Divergence-conforming methods for transient double-diffusive flows:
A priori and *a posteriori* error analysis†

RAIMUND BÜRGER‡

CI²MA and Departamento de Ingeniería Matemática, Universidad de Concepción,
Casilla 160-C, Concepción, Chile

ARBAZ KHAN§

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, India

PAUL E. MÉNDEZ¶

Research Centre on Mathematical Modelling (MODEMAT), Escuela Politécnica Nacional, Quito,
Ecuador

AND

RICARDO RUIZ-BAIER‖

School of Mathematics, Monash University, 9 Rainforest Walk, Melbourne, VIC 3800, Australia; and
World-Class Research Center "Digital biodesign and personalized healthcare", Sechenov First Moscow
State Medical University, Moscow, Russia; and Universidad Adventista de Chile, Casilla 7-D, Chillán,
Chile

The analysis of an $\mathbf{H}(\mathrm{div})$-conforming method for a model of double-diffusive flow in porous media introduced in [Bürger, Méndez, Ruiz-Baier, SINUM (2019), 57:1318–1343] is extended to the time-dependent case. In addition, the efficiency and reliability of residual-based *a posteriori* error estimators for the steady, semi-discrete, and fully discrete problems are established. The resulting methods are applied to simulate the sedimentation of small particles in salinity-driven flows. The method consists of Brezzi–Douglas–Marini approximations for velocity and compatible piecewise discontinuous pressures, whereas Lagrangian elements are used for concentration and salinity distribution. Numerical tests confirm the properties of the proposed family of schemes and of the adaptive strategy guided by the *a posteriori* error indicators.

Keywords: *A priori* and *a posteriori* error bounds; Mixed $\mathbf{H}(\mathrm{div})$-conforming methods; Coupled Navier–Stokes and double-diffusion; Sedimentation and salinity variations.

‡Email: rburger@ing-mat.udec.cl
§Email: arbaz@ma.iitr.ac.in
¶Email: paul.mendez01@epn.edu.ec
‖Author for correspondence. Email: ricardo.ruizbaier@monash.edu

1. Introduction and problem formulation

1.1 *Scope*

A number of physical problems of relevance in industrial applications involve coupled incompressible flow and double-diffusion transport. We are interested in numerical schemes for the approximation of a class of coupled equations that arise as models of sedimentation of small particles under the effect of salinity of the ambient fluid. The governing model can be stated as follows (cf., e.g., Burns & Meiburg (2015); Reali *et al.* (2017)):

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = \mathbf{div}\big(\nu(c)\boldsymbol{\nabla}\boldsymbol{u}\big) - \frac{1}{\rho_{\mathrm{m}}}\nabla p + \frac{\rho}{\rho_{\mathrm{m}}}\boldsymbol{g}, \tag{1.1a}$$

$$\mathrm{div}\,\boldsymbol{u} = 0, \tag{1.1b}$$

$$\partial_t s + \boldsymbol{u} \cdot \nabla s = \frac{1}{\mathrm{Sc}}\Delta s, \tag{1.1c}$$

$$\partial_t c + (\boldsymbol{u} - v_{\mathrm{p}}\boldsymbol{e}_z) \cdot \nabla c = \frac{1}{\tau\mathrm{Sc}}\Delta c, \tag{1.1d}$$

posed on a spatial domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, where $t \in (0, t_{\mathrm{end}}]$ is time, $\boldsymbol{u}$ is the fluid velocity, ν is the concentration-dependent viscosity, ρ_{m} is the mean density of the fluid, p is the fluid pressure, ρ is density, $\boldsymbol{g}$ is the gravity acceleration, s is the salinity concentration, and c is the concentration of solid particles. These equations state momentum (1.1a), and mass (1.1b) conservation of the fluid, and transport of the species (1.1c)-(1.1d). The Schmidt number $\mathrm{Sc} = v_{\mathrm{ref}}/\kappa_{\mathrm{s}}$ is the ratio between kinematic viscosity and diffusivity (here assumed relatively small, e.g. $\mathrm{Sc} = \mathscr{O}(10)$, so that concentration and salinity boundary layers are fully resolved; see, e.g., Kadoch *et al.*, 2012; Rasthofer & Gravemeier, 2018), where κ_{s} is the diffusivity of salinity, and v_{ref} is a reference viscosity in the absence of solid particles. Finally, $\tau = \kappa_{\mathrm{s}}/\kappa_{\mathrm{c}}$ is the inverse of the diffusivity ratio, where κ_{c} is the diffusivity of solid particles, and $\boldsymbol{e}_z$ is the upward-pointing unit vector. We relate the densities through a linearised equation of state

$$\rho = \rho_{\mathrm{m}}(\alpha s + \beta c),$$

where α and β are model constants that approximate the derivatives of the equation of state. Again as in the works Burns & Meiburg (2015); Reali *et al.* (2017); Ruiz-Baier & Lunati (2016), the solid particles are assumed to be mono-sized with radius r, and they settle at dimensionless velocity $v_{\mathrm{p}} = \frac{v_{\mathrm{St}}}{(v_{\mathrm{ref}}g')^{1/3}}$, where $v_{\mathrm{St}} = \frac{2r^2(\rho_{\mathrm{p}} - \rho_{\mathrm{m}})g}{9\rho_{\mathrm{m}}v_{\mathrm{ref}}}$ is the Stokes velocity (settling velocity of a single particle in an unbounded fluid). The coupling mechanisms between flow and transport are only due to advection for concentration and salinity (where the advecting velocity for concentration, $\boldsymbol{u} - v_{\mathrm{p}}\boldsymbol{e}_z$, is also divergence-free), and through the concentration-dependent viscosity. Further details (including boundary and initial conditions) are provided in later parts of the paper.

To put the paper further into the proper perspective, we mention that there exists an abundant body of literature devoted to constructing accurate finite element and related schemes for double-diffusive flows. Some recent contributions include variational multiscale stabilised schemes, least-squares methods, divergence-conforming mixed methods, volume-averaging discretisations, spectral elements, vorticity-based finite element formulations, and similar methods applied to, e.g., flows with heat and mass transport (Abedi & Aliabadi, 2003), reactive Boussinesq flows (Agouzal & Allali, 2003), nonlinear advection-reaction-diffusion in the context of bioconvective flows (Anaya *et al.*, 2018; Lenarda *et al.*, 2017), cross-diffusion and boundary layer effects in double-diffusive Navier–Stokes–Brinkman equations (Baird *et al.*, 2021; Bürger *et al.*, 2019; Dallmann & Arndt, 2016) and in Darcy–Brinkman equations (Yang & Jiang, 2018), or phase change models (Danaila *et al.*, 2014; Woodfield *et al.*, 2019; Zabaras & Samanta, 2004; Zimmerman & Kowalski, 2018); where the list is far from exhaustive.

The solvability analysis for the continuous and discrete problems usually follows energy and fixed-point schemes as done for classical Boussinesq equations, and this is also the approach we follow here. The discretisation in space

uses an interior penalty divergence-conforming method for the flow equations (in this case, Brezzi–Douglas–Marini (BDM) elements of degree $k \geqslant 1$ for the velocity and discontinuous elements of degree $k - 1$ for the pressure following Arnold *et al.* (2001); Könnö & Stenberg (2011)), combined with Lagrangian elements for the diffusive quantities. The present formulation and its analysis stand as a natural extension of the formulation in Bürger *et al.* (2019) to the transient case. As such, the proposed method also features exactly divergence-free velocity approximations ensuring local conservativity and energy stability, and the error estimates of velocity are pressure-robust. The chosen time discretisation is the backward differentiation formula of degree 2 (BDF2), which for $k = 2$ gives a method of order 2 in space and time. Existence of discrete solutions is established by the Brouwer fixed-point theory similarly as in Bürger *et al.* (2019), and the error analysis in the semi-discrete and fully-discrete settings is adapted from the theory of Aldbaissy *et al.* (2018) for the Boussinesq equations and that of Baird *et al.* (2021) for filtration in axisymmetric domains.

In many applications where double-diffusion effects occur, complicated flow patterns exist in zones far from boundary layers and sufficiently refined meshes are needed essentially in the whole spatial domain. However, for salinity-driven settling of solid particles that result in mathematical models such as (1.1), many of the flow features are clustered near zones of high gradients of concentration, which is where the typical *plumes* are observed (Burns & Meiburg, 2015; Lenarda *et al.*, 2017). This motivates the use of adaptive mesh refinement guided by *a posteriori* error indicators. For instance, in the context of phase change models there are some results based on error-related metric change (Danaila *et al.*, 2014; Rakotondrandisa *et al.*, 2020) and on goal-oriented adaptivity (Zimmerman & Kowalski, 2018). Regarding the design and rigorous analysis of residual-based *a posteriori* error estimators for flow-transport couplings, the literature is predominantly focused on the stationary case (see, e.g., Agroum (2017); Allali (2005); Allendes *et al.* (2020); Alvarez *et al.* (2016, 2018); Becker & Braack (2002); Dib *et al.* (2019); Wilfrid (2019); Zhang *et al.* (2011) and the references therein). Only a few results are available for the time-dependent regime, from which we mention the adaptive mixed method for Richards equation in porous media (Bernardi *et al.*, 2014), the remeshing scheme based on goal-oriented adaptivity for solidification problems advanced in Belhamadia *et al.* (2019), the collection of adaptive schemes for reactive flow discussed in Braack & Richter (2007) and for heat transfer in Larson *et al.* (2008). However, none of these theoretical frameworks directly handles divergence-conforming approximations to (1.1).

The *a posteriori* error analysis we advance here is of residual type, and its analysis uses ideas from the abstract results related to spatial estimators for discontinuous Galerkin schemes applied to parabolic problems in Georgoulis *et al.* (2011). The approach hinges on a decomposition of the discrete solution into a conforming and a non-conforming contribution, along with a reconstruction technique (see also Memon *et al.*, 2012). This has also been exploited for the construction of *a posteriori* estimators of time-dependent Stokes and Navier–Stokes equations (Bänsch & Brenner, 2019; Zhang & Li, 2017). Our *a posteriori* error analysis is divided into three parts. In the first part, we present the error estimator for the steady coupled problem, under the assumption of $\mathbf{L}^2$ distributed force and constant source terms. In second part, we extend the *a posteriori* error estimation to the semi-discrete method, and finally we present the *a posteriori* error estimator for the unsteady coupled problem. For the sake of simplicity, we restrict the latter part of the analysis to the backward Euler method. To the best of our knowledge, the *a posteriori* error estimation advanced herein is the first comprehensive study targeted for transient double-diffusive flows.

1.2 *Outline*

The remainder of this paper is organised as follows. In what is left of this section we outline the weak formulation of (1.1), state preliminary results and notation to be used in throughout the manuscript, and announce the stability of the continuous problem. In Section 2 we introduce the divergence-conforming method in fully discrete form, show existence of discrete solutions using fixed-point arguments, and rigorously establish *a priori* error estimates. Section 3 is devoted to the construction and analysis of efficiency and reliability for a residual-based *a posteriori* error estimator tailored for the stationary problem. In turn, these upper and lower bounds are used to establish properties of a second family of estimators for the transient case, and addressed in Sections 4 and 5. In Section 6 we collect numerical tests that verify the theoretical convergence rates predicted by the *a priori* error analysis, confirm the robustness of the proposed *a posteriori* error estimators, and illustrate the use of adaptive methods in the simulation of double-diffusive flows.

1.3 *Preliminaries*

Let Ω be an open and bounded domain in $\mathbb{R}^d$, $d = 2, 3$ with Lipschitz boundary $\Gamma = \partial\Omega$. We denote by $L^p(\Omega)$ and $W^{r,p}(\Omega)$ the usual Lebesgue and Sobolev spaces with respective norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{r,p}(\Omega)}$. If $p = 2$ we write $H^r(\Omega)$ in place of $W^{r,p}(\Omega)$, and denote the corresponding norm by $\|\cdot\|_{r,\Omega}$, ($\|\cdot\|_{0,\Omega}$ for $H^0(\Omega) = L^2(\Omega)$). The space $L_0^2(\Omega)$ denotes the restriction of $L^2(\Omega)$ to functions with zero mean value over Ω. For $r \geqslant 0$, we write the H^r-seminorm as $|\cdot|_{r,\Omega}$ and we denote by $(\cdot,\cdot)_\Omega$ the usual inner product in $L^2(\Omega)$. Spaces of vector-valued functions (in dimension d) are denoted in bold face, i.e., $\mathbf{H}^r(\Omega) = [H^r(\Omega)]^d$, and we use the vector-valued Hilbert spaces

$$\mathbf{H}(\mathrm{div};\Omega) := \left\{ \boldsymbol{w} \in \mathbf{L}^2(\Omega) : \mathrm{div}\,\boldsymbol{w} \in L^2(\Omega) \right\},$$
$$\mathbf{H}_0(\mathrm{div};\Omega) := \left\{ \boldsymbol{w} \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{w}\cdot\boldsymbol{n}_{\partial\Omega} = 0 \text{ on } \partial\Omega \right\},$$
$$\mathbf{H}_0(\mathrm{div}^0;\Omega) := \left\{ \boldsymbol{w} \in \mathbf{H}_0(\mathrm{div};\Omega) : \mathrm{div}\,\boldsymbol{w} = 0 \text{ in } \Omega \right\},$$

where $\boldsymbol{n}_{\partial\Omega}$ denotes the outward normal on $\partial\Omega$; and we endow these spaces with the norm $\|\cdot\|_{\mathrm{div},\Omega}$ defined by $\|\boldsymbol{w}\|_{\mathrm{div},\Omega}^2 := \|\boldsymbol{w}\|_{0,\Omega}^2 + \|\mathrm{div}\,\boldsymbol{w}\|_{0,\Omega}^2$. We denote by $L^s(0,t_{\mathrm{end}};W^{m,p}(\Omega))$ the Banach space of all L^s-integrable functions from $[0,t_{\mathrm{end}}]$ into $W^{m,p}(\Omega)$, with norm

$$\|v\|_{L^s(0,t_{\mathrm{end}};W^{m,p}(\Omega))} := \begin{cases} \left(\displaystyle\int_0^{t_{\mathrm{end}}} \|v(t)\|_{W^{m,p}(\Omega)}^s \, \mathrm{d}t \right)^{1/s} & \text{if } 1 \leqslant s < \infty, \\ \mathrm{ess\,sup}_{t\in[0,t_{\mathrm{end}}]} \|v(t)\|_{W^{m,p}(\Omega)} & \text{if } s = \infty. \end{cases}$$

1.4 *Additional assumptions and weak formulation*

As in, e.g., Dib *et al.* (2019), we assume that viscosity is a Lipschitz continuous and uniformly bounded function of concentration, i.e.,

$$\left| \nu(c_1) - \nu(c_2) \right| \leqslant L_\nu |c_1 - c_2| \quad \text{and} \quad \nu_1 \leqslant \nu(c) \leqslant \nu_2,$$

for any $c, c_1, c_2 \in \mathbb{R}$, and where L_ν, ν_1, ν_2 are positive constants.

For simplicity of notation in presenting the analysis we restrict the weak form to the case of homogeneous Dirichlet boundary conditions for velocity, concentration, and salinity:

$$\boldsymbol{u} = \boldsymbol{0}, \quad s = 0, \quad c = 0 \qquad \text{on } \partial\Omega.$$

Let us define the following spaces

$$\mathbf{V}^t := \left\{ \boldsymbol{w} \in \mathbf{L}^2(0,t_{\mathrm{end}};\mathbf{H}_0^1(\Omega)) : \partial_t \boldsymbol{w} \in L^2(0,t_{\mathrm{end}};\mathbf{L}^2(\Omega)) \right\},$$
$$Q^t := L^2(0,t_{\mathrm{end}};L_0^2(\Omega)),$$
$$M^t := \left\{ s \in L^2(0,t_{\mathrm{end}};H_0^1(\Omega)) : \partial_t s \in L^2(0,t_{\mathrm{end}};L^2(\Omega)) \right\}.$$

For ease of the subsequent presentation, as in Schroeder *et al.* (2018), velocity, pressure, concentration and salinity solutions will be assumed to belong to the spaces $\mathbf{V}^t$, Q^t, M^t and M^t, respectively. Testing each equation in problem (1.1) against suitable functions and integrating by parts whenever adequate, gives the following weak formulation: Find $(\boldsymbol{u}, p, s, c) \in \mathbf{V}^t \times Q^t \times M^t \times M^t$ such that $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \in \mathbf{H}_0(\mathrm{div}^0;\Omega)$, $s(\cdot,0) = 0$, $c(\cdot,0) = 0$ in Ω and for a.e. $t \in [0,t_{\mathrm{end}}]$,

$$(\partial_t \boldsymbol{u}, \boldsymbol{v})_\Omega + a_1(c; \boldsymbol{u}, \boldsymbol{v}) + c_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = F(s, c, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \tag{1.2a}$$

$$b(\boldsymbol{u}, q) = 0 \quad \text{for all } q \in L_0^2(\Omega), \tag{1.2b}$$

$$(\partial_t s, \varphi)_\Omega + a_2(s, \varphi) + c_2(\boldsymbol{u}; s, \varphi) = 0 \quad \text{for all } \varphi \in H_0^1(\Omega), \tag{1.2c}$$

$$(\partial_t c, \psi)_\Omega + \frac{1}{\tau} a_2(c, \psi) + c_2(\boldsymbol{u} - v_{\mathrm{p}}\boldsymbol{e}_z; c, \psi) = 0 \quad \text{for all } \psi \in H_0^1(\Omega), \tag{1.2d}$$

where the variational forms $a_1 : H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$, $a_2 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, $b : \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \to \mathbb{R}$, $c_1 : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$, $c_2 : \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, $F : \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, are defined as follows for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$, $q \in L_0^2(\Omega)$, and $\varphi, \psi \in H_0^1(\Omega)$:

$$a_1(c; \boldsymbol{u}, \boldsymbol{v}) := \big(\nu(c)\boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\nabla}\boldsymbol{v}\big)_\Omega, \quad c_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) := \big((\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \boldsymbol{v}\big)_\Omega, \quad F(s, c, \boldsymbol{v}) := \big((\alpha s + \beta c)\boldsymbol{g}, \boldsymbol{v}\big)_\Omega,$$

$$b(\boldsymbol{v}, q) := -\frac{1}{\rho_{\mathrm{m}}}(q, \operatorname{div}\boldsymbol{v})_\Omega, \quad a_2(\varphi, \psi) := \frac{1}{\mathrm{Sc}}(\nabla\varphi, \nabla\psi)_\Omega, \quad c_2(\boldsymbol{v}; \varphi, \psi) := \big((\boldsymbol{v}\cdot\nabla)\varphi, \psi\big)_\Omega.$$

1.5 *Stability of the continuous problem*

We begin by noting that the variational forms defined above are continuous for all $\boldsymbol{u}, \boldsymbol{v}, \in \mathbf{H}_0^1(\Omega)$, $q \in L_0^2(\Omega)$, and $\varphi, \psi \in H_0^1(\Omega)$:

$$\big|a_1(\cdot, \boldsymbol{u}, \boldsymbol{v})\big| \leqslant C_a \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{v}\|_{1,\Omega}, \qquad \big|a_2(\varphi, \psi)\big| \leqslant \hat{C}_a \|\varphi\|_{1,\Omega} \|\psi\|_{1,\Omega}, \tag{1.3a}$$

$$\big|b(\boldsymbol{v}, q)\big| \leqslant C_b \|\boldsymbol{v}\|_{1,\Omega} \|q\|_{0,\Omega}, \tag{1.3b}$$

$$\big|c_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})\big| \leqslant C_c \|\boldsymbol{w}\|_{1,\Omega} \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{v}\|_{1,\Omega}, \qquad \big|c_2(\boldsymbol{u}; \varphi, \psi)\big| \leqslant \hat{C}_c \|\boldsymbol{u}\|_{1,\Omega} \|\varphi\|_{1,\Omega} \|\psi\|_{1,\Omega}. \tag{1.3c}$$

We also recall (from Girault *et al.*, 2005, Chapter I, Lemma 3.1, for instance) the following Poincaré–Friedrichs inequality:

$$\|\varphi\|_{0,\Omega} \leqslant C_p |\varphi|_{1,\Omega} \qquad \text{for all } \varphi \in H_0^1(\Omega). \tag{1.4}$$

Next, inequality (1.4) readily gives the coercivity of a_2 and also, for a fixed concentration, that of a_1, i.e., there exist positive constants α_a and $\hat{\alpha}_a$ such that

$$a_1(\cdot, \boldsymbol{v}, \boldsymbol{v}) \geqslant \alpha_a \|\boldsymbol{v}\|_{1,\Omega}^2 \qquad \text{for all } \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \tag{1.5a}$$

$$a_2(\varphi, \varphi) \geqslant \hat{\alpha}_a \|\varphi\|_{1,\Omega}^2 \qquad \text{for all } \varphi \in H_0^1(\Omega). \tag{1.5b}$$

Using the definition and characterisation of the kernel of the bilinear form $b(\cdot, \cdot)$, we can write

$$\mathbf{X} := \big\{\boldsymbol{v} \in \mathbf{H}_0^1(\Omega) : b(\boldsymbol{v}, q) = 0 \quad \forall q \in L_0^2(\Omega)\big\} = \big\{\boldsymbol{v} \in \mathbf{H}_0^1(\Omega) : \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega\big\},$$

and applying integration by parts we can readily observe that

$$c_1(\boldsymbol{w}; \boldsymbol{v}, \boldsymbol{v}) = 0 \quad \text{and} \quad c_2(\boldsymbol{w}; \varphi, \varphi) = 0 \quad \text{for all } \boldsymbol{w} \in \mathbf{X}, \boldsymbol{v} \in \mathbf{H}^1(\Omega), \varphi \in H^1(\Omega). \tag{1.6}$$

It is well known that the bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition (see, e.g., Temam, 2001):

$$\sup_{\boldsymbol{v} \in \mathbf{H}_0^1(\Omega) \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\Omega}} \geqslant \beta \|q\|_{0,\Omega} \quad \text{for all } q \in L_0^2(\Omega).$$

Finally, for $\boldsymbol{v} \in \mathbf{W}^{1,\infty}(\Omega)$ and $\varphi \in W^{1,\infty}(\Omega)$, one can show that there exists an embedding constant $C_\infty > 0$ such that

$$\|\boldsymbol{v}\|_{1,\Omega} \leqslant C_\infty \|\boldsymbol{v}\|_{\mathbf{W}^{1,\infty}(\Omega)} \quad \text{and} \quad \|\varphi\|_{1,\Omega} \leqslant C_\infty \|\varphi\|_{W^{1,\infty}(\Omega)}.$$

LEMMA 1.1 (Stability) If $\boldsymbol{g} \in L^\infty(0, t_{\mathrm{end}}; \mathbf{L}^\infty(\Omega))$, $\boldsymbol{u}_0 \in \mathbf{L}^2(\Omega)$ and $s_0, c_0 \in L^2(\Omega)$, then, for any solution $\boldsymbol{u}, s, c$ of (1.2) and for $t \in (0, t_{\mathrm{end}}]$, there exists a constant $\gamma > 0$ such that

$$\|\boldsymbol{u}\|_{L^2(0,t;\mathbf{H}^1(\Omega))} + \|s\|_{L^2(0,t;H^1(\Omega))} + \|c\|_{L^2(0,t;H^1(\Omega))} \leqslant \gamma\big(\|\boldsymbol{u}_0\|_{0,\Omega} + \|s_0\|_{0,\Omega} + \|c_0\|_{0,\Omega}\big),$$

where γ might depend on η_1, τ, Sc, ρ, ρ_{m}, C_p, $\|\boldsymbol{g}\|_{\infty,\Omega}$, α, and β.

Proof. We can take $\boldsymbol{u} \in \mathbf{X}$ and due to the inf-sup condition we can solve an equivalent reduced problem where $b(\cdot,\cdot)$ is removed from the variational form (1.2). Setting $\boldsymbol{v} = \boldsymbol{u}$ and using (1.6), (1.5a), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{0,\Omega}^2 + \alpha_a\|\boldsymbol{u}\|_{1,\Omega}^2 \leqslant \|\boldsymbol{g}\|_{\infty,\Omega}C\big(\|s\|_{0,\Omega} + \|c\|_{0,\Omega}\big)\|\boldsymbol{u}\|_{0,\Omega}.$$

Now we use Young's inequality with $\varepsilon = \alpha_a/4$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{0,\Omega}^2 + \frac{\alpha_a}{2}\|\boldsymbol{u}\|_{1,\Omega}^2 \leqslant \frac{C\|\boldsymbol{g}\|_{\infty,\Omega}^2}{\alpha_a}\big(\|s\|_{0,\Omega}^2 + \|c\|_{0,\Omega}^2\big).$$

Integrating this inequality between 0 and t yields, in particular

$$\|\boldsymbol{u}(\cdot,t)\|_{0,\Omega}^2 + \alpha_a\int_0^t\|\boldsymbol{u}(\cdot,z)\|_{1,\Omega}^2\,\mathrm{d}z \leqslant \|\boldsymbol{u}(\cdot,0)\|_{0,\Omega}^2 + \frac{C}{\alpha_a}\int_0^t\|c(\cdot,z)\|_{0,\Omega}^2\,\mathrm{d}z + \frac{C}{\alpha_a}\int_0^t\|s(\cdot,z)\|_{0,\Omega}^2\,\mathrm{d}z. \qquad (1.7)$$

Analogously, applying (1.5b) and (1.6) to (1.2c) and (1.2d), after integrating from 0 to t we find that

$$\|s(\cdot,t)\|_{0,\Omega}^2 + 2\hat{\alpha}_a\int_0^t\|s(\cdot,z)\|_{1,\Omega}^2\,\mathrm{d}z \leqslant \|s(\cdot,0)\|_{0,\Omega}^2, \qquad (1.8a)$$

$$\|c(\cdot,t)\|_{0,\Omega}^2 + 2\hat{\alpha}_a\int_0^t\|c(\cdot,z)\|_{1,\Omega}^2\,\mathrm{d}z \leqslant \|c(\cdot,0)\|_{0,\Omega}^2. \qquad (1.8b)$$

Finally, we derive the sought result from (1.7), (1.8a) and (1.8b). □

A problem similar to (1.2) was studied by Agroum *et al.* (2015). Assuming that $F \in L^2(0,t_{\mathrm{end}};\mathbf{H}^{-1}(\Omega))$, that the initial velocity $\boldsymbol{u}_0$ belongs to $\mathbf{L}^2(\Omega)$ and that the initial data for the coupled species (s and c in our context) belong to $L^2(\Omega)$, the authors showed existence of a solution by using the Galerkin method and applying the Cauchy–Lipschitz theorem, and proved its uniqueness in two dimensions. Such analysis can be applied to (1.2) by noting that F is a Lipschitz-continuous function of c and s, and assuming the initial data belong to appropriate spaces. This is, however, not the focus of the paper and, instead, we move on to the discrete analysis.

2. Finite element discretisation and *a priori* error bounds

2.1 *Preliminaries*

We discretise in space by a family of regular partitions, denoted $\mathcal{T}_h$, of $\Omega \subset \mathbb{R}^d$ into simplices K (triangles in 2D or tetrahedra in 3D) of diameter h_K. We label by K^- and K^+ the two elements adjacent to a facet (an edge in 2D or a face in 3D), while h_e stands for the maximum diameter of the facet. Let $\mathcal{E}_h$ denote the set of all facets and $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^\partial$ where $\mathcal{E}_h^i$ and $\mathcal{E}_h^\partial$ are the subset of interior facets and boundary facets, respectively. If $\boldsymbol{v}$ and w are smooth vector and scalar fields defined on $\mathcal{T}_h$, then $(\boldsymbol{v}^\pm, w^\pm)$ denote the traces of $(\boldsymbol{v}, w)$ on e that are the extensions from the interior of K^+ and K^-, respectively. Let $\boldsymbol{n}_e^+, \boldsymbol{n}_e^-$ be the outward unit normal vectors on the boundaries of two neighbouring elements sharing the facet e, K^+ and K^-, respectively. We also use the notation $(\boldsymbol{w}_e \cdot \boldsymbol{n}_e)|_e = (\boldsymbol{w}^+ \cdot \boldsymbol{n}_e^+)|_e$. The average $\{\!\{\cdot\}\!\}$ and jump $[\![\cdot]\!]$ operators are defined as

$$\{\!\{\boldsymbol{v}\}\!\} := (\boldsymbol{v}^- + \boldsymbol{v}^+)/2, \quad \{\!\{w\}\!\} := (w^- + w^+)/2, \quad [\![\boldsymbol{v}]\!] := (\boldsymbol{v}^- - \boldsymbol{v}^+), \quad [\![w]\!] := (w^- - w^+),$$

whereas for boundary jumps and averages we adopt the conventions $\{\!\{\boldsymbol{v}\}\!\} = [\![\boldsymbol{v}]\!] = \boldsymbol{v}$, and $\{\!\{w\}\!\} = [\![w]\!] = w$. In addition, ∇_h will denote the broken gradient operator.

