Notes on rigid body dynamics - algorithms

Dynamics of a constrained rigid body

Consider a rigid body constrained to move in subspace $S \subseteq M^6$

Let n_c be the number of constraints, and n_f the number of degrees of freedom

$$n_f = \dim(S) \quad \text{(no. of basis vectors in } S\text{)}$$
$$n_c = 6 - n_f = \dim(S^\perp)$$

Equation of motion

$$\underbrace{f}_{\text{applied force}} + \underbrace{f_c}_{\text{constraint force}} + \underbrace{f_g}_{\text{gravity}} = I a + v \times^* I v$$

Define <u>bias force</u> $p = v \times^* I v - f_g$, so equation is

$$f + f_c = I a + p$$

subject to $v \in S$, $f_c \in S^\perp$

Method 1:

Define $6 \times n_f$ matrix $\underline{S}$ with columns that span S.

$$v = S \dot{q} \quad \text{and} \quad S^T f_c = 0$$

Acceleration constraint $a = S \ddot{q} + \dot{S} \dot{q}$

Now eliminate f_c by projecting equations onto S:

$$\underline{S}^T f + \cancel{\underline{S}^T f_c}^{\,0} = S^T I \underbrace{a}_{S\ddot{q} + \dot{S}\dot{q}} + S^T p \longrightarrow S^T f = S^T I S \ddot{q} + S^T I \dot{S} \dot{q} + S^T p$$

$$\therefore \boxed{\ddot{q} = (S^T I S)^{-1} S^T (f - I \dot{S} \dot{q} - p)}$$

(This is approach taken in Eldredge JCP 2008)

Then $a = \Phi f + b$

$\Phi = S (S^T I S)^{-1} S^T$ apparent inverse inertia

$b = \dot{S} \dot{q} - \Phi (I \dot{S} \dot{q} + p)$ bias acceleration

range$(\Phi) = S$, null$(\Phi) = S^\perp$, rank$(\Phi) = n_f$

Method 2:

Introduce $6 \times n_c$ matrix $\underline{T}$ whose columns span $S^{\perp}$

$$T^T v = 0 \quad , \quad f_c = T\lambda$$

Then acceleration constraint is $T^T a + \dot{T}^T v = 0$

Substitute: $f + T\lambda = I a + p \longrightarrow \begin{bmatrix} I & T \\ T^T & 0 \end{bmatrix} \begin{bmatrix} a \\ -\lambda \end{bmatrix} = \begin{Bmatrix} f - p \\ -\dot{T}^T v \end{Bmatrix}$

Method 3:

Solve in generalized coordinates. Define τ as generalized force.

Both τ and f must deliver same power for any velocity in S

$$\tau^T \dot{q} = f^T v = f^T S \dot{q} \longrightarrow \tau = S^T f$$

removes f_c from f

Then, equation for $\ddot{q}$ from method 1 is

$$\ddot{q} = (S^T I S)^{-1} (\tau - S^T (I S \dot{q} + p))$$

Write this in standard generalized form

$$H \ddot{q} + C = \tau$$

$S^T I S$ $\quad$ $S^T (I S \dot{q} + p)$

Questions:

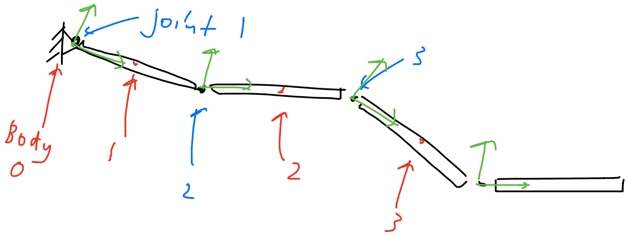
(1) How do we classify the nullspace method in this list?
Can we identify the nullspace matrix Z with matrices presented here?

