\mathbf{X}\times\mathbf{Q}}} \geq \boldsymbol{\alpha}_{s,\mathrm{d}}\, \|(\boldsymbol{u}_h, \theta_h)\|_{\mathbf{X}\times\mathbf{Q}} \qquad \forall\, (\boldsymbol{u}_h, \theta_h) \in \mathbf{X}_h \times \mathbf{Q}_h\,. \tag{5.3}$$

On the other hand, we now introduce the discrete kernels of $\mathbf{d}_i$, $i \in \{1,2\}$, namely

$$\mathbf{K}_{1,h} := \Big\{ \boldsymbol{w}_h \in \mathbf{X}_{1,h} : \quad \mathbf{d}_1(\boldsymbol{w}_h, q_h) = 0 \quad \forall\, q_h \in \mathbf{Q}_{1,h} \Big\} \quad \text{and}$$

$$\mathbf{K}_{2,h} := \Big\{ \boldsymbol{w}_h \in \mathbf{X}_{2,h} : \quad \mathbf{d}_2(\boldsymbol{w}_h, q_h) = 0 \quad \forall\, q_h \in \mathbf{Q}_{2,h} \Big\},$$

and consider the following additional hypotheses:

(H.2) there exists a positive constant $\alpha_{f,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{w}_h \in \mathbf{K}_{1,h} \\ \boldsymbol{w}_h \neq \mathbf{0}}} \frac{\mathbf{a}_f(\boldsymbol{z}_h, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_{\mathbf{X}_1}} \,\geq\, \alpha_{f,\mathrm{d}} \,\|\boldsymbol{z}_h\|_{\mathbf{X}_2} \qquad \forall\, \boldsymbol{z}_h \in \mathbf{K}_{2,h}\,,$$

$$\sup_{\boldsymbol{z}_h \in \mathbf{K}_{2,h}} \mathbf{a}_f(\boldsymbol{z}_h, \boldsymbol{w}_h) \,>\, 0 \qquad \forall\, \boldsymbol{w}_h \in \mathbf{K}_{1,h}\,, \boldsymbol{w}_h \neq \mathbf{0}\,, \quad \text{and}$$

(H.3) for each $i \in \{1,2\}$ there exists a positive constant $\beta_{i,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{w}_h \in \mathbf{X}_{i,h} \\ \boldsymbol{w}_h \neq \mathbf{0}}} \frac{\mathbf{d}_i(\boldsymbol{w}_h, q_h)}{\|\boldsymbol{w}_h\|_{\mathbf{X}_i}} \,\geq\, \beta_{i,\mathrm{d}} \|q_h\|_{\mathbf{Q}_i} \quad \forall\, q_h \in \mathbf{Q}_{i,h}\,.$$

In this way, thanks to **(H.2)** and **(H.3)**, and similarly to (4.12b), we derive its discrete analogue as a straightforward consequence of [6, Corollary 2.2, Section 2.2], which means that there exists a positive constant $\boldsymbol{\alpha}_{f,\mathrm{d}}$ depending only on $\|\mathbf{a}_f\|$, $\alpha_{f,\mathrm{d}}$, $\beta_{1,\mathrm{d}}$, and $\beta_{2,\mathrm{d}}$, such that

$$\sup_{\substack{(\boldsymbol{w}_h, q_h) \in \mathbf{X}_{1,h} \times \mathbf{Q}_{2,h} \\ (\boldsymbol{w}_h, q_h) \neq \mathbf{0}}} \frac{\mathbf{A}_f((\boldsymbol{z}_h, p_h), (\boldsymbol{w}_h, q_h))}{\|(\boldsymbol{w}_h, q_h)\|_{\mathbf{X}_1 \times \mathbf{Q}_2}} \,\geq\, \boldsymbol{\alpha}_{f,\mathrm{d}} \,\|(\boldsymbol{z}_h, p_h)\|_{\mathbf{X}_2 \times \mathbf{Q}_1} \qquad \forall\, (\boldsymbol{z}_h, p_h) \in \mathbf{X}_{2,h} \times \mathbf{Q}_{1,h}\,. \tag{5.4}$$

Having established (5.3) and (5.4), a direct application of [25, Proposition 2.42], along with the discrete version of (4.11), imply, in turn, the discrete version of (4.9a) with the constant $\boldsymbol{\alpha}_{\mathbf{A},\mathrm{d}} := \frac{1}{2} \min\big\{\boldsymbol{\alpha}_{s,\mathrm{d}}, \boldsymbol{\alpha}_{f,\mathrm{d}}\big\}$. Moreover, using again the symmetry of $\mathbf{A}_s$ and the transpose $\mathbf{A}_f^{\mathbf{t}}$ of $\mathbf{A}_f$, as we did in the continuous analysis, we are able to prove the discrete analogue of (4.9b) as well. Consequently, under the discrete counterpart of (4.19), that is

$$\max\big\{\|\mathbf{e}_s\|, \|\mathbf{e}_f\|\big\} \,:=\, C_r(\Omega)\, \max\left\{c_0 + \frac{\alpha^2}{\lambda}, \frac{\alpha}{\lambda}\right\} \,\leq\, \frac{\boldsymbol{\alpha}_{\mathbf{A},\mathrm{d}}}{2}\,, \tag{5.5}$$

we arrive at the discrete versions of (4.20) and (4.21), and hence we can state the following result.

Lemma 5.1. *Assume that the data satisfy (5.5). Then, for each $(\boldsymbol{\phi}_h, \boldsymbol{\eta}_h, \boldsymbol{r}_h) \in \mathrm{X}_{2,h} \times \boldsymbol{\mathcal{M}}_h \times \boldsymbol{\mathcal{H}}_{1,h}$, there exists a unique $((\boldsymbol{u}_h, \theta_h), (\boldsymbol{z}_h, p_h)) \in (\mathbf{X}_h \times \mathbf{Q}_h) \times (\mathbf{X}_{2,h} \times \mathbf{Q}_{1,h})$ solution of the first four rows of (5.1), and hence one can define $\mathbf{S}_h(\boldsymbol{\phi}_h, \boldsymbol{\eta}_h, \boldsymbol{r}_h) := \boldsymbol{z}_h \in \mathbf{X}_{2,h}$. Moreover, there exists a positive constant $C_{\mathbf{S},\mathrm{d}}$, depending only on $\boldsymbol{\alpha}_{\mathbf{A},\mathrm{d}}$, ε, and δ, such that*

$$\begin{aligned} \|\mathbf{S}_h(\boldsymbol{\phi}_h, \boldsymbol{\eta}_h, \boldsymbol{r}_h)\|_{\mathbf{X}_2} \,&=\, \|\boldsymbol{z}_h\|_{\mathbf{X}_2} \,\leq\, \|(\boldsymbol{u}_h, \theta_h)\|_{\mathbf{X} \times \mathbf{Q}} \,+\, \|(\boldsymbol{z}_h, p_h)\|_{\mathbf{X}_2 \times \mathbf{Q}_1} \\ &\leq\, C_{\mathbf{S},\mathrm{d}} \Big\{ \|\boldsymbol{f}\|_{0,\Omega} \,+\, \|g\|_{0,\Omega} \,+\, \|\boldsymbol{\eta}_h\|_{\boldsymbol{\mathcal{M}}} \,\|\boldsymbol{\phi}_h\|_{0,r;\Omega} \,+\, \|\boldsymbol{r}_h\|_{\boldsymbol{\mathcal{H}}_1} \Big\}. \end{aligned} \tag{5.6}$$

Proof. Similarly to the proof of Lemma 4.3, the result follows as a direct application of the discrete Banach–Nečas–Babuška Theorem (cf. [25, Theorem 2.22]). $\square$

5.4 Well-definedness of the operator $\widetilde{\mathrm{S}}_h$

We begin by letting $K_{i,h}$ be the discrete kernel of b_i, $i \in \{1,2\}$, that is

$$K_{i,h} := \Big\{ \psi_h \in \mathrm{X}_{i,h} : \quad b_i(\psi_h,\gamma_h) \,=\, 0 \quad \forall\, \gamma \in \mathrm{M}_{i,h} \Big\},$$

and by assuming the following hypotheses

(H.4) there exists a positive constant $\widetilde{\alpha}_{\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\psi_h\in K_{1,h}\\ \psi_h\neq \mathbf{0}}} \frac{a(\phi_h,\psi_h)}{\|\psi_h\|_{\mathrm{X}_1}} \,\geq\, \widetilde{\alpha}_{\mathrm{d}}\,\|\phi_h\|_{\mathrm{X}_2} \qquad \forall\, \phi_h \in K_{2,h}\,,$$

$$\sup_{\phi_h\in K_{2,h}} a(\phi_h,\psi_h) \,>\, 0 \qquad \forall\, \psi_h \in K_{1,h}\,, \quad \psi_h \neq \mathbf{0}\,, \quad \text{and}$$

(H.5) for each $i \in \{1,2\}$ there exists a positive constant $\widetilde{\beta}_{i,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\psi_h\in \mathrm{X}_{i,h}\\ \psi_h\neq \mathbf{0}}} \frac{b_i(\psi_h,\gamma_h)}{\|\psi_h\|_{\mathrm{X}_i}} \,\geq\, \widetilde{\beta}_{i,\mathrm{d}}\,\|\gamma_h\|_{\mathrm{M}_i} \qquad \forall\, \gamma_h \in \mathrm{M}_{i,h}\,.$$

As a consequence of **(H.4)** and **(H.5)** we are able to state now the discrete version of Lemma 4.4

Lemma 5.2. *For each $\boldsymbol{\eta}_h = (\eta_{1,h},\eta_{2,h}) \in \boldsymbol{\mathcal{M}}_h$, there exists a unique $(\boldsymbol{\varphi}_h,\chi_h) \in \mathrm{X}_{2,h}\times \mathrm{M}_{1,h}$ solution of the fifth and sixth rows of (5.1), and hence one can define $\widetilde{\mathbf{S}}_h(\boldsymbol{\eta}_h) := \boldsymbol{\varphi}_h \in \mathrm{X}_{2,h}$. Moreover, there exist positive constants $C_{\widetilde{\mathbf{S}},\mathrm{d}}$ and $\widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}$, which depend on ε, c_r (cf. (3.27)), $|\Omega|$, ρ, r, $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{1,\mathrm{d}}$, and $\widetilde{\beta}_{2,\mathrm{d}}$, such that*

$$\|\widetilde{\mathbf{S}}_h(\boldsymbol{\eta}_h)\|_{\mathrm{X}_2} = \|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} \,\leq\, C_{\widetilde{\mathbf{S}},\mathrm{d}} \Big\{ \|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} \,+\, \|\boldsymbol{\eta}_h\|_{0,\rho;\Omega} \Big\}, \quad \textit{and} \tag{5.7a}$$

$$\|\chi_h\|_{\mathrm{M}_1} \,\leq\, \widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}} \Big\{ \|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} \,+\, \|\boldsymbol{\eta}_h\|_{0,\rho;\Omega} \Big\}\,. \tag{5.7b}$$

Proof. It reduces to a direct application of [6, Corollary 2.2, eqns. (2.24), (2.25)]. □