2.2 *Galerkin method*

For $k \geqslant 1$ and a mesh $\mathcal{T}_h$ on Ω, let us consider the discrete spaces (see e.g. Brezzi *et al.*, 1985)

$$\mathbf{V}_h := \big\{\boldsymbol{v}_h \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{v}_h|_K \in [\mathcal{P}_k(K)]^{\mathrm{d}} \quad \forall K \in \mathcal{T}_h; \quad \boldsymbol{v}_h \cdot \boldsymbol{n}_{\partial\Omega} = 0 \text{ on } \partial\Omega\big\},$$

$$\mathbf{V}_h^t := \left\{ \boldsymbol{v}_h \in \mathbf{L}^2(0, t_{\text{end}}; \mathbf{V}_h) : \partial_t \boldsymbol{v}_h \in \mathbf{L}^2(0, t_{\text{end}}; \mathbf{V}_h) \right\},$$

$$\mathscr{Q}_h := \left\{ q_h \in L_0^2(\Omega) : q_h|_K \in \mathscr{P}_{k-1}(K) \quad \forall K \in \mathscr{T}_h \right\}, \qquad \mathscr{Q}_h^t := L^2(0, t_{\text{end}}; \mathscr{Q}_h),$$

$$\mathscr{M}_h := \left\{ s_h \in C(\bar{\Omega}) : l_h|_K \in \mathscr{P}_k(K) \quad \forall K \in \mathscr{T}_h \right\}, \qquad \mathscr{M}_{h,0} := \mathscr{M}_h \cap H_0^1(\Omega),$$

$$\mathscr{M}_{h,0}^t := \left\{ s_h \in L^2(0, t_{\text{end}}; \mathscr{M}_{h,0}) : \partial_t s_h \in L^2(0, t_{\text{end}}; \mathscr{M}_{h,0}) \right\},$$

which, in particular, satisfy $\operatorname{div} \mathbf{V}_h \subset \mathscr{Q}_h$ (cf. Könnö & Stenberg, 2011). Here $\mathscr{P}_k(K)$ denotes the local space spanned by polynomials of degree up to k and $\mathbf{V}_h$ is the space of divergence-conforming BDM elements. Associated with these finite-dimensional spaces, we state the following semi-discrete Galerkin formulation for problem (1.2): Find $(\boldsymbol{u}_h, p_h, s_h, c_h) \in \mathbf{V}_h^t \times \mathscr{Q}_h^t \times \mathscr{M}_{h,0}^t \times \mathscr{M}_{h,0}^t$ such that:

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h)_\Omega + a_1^h(c_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + c_1^h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = F(s_h, c_h, \boldsymbol{v}_h) \quad \text{for all } \boldsymbol{v}_h \in \mathbf{V}_h,$$

$$b(\boldsymbol{u}_h, q_h) = 0 \quad \text{for all } q_h \in \mathscr{Q}_h,$$

$$(\partial_t s_h, \varphi_h)_\Omega + a_2(s_h, \varphi_h) + c_2(\boldsymbol{u}_h; s_h, \varphi_h) = 0 \quad \text{for all } \varphi_h \in \mathscr{M}_{h,0},$$

$$(\partial_t c_h, \psi_h)_\Omega + \frac{1}{\tau} a_2(c_h, \psi_h) + c_2(\boldsymbol{u}_h - v_{\text{p}} \boldsymbol{e}_z; c_h, \psi_h) = 0 \quad \text{for all } \psi_h \in \mathscr{M}_{h,0}. \tag{2.1}$$

The discrete versions of the variational forms $a_1^h(\cdot; \cdot, \cdot)$ and $c_1^h(\cdot; \cdot, \cdot)$ are defined using a symmetric interior penalty and an upwind approach, respectively (see, e.g., Arnold *et al.*, 2001; Könnö & Stenberg, 2011):

$$a_1^h(c_h; \boldsymbol{u}_h, \boldsymbol{v}_h) := \int_\Omega \nu(c_h) \boldsymbol{\nabla}_h(\boldsymbol{u}_h) : \boldsymbol{\nabla}_h(\boldsymbol{v}_h) \, \mathrm{d}\boldsymbol{x} + \sum_{e \in \mathscr{E}_h} \int_e \left(-\{\!\!\{\nu(c_h) \boldsymbol{\nabla}_h(\boldsymbol{u}_h) \boldsymbol{n}_e\}\!\!\} \cdot [\![\boldsymbol{v}_h]\!] \right.$$

$$\left. - \{\!\!\{\nu(c_h) \boldsymbol{\nabla}_h(\boldsymbol{v}_h) \boldsymbol{n}_e\}\!\!\} \cdot [\![\boldsymbol{u}_h]\!] + \frac{a_0}{h_e} \nu(c_h) [\![\boldsymbol{u}_h]\!] \cdot [\![\boldsymbol{v}_h]\!] \right) \mathrm{d}S, \tag{2.2a}$$

$$c_1^h(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) := \int_\Omega (\boldsymbol{w}_h \cdot \nabla) \boldsymbol{u}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} + \sum_{e \in \mathscr{E}_h^i} \int_e (\boldsymbol{w}_e \cdot \boldsymbol{n}_e) [\![\boldsymbol{u}_h]\!] \cdot \{\!\!\{\boldsymbol{v}_h\}\!\!\} \, \mathrm{d}S + \sum_{e \in \mathscr{E}_h^i} \int_e \frac{1}{2} |\boldsymbol{w}_e \cdot \boldsymbol{n}_e| [\![\boldsymbol{u}_h]\!] \cdot [\![\boldsymbol{v}_h]\!] \, \mathrm{d}S, \tag{2.2b}$$

where $a_0 > 0$ is a jump penalisation parameter.

We partition the interval $[0, t_{\text{end}}]$ into N subintervals $[t_{n-1}, t_n]$ of length Δt. We use the implicit BDF2 scheme where all first-order time derivatives are approximated using the centred operator

$$\partial_t \boldsymbol{u}_h(t^{n+1}) \approx \frac{1}{\Delta t} \left(\frac{3}{2} \boldsymbol{u}_h^{n+1} - 2 \boldsymbol{u}_h^n + \frac{1}{2} \boldsymbol{u}_h^{n-1} \right), \tag{2.3}$$

(similarly for $\partial_t c$) and for the first time step a first-order backward Euler method is used from t^0 to t^1, starting from the interpolates $\boldsymbol{u}_h^0, s_h^0$ of the initial data:

$$\left(\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0, \boldsymbol{v}_h \right)_\Omega = \Delta t \left(-a_1^h(c_h^1; \boldsymbol{u}_h^1, \boldsymbol{v}_h) - c_1^h(\boldsymbol{u}_h^1; \boldsymbol{u}_h^1, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^1) + F(s_h^1, c_h^1, \boldsymbol{v}_h) \right) \quad \text{for all } \boldsymbol{v}_h \in \mathbf{V}_h,$$

$$b(\boldsymbol{u}_h^1, q_h) = 0 \quad \text{for all } q_h \in \mathscr{Q}_h,$$

$$\left(s_h^1 - s_h^0, \varphi_h \right)_\Omega = \Delta t \left(-a_2(s_h^1, \varphi_h) - c_2(\boldsymbol{u}_h^1; s_h^1, \varphi_h) \right) \quad \text{for all } \varphi_h \in \mathscr{M}_{h,0},$$

$$\left(c_h^1 - c_h^0, \psi_h \right)_\Omega = \Delta t \left(-\frac{1}{\tau} a_2(c_h^1, \psi_h) - c_2(\boldsymbol{u}_h^1 - v_{\text{p}} \boldsymbol{e}_z; c_h^1, \psi_h) \right) \quad \text{for all } \psi_h \in \mathscr{M}_{h,0}. \tag{2.4}$$

In what follows, we define the difference operator

$$\mathscr{D} y^{n+1} := 3 y^{n+1} - 4 y^n + y^{n-1},$$

for any quantity indexed by the time step n. For instance, (2.3) can be written as $\partial_t \boldsymbol{u}_h(t^{n+1}) \approx \frac{1}{2\Delta t} \mathscr{D} \boldsymbol{u}_h^{n+1}$.

The resulting set of nonlinear equations is solved by an iterative Newton–Raphson method with exact Jacobian. Hence for $1 \leqslant n \leqslant N-1$, the complete discrete system is given by

$$
\frac{1}{3}\big(\mathscr{D}\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h\big)_{\Omega} = \frac{2}{3}\Delta t\big(-a_1^h(c_h^{n+1};\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h) - c_1^h(\boldsymbol{u}_h^{n+1};\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h) - b(\boldsymbol{v}_h,p_h^{n+1}) + F(s_h^{n+1},c_h^{n+1},\boldsymbol{v}_h)\big)
$$
$$
\text{for all } \boldsymbol{v}_h \in \mathbf{V}_h,
$$
$$
b(\boldsymbol{u}_h^{n+1},q_h) = 0 \quad \text{for all } q_h \in \mathscr{Q}_h, \tag{2.5}
$$
$$
\frac{1}{3}\big(\mathscr{D}s_h^{n+1},\varphi_h\big)_{\Omega} = \frac{2}{3}\Delta t\big(-a_2(s_h^{n+1},\varphi_h) - c_2(\boldsymbol{u}_h^{n+1};s_h^{n+1},\varphi_h)\big) \quad \text{for all } \varphi_h \in \mathscr{M}_{h,0},
$$
$$
\frac{1}{3}\big(\mathscr{D}c_h^{n+1},\psi_h\big)_{\Omega} = \frac{2}{3}\Delta t\left(-\frac{1}{\tau}a_2(c_h^{n+1},\psi_h) - c_2(\boldsymbol{u}_h^{n+1} - v_{\mathrm{p}}\boldsymbol{e}_z;c_h^{n+1},\psi_h)\right) \quad \text{for all } \psi_h \in \mathscr{M}_{h,0}.
$$

2.3 *Properties of the discrete problem*

For the subsequent analysis, we introduce for $r \geqslant 0$ the broken $\mathbf{H}^r$ space

$$
\mathbf{H}^r(\mathscr{T}_h) := \big\{\boldsymbol{v} \in \mathbf{L}^2(\Omega) : \boldsymbol{v}|_K \in \mathbf{H}^r(K),\ K \in \mathscr{T}_h\big\}
$$

as well as the mesh-dependent broken norms

$$
\|\boldsymbol{v}\|_{*,\mathscr{T}_h}^2 := \sum_{K \in \mathscr{T}_h} \|\boldsymbol{\nabla}_h(\boldsymbol{v})\|_{\mathbf{L}^2(K)}^2 + \sum_{e \in \mathscr{E}_h} \frac{1}{h_e}\|[\![\boldsymbol{v}]\!]\|_{\mathbf{L}^2(e)}^2,
$$
$$
\|\boldsymbol{v}\|_{1,\mathscr{T}_h}^2 := \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\boldsymbol{v}\|_{*,\mathscr{T}_h}^2 \quad \text{for all } \boldsymbol{v} \in \mathbf{H}^1(\mathscr{T}_h), \quad \|\boldsymbol{v}\|_{2,\mathscr{T}_h}^2 := \|\boldsymbol{v}\|_{1,\mathscr{T}_h}^2 + \sum_{K \in \mathscr{T}_h} h_K^2 |\boldsymbol{v}|_{H^2(K)}^2 \quad \text{for all } \boldsymbol{v} \in \mathbf{H}^2(\mathscr{T}_h),
$$

where the stronger norm $\|\cdot\|_{2,\mathscr{T}_h}$ is used to show continuity. By using the inverse estimate

$$
|\boldsymbol{w}|_{2,K} \leqslant Ch_K^{-1}|\boldsymbol{w}|_{1,K} \quad \text{for all } K \in \mathscr{T}_h, \boldsymbol{w} \in [\mathscr{P}_k(K)]^{\mathrm{d}},
$$

it can be seen that this norm is equivalent to $\|\cdot\|_{1,\mathscr{T}_h}$ on $\mathbf{V}_h$ (see, e.g., Arnold *et al.*, 2001). We also define the discrete kernel of the bilinear form $b(\cdot,\cdot)$ as

$$
\mathbf{X}_h := \{\boldsymbol{v}_h \in \mathbf{V}_h : b(\boldsymbol{v}_h,q_h) = 0 \ \forall q_h \in \mathscr{Q}_h\} = \{\boldsymbol{v}_h \in \mathbf{V}_h : \mathrm{div}\,\boldsymbol{v}_h = 0 \text{ in } \Omega\}, \tag{2.6}
$$

Finally, adapting the argument used in (Karakashian & Jureidini, 1998, Proposition 4.5), we can state the following version of the discrete Sobolev embedding: for $r = 2,4$ there exists a constant $C_{\mathrm{emb}} > 0$ such that

$$
\|\boldsymbol{v}\|_{\mathbf{L}^r(\Omega)} \leqslant C_{\mathrm{emb}}\|\boldsymbol{v}\|_{1,\mathscr{T}_h} \quad \text{for all } \boldsymbol{v} \in \mathbf{H}^1(\mathscr{T}_h). \tag{2.7}
$$

With these norms, we can establish continuity of the trilinear and bilinear forms constituting the variational formulation. The proof follows from (Arnold *et al.*, 2001, Section 4).

LEMMA 2.1 The following properties hold:

$$
\begin{aligned}
\big|a_1^h(\cdot,\boldsymbol{u},\boldsymbol{v})\big| &\leqslant C\|\boldsymbol{u}\|_{2,\mathscr{T}_h}\|\boldsymbol{v}\|_{1,\mathscr{T}_h} && \text{for all } \boldsymbol{u} \in \mathbf{H}^2(\mathscr{T}_h),\ \boldsymbol{v} \in \mathbf{V}_h, \\
\big|a_1^h(\cdot,\boldsymbol{u},\boldsymbol{v})\big| &\leqslant \tilde{C}_a\|\boldsymbol{u}\|_{1,\mathscr{T}_h}\|\boldsymbol{v}\|_{1,\mathscr{T}_h} && \text{for all } \boldsymbol{u},\boldsymbol{v} \in \mathbf{V}_h, \\
\big|b(\boldsymbol{v},q)\big| &\leqslant \tilde{C}_b\|\boldsymbol{v}\|_{1,\mathscr{T}_h}\|q\|_{0,\Omega} && \text{for all } \boldsymbol{v} \in \mathbf{H}^1(\mathscr{T}_h),\ q \in L^2(\Omega),
\end{aligned}
$$

and for all $\boldsymbol{w} \in \mathbf{H}^1(\mathscr{T}_h)$ and $\varphi,\psi \in H^1(\Omega)$, there holds

$$
\big|c_2(\boldsymbol{w};\varphi,\psi)\big| \leqslant \tilde{C}\|\boldsymbol{w}\|_{1,\mathscr{T}_h}\|\varphi\|_{1,\Omega}\|\psi\|_{1,\Omega}. \tag{2.8}
$$

Moreover, for $\gamma_1, \gamma_2 \in H^1(\Omega)$, $\boldsymbol{u} \in \mathbf{C}^1(\mathscr{T}_h) \cap \mathbf{H}_0^1(\Omega)$ and $\boldsymbol{v} \in \mathbf{V}_h$, there holds

$$\left| a_1^h(\gamma_1; \boldsymbol{u}, \boldsymbol{v}) - a_1^h(\gamma_2; \boldsymbol{u}, \boldsymbol{v}) \right| \leqslant \tilde{C}_{\mathrm{Lip}} \|\gamma_1 - \gamma_2\|_{1,\Omega} \|\boldsymbol{u}\|_{\mathbf{W}^{1,\infty}(\mathscr{T}_h)} \|\boldsymbol{v}\|_{1,\mathscr{T}_h}, \tag{2.9}$$

where the constant $\tilde{C}_{\mathrm{Lip}} > 0$ is independent of h (cf. Bürger *et al.*, 2019). Note that while the coercivity of the form $a_2(\cdot, \cdot)$ in the discrete setting is readily implied by (1.5b), there also holds (cf. Könnö & Stenberg, 2011, Lemma 3.2)

$$a_1^h(\cdot, \boldsymbol{v}, \boldsymbol{v}) \geqslant \tilde{\alpha}_a \|\boldsymbol{v}\|_{1,\mathscr{T}_h}^2 \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h, \tag{2.10}$$

provided that the stabilisation parameter $a_0 > 0$ in (2.2a) is sufficiently large and independent of the mesh size.

Let $\boldsymbol{w} \in \mathbf{H}_0(\mathrm{div}^0; \Omega)$ and let us introduce the jump seminorm

$$|\boldsymbol{u}_h|_{\boldsymbol{w},\mathrm{upw}} := \sum_{e \in \mathscr{E}_h^i} \int_e \frac{1}{2} |\boldsymbol{w}_e \cdot \boldsymbol{n}_e| |[\![\boldsymbol{u}_h]\!]|^2 \, \mathrm{d}S.$$

Then, due to the skew-symmetric form of the operators c_1^h and c_2, and the positivity of the nonlinear upwind term of c_1^h, we can write

$$c_1^h(\boldsymbol{w}; \boldsymbol{u}_h, \boldsymbol{u}_h) = |\boldsymbol{u}_h|_{\boldsymbol{w},\mathrm{upw}}^2 \geqslant 0 \quad \text{for all } \boldsymbol{u}_h \in \mathbf{V}_h, \tag{2.11a}$$

$$c_2(\boldsymbol{w}; \psi_h, \psi_h) = 0 \quad \text{for all } \psi_h \in \mathscr{M}_h, \tag{2.11b}$$

as well as the following relation (which is based on (2.7) and follows by the same method as in Karakashian & Jureidini, 1998): For any $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{u} \in \mathbf{H}^2(\mathscr{T}_h)$ there holds

$$\left| c_1^h(\boldsymbol{w}_1; \boldsymbol{u}, \boldsymbol{v}) \right| - \left| c_1^h(\boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v}) \right| \leqslant \tilde{C}_c \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{1,\mathscr{T}_h} \|\boldsymbol{v}\|_{1,\mathscr{T}_h} \|\boldsymbol{u}\|_{1,\mathscr{T}_h} \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h. \tag{2.12}$$

We also have

$$F(\psi, \phi, \boldsymbol{v}) \leqslant C_f \big(\|\psi\|_{0,\Omega} + \|\phi\|_{0,\Omega} \big) \|\boldsymbol{v}\|_{0,\Omega} \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h.$$

Finally, we recall from Könnö & Stenberg (2011) the following discrete inf-sup condition for $b(\cdot, \cdot)$, where $\tilde{\beta}$ is independent of h:

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{1,\mathscr{T}_h}} \geqslant \tilde{\beta} \|q_h\|_{0,\Omega} \quad \text{for all } q_h \in \mathscr{Q}_h. \tag{2.13}$$

2.4 *Stability*

As an auxiliary technical tool, we will also require the following algebraic relation: for any real numbers a^{n+1}, a^n, a^{n-1} and defining $\Lambda a^n := a^{n+1} - 2a^n + a^{n-1}$, we have

$$2(3a^{n+1} - 4a^n + a^{n-1}, a^{n+1}) = |a^{n+1}|^2 + |2a^{n+1} - a^n|^2 + |\Lambda a^n|^2 - |a^n|^2 - |2a^n - a^{n-1}|^2. \tag{2.14}$$

THEOREM 2.1 Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}) \in \mathbf{V}_h \times \mathscr{Q}_h \times \mathscr{M}_{h,0} \times \mathscr{M}_{h,0}$ be a solution of problem (2.5), with initial data $(\boldsymbol{u}_h^1, s_h^1, c_h^1)$ and $(\boldsymbol{u}_h^0, s_h^0, c_h^0)$. Then the following bounds are satisfied, where C_1, C_2 and C_3 are constants that are independent of h and of Δt:

$$\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda \boldsymbol{u}_h^j\|_{0,\Omega}^2 + \sum_{j=1}^n \Delta t \|\boldsymbol{u}_h^{j+1}\|_{1,\mathscr{T}_h}^2 + \sum_{j=1}^n \Delta t |\boldsymbol{u}_h^j|_{\boldsymbol{u}_h^j,\mathrm{upw}}^2$$

$$\leqslant C_1 \big(\|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2 + \|c_h^1\|_{0,\Omega}^2 + \|2c_h^1 - c_h^0\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^1\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|_{0,\Omega}^2 \big),$$

$$\|s_h^{n+1}\|_{0,\Omega}^2 + \|2s_h^{n+1} - s_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda s_h^j\|_{0,\Omega}^2 + \sum_{j=1}^n \Delta t \|s_h^{j+1}\|_{1,\Omega}^2 \leqslant C_2 \big(\|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2 \big), \tag{2.15}$$

$$\|c_h^{n+1}\|_{0,\Omega}^2 + \|2c_h^{n+1} - c_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda c_h^j\|_{0,\Omega}^2 + \sum_{j=1}^n \Delta t \|c_h^{j+1}\|_{1,\Omega}^2 \leqslant C_3 \big(\|c_h^1\|_{0,\Omega}^2 + \|2c_h^1 - c_h^0\|_{0,\Omega}^2 \big).$$

Proof. First we take $\boldsymbol{v}_h = 4\boldsymbol{u}_h^{n+1}$ and $q_h = 4p_h^{n+1}$ in the first and second equation of (2.5), respectively and apply (2.14), (2.10) and (2.11a) to deduce the estimate

$$\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{0,\Omega}^2 + \|\Lambda\boldsymbol{u}_h^n\|_{0,\Omega}^2 + 4\Delta t\tilde{\alpha}_a\|\boldsymbol{u}_h^{n+1}\|_{1,\mathcal{T}_h}^2 + 4\Delta t|\boldsymbol{u}_h^{n+1}|_{\boldsymbol{u}_h^{n+1},\text{upw}}^2$$

$$\leqslant 4\Delta tC_f(\|s_h^{n+1}\|_{0,\Omega} + \|c_h^{n+1}\|_{0,\Omega})\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega} + \|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{0,\Omega}^2.$$

Using Young's inequality with $\varepsilon = \tilde{\alpha}_a/2$ and summing over n we can assert that

$$\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda\boldsymbol{u}_h^j\|_{0,\Omega}^2 + 2\tilde{\alpha}_a\sum_{j=1}^n \Delta t\|\boldsymbol{u}_h^{j+1}\|_{1,\mathcal{T}_h}^2 + \sum_{j=1}^n 4\Delta t|\boldsymbol{u}_h^{j+1}|_{\boldsymbol{u}_h^{j+1},\text{upw}}^2$$

$$\leqslant \frac{C}{\tilde{\alpha}_a}\sum_{j=1}^n \Delta t\|s_h^j\|_{0,\Omega}^2 + \frac{C}{\tilde{\alpha}_a}\sum_{j=1}^n \Delta t\|c_h^j\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^1\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|_{0,\Omega}^2. \tag{2.16}$$

Similarly in the third equation of (2.5), we take $\varphi_h = 4s_h^{n+1}$ and use property (2.11b) and relation (2.14) to arrive at the inequality

$$\|s_h^{n+1}\|_{0,\Omega}^2 + \|2s_h^{n+1} - s_h^n\|_{0,\Omega}^2 + \|\Lambda s_h^n\|_{0,\Omega}^2 + 4\alpha_2\Delta t\|s_h^{n+1}\|_{1,\Omega}^2 \leqslant \|s_h^n\|_{0,\Omega}^2 + \|2s_h^n - s_h^{n-1}\|_{0,\Omega}^2.$$

Hence, summing over n, we get

$$\|s_h^{n+1}\|_{0,\Omega}^2 + \|2s_h^{n+1} - s_h^n\|_{0,\Omega}^2 + \sum_{j=1}^n \|\Lambda s_h^j\|_{0,\Omega}^2 + 4\hat{\alpha}_a\sum_{j=1}^n \Delta t\|s_h^{j+1}\|_{1,\Omega}^2 \leqslant \|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2.$$

We proceed in the same way taking $\psi_h = 4c_h^{n+1}$ in the fourth equation of (2.5) to get the third result. Finally, we can assert the first result by substituting the bounds for c_h and s_h into (2.16). □

Note that, in contrast to the linear case, the stability result and the existence of a discrete solution do not guarantee, in general, uniqueness of solution.

2.5 *Solvability*

THEOREM 2.2 (Existence of a discrete solution) Assume that $\min\{\hat{\alpha}_a, \hat{\alpha}_a/\tau\} > \frac{C_f^2}{2\tilde{\alpha}_a}$. Then problem (2.5), with initial data $(\boldsymbol{u}_h^1, s_h^1, c_h^1)$ and $(\boldsymbol{u}_h^0, s_h^0, c_h^0)$ (where $(\boldsymbol{u}_h^1, s_h^1, c_h^1)$ is obtained from $(\boldsymbol{u}_h^0, s_h^0, c_h^0)$ by a backward Euler method), admits a (not necessarily unique) solution

$$(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}) \in \mathbf{V}_h \times \mathcal{Q}_h \times \mathcal{M}_{h,0} \times \mathcal{M}_{h,0}.$$

The proof of Theorem 2.2 is conducted using a fixed-point argument that employs Brouwer's fixed-point theorem in the following form (given by Girault & Raviart, 1986, Corollary 1.1, Chapter IV):

THEOREM 2.3 (Brouwer's fixed-point theorem) Let H be a finite-dimensional Hilbert space with scalar product $(\cdot,\cdot)_H$ and corresponding norm $\|\cdot\|_H$. Let $\Phi: H \to H$ be a continuous mapping for which there exists $\mu > 0$ such that $(\Phi(u), u)_H \geqslant 0$ for all $u \in H$ with $\|u\|_H = \mu$. Then there exists $u \in H$ such that $\Phi(u) = 0$ and $\|u\|_H \leqslant \mu$.

Proof of Theorem 2.2. To simplify the proof we introduce the constants

$$C_u := C_1\big(\|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2 + \|c_h^1\|_{0,\Omega}^2 + \|2c_h^1 - c_h^0\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^1\|_{0,\Omega}^2 + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|_{0,\Omega}^2\big),$$

$$C_s := C_2\big(\|s_h^1\|_{0,\Omega}^2 + \|2s_h^1 - s_h^0\|_{0,\Omega}^2\big), \qquad C_c := C_3\big(\|c_h^1\|_{0,\Omega}^2 + \|2c_h^1 - c_h^0\|_{0,\Omega}^2\big).$$

We proceed by induction on $n \geqslant 1$. We define the mapping

$$\Phi : \mathbf{V}_h \times \mathcal{Q}_h \times \mathcal{M}_{h,0} \times \mathcal{M}_{h,0} \to \mathbf{V}_h \times \mathcal{Q}_h \times \mathcal{M}_{h,0} \times \mathcal{M}_{h,0}, \tag{2.17}$$

using the relation

$$
\begin{aligned}
&\big(\Phi(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}), (\boldsymbol{v}_h, q_h, \varphi_h, \psi_h)\big)_\Omega \\
&= \frac{(\mathscr{D}\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)_\Omega}{2\Delta t} + a_1^h\big(c_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\big) + c_1^h\big(\boldsymbol{u}_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\big) + b(\boldsymbol{v}_h, p_h^{n+1}) - b(\boldsymbol{u}_h^{n+1}, q_h) - F\big(s_h^{n+1}, c_h^{n+1}, \boldsymbol{v}_h\big) \\
&\quad + \frac{(\mathscr{D}s_h^{n+1}, \varphi_h)_\Omega}{2\Delta t} + a_2\big(s_h^{n+1}, \varphi_h\big) + c_2\big(\boldsymbol{u}_h^{n+1}; s_h^{n+1}, \varphi_h\big) + \frac{(\mathscr{D}c_h^{n+1}, \psi_h)_\Omega}{2\Delta t} + \frac{a_2(c_h^{n+1}, \psi_h)}{\tau} \\
&\quad + c_2\big(\boldsymbol{u}_h^{n+1} - v_{\mathrm{p}}\boldsymbol{e}_z; c_h^{n+1}, \psi_h\big).
\end{aligned}
$$

Note that this map is well-defined and continuous on $\mathbf{V}_h \times \mathscr{Q}_h \times \mathscr{M}_{h,0} \times \mathscr{M}_{h,0}$. On the other hand, if we take

$$
(\boldsymbol{v}_h, q_h, \varphi_h, \psi_h) = (\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}),
$$

and employ (2.11a), (2.11b) and (2.10), we obtain

$$
\begin{aligned}
&\big(\Phi(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}), (\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1})\big)_\Omega \\
&\geqslant -\frac{1}{2\Delta t}\|4\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{0,\Omega}\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega} + \tilde{\alpha}_a\|\boldsymbol{u}_h^{n+1}\|_{1,\mathscr{T}_h}^2 + |\boldsymbol{u}_h^{n+1}|_{\boldsymbol{u}_h^{n+1},\mathrm{upw}}^2 - C_f\big(\|s_h^{n+1}\|_{0,\Omega} + \|c_h^{n+1}\|_{0,\Omega}\big)\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega} \\
&\quad - \frac{1}{2\Delta t}\|4s_h^n - s_h^{n-1}\|_{0,\Omega}\|s_h^{n+1}\|_{0,\Omega} + \hat{\alpha}_a\|s_h^{n+1}\|_{1,\Omega}^2 - \frac{1}{2\Delta t}\|4c_h^n - c_h^{n-1}\|_{0,\Omega}\|c_h^{n+1}\|_{0,\Omega} + \frac{\hat{\alpha}_a}{\tau}\|c_h^{n+1}\|_{1,\Omega}^2.
\end{aligned}
$$

Next, we use (2.15) and Young's inequality with $\varepsilon = \tilde{\alpha}_a$. Under the assumption that $\min\{\hat{\alpha}_a, \hat{\alpha}_a/\tau\} > \frac{C_f^2}{\tilde{\alpha}_a}$, we deduce that

$$
\begin{aligned}
&\big(\Phi(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}), (\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1})\big)_\Omega \\
&\geqslant \frac{\tilde{\alpha}_a}{2}\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \frac{\hat{\alpha}_a}{2}\|s_h^{n+1}\|_{0,\Omega}^2 + \frac{\hat{\alpha}_a}{2\tau}\|c_h^{n+1}\|_{0,\Omega}^2 - \frac{5}{2\Delta t}C_u\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega} - \frac{5}{2\Delta t}C_s\|s_h^{n+1}\|_{0,\Omega} - \frac{5}{2\Delta t}C_c\|c_h^{n+1}\|_{0,\Omega}.
\end{aligned}
$$

Then, setting

$$
C_R = \min\left\{\frac{\tilde{\alpha}_a}{2}, \frac{\hat{\alpha}_a}{2\tau}, \frac{\hat{\alpha}_a}{2}\right\}, \quad C_r = \sqrt{2}\max\left\{\frac{5}{2\Delta t}C_u, \frac{5}{2\Delta t}C_s, \frac{5}{2\Delta t}C_c\right\},
$$

we proceed to apply the inequality $a + b \leqslant \sqrt{2}(a^2 + b^2)^{1/2}$, valid for all $a, b \in \mathbb{R}$, to obtain

$$
\begin{aligned}
&\big(\Phi(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}), (\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1})\big)_\Omega \\
&\geqslant C_R\big(\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|s_h^{n+1}\|_{0,\Omega}^2 + \|c_h^{n+1}\|_{0,\Omega}^2\big) - C_r\big(\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|s_h^{n+1}\|_{0,\Omega}^2 + \|c_h^{n+1}\|_{0,\Omega}^2\big)^{1/2}.
\end{aligned}
$$

Hence, the right-hand side is nonnegative on a sphere of radius $r := C_r/C_R$. Consequently, by Theorem 2.3, there exists a solution to the fixed-point problem $\Phi(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, s_h^{n+1}, c_h^{n+1}) = 0$, where the fixed-point map (2.17) is the solution operator for the fully discrete problem (2.5). □

2.6 *A priori error estimates*

The analysis in this section follows from standard arguments applicable to the approximation and error bounds for isolated solutions. For this we require to assume the uniqueness of discrete solution.