(2) Relate link assembly and equations in Eldredge JCP 2008 to these (Meth 1)

(3) Develop free swimming linkage method, with specific details

(4) Relate linkage construction in DynPlate to these methods

Assembly of linked rigid bodies



Given v_0, a_0,

$$I_j = {}^jX_c^* \, I_{c,j} \, {}^cX_j$$

$$v_{J,j} = S\dot{\theta}_j \longrightarrow v_j = {}^jX_{j-1} v_{j-1} + S\dot{\theta}_j$$

$$c_j = v_j \times S\dot{\theta}_j \longleftarrow a_j = S\ddot{\theta}_j + \underbrace{{}^jX_{j-1} a_{j-1} + \overset{\dot{S}\dot{\theta}_j}{c_j}}_{a'}$$

$$p_j = v_j \times^* I_j v_j - {}^jX_0^* f_{ex_j}$$

$${}^{j+1}X_0 = {}^{j+1}X_j \, {}^jX_0$$

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for $j = N_b, \dots, 1$

$$U_j = I_j^A S$$

$$D_j = S^T I_j^A S$$

$$\tau_j = -R_j \dot{\theta}_j - K_j \theta_j$$

$$u_j = \tau_j - S^T p_j^A$$

if $j > 1$

$$I^a = I_j^A - U_j D_j^{-1} U_j^T = I_j^A - I_j^A S (S^T I_j^A S)^{-1} S^T I_j^{AT}$$

$$I_j^A$$

$$p^a = p_j^A + I^a c_j + U_j D_j^{-1} u_j = p_j^A + I^a c_j + I_j^A S (S^T I_j^A S)^{-1} (\tau_j - S^T p_j^A)$$

$$I_{j-1}^A = I_{j-1} + {}^{j-1}X_j^* \, I^a \, {}^jX_{j-1}$$

$$p_{j-1}^A = p_{j-1} + {}^{j-1}X_j^* \, p^a$$

end

end

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for $j = 1, \dots, N_b$

$$a' = {}^jX_{j-1} a_{j-1} + c_j$$

$$\ddot{\theta}_j = D_j^{-1} (u_j - U_j^T a')$$

$$a_j = a' + S\ddot{\theta}_j$$

end

$j = N_b$

$$f_j = I_j \underbrace{a_j}_{a' + S\ddot{\theta}_j} + p_j$$

$$= I_j S \ddot{\theta}_j + I_j a' + p_j$$

Project onto S:

$$0 = \underbrace{S^T I_j S}_{D_j} \ddot{\theta}_j + \underbrace{S^T I_j}_{U_j^T} a' + \underbrace{S^T p_j - S^T \underbrace{f_j}_{\tau_j}}_{-u_j} \longrightarrow D_j \ddot{\theta}_j = u_j - U_j^T a'$$

$$\boxed{\ddot{\theta}_j = D_j^{-1}(u_j - U_j^T a')}$$

Solve for f_j:

$$\longrightarrow f_j = I_j S \ddot{\theta}_j + I_j a' + p_j$$

$$= I_j S D_j^{-1}(u_j - U_j^T a') + I_j a' + p_j$$

$$= \underbrace{(I_j - I_j S D_j^{-1} U_j^T)}_{I_j^a} \underbrace{a'}_{{}^jX_{j-1} a_{j-1} + c_j} + I_j S D_j^{-1} u_j + p_j$$

$$\longrightarrow f_j = I_j^a \, {}^jX_{j-1} a_{j-1} + \underbrace{p_j + I_j^a c_j + I_j S D_j^{-1} u_j}_{p_j^a} \longrightarrow \boxed{{}^{j-1}X_j^* f_j = {}^{j-1}X_j^* I_j^a \, {}^jX_{j-1} a_{j-1} + {}^{j-1}X_j^* p_j^a}$$

Each body $j < N_b \longrightarrow$

$$f_{j-1} = I_{j-1} a_{j-1} + p_{j-1} + {}^{j-1}X_j^* f_j$$

$$= \underbrace{(I_{j-1} + {}^{j-1}X_j^* I_j^a \, {}^jX_{j-1})}_{I_{j-1}^A} a_{j-1} + \underbrace{p_{j-1} + {}^{j-1}X_j^* p_j^a}_{p_{j-1}^A}$$

$$\boxed{f_{j-1} = I_{j-1}^A a_{j-1} + p_{j-1}^A}$$

Multiply by $Z_p^T = P_{Up}^T D^{-T}$

$$\ddot{q} = Z_p \ddot{\tilde{q}}_p + Z_A \ddot{\tilde{q}}_A + D^{-1} c$$

$$(Z_p = D^{-1} P_{Up}, \quad Z_A = D^{-1} P_{UA})$$

$$\underbrace{Z_p^T H Z_p}_{\tilde{H}_p}$$

$$\boxed{\tilde{H}_p \ddot{\tilde{q}}_p = Z_p^T(\tau - H D^{-1} c - H Z_A \ddot{\tilde{q}}_A) + \tau_p}$$

