Matrix representation of linked system

We seek form $\begin{bmatrix} H & K^T \\ K & 0 \end{bmatrix} \begin{Bmatrix} \ddot q \\ -\lambda \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$

The vector $\ddot q$ contains accelerations of all bodies

Start with each body's equation. $v_j \times^* I_j v_j - f_j^e$

gravity, fluid

$$f_j = I_j a_j + p_j + {}^jX^*_{j+1} f_{j+1}$$

We write $f_j = f_{a_j} + f_{c_j}$, where f_{a_j} contains, e.g., springs/dampers, and f_{c_j} is constraint force.

also called P_c

Define $6 \times n_c$ matrix T whose columns span $S^\perp$

also called P_u

$$P_c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow S^T T = 0$$

5×1

We can represent $f_{c_j} = T \lambda_j$. Then,   (Also, $f_{a_j} = S\tau_j$)

↳ $D^T S \tau$

$$f_{a_j} + T\lambda_j = I_j a_j + p_j + {}^jX^*_{j+1}(f_{a_{j+1}} + T\lambda_{j+1})$$

$v_j \times S\dot\theta_j$

Also, $a_j = {}^jX_{j-1} a_{j-1} + S\ddot\theta_j + c_j \longrightarrow T^T a_j = T^T\,{}^jX_{j-1} a_{j-1} + T^T c_j$

Note that $S^T f_{a_j} = \tau_j$.

$$\underbrace{\begin{bmatrix} T^T & 0 & \cdots & 0 \\ 0 & T^T & \cdots & 0 \\ & & & \\ \cdots & & & T^T \end{bmatrix} \overbrace{\begin{bmatrix} 1 & \cdots & & 0 \\ -{}^2X_1 & 1 & \cdots & 0 \\ & & & \\ & \cdots & -{}^{N_b}X_{N_b-1} & 1 \end{bmatrix}}^{D}}_{K = T^T D} \overbrace{\begin{Bmatrix} a_1 \\ \vdots \\ a_{N_b} \end{Bmatrix}}^{\ddot q} = \underbrace{+\begin{bmatrix} T^T & \cdots & 0 \\ 0 \; T^T & \cdots & 0 \\ & & \\ \cdots & & T^T \end{bmatrix} \begin{Bmatrix} c_1 + {}^1X_0 a_0 \\ c_2 \\ \vdots \\ c_{N_b} \end{Bmatrix}}_{R_2}$$

Each row of momentum equation is

$$I_j a_j - T\lambda_j + {}^jX^*_{j+1} T\lambda_{j+1} = f_{a_j} - {}^jX^*_{j+1} f_{a_{j+1}} - p_j$$

$$\underbrace{\begin{bmatrix} I_1 & & 0 \\ & \ddots & \\ 0 & & I_{N_b} \end{bmatrix}}_{H} \underbrace{\begin{Bmatrix} a_1 \\ \vdots \\ a_{N_b} \end{Bmatrix}}_{\ddot q} + \underbrace{\begin{bmatrix} 1 & -{}^1X_2^* & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} T & & 0 \\ & \ddots & \\ 0 & & T \end{bmatrix}}_{K^T} \underbrace{\begin{Bmatrix} -\lambda_1 \\ \vdots \\ -\lambda_{N_b} \end{Bmatrix}}_{-\lambda} = \underbrace{\begin{Bmatrix} f_{a_1} - {}^1X_2^* f_{a_2} - p_1 \\ \vdots \\ f_{a_{N_b}} - p_{N_b} \end{Bmatrix}}_{R_1}$$

Inverse of D . . .

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & & 0 \\ -{}^2X_1 & 1 & \cdots & & 0 \\ 0 & -{}^3X_2 & 1 & & \\ & & & \ddots & \\ \cdot & \cdots & & -{}^{N_1}X_{N_b-1} & 1 \end{bmatrix}}_{D} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & \cdots & & 0 & 0 \\ {}^2X_1 & 1 & 0 & & & 0 \\ {}^3X_1 & {}^3X_2 & 1 & & & \vdots \\ {}^4X_1 & {}^4X_2 & {}^4X_3 & 1 & & \vdots \\ & & & & \ddots & 0 \\ {}^{N_b}X_1 & \cdots & & & {}^{N_b}X_{N_b-1} & 1 \end{bmatrix}}_{D^{-1}}$$

(Both left + right inverse)

So
$$\ddot{q} = D^{-1} S \underbrace{\ddot{\theta}}_{\ddot{\tilde{q}}} + D^{-1} C$$

Can we show that $\underset{T^T D}{K} D^{-1} S = 0$? $\Longrightarrow$ $\underline{T^T D D^{-1} S = T^T S = 0}$

Let us define $Z = D^{-1} S$. $\Longrightarrow$ $\ddot{q} = Z \ddot{\theta} + \ddot{q}_p$

$\ddot{q}_p = D^{-1} C$

Now, we can replace $\ddot{q}$ in equation, and project onto Z

$$H \ddot{q} - K^T \lambda = R_1 \longrightarrow \underbrace{Z^T H Z}_{\tilde{H}} \ddot{\theta} - \cancel{Z^T K^T} \lambda = \underbrace{Z^T (R_1 - H \ddot{q}_p)}_{\tilde{R}_1}$$

$\ddot{q} \leftarrow Z\ddot{\theta} + \ddot{q}_p$

$$\therefore \boxed{\tilde{H} \ddot{\theta} = \tilde{R}_1}$$

To recover λ, note that $K = T^T D$, so $K^T = D^T T$.

Thus, $K^{-1} = D^{-1} T$. Thus, we can write

$$K^{-T} = T^T D^{-T}$$

$$T^T D^{-T} (H \ddot{q} - D^T T \lambda = \tau + D^T S \tau_J)$$

and

$$\boxed{\lambda = T^T D^{-T} (H \ddot{q} - \tau)}$$

$$T^T D^{-T} H \ddot{q} - \lambda = T^T D^{-T} \tau$$

$\ddot{q} \leftarrow D^{-1} S \ddot{\tilde{q}} + D^{-1} a'$

$$T \lambda = T T^T D^{-T} (H \ddot{q} - \tau)$$

$$D^T T \lambda = D^T T T^T D^{-T} (H \ddot{q} - \tau)$$

Split unconstrained degrees of freedom into two types:

- active – motion of these are prescribed
- passive – motion of these are to be solved for

Rewrite constraint equations, motion vector, dynamical equations to include $\underbrace{\text{body } 0}$.

$$\dot{q} = \left.\begin{bmatrix} \dot{q}_0 \\ \vdots \\ \dot{q}_{N_b} \end{bmatrix}\right\} N = 6(N_b+1) \qquad D = \begin{bmatrix} 1 & & & & & 0 \\ -{}^{J_1}X_0 & {}^{J_1}X_1 & & & & \\ & -{}^{J_2}X_1 & {}^{J_2}X_2 & & & \\ & & \ddots & \ddots & & \\ & 0 & & & -{}^{J_{N_b}}X_{N_b-1} & {}^{J_{N_b}}X_{N_b} \end{bmatrix} \in \mathbb{R}^{N\times N}$$

