Computing an Affine Model for a K_9-Dessin

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Abstract

- Biggs [1] constructed complete regular maps $K_n \hookrightarrow \Sigma_g$ for any prime power $n = p^f$ as Cayley maps associated to the finite fields $\mathbb{F}_n = \mathbb{F}_{p^f}$. James and Jones [4] proved Biggs' construction gives all complete regular maps. We refer to the bipartification dessin of a complete regular map with n verticies as a K_n-*dessin*. Affine models for K_n-dessins for $n = 2, 3, 4, 5, 7$ can easily be obtained. Hidalgo [3] computed an affine model for a K_8-dessin, defined over its minimal field of definition. In this work, we compute an affine model for a K_9-dessin, defined over its minimal field of definition, using both algebraic and complex analytic methods. We also present a visualization of a K_9 dessin.

Background

- A **topological map** is an embedding of a graph into a surface such that the complement of the image is a disjoint union of topological disks.
- A map is called **regular** if its group of automorphisms acts transitively on the set of flags, i.e. mutually incident vertex-edge-face triples.
- A **complete regular map** is a regular map whose underlying graph is complete, i.e. every vertex is connected to every other vertex by exactly one edge.
- **Theorem 1** (Biggs, [1]): A complete regular map with n vertices exists if and only if n is a prime power.
- **Theorem 2** (James & Jones, [4]): If $n = p^f$ is a prime power, there are $\phi(n-1)/f$ isomorphism classes of complete regular maps with n vertices, where ϕ is Euler's totient function.
- **Corollary**: The genus of any complete regular map with n vertices is

$$g(n) = \begin{cases} (n^2 - 7n + 4)/4 & \text{if } n \equiv 3 \mod 4 \\ (n-1)(n-4)/4 & \text{otherwise.} \end{cases}$$

- A **dessin** is a topological map where the underlying graph is equipped with a {black, white}-coloring. There is a bijective correspondence between between equivalence classes of dessins and equivalence classes of Belyi pairs [2].
- A **Belyi pair** is a pair (S, β) where S is a compact connected Riemann surface and β is a meromorphic function on S which is branched over (at most) $0, 1, \infty$.
- The **bipartification dessin** of a map $M : G \hookrightarrow \Sigma$ is obtained by coloring all existing vertices of G black, and adding new white verticies at the midpoints of edges.
- The **complete regular dessin** or K_n-dessin is the bipartification dessin of a complete regular map $K_n \hookrightarrow \Sigma$.
- An **affine model** of a dessin is a Belyi pair (C, β) where C is an affine algebraic curve and β is a rational function on C.

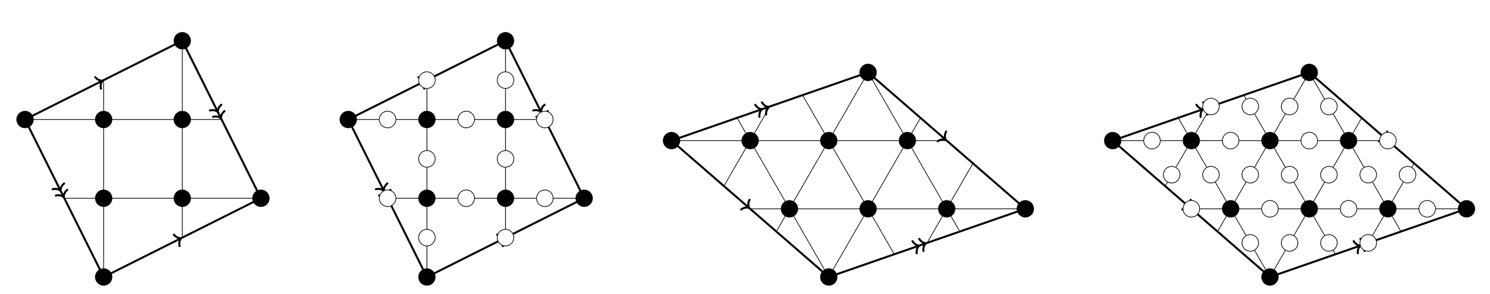


Figure 1. Two complete regular maps of genus one and their associated bipartification dessins.