Note that

$$D^{-1} = \begin{bmatrix} 1 & & & & \\ {}^1X_0 & {}^1X_{J_1} & & & \\ {}^2X_0 & {}^2X_{J_1} & {}^2X_{J_2} & & \\ \vdots & & & \ddots & \\ {}^{N_b}X_0 & {}^{N_b}X_{J_1} & \cdots & \cdots & {}^{N_b}X_{J_{N_b}} \end{bmatrix}$$

$$\underline{H \ddot{q} - K^T \lambda} = \tau + D^T P_{Up} \tau_p + D^T P_{UA} \tau_A$$

The articulated body inertia I_j^A and the bias force p_j^A allow us to write

$$f_j = I_j^A a_j + p_j^A$$

where f_j is force transmitted across joint j (the parent joint)

Also, $P_{Up_j}^T f_j = \tau_{p_j}$ (passive force)

so $\tau_{p_j} = P_{Up_j}^T I_j^A a_j + P_{Up_j}^T p_j^A$

If $j=1$, then we have

$$\tau_{p_1} = P_{Up_1}^T I_1^A a_1 + P_{Up_1}^T p_1^A \longrightarrow P_{Up_1}^T I_1^A P_{Up_1} \ddot{\tilde{q}}_{1,p} = \tau_{p_1} - P_{Up_1}^T I_1^A (P_{UA} \ddot{\tilde{q}}_{1,A} + c_1) - P_{Up_1}^T p_1^A$$

$$a_1 = \cancel{a_0} + \underbrace{P_U \ddot{\tilde{q}}_1}_{P_{Up} \ddot{\tilde{q}}_{1,p} + P_{UA} \ddot{\tilde{q}}_{1,A}} + c_1$$

Thus

$$\ddot{\tilde{q}}_{1,p} = (P_{Up_1}^T I_1^A P_{Up_1})^{-1} \left[\tau_{p_1} - P_{Up_1}^T \left(p_1^A + I_1^A (c_1 + P_{UA_1} \ddot{\tilde{q}}_{1,A}) \right) \right]$$

In general,

$$\boxed{\ddot{\tilde{q}}_{j,p} = (P_{Up_j}^T I_j^A P_{Up_j})^{-1} \left[\tau_{p_j} - P_{Up_j}^T \left(p_j^A + I_j^A (a_{\lambda(j)} + \underbrace{c_j + P_{UA_j} \ddot{\tilde{q}}_{j,A}}_{c_j'}) \right) \right]}$$

$$\boxed{a_j = a_{\lambda(j)} + c_j' + P_{Up_j} \ddot{\tilde{q}}_{j,p}}$$

Define $U_j = I_j^A P_{up_j}$ $\quad \Phi_j = \left(P_{up_j}^T U_j\right)^{-1}$ $\quad u_j = \tau_{p_j} - P_{up_j}^T p_j^A$ $\quad a_j' = a_{\lambda(j)} + c_j'$

$$\boxed{\begin{aligned} \ddot{\tilde{q}}_{j,p} &= \Phi_j \left(u_j - U_j^T a_j'\right) \\ a_j &= a_j' + P_{up_j} \ddot{\tilde{q}}_{j,p} \end{aligned}}$$

Also, note that

$$I_j^A = I_j + \sum_{i \in \mu(j)} I_i^a \qquad I_j^a = I_j^A - U_j \Phi_j U_j^T$$

$$p_j^A = p_j + \sum_{i \in \mu(j)} p_i^a \qquad p_j^a = p_j^A + I_j^a c_j' + U_j \Phi_j u_j$$

Check this...

