Advanced Microeconomics Theory

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Contents

Chapter 4

Comparative Statics Analysis

A central question in economics is how an endogenous variable changes with an exogenous variable. For instance, in the indirect approach of single-product profit maximization:

$$\max_{y \geq 0} py - c(\mathbf{w}, y)$$

Holding the input prices $\mathbf{w}$ as given, we are interested in how the firm's optimal supply (correspondence) y^* changes with the output price p.
More generally, let $F : X \times \Theta \to \mathbb{R}$, where $X, \Theta \subset \mathbb{R}$, we are interested in how the maximizer

$$x^*(\theta) = \arg\max_{x \in X} F(x, \theta)$$

would change with the parameter θ.

4.1 Univariate Comparative Statics

The classical approach is to apply the implicit function theorem. We need at least two sets of assumptions:

- $F(\cdot, \cdot)$ is twice continuously differentiable.
- The maximizer $x^*(\theta)$ is unique and is characterized by the first-order condition.

Remark.

> A set of sufficient conditions for the second assumption is needed:
>
> - The choice set X is convex.
> - $F(\cdot, \theta)$ is strictly concave in x.
> - The solution is interior.

Under the two sets of assumptions, the first-order condition is:

$$F_x(x(\theta), \theta) = 0.$$

Differentiating on both sides with respect to θ, we have:

$$F_{xx}(x(\theta),\theta)\cdot x'(\theta)+F_{x\theta}(x(\theta),\theta)=0$$
$$\Longrightarrow x'(\theta)=-\frac{F_{x\theta}(x(\theta),\theta)}{F_{xx}(x(\theta),\theta)}$$

If $F(\cdot,\theta)$ is strictly concave in x, then $F_{xx}(\cdot,\cdot)<0$ (assuming no reflection point), so $x(\cdot)$ is strictly increasing in θ if $F_{x\theta}(\cdot,\cdot)>0$.

The advantage for the classical method is straightforward that, it provides explicit expression for $x'(\theta)$ and makes it possible for quantitative analysis. The disadvantages, however, grow out from its proposed advantage:

- Technical issue: Strong assumptions and tedious calculation in some examples.
- Substantive issue: The requirement of strict concavity in x counters our intuition. (Recall from consumer theory that the set of maximizers is unaffected by any positive monotonic transformation.)

Example.

Taking $\mathbf{w}$ as given, the profit maximization problem is given by

$$\max_{y\geq 0} py-c(y)$$

We use the classical approach to determine whether (and when) the firm's supply curve is (weakly) upward sloping.

Notice that the choice set $Y=[0,+\infty)$ is convex, and when $c(\cdot)$ is strictly convex, the objective function is strictly concave. Assuming interiority, the first-order condition is given by:

$$p=c'(y(p))$$

Differentiating on both sides with respect to p, we get

$$y'(p)=\frac{1}{c''(y(p))}$$

Thus, $c(\cdot)$ being strictly convex is a sufficient condition for the supply curve being upward sloping. However, the requirement of $c(\cdot)$ being strictly convex is not always a reasonable assumption. Moreover, this assumption is not necessary. Recall that we have *Law of Supply*, which states that the firm's supply curve is weakly upward sloping without any other assumption.

In conclusion, we argue that the differentiability and concavity of $F(\cdot,\theta)$ are not indispensable for comparative statics analysis. By comparison, $F_{x\theta}(\cdot,\cdot)>0$ captures the *complementarity* between x and θ. Naturally, we would like to ask whether $F_{x\theta}(\cdot,\cdot)>0$ alone is sufficient for $x^*(\theta)$ to be increasing in θ. Our goal in this part is to formulate a discrete analogue of this condition and show that $F_{x\theta}(\cdot,\cdot)>0$ is sufficient for $x^*(\theta)$ to be increasing in θ.

That $F(\cdot,\cdot)$ is twice *continuously* differentiable is too strong an assumption. For the general case, we introduce the following definition of (strict) increasing differences.

