LIFTING POINTS FROM INITIAL DEGENERATION OF LINEAR SPACES

RUIZHEN LIU

ABSTRACT. We give an elementary way of lifting points from initial degeneration of linear spaces defined by Plücker coordinates. We use this to give another proof of Speyer's theorem on tropical linear spaces [Spe08, Prop 4.2].

1. INTRODUCTION

In his tropical linear space papers [Spe08], Speyer invoked the fundamental theorem of tropicalisation [SS04, Theorem 2.1, Corollary 2.2] to show two constructions of tropicalisation of linear spaces coincide (c.f. Proposition 4.2, [Spe08]). However, the original proof of the fundamental theorem contains a gap. Although there have been various later proofs of the fundamental theorem involving uniqueness of flat limits [JMM08; Pay12] as well as rigid analytic geometry [Dra08], no elementary fix in the case of linear spaces exists to the best of our knowledge. This note will record a linear algebraic way of lifting points from initial degenerations of linear spaces, which gives rise to simple proofs of fundamental theorem in the linear case and Speyer's theorem on tropical linear spaces.

We will work over an algebraically closed real-valued field (K, val) with nontrivial valuation. Let $R = \mathrm{val}^{-1}(\mathbb{R}_{\geq 0}) \cup 0$ be the associated local ring and $\mathfrak{m}_R = \mathrm{val}^{-1}(\mathbb{R}_{>0}) \cup \{0\}$ be the maximal ideal. We will denote the residue map $R \to R/\mathfrak{m}_R =: k$ as $x \mapsto \overline{x}$. The residue field k is algebraically closed, since K is. Put $G_{\mathrm{val}} = \mathrm{im}(\mathrm{val}) \subseteq \mathbb{R}$ as the additive group of valuations. Since K is algebraically closed, the valuation map admits a splitting $G_{\mathrm{val}} \to K^*$, which we denote by $v \mapsto t^v$.

Let $f = \sum_u a_u X_1^{u_1} \cdots X_n^{u_n}$ be a polynomial in $K[X_1, \cdots, X_n]$, we define the *tropical hypersurface* $\mathrm{Trop}(f)$ as

$$\mathrm{Trop}(f) := \{w \in G_{\mathrm{val}}^n \colon \min_{u \in \mathbb{Z}_{\geq 0}^n} \{\mathrm{val}(a_u) + \langle u, w \rangle\} \text{ is attained twice}\}.$$

We also define the *initial degeneration* of f with weight $w \in \mathbb{R}^n$ as the polynomial

$$\mathrm{in}_w f = \sum_u \overline{t^{-\mathrm{val}(a_u)} a_u} X_1^{u_1} \cdots X_n^{u_n} \in k[X_1, \cdots, X_n],$$

where the summation is taken over all u where $\min_{u \in \mathbb{Z}_{\geq 0}^n} \{\mathrm{val}(a_u) + \langle u, w \rangle\}$ is attained. If $w \in G_{\mathrm{val}}^n$, the polynomial $\mathrm{in}_w f$ coincides with the lowest order term f_0 in the expansion $f(t^{w_1} X_1, \cdots, t^{w_n} X_n) = \sum t^{v_i} f_i$, where v_i's are increasing and $f_i \in k[X_1, \cdots, X_n]$.

Let $A \in \mathrm{Mat}_{r \times n} K$ be an r-by-n matrix with row vectors v_i spanning an r-dimensional linear space $L \subseteq K^n$. The (r, r)-minors $(P_I) = (\det A_I)$, where I runs through all r-element subset of $[n]$ and A_I is the submatrix obtained by picking out columns indexed by I, give rises to the Plücker coordinates associated to L. Speyer's theorem of tropical linear space (c.f. [Spe08, Prop. 4.2]) is:

The author thanks Hunter Spink for extremely helpful discussions and comments.