5.5 Well-definedness of the operators $\mathbf{T}_h$ and $\boldsymbol{\Xi}_h$

In what follows we proceed similarly as in Section 4.3. We begin by noticing that the positive semi-definiteness and symmetry properties involving the bilinear forms $\mathcal{A}$ and $\mathcal{C}$ are certainly valid at the present discrete context as well. In turn, it is easily seen that the discrete Kernel V_h of $\mathcal{B}$ is given by

$$V_h \,:=\, \mathcal{H}_{1,h}\times V_{0,h}\,, \quad \text{where} \quad V_{0,h} \,:=\, \Big\{ \boldsymbol{\tau}_{i,h} \in \mathcal{H}_{2,h} : \quad \int_\Omega \eta_{i,h}\,\mathrm{div}(\boldsymbol{\tau}_{i,h}) \,=\, 0 \quad \forall\, \eta_{i,h} \in \mathcal{M}_h \Big\}.$$

Thus, assuming the hypothesis

(H.6) $\mathrm{div}(\mathcal{H}_{2,h}) \,\subseteq\, \mathcal{M}_h\,,$

we readily deduce that $V_{0,h}$ becomes

$$V_{0,h} \,:=\, \Big\{ \boldsymbol{\tau}_{i,h} \in \mathcal{H}_{2,h} : \quad \mathrm{div}(\boldsymbol{\tau}_{i,h}) = 0 \quad \text{in} \quad \Omega \Big\},$$

which constitutes the discrete version of (4.25). Next, regarding the bilinear forms defining $\mathcal{A}$ (cf. (4.26)), we first let $\mathcal{K}_{1,h}$ and $\mathcal{K}_{2,h}$ be the kernels of $B_1|_{\mathcal{H}_{1,h}\times V_{0,h}}$ and $B_2|_{\mathcal{H}_{1,h}\times V_{0,h}}$, respectively. Then, similarly as for the continuous case (cf. (4.28)), we find that

$$\mathcal{K}_{1,h} = \mathcal{K}_{2,h} = \mathcal{K}_h := \left\{ \boldsymbol{s}_{i,h} \in \mathcal{H}_{1,h} : \quad \int_\Omega \boldsymbol{s}_{i,h} \cdot \boldsymbol{\tau}_{i,h} = 0 \quad \forall\, \boldsymbol{\tau}_{i,h} \in V_{0,h} \right\}.$$

In this way, since the $\mathcal{H}_1$-ellipticity of A (cf. (4.27)) is naturally inherited by the subspace $\mathcal{H}_{1,h}$, we conclude that the discrete inf-sup conditions specified in [6, eqns. (2.19) and (2.20)] are clearly satisfied by A. Analogously to the continuous case, note that the above holds irrespective of the specific conditions defining $\mathcal{K}_h$, except being a subspace of $\mathcal{H}_{1,h}$.

In order to proceed, we now assume that

(H.7) $V_{0,h} \subseteq \mathcal{H}_{1,h}$,

which implies that B_1 and B_2 satisfy the discrete inf-sup condition specified in [6, eqn. (2.22)]. In fact, given $\boldsymbol{\tau}_{i,h} \in V_{0,h} \subseteq \mathcal{H}_{1,h}$, and similarly as in the continuous case (cf. (4.29)), we can bound the supremum by below with $\boldsymbol{s}_{i,h} = -\boldsymbol{\tau}_{i,h}$ (for B_1) and $\boldsymbol{s}_{i,h} = \boldsymbol{\tau}_{i,h}$ (for B_2), so that we obtain for each $j \in \{1,2\}$

$$\sup_{\substack{\boldsymbol{s}_{i,h}\in\mathcal{H}_{1,h}\\ \boldsymbol{s}_{i,h}\neq\mathbf{0}}} \frac{B_j(\boldsymbol{s}_{i,h},\boldsymbol{\tau}_{i,h})}{\|\boldsymbol{s}_{i,h}\|_{\mathcal{H}_1}} = \sup_{\substack{\boldsymbol{s}_{i,h}\in\mathcal{H}_{1,h}\\ \boldsymbol{s}_{i,h}\neq\mathbf{0}}} \frac{B_j(\boldsymbol{s}_{i,h},\boldsymbol{\tau}_{i,h})}{\|\boldsymbol{s}_{i,h}\|_{0,\Omega}} \geq \|\boldsymbol{\tau}_{i,h}\|_{0,\Omega} = \|\boldsymbol{\tau}_{i,h}\|_{\mathcal{H}_2} \qquad \forall\, \boldsymbol{\tau}_{i,h} \in V_{0,h}\,.$$

It remains to assume the discrete inf-sup condition for $\mathcal{B}$, namely

(H.8) there exists a positive constant $\beta_{\mathcal{B},\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\boldsymbol{s}}_{i,h}\in\mathbf{H}_h\\ \vec{\boldsymbol{s}}_{i,h}\neq\mathbf{0}}} \frac{\mathcal{B}(\vec{\boldsymbol{s}}_{i,h},\xi_{i,h})}{\|\vec{\boldsymbol{s}}_{i,h}\|_{\boldsymbol{\mathcal{H}}}} \geq \beta_{\mathcal{B},\mathrm{d}}\,\|\xi_{i,h}\|_{\mathcal{M}} \qquad \forall\, \xi_{i,h} \in \mathcal{M}_h\,.$$

Therefore, having A, B_1 and B_2 satisfied the hypotheses of [6, Corollary 2.2], we conclude the discrete analogue of the inf-sup condition (4.30) for $\mathcal{A}$ with the same constant $\widehat{\alpha}$. This inequality, along with **(H.8)**, imply the discrete version of the inf-sup condition (4.24) for $\boldsymbol{\mathcal{A}}$ with a constant $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}},\mathrm{d}}$, depending only on $\widehat{\alpha}$, $\beta_{\mathcal{B},\mathrm{d}}$, $\|\mathcal{A}\|$, and $\|\mathcal{C}\|$. The same property is carried over to $\boldsymbol{\mathcal{A}}_{\boldsymbol{w}_h,\boldsymbol{\phi}_h}$ (cf. (4.32)) for each $(\boldsymbol{w}_h,\boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ satisfying the discrete version of (4.34), that is

$$\|(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\|_{\mathbf{X}_2\times\mathrm{X}_2} \leq \mathtt{R}_{\mathrm{d}} := \frac{\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}},\mathrm{d}}}{2\max\{\|\mathcal{D}\|,\|\mathcal{E}\|\}}\,, \tag{5.8}$$

thus yielding the discrete analogue of (4.35).

Consequently, we can state the well-definedness of the components of $\mathbf{T}_h$ and $\boldsymbol{\Xi}_h$ as follows.

Lemma 5.3. *For each $i \in \{1,2\}$, and for each $(\boldsymbol{w}_h,\boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ satisfying (5.8), there exists a unique $(\vec{\boldsymbol{t}}_{i,h},\xi_{i,h}) = \big((\boldsymbol{t}_{i,h},\boldsymbol{\sigma}_{i,h}),\xi_{i,h}\big) \in \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h$ solution of the seventh and eighth rows of (5.1), and hence we can define $\mathrm{T}_{i,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h) := \boldsymbol{t}_{i,h} \in \mathcal{H}_{1,h}$ and $\Xi_{i,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h) := \xi_{i,h} \in \mathcal{M}_h$. Moreover, there exists a positive constant $C_{\mathbf{T},\mathrm{d}}$, independent of $(\boldsymbol{w}_h,\boldsymbol{\phi}_h)$, such that*

$$\begin{aligned}\|\mathrm{T}_{i,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\|_{\mathcal{H}_1} + \|\Xi_{i,h}(\boldsymbol{w}_h,\boldsymbol{\phi}_h)\|_{\mathcal{M}} &= \|\boldsymbol{t}_{i,h}\|_{\mathcal{H}_1} + \|\xi_{i,h}\|_{\mathcal{M}}\\ &\leq \|(\vec{\boldsymbol{t}}_{i,h},\xi_{i,h})\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} \leq C_{\mathbf{T},\mathrm{d}}\left\{\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\right\}.\end{aligned} \tag{5.9}$$

Proof. It is a straightforward application of [25, Theorem 2.22]. $\square$

5.6 Solvability analysis of the discrete fixed-point equation

Having established that the discrete operators $\mathbf{S}_h$, $\widetilde{\mathbf{S}}_h$, $\mathbf{T}_h$, $\mathbf{\Xi}_h$, and hence $\mathbf{\Pi}_h$, are all well-defined, we now proceed as in Section 4.4 to address the solvability of the discrete fixed-point equation (5.2). To this end, we first introduce the ball

$$\mathrm{W}(\mathtt{R}_\mathrm{d}) := \Big\{ (\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} : \quad \|(\boldsymbol{w}_h, \boldsymbol{\phi}_h)\|_{\mathbf{X}_2 \times \mathrm{X}_2} \le \mathtt{R}_\mathrm{d} \Big\}.$$

Then, analogously to the derivation of Lemma 4.6, we deduce that $\mathbf{\Pi}_h$ maps $\mathrm{W}(\mathtt{R}_\mathrm{d})$ into itself under the same assumption (4.40), except that $C(\mathtt{R})$ is replaced by a constant $C(\mathtt{R}_\mathrm{d})$ depending on $C_{\mathbf{S},\mathrm{d}}$, $C_{\widetilde{\mathbf{S}},\mathrm{d}}$, $C_{\mathbf{T},\mathrm{d}}$ and $\mathtt{R}_\mathrm{d}$. Moreover, following analogous arguments to those employed in the proofs of Lemmas 4.7, 4.8, and 4.9, we are able to prove the continuity properties of $\mathbf{S}_h$, $\widetilde{\mathbf{S}}_h$, $\mathbf{T}_h$, and $\mathbf{\Xi}_h$, with corresponding constants denoted by $L_{\mathbf{S},\mathrm{d}}$, $L_{\widetilde{\mathbf{S}},\mathrm{d}}$, and $L_{\mathbf{T},\mathrm{d}}$, respectively. Hence, proceeding analogously to the derivation of (4.51), we find that there exists a positive constant $L_{\mathbf{\Pi},\mathrm{d}}$, depending only on $L_{\mathbf{S},\mathrm{d}}$, $L_{\widetilde{\mathbf{S}},\mathrm{d}}$, $L_{\mathbf{T},\mathrm{d}}$, and $\mathtt{R}_\mathrm{d}$, such that

$$\|\mathbf{\Pi}_h(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) - \mathbf{\Pi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h)\| \le L_{\mathbf{\Pi},\mathrm{d}} \sum_{i=1}^{2} \Big\{ \|\xi_{i,\mathtt{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big\} \|(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) - (\boldsymbol{w}_h, \boldsymbol{\phi}_h)\| . \tag{5.10}$$

for all $(\boldsymbol{z}_h, \boldsymbol{\varphi}_h)$, $(\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathrm{W}(\mathtt{R}_\mathrm{d})$.

According to the above, the main result of this section is established as follows.