Let us denote by $\mathscr{I}_h : H^2(\Omega) \to \mathscr{M}_h$ the nodal interpolator with respect to a unisolvent set of Lagrangian interpolation nodes associated with $\mathscr{M}_h$. Furthermore, $\Pi_h \boldsymbol{u}$ denotes the BDM projection of $\boldsymbol{u}$, and $\mathscr{L}_h p$ is the L^2-projection of p onto $\mathscr{Q}_h$. Under usual assumptions, the following approximation properties hold (see Könnö & Stenberg, 2011):

$$
\|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{1,\mathscr{T}_h} \leqslant C^* h^k \|\boldsymbol{u}\|_{k+1,\Omega}, \quad \|c - \mathscr{I}_h c\|_{1,\Omega} \leqslant C^* h^k \|c\|_{k+1,\Omega},
$$

$$\|s - \mathscr{I}_h s\|_{1,\Omega} \leqslant C^* h^k \|s\|_{k+1,\Omega}, \quad \|p - \mathscr{L}_h p\|_{0,\Omega} \leqslant C^* h^k \|p\|_{k,\Omega}. \tag{2.18}$$

The following development follows the structure adopted in Aldbaissy *et al.* (2018). We begin with a property of the discrete bilinear forms and the continuous variational formulation.

LEMMA 2.2 Assume that $\boldsymbol{u} \in \mathbf{L}^2(0, t_{\mathrm{end}}; \mathbf{H}^2(\Omega))$, $\partial_t \boldsymbol{u} \in \mathbf{L}^2(0, t_{\mathrm{end}}; \mathbf{L}^2(\Omega))$, $p \in Q^t$ and $s, c \in M^t$. Then for a.e $t \in [0, t_{\mathrm{end}}]$, we have

$$(\partial_t \boldsymbol{u}, \boldsymbol{v})_\Omega + a_1^h(c; \boldsymbol{u}, \boldsymbol{v}) + c_1^h(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = F(s, c, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h, \tag{2.19a}$$

$$b(\boldsymbol{u}, q) = 0 \quad \text{for all } q \in \mathscr{Q}_h, \tag{2.19b}$$

$$(\partial_t s, \varphi)_\Omega + a_2(s, \varphi) + c_2(\boldsymbol{u}; s, \varphi) = 0 \quad \text{for all } \varphi \in \mathscr{M}_{h,0}, \tag{2.19c}$$

$$(\partial_t c, \psi)_\Omega + \frac{1}{\tau} a_2(c, \psi) + c_2(\boldsymbol{u} - v_{\mathrm{p}} \boldsymbol{e}_z; c, \psi) = 0 \quad \text{for all } \psi \in \mathscr{M}_{h,0}. \tag{2.19d}$$

Proof. Since we have assumed that $\boldsymbol{u} \in \mathbf{H}^2(\Omega)$, integration by parts yields the required result. See also Arnold *et al.* (2001). The third and fourth equations are a straightforward consequence of properties of the continuous weak form. □

Since for the following theorems we will assume the exact c and s belong to $H^2(\Omega)$, we have $c, s \in C(\bar{\Omega})$. Now we decompose the errors as follows:

$$\boldsymbol{u} - \boldsymbol{u}_h = E_{\boldsymbol{u}} + \xi_{\boldsymbol{u}} = (\boldsymbol{u} - \Pi_h \boldsymbol{u}) + (\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), \quad p - p_h = E_p + \xi_p = (p - \mathscr{L}_h p) + (\mathscr{L}_h p - p_h),$$

$$s - s_h = E_s + \xi_s = (s - \mathscr{I}_h s) + (\mathscr{I}_h s - s_h), \quad c - c_h = E_c + \xi_c = (c - \mathscr{I}_h c) + (\mathscr{I}_h c - c_h).$$

Assuming that $\boldsymbol{u}_h^0 = \Pi_h \boldsymbol{u}(0)$, $s_h^0 = \mathscr{I}_h s(0)$ and $c_h^0 = \mathscr{I}_h c(0)$, we also use the notation $E_{\boldsymbol{u}}^n = (\boldsymbol{u}(t_n) - \Pi_h \boldsymbol{u}(t_n))$ and $\xi_{\boldsymbol{u}}^n = (\Pi_h \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n)$, and similar notation for other variables. Since for the first time iteration of system (2.1) we adopt a backward Euler scheme, a dedicated error estimate is required for this step.

For simplicity of notation, in what follows, we write down $\boldsymbol{u}'$ instead of $\partial_t \boldsymbol{u}$, $\boldsymbol{u}''$ instead of $\partial_{tt} \boldsymbol{u}$, and so on.

THEOREM 2.4 Let us assume that

$$\boldsymbol{u} \in \begin{cases} L^\infty(0, t_{\mathrm{end}}; \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \cap \mathbf{C}^1(\bar{\Omega})), & \text{if } k = 1, \\ L^\infty(0, t_{\mathrm{end}}; \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0^1(\Omega)), & \text{if } k \geqslant 2, \end{cases} \quad \boldsymbol{u}'' \in C^0(0, t_{\mathrm{end}}; \mathbf{L}^2(\Omega)),$$

$$p \in L^\infty(0, t_{\mathrm{end}}; H^2(\Omega)), \quad s \in W^{2,\infty}(0, t_{\mathrm{end}}; H^{k+1}(\Omega) \cap H_0^1(\Omega)), \quad c \in W^{2,\infty}(0, t_{\mathrm{end}}; H^{k+1}(\Omega) \cap H_0^1(\Omega)), \tag{2.20}$$

and also that $\|\boldsymbol{u}\|_{L^\infty(0, t_{\mathrm{end}}; \mathbf{W}^{1,\infty}(\Omega))} < M$, with $0 < M \leqslant \min\left\{ \frac{\tilde{\alpha}_a}{4 \tilde{C}_c C^*}, \frac{\sqrt{\tilde{\alpha}_a}}{4 C_{\mathrm{Lip}} \sqrt{2}} \right\}$. Then there exist positive constants $C_{\boldsymbol{u}}^1$, C_s^1, C_c^1, independently of h and Δt, such that

$$\frac{1}{4} \|\xi_{\boldsymbol{u}}^1\|_{0,\Omega}^2 + \frac{1}{2} \Delta t \tilde{\alpha}_a \|\xi_{\boldsymbol{u}}^1\|_{1,\mathscr{T}_h}^2 + \Delta t |\xi_{\boldsymbol{u}}^1|_{\boldsymbol{u}_h^1, \mathrm{upw}}^2 \leqslant C_{\boldsymbol{u}}^1 (h^{2k} + [\Delta t]^4),$$

$$\frac{1}{4} \|\xi_s^1\|_{0,\Omega}^2 + \frac{1}{2} \Delta t \hat{\alpha}_a \|\xi_s^1\|_{1,\Omega}^2 \leqslant C_s^1 (h^{2k} + [\Delta t]^4),$$

$$\frac{1}{4} \|\xi_c^1\|_{0,\Omega}^2 + \frac{1}{2} \Delta t \hat{\alpha}_a \|\xi_c^1\|_{0,\Omega}^2 \leqslant C_c^1 (h^{2k} + [\Delta t]^4).$$

The proof of this result is postponed to Appendix A.

THEOREM 2.5 Let $(\boldsymbol{u}, p, s, c)$ be the unique solution of (1.2) under the assumptions of Section 1.5, and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ be a solution of (2.5). Suppose, in addition to (2.20), that

$$\boldsymbol{u} \in W^{2,\infty}(0, t_{\mathrm{end}}; \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}_0^1(\Omega)), \quad \boldsymbol{u}' \in L^\infty(0, t_{\mathrm{end}}; \mathbf{H}^k(\Omega)), \quad \boldsymbol{u}^{(3)} \in L^2(0, t_{\mathrm{end}}; \mathbf{L}^2(\Omega)),$$

and $\|\boldsymbol{u}\|_{L^\infty(0,t_{\text{end}};\mathbf{H}^1(\Omega))} < M$ for a sufficiently small constant $0 < M \leqslant \min\left\{\frac{\tilde{\alpha}_a}{4\tilde{C}_c C^*}, \frac{\sqrt{\tilde{\alpha}_a}}{4C_{\text{Lip}}\sqrt{2}}\right\}$. Then there exist constants $C, 0 < \gamma_1 = \frac{16M^2 C_{\text{Lip2}}}{\tilde{\alpha}_a} \leqslant \frac{\alpha_a \tilde{\alpha}_a}{2}$, independent of h and Δt, such that, for all $m+1 \leqslant N$,

$$\|\xi_{\boldsymbol{u}}^{m+1}\|_{0,\Omega}^2 + \|2\xi_{\boldsymbol{u}}^{m+1} - \xi_{\boldsymbol{u}}^m\|_{0,\Omega}^2 + \sum_{n=1}^m \|\Lambda \xi_{\boldsymbol{u}}^n\|_{0,\Omega}^2 + \sum_{n=1}^m \Delta t \tilde{\alpha}_a \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 + \sum_{n=1}^m \Delta t |\xi_{\boldsymbol{u}}^{n+1}|_{\boldsymbol{u}_h^{n+1},\text{upw}}^2$$

$$\leqslant C([\Delta t]^4 + h^{2k}) + \sum_{n=1}^m \gamma_1 \Delta t \|\xi_c^{n+1}\|_{1,\Omega}^2.$$

For a detailed proof, see Appendix B.

Note that, differently from the robust error analysis in, e.g., Han & Hou (2021) or in Schroeder *et al.* (2018), Theorems 2.4 and 2.5, require a smallness assumption on the velocity solution of the continuous problem. The assumption is needed due to the coupling through the viscosity with the balance equation of concentration. If such dependency is removed, for instance when a constant viscosity value is used, the smallness assumption is no longer required.

THEOREM 2.6 Let $(\boldsymbol{u}, p, s, c)$ be the solution of (1.2) under the assumptions of Section 1.5, and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ be a solution of (2.5). Assuming, in addition to (2.20), that

$$s' \in L^\infty(0, t_{\text{end}}; H^k(\Omega)), \quad s^{(3)} \in L^2(0, t_{\text{end}}; L^2(\Omega)),$$

then there exist constants $C, 0 < \gamma_2 := \frac{28\tilde{C}_c^2 C^{*2}}{3\hat{\alpha}_a}\|s\|_{L^\infty(0,t_{\text{end}};H^1(\Omega))}^2$, independent of h and Δt, such that for all $m+1 \leqslant N$

$$\|\xi_s^{m+1}\|_{0,\Omega}^2 + \|2\xi_s^{m+1} - \xi_s^m\|_{0,\Omega}^2 + \sum_{n=1}^m \|\Lambda \xi_s^{n+1}\|_{0,\Omega}^2 + \sum_{n=1}^m \Delta t \hat{\alpha}_a \|\xi_s^{n+1}\|_{1,\Omega}^2$$

$$\leqslant C([\Delta t]^4 + h^{2k}) + \sum_{n=1}^m \gamma_2 \Delta t \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2.$$

The proof is found in Appendix C.

THEOREM 2.7 Let $(\boldsymbol{u}, p, s, c)$ be the solution of (1.2) under the assumptions of Section 1.5, and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ be a solution of (2.5). If, in addition to (2.20) we have

$$c' \in L^\infty(0, t_{\text{end}}; H^k(\Omega)), \quad c^{(3)} \in L^2(0, t_{\text{end}}; L^2(\Omega)),$$

then there exist constants $C, \gamma_3 > 0$ that are independent of h and Δt, such that for all $m+1 \leqslant N$

$$\|\xi_c^{m+1}\|_{0,\Omega}^2 + \|2\xi_c^{m+1} - \xi_c^m\|_{0,\Omega}^2 + \sum_{n=1}^m \|\Lambda \xi_c^{n+1}\|_{0,\Omega}^2 + \sum_{n=1}^m \Delta t \hat{\alpha}_a \|\xi_c^{n+1}\|_{1,\Omega}^2$$

$$\leqslant C([\Delta t]^4 + h^{2k}) + \sum_{n=1}^m \gamma_3 \Delta t \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2.$$

Proof. It follows along the same lines of the proof of Theorem 2.6, with constant γ_3 given by

$$\gamma_3 = \frac{28\tilde{C}_c^2 C^{*2}}{3\hat{\alpha}_a}\|c\|_{L^\infty(0,t_{\text{end}};H^1(\Omega))}^2.$$

$\square$

THEOREM 2.8 Under the same assumptions of Theorems 2.5 to 2.7, there exist positive constants γ_u, γ_s and γ_c independent of Δt and h, such that, for a sufficiently small Δt and all $m+1 \leqslant N$, there hold

$$\left(\|\xi_{\boldsymbol{u}}^{m+1}\|_{0,\Omega}^2 + \|2\xi_{\boldsymbol{u}}^{m+1} - \xi_{\boldsymbol{u}}^m\|_{0,\Omega}^2 + \sum_{n=1}^m \left(\|\Lambda \xi_{\boldsymbol{u}}^n\|_{0,\Omega}^2 + \Delta t \tilde{\alpha}_a \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 + \Delta t |\xi_{\boldsymbol{u}}^{n+1}|_{\boldsymbol{u}_h^{n+1},\text{upw}}^2\right)\right)^{1/2} \leqslant \gamma_u([\Delta t]^2 + h^k),$$

$$\left(\|\xi_s^{m+1}\|_{0,\Omega}^2 + \|2\xi_s^{m+1} - \xi_s^m\|_{0,\Omega}^2 + \sum_{n=1}^{m} \left(\|\Lambda \xi_s^n\|_{0,\Omega}^2 + \Delta t \hat{\alpha}_a \|\xi_s^{n+1}\|_{1,\Omega}^2 \right) \right)^{1/2} \leqslant \gamma_s([\Delta t]^2 + h^k),$$

$$\left(\|\xi_c^{m+1}\|_{0,\Omega}^2 + \|2\xi_c^{m+1} - \xi_c^m\|_{0,\Omega}^2 + \sum_{n=1}^{m} \left(\|\Lambda \xi_c^n\|_{0,\Omega}^2 + \Delta t \hat{\alpha}_a \|\xi_c^{n+1}\|_{1,\Omega}^2 \right) \right)^{1/2} \leqslant \gamma_c([\Delta t]^2 + h^k).$$

Proof. From Theorems 2.5 and 2.7, since $\gamma_1 \leqslant \frac{\hat{\alpha}_a \tilde{\alpha}_a}{2}$, we have the estimate

$$\sum_{n=1}^{m} \Delta t \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \leqslant C([\Delta t]^4 + h^{2k}) + \sum_{n=1}^{m} \Delta t \frac{\hat{\alpha}_a}{2} \|\xi_c^{n+1}\|_{1,\mathscr{T}_h}^2,$$

which, after substitution back into Theorem 2.7, yields

$$\sum_{n=1}^{m} \Delta t \|\xi_c^{n+1}\|_{1,\Omega}^2 \leqslant C([\Delta t]^4 + h^{2k}). \tag{2.21}$$

The first bound follows by combining (2.21) and Theorem 2.5, whereas the second and third bounds follow directly from the first bound in combination with Theorems 2.6 and 2.7. $\qquad\square$

LEMMA 2.3 Under the same assumptions of Theorem 2.8, and $p \in L^\infty(0, t_{\text{end}}; H^2(\Omega))$, we have

$$\left(\sum_{n=1}^{m} \Delta t \|p(t_{n+1}) - p_h^{n+1}\|_{0,\Omega}^2 \right)^{1/2} \leqslant \hat{\gamma}_p([\Delta t]^2 + h^k).$$

Proof. Owing to the inf-sup condition (2.13), there exists $\boldsymbol{w}_h \in \mathbf{X}_h^{\perp}$ such that

$$b(\boldsymbol{w}_h, \mathscr{L}_h p^{n+1} - p_h^{n+1}) = \|\mathscr{L}_h p^{n+1} - p_h^{n+1}\|_{0,\Omega}^2,$$

$$\|\boldsymbol{w}_h\|_{1,\mathscr{T}_h} \leqslant \frac{1}{\tilde{\beta}} \|\mathscr{L}_h p^{n+1} - p_h^{n+1}\|_{0,\Omega}. \tag{2.22}$$

From (2.5) and Lemma 2.2, proceeding as in the proof of Theorem 2.5, we obtain

$$\Delta t b(\boldsymbol{w}_h, \mathscr{L}_h p^{n+1} - p_h^{n+1})$$
$$= -\Delta t \left(\boldsymbol{u}'(t_{n+1}) - \frac{1}{2\Delta t} \mathscr{D} \boldsymbol{u}_h^{n+1}, \boldsymbol{w}_h \right)_\Omega + \Delta t \left(a_1^h(c_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{w}_h) - a_1^h(c(t_{n+1}); \boldsymbol{u}(t_{n+1}), \boldsymbol{w}_h) \right)$$
$$\quad + \Delta t \left(c_1^h(\boldsymbol{u}_h^{n+1}; \boldsymbol{u}_h^{n+1}, \boldsymbol{w}_h) - c_1^h(\boldsymbol{u}(t_{n+1}); \boldsymbol{u}(t_{n+1}), \boldsymbol{w}_h) \right)$$
$$\quad + \Delta t \left(F(s_h^{n+1}, c_h^{n+1}, \boldsymbol{w}_h) - F(s(t_{n+1}), c(t_{n+1}), \boldsymbol{w}_h) \right) + \Delta t b(\boldsymbol{w}_h, \mathscr{L}_h p^{n+1} - p(t_{n+1}))$$
$$\leqslant \frac{[\Delta t]^2}{2\sqrt{3}} \|\boldsymbol{u}^{(3)}\|_{L^2(t_{n-1}, t_{n+1}, \mathbf{L}^2(\Omega))} \sqrt{\Delta t} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h} + \Delta t C_f \|\xi_s^{n+1}\|_{0,\Omega} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h}$$
$$\quad + \tilde{C}_a C^* h^k \Delta t \|\boldsymbol{u}\|_{L^\infty(0, t_{\text{end}}; \mathbf{H}^{k+1}(\Omega))} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h} + \tilde{C}_{\text{lip}} M \Delta t \|\xi_c^{n+1}\|_{1,\Omega} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h}$$
$$\quad + \tilde{C}_{\text{lip}} \Delta t M \|\xi_c^{n+1}\|_{1,\Omega} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h} + \Delta t C \tilde{C}_c C^* C_\infty M \|\xi_c^{n+1}\|_{1,\Omega} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h}$$
$$\quad + 2\Delta t C^* \tilde{C}_c h^k \|\boldsymbol{u}\|_{L^\infty(0, t_{\text{end}}; \mathbf{H}^1(\Omega))} \|\boldsymbol{u}\|_{L^\infty(0, t_{\text{end}}; \mathbf{H}^{k+1}(\Omega))} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h} + \Delta t C_f \|\xi_s^{n+1}\|_{0,\Omega} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h}$$
$$\quad + \Delta t C^* C_f h^k \|s\|_{L^\infty(0, t_{\text{end}}; H^1(\Omega))} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h} + \Delta t C^* C_f h^k \|c\|_{L^\infty(0, t_{\text{end}}; H^1(\Omega))} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h}$$
$$\quad + C \Delta t h^k \|p\|_{L^2(0, t_{\text{end}}; H^k(\Omega))} \|\boldsymbol{w}_h\|_{1,\mathscr{T}_h}.$$

Summing over $1 \leqslant n \leqslant m$ for all $m + 1 \leqslant N$, using (2.22), and triangle inequality, we obtain

$$\left(\sum_{n=1}^{m} \Delta t \|p(t_{n+1}) - p_h^{n+1}\|_{0,\Omega}^2 \right)^{1/2} \leqslant \frac{C}{\tilde{\beta}} \left([\Delta t]^2 + h^k + \left(\sum_{n=1}^{m} \Delta t \|\xi_c^{n+1}\|_{0,\Omega}^2 \right)^{1/2} + \left(\sum_{n=1}^{m} \Delta t \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \right)^{1/2} \right),$$

and the desired result readily follows from Theorem 2.8. $\qquad\qquad\square$

We next proceed to derive and analyse *a posteriori* error estimators. We split the presentation into three cases of increasing complexity, starting with an estimator focusing on the steady coupled problem.

3. *A posteriori* error estimation for the stationary double-diffusive flow problem

3.1 *Defining the estimator*

Let us consider the following nonlinear coupled problem in weak form, associated with a steady counterpart of (1.2). Find $(\boldsymbol{u}, p, s, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$a_1(c, \boldsymbol{u}, \boldsymbol{v}) + c_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \quad \text{for all } \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \tag{3.1a}$$

$$b(\boldsymbol{u}, q) = 0 \quad \text{for all } q \in L_0^2(\Omega), \tag{3.1b}$$

$$a_2(s, \phi) + c_2(\boldsymbol{u}; s, \phi) = (f_1, \phi)_{0,\Omega} \quad \text{for all } \phi \in H_0^1(\Omega), \tag{3.1c}$$

$$\frac{1}{\tau} a_2(c, \psi) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; c, \psi) = (f_2, \psi)_{0,\Omega} \quad \text{for all } \psi \in H_0^1(\Omega), \tag{3.1d}$$

where $(\alpha s + \beta c)\boldsymbol{g} = (\rho/\rho_{\mathrm{m}})\boldsymbol{g} = \boldsymbol{f} \in \mathbf{L}^2(\Omega)$, and f_1, f_2 are taken as constant.

Let us also consider its discrete counterpart: Find $(\boldsymbol{u}_h, p_h, s_h, c_h) \in \mathbf{V}_h \times \mathscr{Q}_h \times \mathscr{M}_{h,0} \times \mathscr{M}_{h,0}$ such that

$$a_1(c_h, \boldsymbol{u}_h, \boldsymbol{v}) + c_1(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}_h, \boldsymbol{v})_{0,\Omega} \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h, \tag{3.2a}$$

$$b(\boldsymbol{u}_h, q) = 0 \quad \text{for all } q \in \mathscr{Q}_h, \tag{3.2b}$$

$$a_2(s_h, \phi) + c_2(\boldsymbol{u}_h; s_h, \phi) = (f_1, \phi)_{0,\Omega} \quad \text{for all } \phi \in \mathscr{M}_{h,0}, \tag{3.2c}$$

$$\frac{1}{\tau} a_2(c_h, \psi) + c_2(\boldsymbol{u}_h - v_p \boldsymbol{e}_z; c_h, \psi) = (f_2, \psi)_{0,\Omega} \quad \text{for all } \psi \in \mathscr{M}_{h,0}, \tag{3.2d}$$

where an assumption similar to that of the continuous case is used, namely $\boldsymbol{f}_h = (\alpha s_h + \beta c_h)\boldsymbol{g}$, and f_1, f_2 constants. Note that, since the dependence of the buoyancy term on the concentration and salinity is linear, the difference $\boldsymbol{f} - \boldsymbol{f}_h$ could be treated as data oscillation. See Remark 3.1 below.

Note also that the well-posedness of the continuous weak formulation (3.1) and discrete weak formulation (3.2) follow as in Bürger *et al.* (2019) and Tushar *et al.* (2023), for data as assumed above. In particular, Theorem 3.2 of Bürger *et al.* (2019) ensures the uniqueness of the solution if $\|\boldsymbol{u}\|_{1,\infty} < M$, $\|s\|_\infty < M$ and $\|c\|_\infty < M$ for a sufficiently small M. In turn, Theorem 3.9 of Tushar *et al.* (2023) proves the uniqueness of the solution with minimum regularity assumptions.

For each $K \in \mathscr{T}_h$ and each $e \in \mathscr{E}_h$ we define element-wise and facet-wise residuals as follows:

$$\mathbf{R}_K := \{\boldsymbol{f}_h + \mathbf{div}(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h) - \boldsymbol{u}_h \cdot \boldsymbol{\nabla}\boldsymbol{u}_h - (\rho_{\mathrm{m}})^{-1}\nabla p_h\}|_K, \tag{3.3a}$$

$$R_{1,K} := \{f_1 + \mathrm{Sc}^{-1}\Delta s_h - \boldsymbol{u}_h \cdot \nabla s_h\}|_K, \tag{3.3b}$$

$$R_{2,K} := \{f_2 + (\tau\mathrm{Sc})^{-1}\Delta c_h - (\boldsymbol{u}_h - v_p \boldsymbol{e}_z) \cdot \nabla c_h\}|_K, \tag{3.3c}$$

$$\mathbf{R}_e := \begin{cases} \frac{1}{2}[\![((\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h)\boldsymbol{n}]\!] & \text{for } e \in \mathscr{E}_h \setminus \Gamma, \\ 0 & \text{for } e \in \Gamma, \end{cases} \tag{3.3d}$$

$$R_{1,e} := \begin{cases} \frac{1}{2}[\![(\mathrm{Sc}^{-1}\nabla s_h) \cdot \boldsymbol{n}]\!] & \text{for } e \in \mathscr{E}_h \setminus \Gamma, \\ 0 & \text{for } e \in \Gamma, \end{cases} \tag{3.3e}$$

$$R_{2,e} := \begin{cases} \frac{1}{2}[\![((\tau\mathrm{Sc})^{-1}\nabla c_h) \cdot \boldsymbol{n}]\!] & \text{for } e \in \mathscr{E}_h \setminus \Gamma, \\ 0 & \text{for } e \in \Gamma. \end{cases} \tag{3.3f}$$

Then we introduce the element-wise error estimator $\Psi_K^2 := \Psi_{R_K}^2 + \Psi_{e_K}^2 + \Psi_{J_K}^2$ with contributions defined as

$$\Psi_{R_K}^2 := h_K^2 \left(\|\mathbf{R}_K\|_{0,K}^2 + \|R_{1,K}\|_{0,K}^2 + \|R_{2,K}\|_{0,K}^2 \right),$$
$$\Psi_{e_K}^2 := \sum_{e \in \partial K} h_e \left(\|\mathbf{R}_e\|_{0,e}^2 + \|R_{1,e}\|_{0,e}^2 + \|R_{2,e}\|_{0,e}^2 \right),$$
$$\Psi_{J_K}^2 := \sum_{e \in \partial K} h_e^{-1} \|[\![\boldsymbol{u}_h]\!]\|_{0,e}^2,$$

so a global *a posteriori* error estimator for the nonlinear coupled and steady problem (3.2) is

$$\Psi := \left(\sum_{K \in \mathscr{T}_h} \Psi_K^2 \right)^{1/2}. \tag{3.4}$$

3.2 *Reliability*

Let us introduce the space

$$\tilde{X}(\mathscr{T}_h) = \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{div}^0, \Omega) : \boldsymbol{v} \in \mathbf{H}^1(K) \ \forall K \in \mathscr{T}_h\}.$$

Then, for a fixed $(\tilde{\boldsymbol{u}}, \tilde{c}) \in \tilde{X}(\mathscr{T}_h) \times H_0^1(\Omega)$, we define the bilinear form $\mathscr{A}_h^{(\tilde{\boldsymbol{u}}, \tilde{c})}(\cdot, \cdot)$ as

$$\mathscr{A}_h^{(\tilde{\boldsymbol{u}}, \tilde{c})} \big((\boldsymbol{u}, p, s, c), (\boldsymbol{v}, q, \phi, \psi) \big) = \tilde{a}_1^h(\tilde{c}, \boldsymbol{u}, \boldsymbol{v}) + c_1^h(\tilde{\boldsymbol{u}}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q) + a_2(s, \phi)$$
$$+ c_2(\tilde{\boldsymbol{u}}; s, \phi) + \frac{1}{\tau} a_2(c, \psi) + c_2(\tilde{\boldsymbol{u}} - v_p \boldsymbol{e}_z; c, \psi),$$

for all $(\boldsymbol{u}, p, s, c), (\boldsymbol{v}, q, \phi, \psi) \in \mathbf{V}_h \times \mathscr{Q}_h \times \mathscr{M}_h \times \mathscr{M}_h$, where

$$\tilde{a}_1^h(\tilde{c}, \boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \nu(\tilde{c}) \boldsymbol{\nabla}_h(\boldsymbol{u}) : \boldsymbol{\nabla}_h(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} + \sum_{e \in \mathscr{E}_h} \int_e \frac{a_0}{h_e} \nu(\tilde{c}) [\![\boldsymbol{u}]\!] \cdot [\![\boldsymbol{v}]\!] \, \mathrm{d}S.$$

Note that $a_1^h(\tilde{c}, \boldsymbol{u}, \boldsymbol{v}) = \tilde{a}_1^h(\tilde{c}, \boldsymbol{u}, \boldsymbol{v}) + K_h(\tilde{c}, \boldsymbol{u}, \boldsymbol{v})$, where

$$K_h(\tilde{c}, \boldsymbol{u}, \boldsymbol{v}) := \sum_{e \in \mathscr{E}_h} \int_e -\{\!\{\nu(\tilde{c}) \boldsymbol{\nabla}_h(\boldsymbol{u}) \boldsymbol{n}_e\}\!\} \cdot [\![\boldsymbol{v}]\!] - \{\!\{\nu(\tilde{c}) \boldsymbol{\nabla}_h(\boldsymbol{v}) \boldsymbol{n}_e\}\!\} \cdot [\![\boldsymbol{u}]\!] \, \mathrm{d}S,$$

and we point out that $\mathscr{A}_h^{(\cdot, \cdot)} \big((\boldsymbol{u}, p, s, c), (\boldsymbol{v}, q, \phi, \psi) \big)$ is well-defined also for every $(\boldsymbol{u}, p, s, c), (\boldsymbol{v}, q, \phi, \psi) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$.