For $i \in \mu(j)$, $f_i = I_i^A a_i + p_i^A$

But $a_i = a_{\lambda(i)} + c_i + P_u \ddot{\tilde{q}}_i$, so

$$f_i = I_i^A a_{\lambda(i)} + I_i^A\left(c_i' + P_{u_{p_i}} \ddot{\tilde{q}}_{i,p}\right) + p_i^A$$

$$\ddot{\tilde{q}}_{i,p} = \Phi_i\left(u_i - U_i^T a_{\lambda(i)} - U_i^T c_i'\right)$$

$$= \Big(I_i^A - \underbrace{I_i^A P_{up_i}}_{U_i} \Phi_i U_i^T\Big) a_{\lambda(i)} + p_i^A + \left(I_i^A - U_i \Phi_i U_i^T\right) c_i' + U_i \Phi_i u_i$$

$$\longrightarrow f_i = I_i^a a_{\lambda(i)} + p_i^a$$

$$f_i = \frac{d}{dt}\left(I_i^A v_i + m_i^A\right) \qquad v_i = v_{\lambda(i)} + \underbrace{P_{ua_i} \dot{\tilde{q}}_{iA} + P_{up_i} \dot{\tilde{q}}_{ip}}_{v_{J_i}}$$

$$f_i = \frac{d}{dt}\left(I_i^A v_{\lambda(i)} + m_i^A + I_i^A P_u \dot{\tilde{q}}_i\right)$$

(if only one child)

$$f_{\lambda(i)} = \frac{d}{dt}\left(I_{\lambda(i)} v_{\lambda(i)} + m_{\lambda(i)}\right) + f_i$$

$$= \frac{d}{dt}\Big[\underbrace{\left(I_{\lambda(i)} + I_i^A\right)}_{I_{\lambda(i)}^A} v_{\lambda(i)} + \underbrace{m_{\lambda(i)} + m_i^A + I_i^A P_{u_i} \dot{\tilde{q}}_i}_{m_{\lambda(i)}^A}\Big]$$

When $\lambda(i) = 1$, then $P_{up,1}^T f_1 = 0 = \frac{d}{dt}\Big[\underbrace{P_{up,1}^T I_1^A P_{up,1} \dot{\tilde{q}}_{1,p} + P_{up,1}^T m_1^A}_{M_{tot,p}}\Big]$

To set $M_{tot,p}(0) = 0 \longrightarrow \boxed{\dot{\tilde{q}}_{1,p}(0) = -\left(P_{up,1}^T I_1^A P_{up,1}\right)^{-1} P_{up,1}^T m_1^A}$

So total momentum $\sum_j {}^iX_{b_j}^* I_j v_j = 0 \longrightarrow \sum_j {}^iX_{c_j}^* I_{c_j} v_{c_j} = 0$

$${}^{b_j}X_{c_j}^* I_{c_j} \underbrace{{}^{c_j}X_{b_j} v_j}_{v_{c_j}}$$

$$\underbrace{\begin{bmatrix} I_{c_j}\omega_j \\ m_j \dot r_{c_j}\end{bmatrix}}_{} \quad \begin{bmatrix}\omega_j \\ \dot r_{c_j}\end{bmatrix} \quad = \frac{d}{dt}\begin{bmatrix} I_{c_j}\, a \\ m_j r_{c_j}\end{bmatrix}$$

$$\sum_j {}^iX_{c_j}^* \underbrace{I_{c_j}\, {}^{c_j}X_i}_{{}^iI_j}\, {}^iv_j = 0$$

$$\frac{d}{dt}{}^iX_{c_j}^* = \left(\frac{d}{dt}{}^{c_j}X_i\right)^T = \left[{}^{c_j}(v_i - v_{c_j})\times {}^{c_j}X_i\right]^T = \left[-v_{c_j}\times {}^{c_j}X_i\right]^T = +{}^iX_{c_j}^* v_{c_j}\times^*$$

$$\rightarrow \frac{d}{dt}\sum_j {}^iX_{c_j}^* I_{c_j}\begin{bmatrix}\omega_j\\ \dot r_{c_j}\end{bmatrix} - \sum_j {}^iX_{c_j}^* v_{c_j}\times^* I_{c_j} v_{c_j}$$

center of mass:

$$M\,\underline{X}_c = \sum_j \int_{V_j} \underline{x}\,\rho\, \underbrace{dV}_{w\,dA} \qquad \left(M_j x_{c_j}\right)$$

$$\frac{dx_c}{dt} = \underline{V}_c = \frac{1}{M}\sum_j\int_{V_j}\rho\,\underline{u}\, dV = \frac{1}{M}\sum_j M_j \underline{V}_{c_j}$$

$$\underline{u} = \underline{V}_{c_j} + \underline{\Omega}_j\times(x - x_{c_j})$$