Now let's re-define P_u to include the 6 basis vectors for the reference point 0, e.g. $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, so that $N_U = \underbrace{N_J}_{\text{no. of joints}} + \underbrace{6}_{\text{no. of ref. body dof}}$.

more precisely $\sum_{j=1}^{N_J} N_{dof_j}$

Split P_u into passive and active columns

$$P_u = [P_{UP} \; P_{UA}] \qquad \begin{matrix} P_{UP} \text{ is } N \times N_{UP} \\ P_{UA} \text{ is } N \times N_{UA} \end{matrix}$$

N_c is still $5N_b$ for the joint constraints. ← or $6N_J - N_U$

We define $\dot{\tilde{q}} = \begin{bmatrix} \dot{q}_0 \\ \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_{N_b} \end{bmatrix} \in \mathbb{R}^{N_U}$. Then the constraint equations can be written as

$$D\dot{q} = P_u \dot{\tilde{q}}$$

This gets split up into passive/active modes:

$$D\dot{q} = P_{UP}\dot{\tilde{q}}_P + P_{UA}\dot{\tilde{q}}_A$$

and then $\dot{q} = \underbrace{D^{-1}P_{UP}}_{Z_P}\dot{\tilde{q}}_P + \underbrace{D^{-1}P_{UA}}_{Z_A}\dot{\tilde{q}}_A$. Also,

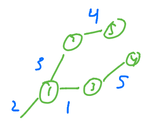
$$\underbrace{P_c^T D}_{K}\dot{q} = 0$$

As usual, there is an additional term for acceleration,

$$D\ddot{q} = P_u\ddot{\tilde{q}} + a'$$

$$\ddot{q} = Z_P\ddot{\tilde{q}}_P + Z_A\ddot{\tilde{q}}_A + D^{-1}a'$$

Note that $D^{-1} = \begin{bmatrix} 1 & & & & \\ {}^1X_0 & {}^1X_{J_1} & & & \\ {}^2X_0 & {}^2X_{J_1} & {}^2X_{J_2} & & \\ \vdots & & & \ddots & \\ {}^{N_b}X_0 & {}^{N_b}X_{J_1} & \cdots & & {}^{N_b}X_{J_{N_b}} \end{bmatrix}$



D: body → joint
D^{-1}: joint → body

$${}^2X_{J_1} = {}^2X_1\,{}^1X_{J_1}$$
$${}^3X_{J_1} = {}^3X_2\,{}^2X_1\,{}^1X_{J_1} = {}^3X_2\,{}^2X_{J_1}$$

Now write the dynamical equations:

$$H\ddot{q} - K^T\lambda = \tau + D^T P_{UP}\tau_P + D^T P_{UA}\tau_A$$

Replace $\ddot{q}$ by $\ddot{\tilde{q}}$:

$$H(Z_P\ddot{\tilde{q}}_P + Z_A\ddot{\tilde{q}}_A + D^{-1}a') - K^T\lambda = \tau + D^T P_{UP}\tau_P + D^T P_{UA}\tau_A$$

Multiply by $Z_P^T = P_{UP}^T D^{-T}$

$$\underbrace{Z_P^T H Z_P}_{\tilde{H}_P}\ddot{\tilde{q}}_P = -Z_P^T H Z_A\ddot{\tilde{q}}_A - Z_P^T H D^{-1}a' - \overbrace{Z_P^T K^T}^{P_{UP}D^{-T}D^TP_c = 0}\lambda = Z_P^T\tau + \overbrace{Z_P^T D^T P_{UP}}^{P_{UP}D^{-T}D^TP_{UP} = I}\tau_P + \underbrace{Z_P^T D^T P_{UA}}_{P_{UP}D^{-T}D^TP_{UA} = 0}\tau_A$$

$$\boxed{\tilde{H}_P\ddot{\tilde{q}}_P = Z_P^T(\tau - HD^{-1}a' - HZ_A\ddot{\tilde{q}}_A) + \tau_P}$$

P_U is $N \times N_U$

Energy equation for rigid body system

Start with the constrained momentum equation

$$H\ddot{q} - K^T\lambda = \tau + D^T P_{Up}\tau_p + D^T P_{UA}\tau_A$$

Define the energy as $\dot{q}^T f$ where $\dot{q} \in M^N$, $f \in F^N$

Take $\dot{q}^T$ times the momentum equation:

$$\dot{q}^T H\ddot{q} - \dot{q}^T K^T\lambda = \dot{q}^T \tau + \dot{q}^T D^T P_{Up}\tau_p + \dot{q}^T D^T P_{UA}\tau_A$$

$\tau = g + p + f \leftarrow$ in body system

g: gravity expressed in each body's system

$p = -\dot{H}\dot{q}$

Note that $-\dot{q}^T p = \dot{q}^T \dot{H}\dot{q}$ has elements $v_j \cdot (v_j \times^* I_j v_j)$, so it is zero.

Also, since H is symmetric,

$$\frac{d}{dt}\left(\tfrac{1}{2}\dot{q}^T H\dot{q}\right) = \tfrac{1}{2}\ddot{q}^T H\dot{q} + \cancel{\tfrac{1}{2}\dot{q}^T\dot{H}\dot{q}}^{\,0} + \tfrac{1}{2}\dot{q}^T H\ddot{q} = \dot{q}^T H\ddot{q}$$

Write

$$\dot{q} = D^{-1}\left(P_{Up}\dot{\tilde{q}}_p + P_{UA}\dot{\tilde{q}}_A\right) = D^{-1}P_U\dot{\tilde{q}}$$

Then

$$\dot{q}^T K^T\lambda = \dot{\tilde{q}}^T P_U^T D^{-T} D^T P_c\,\lambda = \dot{\tilde{q}}^T \cancel{P_U^T P_c}^{\,0}\,\lambda$$

$$\dot{q}^T D^T P_{UA}\tau_A = \dot{\tilde{q}}^T P_U^T D^{-T} D^T P_{UA}\tau_A = \dot{\tilde{q}}_A^T \tau_A$$

$$\dot{q}^T D^T P_{Up}\tau_p = \dot{\tilde{q}}_p^T \tau_p = \underbrace{-\dot{\tilde{q}}_p^T S\tilde{q}_p}_{-\frac{d}{dt}\left(\frac{1}{2}\tilde{q}_p^T S\tilde{q}_p\right)} - \dot{\tilde{q}}_p^T R\dot{\tilde{q}}_p$$

$\tau_p = -S\tilde{q}_p - R\dot{\tilde{q}}_p$

$$\tfrac{1}{2}\dot{q}^T H\dot{q} = \tfrac{1}{2}\left(\dot{\tilde{q}}_p^T P_{Up}^T + \dot{\tilde{q}}_A^T P_{UA}^T\right)D^{-T} H D^{-1}\left(P_{Up}\dot{\tilde{q}}_p + P_{UA}\dot{\tilde{q}}_A\right)$$

$$= \tfrac{1}{2}\dot{\tilde{q}}_p^T \underbrace{Z_p^T H Z_p}_{\tilde{H}_p}\dot{\tilde{q}}_p + \tfrac{1}{2}\dot{\tilde{q}}_A^T \underbrace{Z_A^T H Z_A}_{\tilde{H}_A}\dot{\tilde{q}}_A + \tfrac{1}{2}\dot{\tilde{q}}_p^T \underbrace{Z_p^T H Z_A}_{\tilde{H}_{pA}}\dot{\tilde{q}}_A + \tfrac{1}{2}\dot{\tilde{q}}_A^T \underbrace{Z_A^T H Z_p}_{\tilde{H}_{Ap} = \tilde{H}_{pA}^T}\dot{\tilde{q}}_p$$