THEOREM 3.1 (Global inf-sup stability) Let the pair $(\tilde{\boldsymbol{u}}, \tilde{c}) \in \tilde{X}(\mathscr{T}_h) \times H_0^1(\Omega)$ satisfy $\|\tilde{\boldsymbol{u}}\|_{1,\mathscr{T}_h} < M$, for a sufficiently small $M > 0$ depending on C_a and C_c. For any $(\boldsymbol{u}, p, s, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$, there exists $(\boldsymbol{v}, q, \phi, \psi) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ with $\|\!|(\boldsymbol{v}, q, \phi, \psi)|\!\| \leqslant 1$ such that

$$\mathscr{A}_h^{(\tilde{\boldsymbol{u}}, \tilde{c})} \big((\boldsymbol{u}, p, s, c), (\boldsymbol{v}, q, \phi, \psi) \big) \geqslant C \|\!|(\boldsymbol{u}, p, s, c)|\!\|,$$

where the norm on the product space is defined as

$$\|\!|(\boldsymbol{v}, q, \phi, \psi)|\!\|^2 := \|\boldsymbol{v}\|_{1,\mathscr{T}_h}^2 + \|q\|_{0,\Omega}^2 + \|\phi\|_{1,\Omega}^2 + \|\psi\|_{1,\Omega}^2.$$

Proof. For any $(\boldsymbol{u}, p, s, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ there holds

$$\mathscr{A}_h^{(\tilde{\boldsymbol{u}}, \tilde{c})} \big((\boldsymbol{u}, p, s, c), (\boldsymbol{u}, -p, s, c) \big) \geqslant \alpha_a \|\boldsymbol{u}\|_{1,\Omega}^2 + \tilde{\alpha}_a \|s\|_{1,\Omega}^2 + \frac{\tilde{\alpha}_a}{\tau} \|c\|_{1,\Omega}^2.$$

Applying the inf-sup condition, we get that for any $p \in L_0^2(\Omega)$, there exists $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ such that $b(\mathbf{v}, p) \geqslant \beta \|p\|_{0,\Omega}^2$ and $\|\mathbf{v}\|_{1,\Omega} \leqslant \|p\|_{0,\Omega}$, where $\beta > 0$ is the inf-sup constant depending only on Ω. Then, we have

$$
\begin{aligned}
\mathscr{A}_h^{(\tilde{u},\tilde{c})}(\mathbf{u}, p, s, c; \mathbf{v}, 0, 0, 0) &= a_1(\tilde{c}; \mathbf{u}, \mathbf{v}) + c_1(\tilde{\mathbf{u}}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) \\
&\geqslant \beta \|p\|_{0,\Omega}^2 - |a_1(\tilde{c}; \mathbf{u}, \mathbf{v})| - |c_1(\tilde{\mathbf{u}}; \mathbf{u}, \mathbf{v})| \\
&\geqslant \beta \|p\|_{0,\Omega}^2 - C_a \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} - C_c \|\tilde{\mathbf{u}}\|_{1,\mathscr{T}_h} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \\
&\geqslant \beta \|p\|_{0,\Omega}^2 - 2C_a \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \\
&\geqslant \beta \|p\|_{0,\Omega}^2 - 2C_a \|\mathbf{u}\|_{1,\Omega} \|p\|_{0,\Omega} \\
&\geqslant \left(\beta - \frac{1}{\varepsilon}\right) \|p\|_{0,\Omega}^2 - \varepsilon C_a^2 \|\mathbf{u}\|_{1,\Omega}^2,
\end{aligned}
$$

where $\varepsilon > 0$. Now, we introduce a $\delta > 0$ such that

$$
\begin{aligned}
\mathscr{A}_h^{(\tilde{u},\tilde{c})}\big((\mathbf{u}, p, s, c), (\mathbf{u} + \delta\mathbf{v}, -p, s, c)\big) &= \mathscr{A}_h^{(\tilde{u},\tilde{c})}(\mathbf{u}, p, s, c, \mathbf{u}, -p, s, c) + \delta \mathscr{A}_h^{(\tilde{u},\tilde{c})}(\mathbf{u}, p, s, c, \mathbf{v}, 0, 0, 0) \\
&\geqslant (\alpha_a - \delta\varepsilon C_a^2) \|\mathbf{u}\|_{1,\Omega}^2 + \delta\left(\beta - \frac{1}{\varepsilon}\right) \|p\|_{0,\Omega}^2 + \tilde{\alpha} \|s\|_{1,\Omega}^2 + \frac{\tilde{\alpha}}{\tau} \|c\|_{1,\Omega}^2.
\end{aligned}
$$

Choosing $\varepsilon = 2/\beta$ and $\delta = \alpha_a/(2\varepsilon C_a^2)$, we obtain

$$
\begin{aligned}
\mathscr{A}_h^{(\tilde{u},\tilde{c})}(\mathbf{u}, p, s, c, \mathbf{u} + \delta\mathbf{v}, -p, s, c) &\geqslant \frac{\alpha_a}{2} \|\mathbf{u}\|_{1,\Omega}^2 + \frac{\beta}{2} \|p\|_{0,\Omega}^2 + \tilde{\alpha} \|s\|_{1,\Omega}^2 + \frac{\tilde{\alpha}}{\tau} \|c\|_{1,\Omega}^2 \\
&\geqslant \min\left\{\frac{\alpha_a}{2}, \frac{\beta}{2}, \tilde{\alpha}, \frac{\tilde{\alpha}}{\tau}\right\} \left(\|\mathbf{u}\|_{1,\Omega}^2 + \|p\|_{0,\Omega}^2 + \|s\|_{1,\Omega}^2 + \|c\|_{1,\Omega}^2\right).
\end{aligned}
$$

Finally, using the triangle inequality, we can assert that

$$
\begin{aligned}
\||(\mathbf{u} + \delta\mathbf{v}, -p, s, c)\||^2 &= \|\mathbf{u} + \delta\mathbf{v}\|_{1,\Omega}^2 + \|p\|_{0,\Omega}^2 + \|s\|_{1,\Omega}^2 + \|c\|_{1,\Omega}^2 \\
&\leqslant 2\left(\|\mathbf{u}\|_{1,\Omega}^2 + \delta^2 \|\mathbf{v}\|_{1,\Omega}^2\right) + \|p\|_{0,\Omega}^2 + \|s\|_{1,\Omega}^2 + \|c\|_{1,\Omega}^2 \\
&\leqslant \max\{2, (1 + 2\delta^2)\} \left(\|\mathbf{u}\|_{1,\Omega}^2 + \|p\|_{0,\Omega}^2 + \|s\|_{1,\Omega}^2 + \|c\|_{1,\Omega}^2\right).
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Let $\widetilde{\mathbf{V}}_h$ be the discontinuous RT/BDM space. Now we define the conforming space $\mathbf{V}_h^c = \widetilde{\mathbf{V}}_h \cap \mathbf{H}_0^1(\Omega)$. Finally, we decompose the $\mathbf{H}(\text{div})$-conforming velocity approximation uniquely into $\mathbf{u}_h = \mathbf{u}_h^c + \mathbf{u}_h^r$, where $\mathbf{u}_h^c \in \mathbf{V}_h^c$ and $\mathbf{u}_h^r \in (\mathbf{V}_h^c)^\perp$, and we note that $\mathbf{u}_h^r = \mathbf{u}_h - \mathbf{u}_h^c \in \mathbf{V}_h$.

LEMMA 3.1 There holds

$$
\|\mathbf{u}_h^r\|_{1,\mathscr{T}_h} \leqslant C_r \left(\sum_{K \in \mathscr{T}_h} \Psi_{J_K}^2\right)^{1/2}.
$$

Proof. It follows straightforwardly from the decomposition $\mathbf{u}_h = \mathbf{u}_h^c + \mathbf{u}_h^r$ and from the facet residual. $\qquad\square$

LEMMA 3.2 If $\|\mathbf{u}\|_{1,\infty} < M$, $\|s\|_\infty < M$ and $\|c\|_\infty < M$ for sufficiently small M, then the following estimate holds:

$$
\begin{aligned}
\frac{C}{2} \||(e^{\mathbf{u}}, e^p, e^s, e^c)\|| &\leqslant \int_\Omega (\mathbf{f} - \mathbf{f}_h) \cdot \mathbf{v} \, d\mathbf{x} + \int_\Omega \mathbf{f}_h \cdot (\mathbf{v} - \mathbf{v}_h) \, d\mathbf{x} + \int_\Omega f_1 (\phi - \phi_h) \, d\mathbf{x} + \int_\Omega f_2 (\psi - \psi_h) \, d\mathbf{x} + K_h(\mathbf{u}_h, \mathbf{v}_h) \\
&\quad - \mathscr{A}_h^{(\mathbf{u}_h, c_h)}(\mathbf{u}_h, p_h, s_h, c_h, \mathbf{v} - \mathbf{v}_h, q, \phi - \phi_h, \psi - \psi_h) + (1 + C) C_r \left(\sum_{K \in \mathscr{T}_h} \Psi_{j_K}^2\right)^{1/2},
\end{aligned}
$$

where
$$\boldsymbol{e^u} := \boldsymbol{u} - \boldsymbol{u}_h, \quad e^p := p - p_h, \quad e^s := s - s_h, \quad e^c := c - c_h.$$

Moreover, $\boldsymbol{v}_h$, ϕ_h and ψ_h denote in this case the Clément interpolations of $\boldsymbol{v} \in \mathbf{H}_0^1(\Omega)$, $\phi_h \in H_0^1(\Omega)$ and $\psi_h \in H_0^1(\Omega)$, respectively.

Proof. Using $\boldsymbol{u}_h = \boldsymbol{u}_h^c + \boldsymbol{u}_h^r$, $\boldsymbol{e}_c^u = \boldsymbol{u} - \boldsymbol{u}_h^c$ and the triangle inequality imply

$$\||(\boldsymbol{e^u}, e^p, e^s, e^c)\|| \leqslant \||(\boldsymbol{e}_c^u, e^p, e^s, e^c)\|| + \|\boldsymbol{u}_h^r\|_{1,\mathscr{T}_h} \leqslant \||(\boldsymbol{e}_c^u, e^p, e^s, e^c)\|| + C_r \left(\sum_{K \in \mathscr{T}_h} \Psi_{J_K}^2 \right)^{1/2},$$

where $(\boldsymbol{e}_c^u, e^p, e^s, e^c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$. Then, Theorem 3.1 gives

$$\begin{aligned}
C\||(\boldsymbol{e}_c^u, e^p, e^s, e^c)\|| &\leqslant \mathscr{A}_h^{(\boldsymbol{u}_h, c_h)}(\boldsymbol{e}_c^u, e^p, e^s, e^c; \boldsymbol{v}, q, \phi, \psi) \\
&\leqslant \mathscr{A}_h^{(\boldsymbol{u}_h, c_h)}(\boldsymbol{e^u}, e^p, e^s, e^c; \boldsymbol{v}, q, \phi, \psi) + \mathscr{A}_h^{(\boldsymbol{u}_h, c_h)}(\boldsymbol{u}_h^r, 0, 0, 0; \boldsymbol{v}, q, \phi, \psi) \\
&\leqslant \mathscr{A}_h^{(\boldsymbol{u}_h, c_h)}(\boldsymbol{e^u}, e^p, e^s, e^c; \boldsymbol{v}, q, \phi, \psi) + C_r \left(\sum_{K \in \mathscr{T}_h} \Psi_{J_K}^2 \right)^{1/2},
\end{aligned}$$

with $\||(\boldsymbol{v}, q, \phi, \psi)\|| \leqslant 1$. Owing to the relation

$$\begin{aligned}
\mathscr{A}_h^{(\boldsymbol{u}_h, c_h)}(\boldsymbol{u}, p, s, c; \boldsymbol{v}, q, \phi, \psi) &= \mathscr{A}_h^{(\boldsymbol{u}, c)}(\boldsymbol{u}, p, s, c; \boldsymbol{v}, q, \phi, \psi) - a_1(c; \boldsymbol{u}, \boldsymbol{v}) + a_1(c_h; \boldsymbol{u}, \boldsymbol{v}) \\
&\quad - c_1(\boldsymbol{e^u}; \boldsymbol{u}, \boldsymbol{v}) - c_2(\boldsymbol{e^u}; s, \phi) - c_2(\boldsymbol{e^u}; c, \psi),
\end{aligned}$$

we then have

$$\begin{aligned}
C\||(\boldsymbol{e^u}, e^p, e^s, e^c)\|| &\leqslant C\||(\boldsymbol{e}_c^u, e^p, e^s, e^c)\|| + C_r \left(\sum_{K \in \mathscr{T}_h} \Psi_{J_K}^2 \right)^{1/2} \\
&\leqslant \mathscr{A}_h^{(\boldsymbol{u}, c)}(\boldsymbol{u}, p, s, c; \boldsymbol{v}, q, \phi, \psi) - a_1(c; \boldsymbol{u}, \boldsymbol{v}) + a_1(c_h, \boldsymbol{u}, \boldsymbol{v}) - c_1(\boldsymbol{e^u}; \boldsymbol{u}, \boldsymbol{v}) - c_2(\boldsymbol{e^u}; s, \phi) \\
&\quad - c_2(\boldsymbol{e^u}; c, \psi) - \mathscr{A}_h^{(\boldsymbol{u}_h, c_h)}(\boldsymbol{u}_h, p_h, s_h, c_h; \boldsymbol{v}, q, \phi, \psi) + (1 + C)C_r \left(\sum_{K \in \mathscr{T}_h} \Psi_{J_K}^2 \right)^{1/2},
\end{aligned}$$

while using the properties

$$\begin{aligned}
\left| a_1(c; \boldsymbol{u}, \boldsymbol{v}) - a_1(c_h; \boldsymbol{u}, \boldsymbol{v}) \right| &\leqslant C_1 \|c - c_h\|_{1,\Omega} \|\boldsymbol{u}\|_{1,\infty} \|\boldsymbol{v}\|_{1,\Omega} \leqslant C_1 M \|e^c\|_{1,\Omega}, \\
c_1(\boldsymbol{e^u}; \boldsymbol{u}, \boldsymbol{v}) &\leqslant C_2 M \|\boldsymbol{e^u}\|_{1,\mathscr{T}_h}, \quad c_2(\boldsymbol{e^u}; s, \phi) \leqslant C_3 M \|\boldsymbol{e^u}\|_{1,\mathscr{T}_h}, \quad c_2(\boldsymbol{e^u}; c, \psi) \leqslant C_4 M \|\boldsymbol{e^u}\|_{1,\mathscr{T}_h}
\end{aligned}$$

yields the bound

$$\begin{aligned}
C\||(\boldsymbol{e^u}, e^p, e^s, e^c)\|| &\leqslant \mathscr{A}_h^{(\boldsymbol{u}, c)}(\boldsymbol{u}, p, s, c; \boldsymbol{v}, q, \phi, \psi) - \mathscr{A}_h^{(\boldsymbol{u}_h, c_h)}(\boldsymbol{u}_h, p_h, s_h, c_h; \boldsymbol{v}, q, \phi, \psi) \\
&\quad + (1 + C)C_r \left(\sum_{K \in \mathscr{T}_h} \Psi_{J_K}^2 \right)^{1/2} - (C_1 + C_2 + C_3 + C_4)M\||(\boldsymbol{e^u}, e^p, e^s, e^c)\||.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\frac{C}{2}\||(\boldsymbol{e^u}, e^p, e^s, e^c)\|| &\leqslant \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x} + \int_\Omega f_1 \phi \, d\boldsymbol{x} + \int_\Omega f_2 \psi \, d\boldsymbol{x} - \mathscr{A}_h^{(\boldsymbol{u}_h, c_h)}(\boldsymbol{u}_h, p_h, s_h, c_h; \boldsymbol{v}, q, \phi, \psi) \\
&\quad + (1 + C)C_r \left(\sum_{K \in \mathscr{T}_h} \Psi_{j_K}^2 \right)^{1/2}, \qquad\qquad\qquad\qquad (3.5)
\end{aligned}$$

and we readily see that after stating the discrete problem as

$$a_1^h(c_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + c_1^h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) - \int_\Omega \boldsymbol{f}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} = 0 \quad \text{for all } \boldsymbol{v}_h \in \mathbf{V}_h,$$

$$a_2(s_h, \phi_h) + c_2(\boldsymbol{u}_h; s_h, \phi_h) - \int_\Omega f_1 \phi_h \, \mathrm{d}\boldsymbol{x} = 0 \quad \text{for all } \phi_h \in \mathscr{M}_{h,0},$$

$$\frac{1}{\tau} a_2(c_h, \psi_h) + c_2(\boldsymbol{u}_h - v_p e_z; c_h, \psi_h) - \int_\Omega f_2 \psi_h \, \mathrm{d}\boldsymbol{x} = 0 \quad \text{for all } \psi_h \in \mathscr{M}_{h,0},$$

and employing (3.5), the sought results follow. □

LEMMA 3.3 For $(\boldsymbol{v}, q, s, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$, there are $\boldsymbol{v}_h \in \mathbf{V}_h$, $s_h \in \mathscr{M}_{h,0}$ and $c_h \in \mathscr{M}_{h,0}$ such that

$$\int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{v} - \boldsymbol{v}_h) \, \mathrm{d}\boldsymbol{x} + \int_\Omega f_1 (\phi - \phi_h) \, \mathrm{d}\boldsymbol{x} + \int_\Omega f_2 (\psi - \psi_h) \, \mathrm{d}\boldsymbol{x}$$

$$- \mathscr{A}_h^{(u_h, c_h)}(\boldsymbol{u}_h, p_h, s_h, c_h, \boldsymbol{v} - \boldsymbol{v}_h, q, \phi - \phi_h, \psi - \psi_h) \leqslant C\left(\Psi + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}\right) \|\|(\boldsymbol{v}, q, s, c)\|\|. \quad (3.6)$$

Proof. Using integration by parts gives

$$\int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}_h \cdot (\boldsymbol{v} - \boldsymbol{v}_h) \, \mathrm{d}\boldsymbol{x} + \int_\Omega f_1 (\phi - \phi_h) \, \mathrm{d}\boldsymbol{x} + \int_\Omega f_2 (\psi - \psi_h) \, \mathrm{d}\boldsymbol{x}$$

$$- \mathscr{A}_h^{(u_h, c_h)}(\boldsymbol{u}_h, p_h, s_h, c_h, \boldsymbol{v} - \boldsymbol{v}_h, q, \phi - \phi_h, \psi - \psi_h) =: T_1 + \cdots + T_5, \quad (3.7)$$

where we define the terms

$$T_1 := \sum_{K \in \mathscr{T}_h} \int_K \left(\boldsymbol{f}_h + \mathbf{div}\left(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h\right) - \boldsymbol{u}_h \cdot \boldsymbol{\nabla}\boldsymbol{u}_h - \frac{1}{\rho_\mathrm{m}}\nabla p_h \right) \cdot (\boldsymbol{v} - \boldsymbol{v}_h) \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x},$$

$$T_2 := \sum_{K \in \mathscr{T}_h} \int_{\partial K} \left(\left(\frac{1}{\rho_\mathrm{m}} p_h \mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h \right) \cdot \boldsymbol{n}_K \right) \cdot (\boldsymbol{v} - \boldsymbol{v}_h) \, \mathrm{d}S,$$

$$T_3 := \sum_{K \in \mathscr{T}_h} \int_{\partial K_{in} \setminus \Gamma} \boldsymbol{u}_h \cdot \boldsymbol{n}_K (\boldsymbol{u}_h - \boldsymbol{u}_h^e) \cdot (\boldsymbol{v} - \boldsymbol{v}_h) \, \mathrm{d}S,$$

$$T_4 := \sum_{K \in \mathscr{T}_h} \int_K \left(f_1 + \frac{1}{\mathrm{Sc}} \Delta s_h - \boldsymbol{u}_h \cdot \nabla s_h \right) (\phi - \phi_h) \, \mathrm{d}\boldsymbol{x} + \sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{1}{\mathrm{Sc}} \nabla s_h \cdot \boldsymbol{n}_K (\phi - \phi_h) \, \mathrm{d}S,$$

$$T_5 := \sum_{K \in \mathscr{T}_h} \int_K \left(f_2 + \frac{1}{\tau \mathrm{Sc}} \Delta c_h - (\boldsymbol{u}_h - v_p e^z) \cdot \nabla c_h \right) (\psi - \psi_h) \, \mathrm{d}\boldsymbol{x} + \sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{1}{\tau \mathrm{Sc}} \nabla c_h \cdot \boldsymbol{n}_K (\psi - \psi_h) \, \mathrm{d}S.$$

Applying the Cauchy–Schwarz inequality to T_1 implies

$$T_1 \leqslant \left(\sum_{K \in \mathscr{T}_h} h_K^2 \|\mathbf{R}_K\|_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathscr{T}_h} h_K^{-2} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,K}^2 \right)^{1/2} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega} \|\boldsymbol{v}\|_{0,\Omega}$$

$$\leqslant \left(\sum_{K \in \mathscr{T}_h} h_K^2 \|\mathbf{R}_K\|_{0,K}^2 \right)^{1/2} \bar{C} \|\boldsymbol{\nabla}\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega} \|\boldsymbol{v}\|_{0,\Omega}.$$

Next, we rewrite T_2 in terms of a sum over interior facets and apply again the Cauchy–Schwarz inequality. Then

$$T_2 = \sum_{e \in \mathscr{E}_h} \int_e [\![((\rho_\mathrm{m})^{-1} p_h \mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h)\boldsymbol{n}]\!] \cdot (\boldsymbol{v} - \boldsymbol{v}_h) \, \mathrm{d}S$$

$$\leqslant \left(\sum_{e \in \mathscr{E}_h} h_e \|\mathbf{R}_e\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathscr{E}_h} h_e^{-1} \|\mathbf{v} - \mathbf{v}_h\|_{0,e}^2 \right)^{1/2}$$

$$\leqslant \left(\sum_{e \in \mathscr{E}_h} h_e \|\mathbf{R}_e\|_{0,e}^2 \right)^{1/2} \bar{C} \|\boldsymbol{\nabla} \mathbf{v}\|_{0,\Omega}.$$

Therefore, owing to the Cauchy–Schwarz inequality, it follows that

$$T_3 \leqslant \left(\sum_{e \in \mathscr{E}_h} \|[\![\boldsymbol{u}_h]\!]\|_{0,e}^2 \right)^{1/2} \bar{C} \|\boldsymbol{\nabla} \mathbf{v}\|_{0,\Omega}.$$

Proceeding in a similar fashion, we are able to establish the following bounds for T_4 and T_5:

$$T_4 \leqslant \left[\left(\sum_{K \in \mathscr{T}_h} h_K^2 \|R_{1,K}\|_{0,K}^2 \right)^{1/2} + \left(\sum_{e \in \mathscr{E}_h} h_e \|R_{1,e}\|_{0,e}^2 \right)^{1/2} \right] \bar{C} \|\nabla \phi\|_{0,\Omega},$$

$$T_5 \leqslant \left[\left(\sum_{K \in \mathscr{T}_h} h_K^2 \|R_{2,K}\|_{0,K}^2 \right)^{1/2} + \left(\sum_{e \in \mathscr{E}_h} h_e \|R_{2,e}\|_{0,e}^2 \right)^{1/2} \right] \bar{C} \|\nabla \psi\|_{0,\Omega}.$$

Finally, (3.6) results as a combination of the bounds derived for T_1, T_2, T_3, T_4 and T_5, together with (3.7). □

THEOREM 3.2 Let $(\boldsymbol{u}, p, s, c)$ be the unique solution to (3.1) and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ a solution to (3.2). Let Ψ be the *a posteriori* error estimator defined in (3.4). If $\|\boldsymbol{u}\|_{1,\infty} < M$, $\|s\|_\infty < M$ and $\|c\|_\infty < M$ for sufficiently small M, then the following estimate holds:

$$\||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, s - s_h, c - c_h)\|| \leqslant C(\Psi + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}),$$

where $C > 0$ is a constant independent of h.

Proof. It suffices to apply Lemmas 3.2 and 3.3. □

REMARK 3.1 Assume that the errors $\|s - s_h\|_{0,\Omega}$ and $\|c - c_h\|_{0,\Omega}$ converge with optimal rate $O(h^{k+1/2})$. Using the triangle inequality, we readily see that

$$\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega} \leqslant C(\|s - s_h\|_{0,\Omega} + \|c - c_h\|_{0,\Omega}),$$

where C is a positive constant. It is then clear that the term $\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}$ decays with optimal rate $O(h^{k+1/2})$. Furthermore, if the energy error $\||\cdot\||$ converges with optimal rate $O(h^k)$, then $\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}$ will be a higher-order term. We remark that similar types of conditions are encountered, for instance, in the *a posteriori* analysis for eigenvalue problems. See, e.g., (Gedicke & Khan, 2020, Lemma 5).

3.3 *Efficiency*

For each $K \in \mathscr{T}_h$, we can define the standard polynomial bubble function b_K. Then, for any polynomial function $\boldsymbol{v}$ on K, the following results hold:

$$\|b_K \boldsymbol{v}\|_{0,K} \leqslant C\|\boldsymbol{v}\|_{0,K}, \qquad \|\boldsymbol{v}\|_{0,K} \leqslant C\|b_K^{1/2} \boldsymbol{v}\|_{0,K}, \qquad \|\nabla(b_K \boldsymbol{v})\|_{0,K} \leqslant Ch_K^{-1} \|\boldsymbol{v}\|_{0,K}, \tag{3.8}$$

where C is a positive constant, independent of K and $\boldsymbol{v}$, see (Verfürth, 1996, Section 3.4).

REMARK 3.2 Another assumption that is incorporated to simplify the presentation is that the viscosity $\nu(c)$ is polynomial. Note however, that we can still prove efficiency of the estimator without using such a simplification. For this, it suffices to define $\nu_h : c \in H^1(K) \to \nu_h(c) \in \mathbb{P}_1$ as

$$\nu_h(c)|_K := \frac{1}{|K|} \int_K \nu(c) \, \mathrm{d}\boldsymbol{x} + \left[\frac{1}{K} \int_K (\nabla \nu(c)) \right] \cdot (\boldsymbol{x} - \boldsymbol{x}_0) \, \mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{x}_0$ is the centre of the element K with $|\boldsymbol{x} - \boldsymbol{x}_0| \leqslant h_K$. For more details, we refer to Dib *et al.* (2019).