Theorem 1.1 (Theorem 3.1)**.** *The following subsets of G_{val}^n coincide:*

(i) The closure of coordinatewise valuations of points in L

$$\mathrm{val}(L) = \{(\mathrm{val}(x_1), \cdots, \mathrm{val}(x_n)) : x \in L\};$$

(ii) The intersection of tropical hyperplanes cut out by cofactor expansions

$$L(P) := \bigcap_{I=\{i_k\}_{1\le k\le r+1}\in\binom{[n]}{r+1}} \mathrm{Trop}\,\Delta_I^P, \quad \Delta_I^P(X) := \sum_{1\le r\le d+1} (-1)^r P_{i_1\cdots\widehat{i_r}\cdots i_{r+1}} X_{i_r} \in K[X_1, \cdots, X_n].$$

2. LIFTING POINTS FROM INITIAL DEGENERATION

Fix a weight $w \in G_{\mathrm{val}}^n$ and a matrix $A \in \mathrm{Mat}_{r\times n}K$ with row vectors spanning a r-dimensional linear subspace $L \subseteq K^n$. Let $y \in \mathbb{A}_k^n$ be a solution to the initial forms $\mathrm{in}_w(\Delta_I^P(X))$, for all $(r+1)$-element subsets $I = \{i_1 < \cdots < i_{r+1}\}$ in $[n]$. Our goal is to lift y to an actual solution

$$x \in V\left(\Delta_I^P; I \in \binom{[n]}{r+1}\right) \subseteq \mathbb{A}_K^n; \quad \overline{t^{-w}x} = y.$$

Lemma 2.1. *Put $A \in \mathrm{Mat}_{r\times n}K$ with row vectors v_i spanning an r-dimensional linear space L and let $P_I \in K^{\binom{n}{r}}$ be the Plücker coordinates. If y solves $\mathrm{in}_0\Delta_I^P$ for $I \in \binom{[n]}{r+1}$, then y is in the row span of $\overline{QA}$ for some $Q \in \mathrm{GL}_r\,K$.*

Proof. The Plücker coordinates are exactly the (r,r)-minors of A. When $Q \in \mathrm{GL}_r$ acts on A on the left, all the Plücker coordinates are multiplied by $\det Q$. If we multiply A by Q, with $\mathrm{val}(\det Q) = c$, we have $\mathrm{in}_0\Delta_I^{\det(Q)P} = \overline{t^{-c}\det Q} \cdot \mathrm{in}_0\Delta_I^P$. Hence we can pick out the submatrix M whose minor has the lowest valuation, multiply A by $Q = M^{-1}$, and permute columns so that $(A_{ij})_{1\le i,j\le r} = \mathrm{Id}_{(r,r)}$, i.e. the first r columns of A forms the identity matrix. By minimality, we also have $\min \mathrm{val}\, P_I = 0$.

Now we can write

$$A = \left(\begin{array}{ccc|ccc} 1 & \dots & 0 & v_{1,r+1} & \dots & v_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & v_{r,r+1} & \dots & v_{r,n} \end{array}\right). \tag{2.1}$$

Here, we notice that v_{ij} must be in R; otherwise $\det(e_1, \cdots, \hat{e}_i, e_r, (v_{i,j})_{1\le i\le r})$ have valuation $\mathrm{val}(v_{ij}) < 0$ contracting the minimality of our choice of submatrix M. Then for $I = [r] \cup l$ with $l \in \{r+1, \cdots, n\}$, the degenerated Plücker equation $\mathrm{in}_0\Delta_I^P$ becomes

$$y_l = \sum_{1\le i\le r} \overline{v_{i,l}}\, y_i. \tag{2.2}$$

These equations are already enough for us to conclude that y is in the row span of $\overline{A}$, as coefficients are given by y_i for $1 \le i \le r$. □

Algorithm 1: Lifting degenerations of a linear space

Data: An (r, n)-matrix $A \in \mathrm{Mat}_{r \times n} K$, with row vectors v_i, row span L, Plücker coordinates P_I; $w \in G_{\mathrm{val}}^n$; a nonzero solution $y \in \mathbb{A}_k^n$ to $\mathrm{in}_w \Delta_I^P$ for all $I \in \binom{[n]}{r+1}$.

Result: A point $x \in L$ such that $\overline{t^{-w} x} = y$.

Reduce to the case $w = 0$: put $X \leftarrow t^w X$ and $A \leftarrow \mathrm{diag}(t_i^{-w_i}) A$;

Apply the reduction in first paragraph of the lemma so that A is in the form Eq. (2.1);

Find $a_i \in k$ such that $y = \sum a_i \overline{v}_i$. These a_i exist by Lemma 2.1;

Pick lifts $\alpha_i \in R$ such that $\overline{\alpha_i} = a_i$;

Set $x \leftarrow t^w \sum \alpha_i v_i$ and output x.