(last two terms together: $\dot{\tilde{q}}_p^T \tilde{H}_{pA}\dot{\tilde{q}}_A$)

Also, the gravity term is

$$\dot{q}^T g = \dot{q}^T\,{}^{b}X^*_{co}\,g = \dot{q}^T\,{}^{b}X^*_{o}\,{}^{o}X^*_{co}\,g = {}^{co}g^T\,{}^{co}\dot{q} = \frac{d}{dt}\left({}^{co}g^T\,{}^{co}q\right) \qquad {}^{o}X^*_{co} = \left({}^{co}X_o\right)^T$$

$g = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$

${}^{b}X^*_{co}$: system based at center of mass, aligned with inertial system (alt: ${}^{b}X^*_{c}\,{}^{c}X^*_{co}$; ${}^{c}X^*_{co}$: pure rotation)

${}^{o}X^*_{co}$: only translation (no rotation)

Thus, our energy equation is

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}\dot{\tilde{q}}_P^T \tilde{H}_P \dot{\tilde{q}}_P}_{\text{kinetic energy in passive dofs}} + \underbrace{\tfrac{1}{2}\dot{\tilde{q}}_A^T \tilde{H}_A \dot{\tilde{q}}_A}_{\text{kinetic energy in active dofs}} + \underbrace{\dot{\tilde{q}}_P^T \tilde{H}_{PA} \dot{\tilde{q}}_A}_{\text{passive-active coupling k.e.}} + \underbrace{\tfrac{1}{2}\tilde{q}_P^T S \tilde{q}_P}_{\text{elastic storage energy}} - \underbrace{{}^{10}g^T\, {}^{10}q_b}_{\text{gravitational potential energy}}\Big)$$

$$= \underbrace{\dot{q}^T f}_{\text{rate of work by external forces (fluid)}} + \underbrace{\dot{\tilde{q}}_A^T \tau_A}_{\text{rate of work by active dofs}} - \underbrace{\dot{\tilde{q}}_P^T R \dot{\tilde{q}}_P}_{\text{rate of dissipation by joint damping}}$$

From the fluid equations

$$\dot{q}^T f = -\underset{\uparrow\, u_b = Eu}{u_b^T} f_b + \frac{d}{dt}\left(\tfrac{1}{2}\dot{q}^T M_f \dot{q}\right) = -u^T \underbrace{E^T f_b}_{\frac{du}{dt} - \left(\frac{1}{Re} Lu + N(u)\right)} + \frac{d}{dt}\left(\tfrac{1}{2}\dot{q}^T M_f \dot{q}\right)$$

so $$\dot{q}^T f = -\frac{d}{dt}\left(\tfrac{1}{2}u^T u\right) + \frac{d}{dt}\left(\tfrac{1}{2}\dot{q}^T M_f \dot{q}\right) + \frac{1}{Re} u^T L u + u^T N(u)$$

Note that we can write this in terms of γ and s (vorticity and streamfunction)

$$u = Cs + U_\infty \longrightarrow u^T u = s^T C^T u + U_\infty^T u$$
$$= s^T \gamma + \cancel{U_\infty^T C s} + \underbrace{U_\infty^T U_\infty}_{\text{if } U_\infty(t)\text{, how do we deal with this in energy?}}$$

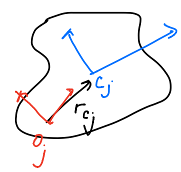
Notes on bias force $v_j \times^* I_j v_j$ and inertia I_j

$$v_j \times^* I_j v_j = \begin{bmatrix} \Omega_j^\times & U_j^\times \\ 0 & \Omega_j^\times \end{bmatrix} I_j v_j$$

Also, $I_j = {}^jX_{c_j}^* \, I_{c_j} \, {}^{c_j}X_j$

defined with axes along principal directions and based at center of mass

(Usually O_j system would be parallel)



$${}^{c_j}X_j = \begin{bmatrix} 1 & 0 \\ -r_{c_j}^\times & 1 \end{bmatrix}$$

$${}^jX_{c_j}^* = \begin{bmatrix} 1 & r_{c_j}^\times \\ 0 & 1 \end{bmatrix}$$

$$I_{c_j} = \begin{bmatrix} \bar{I}_{c_j} & 0 \\ 0 & m_j 1 \end{bmatrix}$$

$$\boxed{I_j = \begin{bmatrix} 1 & r_{c_j}^\times \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{I}_{c_j} & 0 \\ -m_j r_{c_j}^\times & m_j 1 \end{bmatrix} = \begin{bmatrix} \overbrace{\bar{I}_{c_j} - m_j r_{c_j}^\times r_{c_j}^\times}^{\bar{I}_j} & m_j r_{c_j}^\times \\ -m_j r_{c_j}^\times & m_j 1 \end{bmatrix}}$$

Then

$$v_j \times^* I_j v_j = \begin{bmatrix} \Omega_j^\times & U_j^\times \\ 0 & \Omega_j^\times \end{bmatrix} \begin{bmatrix} (\bar{I}_{c_j} - m_j r_{c_j}^\times r_{c_j}^\times)\Omega_j + m_j r_{c_j}^\times U_j \\ -m_j r_{c_j}^\times \Omega_j + m_j U_j \end{bmatrix}$$

$m_j r_{c_j}^\times \Omega_j^\times U_j$

$$= \begin{bmatrix} \Omega_j^\times (\underbrace{\bar{I}_{c_j} - m_j r_{c_j}^\times r_{c_j}^\times}_{\bar{I}_j})\Omega_j + m_j \Omega_j^\times r_{c_j}^\times U_j - m_j U_j^\times r_{c_j}^\times \Omega_j + \cancel{m_j U_j^\times U_j} \\ -m_j \Omega_j^\times r_{c_j}^\times \Omega_j + m_j \Omega_j^\times U_j \end{bmatrix}$$

0

$(\frac{1}{2}\ell, 0)$

$r \times (-\Omega V, \Omega U)$

$\frac{1}{2}\ell \Omega U$

$$\boxed{v_j \times^* I_j v_j = \begin{bmatrix} \Omega_j^\times \bar{I}_j \Omega_j + m_j r_{c_j}^\times \Omega_j^\times U_j \\ m_j \Omega_j^\times \Omega_j^\times r_{c_j} + m_j \Omega_j^\times U_j \end{bmatrix}}$$

Important note:
The spatial acceleration a_j differs from the classical acceleration by an additional term

$$a = \begin{bmatrix} \dot{\Omega} \\ \ddot{x} - \Omega^\times \dot{x} \end{bmatrix} = \underbrace{a'}_{\text{classical acceleration}} - \begin{bmatrix} 0 \\ \Omega^\times U \end{bmatrix}$$