Definition 4.1.1: Increasing Differences

Let $F : X \times \Theta \to \mathbb{R}$, where $X, \Theta \subset \mathbb{R}$. We say that $F(\cdot,\cdot)$ has *increasing differences* in (x, θ) if for any $x, x' \in X$ and $\theta, \theta' \in \Theta$ such that $x' > x$ and $\theta' > \theta$, we have

$$F(x', \theta') - F(x, \theta') \geq F(x', \theta) - F(x, \theta)$$

If the inequalities are strict for all such $x, x' \in X$ and $\theta, \theta' \in \Theta$, $F(\cdot,\cdot)$ has *strictly increasing differences* in (x, θ).

Intuitively, *increasing differences* captures the case where the *marginal value* of x is higher at a higher value of θ. In other words, there is *complementarity* between x and θ.

Proposition 4.1.2: Increasing Differences for Smooth Functions

Suppose $X = [\underline{x}, \bar{x}]$ and $\Theta = \left[\underline{\theta}, \bar{\theta}\right]$, where $X, \Theta \subset \mathbb{R}$.

1. If $F(\cdot,\cdot)$ is continuously differentiable in both x and θ, $F(\cdot,\cdot)$ has increasing differences in (x, θ) if and only if either of the two conditions holds:
   - $F_x(x, \cdot)$ is non-decreasing in θ for all x.
   - $F_\theta(\cdot, \theta)$ is non-decreasing in x for all θ.
2. If $F(\cdot,\cdot)$ is twice continuously differentiable in both x and θ, $F(\cdot,\cdot)$ has increasing differences in (x, θ) if and only if $F_{x\theta}(\cdot,\cdot) \geq 0$ for all (x, θ).

There is one technical question we need to address. When $F(\cdot, \theta)$ is not strictly concave in x, the maximizer $x^*(\theta)$ need not be unique. We shall introduce two natural ways for comparison of sets.

Definition 4.1.3: Comparison of Sets

For any two sets A and B, we say that:

- $A \leq B$ *in the strong set order* if for any $a \in A$ and $b \in B$, we have $\min\{a, b\} \in A$ and $\max\{a, b\} \in B$.
- $A \leq B$ *pointwise* if for any $a \in A$ and $b \in B$, we have $a \leq b$.

Intuitively, $A \leq B$ in the strong set order allows for a common range of A and B, while for the disjoint parts, any element in A is strictly less than that in B. $A \leq B$ pointwise requires every element in B be no less than any element in A.

Theorem 4.1.4: Univariate Topkis' Theorem

Let $F : X \times \Theta \to \mathbb{R}$, where $X, \Theta \subset \mathbb{R}$, and $x^* (\theta) = \arg\max_{x \in X} F (x, \theta)$. Then for any $\theta' > \theta$,

1. If $F (\cdot, \cdot)$ has increasing differences in (x, θ), then $x^* (\theta) \leq x^* (\theta')$ in the strong set order.
2. If $F (\cdot, \cdot)$ has strictly increasing differences in (x, θ), then $x^* (\theta) \leq x^* (\theta')$ pointwise.

Proof for Theorem

Take any $x \in x^* (\theta)$ and $x' \in x^* (\theta')$. Suppose $x > x'$, then by revealed preference,

$$\begin{aligned} F (x, \theta) &\geq F (x', \theta) \\ F (x', \theta') &\geq F (x, \theta') \end{aligned}$$

By increasing differences,

$$F (x, \theta) - F (x', \theta) \leq F (x, \theta') - F (x', \theta')$$

Jointly we have:

$$0 \leq F (x, \theta) - F (x', \theta) \leq F (x, \theta') - F (x', \theta') \leq 0$$
$$\implies \begin{cases} F (x, \theta) = F (x', \theta) \\ F (x, \theta') = F (x', \theta') \end{cases} \implies \begin{cases} x \in x^* (\theta') \\ x' \in x^* (\theta) \end{cases}$$

If $F (\cdot, \cdot)$ has strictly increasing differences in (x, θ), then the combined inequality cannot hold, so it must be that $x^* (\theta) \leq x^* (\theta')$ pointwise. In particular, if $F (\cdot, \cdot)$ is twice continuously differentiable in both x and θ and $x^* (\theta)$ is single-valued, then $F_{x\theta} (\cdot, \cdot) > 0$ is sufficient for $x^* (\theta)$ to be (weakly) increasing in θ. ∎

Example.