Theorem 5.4. *Assume that the data satisfy (5.5) and the discrete version of (4.40), that is*

$$C(\mathtt{R}_\mathrm{d}) \Big\{ \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_\mathtt{D}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \Big(\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big) \Big\} \le \mathtt{R}_\mathrm{d} . \tag{5.11}$$

In addition, assume that

$$L_{\mathbf{\Pi},\mathrm{d}} \sum_{i=1}^{2} \Big\{ \|\xi_{i,\mathtt{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big\} < 1 . \tag{5.12}$$

Then, the discrete fixed point equation (5.2) has a unique solution $(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) \in \mathrm{W}(\mathtt{R}_\mathrm{d})$. Equivalently, the coupled problem (5.1) has a unique solution $\big((\boldsymbol{u}_h, \theta_h), (\boldsymbol{z}_h, p_h)\big) \in (\mathbf{X}_h \times \mathbf{Q}_h) \times (\mathbf{X}_{2,h} \times \mathbf{Q}_{1,h})$, $(\boldsymbol{\varphi}_h, \xi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $(\vec{\boldsymbol{t}}_i, \xi_i) \in \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h$, $i \in \{1, 2\}$. Moreover, there hold the following a priori estimates

$$\begin{aligned}
\|(\boldsymbol{u}_h, \theta_h)\|_{\mathbf{X} \times \mathbf{Q}} + \|(\boldsymbol{z}_h, p_h)\|_{\mathbf{X}_2 \times \mathbf{Q}_1} &\le \widetilde{C}_{\mathbf{S},\mathrm{d}} \Big\{ \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \sum_{i=1}^{2} \big(\|\xi_{i,\mathtt{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \big) \Big\}, \\
\|(\boldsymbol{\varphi}_h, \xi_h)\|_{\mathrm{X}_2 \times \mathrm{M}_1} &\le \widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}} \Big\{ \|\chi_\mathtt{D}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \big(\|\xi_{i,\mathtt{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \big) \Big\}, \\
\|(\vec{\boldsymbol{t}}_i, \xi_i)\|_{\boldsymbol{\mathcal{H}} \times \mathcal{M}} &\le C_{\mathbf{T},\mathrm{d}} \Big\{ \|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big\}, \quad i \in \{1, 2\},
\end{aligned}$$

where $\widetilde{C}_{\mathbf{S},\mathrm{d}}$ and $\widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}$ are positive constants depending only on $C_{\mathbf{S},\mathrm{d}}$, $C_{\widetilde{\mathbf{S}},\mathrm{d}}$, $C_{\mathbf{T},\mathrm{d}}$, and $\mathtt{R}_\mathrm{d}$.

Proof. We recall that (5.11) guarantees that $\mathbf{\Pi}_h$ maps $\mathrm{W}(\mathtt{R}_\mathrm{d})$ into itself, and knowing from (5.10) and (5.12) that $\mathbf{\Pi}_h : \mathrm{W}(\mathtt{R}_\mathrm{d}) \to \mathrm{W}(\mathtt{R}_\mathrm{d})$ is a contraction, a straightforward application of the Banach fixed-point theorem yields the existence of a unique solution $(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) \in \mathrm{W}(\mathtt{R}_\mathrm{d})$ of (5.2), and thus a unique solution $\big((\boldsymbol{u}_h, \theta_h), (\boldsymbol{z}_h, p_h)\big) \in (\mathbf{X}_h \times \mathbf{Q}_h) \times (\mathbf{X}_{2,h} \times \mathbf{Q}_{1,h})$, $(\boldsymbol{\varphi}_h, \xi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $(\vec{\boldsymbol{t}}_i, \xi_i) \in \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h$, $i \in \{1, 2\}$, of (5.1). Finally, the a priori estimates are consequence of (5.6), (5.7a), (5.7b), (5.9), and the fact that $\|\boldsymbol{\varphi}_h\|_{0,r;\Omega} \le \mathtt{R}_\mathrm{d}$. $\square$

We end the section by stressing that the assumption (5.12) could be dropped from the statement of Theorem 5.4, in which case Brouwer's fixed-point theorem (cf. [20, Theorem 9.9-2]) would imply only existence of solution of (5.2) (and hence of (5.1)).

5.7 A priori error analysis

In this section, we derive an a priori error estimate for the Galerkin scheme (5.1) with arbitrary finite element subspaces satisfying the hypotheses introduced in Sections 5.3, 5.4, and 5.5. More precisely, recalling that $\big((\boldsymbol{u},\theta),(\boldsymbol{z},p)\big) \in (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$, $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2\times\mathrm{M}_1$, and $(\vec{\boldsymbol{t}}_i,\xi_i) \in \boldsymbol{\mathcal{H}}\times\mathcal{M}$, $i \in \{1,2\}$, with $(\boldsymbol{z},\boldsymbol{\varphi}) \in \mathrm{W}(\mathtt{R})$, constitute the unique solution of (3.37), and that, in turn, $\big((\boldsymbol{u}_h,\theta_h),(\boldsymbol{z}_h,p_h)\big) \in (\mathbf{X}_h\times\mathbf{Q}_h)\times(\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h})$, $(\boldsymbol{\varphi}_h,\xi_h) \in \mathrm{X}_{2,h}\times\mathrm{M}_{1,h}$, and $(\vec{\boldsymbol{t}}_i,\xi_i) \in \boldsymbol{\mathcal{H}}_h\times\mathcal{M}_h$, $i \in \{1,2\}$, with $(\boldsymbol{z}_h,\boldsymbol{\varphi}_h) \in \mathrm{W}(\mathtt{R}_\mathrm{d})$, is the unique solution of (5.1), we establish a Céa estimate for the global error split as

$$\mathtt{E} := \mathtt{E}_1 + \mathtt{E}_2 + \mathtt{E}_3\,,$$

where

$$\begin{aligned}
\mathtt{E}_1 &:= \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{X}} + \|\theta-\theta_h\|_{\mathbf{Q}} + \|\boldsymbol{z}-\boldsymbol{z}_h\|_{\mathbf{X}_2} + \|p-p_h\|_{\mathbf{Q}_1}\,,\\
\mathtt{E}_2 &:= \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} + \|\chi-\chi_h\|_{\mathrm{M}_1}\,, \quad \text{and}\\
\mathtt{E}_3 &:= \sum_{i=1}^{2}\Big\{\|\vec{\boldsymbol{t}}_i-\vec{\boldsymbol{t}}_{i,h}\|_{\boldsymbol{\mathcal{H}}} + \|\xi_i-\xi_{i,h}\|_{\mathcal{M}}\Big\}\,.
\end{aligned}$$

In what follows, given a subspace Z_h of a generic Banach space $(Z,\|\cdot\|_Z)$, we set

$$\operatorname{dist}(z,Z_h) := \inf_{z_h\in Z_h}\|z-z_h\|_Z \qquad \forall\, z\in Z\,.$$

We begin the analysis by applying the Strang estimate from [25, Lemma 2.27] to the first four rows of equations (3.37) and (5.1). As a consequence, we obtain that there exists a positive constant $\widetilde{C}_1(\mathtt{E})$, depending on $\boldsymbol{\alpha}_{\mathbf{A},\mathrm{d}}$, $\|\mathbf{A}\|$ (cf. (4.7)), $\|\mathbf{e}_s\|$, and $\|\mathbf{e}_f\|$ (cf. (3.21)), such that there holds

$$\mathtt{E}_1 \le \widetilde{C}_1(\mathtt{E})\Big\{\operatorname{dist}((\boldsymbol{u},\theta),\mathbf{X}_h\times\mathbf{Q}_h) + \operatorname{dist}((\boldsymbol{z},p),\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h}) + \|\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}-\mathbf{F}_{\boldsymbol{\varphi}_h,\boldsymbol{\xi}_h,\boldsymbol{t}_h}\|_{\mathbf{X}_h'}\Big\}\,. \tag{5.13}$$

Then, bearing in mind the definition of $\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}$ (cf. (3.17)), and proceeding as in (4.44), (4.45), and (4.46), we find that

$$\|\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}-\mathbf{F}_{\boldsymbol{\varphi}_h,\boldsymbol{\xi}_h,\boldsymbol{t}_h}\|_{\mathbf{X}_h'} \le \max\big\{\varepsilon^{-1},\delta\big\}\Big\{\|\boldsymbol{\xi}\|_{\boldsymbol{\mathcal{M}}}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} + \|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2}\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{\boldsymbol{\mathcal{M}}} + \|\boldsymbol{t}-\boldsymbol{t}_h\|_{\boldsymbol{\mathcal{H}}_1}\Big\}\,,$$

which, replaced back into (5.13), yields

$$\begin{aligned}
\mathtt{E}_1 \le C_1(\mathtt{E})\Big\{&\operatorname{dist}((\boldsymbol{u},\theta),\mathbf{X}_h\times\mathbf{Q}_h) + \operatorname{dist}((\boldsymbol{z},p),\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h})\\
&+ \|\boldsymbol{\xi}\|_{\boldsymbol{\mathcal{M}}}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} + \|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2}\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{\boldsymbol{\mathcal{M}}} + \|\boldsymbol{t}-\boldsymbol{t}_h\|_{\boldsymbol{\mathcal{H}}_1}\Big\}\,,
\end{aligned} \tag{5.14}$$

with $C_1(\mathtt{E}) := \widetilde{C}_1(\mathtt{E})\max\big\{1,\varepsilon^{-1},\delta\big\}$. Next, applying again [25, Lemma 2.27], but now to the fifth and sixth rows of equations (3.37) and (5.1), and using that (cf. (3.26), see also [24, eqn. (95)])

$$\|F_{\boldsymbol{\xi}}-F_{\boldsymbol{\xi}_h}\|_{\mathrm{M}_2'} = \|F_{\boldsymbol{\xi}-\boldsymbol{\xi}_h}\|_{\mathrm{M}_2'} \le |\Omega|^{(\rho-r)/\rho r}\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{\boldsymbol{\mathcal{M}}}\,,$$

we arrive at

$$\mathtt{E}_2 \le C_2(\mathtt{E})\Big\{\operatorname{dist}((\boldsymbol{\varphi},\chi),\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}) + \|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{\boldsymbol{\mathcal{M}}}\Big\}\,, \tag{5.15}$$

with a positive constant $C_2(\mathtt{E})$ depending only on ε, $\widetilde{\alpha}_\mathrm{d}$, $\widetilde{\beta}_{1,\mathrm{d}}$, $\widetilde{\beta}_{2,\mathrm{d}}$, $|\Omega|$, ρ, and r. Furthermore, regarding the last two rows of (3.37) and (5.1), we employ the same Strang estimate from [25, Lemma 2.27] to conclude the existence of a positive constant $\widetilde{C}_3(\mathtt{E})$, depending only on $\boldsymbol{\alpha}_{\boldsymbol{\mathcal{A}}}$, $\|\mathcal{A}\|$, $\|\mathcal{B}\|$, and $\|\mathcal{C}\|$ (cf. (3.34)), such that

$$\begin{aligned}
\mathtt{E}_3 \le \widetilde{C}_3(\mathtt{E})\sum_{i=1}^{2}\Big\{&\operatorname{dist}((\vec{\boldsymbol{t}}_i,\xi_i),\boldsymbol{\mathcal{H}}_h\times\mathcal{M}_h) + \|\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\cdot,\xi_i)-\mathcal{E}_{\boldsymbol{z}_h,\boldsymbol{\varphi}_h}(\cdot,\xi_{i,h})\|_{\boldsymbol{\mathcal{H}}_h'}\\
&+ \|\mathcal{D}_{\boldsymbol{z}}(\xi_i,\cdot)-\mathcal{D}_{\boldsymbol{z}_h}(\xi_{i,h},\cdot)\|_{\mathcal{M}_h'}\Big\}\,.
\end{aligned} \tag{5.16}$$

