Notes on rigid body dynamics

Plücker coordinates

Motion

$$\underline{m}_o = \begin{Bmatrix} m_x \\ m_y \\ m_z \\ m_{ox} \\ m_{oy} \\ m_{oz} \end{Bmatrix} = \begin{Bmatrix} \underline{m} \\ \underline{m}_o \end{Bmatrix}$$

rotation

translation

These are Plücker coordinates for $m \in M^6$

Force

$$\underline{f}_o = \begin{Bmatrix} f_{ox} \\ f_{oy} \\ f_{oz} \\ f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{Bmatrix} \underline{f}_o \\ \underline{f} \end{Bmatrix} \in F^6$$

Moment about O

force

These are Plücker coordinates for $f \in F^6$

Inner product: $m \cdot f = \underline{m}_o^T \underline{f}_o$

Coordinate transformation

Denote ${}^B X_A$ as transform from A to B coordinates (for motion vector)

and ${}^B X_A^*$ for force vector

For rotation, we use a 3x3 rotation matrix E that transforms 3d vectors from A to B coordinates. Then

$${}^B X_A = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$$

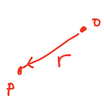
e.g. $\underline{v} = v_x^A \underline{e}_x^A + v_y^A \underline{e}_y^A + v_z^A \underline{e}_z^A$

$= v_x^B \underline{e}_x^B + v_y^B \underline{e}_y^B + v_z^B \underline{e}_z^B$

$$v_x^B = (\underline{e}_x^B \cdot \underline{e}_x^A) v_x^A + (\underline{e}_x^B \cdot \underline{e}_y^A) v_y^A + (\underline{e}_x^B \cdot \underline{e}_z^A) v_z^A$$

$$E = \begin{bmatrix} \underline{e}_x^B \cdot \underline{e}_x^A & \underline{e}_x^B \cdot \underline{e}_y^A & \underline{e}_x^B \cdot \underline{e}_z^A \\ \underline{e}_y^B \cdot \underline{e}_x^A & \underline{e}_y^B \cdot \underline{e}_y^A & \underline{e}_y^B \cdot \underline{e}_z^A \\ \underline{e}_z^B \cdot \underline{e}_x^A & \underline{e}_z^B \cdot \underline{e}_y^A & \underline{e}_z^B \cdot \underline{e}_z^A \end{bmatrix}$$

For translation by r (from O to P)



$$\left.\begin{aligned} m &= m \\ m_p &= m_o + m \times r \end{aligned}\right\} \quad {}^B X_A = \begin{bmatrix} 1 & 0 \\ -r^\times & 1 \end{bmatrix}$$

$r^\times a \equiv r \times a$, $(r^\times)^T = -r^\times$

or for force vectors, since

$$\begin{aligned} f_p &= f_o - r \times f \\ f &= f \end{aligned} \longrightarrow {}^B X_A^* = \begin{bmatrix} 1 & -r^\times \\ 0 & 1 \end{bmatrix}$$

since moment arm is $\overrightarrow{PO}$

In general, ${}^B X_A = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -r^\times & 1 \end{bmatrix}$ translate from center of A to center of B, then rotate

$${}^B X_A^* = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} 1 & -r^\times \\ 0 & 1 \end{bmatrix}$$

The inverses are

$$ {}^{A}X_{B} = \begin{bmatrix} 1 & 0 \\ r\times & 1 \end{bmatrix} \begin{bmatrix} E^{T} & 0 \\ 0 & E^{T} \end{bmatrix} = \left({}^{B}X_{A}^{*}\right)^{T} \qquad {}^{A}X_{B}^{*} = \begin{bmatrix} 1 & r\times \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E^{T} & 0 \\ 0 & E^{T} \end{bmatrix} = \left({}^{B}X_{A}\right)^{T} $$

Spatial cross products

$$ \left.\begin{aligned} \dot{m} &= \hat{v} \times m \\ \dot{f} &= \hat{v} \times^{*} f \end{aligned}\right\} $$

vectors moving with velocity v (i.e. fixed to a rigid body with that velocity)