Thus,

$$I_j a_j + v_j \times^* I_j v_j = \begin{bmatrix} \bar{I}_j & m_j r_{c_j}^\times \\ -m_j r_{c_j}^\times & m_j 1 \end{bmatrix} \begin{Bmatrix} \dot{\Omega}_j \\ \ddot{x}_j - \Omega_j^\times U_j \end{Bmatrix} + \begin{pmatrix} \Omega_j^\times \bar{I}_j \Omega_j + m_j r_{c_j}^\times \Omega_j^\times U_j \\ m_j \Omega_j^\times \Omega_j^\times r_{c_j} + m_j \Omega_j^\times U_j \end{pmatrix}$$

There are some cancellations in this. The result is

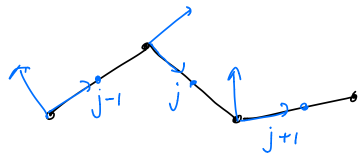
$$\boxed{I_j a_j + v_j \times^* I_j v_j = \begin{bmatrix} \bar{I}_j \dot{\Omega}_j + m_j r_{c_j}^\times \ddot{x}_j + \Omega_j^\times \bar{I}_j \Omega_j \\ m_j \ddot{x}_j - m_j r_{c_j}^\times \dot{\Omega}_j + m_j \Omega_j^\times \Omega_j^\times r_{c_j} \end{bmatrix}}$$

Notes on $c_j = v_j \times S\dot{\theta}_j$

This term is

$$c_j = v_j \times S\dot{\theta}_j = \begin{bmatrix} \Omega_j^{\times} & 0 \\ v_j^{\times} & \Omega_j^{\times} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_j = \begin{pmatrix} \Omega_j^{\times} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ v_j^{\times} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{pmatrix} \dot{\theta}_j = \begin{bmatrix} 0 \\ v_j^{\times} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_j \end{bmatrix} \end{bmatrix}$$

Here, we have used a coordinate system aligned with each body and based at the corresponding hinge



Note that

$$
{}^{j}X_{j-1} = \begin{bmatrix} E(\theta_j) & 0 \\ 0 & E(\theta_j) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -{}^{j}r_{j-1}^{\times} & 1 \end{bmatrix} \qquad {}^{j}r_{j-1} = (\ell_{j-1}, 0, 0)
$$

$$
= \begin{bmatrix} \cos\theta_j & \sin\theta_j & 0 & 0 & 0 & 0 \\ -\sin\theta_j & \cos\theta_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_j & \sin\theta_j & 0 \\ 0 & 0 & 0 & -\sin\theta_j & \cos\theta_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \ell_{j-1} & 0 & 1 & 0 \\ 0 & -\ell_{j-1} & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{j}X_{j-1} = \begin{bmatrix} \cos\theta_j & \sin\theta_j & 0 & 0 & 0 & 0 \\ -\sin\theta_j & \cos\theta_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ell_{j-1}\sin\theta_j & \cos\theta_j & \sin\theta_j & 0 \\ 0 & 0 & \ell_{j-1}\cos\theta_j & -\sin\theta_j & \cos\theta_j & 0 \\ 0 & -\ell_{j-1} & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{j}X_{j-2} = {}^{j}X_{j-1}\,{}^{j-1}X_{j-2} = \begin{bmatrix} \cos\theta_j & \sin\theta_j & 0 & 0 & 0 & 0 \\ -\sin\theta_j & \cos\theta_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ell_{j-1}\sin\theta_j & \cos\theta_j & \sin\theta_j & 0 \\ 0 & 0 & \ell_{j-1}\cos\theta_j & -\sin\theta_j & \cos\theta_j & 0 \\ 0 & -\ell_{j-1} & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\cdot \begin{bmatrix} \cos\theta_{j-1} & \sin\theta_{j-1} & 0 & 0 & 0 & 0 \\ -\sin\theta_{j-1} & \cos\theta_{j-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ell_{j-2}\sin\theta_{j-1} & \cos\theta_{j-1} & \sin\theta_{j-1} & 0 \\ 0 & 0 & \ell_{j-2}\cos\theta_{j-1} & -\sin\theta_{j-1} & \cos\theta_{j-1} & 0 \\ 0 & -\ell_{j-2} & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta_j+\theta_{j+1}) & \sin(\theta_j+\theta_{j+1}) & 0 & 0 & 0 & 0 \\ -\sin(\theta_j+\theta_{j+1}) & \cos(\theta_j+\theta_{j+1}) & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 0 \\ & 0 & \ell_{j-1}\sin\theta_j + \ell_{j-2}\sin(\theta_j+\theta_{j+1}) & \cos(\theta_j+\theta_{j+1}) & \sin(\theta_j+\theta_{j+1}) & 0 \\ & 0 & \ell_{j-1}\cos\theta_j + \ell_{j-2}\cos(\theta_j+\theta_{j+1}) & -\sin(\theta_j+\theta_{j+1}) & \cos(\theta_j+\theta_{j+1}) & 0 \\ \ell_{j-1}\sin\theta_{j-1} & -\ell_{j-1}\cos\theta_{j-1} - \ell_{j-2} & 0 & 0 & 0 & 1 \end{bmatrix}
$$

The distribution and resultant operators are defined by

$$u_b = \mathcal{B}\dot{q} \qquad\qquad f = -\mathcal{B}^T f_b + \mathcal{M}_f \ddot{q}$$

where $u_b \in \mathcal{M}^{J_B}$ and is arranged as $[u_{b1}, \dots, u_{bM}, v_{b1}, \dots, v_{bM}, w_{b1}, \dots, w_{bM}]^T$ (and similarly for $f_b \in \mathcal{F}^{J_B}$)

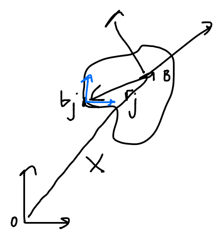
This is meant to describe $\underline{u}_j = \underline{U} + \underline{\Omega} \times (\underline{x}_j - \underline{X})$

and $\underline{F} = \sum_{j=1}^{M} \underline{f}_j$, $\underline{M} = \sum_{j=1}^{M} (\underline{x}_j - \underline{X}) \times \underline{f}_j$

If we express $\dot{q}$ and f in body coordinates, as usual, then

$${}^0x_j = {}^0X + \underbrace{{}^0E_B}_{\text{rotation from body frame to inertial frame}} {}^B r_j$$

r_j in B coordinates



and thus, ${}^0u_j = {}^0U + \underbrace{\frac{d\,{}^0E_B}{dt}}_{{}^0\Omega \times {}^0E_B} {}^B r_j = {}^0U + {}^0\Omega \times (\overbrace{{}^0x_j - {}^0X}^{{}^0r_j})$

or

$$= {}^0E_B\,{}^jU + ({}^0E_B\,{}^j\Omega) \times {}^0E_B\,{}^B r_j$$

$$= {}^0E_B\,({}^BU + \underbrace{{}^B\Omega \times {}^B r_j}_{-{}^B r_j \times {}^B\Omega})$$

or $$\begin{Bmatrix} u_j \\ v_j \\ w_j \end{Bmatrix} = {}^{3}\Big\{ \underbrace{{}^0E_B \Big[\overbrace{-{}^B r_j^{\times} \qquad 1}^{6} \Big]}_{\text{lower blocks of } {}^jX_B} \begin{Bmatrix} {}^P\Omega \\ {}^BU \end{Bmatrix}$$