In turn, from the definitions of $\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}$ and $\mathcal{D}_{\boldsymbol{z}}$ (cf. (3.33)), we readily get that

$$\begin{aligned}\|\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\cdot,\xi_i) - \mathcal{E}_{\boldsymbol{z}_h,\boldsymbol{\varphi}_h}(\cdot,\xi_{i,h})\|_{\boldsymbol{\mathcal{H}}'_h} \,\le\, \|\mathcal{E}\| \Big\{ & \|\xi_{i,h}\|_{0,\rho;\Omega} \left(\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{r,\mathrm{div}_r;\Omega} \,+\, \|\boldsymbol{z} - \boldsymbol{z}_h\|_{0,r;\Omega}\right) \\ & + \quad \left(\|\boldsymbol{z}\|_{0,r;\Omega} \,+\, \|\boldsymbol{\varphi}\|_{r,\mathrm{div}_r;\Omega}\right) \|\xi_i - \xi_{i,h}\|_{0,\rho;\Omega} \Big\},\end{aligned}$$

and

$$\|\mathcal{D}_{\boldsymbol{z}}(\xi_i,\cdot) - \mathcal{D}_{\boldsymbol{z}_h}(\xi_{i,h},\cdot)\|_{\mathcal{M}'_h} \,\le\, \|\mathcal{D}\| \Big\{ \|\mathrm{div}(\boldsymbol{z})\|_{0,\Omega}\, \|\xi_i - \xi_{i,h}\|_{0,\rho;\Omega} \,+\, \|\xi_{i,h}\|_{0,\rho;\Omega}\, \|\mathrm{div}(\boldsymbol{z}) - \mathrm{div}(\boldsymbol{z}_h)\|_{0,\Omega} \Big\},$$

which, jointly with (5.16), imply

$$\begin{aligned}\mathtt{E}_3 \,\le\, C_3(\mathtt{E}) \sum_{i=1}^{2} \Big\{ & \mathrm{dist}((\vec{\boldsymbol{t}}_i,\xi_i), \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h) + \|\xi_{i,h}\|_{\mathcal{M}} \left(\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{\mathrm{X}_2} + \|\boldsymbol{z} - \boldsymbol{z}_h\|_{\mathbf{X}_2}\right) \\ & + \quad \left(\|\boldsymbol{z}\|_{\mathbf{X}_2} \,+\, \|\boldsymbol{\varphi}\|_{\mathrm{X}_2}\right) \|\xi_i - \xi_{i,h}\|_{\mathcal{M}} \Big\},\end{aligned} \tag{5.17}$$

with a positive constant $C_3(\mathtt{E})$ depending only on $\widetilde{C}_3(\mathtt{E})$, $\|\mathcal{E}\|$, and $\|\mathcal{D}\|$. Consequently, adding the inequalities (5.14), (5.15), and (5.17), performing basic algebraic manipulations, and employing the bounds for the terms $\|\boldsymbol{z}\|_{\mathbf{X}_2}$, $\|\boldsymbol{\varphi}\|_{\mathrm{X}_2}$, $\|\boldsymbol{\xi}\|_{\mathcal{M}}$, $\|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2}$, and $\|\xi_{i,h}\|_{\mathcal{M}}$ provided by Theorems 4.10 and 5.4, we deduce the existence of a positive constant $\widetilde{C}(\mathtt{E})$, depending only on $\widetilde{C}_{\mathbf{S}}$, $\widetilde{C}_{\widetilde{\mathbf{S}}}$, $C_{\mathbf{T}}$, $\widetilde{C}_{\mathbf{S},\mathrm{d}}$, $\widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}$, and $C_{\mathbf{T},\mathrm{d}}$, and hence independent of h, such that

$$\begin{aligned}\mathtt{E} \,\le\, \widetilde{C}(\mathtt{E}) \Big\{ & \mathrm{dist}((\boldsymbol{u},\theta), \mathbf{X}_h \times \mathbf{Q}_h) \,+\, \mathrm{dist}((\boldsymbol{z},p), \mathbf{X}_{2,h} \times \mathbf{Q}_{1,h}) \\ & + \mathrm{dist}((\boldsymbol{\varphi},\chi), \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}) + \sum_{i=1}^{2} \mathrm{dist}((\vec{\boldsymbol{t}}_i,\xi_i), \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h) \Big\} \\ & + \widetilde{C}(\mathtt{E}) \Big\{ \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \left(\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\right) \Big\}\, \mathtt{E}.\end{aligned} \tag{5.18}$$

We summarize our findings with the next result.

Theorem 5.5. *In addition to the hypotheses of Theorems 4.10 and 5.4, assume that*

$$\widetilde{C}(\mathtt{E}) \Big\{ \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathtt{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \left(\|\xi_{i,\mathtt{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\right) \Big\} \,\le\, \frac{1}{2}. \tag{5.19}$$

Then, letting $C(\mathtt{E}) := 2\,\widetilde{C}(\mathtt{E})$, *there holds*

$$\begin{aligned}\mathtt{E} \,\le\, C(\mathtt{E}) \Big\{ & \mathrm{dist}((\boldsymbol{u},\theta), \mathbf{X}_h \times \mathbf{Q}_h) \,+\, \mathrm{dist}((\boldsymbol{z},p), \mathbf{X}_{2,h} \times \mathbf{Q}_{1,h}) \\ & + \mathrm{dist}((\boldsymbol{\varphi},\chi), \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}) \,+\, \sum_{i=1}^{2} \mathrm{dist}((\vec{\boldsymbol{t}}_i,\xi_i), \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h) \Big\}.\end{aligned} \tag{5.20}$$

Proof. It follows straightforwardly from (5.18) and (5.19). $\square$

6 Specific finite element subspaces

In this section, we define specific finite element subspaces satisfying the conditions **(H.1) - (H.8)** introduced in Sections 5.3, 5.4, and 5.5, collect their respective approximation properties, and provide the associated rates of convergence of the resulting method.

6.1 Preliminaries

Given an integer $\ell \geq 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_\ell(K)$ (resp. $\widetilde{\mathrm{P}}_k(K)$) be the space of polynomials of degree $\leq k$ (resp. $= k$) defined on K, and denote its vector version by $\mathbf{P}_\ell(K)$. In addition, we let $\mathbf{RT}_\ell(K) = \mathbf{P}_\ell(K) + \widetilde{\mathrm{P}}_\ell(K)\,\mathbf{x}$ be the local Raviart–Thomas space of order ℓ defined on K, where $\mathbf{x}$ stands for a generic vector in R^d. In turn, we let $\mathrm{P}_\ell(\mathcal{T}_h)$, $\mathbf{P}_\ell(\mathcal{T}_h)$, and $\mathbf{RT}_\ell(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_\ell(K)$, $\mathbf{P}_\ell(K)$ and $\mathbf{RT}_\ell(K)$, respectively, that is

$$\begin{aligned}
\mathrm{P}_\ell(\mathcal{T}_h) &:= \Big\{\theta_h \in \mathrm{L}^2(\Omega): \quad \theta_h|_K \in \mathrm{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\}, \\
\mathbf{P}_\ell(\mathcal{T}_h) &:= \Big\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega): \quad \boldsymbol{v}_h|_K \in \mathbf{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\}, \quad \text{and} \\
\mathbf{RT}_\ell(\mathcal{T}_h) &:= \Big\{\boldsymbol{q}_h \in \mathbf{H}(\mathrm{div};\Omega): \quad \boldsymbol{q}_h|_K \in \mathbf{RT}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\}.
\end{aligned}$$

We stress that for each $t \in (1, +\infty)$, there hold $\mathrm{P}_\ell(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{P}_\ell(\mathcal{T}_h) \subseteq \mathbf{H}^1(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$, $\mathbf{RT}_\ell(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div};\Omega)$, and $\mathbf{RT}_\ell(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div}_t;\Omega)$, inclusions that are implicitly utilized below to introduce specific finite element subspaces. Indeed, bearing in mind the notation from Section 5.1, and given an integer $k \geq 0$, we now define for $n = 2$:

$$\begin{aligned}
&\mathbf{X}_h := \mathbf{P}_{k+2}(\mathcal{T}_h) \cap \mathbf{H}^1_0(\Omega), \quad \mathbf{Q}_h := \mathrm{P}_k(\mathcal{T}_h), \\
&\mathbf{X}_{2,h} := \mathbf{RT}_k(\mathcal{T}_h) \cap \mathbf{H}^r_0(\mathrm{div};\Omega), \quad \mathbf{X}_{1,h} := \mathbf{RT}_k(\mathcal{T}_h) \cap \mathbf{H}^s_0(\mathrm{div}_s;\Omega), \\
&\mathbf{Q}_{1,h} := \mathrm{P}_k(\mathcal{T}_h) \cap \mathrm{L}^r_0(\Omega), \quad \mathbf{Q}_{2,h} := \mathrm{P}_k(\mathcal{T}_h) \cap \mathrm{L}^2_0(\Omega), \\
&\mathrm{X}_{2,h} = \mathrm{X}_{1,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad \mathrm{M}_{1,h} = \mathrm{M}_{2,h} := \mathrm{P}_k(\mathcal{T}_h), \\
&\mathcal{H}_{1,h} := \mathbf{P}_k(\mathcal{T}_h), \quad \mathcal{H}_{2,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad \text{and} \quad \mathcal{M}_h := \mathrm{P}_k(\mathcal{T}_h).
\end{aligned} \tag{6.1}$$

In turn, for $n = 3$, and since the pair $\mathbf{P}_{k+2}(\mathcal{T}_h) - \mathrm{P}_k(\mathcal{T}_h)$ is not inf-sup stable in 3D, we consider the generalized Taylor–Hood elements

$$\mathbf{X}_h := \mathbf{P}_{k+2}(\mathcal{T}_h) \cap \mathbf{H}^1_0(\Omega), \quad \mathbf{Q}_h := \mathrm{P}_{k+1}(\mathcal{T}_h) \cap \mathcal{C}(\Omega), \tag{6.2}$$

whereas the remaining subspaces remain as specified in (6.1).

6.2 Verification of the stability conditions

In what follows we make sure that the spaces (6.1) and (6.2) satisfy the assumptions **(H.1)** - **(H.8)**. Indeed, the fact that the pair $(\mathbf{X}_h, \mathbf{Q}_h)$ verifies the inf-sup condition **(H.1)** was already proved in [8, Section 8.4.3] and [7] for the two and three-dimensional case, respectively. In turn, the proof of **(H.2)** for the 2D case was established in [27, Lemma 4.3] thanks to the boundedness of the L^2-type projector onto the discrete kernel of the bilinear forms $\mathbf{d}_1$ and $\mathbf{d}_2$. Whether this boundedness property holds in the 3D case remains still an open question, and hence, up to the authors' knowledge, there is no proof yet for **(H.2)** in 3D. As we will see in what follows, all other hypotheses hold in both dimensions.

Regarding **(H.3)**, the discrete inf-sup condition for $\mathbf{d}_1$ is available in [27, Lemma 4.4], and that for $\mathbf{d}_2$ is shown next employing some results provided in [18, Appendix A].