Then, $\hat{v}_o^{\times} = \begin{bmatrix} \omega \\ v_o \end{bmatrix}^{\times} = \begin{bmatrix} \omega^{\times} & 0 \\ v_o^{\times} & \omega^{\times} \end{bmatrix}$ $\qquad \hat{v}_o^{\times *} = \begin{bmatrix} \omega \\ v_o \end{bmatrix}^{\times} = \begin{bmatrix} \omega^{\times} & v_o^{\times} \\ 0 & \omega^{\times} \end{bmatrix} = -\left(\hat{v}_o^{\times}\right)^{T}$

So, for example

$$ \hat{v}_0 \times \hat{m} = \begin{bmatrix} \omega^{\times} & 0 \\ v_o^{\times} & \omega^{\times} \end{bmatrix} \begin{bmatrix} m \\ m_o \end{bmatrix} = \begin{bmatrix} \omega \times m \\ v_o \times m + \omega \times m_o \end{bmatrix} $$

$$ \hat{v}_o \times^{*} \hat{f} = \begin{bmatrix} \omega^{\times} & v_o^{\times} \\ 0 & \omega^{\times} \end{bmatrix} \begin{Bmatrix} f_o \\ f \end{Bmatrix} = \begin{Bmatrix} \omega \times f_o + v_o \times f \\ \omega \times f \end{Bmatrix} $$

Note that

$$ {}^{B}X_{A}^{*}\left({}^{A}v \times^{*} \underbrace{\left({}^{B}X_{A}^{*}\right)^{-1}}_{{}^{A}X_{B}^{*}}\right) = \left({}^{B}X_{A}\,{}^{A}v\right) \times^{*} $$

Thus, ${}^{B}X_{A}^{*}\left({}^{A}v \times^{*} {}^{A}f\right) = {}^{B}v \times^{*} {}^{B}f$

Differentiation in Moving Coordinates

If A is coordinate system associated with a frame moving with velocity v_A,

v_A in A coordinates

$$ {}^{A}\left(\frac{dm}{dt}\right) = \underbrace{\frac{d\,{}^{A}m}{dt}}_{\text{Componentwise derivatives of } {}^{A}m} + \underbrace{{}^{A}v_{A} \times {}^{A}m}_{\text{rate of change due to frame motion}} $$

Similarly,

$$ {}^{A}\left(\frac{df}{dt}\right) = \frac{d\,{}^{A}f}{dt} + {}^{A}v_{A} \times^{*} {}^{A}f $$

This leads to the expression for rate of change of ${}^{B}X_{A}$:

$$ \frac{d}{dt}\,{}^{B}X_{A} = {}^{B}(v_A - v_B) \times {}^{B}X_{A} $$

Acceleration

Define the spatial acceleration

$$ \hat{a}_o = \frac{d}{dt}\begin{Bmatrix} \omega \\ v_o \end{Bmatrix} = \begin{Bmatrix} \dot{\omega} \\ \dot{v}_o \end{Bmatrix} $$

Keep in mind that v_o is not velocity of one particular body-fixed point, but a measure of flow of points through O (i.e. it is Eulerian, not Lagrangian)

Thus $\dot{v}_o$ is not acceleration of body-fixed point but a measure of rate of change of flow. (It is not the material derivative, but the Eulerian derivative)

If $r(t)$ is position of a fixed point O' on body, and $O = O'$ at t, then

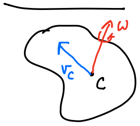
$$(\ddot{r}, \dot{\omega}) \to \text{classical rigid body acceleration} \qquad \boxed{a_o' = \begin{Bmatrix} \dot{\omega} \\ \ddot{r} \end{Bmatrix}}$$

But
$$\dot{v}_o = \frac{v_o(t+\delta t) - v_o(t)}{\delta t} = \frac{[\dot{r}(t+\delta t) + \omega \times (-\delta t\, \dot{r}(t))] - \dot{r}(t)}{\delta t} \to \ddot{r} - \omega \times \dot{r}$$

$$\boxed{\hat{a}_o = \begin{Bmatrix} \dot{\omega} \\ \ddot{r} - \omega \times \dot{r} \end{Bmatrix}} \leftarrow \text{a true vector!}$$

(e.g. $\hat{a}_{rel} = \hat{a}_2 - \hat{a}_1$, for bodies 1,2)