Again consider the firm's profit maximization problem. How would the firm's supply (correspondence) change with the output price p?

The objective function is $F (y, p) = py - c (\mathbf{w}, y)$. $Y = [0, +\infty)$ and $P = [0, +\infty)$ are both intervals in $\mathbb{R}$. Moreover, $F_p (y, p) = y$, which is strictly increasing in y, so $F (\cdot, \cdot)$ has strictly increasing differences. Therefore, the firm's supply increases with the output price p pointwise.

Example.

Consider a monopolist that faces a downward sloping demand curve $Q^D (p)$. Now suppose the government levies a unit tax t on the firm. How would the before-tax price p received by the firm change with the unit tax t?

The objective function is $F(p,t) = (p-t)\,Q^D(p) - c\left(Q^D(p)\right)$. $P = [0,+\infty)$ and $T = [0,+\infty)$ are both intervals in $\mathbb{R}$. Moreover, $F_t(p,t) = -Q^D(p)$, which is strictly increasing in p (since the demand curve is downward sloping), so $F(\cdot,\cdot)$ has strictly increasing differences in (p,t). Consequently, the firm's before-tax price p increases with t pointwise.

Notice that whether $x^*(\theta)$ increases/decreases with the parameter θ is an *ordinal* property, while (strictly) increasing differences is still a *cardinal* property. Indeed, we know from the discussion on consumer theory that $\max_x F(x,\theta)$ and $\max_x G(x,\theta)$ have the same set of maximizers if $G = \varphi \circ F$ for $\varphi(\cdot)$ strictly increasing. Nevertheless, G having increasing differences in (x,θ) does not necessarily mean F having (strictly) increasing differences in (x,θ). In other words, the requirement of (strictly) increasing differences is still too strong for monotone comparative statics. For our purpose, if we can find a positive and monotonic transformation φ such that $G = \varphi \circ F$ has increasing differences or strictly increasing differences in (x,θ), then we know $x^*(\theta)$ increases with θ in the strong set order or pointwise.

Example.

Consider the effects of an increase in the market size on monopoly quantity (and monopoly price). Each consumer in the market has an identical inverse function given by $p^D(q)$. Suppose the number of consumers N is exogenously given, and that the firm's cost function is $c(Q)$, where $Q = Nq$ is the total quantity sold (i.e., the number of consumers times per unit purchase). Discuss how the firm's cost function $c(\cdot)$ would affect the optimal *per-consumer* quantity $q^*(N)$ as the number of consumers N increases.

Solution.

The objective function is

$$F(q,N) = N \cdot p^D(q) \cdot q - c(Nq).$$

Notice that it is hard to check increasing differences of $F(\cdot,\cdot)$. (You may try it yourself and find the process blocked by some terms that need additional information to push forward the computation.) Consider $G(q,N) = \dfrac{F(q,N)}{N}$. Then if $G(\cdot,\cdot)$ is twice continuously differentiable (which indeed can be relaxed), we have

$$\begin{aligned} G(q,N) &= \frac{F(q,N)}{N} = p^D(q) \cdot q - \frac{c(Nq)}{N} \\ G_N(q,N) &= -\frac{q \cdot c'(Nq) \cdot N - c(Nq)}{N^2} \\ G_{Nq}(q,N) &= -qc''(Nq) \end{aligned}$$

If $c(\cdot)$ is concave, then $G_{Nq}(q,N) \geq 0$ and $G(q,N)$ has increasing differences in (q,N), so $q^*(\cdot)$ weakly increases with N. If $c(\cdot)$ is convex, then $G_{Nq}(q,N) \leq 0$ and $G(q,N)$ has increasing differences in $(q,-N)$, so $q^*(\cdot)$ weakly decreases with N.