Lemma 6.1. *There exists a positive constant $\beta_{2,\mathrm{d}}$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{z}_h \in \mathbf{X}_{2,h} \\ \boldsymbol{z}_h \neq \mathbf{0}}} \frac{\mathbf{d}_2(\boldsymbol{z}_h, q_h)}{\|\boldsymbol{z}_h\|_{\mathbf{X}_{2,h}}} \geq \beta_{2,\mathrm{d}}\, \|q_h\|_{\mathbf{Q}_{2,h}} \qquad \forall\, q_h \in \mathbf{Q}_{2,h}. \tag{6.3}$$

Proof. Given $q_h \in \mathbf{Q}_{2,h}$, we first proceed as in the proof of Lemma 4.2, and apply again [32, Theorem 1.1] to deduce that there exists a unique $u \in \mathrm{H}^2(\Omega)$ such that

$$\Delta u \,=\, q_h \quad \text{in} \quad \Omega\,, \quad \nabla u \cdot \boldsymbol{n} \,=\, 0 \quad \text{on} \quad \Gamma\,, \quad \int_\Omega u \,=\, 0\,, \quad \text{and}$$

$$\|u\|_{2,\Omega} \,\leq\, C_{\mathtt{reg}}\, \|q_h\|_{0,\Omega}\,, \tag{6.4}$$

where $C_{\mathtt{reg}}$ is a positive constant depending only on Ω. Then, defining $\boldsymbol{w} \,:=\, \nabla u \,\in\, \mathbf{H}^1(\Omega)$, it follows that $\mathrm{div}(\boldsymbol{w}) \,=\, q_h$ in Ω and $\boldsymbol{w} \cdot \boldsymbol{n} \,=\, 0$ on Γ, whereas using (6.4) we obtain

$$\|\boldsymbol{w}\|_{1,\Omega} \,\leq\, \|u\|_{2,\Omega} \,\leq\, C_{\mathtt{reg}}\, \|q_h\|_{0,\Omega}\,. \tag{6.5}$$

Next, let $\Pi_h^k : \mathbf{H}^1(\Omega) \to \mathbf{RT}_k(\mathcal{T}_h)$ and $\mathcal{P}_h^k : \mathrm{L}^2(\Omega) \to \mathrm{P}_k(\mathcal{T}_h)$ be the global Raviart–Thomas interpolator and the $\mathrm{L}^2(\Omega)$-orthogonal projector, respectively. Then, setting $\boldsymbol{w}_h \,:=\, \Pi_h^k(\boldsymbol{w}) \,\in\, \mathbf{RT}_k(\mathcal{T}_h)$, well-known properties of Π_h^k guarantee that

$$\mathrm{div}(\boldsymbol{w}_h) \,=\, \mathcal{P}_h^k(\mathrm{div}(\boldsymbol{w})) \,=\, \mathcal{P}_h^k(q_h) \,=\, q_h \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{w}_h \cdot \boldsymbol{n} \,=\, 0 \quad \text{on} \quad \Gamma\,, \tag{6.6}$$

so that $\boldsymbol{w}_h \in \mathbf{X}_{2,h}$. In turn, since r belongs to the range specified by (3.7), it is easy to see that $t := r$ and $s := 2$ satisfy the constraints specified in [18, Lemma A.3], whence, along with (6.5), we get

$$\|\boldsymbol{w}_h\|_{0,r;\Omega} \,=\, \|\Pi_h^k(\boldsymbol{w})\|_{0,r;\Omega} \,\leq\, C_\Pi\, \|\boldsymbol{w}\|_{1,\Omega} \,\leq\, C_\Pi\, C_{\mathtt{reg}}\, \|q_h\|_{0,\Omega}\,, \tag{6.7}$$

where C_Π is the positive constant indicated in [18, Lemma A.3]. In this way, from the first identity in (6.6) and (6.7), we conclude that

$$\|\boldsymbol{w}_h\|_{r,\mathrm{div};\Omega} \,=\, \|\boldsymbol{w}_h\|_{0,r;\Omega} \,+\, \|\mathrm{div}(\boldsymbol{w}_h)\|_{0,\Omega} \,\leq\, \big(C_\Pi\, C_{\mathtt{reg}} + 1\big)\, \|q_h\|_{0,\Omega}\,. \tag{6.8}$$

Finally, bounding from below the supremum of (6.3) with $\boldsymbol{w}_h \in \mathbf{X}_{2,h}$, and using again the first identity in (6.6), and (6.8), we arrive at (6.3) with $\beta_{2,\mathrm{d}} \,:=\, \big(C_\Pi\, C_{\mathtt{reg}} + 1\big)^{-1}$. $\square$

Furthermore, for the proof of **(H.4)** we refer to [19, Lemma 5.2], which corresponds to the preprint version of [18]. More precisely, the proof there follows analogously to the one of [27, Lemma 4.3], except that, instead of the operator defined in [27, Lemma 2.3], one employs the slight modification of it derived in [18, Lemma 3.3]. In turn, the proofs of the discrete inf-sup conditions required by **(H.5)**, which adapt the continuous analysis from [24, Lemma 4.4] to the present discrete setting, reduce basically to slight modifications of those of [27, Lemma 4.5] (or [18, Lemma 5.3]).

On the other hand, we readily observe from (6.1) that $\mathrm{div}(\mathcal{H}_{2,h}) \subseteq \mathcal{M}_h$ and $V_{0,h} \subseteq \mathcal{H}_{1,h}$, which confirms the verification of **(H.6)** and **(H.7)**. Finally, we notice that **(H.8)** is proved in [27, Lemma 4.5].

6.3 Rates of convergence

Here we provide the rates of convergence of the Galerkin schemes (5.1) with the specific finite element subspaces introduced in Section 6.1, for which we first collect the respective approximation properties.

($\mathbf{AP}_h^{\boldsymbol{u}}$) there exists a positive constant C, independent of h, such that for each $s \in [1, k\,+\,1]$, and for each $\boldsymbol{v} \in \mathbf{H}^{s\,+\,2}(\Omega)$, there holds

$$\mathrm{dist}\big(\boldsymbol{v}, \mathbf{X}_h\big) \,:=\, \inf_{\boldsymbol{v}_h \in \mathbf{X}_h} \|\boldsymbol{v} \,-\, \boldsymbol{v}_h\|_{1,\Omega} \,\leq\, C\, h^{s+1}\, \|\boldsymbol{v}\|_{s+2,\Omega}\,,$$

$(\mathbf{AP}_h^{\theta})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$ when $n = 2$ (resp. $s \in [1, k+1]$ when $n = 3$), and for each $\vartheta \in \mathrm{H}^l(\Omega)$ when $n = 2$ (resp. $\vartheta \in \mathrm{H}^{s+1}(\Omega)$ when $n = 3$), there holds

$$\operatorname{dist}(\vartheta, \mathbf{Q}_h) \,:=\, \inf_{\vartheta_h \in \mathbf{Q}_h} \|\vartheta - \vartheta_h\|_{0,\Omega} \,\leq\, C\, h^l\, \|\vartheta\|_{l,\Omega}\,,$$

and when $n = 3$

$$\operatorname{dist}(\vartheta, \mathbf{Q}_h) \,:=\, \inf_{\vartheta_h \in \mathbf{Q}_h} \|\vartheta - \vartheta_h\|_{0,\Omega} \,\leq\, C\, h^{s+1}\, \|\vartheta\|_{s+1,\Omega}\,.$$

In turn, thanks to the properties of the Raviart–Thomas interpolator (see, e.g., [27, Section 4.1, eqns. (4.6) and (4.7)] and [18, Appendix A]) and the scalar and vector versions of the L^2-type projector onto piecewise polynomials ([25, Proposition 1.135]), along with interpolation estimates of Sobolev spaces, we have the following statements:

$(\mathbf{AP}_h^{\boldsymbol{z}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{w} \in \mathbf{W}^{l,r}(\Omega) \cap \mathbf{H}_0^r(\mathrm{div};\Omega)$ with $\operatorname{div}(\boldsymbol{w}) \in \mathrm{H}^l(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{w}, \mathbf{X}_{2,h}) \,:=\, \inf_{\boldsymbol{w}_h \in \mathbf{X}_{2,h}} \|\boldsymbol{w} - \boldsymbol{w}_h\|_{r,\mathrm{div};\Omega} \,\leq\, C\, h^l \Big\{ \|\boldsymbol{w}\|_{0,r;\Omega} + \|\operatorname{div}(\boldsymbol{w})\|_{0,\Omega} \Big\}\,,$$

$\left(\mathbf{AP}_h^{p}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $q \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}(q, \mathbf{Q}_{1,h}) \,:=\, \inf_{q_h \in \mathbf{Q}_{1,h}} \|q - q_h\|_{0,r;\Omega} \,\leq\, C\, h^l\, \|q\|_{l,r;\Omega}\,,$$

$\left(\mathbf{AP}_h^{\boldsymbol{\varphi}}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\psi} \in \mathbf{W}^{l,r}(\Omega)$ with $\operatorname{div}(\boldsymbol{\psi}) \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\psi}, \mathrm{X}_{2,h}) \,:=\, \inf_{\boldsymbol{\psi}_h \in \mathrm{X}_{2,h}} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{r,\mathrm{div}_r;\Omega} \,\leq\, C\, h^l \Big\{ \|\boldsymbol{\psi}\|_{0,r;\Omega} + \|\operatorname{div}(\boldsymbol{\psi})\|_{0,r;\Omega} \Big\}\,,$$

$\left(\mathbf{AP}_h^{\chi}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\eta \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}(\eta, \mathrm{M}_{1,h}) \,:=\, \inf_{\eta_h \in \mathrm{M}_{1,h}} \|\eta - \eta_h\|_{0,r;\Omega} \,\leq\, C\, h^l\, \|\eta\|_{l,r;\Omega}\,.$$

$\left(\mathbf{AP}_h^{\boldsymbol{t}_i}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\boldsymbol{s}_i \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{s}_i, \mathcal{H}_{1,h}) \,:=\, \inf_{\boldsymbol{s}_{i,h} \in \mathcal{H}_{1,h}} \|\boldsymbol{s}_i - \boldsymbol{s}_{i,h}\|_{0,\Omega} \,\leq\, C\, h^l\, \|\boldsymbol{s}_i\|_{l,\Omega}\,,$$

$\left(\mathbf{AP}_h^{\boldsymbol{\sigma}_i}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau}_i \in \mathbf{H}^l(\Omega)$ with $\operatorname{div}(\boldsymbol{\tau}_i) \in \mathrm{W}^{l,\varrho}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\tau}_i, \mathcal{H}_{2,h}) \,:=\, \inf_{\boldsymbol{\tau}_{i,h} \in \mathcal{H}_{2,h}} \|\boldsymbol{\tau}_i - \boldsymbol{\tau}_{i,h}\|_{\mathrm{div}_\varrho;\Omega} \,\leq\, C\, h^l \Big\{ \|\boldsymbol{\tau}_i\|_{0,\Omega} + \|\operatorname{div}(\boldsymbol{\tau}_i)\|_{0,\varrho;\Omega} \Big\}\,, \quad \text{and}$$

$\left(\mathbf{AP}_h^{\xi_i}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\eta_i \in \mathrm{W}^{l,\rho}(\Omega)$, there holds

$$\mathrm{dist}(\eta_i, \mathcal{M}_h) := \inf_{\eta_{i,h} \in \mathcal{M}_h} \|\eta_i - \eta_{i,h}\|_{0,\rho;\Omega} \leq C\, h^l \, \|\eta_i\|_{l,\rho;\Omega}\,.$$

Hence, we can state the following main theorem.