Momentum



Center of mass coincides with C at one instant

mass m rotational inertia $\bar{I}_c$ (about center of mass)

Momentum $\begin{cases} \text{linear} & h = m v_c \\ \text{angular} & h_c = \bar{I}_c \omega \end{cases}$ $h_o = h_c + \vec{OC} \times h$

Spatial momentum $\hat{h}_c = \begin{Bmatrix} h_c \\ h \end{Bmatrix} = \begin{bmatrix} \bar{I}_c \omega \\ m v_c \end{bmatrix}$ $\hat{h}_o = \begin{pmatrix} h_o \\ h \end{pmatrix} = \begin{bmatrix} 1 & \vec{OC}^\times \\ 0 & 1 \end{bmatrix} \hat{h}_c$

(In general, ${}^B\hat{h} = {}^B X_A^{*} {}^A\hat{h}$)

Kinetic energy $\frac{1}{2} \hat{h} \cdot \hat{v}$

Inertia

$I : M^6 \mapsto F^6$ (from velocity to momentum)

$\hookrightarrow h = I v$

In particular, $h_c = \underbrace{\begin{bmatrix} \bar{I}_c & 0 \\ 0 & m1 \end{bmatrix}}_{I_c} v_c$

Then,
$$h_o = \begin{bmatrix} 1 & c^\times \\ 0 & 1 \end{bmatrix} I_c v_c = \begin{bmatrix} 1 & c^\times \\ 0 & 1 \end{bmatrix} I_c \begin{bmatrix} 1 & 0 \\ (c^\times)^T & 1 \end{bmatrix} v_o \longrightarrow I_o = \begin{bmatrix} \bar{I}_c + m c^\times c^{\times T} & m c^\times \\ m c^{\times T} & m1 \end{bmatrix}$$

$c = \vec{OC}$

We can write $\underline{I}$ as a symmetric dyadic tensor

$$\underline{I} = \sum_{i=1}^{6} g_i g_i \cdot \qquad g_i \in F^6$$

Note that $\dot{g}_i = v \times^* g_i$ (change only due to body motion)

Thus,

$$\boxed{\frac{d\underline{I}}{dt} = v \times^* \underline{I} - \underline{I}\, v \times}$$

Can also show that $\boxed{{}^B\underline{I} = {}^B X_A^* \, {}^A\underline{I} \, {}^A X_B}$

Kinetic energy $T = \frac{1}{2} v \cdot \underline{I} v$

Equation of motion

$$f = \frac{d}{dt}(\underline{I} v) = \underline{I} a + (v \times^* \underline{I} - \underline{I} v \times) v$$

$$= \underline{I} a + \underbrace{v \times^* \underline{I} v}_{p\text{, bias force}}$$

p could also include other known forces (gravity, etc.)

Inverse inertia

$\underline{\Phi} = \underline{I}^{-1} \longrightarrow$ range of $\underline{\Phi}$ is subspace of M^6 in which the body is free to move

(rank of $\underline{\Phi}$ = no. degrees of motion freedom)

A note on reference frames

Technically, each tensor and vector, when expressed in any body-fixed coordinates, e.g. body system or center-of-mass system, can be regarded as transformed to a system that is fixed, but instantaneously coincides with the body-fixed system.

e.g. frames b and b_i $\longrightarrow$ $I_{b_i} = {}^{b_i}X_b^* \, I_b \, {}^bX_{b_i}$ is inertia in body system

moving with body

fixed but instantaneously coincident with b

$$\frac{d\,{}^bX_{b_i}}{dt} = {}^b(\cancel{v_{b_i}} - v_b) \times {}^bX_{b_i} = -v_b \times {}^bX_{b_i}$$

$$\frac{d\,I_{b_i}}{dt} = \frac{d\,{}^{b_i}X_b^*}{dt} I_b \, {}^bX_{b_i} + {}^{b_i}X_b^* \, I_b \frac{d\,{}^bX_{b_i}}{dt}$$

$$= {}^{b_i}X_b^* \, v_b \times^* I_b \, {}^bX_{b_i} - {}^{b_i}X_b^* \, I_b \, v_b \times {}^bX_{b_i}$$