4.2 Multivariate Comparative Statics

Consider a two-variable maximization problem:

$$(x_1^*(\theta) . x_2^*(\theta)) = \arg \max_{(x_1,x_2)\in X\subset\mathbb{R}^2} F(x_1, x_2, \theta).$$

If we know F has (strictly) differences in (x_1, θ), we cannot conclude that $x_1^*(\theta)$ is weakly increasing in θ unless x_2^* is independent of θ. Intuitively, as θ varies, change in θ has direct effect on x_1, but there is also the effect of θ on x_2 and consequently on x_1, which constitutes the indirect effect of θ on x_1.

Recall from univariate Topkis's theorem that $A \leq B$ in the strong set order if for any $a \in A$ and $b \in B$, we have $\min\{a, b\} \in A$ and $\max\{a, b\} \in B$. In the multivariate case, technically the definition must be properly extended to answer

- For two vectors $\mathbf{x}$ and $\mathbf{y}$, how should we define $\min\{\mathbf{x}, \mathbf{y}\}$ and $\max\{\mathbf{x}, \mathbf{y}\}$?
- Under the proposed definition, is there any guarantee that $\min\{\mathbf{x}, \mathbf{y}\}$ and $\max\{\mathbf{x}, \mathbf{y}\}$ fall within the choice set X?

For the first question, we introduce the definition of meet and *join*, the first of which sets the greatest lower bound of the two vectors, and the second of which sets the smallest upper bound of the two vectors.

Definition 4.2.1: Meet; Join

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

- *meet* of $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \cdots, \min\{x_n, y_n\}).$$

- *join* of $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \cdots, \max\{x_n, y_n\}).$$

where $\mathbf{x} \wedge \mathbf{y}$ is called the greatest lower bound of $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{x} \vee \mathbf{y}$ is called the smallest upper bound of $\mathbf{x}$ and $\mathbf{y}$.

For the second concern, we delineate the boundary of our discussion by defining a structure called *sublattice*.

Definition 4.2.2: Sublattice

A set $X \subset \mathbb{R}^n$ is a *sublattice* if for any $\mathbf{x}, \mathbf{y} \in X$, both $\mathbf{x} \wedge \mathbf{y} \in X$ and $\mathbf{x} \vee \mathbf{y} \in X$.

$\mathbb{R}^n$ *itself is a* lattice, *so any set in* $\mathbb{R}^n$ *is called a* sublattice.

Example.

- $X = X_1 \times X_2 \times \cdots \times X_n$, where $X_i \subset \mathbb{R}$, for $i = 1, 2, \cdots, n$.
- $X = \left\{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{p} \cdot \mathbf{x} \leq m\right\}$, where $\mathbf{p} \gg \mathbf{0}$ is the price vector and $m > 0$ is the income. (NOT a sublattice)
- $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq g(x_1), \text{ for } g(\cdot) \text{ strictly increasing}\right\}$.

Similar to properties of complimentarity of two variables, in the multivariate case we introduce the definition of *supermodularity*.

Definition 4.2.3: Supermodularity

Let $F : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice. F is *supermodular* if for any $\mathbf{x}, \mathbf{y} \in X$,

$$F(\mathbf{x} \wedge \mathbf{y}) + F(\mathbf{x} \vee \mathbf{y}) \geq F(\mathbf{x}) + F(\mathbf{y}).$$

Remark.

- Note when $X = X_1 \times X_2 \subset \mathbb{R}^2$, supermodularity is equivalent to increasing differences in (x_1, x_2).
- More generally, when $X = X_1 \times X_2 \times \cdots \times X_n \subset \mathbb{R}^n$, **supermodularity is equivalent to increasing differences in** (x_i, x_j) **for all pairs of** $i \neq j$.

Putting all together, we have the following multivariate Topkis's Theorem.

Proposition 4.2.4: Multivariate Topkis's Theorem

Let $F : X \times \Theta \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice and $\Theta \subset \mathbb{R}$. Consider $\mathbf{x}^*(\theta) = \arg\max_{\mathbf{x} \in X} F(\mathbf{x}, \theta)$. If F is supermodular, then for any $\theta' > \theta$ and $\mathbf{x} \in \mathbf{x}^*(\theta)$ and $\mathbf{x}' \in \mathbf{x}^*(\theta')$, we have

$$\begin{aligned} \mathbf{x} \wedge \mathbf{x}' &\in \mathbf{x}^*(\theta), \\ \mathbf{x} \vee \mathbf{x}' &\in \mathbf{x}^*(\theta'). \end{aligned}$$

Proof for Proposition.