Theorem 6.2. *Let $((\boldsymbol{u}, \theta), (\boldsymbol{z}, p)) \in (\mathbf{X} \times \mathbf{Q}) \times (\mathbf{X}_2 \times \mathbf{Q}_1)$, $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$, and $(\vec{\boldsymbol{t}}_i, \xi_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}$, be the unique solution of (3.37), with $(\boldsymbol{z}, \boldsymbol{\varphi}) \in \mathrm{W}(\mathtt{R})$, and let $((\boldsymbol{u}_h, \theta_h), (\boldsymbol{z}_h, p_h)) \in (\mathbf{X}_h \times \mathbf{Q}_h) \times (\mathbf{X}_{2,h} \times \mathbf{Q}_{1,h})$, $(\boldsymbol{\varphi}_h, \xi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $(\vec{\boldsymbol{t}}_{i,h}, \xi_{i,h}) \in \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h$, be the unique solution of (5.1), with $(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) \in \mathrm{W}(\mathtt{R}_\mathrm{d})$, which is guaranteed by Theorems 4.10 and 5.4, respectively. Assume the hypotheses of Theorem 5.5, and that there exist $s, l \in [1, k+1]$, such that $\boldsymbol{u} \in \mathbf{H}^{s+2}(\Omega)$, $\theta \in \mathrm{H}^l(\Omega)$ (resp. $\theta \in \mathrm{H}^{s+1}(\Omega)$ when $n = 3$), $\boldsymbol{z} \in \mathbf{W}^{l,r}(\Omega)$, $\mathrm{div}(\boldsymbol{z}) \in \mathrm{H}^l(\Omega)$, $p \in \mathrm{H}^l(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{W}^{l,r}(\Omega)$, $\mathrm{div}(\boldsymbol{\varphi}) \in \mathrm{W}^{l,r}(\Omega)$, $\chi \in \mathrm{W}^{l,r}(\Omega)$, $\boldsymbol{t}_i \in \mathbf{H}^l(\Omega)$, $\boldsymbol{\sigma}_i \in \mathbf{H}^l(\Omega)$, $\mathrm{div}(\boldsymbol{\sigma}_i) \in \mathrm{W}^{l,\varrho}(\Omega)$, and $\xi_i \in \mathrm{W}^{l,\rho}(\Omega)$, $i \in \{1, 2\}$. Then, there exists a positive constant C, independent of h, such that, when $n = 2$*

$$\begin{aligned}\mathtt{E} \leq h^{\min\{s+1,l\}}\Big\{&\|\boldsymbol{u}\|_{s+2,\Omega} + \|\theta\|_{l,\Omega} + \|\boldsymbol{z}\|_{l,r;\Omega} + \|\mathrm{div}(\boldsymbol{z})\|_{l,\Omega} + \|p\|_{l,r;\Omega} + \|\boldsymbol{\varphi}\|_{l,r;\Omega} \\ &+ \|\mathrm{div}(\boldsymbol{\varphi})\|_{l,r;\Omega} + \|\chi\|_{l,r;\Omega} + \sum_{i=1}^{2} \big(\|\boldsymbol{t}_i\|_{l,\Omega} + \|\boldsymbol{\sigma}_i\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\sigma}_i)\|_{l,\varrho;\Omega} + \|\xi_i\|_{l,\rho;\Omega}\big)\Big\},\end{aligned}$$

and when $n = 3$

$$\begin{aligned}\mathtt{E} \leq h^{\min\{s+1,l\}}\Big\{&\|\boldsymbol{u}\|_{s+2,\Omega} + \|\theta\|_{s+1,\Omega} + \|\boldsymbol{z}\|_{l,r;\Omega} + \|\mathrm{div}(\boldsymbol{z})\|_{l,\Omega} + \|p\|_{l,r;\Omega} + \|\boldsymbol{\varphi}\|_{l,r;\Omega} \\ &+ \|\mathrm{div}(\boldsymbol{\varphi})\|_{l,r;\Omega} + \|\chi\|_{l,r;\Omega} + \sum_{i=1}^{2} \big(\|\boldsymbol{t}_i\|_{l,\Omega} + \|\boldsymbol{\sigma}_i\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\sigma}_i)\|_{l,\varrho;\Omega} + \|\xi_i\|_{l,\rho;\Omega}\big)\Big\},\end{aligned}$$

Proof. It follows directly from the Céa estimate (5.20) and the above approximation properties. □

7 Numerical tests

For the computational results that verify the error estimates from Section 6 we employ the open source finite element library `GridapDistributed` [4]. Rather than separating the coupled problem by fixed-point iterations between three subproblems, we solve the nonlinear algebraic system (5.1) with Newton–Raphson's method with exact Jacobian. We set a tolerance of 10^{-8} on either the ℓ^∞-norm of the nonlinear residual or the ℓ^2-norm of the incremental solution vector, and the resulting linear systems are solved with the unsymmetric multifrontal direct method for sparse matrices `UMFPACK`.

Example 1. We carry out the error history associated with the family of discretizations specified in Section 6.1, using polynomial degrees $k = 0, 1, 2$ (additional tests conducted with Brezzi–Douglas–Marini elements for the Darcy flux, electric field and ionic fluxes, not shown here, showed the same qualitative behavior as the one reported here). We take the unit square and unit cube domains $\Omega = (0, 1)^d$ ($d = 2, 3$), with unity model parameters. The Lebesgue exponents in (3.6) are chosen as $r = 3$, $s = \frac{3}{2}$, $\rho = 6$, $\varrho = \frac{6}{5}$ (and they are valid for both 2D and 3D cases). We manufacture the right-hand side and non-homogeneous boundary data $\boldsymbol{f}, g, f_i, \chi_\mathtt{D}, \xi_{i,\mathtt{D}}$ in such a way that the governing equations have the following smooth exact solutions to the primal strong form (2.1)-(2.3)

$$\text{in 2D: } \begin{cases} \boldsymbol{u}_{\mathrm{ex}}(x, y) = \begin{pmatrix} \sin(\pi[x+y]) \\ \cos(\pi[x^2+y^2]) \end{pmatrix}, \quad p_{\mathrm{ex}}(x, y) = \sin(\pi x)\sin(\pi y), \quad \chi_{\mathrm{ex}}(x, y) = \cos(\pi x)\cos(\pi y), \\ \xi_{1,\mathrm{ex}}(x, y) = \cos(\pi[x+y]), \quad \xi_{2,\mathrm{ex}}(x, y) = \sin(\pi[x+y]), \end{cases}$$

Biot unknowns									
DoF	h	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(\theta)$	$r(\theta)$	$e(\boldsymbol{z})$	$r(\boldsymbol{z})$	$e(p)$	$r(p)$
Discretization with $k=0$									
154	0.7071	1.42e+0	$\star$	1.98e+0	$\star$	6.50e+0	$\star$	3.58e-01	$\star$
612	0.3536	6.09e-01	1.221	1.12e+0	0.820	3.48e+0	0.902	1.61e-01	1.151
2440	0.1768	2.86e-01	1.091	5.68e-01	0.980	1.76e+0	0.980	7.79e-02	1.050
9744	0.0884	1.42e-01	1.012	2.85e-01	0.995	8.85e-01	0.995	3.86e-02	1.013
38944	0.0442	7.09e-02	0.998	1.42e-01	0.999	4.43e-01	0.999	1.93e-02	1.003
155712	0.0221	3.55e-02	0.997	7.12e-02	1.000	2.22e-01	1.000	9.63e-03	1.001
Discretization with $k=1$									
456	0.7071	3.63e-01	$\star$	6.74e-01	$\star$	2.06e+0	$\star$	9.30e-02	$\star$
1808	0.3536	8.16e-02	2.155	1.64e-01	2.035	5.46e-01	1.918	2.47e-02	1.910
7200	0.1768	2.03e-02	2.005	4.16e-02	1.984	1.39e-01	1.975	6.32e-03	1.970
28736	0.0884	5.07e-03	2.004	1.04e-02	1.994	3.49e-02	1.994	1.59e-03	1.992
114816	0.0442	1.27e-03	2.001	2.61e-03	1.998	8.73e-03	1.998	3.97e-04	1.998
459008	0.0221	3.16e-04	2.000	6.53e-04	2.000	2.18e-03	2.000	9.94e-05	2.000
Discretization with $k=2$									
910	0.7071	6.03e-02	$\star$	1.15e-01	$\star$	4.69e-01	$\star$	2.16e-02	$\star$
3612	0.3536	9.05e-03	2.736	1.72e-02	2.734	6.09e-02	2.946	2.83e-03	2.934
14392	0.1768	1.16e-03	2.967	2.33e-03	2.889	7.73e-03	2.978	3.60e-04	2.974
57456	0.0884	1.46e-04	2.987	2.96e-04	2.974	9.70e-04	2.994	4.53e-05	2.993
229600	0.0442	1.83e-05	2.993	3.72e-05	2.994	1.21e-04	2.998	5.66e-06	2.998
917952	0.0221	2.30e-06	2.996	4.65e-06	2.998	1.52e-05	3.000	7.08e-07	3.000

Table 7.1: Example1. Error history for the primal-mixed scheme in 2D, showing here only the Biot unknowns (while DoF refers to the total number of degrees of freedom).

$$\text{in 3D: } \begin{cases} \boldsymbol{u}_{\rm ex}(x,y,z) = \begin{pmatrix} \sin(\pi[x\,+\,y\,+\,z]) \\ \cos(\pi[x^2\,+\,y^2\,+\,z^2]) \\ \cos(\pi[x\,+\,y\,+\,z]) \end{pmatrix}, \quad p_{\rm ex}(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z), \\ \chi_{\rm ex}(x,y,z) = \cos(\pi x)\cos(\pi y)\cos(\pi z), \quad \xi_{1,\rm ex}(x,y,z) = \cos(\pi[x\,+\,y\,+\,z]), \\ \xi_{2,\rm ex}(x,y,z) = \sin(\pi[x\,+\,y\,+\,z]), \end{cases}$$

and the exact values of the mixed variables are assigned from the primal ones as

$$\theta_{\rm ex} = \alpha\, p_{\rm ex} - \lambda \operatorname{div} \boldsymbol{u}_{\rm ex}, \quad \boldsymbol{z}_{\rm ex} = -\frac{\kappa}{\nu}\nabla p_{\rm ex}, \quad \boldsymbol{\varphi}_{\rm ex} = \varepsilon \nabla \chi_{\rm ex}, \quad \boldsymbol{t}_i = \nabla \xi_{i,\rm ex},$$
$$\boldsymbol{\sigma}_{i,\rm ex} = \kappa_i \nabla \xi_{i,\rm ex} \,+\, q_i\, \kappa_i \nabla \chi_{\rm ex} \,+\, \frac{\kappa}{\nu}\xi_{i,\rm ex}\nabla p_{\rm ex}.$$

For the numerical tests we consider mixed boundary conditions. Therefore, and due to the properties of the smooth manufactured solutions, the formulation requires non-homogeneous traction and Darcy flux terms

$$\langle [2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}_{\rm ex}) - \theta_{\rm ex}\mathbb{I}]\boldsymbol{n}, \boldsymbol{v}\rangle_{\Gamma_p} \quad \text{and} \quad -\langle \frac{\nu}{\kappa}\boldsymbol{w}\cdot\boldsymbol{n}, p_{\rm ex}\rangle_{\Gamma_p},$$

that appear as right-hand side functionals in the first and third equations of (3.20), respectively. Similarly, we require the additional non-homogeneous source term

$$\int_\Omega (\operatorname{div} \boldsymbol{\varphi}_{\rm ex} \,+\, \xi_{1,\rm ex} - \xi_{2,\rm ex})\gamma,$$

on the right-hand side of the second equation in (3.24). We construct a sequence of six successively refined structured grids $l = 0, 1, \ldots$ of maximum mesh size $h_l = 2^{-l}\sqrt{2}$ (in 2D) on which we generate approximate