Thus,

$$\frac{dh_{b_i}}{dt} = \frac{d}{dt}(I_{b_i} v_{b_i}) = {}^{b_i}X_b^* \, v_b \times^* I_b v_b - {}^{b_i}X_b^* \, I_b \, \cancel{v_b \times v_b} + {}^{b_i}X_b^* \, I_b \, {}^bX_{b_i} \, a_{b_i}$$

$$= {}^{b_i}X_b^*(v_b \times^* h_b + I_b a_b) = v_{b_i} \times^* h_{b_i} + I_{b_i} a_{b_i}$$

$$\boxed{\frac{dh_{b_i}}{dt} = \underbrace{v_{b_i} \times^* h_{b_i}}_{p_{b_i}} + I_{b_i} a_{b_i}}$$

So ${}^bX_{b_i}$ is instantaneously the identity, but has non-zero rate of change.

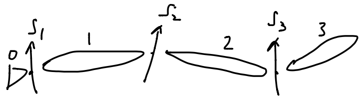
Also, in a joint constraint, $v_j = {}^jX_{j-1} v_{j-1} + v_{J_{i,j}}$, we differentiate to find acceleration with the implicit understanding that these are expressed in j_i coordinates.

Then
$$\frac{d}{dt}({}^{j_i}X_j \, v_{J_{i,j}}) = \underbrace{\frac{d\,{}^{j_i}X_j}{dt}}_{{}^{j_i}v_j \times {}^{j_i}X_j} v_{J_{i,j}} + {}^{j_i}X_j \, a_{J_{i,j}} = {}^{j_i}X_j \, a_{J_{i,j}} + \underbrace{{}^{j_i}v_j \times {}^{j_i}v_{J_{i,j}}}_{c_j}$$

Also, by definition, for two systems j_i and k_i

$$\boxed{\frac{d}{dt}\,{}^{k_i}X_{j_i} = 0}$$

Linked rigid body motion



Velocity across joint is "to" velocity relative to "from" velocity

$$v_{J_i} = v_i - v_{i-1}$$

Assume each joint allows only single degree of freedom

$$v_{J_i} = s_i \dot{q}_i$$

axis vector

Thus $v_i = v_{i-1} + s_i \dot{q}_i$

$$= v_0 + \sum_{j=1}^{i} s_j \dot{q}_j = v_0 + J_i \dot{q}$$

$$J_i = [s_1 \; s_2 \; \cdots \; s_i \; 0 \; 0 \; \cdots 0]$$

$$\dot{q} = [\dot{q}_1 \; \cdots \; \dot{q}_N]^T$$

Acceleration

$$a_{J_i} = a_i - a_{i-1}$$

and $a_{J_i} = s_i \ddot{q}_i + \dot{s}_i \dot{q}_i$

$$\dot{s}_i = v_i \times s_i$$

$$\longrightarrow \quad a_i = a_{i-1} + s_i \ddot{q}_i + v_i \times s_i \dot{q}_i$$

$$= a_0 + \sum_{j=1}^{i} (s_j \ddot{q}_j + v_j \times s_j \dot{q}_j)$$

$$= a_0 + \sum_{j=1}^{i} s_j \ddot{q}_j + \sum_{j=1}^{i} \sum_{k=1}^{j-1} s_k \times s_j \dot{q}_j \dot{q}_k$$

$$+ v_0 \times \sum_{j=1}^{i} s_j \dot{q}_j$$

So we can write $a_i = a_0 + J_i \ddot{q} + \dot{J}_i \dot{q}$

$$+ v_0 \times J_i \dot{q}$$

Equations of Motion

General system

generalized acceleration

generalized velocity

$$\underbrace{H(q)}_{\text{generalized inertia}}\ddot{q} + \underbrace{C(q,\dot{q})}_{\text{bias force}} = \underbrace{\tau}_{\text{generalized forces}}$$

bias force (value of τ that produces zero accel.)
[Coriolis, centrifugal, gravity, etc.]