Since X is a sublattice, $\mathbf{x} \wedge \mathbf{x}' \in X$ and $\mathbf{x} \vee \mathbf{x}' \in X$. By revealed preference,

$$\begin{cases} F(\mathbf{x}, \theta) \geq F(\mathbf{x} \wedge \mathbf{x}', \theta) \\ F(\mathbf{x}', \theta') \geq F(\mathbf{x} \vee \mathbf{x}', \theta') \end{cases} \Longrightarrow F(\mathbf{x}, \theta) + F(\mathbf{x}, \theta') \geq F(\mathbf{x} \wedge \mathbf{x}', \theta) + F(\mathbf{x} \vee \mathbf{x}', \theta')$$

Since F is supermodular,

$$\begin{aligned} F(\mathbf{x}, \theta) + F(\mathbf{x}', \theta) &\leq F((\mathbf{x}, \theta) \wedge (\mathbf{x}', \theta')) + F((\mathbf{x}, \theta) \vee (\mathbf{x}', \theta')) \\ &= F(\mathbf{x} \wedge \mathbf{x}', \theta) + F(\mathbf{x} \vee \mathbf{x}', \theta') \end{aligned}$$

Jointly, it must be that

$$F(\mathbf{x}, \theta) + F(\mathbf{x}, \theta') = F(\mathbf{x} \wedge \mathbf{x}', \theta) + F(\mathbf{x} \vee \mathbf{x}', \theta')$$

Consequently, we have

$$F(\mathbf{x} \wedge \mathbf{x}', \theta) = F(\mathbf{x}, \theta)$$
$$F(\mathbf{x} \vee \mathbf{x}', \theta') = F(\mathbf{x}, \theta')$$

which indicates that $\mathbf{x} \wedge \mathbf{x}' \in \mathbf{x}^*(\theta)$ and $\mathbf{x} \vee \mathbf{x}' \in \mathbf{x}^*(\theta')$.

Example.

Suppose a firm uses two inputs (L and K) to produce a single output y, and the production function is given by

$$y = f(L, K)$$

where $f(\cdot, \cdot)$ is twice continuously differentiable.

1. Discuss how the optimal demand for capital $K^*(w, r, p)$ would be affected by an increase in the rental rate of the capital r.
2. Discuss how the optimal demand for labor $L^*(w, r, p)$ would be affected by an increase in the rental rate of the capital r.

Solution.

$$\pi(L, K, r) = pf(L, K) - wL - rK.$$

Immediately, $\pi_{Lr} = 0$ and $\pi_{Kr} = -1$, π is supermodular in $(-K, r)$, so $K^*(w, r, p)$ weakly decreases with r. Next consider the indirect path of r to L through K.

- If $f_{LK} > 0$, π is supermodular in $(-L, -K, r)$, so $L^*(w, r, p)$ weakly decreases with r.
- If $f_{LK} < 0$, π is supermodular in $(L, -K, r)$, so $L^*(w, r, p)$ weakly increases with r.

Similar to increasing differences, supermodularity is a *cardinal* property, which is again too strong. Indeed, the weaker requirement, *quasi-supermodularity* serves our purpose.

Definition 4.2.5: Quasi-Supermodularity

Let $F : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a sublattice. F is *quasi-supermodular* if for any $\mathbf{x}, \mathbf{y} \in X$,

$$F(\mathbf{x}) \geq F(\mathbf{x} \wedge \mathbf{y}) \Longrightarrow F(\mathbf{x} \vee \mathbf{y}) \geq F(\mathbf{y}),$$
$$F(\mathbf{x}) > F(\mathbf{x} \wedge \mathbf{y}) \Longrightarrow F(\mathbf{x} \vee \mathbf{y}) > F(\mathbf{y}).$$

Under such extension, the analysis can be further extended beyond the case of $X \subset \mathbb{R}^n$ and X forming a lattice structure.