Mixed Poisson unknowns					
DoF	h	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\chi)$	$r(\chi)$
Discretization with $k=0$					
154	0.7071	6.60e+0	$\star$	2.79e-01	$\star$
612	0.3536	3.56e+0	0.890	1.51e-01	0.888
2440	0.1768	1.81e+0	0.978	7.66e-02	0.978
9744	0.0884	9.08e-01	0.994	3.85e-02	0.994
38944	0.0442	4.54e-01	0.999	1.93e-02	0.999
155712	0.0221	2.27e-01	1.000	9.63e-03	1.000
Discretization with $k=1$					
456	0.7071	2.08e+0	$\star$	9.25e-02	$\star$
1808	0.3536	5.52e-01	1.914	2.47e-02	1.902
7200	0.1768	1.41e-01	1.974	6.32e-03	1.970
28736	0.0884	3.53e-02	1.993	1.59e-03	1.992
114816	0.0442	8.85e-03	1.998	3.97e-04	1.998
459008	0.0221	2.21e-03	1.999	9.94e-05	2.000
Discretization with $k=2$					
910	0.7071	4.72e-01	$\star$	2.16e-02	$\star$
3612	0.3536	6.14e-02	2.941	2.83e-03	2.930
14392	0.1768	7.81e-03	2.975	3.60e-04	2.973
57456	0.0884	9.81e-04	2.993	4.53e-05	2.993
229600	0.0442	1.23e-04	2.998	5.66e-06	2.998
917952	0.0221	1.54e-05	2.999	7.08e-07	3.000

Table 7.2: Example 1. Error history for the primal-mixed scheme in 2D, showing here only the mixed Poisson unknowns (DoF here refers to the total number of degrees of freedom).

solutions, and we compute errors for each unknown $e_l(\cdot)$ and experimental orders of convergence

$$r_{l+1}(\cdot) = \frac{\log(e_{l+1}(\cdot)/e_l(\cdot))}{\log(h_{l+1}/h_l)}, \quad l = 0, 1, \ldots$$

Tables 7.1,7.2,7.3 portray the error history in 2D (we break it into Biot, mixed Poisson, and Nernst–Planck unknowns), from which we can readily confirm a convergence of $O(h^{k+1})$ for all field variables. The symbol $\star$ in the first refinement level indicates that no convergence rate is computed.

For every mesh refinement and polynomial degree, the Newton–Raphson algorithm has taken no more than four iterations to achieve the desired converge criterion. We can also observe that the error associated with the Raviart–Thomas vector fields of Darcy flux, electric field, and ionic fluxes $(\boldsymbol{z}, \boldsymbol{\varphi}, \boldsymbol{\sigma}_i)$ are slightly higher than that in the remaining unknowns. Convergence results are also optimal in the 3D case, which we report in Figure 7.1 for the second-order method (using, in particular, Taylor–Hood elements for displacement-total pressure pair), where we see agreement with Theorem 6.2. Sample approximate solutions are depicted in Figure 7.2.

Example 2. After the numerical verification of optimal convergence rates we address the simulation of electrochemically coupled poroelasticity in radially unconfined compression. This type of tests are typical in poromechanics [3, 17, 39], and have also been used for coupling with PNP equations in [36, 43, 47] (where that model includes additional mechanical nonlinearities). The domain is the 2D cut of a disk of cartilage tissue confined between two impermeable rigid plates, giving $\Omega = (0, 1.5) \times (0, 0.5)$ mm^2. On the radial surface (the right edge of the 2D domain) we set zero fluid pressure, zero electrostatic potential, prescribe the potential and ionic concentrations, as well as zero normal total stress. This allows free flow of fluid and current along that boundary. On the left edge we impose zero normal displacement, zero tangential total stress, and zero ionic fluxes and electric field. On the bottom plate we set zero normal

Nernst–Planck unknowns													
DoF	h	$e(\boldsymbol{t}_1)$	$r(\boldsymbol{t}_1)$	$e(\boldsymbol{t}_2)$	$r(\boldsymbol{t}_2)$	$e(\boldsymbol{\sigma}_1)$	$r(\boldsymbol{\sigma}_1)$	$e(\boldsymbol{\sigma}_2)$	$r(\boldsymbol{\sigma}_2)$	$e(\xi_1)$	$r(\xi_1)$	$e(\xi_2)$	$r(\xi_2)$
Discretization with $k=0$													
154	0.707	1.54e+0	$\star$	1.38e+0	$\star$	1.17e+1	$\star$	1.03e+1	$\star$	4.18e-01	$\star$	3.57e-01	$\star$
612	0.353	7.65e-01	1.00	7.84e-01	0.81	5.33e+0	1.13	6.53e+0	0.65	2.18e-01	0.93	2.13e-01	0.74
2440	0.176	3.84e-01	0.99	3.91e-01	1.00	2.65e+0	1.00	3.24e+0	1.01	1.09e-01	0.99	1.09e-01	0.96
9744	0.088	1.92e-01	0.99	1.95e-01	1.00	1.32e+0	1.00	1.63e+0	0.99	5.48e-02	0.99	5.47e-02	0.99
38944	0.044	9.62e-02	0.99	9.77e-02	1.00	6.62e-01	1.00	8.16e-01	1.00	2.74e-02	0.99	2.74e-02	0.99
155712	0.022	4.81e-02	1.00	4.88e-02	1.00	3.31e-01	1.00	4.08e-01	1.00	1.37e-02	1.00	1.37e-02	0.99
Discretization with $k=1$													
456	0.707	2.43e-01	$\star$	3.68e-01	$\star$	1.84e+0	$\star$	4.83e+0	$\star$	9.74e-02	$\star$	9.07e-02	$\star$
1808	0.353	6.66e-02	1.86	8.57e-02	2.10	6.93e-01	1.41	1.03e+00	2.22	2.75e-02	1.82	2.74e-02	1.76
7200	0.176	1.70e-02	1.96	2.13e-02	2.00	1.74e-01	1.99	2.52e-01	2.03	6.96e-03	1.98	6.95e-03	1.97
28736	0.088	4.29e-03	1.98	5.32e-03	2.00	4.38e-02	1.99	6.29e-02	2.00	1.74e-03	1.99	1.74e-03	1.95
114816	0.044	1.08e-03	1.99	1.33e-03	2.00	1.10e-02	1.99	1.57e-02	1.99	4.36e-04	1.99	4.36e-04	1.99
459008	0.022	2.69e-04	1.99	3.32e-04	2.00	2.74e-03	2.00	3.93e-03	2.00	1.09e-04	2.00	1.09e-04	2.00
Discretization with $k=2$													
910	0.707	4.08e-02	$\star$	6.06e-02	$\star$	8.26e-01	$\star$	4.92e-01	$\star$	1.34e-02	$\star$	1.63e-02	$\star$
3612	0.353	6.23e-03	2.71	7.25e-03	3.06	8.88e-02	3.21	1.25e-01	1.97	2.13e-03	2.65	2.16e-03	2.91
14392	0.176	8.02e-04	2.95	9.09e-04	2.99	1.11e-02	3.00	1.51e-02	3.04	2.69e-04	2.98	2.70e-04	3.00
57456	0.088	1.01e-04	2.98	1.14e-04	2.99	1.39e-03	2.99	1.88e-03	3.00	3.37e-05	2.97	3.38e-05	2.99
229600	0.044	1.27e-05	2.99	1.43e-05	2.99	1.74e-04	2.99	2.35e-04	3.00	4.22e-06	2.99	4.22e-06	3.00
917952	0.022	1.59e-06	2.99	1.78e-06	2.99	2.17e-05	3.00	2.94e-05	2.99	5.28e-07	3.00	5.28e-07	3.00

Table 7.3: Example 1. Error history for the primal-mixed scheme in 2D, showing here only the mixed Nernst–Planck unknowns (DoF here refers to the total number of degrees of freedom).

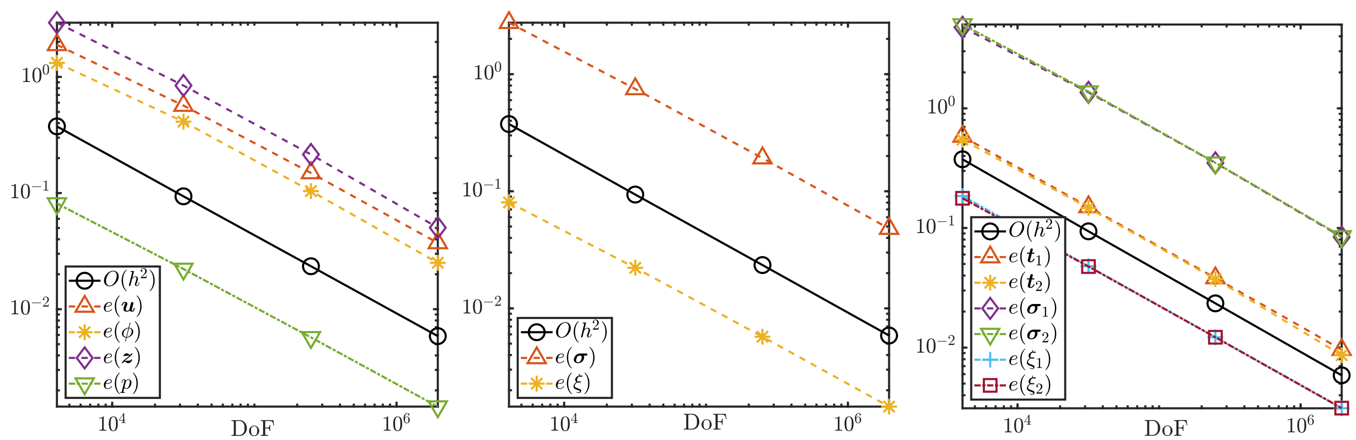

Figure 7.1: Error history for the primal-mixed scheme in 3D, showing the convergence of all individual errors for the Biot, mixed Poisson, and Nernst–Planck sub-systems (left, center, and right panels, respectively).

displacement, zero tangential stress, and zero fluxes, whereas on the top plate we prescribe a given normal traction $\boldsymbol{\sigma n} = (0, -M)^{\mathrm{t}}$ with $M = 0.1\,\mathrm{N/mm^2}$, together with zero fluxes. The model parameters are as follows $E_{\mathrm{Y}} = 0.5\,\mathrm{N/mm^2}$, $\nu_{\mathrm{P}} = 0.1$ (Young modulus and Poisson ratio), $\kappa = 10^{-9}\,\mathrm{mm^2}$ (permeability), $\kappa_1 = 1.28 \times 10^{-2}\,\mathrm{mm^2/s}$, $\kappa_1 = 1.77 \times 10^{-2}\,\mathrm{mm^2/s}$ (ionic diffusivities), $\alpha = 0.8$ (Biot–Willis coefficient), $c_0 = 4 \times 10^{-4}\,1/(\mathrm{N/mm^2})$ (storativity), $\nu = 10^{-4}\,\mathrm{N/mm^2\,s}$ (fluid viscosity). As outputs, in Figure 7.3 we report on the total stress tensor magnitude, cation and anion fluxes, and electric field. All quantities are plotted on the deformed domain. We see the typical deformation of the rightmost part of the domain and the Darcy flux moving in the horizontal direction. For this test we have used the second-order scheme