Problem Statement

- **Problem:** Compute **affine models** of K_n-dessins.
- Jones, Streit & Wolfart [5] proved that the minimal field of definition of a K_n-dessin, where $n = p^f$ for some prime p, is the splitting field of p in the cyclotomic extension $\mathbf{Q}(\zeta_{n-1})/\mathbf{Q}$.
- When $n = 2, 3, 4, 5, 7$, the genus $g(n) \leq 1$ and computing affine models is easy.
- Hidalgo [3] computed explicit affine models when $n = 8$.
- In this work, we compute affine models when $n = 9$.

Main Result

- **Theorem**: An affine model for a K_9-dessin over its minimal field of definition $\mathbb{Q}(\sqrt{-2})$ is (C_{K_9}, β_{K_9}) where C_{K_9} is the genus ten curve given by the equations

$$\frac{x^3 + (4 - 2i\sqrt{2})x^2 + (-46 - 28i\sqrt{2})x - 80i\sqrt{2} - 112}{(i\sqrt{2}x + x + i\sqrt{2} + 4)^2} = -\frac{(32 - 26i\sqrt{2})u^4 - (88 + 32i\sqrt{2})u^3 - (-24 - 60i\sqrt{2})u^2 + 8(1 - 4i\sqrt{2})u + 6i\sqrt{2} - 8}{(-((-4 + i\sqrt{2})u^2) + (-4 - 2i\sqrt{2})u + i\sqrt{2})^2}$$

$$\frac{(x^3 + (6 - 3i\sqrt{2})x^2 + 6(9 + 2i\sqrt{2})x + 150i\sqrt{2} + 76)y}{(i\sqrt{2}x + x + i\sqrt{2} + 4)^3} = \frac{(u + \frac{1}{3}(1 + i\sqrt{2}))^3 v}{((-4 + i\sqrt{2})u^2 - (-4 - 2i\sqrt{2})u - i\sqrt{2})^3}$$

$$\frac{z^3 + (4 - 2i\sqrt{2})z^2 + (-46 - 28i\sqrt{2})z - 80i\sqrt{2} - 112}{(i\sqrt{2}z + z + i\sqrt{2} + 4)^2} = -\frac{(-8 - 6i\sqrt{2})u^4 + 8(1 + 4i\sqrt{2})u^3 - (-24 + 60i\sqrt{2})u^2 - 8(11 - 4i\sqrt{2})u + 26i\sqrt{2} + 32}{(-i\sqrt{2}u^2 + 2(-2 + i\sqrt{2})u + i\sqrt{2} + 4)^2}$$

$$\frac{(2 - 3\sqrt{2})(z^3 + (6 - 3i\sqrt{2})z^2 + 6(9 + 2i\sqrt{2})z + 150i\sqrt{2} + 76)}{(i\sqrt{2}z + z + i\sqrt{2} + 4)^3} = -\frac{(u - i\sqrt{2} - 1)^3 v}{(-i\sqrt{2}u^2 + 2(-2 + i\sqrt{2})u + i\sqrt{2} + 4)^3}$$

$$v^2 = u^6 - 5u^4 - 5u^2 + 1, \quad (-23 - 10i\sqrt{2})y^2 = x^3 - 30x - 56, \quad -1728w^2 = z^3 - 30z - 56$$

and β_{K_9} is the rational function

$$\beta_{K_9}(x, y, z, w, u, v) = -\frac{(u+1)^4}{(u-1)^4}.$$

Big Picture

- Let $n = p^f$ be odd prime power. According to the results of the arithmetic group, each K_n-dessin is an unramified abelian cover of a Wiman surface.
- In particular, each K_9-dessin is a $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$-cover of the **Bolza surface**

$$B\colon y^2 = x(x^4 - 1) \quad \text{equipped with the Belyi function} \quad \beta_B = 1/x^4.$$