Kinetic energy $T = \frac{1}{2}\dot{q}^T H \dot{q}$

Note that $\dot{q}, \ddot{q} \in M^n$ $\qquad H : M^n \mapsto F^n$

$\tau, C \in F^n$ $\qquad$ (q is not a vector, but a point in configuration space)

Generalized motion, force vectors share many properties with spatial vectors (except cross products)

Even more general (without assuming q is integral of velocity)

$$H(q)\,\dot{\alpha} + C(q,\alpha) = \tau$$

alternative velocity vector

$$\dot{q} = Q(q)\,\alpha$$

Motion constraints

implicit constraints: $\phi(q) = 0 \xrightarrow{\frac{d}{dt}} \underbrace{K}_{\frac{\partial\phi}{\partial q}}\dot{q} = 0 \xrightarrow{\frac{d}{dt}} K\ddot{q} = \underbrace{k}_{-\dot{K}\dot{q}}$

The constrained system has the form

$$H\ddot{q} + C = \tau + \tau_c$$

From Jourdain's principle of virtual power, τ_c delivers zero power along every free velocity direction:

$$\tau_c \cdot \dot{q} = 0 \quad \text{for any } \dot{q} \text{ allowed by constraint.}$$

Let $\tau_c = K^T\lambda$.

$\rightarrow \tau_c \cdot \dot{q} = K^T\lambda \cdot \dot{q} = \lambda^T \cancel{K\dot{q}}^{\,0} = 0$ $\qquad$ λ are Lagrange multipliers

Apply constraints:

$$\begin{array}{c} n+n_c \end{array} \left\{ \underbrace{\begin{bmatrix} H & K^T \\ K & 0 \end{bmatrix}}_{n+n_c} \right\} \begin{Bmatrix} \ddot{q} \\ -\lambda \end{Bmatrix} = \begin{Bmatrix} \tau - C \\ k \end{Bmatrix}$$

$n = \dim(\ddot{q})$
$n_c = \dim(\lambda)$ (no. of constraints)

$n_{ic} = \text{rank}(K)$ (no. of indep constraints)
$(n_{ic} \le n_c)$

If $n_{ic} = n_c$, system is nonsingular

If $n_{ic} < n_c$, system is overconstrained (?)

Vector subspaces

If system is constrained to move only in subspace $S \subset M^n$, then

$$\dot{q} \in S = \text{null}(K)$$

If $V = S_1 \oplus S_2$ where V is a vector space and S_1, S_2 are subspaces
$(S_1 \cap S_2 = \{0\},\ \dim(S_1) + \dim(S_2) = \dim(V))$

then any $v \in V$ can be written

$$\underline{v} = \underline{\underline{S}}_1 \underline{\alpha}_1 + \underline{\underline{S}}_2 \underline{\alpha}_2 = [\underline{\underline{S}}_1\ \underline{\underline{S}}_2] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$\underline{\underline{S}}_1$ columns are basis of S_1
$\underline{\underline{S}}_2$ " " " " S_2

Note that $[S_1\ S_2]$ is basis for V, so

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = [S_1\ S_2]^{-1} v$$

Suppose U and V are vector spaces, where $U = V^*$ (the dual of V)
Then, for $u \in U$ and $v \in V$, if $u \cdot v = 0$, they are orthogonal

If $Y_1 \subseteq U$, $Y_2 \subseteq V$, and every element of Y_1 is orthogonal to every element in Y_2, then $Y_1 \perp Y_2$.

The set of all $v \in V$ for which $v \cdot u = 0$ for all $u \in Y \subseteq U$, is the orthogonal complement of Y.

If $\dot{q}$ is restricted to $S \subseteq M^n$, then $\tau_c \in S^\perp \subseteq F^n$

Example:
Given motion subspace $S \subseteq M^6$, and spatial inertia $I : M^6 \mapsto F^6$
then can define two subspaces $T_a = IS$ and $T_c = S^\perp$, with property

$$F^6 = T_a \oplus T_c$$

Thus, can decompose any f into $f = f_a + f_c$.

To find f_a and f_c from f

$$S^T f = S^T f_a + \cancel{S^T f_c}^{\,0} = S^T \underbrace{f_a}_{IS\alpha} = S^T I S \alpha \longrightarrow \alpha = (S^T I S)^{-1} S^T f$$

$$\therefore \boxed{f_a = I \underbrace{S (S^T I S)^{-1} S^T}_{\text{apparent inverse inertia}} f}$$

and $f_c = f - f_a$.

$f_a \rightarrow$ part that creates acceleration of body
$f_c \rightarrow$ part opposed by constraint force $(-f_c)$