Thus, if the interface velocity were organized in M blocks $(u_j, v_j, w_j)^T$, then $\mathcal{B}$ would be a block diagonal matrix with entries ${}^0E_B\,[-{}^B r_j^{\times} \quad 1]$ (so that $\mathcal{B}$ is $3M \times 6M$)

We could re-arrange the rows of interface velocity into $(u_1, u_2, \dots, v_1, \dots, w_1, \dots, w_M)$
This implies left multiplication by a permutation matrix

$$P_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} M \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} M \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} M \end{matrix}$$

$$\mathcal{B} = P_b \begin{bmatrix} {}^0E_B & & 0 \\ & \ddots & \\ 0 & & {}^0E_B \end{bmatrix} \begin{bmatrix} -{}^B r_1^{\times} \;\; 1 & & & 0 \\ & -{}^B r_2^{\times} \;\; 1 & & \\ & & \ddots & \\ 0 & & & -{}^B r_M^{\times} \;\; 1 \end{bmatrix}$$

Extracting planar problem results from 3-d equations

A basic element $\dot{q} \in \mathcal{M}^B$ has 6 entries $\begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \\ u \\ v \\ w \end{bmatrix}$

We can obtain planar entries by multiplying by

$$P_{2d}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Note that we can subdivide T into T_{2d}, T_z, where

$$T_{2d} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T, \quad T_z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T$$

Then, the overall permutation of identity I_6 can be written

$$I_6 = [\underbrace{S}_{\text{unconstrained dofs}} \quad \underbrace{T_{2d}}_{\text{planar constrained dofs}} \quad \underbrace{T_z}_{\text{out-of-plane dofs}}]$$

All are orthogonal, of course.
The 2d reduction operator is then

$$P_{2d} = [S \quad T_{2d}]$$

e.g.

$$\dot{q}_{2d} = P_{2d}^T \dot{q}$$

Alternatively, $\dot{q} = P_{2d}\, q_{2d}$ puts 0 values in out-of-plane entries

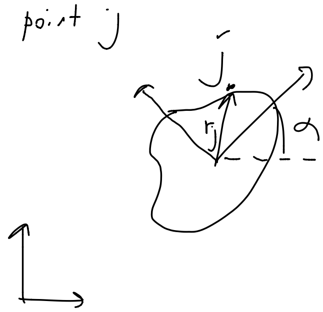
Thus, we can obtain planar equations by multiplying all 6-dof vectors by P_{2d}^T and using, e.g.,

$$I_j^{2d} = P_{2d}^T I_j P_{2d}$$

Similarly for any transformation matrix X

Expressions for 2-d example

$\mathcal{B} \longrightarrow$ Find lower blocks (2×3) of ${}^{j}X_{B}$ from body to interface point j



$${}^{0}E_{B}\left[-{}^{B}r_{j}^{\times} \quad 1\right] = \left[-{}^{0}r_{j}^{\times}\,{}^{0}E_{B} \quad {}^{0}E_{B}\right]$$

Recall that
$$a^{\times} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Also
$${}^{0}E_{B} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{rotz(-\alpha)}$$

Thus
$$\left[-{}^{0}r_{j}^{\times}\,{}^{0}E_{B} \quad {}^{0}E_{B}\right] = \begin{bmatrix} 0 & 0 & -\Delta y_j & \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & \Delta x_j & \sin\alpha & \cos\alpha & 0 \\ \Delta y\cos\alpha - \Delta x\sin\alpha & -\Delta y\sin\alpha - \Delta x\cos\alpha & 0 & 0 & 0 & 1 \end{bmatrix}$$

2d block

$\Delta x_j \equiv x_j - X$
$\Delta y_j \equiv y_j - Y$

The ${}^{j}X_{j-1}$ operator in 2-d form is

$$P_{2d}^{T}\,{}^{j}X_{j-1}\,P_{2d} = \begin{bmatrix} 1 & 0 & 0 \\ l_{j-1}\sin\theta_j & \cos\theta_j & \sin\theta_j \\ l_{j-1}\cos\theta_j & -\sin\theta_j & \cos\theta_j \end{bmatrix}$$

${}^{j}X_{j-1}\,{}^{j-1}X_{j-2}$

$$P_{2d}^{T}\,{}^{j}X_{j-2}\,P_{2d} = \begin{bmatrix} 1 & 0 & 0 \\ l_{j-1}\sin\theta_j + l_{j-2}\sin(\theta_j+\theta_{j-1}) & E(\theta_j+\theta_{j-1}) & \\ l_{j-1}\cos\theta_j + l_{j-2}\cos(\theta_j+\theta_{j-1}) & & \end{bmatrix}$$

$\alpha_j - \alpha_{j-2} = \theta_j + \theta_{j-1}$

$$K = T_{2d}^{T} D P_{2d} = \begin{bmatrix} T_{2d}^{T} & & & \\ & T_{2d}^{T} & & \\ & & \ddots & \\ & & & T_{2d}^{T} \end{bmatrix} \begin{bmatrix} 1_6 & \cdots & & 0 \\ -{}^{2}X_{1} & 1 & \cdots & 0 \\ & & \ddots & \\ & \cdots & -{}^{N_b}X_{N_b-1} & 1 \end{bmatrix} P_{2d} = \begin{bmatrix} T_{2d}^{T}P_{2d} & 0 & & \\ -T_{2d}^{T}\,{}^{2}X_{1}P_{2d} & T_{2d}^{T}P_{2d} & & \\ & & \ddots & 0 \\ & & -T_{2d}^{T}\,{}^{N_b}X_{N_b-1}P_{2d} & T_{2d}^{T}P_{2d} \end{bmatrix}$$

$$T_{2d}^{T}\,{}^{j}X_{j-1}\,P_{2d} = \begin{bmatrix} l_{j-1}\sin\theta_j & \cos\theta_j & \sin\theta_j \\ l_{j-1}\cos\theta_j & -\sin\theta_j & \cos\theta_j \end{bmatrix}, \quad T_{2d}^{T}P_{2d} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{2d}^{T}\,{}^{j}X_{k}\,P_{2d} = \begin{bmatrix} 1 & 0 & 0 \\ \sum_{m=k}^{j-1} l_m\sin(\alpha_j-\alpha_m) & \cos(\alpha_j-\alpha_k) & \sin(\alpha_j-\alpha_k) \\ \sum_{m=k}^{j-1} l_m\cos(\alpha_j-\alpha_m) & -\sin(\alpha_j-\alpha_k) & \cos(\alpha_j-\alpha_k) \end{bmatrix}$$

$$Z = P_{2d}^{T} D^{-1} S = \begin{bmatrix} 1 & 0 & \\ 0 & 0 & \\ 0 & 0 & \\ 1 & 1 & \\ l_1\sin(\alpha_2-\alpha_1) & 0 & \\ l_1\cos(\alpha_2-\alpha_1) & 0 & \\ & 1 & \\ & l_2\sin(\alpha_3-\alpha_2) & \\ & l_2\cos(\alpha_3-\alpha_2) & \\ & \vdots & \\ 1 & & \\ \sum_{m=1}^{N_b-1} l_m\sin(\alpha_{N_b}-\alpha_m) & & \\ \sum_{m=1}^{N_b-1} l_m\cos(\alpha_{N_b}-\alpha_m) & & \end{bmatrix} \longleftarrow$$

entries with blocks
$$\begin{pmatrix} 1 \\ \sum_{m=k}^{j-1} l_m\sin(\alpha_j-\alpha_m) \\ \sum_{m=k}^{j-1} l_m\cos(\alpha_j-\alpha_m) \end{pmatrix}$$
in row j, column $k \le j$