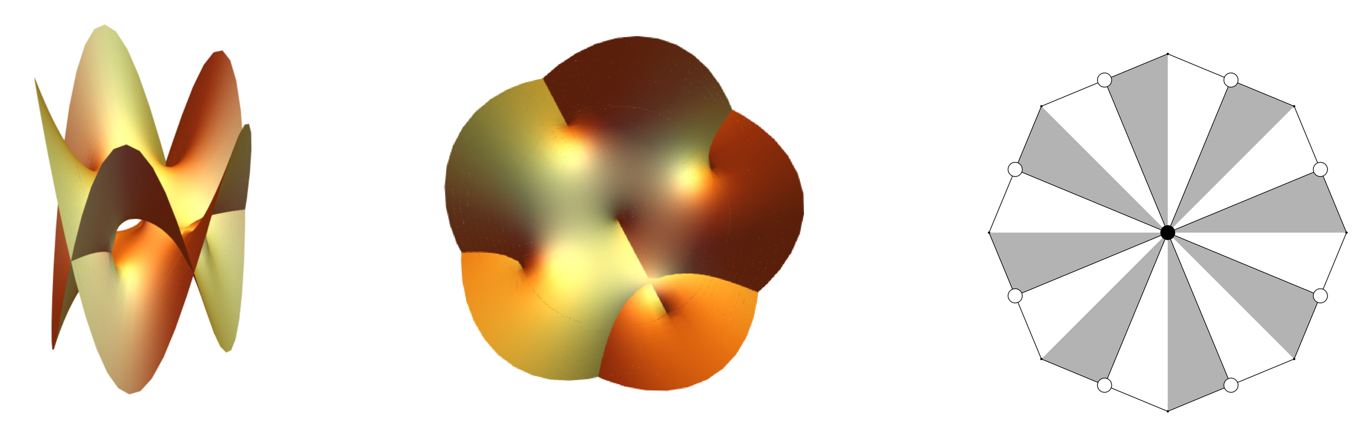


Figure 2. Real part of Bloza Surface and visualization of the dessin associated to Belyi pair (B, β_B).

- The Jacobian of the Bolza surface splits as a square of an elliptic curve.
- The K_9-dessin fits into the following Cartesian square

$$\begin{tikzcd} C_{\mathfrak{p}} \arrow[r] \arrow[d, "f_{\mathfrak{p}}"] \arrow[dr, phantom, "\lrcorner", very near start] & E \times E \arrow[d, "\phi_{\mathfrak{p}}"] \\ B \arrow[r, "AJ"] & E \times E \end{tikzcd}$$

- We are interested in the left-hand vertical arrow, so we need to compute explicit equations for AJ, the Abel-Jacobi map, and $\phi_{\mathfrak{p}}$, a degree-9 isogeny.
- Then, an affine model (C_{K_9}, β_{K_9}) can be obtained as $C_{K_9} = C_{\mathfrak{p}}$ and $\beta_{K_9} = \beta_B \circ f_{\mathfrak{p}}$.

Period Matrix of Bolza Surface

- The **period matrix** A of the Bolza surface B has entries $A_{ij} = \int_{\beta_j} \omega_i$, where $\beta_i, i = 0, 1, 2, 3$ is a set of generators for the fundamental group of B, and $\omega_0 = dx/y, \omega_1 = \zeta^3 x\, dx/y$ is a basis of holomorphic 1-forms on B.
- This is computed by Quine [8]:

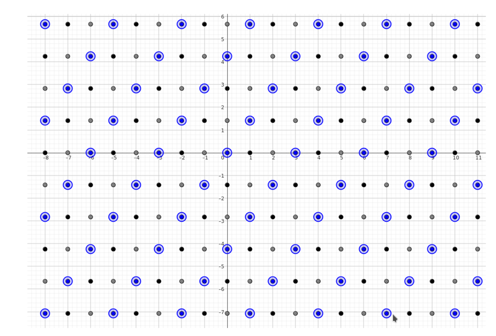
$$A = \alpha \begin{pmatrix} \zeta + \zeta^2 & -1 - \zeta & -\zeta^3 + 1 & \zeta^2 + \zeta^3 \\ \zeta^3 - \zeta^2 & -1 - \zeta^3 & -\zeta + 1 & \zeta^6 + \zeta \end{pmatrix}, \quad \alpha = \frac{\pi \Gamma(\frac{1}{8})\Gamma(\frac{3}{8}) e^{i\pi/8}}{4\pi i}$$

- The **Minkowski embedding** $M : \mathbb{Z}[\zeta_8] \to \mathbb{C}^2$ is $M = (\mathrm{id}, \tau)$ where $\tau(\zeta_8) = \zeta_8^3$.
- **Proposition**: We have an isomorphism of abelian surfaces:

$$J_B(\mathbb{C}) \cong \mathbb{C}^2 / M(\mathbb{Z}[\zeta_8]) \cong (\mathbb{C}/\mathbb{Z}[\sqrt{-2}])^2.$$

