# Advanced Microeconomics Theory

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# Contents

# Chapter 2

# Consumer Theory

Choice theory assumes that a rational decision-maker selects their most preferred option from their choice set. Consumer theory can be viewed as an application of choice theory. We will focus on the *preference-based approach*, with three goals in mind:

1. Formalize the “choice set” and ”preferences” of a consumer.
   - Set of alternatives: consumption set.
   - Choice set: budget set.
   - Preferences: utility representation of a preference relation.
2. Derive a consumer’s optimal choice(s) based on the information of their choice set and the preference relation.
3. Analyze the properties of optimal choices (demand).

## 2.1 Setups

At the beginning, a few assumptions are needed in consumer theory:

**Assumption 2.1.1**

1. *Perfect* information.
2. Consumers are *price takers*.
3. Prices are *linear*.
4. Goods are *divisible*.

**Remark.**

- By assuming consumers to be price takers, we assume that prices $\mathbf{p}$ are taken as known, fixed and exogenous, and there is no searching or bargaining for discounts.
- Linear price means that no quantity discount is offered.
- The idea that goods are divisible can be formally represented by the condition of

> $x \in \mathbb{R}^n_+$. Note that the divisibility assumption does not prevent us from applying the model to situations with discrete and indivisible goods.

For simplicity, we will view the entire positive quadrant as the consumption set.

**Definition 2.1.2: Consumption Set**

With $n$ goods, the *consumption set* is given by:

$$X = \mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, 2, \cdots, n\}.$$

The budget set differs from consumption set in that the former is additionally defined by the agent's income (and thus the price vector).

**Definition 2.1.3: Budget Set**

With $n$ goods and income of $m$, given a price vector $p = (p_1, \cdots, p_n)' \geq 0, p \neq \mathbf{0}$, the *budget set* is given by:

$$B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{x} \leq m, \mathbf{x} \geq \mathbf{0}\}.$$

*By reinterpreting the consumption goods and the budget, the derivation of the budget set can be extended to encompass other economic problems. For example, consumption-leisure choice, and inter-temporal choice.*

## 2.2 Utility Representation

The preference relation is intuitive, but difficult to work with. Having a utility representation for preferences is convenient because it transforms the problem of preference maximization into a relatively familiar mathematical problem.

**Definition 2.2.1: Utility Function**

A preference relation $\succeq$ on $X$ is represented by a *utility function* $u : X \to \mathbb{R}$ if

$$x \succeq y \iff u(x) \geq u(y).$$

Under this definition, the utility function is an *ordinal* representation. The utility level per se has no economic meaning and is just a convenient mathematical tool to represent the consumer's *relative rankings* of different options.

If $u$ represents $\succeq$, then the choice rule defined upon budget set is:

$$C(B(\mathbf{p}, m); \succeq) = \left\{ \mathbf{x} \mid \mathbf{x} \text{ solves } \max_{\mathbf{x} \in B(\mathbf{p}, m)} u(\mathbf{x}) \right\}.$$

*An equivalent notation of such choice rule is $C_{\succeq}(B)$.*

A natural question is that, given a preference relation $\succeq$, whether or not we can always find a function $u(\cdot)$ to represent $\succeq$. Luckily, in cases where preference relation is defined upon a *finite* consumption set $X$, the answer is yes.

### Proposition 2.2.2

If $X$ is finite, then any rational (complete and transitive) preference relation $\succeq$ on $X$ can be represented by a utility function $u : X \to \{1, \cdots, n\}$, where $n = |X|$.

***Proof for Proposition.***

- For $|X| = n = 1$, say $X = \{x\}$, let $u(x) = 1$. . The conclusion is trivial.
- Next, suppose that any preference can be represented as described for any set with at most $n$ elements. We move on to consider a set $X$ with $n + 1$ elements.
  - Since $C_{\succeq}(X) \neq \varnothing$, the set $Y = X \backslash C_{\succeq}(X)$ has no more than $n$ elements. By doing so, we can limit our discuss and deduction back to the hypothesized case. Thus, any preference restricted to that set can be represented by a utility function $u : Y \to \{1, 2, \cdots, n\}$.
  - We extend the domain of $u$ to $X$ by setting $u(x) = n + 1$ for each $x \in C_{\succeq}(X)$. By construction, we have $u(x) \in \{1, \cdots, n, n + 1\}$ for all $x \in X$.
  - Now we show that the constructed $u$ represents $\succeq$, i.e., for any $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$. Suppose $x \succeq y$. There are three possibilities:
    - $x \in C_{\succeq}(X), y \in Y \iff u(x) = n + 1 \geq u(y)$.
    - $x, y \in C_{\succeq}(X) \iff u(x) = u(y) = n + 1$, also, $u(x) \geq u(y)$.
    - $x, y \in Y$. Then, since by construction $u$ represents $\succeq$ on $Y$, $x \succeq y$ if and only if $u(x) \geq u(y)$.

However, if $X$ is infinite, things are a bit more complicated. In general, given any rational preference relation $\succeq$ on $\mathbb{R}^n_+$, it cannot always be represented by a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$.

**Example.**

Consider the lexicographic preferences on the square $X = \mathbb{R}^2$, where $(x_1, x_2) \succ (y_1, y_2)$ if either (1) $x_1 > y_1$ or (2) $x_1 = y_1$ and $x_2 > y_2$. These preferences cannot be represented by a utility function, and this is also an example for which "indifference curves" do not exist, because the agent is never indifferent between any two choices.