The planar inertia for body j is given by

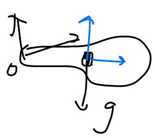
$$P_{2d}^{T} \begin{bmatrix} \overbrace{\bar{I}_{c_j} - m_j r_{c_j}^{\times} r_{c_j}^{\times}}^{\bar{I}_j} & m_j r_{c_j}^{\times} \\ -m_j r_{c_j}^{\times} & m_j 1 \end{bmatrix} P_{2d} \qquad r_{c_j}^{\times} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}l \\ 0 & \frac{1}{2}l & 0 \end{bmatrix}$$

$$r_{c_j}^{\times} r_{c_j}^{\times} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{4}l^2 & 0 \\ 0 & 0 & -\frac{1}{4}l^2 \end{bmatrix}$$

$$\bar{I}_j = \bar{I}_{c_j} + \frac{1}{4} m_j l_j^2$$

$$\begin{pmatrix} \vdots & \vdots & \vdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & -\frac{1}{2}ml \\ \vdots & \vdots & \vdots & 0 & \frac{1}{2}ml & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & \frac{1}{2}ml & 0 & m & 0 \\ 0 & -\frac{1}{2}ml & 0 & 0 & 0 & m \end{pmatrix} \xrightarrow{2\text{-}d} \begin{bmatrix} \bar{I}_{z_j} & 0 & \frac{1}{2}ml \\ 0 & m & 0 \\ \frac{1}{2}ml & 0 & m \end{bmatrix}$$

Gravity in each body



At center of mass, in coordinate system parallel to inertial system

$${}^{0}mg = \begin{pmatrix} 0 \\ mg_x \\ mg_y \end{pmatrix} \longrightarrow {}^{j}X^{*}_{c_j}\,{}^{c_j}X^{*}_{0}\,{}^{0}mg$$

$$\begin{bmatrix} 1 & r_{c_j}^{\times} \\ 0 & 1 \end{bmatrix} \qquad {}^{j}E_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_j & \sin\alpha_j \\ 0 & -\sin\alpha_j & \cos\alpha_j \end{bmatrix}$$

$$\begin{Bmatrix} 0 \\ M(g_x \cos\alpha_j + g_y \sin\alpha_j) \\ M(-g_x \sin\alpha_j + g_y \cos\alpha_j) \end{Bmatrix}$$

$$(\tfrac{1}{2}l_j, 0) \downarrow \; r_{c_j} \times$$

$$\Rightarrow \begin{Bmatrix} \frac{1}{2} M_j l_j (-g_x \sin\alpha_j + g_y \cos\alpha) \\ M_j (g_x \cos\alpha_j + g_y \sin\alpha_j) \\ M_j (-g_x \sin\alpha_j + g_y \cos\alpha_j) \end{Bmatrix}$$

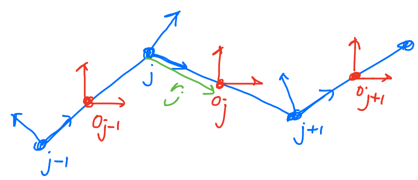
Let's suppose we have a coordinate system at centroid of body j that is aligned with inertial system. Note that neither system j or o_j actually moves as body does. Denote o_j as this system. Then

$$^{o_j}X_j = \begin{bmatrix} E(-\alpha_j) & 0 \\ 0 & E(-\alpha_j) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -r_j^\times & 1 \end{bmatrix}$$

$\boxed{r_j \text{ is vector from } j \text{ to } o_j \text{ (in } j \text{ coordinates)}}$

Denote, e.g., $^o v_j$ as velocity of body j in system o_j.

$$^o v_j = {}^{o_j}X_j\, v_j \quad , \quad {}^o a_j = {}^{o_j}X_j\, a_j$$



$r_j = (\frac{1}{2} \ell_j, 0, 0)$

$$^{o_j}X_j = \begin{bmatrix} \cos\alpha_j & -\sin\alpha_j & 0 & 0 & 0 & 0 \\ \sin\alpha_j & \cos\alpha_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha_j & -\sin\alpha_j & 0 \\ 0 & 0 & 0 & \sin\alpha_j & \cos\alpha_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\ell_j & 0 & 1 & 0 \\ 0 & \frac{1}{2}\ell_j & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{o_j}X_j = \begin{bmatrix} \cos\alpha_j & -\sin\alpha_j & 0 & 0 & 0 & 0 \\ \sin\alpha_j & \cos\alpha_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\ell_j \sin\alpha_j & \cos\alpha_j & -\sin\alpha_j & 0 \\ 0 & 0 & -\frac{1}{2}\ell_j \cos\alpha_j & \sin\alpha_j & \cos\alpha_j & 0 \\ 0 & \frac{1}{2}\ell_j & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{j}X_{o_j} = \begin{bmatrix} \cos\alpha_j & \sin\alpha_j & 0 & 0 & 0 & 0 \\ -\sin\alpha_j & \cos\alpha_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha_j & \sin\alpha_j & 0 \\ 0 & 0 & 0 & -\sin\alpha_j & \cos\alpha_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\ell_j \sin\alpha_j & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\ell_j \cos\alpha_j & 0 & 1 & 0 \\ \frac{1}{2}\ell_j \sin\alpha_j & -\frac{1}{2}\ell_j \cos\alpha_j & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{j}X_{o_j} = \begin{bmatrix} \cos\alpha_j & \sin\alpha_j & 0 & 0 & 0 & 0 \\ -\sin\alpha_j & \cos\alpha_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha_j & \sin\alpha_j & 0 \\ 0 & 0 & \frac{1}{2}\ell_j & -\sin\alpha_j & \cos\alpha_j & 0 \\ \frac{1}{2}\ell_j \sin\alpha_j & -\frac{1}{2}\ell_j \cos\alpha_j & 0 & 0 & 0 & 1 \end{bmatrix}$$

Also,

$$
{}^{j}\dot{X}_{o_{j-1}} = \begin{bmatrix}
\cos\alpha_j & \sin\alpha_j & 0 & 0 & 0 & 0 \\
-\sin\alpha_j & \cos\alpha_j & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos\alpha_j & \sin\alpha_j & 0 \\
0 & 0 & -\frac{1}{2}\ell_{j-1} & -\sin\alpha_j & \cos\alpha_j & 0 \\
-\frac{1}{2}\ell_{j-1}\sin\alpha_j & +\frac{1}{2}\ell_{j-1}\cos\alpha_j & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
{}^{o_{j-1}}_{\ j}X = \begin{bmatrix}
\cos\alpha_j & -\sin\alpha_j & 0 & 0 & 0 & 0 \\
\sin\alpha_j & \cos\alpha_j & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}\ell_{j-1}\sin\alpha_j & \cos\alpha_j & -\sin\alpha_j & 0 \\
0 & 0 & +\frac{1}{2}\ell_{j-1}\cos\alpha_j & \sin\alpha_j & \cos\alpha_j & 0 \\
0 & -\frac{1}{2}\ell_{j-1} & 0 & 0 & 0 & 1
\end{bmatrix}
$$