LEMMA 3.4 Let $(\boldsymbol{u}, p, s, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ be the weak solution to (3.1). Then, the following estimates hold, where C is a positive constant:

$$h_K\|\mathbf{R}_K\|_{0,K} \leqslant C\big(\|c - c_h\|_{1,K} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K} + \|p - p_h\|_{0,K} + h_K\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}\big),$$
$$h_K\|R_{1,K}\|_{0,K} \leqslant C\big(\|s - s_h\|_{1,K} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}\big),$$
$$h_K\|R_{2,K}\|_{0,K} \leqslant C\big(\|c - c_h\|_{1,K} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}\big).$$

Moreover, it also follows that

$$\Psi_K \leqslant C\|\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, s - s_h, c - c_h)\|\|_K.$$

Proof. For each $K \in \mathscr{T}_h$, we define $\mathbf{W}_b = b_K \mathbf{R}_K$. Then, using (3.8), we have

$$\frac{1}{C^2}\|\mathbf{R}_K\|_{0,K}^2 \leqslant \|b_K^{1/2}\mathbf{R}_K\|_{0,K}^2 = \int_K \mathbf{R}_K \cdot \mathbf{W}_b\, \mathrm{d}\boldsymbol{x}$$
$$= \int_K \left(\boldsymbol{f}_h + \mathbf{div}\big(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h\big) - (\boldsymbol{u}_h \cdot \boldsymbol{\nabla})\boldsymbol{u}_h - \frac{1}{\rho_m}\nabla p_h\right) \cdot \mathbf{W}_b\, \mathrm{d}\boldsymbol{x} =: T_1 + T_2,$$

where

$$T_1 := \int_K \left(\big((\nu(c) - \nu(c_h))\boldsymbol{\nabla}\boldsymbol{u} + \nu(c_h)\boldsymbol{\nabla}(\boldsymbol{u} - \boldsymbol{u}_h)\big) : \boldsymbol{\nabla}\mathbf{W}_b - \frac{1}{\rho_m}(p - p_h)\,\mathrm{div}\,\mathbf{W}_b\right) \mathrm{d}\boldsymbol{x} + \int_K (\boldsymbol{f}_h - \boldsymbol{f}) \cdot \mathbf{W}_b\, \mathrm{d}\boldsymbol{x},$$
$$T_2 := \int_K \big(((\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{\nabla})\boldsymbol{u} + (\boldsymbol{u}_h \cdot \boldsymbol{\nabla})(\boldsymbol{u} - \boldsymbol{u}_h)\big) \cdot \mathbf{W}_b\, \mathrm{d}\boldsymbol{x}.$$

Using the Cauchy–Schwarz inequality and (3.8) we obtain

$$T_1 \leqslant C_1\big(\|c - c_h\|_{1,K} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K} + \|p - p_h\|_{0,K} + h_K\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}\big)h_K^{-1}\|\mathbf{R}_K\|_{0,K},$$
$$T_2 \leqslant C_2\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}h_K^{-1}\|\mathbf{R}_K\|_{0,K},$$

and combining these bounds leads to the first stated result. The other two bounds follow similarly. $\qquad\square$

Let e denote an interior facet that is shared by two elements K and K'. Let ω_e be the patch which is the union of K and K'. Next, we define the facet bubble function ζ_e on e with the property that it is positive in the interior of the patch ω_e and zero on the boundary of the patch. From Verfürth (1996), the following results hold:

$$\|q\|_{0,e} \leqslant C\|\zeta_e^{1/2}q\|_{0,e}, \tag{3.9a}$$
$$\|\zeta_e q\|_{0,K} \leqslant Ch_e^{1/2}\|q\|_{0,e}, \quad \|\nabla(\zeta_e q)\|_{0,K} \leqslant Ch_e^{-1/2}\|q\|_{0,e} \qquad \text{for all } K \in \omega_e. \tag{3.9b}$$

LEMMA 3.5 The following estimates hold:

$$h_e\|\mathbf{R}_e\|_{0,e}^2 \leqslant C \sum_{K \in \omega_e}\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}^2 + \|c - c_h\|_{1,K}^2 + \|p - p_h\|_{0,K}^2 + h_K^2\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}^2\big),$$
$$h_e\|R_{1,e}\|_{0,e}^2 \leqslant C \sum_{K \in \omega_e}\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}^2 + \|s - s_h\|_{1,K}^2\big),$$
$$h_e\|R_{2,e}\|_{0,e}^2 \leqslant C \sum_{K \in \omega_e}\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,K}^2 + \|c - c_h\|_{1,K}^2\big).$$

Moreover, we also have

$$\Psi_{e_K}^2 \leqslant C \sum_{e \in \partial K}\sum_{K \in \omega_e}\big(\|\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h, s - s_h, c - c_h)\|\|_K^2 + h_K^2\|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}^2\big).$$

Proof. Let e be an interior facet and let us define a rescaling of the facet bubble function in the form

$$\boldsymbol{\vartheta}_e := \sum_{e\in\partial K} \frac{h_e}{2} \mathbf{R}_e \zeta_e.$$

Using (3.9) gives

$$\begin{aligned}
h_e \|\mathbf{R}_e\|_{0,e}^2 &\leqslant C\big([\![(\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h]\!], \boldsymbol{\vartheta}_e\big)_e \\
&\leqslant C\big([\![(\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h]\!] - [\![(\rho_{\mathrm{m}})^{-1} p \mathbf{I} - \nu(c)\boldsymbol{\nabla}\boldsymbol{u}]\!], \boldsymbol{\vartheta}_e\big)_e.
\end{aligned} \tag{3.10}$$

Using integration by parts on each element of patch ω_e implies

$$\begin{aligned}
\big([\![(\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h]\!], \boldsymbol{\vartheta}_e\big)_e &= \sum_{K\in\omega_e} \int_K \left(\mathbf{div}\big(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h\big) - \mathbf{div}\big(\nu(c)\boldsymbol{\nabla}\boldsymbol{u}\big) + \frac{1}{\rho_{\mathrm{m}}}\nabla(p-p_h) \right) \cdot \boldsymbol{\vartheta}_e \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_K \left(\frac{1}{\rho_{\mathrm{m}}}(p-p_h)\mathbf{I} + \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h - \nu(c)\boldsymbol{\nabla}\boldsymbol{u} \right) : \boldsymbol{\nabla}\boldsymbol{\vartheta}_e \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Note that $(\boldsymbol{u}, p, s, c)$ solves the underlying problem, so we then have

$$\begin{aligned}
\big([\![(\rho_{\mathrm{m}})^{-1} p_h \mathbf{I} - \nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h]\!], \boldsymbol{\vartheta}_e\big)_e &= \sum_{K\in\omega_e} \int_K \left(\boldsymbol{f} + \mathbf{div}\big(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h\big) - \boldsymbol{u}_h\cdot\boldsymbol{\nabla}\boldsymbol{u}_h - \frac{1}{\rho_{\mathrm{m}}}\nabla p_h \right) \cdot \boldsymbol{\vartheta}_e \,\mathrm{d}\boldsymbol{x} \\
&\quad + \sum_{K\in\omega_e} \int_K (\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u}_h\cdot\boldsymbol{\nabla}\boldsymbol{u}_h) \cdot \boldsymbol{\vartheta}_e \,\mathrm{d}\boldsymbol{x} \\
&\quad + \sum_{K\in\omega_e} \int_K \left(\frac{p-p_h}{\rho_{\mathrm{m}}}\mathbf{I} - \big(\nu(c)-\nu(c_h)\big)\boldsymbol{\nabla}\boldsymbol{u}_h - \nu(c)\boldsymbol{\nabla}(\boldsymbol{u}-\boldsymbol{u}_h) \right) : \boldsymbol{\nabla}\boldsymbol{\vartheta}_e \,\mathrm{d}\boldsymbol{x} \\
&=: T_1 + T_2 + T_3. \tag{3.11}
\end{aligned}$$

Next, applying the Cauchy–Schwarz inequality together with Lemma 3.4 and (3.9) gives

$$\begin{aligned}
T_1 &\leqslant C_1 \left(\sum_{K\in\omega_e} h_K^2 \|\mathbf{R}_K\|_{0,K}^2 + h_K^2 \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}^2 \right)^{1/2} \left(\sum_{K\in\omega_e} h_K^{-2} \|\boldsymbol{\vartheta}_e\|_{0,K}^2 \right)^{1/2} \\
&\leqslant C_1 \left(\sum_{K\in\omega_e} |\!|\!|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, s-s_h, c-c_h)|\!|\!|_K^2 \right)^{1/2} h_e^{1/2} \|\mathbf{R}_e\|_{0,e}, \\
T_2 &\leqslant C_2 \left(\sum_{K\in\omega_e} \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,K}^2 \right)^{1/2} h_e^{1/2} \|\mathbf{R}_e\|_{0,e}, \\
T_3 &\leqslant C_3 \left(\sum_{K\in\omega_e} |\!|\!|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, s-s_h, c-c_h)|\!|\!|_K^2 \right)^{1/2} h_e^{1/2} \|\mathbf{R}_e\|_{0,e}.
\end{aligned}$$

Combining the bounds of T_1, T_2 and T_3 with (3.10) and (3.11) implies the first stated result. Similarly, we can prove the other two bounds. $\qquad\square$

THEOREM 3.3 Let $(\boldsymbol{u}, p, s, c)$ be the unique solution to (3.1) and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ a solution of problem (3.2). Let Ψ be defined as in (3.4). Then, there exists a constant $C > 0$ that is independent of h such that

$$\Psi \leqslant C\left(|\!|\!|(\boldsymbol{u}-u_h, p-p_h, s-s_h, c-c_h)|\!|\!| + \left(\sum_{K\in\mathscr{T}_h} h_K^2 \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}^2 \right)^{1/2} \right).$$

Proof. Combining Lemmas 3.4 and 3.5 implies the stated result. $\qquad\square$

4. *A posteriori* error bound for the semidiscrete method

For $t \in (0, t_{\text{end}}]$, let us consider the problem: find $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{c}, \tilde{s}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$a_1(c_h, \tilde{\boldsymbol{u}}, \boldsymbol{v}) + c_1(\boldsymbol{u}_h; \tilde{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}) = (\widetilde{\boldsymbol{f}}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \tag{4.1a}$$

$$b(\tilde{\boldsymbol{u}}, q) = 0 \qquad \text{for all } q \in L_0^2(\Omega), \tag{4.1b}$$

$$a_2(\tilde{s}, \phi) + c_2(\boldsymbol{u}_h; \tilde{s}, \phi) = (f_1, \phi) \quad \text{for all } \phi \in H_0^1(\Omega), \tag{4.1c}$$

$$\frac{1}{\tau} a_2(\tilde{c}, \psi) + c_2(\boldsymbol{u}_h - v_p \boldsymbol{e}_z; \tilde{c}, \psi) = (f_2, \psi) \quad \text{for all } \psi \in H_0^1(\Omega), \tag{4.1d}$$

where

$$\widetilde{\boldsymbol{f}} = (\alpha s_h + \beta c_h)\mathbf{g} - \partial_t \boldsymbol{u}_h \in \mathbf{L}^2(\Omega), \quad f_1 = -\partial_t s_h \in L^2(\Omega), \quad f_2 = -\partial_t c_h \in L^2(\Omega). \tag{4.2}$$

The time derivatives of the semi-discrete velocity, salinity and concentration are considered on the right-hand sides as in the so-called elliptic reconstruction approach from, e.g., Cangiani *et al.* (2014) and Cangiani *et al.* (2020). Also, for each $t \in (0, t_{\text{end}}]$, we write the discrete weak formulation: find $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{c}_h, \tilde{s}_h) \in C^{0,1}(0, t_{\text{end}}; \mathbf{V}_h) \times C^{0,0}(0, t_{\text{end}}; \mathscr{Q}_h) \times C^{0,1}(0, t_{\text{end}}; \mathscr{M}_{h,0}) \times C^{0,1}(0, t_{\text{end}}; \mathscr{M}_{h,0})$ such that

$$a_1(c_h, \tilde{\boldsymbol{u}}_h, \boldsymbol{v}) + c_1(\boldsymbol{u}_h; \tilde{\boldsymbol{u}}_h, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}) = (\widetilde{\boldsymbol{f}}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h, \tag{4.3a}$$

$$b(\tilde{\boldsymbol{u}}_h, q) = 0 \qquad \text{for all } q \in \mathscr{Q}_h, \tag{4.3b}$$

$$a_2(\tilde{s}_h, \phi) + c_2(\boldsymbol{u}_h; \tilde{s}_h, \phi) = (f_1, \phi) \quad \text{for all } \phi \in \mathscr{M}_{h,0}, \tag{4.3c}$$

$$\frac{1}{\tau} a_2(\tilde{c}_h, \psi) + c_2(\boldsymbol{u}_h - v_p \boldsymbol{e}_z; \tilde{c}_h, \psi) = (f_2, \psi) \quad \text{for all } \psi \in \mathscr{M}_{h,0}, \tag{4.3d}$$

where (4.2) remains in effect.

REMARK 4.1 For given $\boldsymbol{u}_h$, s_h and c_h, the well-posedness of the continuous weak formulation (4.1) and of the discrete weak formulation (4.3) follow from Bürger *et al.* (2019), for each $t \in (0, t_{\text{end}}]$, and using the data (4.2).

REMARK 4.2 From (2.1), we have that $(\boldsymbol{u}_h, p_h, c_h, s_h)$ is also a discrete solution of (4.3) for each $t \in (0, t_{\text{end}}]$. But, since the discrete weak formulation (4.3) is well-posed, we also conclude that $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{c}_h, \tilde{s}_h) = (\boldsymbol{u}_h, p_h, c_h, s_h)$ for each $t \in (0, t_{\text{end}}]$.

LEMMA 4.1 For each $t \in (0, t_{\text{end}}]$ and for all $(\boldsymbol{v}, q, \phi, \psi) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ we have

$$(\partial_t e_{\boldsymbol{u}}, \boldsymbol{v}) + a_1(c, \rho_{\boldsymbol{u}}, \boldsymbol{v}) + c_1(\boldsymbol{u}; \rho_{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, p - \tilde{p}) = a_1(c_h, \tilde{\boldsymbol{u}}, \boldsymbol{v}) - a_1(c, \tilde{\boldsymbol{u}}, \boldsymbol{v}) - c_1(e_{\boldsymbol{u}}; \tilde{\boldsymbol{u}}, \boldsymbol{v}) + (\boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{v})$$
$$\text{for all } \boldsymbol{v} \in \mathbf{H}_0^1(\Omega),$$

$$b(\boldsymbol{u} - \tilde{\boldsymbol{u}}, q) = 0 \qquad \text{for all } q \in L_0^2(\Omega),$$

$$(\partial_t e_s, \phi) + a_2(\rho_s, \phi) + c_2(\boldsymbol{u}; \rho_s, \phi) = -c_2(e_{\boldsymbol{u}}; \tilde{s}, \phi) \quad \text{for all } \phi \in H_0^1(\Omega),$$

$$(\partial_t e_c, \psi) + \frac{1}{\tau} a_2(\rho_c, \psi) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; \rho_c, \psi) = -c_2(e_{\boldsymbol{u}}; \tilde{c}, \psi) \quad \text{for all } \psi \in H_0^1(\Omega),$$

where $e_{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_h$, $e_s = s - s_h$, $e_c = c - c_h$, $\rho_{\boldsymbol{u}} = \boldsymbol{u} - \tilde{\boldsymbol{u}}$, $\rho_s = s - \tilde{s}$ and $\rho_c = c - \tilde{c}$.

Next we introduce the semi-discrete error indicator Θ as

$$\Theta^2 := \|e_{\boldsymbol{u}}(0)\|_{0,\Omega}^2 + \|e_c(0)\|_{0,\Omega}^2 + \|e_s(0)\|_{0,\Omega}^2 + \int_0^{t_{\text{end}}} \Psi^2 \, \mathrm{d}t + \int_0^{t_{\text{end}}} \Theta_2^2 \, \mathrm{d}t + \max_{0 \leqslant t \leqslant T} \Theta_3^2, \tag{4.4}$$

where

$$\Theta_2^2 := \sum_{e \in \mathscr{E}_h} h_e \|[\![\partial_t \boldsymbol{u}_h]\!]\|_{0,e}^2, \qquad \Theta_3^2 := \sum_{e \in \mathscr{E}_h} h_e \|[\![\boldsymbol{u}_h]\!]\|_{0,e}^2,$$

whereas Ψ is the global *a posteriori* error estimator for the steady problem with element and facet residual contributions defined in (3.3). In this case we now replace $\boldsymbol{f}$ and f_1, f_2 by (4.2).

THEOREM 4.1 Let $(\boldsymbol{u}, p, s, c)$ and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ be the solutions to (1.2) and (4.3), respectively. Let Θ be the *a posteriori* error estimator defined in (4.4). If $\boldsymbol{u}$, s and c satisfy the bounds

$$\|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};W^{1,\infty}(\Omega))} < M, \quad \|s\|_{L^\infty(0,t_{\mathrm{end}};L^\infty(\Omega))} < M, \quad \text{and} \quad \|c\|_{L^\infty(0,t_{\mathrm{end}};L^\infty(\Omega))} < M, \tag{4.5}$$

for sufficiently small M, then there exists $C > 0$, independent of h, such that

$$\left(\|e_{\boldsymbol{u}}\|_\star^2 + \|e_s\|_\star^2 + \|e_c\|_\star^2\right)^{1/2} \leqslant C\left(\Theta^2 + \int_0^{t_{\mathrm{end}}} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}^2 \, \mathrm{d}t\right)^{1/2},$$

$$\|\partial_t e_{\boldsymbol{u}} + \nabla(p - p_h)\|_{L^2(0,t_{\mathrm{end}};\mathbf{H}^{-1}(\Omega))} + \|\partial_t e_s\|_{L^2(0,t_{\mathrm{end}};H^{-1}(\Omega))} + \|\partial_t e_c\|_{L^2(0,t_{\mathrm{end}};H^{-1}(\Omega))}$$

$$\leqslant C\left(\Theta^2 + \int_0^{t_{\mathrm{end}}} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}^2 \, \mathrm{d}t\right)^{1/2},$$

where we define

$$\|\boldsymbol{v}\|_\star^2 := \|\boldsymbol{v}\|_{L^\infty(0,t_{\mathrm{end}};L^2(\Omega))}^2 + \int_0^{t_{\mathrm{end}}} \|\boldsymbol{v}\|_{1,\mathscr{T}_h}^2 \, \mathrm{d}t, \quad \|\phi\|_\star^2 := \|\phi\|_{L^\infty(0,t_{\mathrm{end}};L^2(\Omega))}^2 + \int_0^{t_{\mathrm{end}}} \|\phi\|_{1,\Omega}^2 \, \mathrm{d}t.$$

The proof is detailed in Appendix D.

5. *A posteriori* error analysis for the fully discrete method

In this section, we develop an *a posteriori* error estimator for the fully discrete problem and focus the presentation on the simpler case of a time discretisation by the backward Euler method. For each time step k ($1 \leqslant k \leqslant N$), we define the (global in space) time indicator Ξ_k as

$$\Xi_k := (\Xi_{k,1}^2 + \Xi_{k,2}^2 + \Xi_{k,3}^2)^{1/2},$$

where

$$\Xi_{k,1}^2 := \tilde{\tau}_k\left(\|\boldsymbol{u}_h^k - I^k \boldsymbol{u}_h^{k-1}\|_{1,\mathscr{T}_{h,k}}^2 + h_e \tilde{\tau}_k^{-2} \|[\![I^k \boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h^{n-1}]\!]\|_{0,e}^2 + h_e \tilde{\tau}_k^{-2} \|[\![\boldsymbol{u}_h^n - I^k \boldsymbol{u}_h^{n-1}]\!]\|_{0,e}^2\right),$$

$$\Xi_{k,2}^2 := \tilde{\tau}_k \|s_h^k - s_h^{k-1}\|_1^2, \qquad \Xi_{k,3}^2 := \tilde{\tau}_k \|c_h^k - c_h^{k-1}\|_1^2,$$

and I^k is a generic data transfer operator which depends on the specific implementation. For more details, see Georgoulis *et al.* (2011).

Here it will be convenient to use the auxiliary norm

$$\|\boldsymbol{u}_h^k\|_{1,\mathscr{T}_{h,k}}^2 := \sum_{K \in \mathscr{T}_{h,k}} \|\boldsymbol{\nabla}_h \boldsymbol{u}_h^k\|_{0,K} + \sum_{e \in \mathscr{E}_h} \frac{1}{h_e} \|[\![\boldsymbol{u}_h^k]\!]\|_{0,e}.$$

Next we define the cumulative time and spatial error indicators as

$$\Xi^2 := \sum_{k=1}^N \Xi_k^2, \qquad \Upsilon^2 := \sum_{k=1}^N \tilde{\tau}_k\left(\Upsilon_k^2(\boldsymbol{u}_h^k, p_h^k, s_h^k, c_h^k) + \Upsilon_k^2(I^k \boldsymbol{u}_h^{k-1}, I_p^k p_h^{k-1}, s_h^{k-1}, c_h^{k-1})\right), \tag{5.1}$$

where the terms Υ_k^2 are constructed with the *a posteriori* error estimator contributions defined as in the steady case (3.3), but at a given time step k. That is,

$$\Upsilon_k^2(\boldsymbol{u}_h^k, p_h^k, s_h^k, c_h^k) = \Upsilon_{K,k}^2 + \Upsilon_{e,k}^2 + \Upsilon_{J,k}^2,$$

with

$$\Upsilon_{K,k}^2 := h_K^2 \left(\|\mathbf{R}_K^k\|_{0,K}^2 + \|R_{1,K}^k\|_{0,K}^2 + \|R_{2,K}^k\|_{0,K}^2 \right),$$

$$\Upsilon_{e,k}^2 := \sum_{e \in \partial K} h_e \left(\|\mathbf{R}_e^k\|_{0,e}^2 + \|R_{1,e}^k\|_{0,e}^2 + \|R_{2,e}^k\|_{0,e}^2 \right), \quad \Upsilon_{J,k}^2 := \sum_{e \in \partial K} h_e^{-1} \|[\![\boldsymbol{u}_h^k]\!]\|_{0,e}^2,$$

and

$$\mathbf{R}_K^k := \left\{ -\frac{1}{\tilde{\tau}_k} (\boldsymbol{u}_h^k - I^k \boldsymbol{u}_h^{k-1}) + \frac{\rho}{\rho_m} \boldsymbol{g} + \mathbf{div}(\nu(c_h)\boldsymbol{\nabla}\boldsymbol{u}_h) - \boldsymbol{u}_h \cdot \boldsymbol{\nabla}\boldsymbol{u}_h - \frac{1}{\rho_m}\nabla p_h \right\} \Big|_K,$$

$$R_{1,K}^k := \left\{ -\frac{s_h^k - s_h^{k-1}}{\tilde{\tau}_k} + \frac{1}{\mathrm{Sc}}\Delta s_h - \boldsymbol{u}_h \cdot \nabla s_h \right\} \Big|_K, \quad R_{1,e}^k := \begin{cases} \frac{1}{2}[\![(\mathrm{Sc}^{-1}\nabla s_h)\cdot\boldsymbol{n}]\!] & \text{for } e \in \mathscr{E}_h \setminus \Gamma, \\ 0 & \text{for } e \in \Gamma, \end{cases}$$

$$R_{2,K}^k := \left\{ -\frac{c_h^k - c_h^{k-1}}{\tilde{\tau}_k} + \frac{1}{\tau\mathrm{Sc}}\Delta c_h - (\boldsymbol{u}_h - v_p\boldsymbol{e}_z)\cdot\nabla c_h \right\} \Big|_K, \quad R_{2,e}^k := \begin{cases} \frac{1}{2}[\![((\tau\mathrm{Sc})^{-1}\nabla c_h)\cdot\boldsymbol{n}]\!] & \text{for } e \in \mathscr{E}_h \setminus \Gamma, \\ 0 & \text{for } e \in \Gamma. \end{cases}$$

For each time step k, we can split again the $\mathbf{H}(\mathrm{div})$-conforming discrete solution $\boldsymbol{u}_h^k$ into a conforming part $\boldsymbol{u}_{hc}^k$ and a non-conforming part $\boldsymbol{u}_{hr}^k$ such that $\boldsymbol{u}_h^k = \boldsymbol{u}_{h,c}^k + \boldsymbol{u}_{h,r}^k$. For each $t \in (t_{k-1}, t_k]$, we introduce a linear interpolant $\boldsymbol{u}_h(t)$ in terms of t as

$$\boldsymbol{u}_h(t) := \frac{t_k - t}{\tilde{\tau}_k} I^k \boldsymbol{u}_h^{k-1} + \frac{t - t_k}{\tilde{\tau}_k} \boldsymbol{u}_h^k,$$

where $\{l_k, l_{k+1}\}$ is the standard linear interpolation basis defined on $[t^k, t^{k+1}]$. Similarly, we may introduce $\boldsymbol{u}_{h,c}(t)$ and $\boldsymbol{u}_{h,r}(t)$. Then, setting $\boldsymbol{e}_{\boldsymbol{u}_c} = \boldsymbol{u} - \boldsymbol{u}_{h,c}$, we have $\boldsymbol{e}^{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_h = \boldsymbol{e}^{\boldsymbol{u}_c} - \boldsymbol{u}_{h,r}$. For $t \in (t_{k-1}, t_k)$, we define

$$\partial_t \boldsymbol{u}_{\tilde{\tau}}(t) := \frac{1}{\tilde{\tau}_k} (\boldsymbol{u}^k - \boldsymbol{u}^{k-1}),$$

and for all $t \in (t_{k-1}, t_k)$, we consider the problem of finding $(\tilde{\boldsymbol{u}}^k, \tilde{p}^k, \tilde{s}^k, \tilde{c}^k) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\left(\partial_t \boldsymbol{u}_h(t), \boldsymbol{v}\right) + a_1(c_h, \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) + c_1(\boldsymbol{u}_h; \tilde{\boldsymbol{u}}^k, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}^k) = (\boldsymbol{f}^k, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \tag{5.2a}$$

$$b(\tilde{\boldsymbol{u}}^k, q) = 0 \quad \text{for all } q \in L_0^2(\Omega), \tag{5.2b}$$

$$\left(\partial_t s_h(t), \phi\right) + a_2(\tilde{s}^k, \phi) + c_2(\boldsymbol{u}_h; \tilde{s}^k, \phi) = 0 \quad \text{for all } \phi \in H_0^1(\Omega), \tag{5.2c}$$

$$\left(\partial_t c_h, \psi\right) + \frac{1}{\tau} a_2(\tilde{c}^k, \psi) + c_2(\boldsymbol{u}_h - v_p\boldsymbol{e}_z; \tilde{c}^k, \psi) = 0 \quad \text{for all } \psi \in H_0^1(\Omega), \tag{5.2d}$$

where $\boldsymbol{f}^k = (\alpha s_h^k + \beta c_h^k)\boldsymbol{g} \in \mathbf{L}^2(\Omega)$.

LEMMA 5.1 If $\boldsymbol{u}$, s and c satisfy the bounds (4.5), then the following estimates hold:

$$\frac{1}{2}\|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2 + \alpha_1 \int_0^{t_n} \|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(t)\|_{1,\Omega}^2 \,\mathrm{d}t + \frac{1}{2}\|e_{\tilde{\tau}}^s(t_n)\|_{0,\Omega}^2 + \alpha_2 \int_0^{t_n} \|e_{\tilde{\tau}}^s(t)\|_{1,\Omega}^2 \,\mathrm{d}t + \frac{1}{2}\|e_{\tilde{\tau}}^c(t_n)\|_{0,\Omega}^2 + \alpha_3 \tau \int_0^{t_n} \|e_{\tilde{\tau}}^c(t)\|_{1,\Omega}^2 \,\mathrm{d}t$$

$$\leqslant C(\Xi^2 + \Upsilon^2 + \Theta^2) + \frac{1}{2}\|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^s(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^c(0)\|_{0,\Omega}^2 + \frac{1}{2}\sum_{k=1}^{n-1}\left(\|\boldsymbol{u}(t_k) - I^{k+1}\tilde{\boldsymbol{u}}_{h,c}^k\|_{0,\Omega}^2 - \|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2\right),$$

$$\sum_{k=1}^{n}\left(\|\partial_t e_{\tilde{\tau}}^{\boldsymbol{u}} + \nabla(p - p_h)\|_{L^2(t_{k-1}, t_k; \mathbf{H}^{-1}(\Omega))} + \|\partial_t e_{\tilde{\tau}}^s\|_{L^2(t_{k-1}, t_k; H^{-1}(\Omega))} + \|\partial_t e_{\tilde{\tau}}^c\|_{L^2(t_{k-1}, t_k; H^{-1}(\Omega))}\right)$$

$$\leqslant C(\Xi^2 + \Upsilon^2 + \Theta^2) + \frac{1}{2}\|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^s(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^c(0)\|_{0,\Omega}^2 + \frac{1}{2}\sum_{k=1}^{n-1}\left(\|\boldsymbol{u}(t_k) - I^{k+1}\tilde{\boldsymbol{u}}_{h,c}^k\|_{0,\Omega}^2 - \|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2\right),$$

where $\Theta^2 = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\boldsymbol{f} - \boldsymbol{f}^k\|_{0,\Omega}^2 \,\mathrm{d}t$.

$1/h$	$e_{\boldsymbol{u}}$	rate	e_p	rate	e_s	rate	e_c	rate	$\|\operatorname{div}\boldsymbol{u}_h\|_{\infty,\Omega}$
$\sqrt{2}$	1.3970	–	5.0910	–	0.03723	–	0.02511	–	2.19e-11
$2\sqrt{2}$	0.5651	1.306	1.9920	1.354	0.01098	1.762	0.00679	1.887	4.08e-12
$4\sqrt{2}$	0.1719	1.717	0.6402	1.637	0.00298	1.882	0.00171	1.990	1.00e-12
$8\sqrt{2}$	0.0456	1.914	0.1695	1.917	0.00080	1.904	0.00046	1.903	5.20e-13
$16\sqrt{2}$	0.0115	1.994	0.0412	2.039	0.00021	1.941	0.00012	1.962	2.23e-13

Table 6.1. Example 1. Experimental errors and convergence rates for the approximate solutions $\boldsymbol{u}_h$, p_h, s_h and c_h, where the polynomial degree $k = 2$ is used. The ℓ^∞-norm of the vector formed by the divergence of the discrete velocity computed at time t_{end} for each discretisation is shown in the last column.

The proof is postponed to Appendix E.

THEOREM 5.1 Let $(\boldsymbol{u}, p, s, c)$ be the solution of (1.2), and $(\boldsymbol{u}_h, p_h, s_h, c_h)$ the corresponding discrete solution. Let Ξ, Υ be the *a posteriori* error estimators defined in (5.1). If $\boldsymbol{u}$, s and c satisfy the bounds (4.5), then the following reliability estimate holds:

$$
\left(\int_0^{t_{\text{end}}} \left[\|e_\tau^{\boldsymbol{u}}\|_{1,\mathscr{T}_h}^2 + \|e_\tau^c\|_{1,\Omega}^2 + \|e_\tau^s\|_{1,\Omega}^2 \right] \mathrm{d}t \right)^{1/2}
$$

$$
\leqslant C \left(\Xi^2 + \Upsilon^2 + \frac{1}{2}\|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^s(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^t(0)\|_{0,\Omega}^2 + \sum_{k=1}^{N-1} \|u_{h,r}^k - I^{k+1}u_{h,r}^k\|_{0,\Omega}^2 + \Theta^2 \right)^{1/2},
$$

$$
\sum_{k=1}^N \left(\|\partial_t e_{\tilde{\tau}}^{\boldsymbol{u}} + \nabla(p - p_h)\|_{L^2(t_{k-1},t_k;\mathbf{H}^{-1}(\Omega))} + \|\partial_t e_{\tilde{\tau}}^s\|_{L^2(t_{k-1},t_k;H^{-1}(\Omega))} + \|\partial_t e_{\tilde{\tau}}^c\|_{L^2(t_{k-1},t_k;H^{-1}(\Omega))} \right)
$$

$$
\leqslant C \left(\Xi^2 + \Upsilon^2 + \frac{1}{2}\|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^s(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^t(0)\|_{0,\Omega}^2 + \sum_{k=1}^{N-1} \|u_{h,r}^k - I^{k+1}u_{h,r}^k\|_{0,\Omega}^2 + \Theta^2 \right)^{1/2}.
$$

Proof. Using $\boldsymbol{u}_h^k = \boldsymbol{u}_{h,c}^k + \boldsymbol{u}_{h,r}^k$ along with the identity in (Georgoulis *et al.*, 2011, (5.59)-(5.60)) that in our context reads

$$
\|\boldsymbol{u}(t_k) - I^{k+1}\tilde{\boldsymbol{u}}_{h,c}^k\|_{0,\Omega}^2 - \|e_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_k)\|_{0,\Omega}^2 = \|\boldsymbol{u}_{h,r}^k - I^{k+1}\boldsymbol{u}_{h,r}^k\|_{0,\Omega}^2 + \langle \boldsymbol{u}_{h,r}^k - I^{k+1}\boldsymbol{u}_{h,r}^k, e_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_k)\rangle,
$$

we can invoke Lemma 5.1 and reuse the strategy applied in Theorem 4.1 to complete the proof. $\qquad\square$

6. Numerical tests

We now present computational examples illustrating the properties of the numerical schemes. All numerical routines have been realised using the open-source finite element libraries FEniCS (Alnæs *et al.*, 2015) (for examples 1,2 and 3), and FreeFem++ (Hecht, 2012) (for example 4).