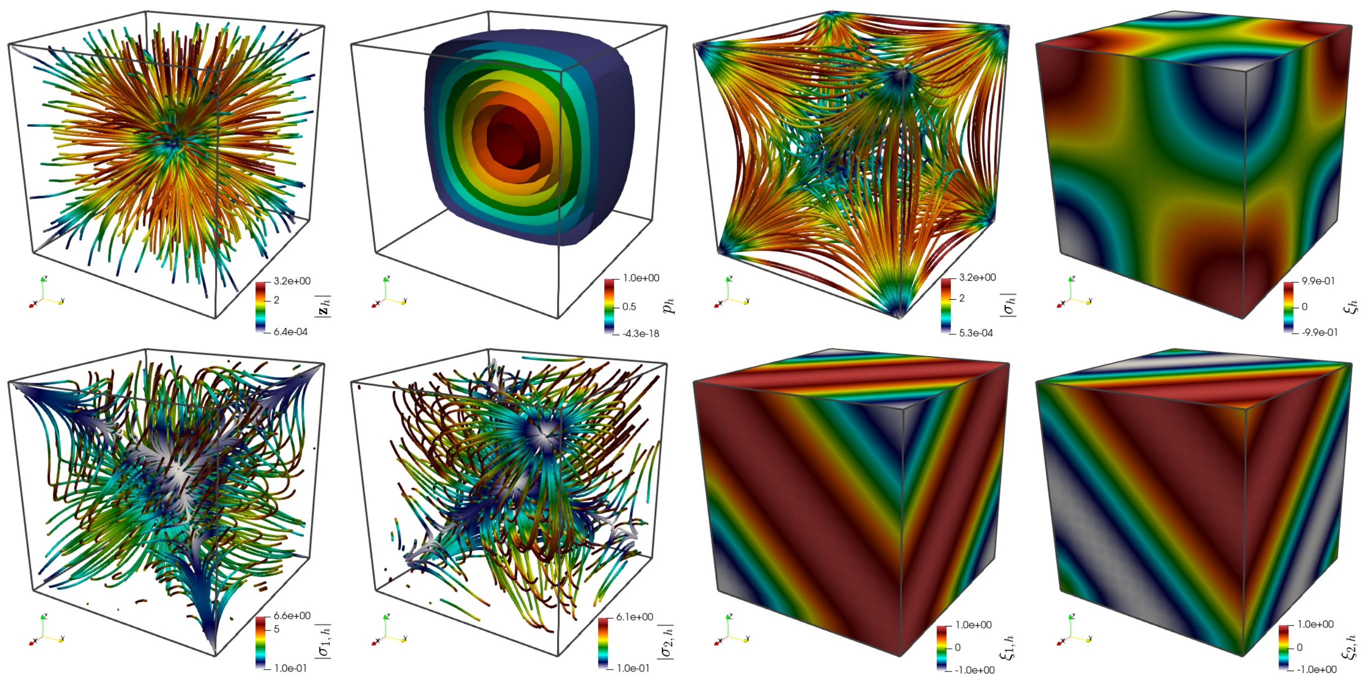

Figure 7.2: Example 1. Sample of approximate solutions for the convergence test in 3D.

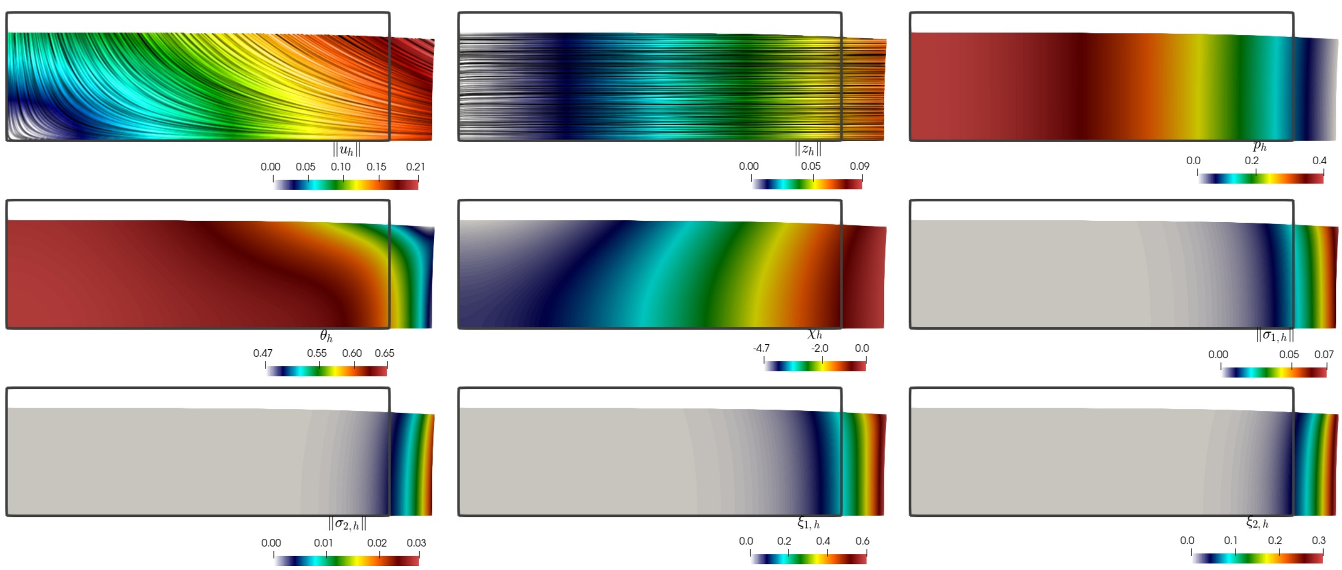

Figure 7.3: Example 2. Unconfined compression of poroelastic material between impermeable plates. Sample of approximate solutions (displacement, Darcy flux, fluid pressure, electrostatic potential, cation and anion fluxes, and cation and anion concentration shown on the deformed configuration).

(setting $k = 1$).

Example 3. Finally, we simulate the ion spreading and the poromechanical response of a fully saturated deformable porous structure. For this we adapt the configuration in [23] and [37, Section 5.2] to the poroelastic regime and use the domain $\Omega = (0,1) \times (0,2)$, which we discretize into a structured mesh of 10'000 triangles. The boundary conditions are as follows: for the solid phase we set clamped conditions $\boldsymbol{u} = \boldsymbol{0}$ on the left boundary ($x = 0$) for the fluid phase we impose slip conditions $\boldsymbol{z} \cdot \boldsymbol{n} = 0$ everywhere on the boundary. For the chemical species we assume that the normal trace of the total fluxes is zero everywhere

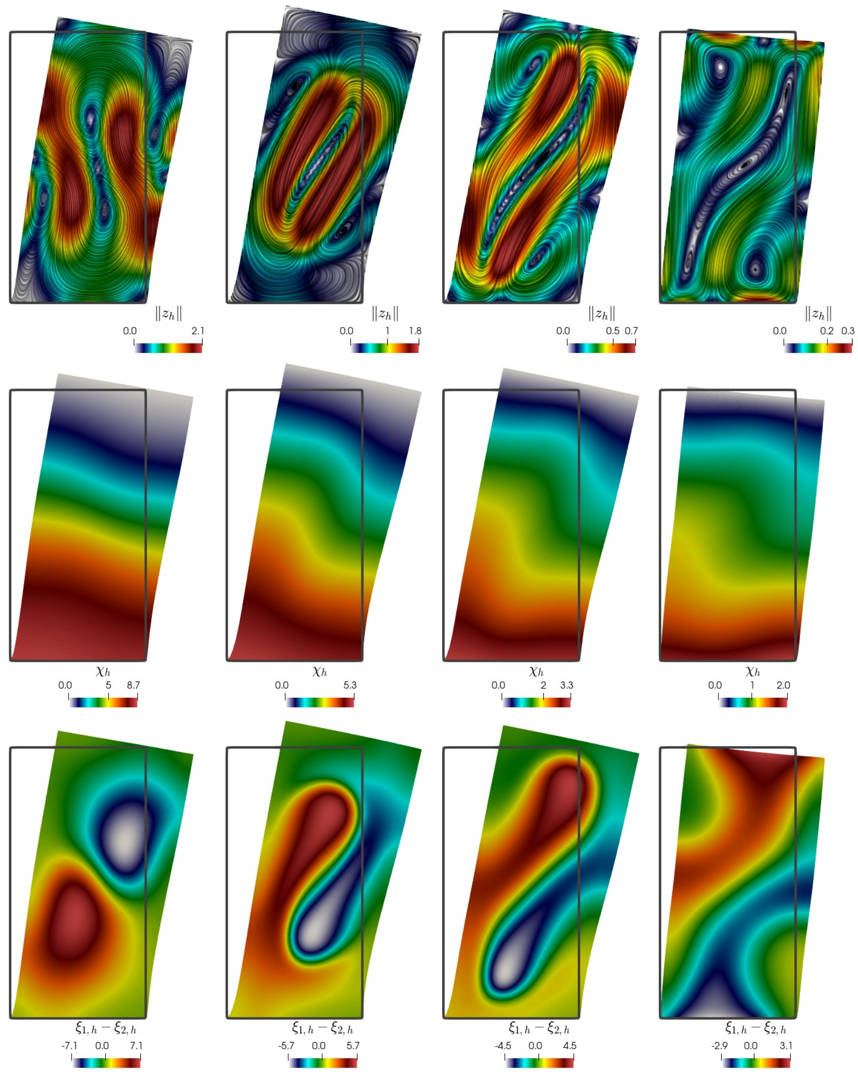

Figure 7.4: Example 3. Ion spreading in a charged deformable cell. Sample of approximate solutions at times $t = 0.1, 0.4, 0.8, 2$ (from left to right). We display the Darcy flux, electrostatic potential, and relative concentration on the deformed configuration.

on the boundary $\boldsymbol{\sigma}_i \cdot \boldsymbol{n} = 0$ (that is, the boundary is considered impenetrable for the ionic quantities), which is imposed essentially. For the electrostatic sub-system we consider two separate sub-boundaries: on the top segment ($y = 2$) we prescribe a given potential χ_0 (representing a ground condition, imposed naturally), on the vertical walls of the reservoir we set zero normal trace of the electric field $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$, and the bottom segment is regarded as a positively charged surface $\boldsymbol{\varphi} \cdot \boldsymbol{n} = s_E$ (the two last conditions are imposed essentially).

Note that just for this test, the drag due to electric field and concentration difference is considered as a right-hand side of the Darcy momentum equation. Also, for this test we consider the time-dependent version of the equations and so we include the term $\frac{1}{\Delta t}\frac{\alpha}{\lambda}(\theta^{m+1} - \theta^m) - \frac{1}{\Delta t}(c_0 + \frac{\alpha^2}{\lambda})(c^{m+1} - c^m)$ in the mass conservation equation and the terms $-\frac{1}{\Delta t}(\xi_1^{m+1} - \xi_1^m) - \frac{1}{\Delta t}(\xi_2^{m+1} - \xi_2^m)$ in the two ion conservation equations, where the superscripts $m, m+1$ denote approximations at time instants t^m, t^{m+1} using backward Euler's method. For this we take a constant time step $\Delta t = 0.01$ and run the system until the final time $t = 2$. The initial pressure and total pressure are zero and the initial concentrations of positively and negatively charged particles are as follows

$$\xi_{i,0}(\boldsymbol{x}) = \frac{\hat{\xi}_0}{2\pi R^2} \exp\left\{-\frac{(x-\frac{1}{2}+\frac{q_i}{8})^2+(y-1+\frac{q_i}{2})^2}{2R^2}\right\},$$

and the remaining parameters adopt the values (all adimensional) $c_0 = 0.01$, $\alpha = 0.9$, $\mu = 10$, $\lambda = 1000$, $\varepsilon = 0.5$, $\nu = 0.08$, $\kappa_1 = \kappa_2 = 0.01$, $s_E = 1$, $\chi_0 = 0$, $\hat{\xi}_0 = 3$, $R = \frac{1}{4}$.

Snapshots of the approximate solutions, computed using the lowest order method with $k = 0$, and taken at four time instants are shown in Figure 7.4. We plot the net charge (difference between concentrations of ionic concentrations), the line integral convolution of the relative fluid velocity, and the electrostatic potential.

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