The original constraint can be written as

$$
{}^{j}X_{o_j}\,{}^{o}v_j = \underbrace{{}^{j}X_{j-1}\,{}^{j-1}X_{o_{j-1}}}_{{}^{j}X_{o_{j-1}}}\,{}^{o}v_{j-1} + S\dot{\theta}_j \quad \text{and} \quad {}^{j}X_{o_j}\,{}^{o}a_j = {}^{j}X_{o_{j-1}}\,{}^{o}a_{j-1} + S\ddot{\theta}_j + \underbrace{c_j}_{({}^{j}X_{o_j}\,{}^{o}v_j)\times S\dot{\theta}_j}
$$

Furthermore,

$$
T^{T}\,{}^{j}X_{o_j}\,{}^{o}a_j = T^{T}\,{}^{j}X_{o_{j-1}}\,{}^{o}a_{j-1} + T^{T}c_j
$$

Note that

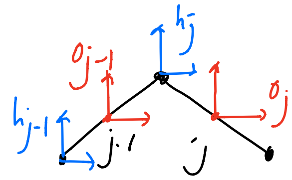
$$
({}^{j}X_{o_j})^{T}\,T = \left[\begin{array}{cc|cc|c}
\cos\alpha_j & -\sin\alpha_j & 0 & 0 & \frac{1}{2}\ell_j\sin\alpha_j \\
\sin\alpha_j & \cos\alpha_j & 0 & 0 & -\frac{1}{2}\ell_j\cos\alpha_j \\
\hline
0 & 0 & 0 & \frac{1}{2}\ell_j & 0 \\
0 & 0 & \cos\alpha_j & -\sin\alpha_j & 0 \\
0 & 0 & \sin\alpha_j & \cos\alpha_j & 0 \\
\hline
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

Each row of momentum equation is

$$
\underbrace{{}^{o}_{j}X^{*}\, I_j\, {}^{j}X_{o_j}}_{{}^{o}I_j}\,{}^{o}a_j - {}^{o}_{j}X^{*}\,T\lambda_j + {}^{o}_{j+1}X^{*}\,T\lambda_{j+1} = {}^{o}_{j}X^{*} f_{a_j} - {}^{o}_{j+1}X^{*} f_{a_{j+1}} - {}^{o}_{j}X^{*} p_j
$$

Note that the relevant entries in $({}^{j}X_{o_j}^{T})\,T$ and its transpose depend on the fact that the coordinate system j is aligned with body. If instead j were parallel to o_j, then this would have different entries. (See, e.g. original manuscript Wang + Eldredge JCP 2014)

Let's denote this system as h_j



$$r_j^{\times} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

Then ${}^{o_j}r_{h_j} = \left(\tfrac{1}{2} l_j \cos\alpha_j,\ \tfrac{1}{2} l_j \sin\alpha_j,\ 0\right)$

$${}^{o_j}X_{h_j} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} l_j \sin\alpha_j & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} l_j \cos\alpha_j & 0 & 1 & 0 \\ \frac{1}{2} l_j \sin\alpha_j & -\frac{1}{2} l_j \cos\alpha_j & 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{h_j}X_{o_j} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & +\frac{1}{2} l_j \sin\alpha_j & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} l_j \cos\alpha_j & 0 & 1 & 0 \\ +\frac{1}{2} l_j \sin\alpha_j & +\frac{1}{2} l_j \cos\alpha_j & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus

$$({}^{h_j}X_{o_j})^T T = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} l_j \sin\alpha_j \\ 0 & 1 & 0 & 0 & \frac{1}{2} l_j \cos\alpha_j \\ 0 & 0 & \frac{1}{2} l_j \sin\alpha_j & -\frac{1}{2} l_j \cos\alpha_j & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is consistent with manuscript, albeit with a slightly different numbering system.

The difference in using this h_j system instead of j is that the components of $\dot{d}$ are now aligned with inertial axes rather than body.

Older version, from original manuscript (Wang + Eldredge JCP)

$$Z = \begin{bmatrix} -K_1^{-1}K_2 \\ 1_{N-S} \end{bmatrix} \qquad N = 3N_b \qquad S = 2N_b$$

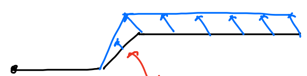
$$\underset{2N_b\times 2N_b}{K_1^{-1}} = -\begin{bmatrix} 1_2 & & & 0 \\ 1_2 & 1_2 & & \\ & & \ddots & \\ 1_2 & 1_2 & \cdots & 1_2 \end{bmatrix} \qquad \underset{2N_b\times N_b}{K_2} = \begin{bmatrix} Q_0 & & & \\ Q_0 & Q_1 & & \\ & Q_1 & \ddots & \\ & & Q_{N_b-2} & Q_{N_b-1} \end{bmatrix}$$

$$\underbrace{-K_1^{-1}K_2}_{2N_b\times N_b} = \begin{bmatrix} Q_0 & & & \\ 2Q_0 & Q_1 & & \\ \vdots & 2Q_1 & \ddots & \\ & \vdots & & \\ 2Q_0 & 2Q_1 & & Q_{N_b-1} \end{bmatrix}$$

So

$$Z = \begin{bmatrix} Q_0 & & & \\ 2Q_0 & Q_1 & \cdots & \\ \vdots & \vdots & & \\ 2Q_0 & 2Q_1 & & Q_{N_b-1} \\ 1 & 0 & & 0 \\ 0 & 1 & \ddots & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

e.g. each column of Z is a motion of the form



$$\ddot{q}_1 = -K_1^{-1}K_2\,\ddot{q}_2$$

Let's relate this back to original constraint equations.
For example,

$$a_j = {}^{j}X_{j-1}\,a_{j-1} + S\ddot{\theta}_j + c_j$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -{}^{2}X_1 & 1 & \cdots & 0 \\ & & \ddots & \\ \cdots & & -{}^{N_b}X_{N_b-1} & 1 \end{bmatrix}}_{J} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_b} \end{Bmatrix}}_{\ddot{q}} = \underbrace{\begin{bmatrix} S & & & \\ & S & & 0 \\ & & \ddots & \\ 0 & & & S \end{bmatrix}}_{S} \underbrace{\begin{Bmatrix} \ddot{\theta}_1 \\ \vdots \\ \ddot{\theta}_{N_b} \end{Bmatrix}}_{\ddot{\theta}} + \underbrace{\begin{Bmatrix} c_1 + {}^{1}X_0\,a_0 \\ \vdots \\ c_{N_b} \end{Bmatrix}}_{c}$$

$$T^T S = 0 \rightarrow$$

$$K = \begin{bmatrix} T^T & 0 & 0 & 0 \\ -T^T\,{}^{2}X_1 & T^T & & \vdots \\ & \ddots & \ddots & \vdots \\ & & T^T & \\ & & -T^T\,{}^{N_b}X_{N_b-1} & T^T \end{bmatrix}$$