But $\underline{P} = \sum_j M_j \underline{V}_{c_j}$

and $\frac{d\underline{P}}{dt} = \underline{F}$ so if $\underline{F} = 0$

then $\underline{P} = \underline{P}(0) = \underline{P}_0$

Let $\underline{P}_0 = 0$. Then

$$\frac{d\underline{X}_c}{dt} = \frac{1}{M}\underline{P} = 0$$

Now return to the spatial vector form

Denote h as momentum ...

$$h = \sum_j I_j v_j \longrightarrow f = \frac{dh}{dt} = \frac{d}{dt}\sum_j I_j v_j = \sum_j\left(v_j\times^* I_j v_j + I_j a_j\right)$$

Note that

$$v_j\times^* I_j v_j = \begin{bmatrix}\Omega_j^\times & V_j^\times\\ 0 & \Omega_j^\times\end{bmatrix}\begin{bmatrix}\bar I_{c_j} & 0\\ 0 & M_j 1\end{bmatrix}\begin{bmatrix}\Omega_j\\ V_j\end{bmatrix} = \begin{bmatrix}\Omega_j^\times & V_j^\times\\ 0 & \Omega_j^\times\end{bmatrix}\begin{bmatrix}\bar I_{c_j}\Omega_j\\ M_j V_j\end{bmatrix} = \begin{bmatrix}\Omega_j\times\bar I_{c_j}\Omega_j + \cancel{V_j\times M_j V_j}\\ \Omega_j\times M_j V_j\end{bmatrix}$$

$$I_j a_j = \begin{bmatrix}\bar I_{c_j} & 0\\ 0 & M_j 1\end{bmatrix}\begin{bmatrix}\dot\Omega_j\\ \ddot c_j - \Omega_j\times V_j\end{bmatrix} = \begin{bmatrix}\bar I_{c_j}\dot\Omega_j\\ M_j\ddot c_j - M_j\Omega_j\times V_j\end{bmatrix}$$

So

$$f = \sum_j\begin{pmatrix}\bar I_c\dot\Omega_j + \Omega_j\times\bar I_{c_j}\Omega_j\\ M_j\ddot c_j\end{pmatrix} = \frac{d}{dt}\sum_j\begin{bmatrix}\bar I_{c_j}\Omega_j\\ M_j V_j\end{bmatrix} \qquad (V_j = \dot c_j)$$

$$h_j = I_{c_j} v_{c_j} = {}^{i_j}X_{c_j}^*\,\tilde I_{c_j}\,\underbrace{{}^{c_j}X_{i_j}\,{}^{i_j}v_{c_j}}_{{}^{c_j}v_{c_j}} =$$

$$I_{c_j} = {}^{i_j}X_{c_j}^*\,\tilde I_{c_j}\,{}^{c_j}X_{i_j}, \qquad v_{c_j} = {}^{i_j}v_{c_j}$$

pure rotation from inertial axes to body axes at center of mass

invariant

Equations of motion

The equations of motion for a body are

$$\underline{f} = \frac{d\underline{h}}{dt} = \frac{d}{dt}(\underline{\underline{I}}\,\underline{v})$$

Note that, we denote o as the origin of some inertial system and c as the center of mass of the body, then

Also, let i denote a system based at c but aligned with the inertial axes.

Then ${}^{i}X_c$ is a pure rotation and ${}^{o}X_i$ is pure translation. (by vector $-\underline{c}$, where $\underline{c}$ is the vector from o to c)

Note that we can write $\underline{\underline{I}}$ in coordinates as ${}^{i}I = \begin{bmatrix} {}^{i}\bar{I}_c & 0 \\ 0 & M1 \end{bmatrix}$

or as ${}^{c}I = \begin{bmatrix} \bar{I}_c & 0 \\ 0 & M1 \end{bmatrix}$, where the coefficients are invariant.