6.1 *Example 1: accuracy verification against smooth solutions*

A known analytical solution example is used to verify theoretical convergence rates of the scheme, focusing on the polynomial degree $k = 2$. We choose $t_{\text{end}} = 2$ and $\Omega = (0,1)^2$. We take the parameter values $\nu = \exp(-c)$, $\rho = \rho_{\text{m}}(\alpha s + \beta c)$, $\alpha = 1$, $\beta = 1$, $\rho_{\text{m}} = 1.5$, $\boldsymbol{g} = (0,-1)^T$, Sc $= 1$, $\tau = 0.5$, $\nu_p = 1$, $a_0 = 50$. Following the approach of manufactured solutions, we prescribe boundary data and additional external forces and adequate source terms so that the closed-form solutions to (1.1) are given by the smooth functions

$$
\boldsymbol{u}(x,y,t) = \begin{pmatrix} (\sin(\pi x))^2 (\sin(\pi y))^2 \cos(\pi y) \sin(t) \\ -1/3 \sin(2\pi x)(\sin(\pi y))^3 \sin(t) \end{pmatrix}, \quad p(x,y,t) = (x^4 - y^4)\sin(t),
$$

$$c(x,y,t) = \frac{1}{2}(1 + \cos(\pi/4(xy)))\exp(-t), \quad s(x,y,t) = \frac{1}{2}(1 + \sin(\pi/2(xy)))\exp(-t).$$

As $\boldsymbol{u}$ is prescribed everywhere on $\partial\Omega$, for sake of uniqueness we impose $p \in L_0^2(\Omega)$ through a real Lagrange multiplier approach. To verify the *a priori* error estimates, we introduce the discrete norms

$$\|\boldsymbol{u}\|_{0,\mathscr{T}_h} := \left(\Delta t \sum_{n=1}^{N} \|\boldsymbol{u}_h^n\|_{1,\mathscr{T}_h}^2\right)^{1/2}, \quad \text{and} \quad \|\chi\|_{0,k} := \left(\Delta t \sum_{n=1}^{N} \|\chi_h^n\|_{k,\Omega}^2\right)^{1/2}.$$

The corresponding individual errors and convergence rates are computed as

$$e_{\boldsymbol{u}} = \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\mathscr{T}_h}, \quad e_p = \|p - p_h\|_{0,0}, \quad e_s = \|s - s_h\|_{0,1}, \quad e_c = \|c - c_h\|_{0,1},$$
$$\texttt{rate} = \log(e_{(\cdot)}/\tilde{e}_{(\cdot)})[\log(\xi/\tilde{\xi})]^{-1}, \xi = \{h, \Delta t\}, \tag{6.1}$$

where $e, \tilde{e}$ denote errors generated on two consecutive pairs of mesh size and time step $(h, \Delta t)$, and $(\tilde{h}, \tilde{\Delta t})$, respectively. Choosing $\Delta t = \sqrt{2}h$ and using scheme (2.5), the results in Table 6.1 confirm that the rates of convergence are optimal, coinciding with the theoretical bounds anticipated in Theorem 2.8.

6.2 *Example 2: adaptive mesh refinement for the stationary problem*

The classical strategy due to Dörfler (1994) is employed for the adaptive algorithm based on the steps of solving, estimating, marking, and refining. Estimation is performed by computing the error indicators and using them to select/mark elements that contribute the most to the error (Larson *et al.*, 2008). The marking is done following the bulk criterion of selecting sufficiently many elements so that they represent a given fraction of the total estimated error. That is, one refines all elements $K \in \mathscr{T}_h$ for which

$$\Psi_K \geqslant \gamma_{\text{ratio}} \max_{L \in \mathscr{T}_h} \Psi_L,$$

where $0 < \gamma_{\text{ratio}} < 1$ is a user-defined constant (that we tune in order to generate a similar number of degrees of freedom, or comparable errors, as those obtained under uniform refinement). The algorithm aims for equidistribution of the local error indicator on the updated mesh.

In the adaptive case, instead of (6.1) the convergence rates (for the spatial errors) are computed as

$$\texttt{rate} = -2\log(e_{(\cdot)}/\tilde{e}_{(\cdot)})[\log(\texttt{DoF}/\widetilde{\texttt{DoF}})]^{-1},$$

where $\texttt{DoF}$ and $\widetilde{\texttt{DoF}}$ are the number of degrees of freedom associated with each refinement level. The robustness of the global estimators is measured using the effectivity index (ratio between the total error and the indicator)

$$\texttt{eff}(\Psi) = \frac{\{e_{\boldsymbol{u}}^2 + e_p^2 + e_s^2 + e_c^2\}^{1/2}}{\Psi}.$$

We start with verifying the robustness of the *a posteriori* error estimator Ψ, and construct closed-form solutions to the stationary counterpart of the coupled problem (1.1). We consider concentration-dependent viscosity, model parameter values, and stabilisation constant as

$$\nu(c) = \frac{1}{10}(1 + \exp(-1/4c)), \quad \rho = 1, \quad \rho_{\text{m}} = 1.5, \quad \boldsymbol{g} = (0, -1)^T, \quad \text{Sc} = 1,$$
$$\tau = 0.5, \quad v_p = 1, \quad \alpha = 0.5, \quad \beta = 0.5, \quad a_0 = 5.$$

The considered exact solutions are defined on the L-shaped domain $\Omega = (-1, 1)^2 \setminus (0, 1)^2$

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p(x,y) = \frac{2 + \sin(xy)}{(x - 0.02)^2 + (y - 0.02)^2},$$
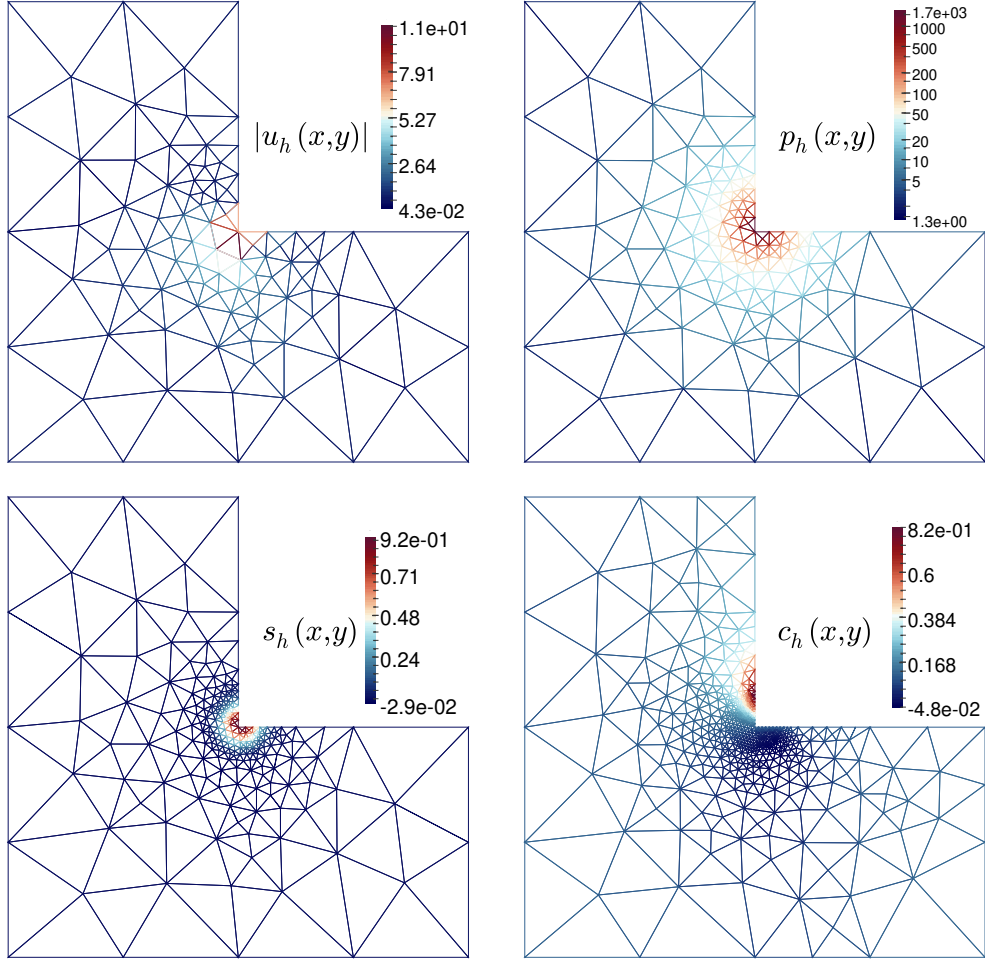
FIG. 6.1. *Example 2. Approximate velocity magnitude (after 3 refinement steps), pressure (after 4 refinement steps), concentration s (after 5 refinement steps), and distribution of c after 6 steps of adaptive refinement.*

$$s(x,y) = \exp(-150(x-0.01)^2 - 150(y-0.01)^2), \quad c(x,y) = \frac{1}{10} + \frac{\cos(\pi x)\sin(\frac{\pi}{2}y)}{25((x-0.1)^2 + (y-0.1)^2)}.$$

These solutions exhibit a generic singularity towards the reentrant corner, and therefore one expects that the error decay is suboptimal when applying uniform mesh refinement. After solving the coupled stationary problem on sequences of uniformly and adaptively refined meshes and using the lowest-order scheme with $k = 1$, the aforementioned behaviour is indeed observed in Table 6.2, where the first part of the table shows deterioration of the convergence due to the high gradients of the exact solutions on the non-convex domain. The results shown in the bottom block of the table confirm that as more degrees of freedom are added, a restored error reduction rate is observed due to adaptive mesh refinement guided by the *a posteriori* error estimator Ψ. The second-last column of the table also indicates that the effectivity index oscillates under uniform refinement, while it is much more steady in the adaptive case. We tabulate as well the Newton-Raphson iteration count (needed to reach the relative residual tolerance of 1e-6), and this number is also systematically smaller for the adaptive case (about four steps in all instances) than for the uniform refinement case (up

DoF	h	$e_{\boldsymbol{u}}$	rate	e_p	rate	e_s	rate	e_c	rate	$\|\text{div}\,\boldsymbol{u}_h\|_{\infty,\Omega}$	eff(Ψ)	iter
Error history under uniform mesh refinement
79	1	26.85	–	284.0	–	1.829	–	1.3724	–	2.22e-12	0.329	5
275	0.5	19.57	0.396	152.9	0.893	1.736	0.075	0.9903	0.499	3.65e-13	0.257	7
1027	0.25	29.59	-0.622	86.54	0.821	1.188	0.547	0.7004	0.441	1.07e-13	0.111	6
3971	0.125	12.53	0.924	72.47	0.256	0.642	0.886	0.4834	0.535	3.21e-14	0.104	7
15619	0.0625	7.561	0.805	53.41	0.440	0.457	0.491	0.2863	0.756	3.39e-14	0.171	7
Error history under adaptive mesh refinement
79	1	26.85	–	284.0	–	2.829	–	1.3724	–	2.22e-12	0.329	4
275	0.5	15.92	0.824	154.8	0.973	1.725	0.937	0.9617	0.573	3.67e-13	0.261	3
943	0.5	10.78	1.172	90.23	0.878	0.822	0.863	0.6793	1.064	2.18e-13	0.260	4
1601	0.5	7.398	2.499	74.35	0.732	0.641	1.428	0.4398	1.642	2.28e-13	0.261	4
2363	0.5	2.139	2.265	53.35	1.706	0.461	1.569	0.2683	2.539	2.15e-13	0.257	3
4253	0.2877	3.420	1.394	29.41	2.027	0.235	2.295	0.1541	1.888	1.05e-13	0.258	4
11662	0.25	1.012	1.267	17.58	1.019	0.118	1.368	0.0873	1.126	1.07e-13	0.258	5
38174	0.1416	0.557	1.006	9.388	1.058	0.059	1.156	0.0464	1.063	9.01e-13	0.261	4

Table 6.2. Example 2. Experimental errors and convergence rates for the approximate solutions $\boldsymbol{u}_h$, p_h, s_h and c_h of the stationary problem under uniform and adaptive mesh refinement following the estimator Ψ, using the lowest-order scheme with $k = 1$. For the adaptive case we employ $\gamma_{\text{ratio}} = 1 \cdot 10^{-4}$.

to seven nonlinear steps for certain refinement levels). As an example we plot in Figure 6.1 solutions on relative coarse meshes and display meshes generated with the adaptive algorithm, indicating significant refinement near the reentrant corner. Let us also remark that the boundary conditions for velocity have been imposed (here and in all other tests) essentially for the normal component, while the tangent component is fixed through Nitsche's penalisation. For this example we use a constant $a_{\text{Nitsche}} = 10^3$.

6.3 *Example 3: robustness of the estimator for the transient problem*

Next we turn to the numerical verification of robustness of the *a posteriori* error estimator for the fully discrete approximations of the time-dependent coupled problem. We consider now the time interval $(0, 0.01]$ and choose $\Delta t = 0.002$. The closed-form solutions on the unit square domain are as follows

$$\boldsymbol{u}(x,y,t) = \sin(t) \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p(x,y,t) = \cos(t)(x^4 - y^4),$$

$$c(x,y,t) = \frac{1}{2}(1 + \cos(\pi/4(xy)))\exp(-t), \quad s(x,y,t) = \frac{1}{2}(1 + \sin(\pi/2(xy)))\exp(-t).$$

Cumulative errors up to t_{final} are computed as

$$E_{\boldsymbol{u}} := \left(\Delta t \sum_{n=1}^{N} \|\boldsymbol{u}_h^n - \boldsymbol{u}(t^n)\|_{1,\mathscr{T}_h}^2 \right)^{1/2}, \quad E_p := \left(\Delta t \sum_{n=1}^{N} \|p_h^n - p(t^n)\|_{0,\Omega}^2 \right)^{1/2},$$

$$E_s := \left(\Delta t \sum_{n=1}^{N} \|s_h^n - s(t^n)\|_{1,\Omega}^2 \right)^{1/2}, \quad E_c := \left(\Delta t \sum_{n=1}^{N} \|c_h^n - c(t^n)\|_{1,\Omega}^2 \right)^{1/2},$$

and the resulting error history, after six steps of uniform mesh refinement, is collected in Table 6.3. Here we have used the lowest-order scheme with $k = 1$. To be consistent with the development in Section 5, the numerical verification in this set of tests was carried out using a backward Euler time discretisation. The *a posteriori* error estimator (5.1) is computed and the effectivity index is also tabulated, showing that the estimator is robust (and confirming the theoretical

DoF	h	$E_{\boldsymbol{u}}$	rate	E_p	rate	E_s	rate	E_c	rate	eff(Υ)
59	0.7071	0.00228	–	0.02080	–	0.01811	–	0.00711	–	0.0859
195	0.3536	0.00081	1.504	0.01191	0.843	0.00989	0.872	0.00303	1.227	0.0970
707	0.1768	0.00034	1.217	0.00621	0.940	0.00486	1.025	0.00149	1.022	0.0973
2691	0.0884	0.00015	1.119	0.00313	0.984	0.00241	1.009	0.00074	1.008	0.0967
10499	0.0442	7.70e-05	1.053	0.00157	0.996	0.00120	1.004	0.00031	1.003	0.0961
41475	0.0221	3.78e-05	1.025	0.00078	0.999	0.00060	1.002	0.00018	1.001	0.0972
164867	0.0110	1.87e-05	1.012	0.00039	0.999	0.00030	1.001	9.26e-05	1.000	0.0966

Table 6.3. Example 3. Experimental errors and convergence rates for the approximate solutions of the transient problem under uniform mesh refinement following the estimator Υ, and using the lowest-order scheme with $k = 1$.

reliability bound as well as giving a heuristic indication of its efficiency). Note that in this case, since the mesh refinement is uniform, the auxiliary interpolation of the solutions at the last time step on the current mesh is not necessary. The average number of Newton-Raphson iterations required for convergence was 3.2.

6.4 *Example 4: adaptive simulation of exothermic flows*

To conclude this section, and to include an illustrative simulation exemplifying that the $\mathbf{H}$(div)-conforming scheme along with the *a posteriori* error estimator perform well for an applicative problem, we address the computation of exothermic flows that develop fingering instabilities. The problem configuration is adapted as a simplification of the problem solved in Lenarda *et al.* (2017) (see also Lee & Kim, 2015; Ruiz-Baier & Lunati, 2016), where the fields c, s represent solutal concentration and temperature, respectively. The model assumes an additional drag term due to porosity so that the momentum equation is of Navier–Stokes–Brinkman type. The domain is the rectangular region $\Omega = (0, L) \times (0, H)$, and the initial solutal and temperature profiles are imposed as

$$c^0(x,y) = \begin{cases} 0.999 + 0.001\zeta_c & \text{if } H - \varepsilon \leqslant y \leqslant H, \\ 0 & \text{otherwise,} \end{cases} \quad s^0(x,y) = \begin{cases} 0.999 + 0.001\zeta_s & \text{if } H - \varepsilon \leqslant y \leqslant H, \\ 0 & \text{otherwise} \end{cases},$$

where ζ_c, ζ_s are random fields uniformly distributed on $[0, 1]$. The geometric and model constants are $H = 1000$, $L = 2000$, $\Delta t = 20$, $t_{\text{end}} = 1500$, $\nu = 1 + 0.25\zeta_\nu$, $\kappa = 1$, $1/\text{Sc} = 8$, $1/(\tau\text{Sc}) = 2.5$, $\rho_{\text{m}} = 1$, $\nu_{\text{p}} = 0$, $\alpha = 5$, $\beta = -1$. The polynomial degree for this example is $k = 2$.

Boundary conditions are of mixed type for solutal and temperature distribution. Both fields are prescribed to 0 and 1 on the bottom and top of the domain, respectively; while on the vertical walls we impose zero-flux boundary conditions. The velocity is of slip type on the whole boundary, and therefore a zero-mean condition for the pressure is considered using a real Lagrange multiplier. The solution algorithm, differently from the previous tests, is based on an inner fixed-point iteration between an Oseen and a transport system, rather than an exact Newton-Raphson method. An initial coarse mesh of 5300 elements is constructed, and an adaptive mesh refinement (only one iteration) guided by the estimator (5.1) is applied at the end of each time step, and the algorithm does allow for mesh coarsening. Figure 6.2 shows snapshots of adapted meshes at different times, and also samples of solute concentration, temperature distribution, velocity, and pressure at the final time.

References

J. Abedi and S. Aliabadi, *Simulation of incompressible flows with heat and mass transfer using parallel finite element method*. Electronic J. Diff. Eq., 10 (2003), pp. 1–11.

A. Agouzal and K. Allali, *Numerical analysis of reaction front propagation model under Boussinesq approximation*. Math. Meth. Appl. Sci., 26 (2003), pp. 1529–1572.
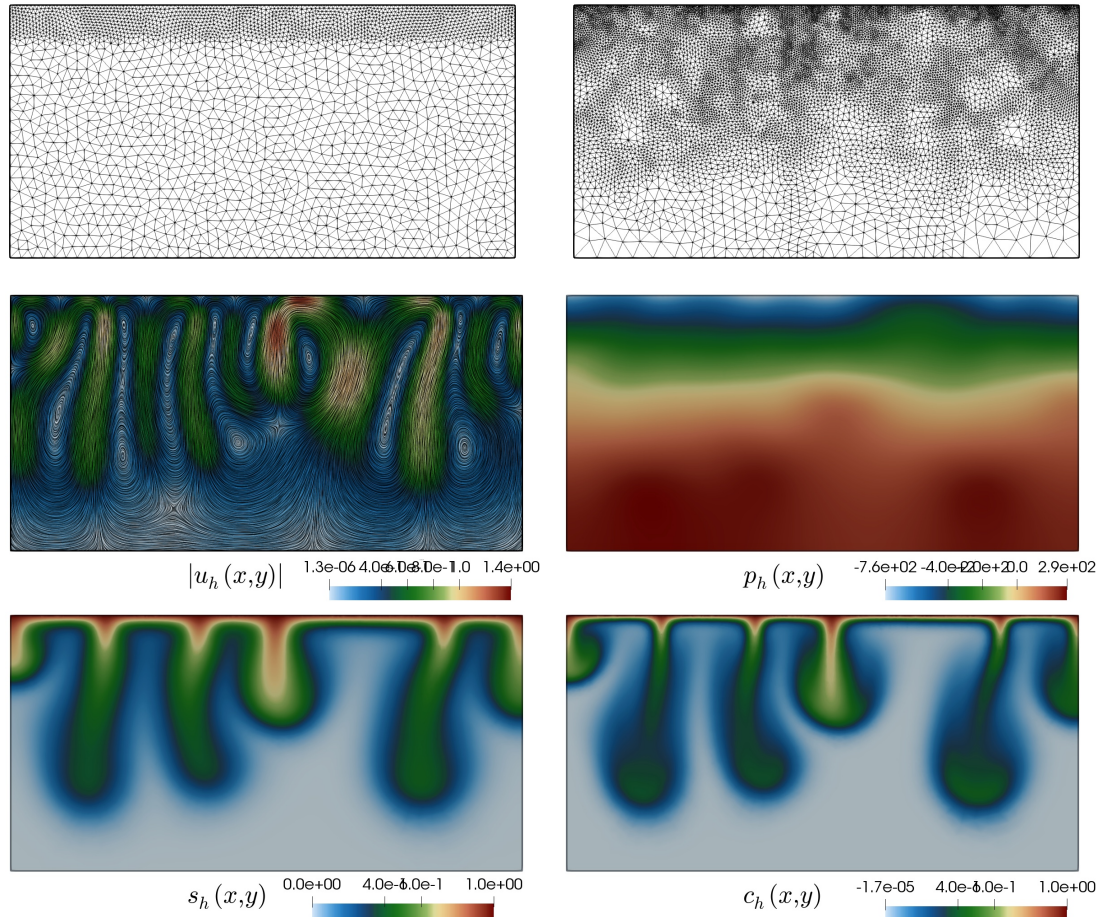
$|u_h(x,y)|$ 1.3e-06 4.0e-06.0e-8.0e-1.0   1.4e+00

$p_h(x,y)$ -7.6e+02 -4.0e-20.0e+20.0   2.9e+02

$s_h(x,y)$ 0.0e+00 4.0e-6.0e-1   1.0e+00

$c_h(x,y)$ -1.7e-05 4.0e-6.0e-1   1.0e+00

FIG. 6.2. *Example 4. Samples of adapted meshes at times $t = 60, 1200$ (top panels), and approximate solutions shown at time $t = 1500$ (bottom rows), and computed with a method using $k = 2$.*

R. AGROUM, *A posteriori error analysis for solving the Navier-Stokes problem and convection-diffusion equation.* Numer. Methods Part. Diff. Eqns., 34 (2017), no. 2, pp. 401–418.

R. AGROUM, C. BERNARDI, AND J. SATOURI, *Spectral discretization of the time-dependent Navier-Stokes problem coupled with the heat equation.* Appl. Math. Comput., 268 (2015), pp. 59–82.

R. ALDBAISSY, F. HECHT, G. MANSOUR, AND T. SAYAH, *A full discretisation of the time-dependent Boussinesq (buoyancy) model with nonlinear viscosity.* Calcolo, 55 (2018), pp. 44–49.

K. ALLALI, *A priori and a posteriori error estimates for Boussinesq equations.* Int. J. Numer. Anal. Model., 2 (2005), pp. 179–196.

A. ALLENDES, C. NARANJO, AND E. OTÁROLA, *Stabilized finite element approximations for a generalized Boussinesq problem: A posteriori error analysis.* Comput. Methods Appl. Mech. Engrg., 361 (2020), e. 112703.

M.S. ALNÆS, J. BLECHTA, J. HAKE, A. JOHANSSON, B. KEHLET, A. LOGG, C. RICHARDSON, J. RING, M.E. ROGNES, AND G.N. WELLS, *The FEniCS project version 1.5*. Arch. Numer. Softw., 3 (2015), pp. 9–23.

M. ALVAREZ, G.N. GATICA, AND R. RUIZ-BAIER, *A posteriori error analysis for a viscous flow-transport problem*. ESAIM: Math. Model. Numer. Anal., 50 (2016), no. 6, pp. 1789–1816.

M. ALVAREZ, G.N. GATICA, AND R. RUIZ-BAIER, *A posteriori error estimation for an augmented mixed-primal method applied to sedimentation-consolidation systems*. J. Comput. Phys., 367 (2018), pp. 322–346.

V. ANAYA, M. BENDAHMANE, D. MORA, AND R. RUIZ-BAIER, *On a primal-mixed vorticity-based formulation for reaction-diffusion-Brinkman systems*. Netw. Heterog. Media, 13 (2018), pp. 69–94.

D.N. ARNOLD, F. BREZZI, B. COCKBURN, AND L.D. MARINI, *Unified analysis of discontinuous Galerkin methods for elliptic problems*. SIAM J. Numer. Anal., 39 (2002), pp. 1749–1779.

G. BAIRD, R. BÜRGER, P.E. MÉNDEZ, AND R. RUIZ-BAIER, *Second-order schemes for axisymmetric Navier-Stokes-Brinkman and transport equations modelling water filters*. Numer. Math., 147 (2021), no. 2, pp. 431–479.

E. BÄNSCH AND A. BRENNER, *A posteriori estimates for the two-step backward differentiation formula and discrete regularity for the time-dependent Stokes equations*. J. Numer. Anal., 39 (2019), no. 2, pp. 713–759.

R. BECKER AND M. BRAACK, *Solution of a stationary benchmark problem for natural convection with large temperature difference*. Int. J. Thermal Sci., 41 (2002), no. 5, pp. 428–439.

Y. BELHAMADIA, A. FORTIN, AND T. BRIFFARD, *A two-dimensional adaptive remeshing method for solving melting and solidification problems with convection*. Numer. Heat Transf. A: Appl., 76 (2019), no. 4, pp. 179–197.

C. BERNARDI, L. EL ALAOUI AND Z. MGHAZLI, *A posteriori analysis of a space and time discretization of a nonlinear model for the flow in partially saturated porous media*. IMA J. Numer. Anal., 34 (2014), no. 3, pp. 1002–1036.

M. BRAACK AND T. RICHTER, *Solving multidimensional reactive flow problems with adaptive finite elements*. In: Reactive Flows, Diffusion and Transport, Springer, Berlin, 2007, pp. 93–112.

F. BREZZI, J. DOUGLAS, AND L.D. MARINI, *Two families of mixed finite elements for second order elliptic problems*. Numer. Math., 47 (1985), pp. 217–235.

R. BÜRGER, P.E. MÉNDEZ, AND R. RUIZ-BAIER, *On H(div)-conforming methods for double-diffusion equations in porous media*. SIAM J. Numer. Anal., 57 (2019), pp. 1318–1343.

P. BURNS AND E. MEIBURG, *Sediment-laden fresh water above salt water: nonlinear simulations*. J. Fluid Mech., 762 (2015), pp. 156–195.

A. CANGIANI, E.H. GEORGOULIS, AND S. METCAFLE, *Adaptive discontinuous Galerkin methods for nonstationary convection–diffusion problems*. IMA J. Numer. Anal., 34 (2014), pp. 1578–1597.

A. CANGIANI, E.H. GEORGOULIS, AND M. SABAWI, *A posteriori error analysis for implicit–explicit hp-discontinuous Galerkin timestepping methods for semilinear parabolic problems*. J. Sci. Comput., 82(2) (2020), e. 26.

H. DALLMANN AND D. ARNDT, *Stabilized finite element methods for the Oberbeck-Boussinesq model*. J. Sci. Comput., 69 (2016), no. 1, pp. 244–273.

I. DANAILA, R. MOGLAN, F. HECHT, AND S. LE MASSON, *A Newton method with adaptive finite elements for solving phase-change problems with natural convection*. J. Comput. Phys., 274 (2014), pp. 826–840.

S. DIB, V. GIRAULT, F. HECHT, AND T. SAYAH, *A posteriori error estimates for Darcy's problem coupled with the heat equation*. ESAIM: Math. Model. Numer. Anal., 53 (2019), pp. 2121–2159.

W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*. SIAM J. Numer. Anal., 33 (1994), no. 3, pp. 1106–1124.

J. GEDICKE AND A. KHAN, *Divergence-conforming discontinuous Galerkin finite elements for Stokes eigenvalue problems*. Numer. Math., 144 (2020), pp. 585–614.

E.H. GEORGOULIS, O. LAKKIS, AND J.M. VIRTANEN, *A posteriori error control for discontinuous Galerkin methods for parabolic problems*. SIAM J. Numer. Anal., 49 (2011), no. 2, pp. 427–458.

V. GIRAULT AND P.A. RAVIART, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*. Springer-Verlag, Berlin, 1986.

V. GIRAULT, B. RIVIÈRE, AND M.F. WHEELER, *A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems*. Math. Comput., 74 (2005), pp. 53–84.

Y. HAN AND Y. HOU, *Robust error analysis of H(div)- conforming DG method for the time-dependent incompressible Navier-Stokes equations*. J. Comput. Appl. Math., 390 (2021), no. 13, e113365.

F. HECHT, *New development in FreeFem++*, J. Numer. Math., 20 (2012), no. 3–4, pp. 251–265.

B. KADOCH, D. KOLOMENSKIY, P. ANGOT, AND K. SCHNEIDER, *A volume penalization method for incompressible flows and scalar advection-diffusion with moving obstacles*. J. Comput. Phys., 231 (2012), no. 12, pp. 4365–4383.

O.A. KARAKASHIAN AND W.N. JUREIDINI, *A nonconforming finite element method for the stationary Navier-Stokes equations*. SIAM J. Numer. Anal., 35 (1998), pp. 93–120.

J. KÖNNÖ AND R. STENBERG, *H(div)-conforming finite elements for the Brinkman problem*. Math. Models Methods Appl. Sci., 21 (2011), pp. 2227–2248.

M.G. LARSON, R. SÖDERLUND, AND F. BENGZON, *Adaptive finite element approximation of coupled flow and transport problems with applications in heat transfer*. Int. J. Numer. Meth. Fluids, 57 (2008), pp. 1397–1420.

H.G. LEE AND J. KIM, *Numerical investigation of falling bacterial plumes caused by bioconvection in a three-dimensional chamber*. Eur. J. Mech. B/Fluids, 52 (2015), pp. 120–130.

P. LENARDA, M. PAGGI, AND R. RUIZ-BAIER, *Partitioned coupling of advection-diffusion-reaction systems and Brinkman flows*. J. Comput. Phys., 344 (2017), pp. 281–302.