We now argue that this algorithm is correct. It suffices to demonstrate the case $w = 0$. After reducing A to form Eq. (2.1), we have that $y = \sum a_i \overline{v_i}$ with $a_i \in k$ not all zero. By construction, x lies in the row span L. To see that x lies over y, we note that

$$\overline{x} = \sum \overline{\alpha_i}\, \overline{v_i} = \sum a_i \overline{v_i} = y.$$

Remark 2.2. In fact, the set of lifts is Zariski-dense in L. The coefficients a_i are given by y_i for $i \in [r]$, all of which are in k^*. We claim the set of lifts of $a = (a_1, \cdots, a_r)$ is dense in $(K^*)^r$: by multiplying by a^{-1}, it suffices to show the result for $1_T = (1, \cdots, 1)$. In dimension 1, the lifts are given by $1 + \mathfrak{m}_R \subseteq R$, which is dense in K^*. For arbitrary n, the lifts of 1_T are given by the product of dense sets $(1 + \mathfrak{m}_R)^{\oplus r} \subseteq (K^*)^r$, which is dense. Finally, two different lifts of a cannot yield the same vector x, as that will imply the row vectors are linearly dependent.

3. PROOF OF MAIN RESULT

Here, we reprove the fundamental theorem of tropicalization in the case of linear spaces.

Theorem 3.1. *Put $A \in \mathrm{Mat}_{r \times n} K$ with row vectors v_i spanning an r-dimensional linear space $L \subseteq K^n$ and let $P_I \in K^{\binom{n}{r}}$ be the Plücker coordinates. The following subsets of G_{val}^n coincide:*

(i) The closure of coordinatewise valuations of points in L

$$\mathrm{val}(L) = \{(\mathrm{val}(x_1), \cdots, \mathrm{val}(x_n)) : x \in L\};$$

(ii) The intersection of tropical hyperplanes $\mathrm{Trop}\, f$ with $f \in \langle \Delta_I^P, I \in \binom{[n]}{d+1} \rangle$

(iii) The intersection of tropical hyperplanes

$$L(P) := \bigcap_{I \in \binom{[n]}{r+1}} \mathrm{Trop}\, \Delta_I^P;$$

(iv) The set of vectors w such that the ideal $J_w = \langle \mathrm{in}_w \Delta_I^P; I \in \binom{[n]}{r+1} \rangle \subseteq k[x_1, \cdots, x_n]$ contains no monomials.

Proof. Here, we imitate the proof of Theorem 2.1 of [SS04]. Clearly (i) is contained in (ii), which is contained in (iii). That (iii) is contained in (iv) is the content of [Spe08, Prop 2.3, Prop. 4.2]. It suffices to show (iv) implies (i), which is where the gap in the original proof exists. The hypothesis that J_w contains no monomials means $\overline{t^w L} \times_{\mathrm{Spec}\, R} k$ intersects the open torus $(k^*)^n$. Since k is algebraically closed, there exists a closed point y in the intersection. Algorithm 1 produces a point $x \in L$ such that $\overline{t^{-w} x} = y$. It follows that $\mathrm{val}(x) = w$. □

REFERENCES

[Dra08] J. DRAISMA, "A tropical approach to secant dimensions," *J. Pure Appl. Algebra*, vol. 212, no. 2, pp. 349–363, 2008 (cit. on p. 1).

[JMM08] A. N. JENSEN, H. MARKWIG, and T. MARKWIG, "An algorithm for lifting points in a tropical variety," *Collect. Math.*, vol. 59, no. 2, pp. 129–165, 2008 (cit. on p. 1).

[Pay12] S. PAYNE, "Erratum to: Fibers of tropicalization [mr2504879]," *Math. Z.*, vol. 272, no. 3-4, pp. 1403–1406, 2012 (cit. on p. 1).

[SS04] D. SPEYER and B. STURMFELS, "The tropical Grassmannian," *Adv. Geom.*, vol. 4, no. 3, pp. 389–411, 2004 (cit. on pp. 1, 3).

[Spe08] D. E. SPEYER, "Tropical linear spaces," *SIAM J. Discrete Math.*, vol. 22, no. 4, pp. 1527–1558, 2008 (cit. on pp. 1, 3).

Email address: `ruizhen.liu@mail.utoronto.ca`