Computing the Isogeny

- Let E be the elliptic curve $E : y^2 = x^3 - 30x - 56$. Let $\kappa = \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{4\sqrt{6\pi}}, \Lambda = \kappa\mathbb{Z}[\sqrt{-2}]$.
- **Proposition**: We have an isomorphism of Riemann surfaces $\mathbb{C}/\Lambda \to E(\mathbb{C})$ given by $z \mapsto [\wp_\Lambda(z) : \frac{\wp'_\Lambda(z)}{2} : 1]$ where $\wp_\Lambda$ is the Weierstrass-$\wp$ function of Λ.



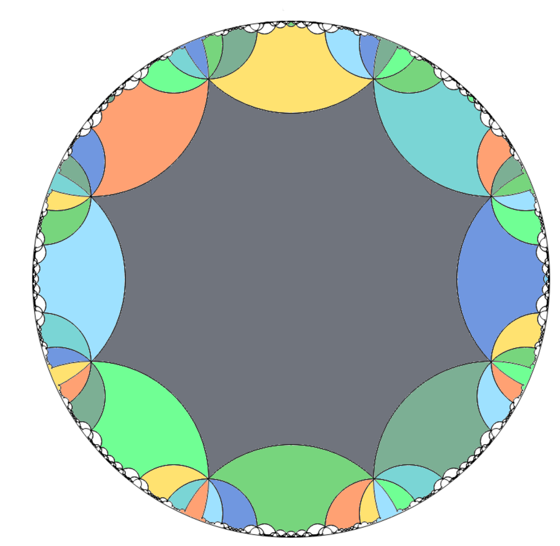
- This lattice picture shows the poles of $\wp_{\alpha\Lambda}(z) + \wp_{\alpha\Lambda}(z + \kappa) + \wp_{\alpha\Lambda}(z - \kappa)$, where $\alpha = 1 + \sqrt{-2}$. Using Liouville's theorem, we can show its equals to $\wp_\Lambda(z) + \frac{2\sqrt{-2}}{1+\sqrt{-2}}$.
- **Proposition:** $\wp_{\alpha\Lambda}(z) + \wp_{\alpha\Lambda}(z + \kappa) + \wp_{\alpha\Lambda}(z - \kappa) = \wp_\Lambda(z) + 2\sqrt{-2}/(1 + \sqrt{-2})$.
- Using this proposition, we can compute the endomorphism ϕ of the elliptic curve E given by complex multiplication by $1 + \sqrt{-2}$. The product map $\phi \times \phi$ is $\phi_{\mathfrak{p}}$.
- Alternatively, the equation for multiplication by $\sqrt{-2}$ has been already computed in Proposition 2.3.1 of Silverman [9]. Combining this with the addition formula on an elliptic curve gives equations for multiplication by $(1 + \sqrt{-2})$.

Concluding and Future Work

- Let $C_{\mathfrak{p}}$ be the fiber product of B and $E \times E$, namely the 6-tuples (x, y, a, b, c, d) satisfying $AJ(x, y) = \phi_{\mathfrak{p}}(a, b, c, d)$. By chapter 9 of Milne [7], $C_{\mathfrak{p}}$ is the cover we want. Thus $(C_{\mathfrak{p}}, \beta_{K_9})$ is an affine model for a K_9 dessin. Finally, we make a suitable coordinate change, to get an affine model over $\mathbb{Q}(\sqrt{-2})$.
- **Future Work**: Compute affine models for K_n dessins for $n > 9$. Try to find a uniform method for the case when n is odd.

Visualization of a K_9-dessin

- The Bolza surface is uniformized by the hyperbolic plane (Poincare disk model) tesselated by regular octagons. We color the octogons according to the surjective group homorphism $\pi_1(P_8/\sim) \to \mathbb{Z}[\zeta_8]/(1 + \sqrt{-2}) \cong \mathbb{F}_9 \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.



Element of $\mathbb{F}_9$	Color
0	Navy
1	Blue
2	Aqua
ζ_8	Teal
$\zeta_8 + 1$	Olive
$\zeta_8 + 2$	Green
$2\zeta_8$	Lime
$2\zeta_8 + 1$	Yellow
$2\zeta_8 + 2$	Orange

Figure 3. Visualization of a complete regular map with nine vertices.

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