S. MEMON, N. NATARAJ, AND A.K. PANI, *An a posteriori error analysis of a mixed finite element Galerkin approximation to second order linear parabolic problems*. SIAM J. Numer. Anal., 50 (2012), pp. 1367–1393.

A. RAKOTONDRANDISA, G. SADAKA, AND I. DANAILA, *A finite-element toolbox for the simulation of solid-liquid phase-change systems with natural convection*. Comput. Phys. Commun., 253 (2020), e. 107188.

U. RASTHOFER AND V. GRAVEMEIER, *Recent developments in variational multiscale methods for large-eddy simulation of turbulent flow*. Arch. Comput. Methods Engrg., 25 (2018), pp. 647–690.

J.F. REALI, P. GARAUD, A. ALSINAN, AND E. MEIBURG, *Layer formation in sedimentary fingering convection*. J. Fluid Mech., 816 (2017), pp. 268–305.

R. RUIZ-BAIER AND I. LUNATI, *Mixed finite element – discontinuous finite volume element discretization of a general class of multicontinuum models*. J. Comput. Phys., 322 (2016), pp. 666–688.

P. W. SCHROEDER, C. LEHRENFELD, A. LINKE, AND G. LUBE. *Towards computable flows and robust estimates for inf- sup stable FEM applied to the time-dependent incompressible Navier–Stokes equations*. SeMA J., 75 (2018), no. 4, pp. 629–653.

R. TEMAM, *Navier-Stokes equations. Theory and numerical analysis*. Reedition in the AMS-Chelsea Series, AMS, Providence, 2001.

J. TUSHAR, A. KHAN, M. T. MOHAN, *Optimal Control of Stationary Doubly Diffusive Flows on Two and Three Dimensional Bounded Lipschitz Domains: A Theoretical Study*, arXiv preprint arXiv:2308.02178, 2023.

R. VERFÜRTH, *A Review of A Posteriori Error Estimation and Adaptive-Mesh-Refinement Techniques*. Wiley-Teubner (Chichester), 1996.

H.K. WILFRID, *An a posteriori error analysis for a coupled continuum pipe-flow/Darcy model in Karst aquifers: Anisotropic and isotropic discretizations*. Results Appl. Math., 4 (2019), e. 100081.

J. WOODFIELD, M. ALVAREZ, B. GOMEZ-VARGAS, AND R. RUIZ-BAIER, *Stability and finite element approximation of phase change models for natural convection in porous media*. J. Comput. Appl. Math., 360 (2019), pp. 117–137.

Y.-B. YANG AND Y.-L. JIANG, *An explicitly uncoupled VMS stabilization finite element method for the time-dependent Darcy-Brinkman equations in double-diffusive convection*. Numer. Algor., 78 (2018), pp. 569–597.

N. ZABARAS AND D. SAMANTA, *A stabilized volume-averaging finite element method for flow in porous media and binary alloy solidification processes*. Int. J. Num. Methods Engrg., 60 (2004), pp. 1103–1138.

A.G. ZIMMERMAN AND J. KOWALSKI, *Monolithic simulation of convection-coupled phase-change - verification and reproducibility*. In: Int. Conf. Comput. Engrg., ICCE, (2017), pp. 177–197.

Y. ZHANG, Y. HOU, AND Z. HONGLIANG, *A posteriori error estimation and adaptive computation of conduction convection problems*. Appl. Math. Model., 35 (2011), no. 5, pp. 2336–2347.

T. ZHANG AND S. LI, *A posteriori error estimates of finite element method for the time-dependent Navier-Stokes equations*. Appl. Math. Comput., 315 (2017), pp. 13–26.

A. Proof of Theorem 2.4

Proof of Theorem 2.4.   As in the continuous case, using the discrete kernel (2.6) and relying on the inf-sup condition (2.13), we can continue with an equivalent discrete problem without pressure.

Taking into account the assumed regularity for $\boldsymbol{u}$, $\boldsymbol{u}'$, and $\boldsymbol{u}''$, we have for all $\boldsymbol{x}$, a $\gamma(\boldsymbol{x}) \in (0,1)$ such that

$$\boldsymbol{u}(0) = \boldsymbol{u}(\Delta t) - \Delta t \boldsymbol{u}'(\Delta t) + \frac{1}{2}[\Delta t]^2 \boldsymbol{u}''(\Delta t \gamma),$$

then $\boldsymbol{u}$ satisfies the error equation

$$
\begin{aligned}
\|\xi_{\boldsymbol{u}}^1\|_{0,\Omega}^2 + \Delta t \tilde{\alpha}_a \|\xi_{\boldsymbol{u}}^1\|_{1,\mathscr{T}_h}^2 + \Delta t |\xi_{\boldsymbol{u}}^1|_{\boldsymbol{u}_h^1,\mathrm{upw}}^2 \leqslant {} & \left(\Pi_h \boldsymbol{u}(\Delta t) - \boldsymbol{u}(\Delta t) + \boldsymbol{u}_h^0 - \boldsymbol{u}(0), \xi_{\boldsymbol{u}}^1\right)_\Omega \\
& + \Delta t \left(a_1^h(c_h^1; \Pi_h \boldsymbol{u}(\Delta t), \xi_{\boldsymbol{u}}^1) - a_1^h(c^1; \boldsymbol{u}(\Delta t), \xi_{\boldsymbol{u}}^1)\right) \\
& - \Delta t \left(c_1^h(\boldsymbol{u}_h^1; \boldsymbol{u}_h^1, \xi_{\boldsymbol{u}}^1) - c_1^h(\boldsymbol{u}(\Delta t), \boldsymbol{u}(\Delta t), \xi_{\boldsymbol{u}}^1)\right) \\
& + \Delta t \left(F(s_h^1, c_h^1, \xi_{\boldsymbol{u}}^1) - F(s(\Delta t), c(\Delta t), \xi_{\boldsymbol{u}}^1)\right),
\end{aligned}
$$

which results after choosing $\xi_{\boldsymbol{u}}^1$ as test function in the first equation of the reduced form of Lemma 2.2 and system (2.1), performing an Euler scheme step, subtracting both equations, and adding $\pm a_1^h(c_1^1; \Pi_h \boldsymbol{u}(\Delta t), \xi_{\boldsymbol{u}}^1)$. Now, invoking the approximation estimates (2.18), Young's inequality, and the stability properties, we get

$$
\begin{aligned}
\frac{1}{4}\|\xi_{\boldsymbol{u}}^1\|_{0,\Omega}^2 + \frac{1}{4}\Delta t \tilde{\alpha}_a \|\xi_{\boldsymbol{u}}^1\|_{1,\mathscr{T}_h}^2 + \Delta t |\xi_{\boldsymbol{u}}^1|_{\boldsymbol{u}_h^1,\mathrm{upw}}^2 &\leqslant Ch^{2k}\Delta t \Big(\|\boldsymbol{u}(\Delta t)\|_{k+1,\Omega}^2 + \|\boldsymbol{u}(0)\|_{k+1,\Omega}^2 + \|c(\Delta t)\|_{k+1,\Omega}^2 \Big) \\
&\quad + C[\Delta t]^4 \|\boldsymbol{u}''\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{L}^2(\Omega))}^2 + \frac{4\tilde{C}_{\mathrm{Lip}}^2 M^2}{\tilde{\alpha}_a}\Delta t \|\xi_c^1\|_{1,\Omega}^2 \\
&\quad + \Delta t C_f^2 \|\xi_c^1\|_{1,\Omega}^2 + \Delta t C_f^2 \|\xi_s^1\|_{1,\Omega}^2.
\end{aligned}
\tag{A.1}
$$

Next, we choose ξ_s^1 as test function in (2.19c) and system (2.1); we follow the same steps as before, adding to the sum of both equations the term $\pm a_2(\mathscr{I}_h s^1, \xi_c^1)$ to obtain

$$
\begin{aligned}
\frac{1}{4}\|\xi_s^1\|_{0,\Omega}^2 + \frac{1}{2}\Delta t \hat{\alpha}_a \|\xi_s^1\|_{1,\Omega}^2 &\leqslant C\Delta t h^{2k}\Big(\|\boldsymbol{u}(\Delta t)\|_{k+1,\Omega}^2 + \|s(\Delta t)\|_{k+1,\Omega}^2 + \|s(0)\|_{k+1,\Omega}^2 \Big) \\
&\quad + C[\Delta t]^4 \|s''\|_{L^\infty(0,t_{\mathrm{end}};L^2(\Omega))}^2 + \frac{4\tilde{C}_c^2}{\hat{\alpha}_a}\Delta t \|s\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}^2 \|\xi_{\boldsymbol{u}}\|_{1,\mathscr{T}_h}^2.
\end{aligned}
\tag{A.2}
$$

In the same way, choosing ξ_c^1 as test function in (2.19d) and in (2.1) we arrive at

$$
\begin{aligned}
\frac{1}{4}\|\xi_c^1\|_{0,\Omega}^2 + \frac{1}{2}\Delta t \hat{\alpha}_a \|\xi_c^1\|_{1,\Omega}^2 &\leqslant C\Delta t h^{2k}\Big(\|\boldsymbol{u}(\Delta t)\|_{k+1,\Omega}^2 + \|c(\Delta t)\|_{k+1,\Omega}^2 + \|c(0)\|_{k+1,\Omega}^2 + v_{\mathrm{p}} \Big) \\
&\quad + C[\Delta t]^4 \|c''\|_{L^\infty(0,t_{\mathrm{end}};L^2(\Omega))}^2 + \frac{4\tilde{C}_c^2}{\hat{\alpha}_a}\Delta t \|c\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}^2 \|\xi_{\boldsymbol{u}}\|_{1,\mathscr{T}_h}^2.
\end{aligned}
\tag{A.3}
$$

Now, from (A.1) we deduce that

$$
\Delta t \|\xi_{\boldsymbol{u}}\|_{1,\mathscr{T}_h}^2 \leqslant C(h^{2k} + [\Delta t]^4) + \frac{16\tilde{C}_{\mathrm{Lip}}^2 M^2}{\tilde{\alpha}_a^2}\Delta t \|\xi_c^1\|_{1,\Omega}^2 + \Delta t \frac{4C_f^2}{\tilde{\alpha}_a}\|\xi_c^1\|_{1,\Omega}^2 + \Delta t \frac{4C_f^2}{\tilde{\alpha}_a}\|\xi_s^1\|_{1,\Omega}^2.
\tag{A.4}
$$

We insert the previous identity into (A.3) and consider M and Δt sufficiently small such that the terms multiplying $\|\xi_c\|_{1,\Omega}^2$, can be absorbed into the left-hand side of the inequality, to get

$$
\frac{1}{4}\|\xi_c^1\|_{0,\Omega}^2 + \frac{1}{4}\Delta t \hat{\alpha}_a \|\xi_c\|_{1,\Omega}^2 \leqslant C_c^1(h^{2k} + [\Delta t]^4) + \Delta t \frac{4C_f^2}{\tilde{\alpha}_a}\|\xi_s^1\|_{1,\Omega}^2.
\tag{A.5}
$$

Substituting this result back into (A.4) and then into (A.2), get us the second estimate. The first estimate follows by directly substituting (A.5) into (A.1). $\qquad\square$

B. Proof of Theorem 2.5

Proof of Theorem 2.5. We appeal to the reduced form of the problem again, taking solutions living in the discrete and continuous kernels, $\boldsymbol{u}_h \in \mathbf{X}_h$ and $\boldsymbol{u} \in \mathbf{X}$. Then we choose as test function $\boldsymbol{v}_h = \xi_{\boldsymbol{u}}^{n+1}$ in the first equation of (2.5) and insert the terms

$$
\pm \frac{(\mathscr{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t}, \quad \pm \frac{(\mathscr{D}\Pi_h \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t}, \quad \pm a_1^h\big(c_h^{n+1}; \Pi_h \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big)_\Omega.
$$

Hence we get

$$
\begin{aligned}
-\frac{(\mathscr{D}\xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t} &- \frac{(\mathscr{D}E_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t} + \frac{(\mathscr{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega}{2\Delta t} + a_1^h\big(c_h^{n+1}; \xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big) \\
&+ a_1^h\big(c_h^{n+1}; \Pi_h \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big) + c_1^h\big(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big) = F\big(s_h^{n+1}, c_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big).
\end{aligned}
\tag{B.1}
$$

We consider (2.19a) (see Lemma 2.2) at $t = t_{n+1}$ and $\boldsymbol{v} = \xi_{\boldsymbol{u}}^{n+1}$. Inserting the term $\pm (\mathscr{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega / (2\Delta t)$, we readily deduce the identity

$$\frac{(\mathscr{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_{1,\Omega}}{2\Delta t} + a_1^h(c^{n+1}; \boldsymbol{u}(t_n), \xi_{\boldsymbol{u}}^{n+1}) + c_1^h(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) = F(s(t_{n+1}), c(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}). \quad \text{(B.2)}$$

We can then subtract (B.2) from (B.1) and multiply both sides by $4\Delta t$, yielding $I_1 + I_2 + \cdots + I_7 = 0$, with

$$I_1 := 2(\mathscr{D}\xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_\Omega, \quad I_2 := 4\Delta t\, a_1^h(c_h^{n+1}; \xi_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_\Omega, \quad I_3 := 4\Delta t\left(\boldsymbol{u}'(t_{n+1}) - \frac{1}{2\Delta t}\mathscr{D}\boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\right)_\Omega,$$

$$I_4 := 2(\mathscr{D}E_{\boldsymbol{u}}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})_\Omega, \quad I_5 := 4\Delta t\left(a_1^h(c^{n+1}; \boldsymbol{u}^{n+1}, \xi_{\boldsymbol{u}}^{n+1}) - a_1^h(c_h^{n+1}; \Pi_h \boldsymbol{u}^{n+1}, \xi_{\boldsymbol{u}}^{n+1})\right),$$

$$I_6 := 4\Delta t\left(c_1^h(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1})\right),$$

$$I_7 := 4\Delta t\left(F(s(t_{n+1}), c(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}) - F(s_h^{n+1}, c_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1})\right).$$

For the first term, using (2.14) we can assert that

$$I_1 = \|\xi_{\boldsymbol{u}}^{n+1}\|_{0,\Omega}^2 + \|2\xi_{\boldsymbol{u}}^{n+1} - \xi_{\boldsymbol{u}}^n\|_{0,\Omega}^2 + \|\Lambda\xi_{\boldsymbol{u}}^{n+1}\|_{0,\Omega}^2 - \|\xi_{\boldsymbol{u}}^n\|_{0,\Omega}^2 - \|2\xi_{\boldsymbol{u}}^n - \xi_{\boldsymbol{u}}^{n-1}\|_{0,\Omega}^2.$$

Using the ellipticity stated in (2.10), we readily get

$$I_2 \geqslant 4\Delta t\, \tilde{\alpha}_a \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2.$$

On the other hand, Taylor's formula with integral remainder allows us to write

$$\left|\boldsymbol{u}'(t_{n+1}) - \frac{1}{2\Delta t}\mathscr{D}\boldsymbol{u}(t_{n+1})\right| = \frac{[\Delta t]^{3/2}}{2\sqrt{3}}\|\boldsymbol{u}^{(3)}\|_{L^2(t^{n-1}, t^{n+1}; \mathbf{L}^2(\Omega))},$$

and then, by combining Cauchy–Schwarz and Young's inequality, we obtain the bound

$$|I_3| \leqslant \frac{[\Delta t]^4}{24\varepsilon_1}\|\boldsymbol{u}^{(3)}\|_{L^2(t_{n-1}, t_{n+1}; \mathbf{L}^2(\Omega))}^2 + \frac{\Delta t\varepsilon_1}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2.$$

Now we insert $\pm 4\Delta t E_{\boldsymbol{u}}'(t_{n+1})$ into the fourth term, which leads to

$$I_4 = -4\Delta t(E_{\boldsymbol{u}}'(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})_\Omega + \left(E_{\boldsymbol{u}}'(t_{n+1}) - \frac{\mathscr{D}E_{\boldsymbol{u}}^{n+1}}{2\Delta t}, \xi_{\boldsymbol{u}}^{n+1}\right)_\Omega.$$

Proceeding as before, and using (2.18) on the first term of I_4, we get

$$|I_4| \leqslant \frac{C}{2\varepsilon_2}h^{2k}\|\boldsymbol{u}'\|_{\mathbf{L}^\infty(0, t_{\text{end}}; \mathbf{H}^k(\Omega))}^2 + \frac{\Delta t\varepsilon_2}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2$$

$$+ \frac{[\Delta t]^4 C}{2\varepsilon_3}\|\boldsymbol{u}^{(3)}\|_{L^2(0, t_{\text{end}}; \mathbf{L}^2(\Omega))}^2 + \frac{\Delta t\varepsilon_3}{2}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2.$$

Again we insert $\pm a_1^h(\mathscr{I}_h c^{n+1}; \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})$ and $\pm a_1^h(c_h^{n+1}; \boldsymbol{u}(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1})$. Then, by virtue of (2.18), (2.9), Lemma 2.1 and Young's inequality we immediately have

$$|I_5| \leqslant \frac{8\tilde{C}_{\text{Lip}}^2 M^2 C^* \Delta t}{\varepsilon_4}h^{2k}\|c\|_{L^\infty(0, t_{\text{end}}; H^{k+1}(\Omega))}^2 + \frac{\varepsilon_4}{2}\Delta t\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h} + \frac{8\tilde{C}_{\text{Lip}}^2 M^2 \Delta t}{\varepsilon_5}\|\xi_c^{n+1}\|_{1,\Omega}^2$$

$$+ \frac{\varepsilon_5}{2}\Delta t\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h} + \frac{8\tilde{C}_a^2 C^* \Delta t}{\varepsilon_6}h^{2k}\|c\|_{L^\infty(0, t_{\text{end}}; H^{k+1}(\Omega))}^2 + \frac{\varepsilon_6}{2}\Delta t\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}.$$

Adding and subtracting suitable terms within I_6, yields

$$
\begin{aligned}
I_6 = 4\Delta t \big(& c_1^h(\boldsymbol{u}(t_{n+1}),\boldsymbol{u}(t_{n+1}),\xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\boldsymbol{u}(t_{n+1}),\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) \\
& + c_1^h(\boldsymbol{u}(t_{n+1}),\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\Pi_h \boldsymbol{u}(t_{n+1}),\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) \\
& + c_1^h(\Pi_h \boldsymbol{u}(t_{n+1}),\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\boldsymbol{u}_h^{n+1},\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) \\
& + c_1^h(\boldsymbol{u}_h^{n+1},\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\boldsymbol{u}_h^{n+1},\boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) \big).
\end{aligned}
$$

We can bound from below the last term using (2.11a). In addition, we define

$$
\begin{aligned}
\tilde{I}_6 := 4\Delta t \big(& c_1^h(\boldsymbol{u}(t_{n+1}),\boldsymbol{u}(t_{n+1}),\xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\boldsymbol{u}(t_{n+1}),\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) \\
& + c_1^h(\boldsymbol{u}(t_{n+1}),\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\Pi_h \boldsymbol{u}(t_{n+1}),\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) \\
& + c_1^h(\Pi_h \boldsymbol{u}(t_{n+1}),\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) - c_1^h(\boldsymbol{u}_h^{n+1},\Pi_h \boldsymbol{u}_h^{n+1},\xi_{\boldsymbol{u}}^{n+1}) \big).
\end{aligned}
$$

The bound (2.12) and (2.18) imply that

$$
\begin{aligned}
|\tilde{I}_6| \leqslant\ & 4\Delta t \tilde{C}_c \big(\|\boldsymbol{u}(t_{n+1})\|_{1,\mathscr{T}_h} \|E_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h} \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h} + \|\Pi_h \boldsymbol{u}(t_{n+1})\|_{1,\mathscr{T}_h} \|E_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h} \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h} \\
& + \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \|\Pi_h \boldsymbol{u}(t_{n+1})\|_{1,\mathscr{T}_h} \big) \\
\leqslant\ & 4\Delta t \bigg(\frac{2Ch^{2k}\tilde{C}_c^2}{\varepsilon_7} \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^{k+1}(\Omega))}^2 \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^1(\Omega))}^2 + \frac{\varepsilon_7}{8}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \\
& + \frac{2h^{2k}C\tilde{C}_c^2}{\varepsilon_8} \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^1(\Omega))}^2 \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}_1^{k+1}(\Omega))}^2 + \frac{\varepsilon_8}{8}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \\
& + \tilde{C}_c C^* \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^1(\Omega))}^2 \bigg) \\
\leqslant\ & 4\Delta t \bigg(\frac{2Ch^{2k}}{\varepsilon_7} \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^{k+1}(\Omega))}^2 \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^1(\Omega))}^2 \\
& + \frac{\varepsilon_7}{8}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 + \frac{h^{2k}C}{2\varepsilon_8} \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^1(\Omega))}^2 \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^{k+1}(\Omega))}^2 \\
& + \frac{\varepsilon_8}{8}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 + C^*\tilde{C}_c M\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \bigg),
\end{aligned}
$$

where C^* is a positive constant coming from (2.18). We also have

$$
\begin{aligned}
|I_7| \leqslant\ & 4\Delta t \bigg(\frac{2C_f^2}{\varepsilon_9}(\|\xi_s^{n+1}\|_{0,\Omega}^2 + \|E_s^{n+1}\|_{0,\Omega}^2) + \frac{2C_f^2}{\varepsilon_{10}}(\|\xi_c^{n+1}\|_{0,\Omega}^2 + \|E_c^{n+1}\|_{0,\Omega}^2) \\
& + \frac{\varepsilon_9}{8}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 + \frac{\varepsilon_{10}}{8}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \bigg) \\
\leqslant\ & 4\Delta t \bigg(\frac{2C_f^2}{\varepsilon_9}(\|\xi_s^{n+1}\|_{0,\Omega}^2 + C^*h^{2k}\|s\|_{L^\infty(0,t_{\mathrm{end}},H^{k+1}(\Omega))}^2) + \frac{2C_f^2}{\varepsilon_{10}}(\|\xi_c^{n+1}\|_{0,\Omega}^2 + + C^*h^{2k}\|c\|_{L^\infty(0,t_{\mathrm{end}},H^{k+1}(\Omega))}^2)) \\
& + \frac{\varepsilon_9}{8}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 + \frac{\varepsilon_{10}}{8}\|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \bigg).
\end{aligned}
$$

Hence, by choosing $\varepsilon_i = 3\tilde{\alpha}_a/5$ for $i = 1,\dots,10$, collecting the above estimates, and summing over $1 \leqslant n \leqslant m$ for all $m+1 \leqslant N$, we get

$$
\|\xi_{\boldsymbol{u}}^{m+1}\|_{0,\Omega}^2 + \|2\xi_{\boldsymbol{u}}^{m+1} - \xi_{\boldsymbol{u}}^m\|_{0,\Omega}^2 + \sum_{n=1}^{m} \|\Lambda \xi_{\boldsymbol{u}}^n\|_{0,\Omega}^2 - 3\|\xi_{\boldsymbol{u}}^1\|_{0,\Omega}^2
$$

$$+ \sum_{n=1}^{m} \Delta t \tilde{\alpha}_a \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 + \sum_{n=1}^{m} \Delta t |\xi_{\boldsymbol{u}}^{n+1}|_{\boldsymbol{u}_h^{n+1},\mathrm{upw}}^2 \leqslant C([\Delta t]^4 + h^{2k}) + \frac{16 M^2 C_{\mathrm{Lip}^2} \Delta t}{\tilde{\alpha}_a} \sum_{n=1}^{m} \|\xi_c^{n+1}\|_{1,\Omega}^2,$$

where

$$M \leqslant \min\left\{ \frac{\tilde{\alpha}_a}{4\tilde{C}_c C^*}, \frac{\sqrt{\tilde{\alpha}_a}}{4 C_{\mathrm{Lip}} \sqrt{2}} \right\}, \quad \gamma_1 = \frac{16 M^2 C_{\mathrm{Lip}^2}}{\tilde{\alpha}_a} \leqslant \frac{\hat{\alpha}_a \tilde{\alpha}_a}{2}.$$

Finally, Theorem 2.4 yields the assertion of the theorem. $\qquad\square$

C. Proof of Theorem 2.6

Proof of Theorem 2.6. Proceeding similarly as for Theorem 2.5, we choose as test function $\varphi_h = \xi_s^{n+1}$ in the second equation of (2.5) and insert suitable additional terms to obtain the following identity (analogous to (B.1))

$$-\frac{(\mathscr{D}\xi_s^{n+1}, \xi_s^{n+1})_\Omega}{2\Delta t} - \frac{(\mathscr{D}E_s^{n+1}, \xi_s^{n+1})_\Omega}{2\Delta t} - a_2(\xi_s^{n+1}, \xi_s^{n+1}) + \frac{(\mathscr{D}s(t_{n+1}), \xi_s^{n+1})_\Omega}{2\Delta t} \tag{C.1}$$
$$+ a_2(\mathscr{I}_h s(t_{n+1}), \xi_s^{n+1}) + c_2(\boldsymbol{u}_h^{n+1}, s_h^{n+1}, \xi_s^{n+1}) = 0.$$

From (2.19b), focusing on $t = t_{n+1}$, using $\varphi = \xi_s^{n+1}$ and proceeding as in the derivation of (B.2), we obtain

$$\frac{(\mathscr{D}s(t_{n+1}), \xi_s^{n+1})_\Omega}{2\Delta t} + a_2\big(s(t_{n+1}), \xi_s^{n+1}\big) + c_2\big(\boldsymbol{u}(t_{n+1}), s(t_{n+1}), \xi_s^{n+1}\big) = -\left(s'(t_{n+1}) - \frac{\mathscr{D}s(t_{n+1})}{2\Delta t}, \xi_s^{n+1}\right)_\Omega. \tag{C.2}$$

Next we subtract (C.2) from (C.1) and multiply both sides by $4\Delta t$. This yields $\hat{I}_1 + \cdots + \hat{I}_6 = 0$, where

$$\hat{I}_1 := 2(\mathscr{D}\xi_s^{n+1}, \xi_s^{n+1})_\Omega, \quad \hat{I}_2 := 4\Delta t a_2(\xi_s^{n+1}, \xi_s^{n+1}), \quad \hat{I}_3 := 4\Delta t\left(s'(t_{n+1}) - \frac{\mathscr{D}s(t_{n+1})}{2\Delta t}, \xi_s^{n+1}\right)_\Omega,$$
$$\hat{I}_4 := 2\big(\mathscr{D}E_s^{n+1}, \xi_s^{n+1}\big)_\Omega, \quad \hat{I}_5 := 4\Delta t a_2\big(E_s^{n+1}, \xi_s^{n+1}\big),$$
$$\hat{I}_6 := 4\Delta t\big(c_2\big(\boldsymbol{u}_h^{n+1}, s_h^{n+1}, \xi_{\boldsymbol{u}}^{n+1}\big) - c_2\big(\boldsymbol{u}(t_{n+1}), s(t_{n+1}), \xi_{\boldsymbol{u}}^{n+1}\big)\big).$$

For $\hat{I}_1$, $\hat{I}_2$ and $\hat{I}_3$ we use (2.14), (1.5b), and Taylor expansion along with Young's inequality, respectively, to obtain

$$\hat{I}_1 = \|\xi_s^{n+1}\|_{0,\Omega}^2 + \|2\xi_s^{n+1} - \xi_s^n\|_{0,\Omega}^2 + \|\Lambda\xi_s^{n+1}\|_{0,\Omega}^2 - \|\xi_s^n\|_{0,\Omega}^2 - \|2\xi_s^n - \xi_s^{n-1}\|_{0,\Omega}^2,$$
$$\hat{I}_2 \geqslant 4\Delta t \hat{\alpha}_a \|\xi_s^{n+1}\|_{1,\Omega}^2, \qquad |\hat{I}_3| \leqslant \frac{[\Delta t]^4}{24\varepsilon_1} \|s^{(3)}\|_{L^2(t_{n-1}, t_{n+1}; L^2(\Omega))}^2 + \frac{\Delta t \varepsilon_1}{2} \|\xi_s^{n+1}\|_{1,\Omega}^2.$$

Inserting $\pm 4\Delta t E_s'(t_{n+1})$ into $\hat{I}_4$ and using (2.18) leads to the bound

$$|\hat{I}_4| \leqslant \frac{C}{2\varepsilon_2} h^{2k} \|s'\|_{L^\infty(0,t_{\mathrm{end}}; H^k(\Omega))}^2 + \frac{\Delta t \varepsilon_2}{2} \|\xi_s^{n+1}\|_{1,\Omega}^2 + \frac{[\Delta t]^4 C}{2\varepsilon_3} \|s^{(3)}\|_{L^2(0,t_{\mathrm{end}}; L^2(\Omega))}^2 + \frac{\Delta t \varepsilon_3}{2} \|\xi_s^{n+1}\|_{1,\Omega}^2.$$

Employing again (2.18) in combination with (1.3a) we have

$$|\hat{I}_5| \leqslant \frac{8\hat{C}_a^2 \Delta t}{\varepsilon_4} h^{2k} \|s\|_{L^\infty(0,t_{\mathrm{end}}; H^{k+1}(\Omega))}^2 + \frac{\Delta t \varepsilon_4}{2} \|\xi_s^{n+1}\|_{1,\Omega}^2.$$

In order to derive a bound for $\hat{I}_6$ we proceed as for the bound on I_7 in the proof of Theorem 2.5; namely adding and subtracting suitable terms in the definition of $\hat{I}_6$, defining $\tilde{I}_6$ in this case by

$$\hat{I}_6 = \tilde{I}_6 + 4\delta_t c_2\big(\boldsymbol{u}_h^{n+1}, \xi_s^{n+1}, \xi_s^{n+1}\big),$$

and applying (2.11b), (2.8), (2.18) and Young's inequality to the result, we get

$$|\tilde{I}_6| \leqslant 4\Delta t \left(\frac{2\tilde{C}_c^2 C^{*2}}{\varepsilon_5} \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2 \|s\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}^2 + \frac{1}{8\varepsilon_5}\|\xi_s\|_{1,\Omega}^2 \right.$$
$$+ \frac{2h^{2k}\tilde{C}_c^2 C^{*2}}{\varepsilon_6} \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}^2 \|s\|_{L^\infty(0,t_{\mathrm{end}};H^{k+1}(\Omega))}^2 + \frac{\varepsilon_6}{8}\|\xi_s^{n+1}\|_{1,\Omega}^2$$
$$\left. + \frac{2h^{2k}\tilde{C}^2 C^{*2}}{\varepsilon_7} \|\boldsymbol{u}\|_{L^\infty(0,t_{\mathrm{end}};\mathbf{H}^{k+1}(\Omega))}^2 \|s\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}^2 + \frac{\varepsilon_7}{8}\|\xi_s^{n+1}\|_{1,\Omega}^2 \right).$$