Then

$${}^{i}I = {}^{i}X_c^{*}\,{}^{c}I\,{}^{c}X_i$$

$${}^{i}X_c = {}^{i}X_b\,{}^{b}X_c = \underbrace{\begin{bmatrix} {}^{i}E_c & 0 \\ 0 & {}^{i}E_c \end{bmatrix}}_{{}^{i}X_{ic}} \overbrace{\begin{bmatrix} 1 & 0 \\ r^{\times} & 1 \end{bmatrix}}^{{}^{ic}X_c}$$

$${}^{c}X_i = {}^{c}X_b\,{}^{b}X_i = \overbrace{\begin{bmatrix} {}^{c}E_i & 0 \\ 0 & {}^{c}E_i \end{bmatrix}}^{{}^{c}X_{c_i}} \overbrace{\begin{bmatrix} 1 & 0 \\ r^{\times} & 1 \end{bmatrix}}^{{}^{ci}X_i}$$

$${}^{ci}v = {}^{ci}X_b\,{}^{b}v = {}^{ci}X_c\,{}^{c}X_b\,{}^{b}v$$

$$I_1^A = I_1$$
$$I_2^A = I_2$$

$$v_1 = v_{J_1}$$
$$v_2 = {}^2X_1 v_1 + v_{J_2}$$

$$c_1 = v_1 \times v_{J_1} + \cancel{P_{v_{a_1}} \ddot{\tilde{q}}_{1c}}$$
$$c_2 = v_2 \times v_{J_2} + \underbrace{P_{v_{a_2}} \ddot{\tilde{q}}_{2c}}_{a_{J_2}}$$

$$p_1^A = v_1 \times^* I_1 v_1$$
$$p_2^A = v_2 \times^* I_2 v_2$$

$$I^a = I_2^A$$
$$p^a = p_2^A + I^a c_2$$
$$I_1^A = I_1^A + {}^1X_2^* I^a \, {}^2X_1 = I_1 + {}^1X_2^* I_2 \, {}^2X_1$$
$$p_1^A = p_1^A + {}^1X_2^* p^a = v_1 \times^* I_1 v_1 + {}^1X_2^* (v_2 \times^* I_2 v_2 + I_2 c_2)$$

$$U_1 = I_1^A P_{v_{p_1}}$$
$$\tilde{H}_{p_1}^{-1} = (P_{v_{p_1}}^T I_1^A P_{v_{p_1}})^{-1}$$
$$u_1 = -P_{v_{p_1}}^T p_1^A$$

free motion:
$$P_{v_{p_1}} = 1_6$$
$$\ddot{\tilde{q}}_{p_1} = \ddot{\tilde{q}}_1$$

$$a_1' = c_1$$
$$\ddot{\tilde{q}}_{p_1} = \tilde{H}_{p_1}^{-1}(u_1 - U_1^T a_1') \longrightarrow (I_1 + {}^1X_2^* I_2 \, {}^2X_1)\ddot{\tilde{q}}_1 = -[v_1 \times^* I_1 v_1 + {}^1X_2^*(v_2 \times^* I_2 v_2 + I_2 c_2)] - (I_1 + {}^1X_2^* I_2 \, {}^2X_1)^T c_1$$
$$a_{J_1} = P_{v_{p_1}} \ddot{\tilde{q}}_{p_1}$$
$$a_1 = a_1' + a_{J_1}$$
$$a_2' = {}^2X_1 a_1 + c_2$$
$$a_2 = a_2'$$

$$\Longrightarrow (I_1 + {}^1X_2^* I_2 \, {}^2X_1)(\underbrace{\ddot{\tilde{q}}_1}_{a_{J_1}} + v_1 \times v_{J_1}) + v_1 \times^* I_1 v_1 + {}^1X_2^*(v_2 \times^* I_2 v_2 + I_2 v_2 \times v_{J_2} + I_2 a_{J_2}) = 0$$