Note that $S^T a_j = S^T\,{}^jX_{j-1}\, a_{j-1} + \ddot\theta_j + S^T c_j$

The permutation of each body system is given by

$$\begin{bmatrix} T^T \\ S^T \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{bmatrix} \longrightarrow \text{moves } \ddot\alpha_j \text{ to last row}$$

Let's define $P^T = \begin{bmatrix} T^T \\ S^T \end{bmatrix} = \begin{bmatrix} T^T & 0 & & 0 \\ 0 & T^T & \cdots & \\ 0 & 0 & & T^T \\ S^T & 0 & & 0 \\ & S^T & & \\ & & \ddots & \\ 0 & & & S^T \end{bmatrix}$

The constraint equations are

$$J\ddot q = S\ddot\theta + C$$

and since $\ddot q = P\begin{Bmatrix}\ddot q_1 \\ \ddot q_2\end{Bmatrix}$

Then

$$\underbrace{\underbrace{P^T J}_{\begin{bmatrix}T^TJ \\ S^TJ\end{bmatrix}} P}_{\begin{bmatrix}K_1 & K_2 \\ 0 & \Delta\end{bmatrix}} \begin{Bmatrix}\ddot q_1 \\ \ddot q_2\end{Bmatrix} = \underbrace{\begin{bmatrix}T^T \\ S^T\end{bmatrix} S\ddot\theta}_{\begin{bmatrix}0 \\ \ddot\theta\end{bmatrix}} + \underbrace{\begin{bmatrix}T^T \\ S^T\end{bmatrix} C}_{\begin{bmatrix}T^TC \\ S^TC\end{bmatrix}}$$

$T^TC \to R_2$

$S^TC \to \begin{Bmatrix}\ddot\alpha_0 \\ 0 \\ \vdots \\ 0\end{Bmatrix}$

$$\underbrace{\Delta_{N_b}}_{N_b \times N_b} = \begin{bmatrix} 1 & 0 & & & 0 \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \\ 0 & & & -1 & 1 \end{bmatrix}$$

Now Multiply by

$$\begin{bmatrix}K_1^{-1} & 0 \\ 0 & \Delta^{-1}\end{bmatrix}\left(\begin{bmatrix}K_1 & K_2 \\ 0 & \Delta\end{bmatrix}\begin{pmatrix}\ddot q_1 \\ \ddot q_2\end{pmatrix} = \begin{pmatrix}0 \\ \ddot\theta\end{pmatrix} + \begin{bmatrix}R_1 \\ S^TC\end{bmatrix}\right)$$

So

$$\begin{bmatrix}1_s & K_1^{-1}K_2 \\ 0 & 1_{N-s}\end{bmatrix}\begin{Bmatrix}\ddot q_1 \\ \ddot q_2\end{Bmatrix} = \begin{pmatrix}0 \\ \Delta^{-1}\ddot\theta\end{pmatrix} + \begin{pmatrix}K_1^{-1}R_2 \\ \Delta^{-1}S^TC\end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \ddot q_1 &= -K_1^{-1}K_2\,\ddot q_2 + K_1^{-1}R_1 \\ \ddot q_2 &= \Delta^{-1}\ddot\theta + \Delta^{-1}S^TC \end{aligned}\right\} \quad \ddot q_1 = -K_1^{-1}K_2(\Delta^{-1}\ddot\theta + \Delta^{-1}S^TC) + K_1^{-1}\underbrace{R_2}_{T^TC}$$

$$\begin{Bmatrix}\ddot q_1 \\ \ddot q_2\end{Bmatrix} = \begin{bmatrix}-K_1^{-1}K_2\Delta^{-1} \\ \Delta^{-1}\end{bmatrix}\ddot\theta + \begin{bmatrix}-K_1^{-1}(K_2\Delta^{-1}S^T - T^T) \\ \Delta^{-1}S^T\end{bmatrix} C$$

Thus, we can write

$$\ddot q = P\begin{pmatrix}\ddot q_1\\ \ddot q_2\end{pmatrix} = \underbrace{P}_{[T\ S]}\begin{bmatrix}-K_1^{-1}K_2\Delta^{-1}\\ \Delta^{-1}\end{bmatrix}\ddot\theta + P\begin{pmatrix}K_1^{-1}(T^T - K_2\Delta^{-1}S^T)\\ \Delta^{-1}S^T\end{pmatrix}C$$

$$\Rightarrow \ddot q = (TK_1^{-1}K_2 + S)\Delta^{-1}\ddot\theta + \underbrace{(TK_1^{-1}T + S\Delta^{-1}S^T - TK_1^{-1}K_2\Delta^{-1}S^T)C}_{\ddot q_p}$$

$$\ddot q = \ddot q_p + PZ\ddot q_2 = \ddot q_p + PZ(\Delta^{-1}\ddot\theta + \Delta^{-1}S^TC) = \tilde{\ddot q}_p + P\tilde Z\ddot\theta$$

$$= \underbrace{\ddot q_p + PZ\Delta^{-1}S^TC}_{\tilde{\ddot q}_p} + \underbrace{PZ}\Delta^{-1}\ddot\theta$$

$$[T\ S]\underbrace{\begin{bmatrix}-K_1^{-1}K_2\\ I_{N-s}\end{bmatrix}}_{\tilde Z}\Delta^{-1} = (-TK_1^{-1}K_2 + S)\Delta^{-1}$$

$$\underbrace{\begin{bmatrix}1 & 0 & & 0\\ 1 & 1 & & \\ & & \ddots & 0\\ 1 & & 1 & 1\end{bmatrix}}_{\Sigma_{N_b} = \Delta_{N_b}^{-1}}\underbrace{\begin{bmatrix}1 & & & 0\\ -1 & 1 & & \\ & -1 & 1 & \\ 0 & & \ddots & \\ & & -1 & 1\end{bmatrix}}_{\Delta_{N_b}} = \mathbb{1}_{N_b}$$

Note that ${}^{j}X_{j-1}S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \ell_{j-1}\sin\theta_j \\ \ell_{j-1}\cos\theta_j \\ 0 \end{bmatrix}$, ${}^{j}X_{j-2}S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \ell_{j-1}\sin\theta_j + \ell_{j-2}\sin(\theta_j+\theta_{j+1}) \\ \ell_{j-1}\cos\theta_j + \ell_{j-2}\cos(\theta_j+\theta_{j+1}) \\ 0 \end{bmatrix}$, etc.

As usual, only the rows $3 \to 5$ matter in 2-d.

Recall that $I_j = {}^{j}X_c^{*}\, I_{c,j}\, {}^{c}X_j$, so

Changes to make to paper:

- denote space of Force and motion vectors in body (bodies) as $\mathcal{F}$ and $\mathcal{M}$, respectively. Let $\mathcal{B}$ denote physical region for body, and ∂B the interface
- Address form of B in Appendix A
- reconcile Z and nullspace treatment with one that produces eqns in terms of $\ddot{\theta}$
- Limit as $l_j \to 0$ and $N_b \to 0$