In this manner, and after choosing $\varepsilon_i = 6\hat{\alpha}_a/7$ for $i = 1,\ldots,7$, we can collect the above estimates and sum over $1 \leqslant n \leqslant m$, for all $m+1 \leqslant N$, to get

$$\|\xi_s^{m+1}\|_{0,\Omega}^2 + \|2\xi_s^{m+1} - \xi_s^m\|_{0,\Omega}^2 + \sum_{n=1}^m \|\Lambda\xi_s^n\|_{0,\Omega}^2 + \sum_{n=1}^m \Delta t\hat{\alpha}_a \|\xi_s^{n+1}\|_{1,\Omega}^2 - 3\|\xi_s^1\|_{0,\Omega}^2$$

$$\leqslant C([\Delta t]^4 + h^{2k}) + \gamma_2 \Delta t \sum_{n=1}^m \|\xi_{\boldsymbol{u}}^{n+1}\|_{1,\mathscr{T}_h}^2, \quad \text{where} \quad \gamma_2 := \frac{28\tilde{C}_c^2 C^{*2}}{3\hat{\alpha}_a}\|s\|_{L^\infty(0,t_{\mathrm{end}};H^1(\Omega))}^2.$$

And this step concludes the proof. $\qquad\square$

D. Proof of Theorem 4.1

Proof of Theorem 4.1. Choosing $\boldsymbol{v} = e_{\boldsymbol{u}}^c$, $q = p - \tilde{p}$, $\phi = e_s$ and $\psi = e_c$ in Lemma 4.1 gives

$$(\partial_t e_{\boldsymbol{u}}, e_{\boldsymbol{u}}^c) + a_1(c,\rho_{\boldsymbol{u}},e_{\boldsymbol{u}}^c) + c_1(\boldsymbol{u};\rho_{\boldsymbol{u}},e_{\boldsymbol{u}}^c) = a_1(c_h,\tilde{\boldsymbol{u}},e_{\boldsymbol{u}}^c) - a_1(c,\tilde{\boldsymbol{u}},e_{\boldsymbol{u}}^c) - c_1(e_{\boldsymbol{u}};\tilde{\boldsymbol{u}},e_{\boldsymbol{u}}^c) + (\boldsymbol{f} - \boldsymbol{f}_h, e_{\boldsymbol{u}}^c),$$
$$(\partial_t e_s, e_s) + a_2(\rho_s, e_s) + c_2(\boldsymbol{u};\rho_s,e_s) = -c_2(e_{\boldsymbol{u}};\tilde{s},e_s),$$
$$(\partial_t e_c, e_c) + \frac{1}{\tau}a_2(\rho_c, e_c) + c_2(\boldsymbol{u} - v_p\boldsymbol{e}_z;\rho_c,e_c) = -c_2(e_{\boldsymbol{u}};\tilde{c},e_c).$$

Moreover, there also holds

$$(\partial_t e_{\boldsymbol{u}}^c, e_{\boldsymbol{u}}^c) + a_1(c,e_{\boldsymbol{u}}^c,e_{\boldsymbol{u}}^c) + c_1(\boldsymbol{u};e_{\boldsymbol{u}}^c,e_{\boldsymbol{u}}^c) = \left(\partial_t \boldsymbol{u}_{h,r}, e_{\boldsymbol{u}}^c\right) + a_1(c_h,\tilde{\boldsymbol{u}},e_{\boldsymbol{u}}^c) - a_1(c,\tilde{\boldsymbol{u}},e_{\boldsymbol{u}}^c) - c_1(e_{\boldsymbol{u}};\tilde{\boldsymbol{u}},e_{\boldsymbol{u}}^c)$$
$$+ a_1(c,\theta_{\boldsymbol{u}}^c,e_{\boldsymbol{u}}^c) + c_1(\boldsymbol{u};\theta_{\boldsymbol{u}}^c,e_{\boldsymbol{u}}^c) + (\boldsymbol{f} - \boldsymbol{f}_h, e_{\boldsymbol{u}}^c),$$
$$(\partial_t e_s, e_s) + a_2(e_s, e_s) + c_2(\boldsymbol{u};e_s,e_s) = -c_2(e_{\boldsymbol{u}};\tilde{s},e_s) + a_2(\theta_s,e_s) + c_2(\boldsymbol{u};\theta_s,e_s),$$
$$(\partial_t e_c, e_c) + \frac{1}{\tau}a_2(e_c, e_c) + c_2(\boldsymbol{u} - v_p\boldsymbol{e}_z;e_c,e_c) = -c_2(e_{\boldsymbol{u}};\tilde{c},e_c) + \frac{1}{\tau}a_2(\theta_c,e_c) + c_2(\boldsymbol{u} - v_p\boldsymbol{e}_z;\theta_c,e_s),$$

where $\theta_{\boldsymbol{u}}^c = \tilde{\boldsymbol{u}} - \boldsymbol{u}_h^c$. Using the Cauchy–Schwarz inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\boldsymbol{u}}^c\|_{0,\Omega}^2 + \alpha_a\|e_{\boldsymbol{u}}^c\|_{1,\mathscr{T}_h}^2 \leqslant \left(C_1\|\theta_{\boldsymbol{u}}^c\|_{1,\mathscr{T}_h} + M\|\boldsymbol{u}_{h,r}\|_{1,\mathscr{T}_h}\right)\|e_{\boldsymbol{u}}^c\|_{1,\mathscr{T}_h} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,\Omega}\|e_{\boldsymbol{u}}^c\|_{0,\Omega}$$
$$+ C_2 M\|e_c\|_{1,\Omega}\|e_{\boldsymbol{u}}^c\|_{1,\mathscr{T}_h} + C_3 M\|e_{\boldsymbol{u}}^c\|_{1,\mathscr{T}_h}^2 + \|\partial_t \boldsymbol{u}_{h,r}\|_{0,\Omega}\|e_{\boldsymbol{u}}^c\|_{0,\Omega},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|e_s\|_{0,\Omega}^2 + \tilde{\alpha}_a\|e_s\|_{1,\Omega}^2 \leqslant \left(C_4\|\theta_s\|_{1,\Omega} + M\|\boldsymbol{u}_{h,r}\|_{1,\mathscr{T}_h}\right)\|e_s\|_{1,\Omega} + C_5 M\|e_s\|_{1,\Omega}\|e_{\boldsymbol{u}}^c\|_{1,\Omega},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|e_c\|_{0,\Omega}^2 + \tilde{\alpha}_a\|e_c\|_{1,\Omega}^2 \leqslant \left(C_6\|\theta_c\|_{1,\Omega} + M\|\boldsymbol{u}_{h,r}\|_{1,\Omega}\right)\|e_c\|_{1,\Omega} + C_7 M\|e_c\|_{1,\Omega}\|e_{\boldsymbol{u}}^c\|_{1,\Omega}.$$

Let us now suppose that $E_c := \|e_{\boldsymbol{u}}^c(T_0)\| = \|e_{\boldsymbol{u}}\|_{L^\infty(0,t_{\mathrm{end}};L^2(\Omega))}$, for some $T_0 \in [0, t_{\mathrm{end}}]$. Then using Poincaré-Friedrichs's inequality, Young's inequality and then combining the three equations implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e_{\boldsymbol{u}}^c\|_{0,\Omega}^2 + \|e_{\boldsymbol{u}}^c\|_{1,\mathscr{T}_h}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|e_c\|_{0,\Omega}^2 + \|e_c\|_{1,\Omega}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|e_s\|_{0,\Omega}^2 + \|e_s\|_{1,\Omega}^2$$

$$\leqslant C\big(\|\theta_{\boldsymbol{u}}^c\|_{1,\mathscr{T}_h}^2 + \|\theta_s\|_{1,\Omega}^2 + \|\theta_c\|_{1,\Omega}^2 + \|\partial_t\boldsymbol{u}_{h,r}\|_{0,\Omega}^2 + M\|\boldsymbol{u}_{h,r}\|_{1,\mathscr{T}_h}^2 + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}^2\big).$$

Integrating with respect to t on $[0,t_{\text{end}}]$ and $[0,T_0]$ yields

$$\|e_{\boldsymbol{u}}^c\|_\star^2 + \|e_c\|_\star^2 + \|e_s\|_\star^2 \leqslant \|e_{\boldsymbol{u}}^c(0)\|_{0,\Omega}^2 + \|e_c(0)\|_{0,\Omega}^2 + \|e_s(0)\|_{0,\Omega}^2$$
$$+ C\bigg(\int_0^{t_{\text{end}}}\big(\|\theta_{\boldsymbol{u}}^c\|_{1,\mathscr{T}_h}^2 + \|\theta_s\|_{1,\Omega}^2 + \|\theta_c\|_{1,\Omega}^2\big)\,\mathrm{d}t + \int_0^{t_{\text{end}}}\|\partial_t\boldsymbol{u}_{h,r}\|_{0,\Omega}^2\,\mathrm{d}t$$
$$+ M\int_0^{t_{\text{end}}}\|\boldsymbol{u}_{h,r}\|_{1,\mathscr{T}_h}^2\,\mathrm{d}t + \int_0^{t_{\text{end}}}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}^2\,\mathrm{d}t\bigg),$$

and we moreover have

$$\|e_{\boldsymbol{u}}\|_\star^2 + \|e_c\|_\star^2 + \|e_s\|_\star^2 \leqslant \|e_{\boldsymbol{u}}(0)\|_{0,\Omega}^2 + \|e_c(0)\|_{0,\Omega}^2 + \|e_s(0)\|_{0,\Omega}^2$$
$$+ C\bigg(\int_0^{t_{\text{end}}}\big(\|\theta_{\boldsymbol{u}}\|_{1,\mathscr{T}_h}^2 + \|\theta_s\|_{1,\Omega}^2 + \|\theta_c\|_{1,\Omega}^2\big)\,\mathrm{d}t + \int_0^{t_{\text{end}}}\|\partial_t\boldsymbol{u}_{h,r}\|_{0,\Omega}^2\,\mathrm{d}t + \|\boldsymbol{u}_{h,r}\|_\star^2$$
$$+ \int_0^{t_{\text{end}}}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\Omega}^2\,\mathrm{d}t\bigg), \tag{D.1}$$

and as a result we can combine Theorem 3.2 and (D.1) to readily obtain the first stated result.

On the other hand, integrating by parts in Lemma 4.1 yields

$$(\partial_t e_{\boldsymbol{u}} + \nabla(p-p_h),\boldsymbol{v}) = -a_1(c,\rho_{\boldsymbol{u}},\boldsymbol{v}) - c_1(\boldsymbol{u};\rho_{\boldsymbol{u}},\boldsymbol{v}) - b(\boldsymbol{v},p_h-\tilde{p}) + a_1(c_h,\tilde{\boldsymbol{u}},\boldsymbol{v}) + (\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{v})$$
$$- a_1(c,\tilde{\boldsymbol{u}},\boldsymbol{v}) - c_1(e_{\boldsymbol{u}};\tilde{\boldsymbol{u}},\boldsymbol{v}) \quad \text{for all } \boldsymbol{v}\in\mathbf{H}_0^1(\Omega),$$
$$(\partial_t e_s,\phi) = -a_2(\rho_s,\phi) - c_2(\boldsymbol{u};\rho_s,\phi) - c_2(e_{\boldsymbol{u}};\tilde{s},\phi) \quad \text{for all } \phi\in H_0^1(\Omega),$$
$$(\partial_t e_c,\psi) = -\frac{1}{\tau}a_2(\rho_c,\psi) - c_2(\boldsymbol{u}-v_p\boldsymbol{e}_z;\rho_c,\psi) - c_2(e_{\boldsymbol{u}};\tilde{c},\psi) \quad \text{for all } \psi\in H_0^1(\Omega).$$

We apply Young's inequality and the definition of the dual norm. Then, we integrate in time the resulting expression. Finally, the second result is a consequence of Theorem 3.2 and (D.1). $\qquad\square$

E. Proof of Lemma 5.1

Proof of Lemma 5.1. Combining (1.2) and (5.2) implies

$$\big(\partial_t(e_{\tilde{\tau}}^{\boldsymbol{u}}),\boldsymbol{v}\big) + a_1(c,\boldsymbol{u}-\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) + c_1(\boldsymbol{u},\boldsymbol{u}-\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) + b(\boldsymbol{v},p-\tilde{p}^k)$$
$$+ a_1(c,\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) - a_1(c_h,\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) + c_1(\boldsymbol{u}-\boldsymbol{u}_h,\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) - (\boldsymbol{f}-\boldsymbol{f}^k,\boldsymbol{v}) = 0 \quad \text{for all } \boldsymbol{v}\in\mathbf{H}_0^1(\Omega), \tag{E.1a}$$
$$b(\boldsymbol{u}-\tilde{\boldsymbol{u}}^k,q) = 0 \quad \text{for all } q\in L_0^2(\Omega), \tag{E.1b}$$
$$\big(\partial_t e_{\tilde{\tau}}^s(t),\phi\big) - a_2(s-\tilde{s}^k,\phi) + c_2(\boldsymbol{u};s-\tilde{s}^k,\phi) = c_2(\boldsymbol{u}-\boldsymbol{u}_h;\tilde{s}^k,\phi) \quad \text{for all } \phi\in H_0^1(\Omega), \tag{E.1c}$$
$$\big(\partial_t e_{\tilde{\tau}}^c,\psi\big) - \frac{1}{\tau}a_2(c-\tilde{c}^k,\psi) + c_2(\boldsymbol{u}-v_p\boldsymbol{e}_z;c-\tilde{c}^k,\psi) = c_2(\boldsymbol{u}-\boldsymbol{u}_h;\tilde{c}^k,\psi) \quad \text{for all } \psi\in H_0^1(\Omega). \tag{E.1d}$$

Moreover, we have

$$(\partial_t e_{\tilde{\tau}}^{\boldsymbol{u}_c},\boldsymbol{v}) + a_1(c,e_{\tilde{\tau}}^{\boldsymbol{u}_c},\boldsymbol{v}) + c_1(\boldsymbol{u},e_{\tilde{\tau}}^{\boldsymbol{u}_c},\boldsymbol{v}) + b(\boldsymbol{v},p-p^k) = (\partial_t\tilde{\boldsymbol{u}}_{h,r},\boldsymbol{v}) - a_1(c,\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) + a_1(c_h,\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) - c_1(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}^k,\boldsymbol{v})$$
$$- a_1(c,\tilde{\boldsymbol{u}}_{h,c}-\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) - c_1(\boldsymbol{u},\tilde{\boldsymbol{u}}_{h,c}-\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) + (\boldsymbol{f}-\boldsymbol{f}^k,\boldsymbol{v}),$$
$$b(e_{\tilde{\tau}}^{\boldsymbol{u}},q) = 0,$$
$$\big(\partial_t e_{\tilde{\tau}}^s(t),\phi\big) - a_2(e_k^s,\phi) + c_2(\boldsymbol{u};e_k^s,\phi) = c_2(\boldsymbol{u}-\boldsymbol{u}_h;\tilde{s}^k,\phi) - a_2(s_h-\tilde{s}^k,\phi) + c_2(\boldsymbol{u};s_h-\tilde{s}^k,\phi),$$

$$(\partial_t e_{\tilde{\tau}}^c, \psi) - \frac{1}{\tau} a_2(e_k^c, \psi) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; e_k^c, \psi) = c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{c}^k, \psi)$$
$$- \frac{1}{\tau} a_2(c_h - \tilde{c}^k, \psi) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; c_h - \tilde{c}^k, \psi).$$

Choosing $\boldsymbol{v} = \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}$, $q = p - p^k$, $\phi = e_{\tilde{\tau}}^s$, $\psi = e_{\tilde{\tau}}^c$ and then combining the first two equations, we have

$$(\partial_t \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + a_1(c, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + c_1(\boldsymbol{u}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) - (\partial_t \tilde{\boldsymbol{u}}_{h,r}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + a_1(c, \tilde{\boldsymbol{u}}^k, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) - a_1(c_h, \tilde{\boldsymbol{u}}^k, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c})$$
$$+ c_1(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + a_1(c, \tilde{\boldsymbol{u}}_{h,c} - \tilde{\boldsymbol{u}}^k, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) + c_1(\boldsymbol{u}, \tilde{\boldsymbol{u}}_{h,c} - \tilde{\boldsymbol{u}}^k, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) - (\boldsymbol{f} - \boldsymbol{f}^k, \boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}) = 0,$$
$$(\partial_t e_{\tilde{\tau}}^s, e_{\tilde{\tau}}^s) + a_2(e_{\tilde{\tau}}^s, e_{\tilde{\tau}}^s) + c_2(\boldsymbol{u}; e_{\tilde{\tau}}^s, e_{\tilde{\tau}}^s) - c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{s}^k, e_{\tilde{\tau}}^s) + a_2(s_h - \tilde{s}^k, e_{\tilde{\tau}}^s) - c_2(\boldsymbol{u}; s_h - \tilde{s}^k, e_{\tilde{\tau}}^s) = 0,$$
$$(\partial_t e_{\tilde{\tau}}^c, e_{\tilde{\tau}}^c) + \frac{1}{\tau} a_2(e_{\tilde{\tau}}^c, e_{\tilde{\tau}}^c) + c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; e_{\tilde{\tau}}^c, e_{\tilde{\tau}}^c) - c_2(\boldsymbol{u} - \boldsymbol{u}_h; \tilde{c}^k, e_{\tilde{\tau}}^c) + \frac{1}{\tau} a_2(c_h - \tilde{c}^k, e_{\tilde{\tau}}^c)$$
$$- c_2(\boldsymbol{u} - v_p \boldsymbol{e}_z; c_h - \tilde{c}^k, e_{\tilde{\tau}}^c) = 0.$$

These identities readily allow us to derive the bounds

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{0,\Omega}^2 + \alpha_a \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{1,\Omega}^2 \leqslant (\|\partial_t \tilde{\boldsymbol{u}}_{h,r}\|_{0,\Omega} + M\|c - c_h\|_{1,\Omega} + (1 + 2M)\|\boldsymbol{u}_{hc} - \tilde{\boldsymbol{u}}^k\|_{1,\Omega})\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{1,\Omega} + M\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{1,\Omega}^2$$
$$+ \|\boldsymbol{f} - \boldsymbol{f}^k\|_{0,\Omega} \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{0,\Omega},$$
$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|e_{\tilde{\tau}}^s\|_{0,\Omega}^2 + \hat{\alpha}_a \|e_{\tilde{\tau}}^s\|_{1,\Omega}^2 \leqslant (M\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\mathscr{T}_h} + (1 + M)\|s_h - \tilde{s}^k\|_{1,\Omega})\|e_{\tilde{\tau}}^s\|_{1,\Omega},$$
$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|e_{\tilde{\tau}}^c\|_{0,\Omega}^2 + \frac{\hat{\alpha}_a}{\tau} \|e_{\tilde{\tau}}^c\|_{1,\Omega}^2 \leqslant (M\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\mathscr{T}_h} + (1 + M)\|c_h - \tilde{c}^k\|_{1,\Omega})\|e_{\tilde{\tau}}^c\|_{1,\Omega}.$$

And owing to Young's inequality, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{0,\Omega}^2 + \alpha_1 \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}\|_{1,\Omega}^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|e_{\tilde{\tau}}^s\|_{0,\Omega}^2 + \alpha_2 \|e_{\tilde{\tau}}^s\|_{1,\Omega}^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|e_{\tilde{\tau}}^c\|_{0,\Omega}^2 + \alpha_3 \|e_{\tilde{\tau}}^c\|_{1,\Omega}^2$$
$$\leqslant C_1 (\|\tilde{\boldsymbol{u}}_{hc} - \tilde{\boldsymbol{u}}^k\|_{1,\mathscr{T}_h}^2 + \|s_h - \tilde{s}^k\|_{1,\Omega}^2 + \|c_h - \tilde{c}^k\|_{1,\Omega}^2 + \|\partial_t \tilde{\boldsymbol{u}}_{h,r}\|_{0,\Omega}^2 + \|\boldsymbol{f} - \boldsymbol{f}^k\|_{0,\Omega}^2)$$
$$\leqslant 2C_1 (\|\tilde{\boldsymbol{u}}_{hc} - \tilde{\boldsymbol{u}}_{\tilde{\tau}}\|_{1,\mathscr{T}_h}^2 + \|s_h - \tilde{s}_{\tilde{\tau}}\|_{1,\Omega}^2 + \|c_h - \tilde{c}_{\tilde{\tau}}\|_{1,\Omega}^2$$
$$+ \|\tilde{\boldsymbol{u}}_{\tilde{\tau}} - \tilde{\boldsymbol{u}}^k\|_{1,\mathscr{T}_h}^2 + \|\tilde{s}_{\tilde{\tau}} - \tilde{s}^k\|_{1,\Omega}^2 + \|\tilde{c}_{\tilde{\tau}} - \tilde{c}^k\|_{1,\Omega}^2) + C_1 (\|\partial_t \tilde{\boldsymbol{u}}_{h,r}\|_{0,\Omega}^2 + \|\boldsymbol{f} - \boldsymbol{f}^k\|_{0,\Omega}^2), \tag{E.2}$$

where $C_1 = \max\{\frac{2}{\alpha_a}(1 + 2M)^2, \frac{1}{2\hat{\alpha}_a}(1 + M)^2, \frac{\tau}{2\hat{\alpha}_a}(1 + M)^2\}$, $\alpha_1 = \frac{\alpha}{2} - M - 2M^2$, $\alpha_2 = \hat{\alpha}_a/2$ and $\alpha_3 = \frac{\hat{\alpha}_a}{2\tau} - \frac{3M^2}{2}$.
Moreover, we have

$$\frac{1}{2} \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2 + \alpha_1 \int_0^{t_n} \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t)\|_{1,\Omega}^2 \,\mathrm{d}t + \frac{1}{2} \|e_{\tilde{\tau}}^s(t_n)\|_{0,\Omega}^2 + \alpha_2 \int_0^{t_n} \|e_{\tilde{\tau}}^s(t)\|_{1,\Omega}^2 \,\mathrm{d}t + \frac{1}{2} \|e_{\tilde{\tau}}^c(t_n)\|_{0,\Omega}^2 + \alpha_3 \int_0^{t_n} \|e_{\tilde{\tau}}^c(t)\|_{1,\Omega}^2 \,\mathrm{d}t$$
$$\leqslant \frac{1}{2} \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2 + 2C_1 \left(\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \{\|(\tilde{\boldsymbol{u}}_{\tilde{\tau}} - \tilde{\boldsymbol{u}}^k)\|_{1,\mathscr{T}_h}^2 + \|(\tilde{s}_{\tilde{\tau}} - \tilde{s}^k)\|_{1,\Omega}^2 + \|(\tilde{c}_{\tilde{\tau}} - \tilde{c}^k)\|_{1,\Omega}^2\} \,\mathrm{d}t \right.$$
$$+ \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \{\|\tilde{\boldsymbol{u}}_{hc} - \tilde{\boldsymbol{u}}_{\tilde{\tau}}\|_{1,\mathscr{T}_h}^2 + \|s_h - \tilde{s}^k\|_{1,\Omega}^2 + \|c_h - \tilde{c}_{\tilde{\tau}}\|_{1,\Omega}^2\} \,\mathrm{d}t \Bigg)$$
$$+ \frac{1}{2} \sum_{k=1}^{n-1} (\|\boldsymbol{u}(t_k) - I^{k+1} \tilde{\boldsymbol{u}}_{h,c}^k\|_{0,\Omega}^2 - \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2) + C_1 \sum_{k=1}^n \left(\int_{t_{k-1}}^{t_k} \{\|\partial_t \tilde{\boldsymbol{u}}_{h,r}\|_{0,\Omega}^2 + \|\boldsymbol{f} - \boldsymbol{f}^k\|_{0,\Omega}^2\} \,\mathrm{d}t \right).$$

In light of the definition of $\boldsymbol{u}_{\tilde{\tau}}$, $s_{\tilde{\tau}}$ and $c_{\tilde{\tau}}$, we get

$$\int_{t_{k-1}}^{t_k} \left(\|\boldsymbol{\nabla}(\boldsymbol{u}_{\tilde{\tau}} - \boldsymbol{u}^k)\|_{0,\mathscr{T}_h}^2 + \|\nabla(s_{\tilde{\tau}} - s^k)\|_{0,\Omega}^2 + \|\nabla(c_{\tilde{\tau}} - c^k)\|_{0,\Omega}^2 \right) \,\mathrm{d}t$$
$$\leqslant \tilde{\tau}_k (\|\boldsymbol{\nabla}(\boldsymbol{u}^k - \boldsymbol{u}^{k-1})\|_{0,\mathscr{T}_h}^2 + \|\nabla(s^k - s^{k-1})\|_{0,\Omega}^2 + \|\nabla(c^k - c^{k-1})\|_{0,\Omega}^2). \tag{E.3}$$

REFERENCES

Then we can apply triangle inequality, which gives

$$
\begin{aligned}
\tilde{\tau}_k\big(\|\boldsymbol{u}^k - \boldsymbol{u}^{k-1}\|_{1,\Omega}^2 &+ \|s^k - s^{k-1}\|_{1,\Omega}^2 + \|c^k - c^{k-1}\|_{1,\Omega}^2\big) \\
&\leqslant \Upsilon_k^2 + \tilde{\tau}_k\big(\|\boldsymbol{u}^k - \boldsymbol{u}_h^k\|_{1,\mathscr{T}_h}^2 + \|s^k - s_h^k\|_{1,\Omega}^2 + \|c^k - c_h^k\|_{1,\Omega}^2\big) \\
&\quad + \tilde{\tau}_k\big(\|\boldsymbol{u}^{k-1} - I_k\boldsymbol{u}_h^{k-1}\|_{1,\mathscr{T}_h}^2 + \|s^{k-1} - s_h^{k-1}\|_{1,\Omega}^2 + \|c^{k-1} - c_h^{k-1}\|_{1,\Omega}^2\big).
\end{aligned}
\tag{E.4}
$$

Combining the results with Theorem 3.2 implies that

$$
\begin{aligned}
\frac{1}{2}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2 &+ \alpha_1\int_0^{t_n}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t)\|_{1,\Omega}^2\,\mathrm{d}t + \frac{1}{2}\|e_{\tilde{\tau}}^{s}(t_n)\|_{0,\Omega}^2 + \alpha_2\int_0^{t_n}\|e_{\tilde{\tau}}^{s}(t)\|_{1,\Omega}^2\,\mathrm{d}t + \frac{1}{2}\|e_{\tilde{\tau}}^{c}(t_n)\|_{0,\Omega}^2 + \alpha_3\int_0^{t_n}\|e_{\tilde{\tau}}^{c}(t)\|_{1,\Omega}^2\,\mathrm{d}t \\
&\leqslant C(\Xi^2 + \Upsilon^2 + \Theta^2) + \frac{1}{2}\|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^{s}(0)\|_{0,\Omega}^2 + \frac{1}{2}\|e_{\tilde{\tau}}^{t}(0)\|_{0,\Omega}^2 + \frac{1}{2}\sum_{k=1}^{n-1}\big(\|\boldsymbol{u}(t_k) - I^{k+1}\tilde{\boldsymbol{u}}_{h,c}^k\|_{0,\Omega}^2 - \|\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}_c}(t_n)\|_{0,\Omega}^2\big).
\end{aligned}
\tag{E.5}
$$

It is then possible to apply integration by parts in (E.1a), which yields

$$
\begin{aligned}
\big(\partial_t\boldsymbol{e}_{\tilde{\tau}}^{\boldsymbol{u}} + \nabla(p - p_h),\boldsymbol{v}\big) &= -a_1(c,\boldsymbol{u} - \tilde{\boldsymbol{u}}^k,\boldsymbol{v}) - c_1(\boldsymbol{u},\boldsymbol{u} - \tilde{\boldsymbol{u}}^k,\boldsymbol{v}) - b(\boldsymbol{v},p_h - \tilde{p}^k) \\
&\quad - a_1(c - c_h,\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) - c_1(\boldsymbol{u} - \boldsymbol{u}_h,\tilde{\boldsymbol{u}}^k,\boldsymbol{v}) + (\boldsymbol{f} - \boldsymbol{f}^k,\boldsymbol{v}) \quad \text{for all } \boldsymbol{v}\in\mathbf{H}_0^1(\Omega), \\
\big(\partial_t e_{\tilde{\tau}}^{s}(t),\phi\big) &= a_2(s - \tilde{s}^k,\phi) - c_2(\boldsymbol{u};s - \tilde{s}^k,\phi) + c_2(\boldsymbol{u} - \boldsymbol{u}_h;\tilde{s}^k,\phi) \quad \text{for all } \phi\in H_0^1(\Omega), \\
\big(\partial_t e_{\tilde{\tau}}^{c},\psi\big) &= \frac{1}{\tau}a_2(c - \tilde{c}^k,\psi) - c_2(\boldsymbol{u} - v_p\boldsymbol{e}_z;c - \tilde{c}^k,\psi) + c_2(\boldsymbol{u} - \boldsymbol{u}_h;\tilde{c}^k,\psi) \quad \text{for all } \psi\in H_0^1(\Omega).
\end{aligned}
$$

Next we invoke Young's inequality and the definition of the dual norm. Then, we integrate the whole expression in time between t_{k-1} and t_k for each $k = 1,2,\ldots,n$ and sum the expression for each k. Finally, it suffices to use (E.2), (E.3), (E.4) and (E.5) to establish the second bound in the Lemma. $\qquad\square$