$$\frac{dh_{1i}}{dt} \rightarrow \boxed{I_1 \underbrace{(\ddot{\tilde{q}}_1 + v_1 \times v_{J_1})}_{a_1} + v_1 \times^* I_1 v_1}$$
$$+ {}^1X_2^* \underbrace{\boxed{I_2 \, {}^2X_1 (\ddot{\tilde{q}}_1 + v_1 \times v_{J_1}) + v_2 \times^* I_2 v_2 + I_2(a_{J_2} + v_2 \times v_{J_2})}}_{a_2} = 0$$
$$\frac{dh_{2i}}{dt}$$

$$\frac{d}{dt}(h_{1i} + {}^{1i}X_{2i}^* h_{2i}) = 0$$

$${}^{1i}X_{1ci}^* \underbrace{I_{1ci}}_{} V_{1ci} \qquad {}^{2i}X_{2ci}^* \underbrace{I_{2ci}}_{} V_{2ci}$$

$$\begin{bmatrix} \bar{I}_{c_1} & 0 \\ 0 & m_1 1 \end{bmatrix} \begin{Bmatrix} \Omega_{1i} \\ V_{1ci} \end{Bmatrix} \qquad \begin{bmatrix} \bar{I}_{c_2} & 0 \\ 0 & m_2 1 \end{bmatrix} \begin{Bmatrix} \Omega_{2i} \\ V_{2ci} \end{Bmatrix}$$

$$\longrightarrow {}^{1i}X_{1ci}^* \left\{ \begin{pmatrix} \bar{I}_{c_1} \Omega_{1i} \\ m_1 V_{1ci} \end{pmatrix} + {}^{1ci}X_{2ci}^* \begin{pmatrix} \bar{I}_{c_2} \Omega_{2i} \\ m_2 V_{2ci} \end{pmatrix} \right\} = \text{const}$$

or equivalently, after multiplying by ${}^iX_{1i}^*$,

$${}^iX_{1ci}^* \begin{pmatrix} \bar{I}_{c_1} \Omega_{1i} \\ m_1 V_{1ci} \end{pmatrix} + {}^iX_{2ci}^* \begin{pmatrix} \bar{I}_{c_2} \Omega_{2i} \\ m_2 V_{2ci} \end{pmatrix} = \text{const}$$

Free swimming

Now we seek equation for a_0.

For any body j,

$$f_j = I_j a_j + p_j + {}^jX^*_{j+1} f_{j+1}$$

We seek to eliminate the forces f_j.

Start at body $j = N_b - 1$,

$$f_j = I_j a_j + p_j + {}^jX^*_{j+1} \big(I_{j+1} \underbrace{a_{j+1}}_{{}^{j+1}X_j a_j + S\ddot{\theta}_{j+1} + c_{j+1}} + p_{j+1}\big)$$

$$= \underbrace{\left(I_j + {}^jX^*_{j+1} I_{j+1} {}^{j+1}X_j\right)}_{I_j^A} a_j + \underbrace{p_j + {}^jX^*_{j+1} p_{j+1} + {}^jX^*_{j+1} c_{j+1} + {}^jX^*_{j+1} S\ddot{\theta}_{j+1}}_{p_j^A}$$

Thus,

$$I_j^A = I_j + {}^jX^*_{j+1} I^A_{j+1} {}^{j+1}X_j \quad , \quad p_j^A = p_j + {}^jX^*_{j+1} p^A_{j+1} + {}^jX^*_{j+1}\left(c_{j+1} + S\ddot{\theta}_{j+1}\right)$$

and

$$\boxed{f_j = I_j^A a_j + p_j^A}$$

But $f_0 = 0$, so $\boxed{a_0 = -(I_0^A)^{-1} p_0^A}$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

If constrained to slide along a rail, then $v_0 = S_x \dot{X}$ and $f_0 \neq 0$, but $S_x^T f_0 = 0$. Then

$$a_0 = S_x \ddot{X} + \cancel{\dot{S}_x}\,\dot{X}$$

$$f_0 = I_0^A a_0 + p_0^A$$

$$0 = \underbrace{S_x^T I_0^A S_x}_{D_0} \ddot{X} + \underbrace{S_x^T p_0^A}_{u_0} \longrightarrow \boxed{\ddot{X} = D_0^{-1} u_0}$$