To see this, suppose by contradiction there exists a utility representation $u(x, y)$. For any $x \in \mathbb{R}_+$, consider the interval $I(x) = \big(\inf_y u(x, y), \sup_y u(x, y)\big)$. $I(x)$ is not degenerate, and $I(x_1), I(x_2)$ do not overlap for $x_1 \neq x_2$. Since rational numbers are dense, we construct a function $r$ to pick a rational number inside $I(x)$, i.e., $r(x) \in I(x)$. Since $I(x_1), I(x_2)$ do not overlap, $r(x_1) \neq r(x_2)$ for $x_1 \neq x_2$. Thus, $r : \mathbb{R}_+ \to \mathbb{Q}_+$ is an injective function. Since $\mathbb{R}_+$ is uncountable and $\mathbb{Q}_+$ is countable, this mapping is impossible.

However, note that if we replace $X = \mathbb{R}^2_+$ with $X' = \mathbb{Q}^2_+$, the lexicographic preference relation would admit a utility representation. More generally, this idea is formalized into the proposition in countable set case.

Take a step forward. If a preference relation $\succeq$ is defined upon a countable set $X$ (possibly infinite), then $\succeq$ admits a utility representation. We can always achieve this by construction, as it goes in the following proposition.

### Proposition 2.2.3

If $X \neq \varnothing$ is countable, then any rational (complete and transitive) preference relation $\succeq$ on $X$ can be represented by a utility function $u : X \to (0, 1)$.

***Proof for Proposition.***

- First, construct a mapping of utility function $u : X \to (0, 1)$.
  - Let $(x_n)_{n=1}^{\infty}$ be a countable enumeration of $X$.
  - Let $u(x_1) = \frac{1}{2}$. Consider $x_{n+1}$, $n \geq 1$.
    * If $x_{n+1} \sim x_i$ for some $1 \leq i \leq n$, then define $u(x_{n+1}) = u(x_i)$.
    * Otherwise, define

$$M_n = \max\{\max\{u(x_i) : x_{n+1} \succ x_i, 1 \leq i \leq n\}, 0\}$$
$$m_n = \min\{\min\{u(x_i) : x_i \succ x_{n+1}, 1 \leq i \leq n\}, 1\}$$

    By construction $M_n > m_n$. Define $u(x_{n+1}) = \frac{M_n + m_n}{2}$.

- Second, prove that $u(\cdot)$ is a utility representation of preference relation $\succeq$.
  - Take any $y, z \in X$. Since $X$ is countable, $\exists i, j$ such that $y = x_i$ and $z = x_j$. Without loss of generality, suppose $i \leq j$.
  - By construction, $u(x_i) = u(x_j)$ if and only if $x_i \sim x_j$. For $x_j \nsim x_i$, $u(x_j) > u(x_i)$ if and only if $x_j \succ x_i$.

Intuitively, the problem with the lexicographic preference relation is that there is sudden preference reversals. For instance, we know $(3, 3) \succ (3, 2)$, but $(3, 2) \succ (x, 3)$ for $x$ slightly less than 3. This motivates the following additional (technical) restriction on preference relations. The "continuity" restriction on preferences is a condition motivated from the simple and compelling reason that any *finite* set of observed choices that is consistent with HARP is also consistent with continuity.

### Definition 2.2.4: Continuity

A preference relation $\succeq$ on $X \subseteq \mathbb{R}^n$ is *continuous* if for any sequence $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \to x, y^n \to y$, and $x^n \succeq y^n$ for all $n$, we have $x \succeq y$.

Continuity condition implies not only that a utility representation exists, but that a *continuous* representation exists.

**Proposition 2.2.5**

Any complete, transitive and continuous preference relation $\succeq$ on $X$ on $X \subseteq \mathbb{R}^n$ can be represented by a continuous utility function $u : X \to \mathbb{R}$.

***Proof for Proposition.***

In order to have a simple, constructive proof, we prove the proposition only for the case of a monotone preference relation $\succeq$ on $X = \mathbb{R}^n_+$.

Let $e = (1, \cdots, 1)$ and consider bundles of the form $\alpha e = (\alpha, \cdots, \alpha)$ where $\alpha \geq 0$. For each $x \in \mathbb{R}^n_+$, we construct a utility number as follows: $u(x) = \max A(x)$, where $A(x) = \{\alpha \in R_+ : \alpha e \preceq x\}$. To see that the set $A(x)$ has a maximal point, note that the set is

- Nonempty, since $0 \in A(x)$ by monotonicity of $\succeq$;
- Closed, by the continuity of $\succeq$;
- Bounded, since by monotonicity of $\succeq$, $\alpha \leq \max\{x_1, \cdots, x_n\}$ for each $\alpha \in A(x)$.

Now we show that we must have $u(x)e \sim x$.

1. $u(x)e \preceq x$. This is satisfied by construction of $u(x) \in A(x)$.
2. $u(x)e \succeq x$. For each $n \geq 1$, we have $u(x) + \frac{1}{n} \notin A(x)$, hence $(u(x) + \frac{1}{n})e \not\preceq x$, therefore by completeness of $\succeq$ we have $(u(x) + \frac{1}{n})e \succeq x$, which by continuity of $\succeq$ implies
$$\lim_{n\to\infty} (u(x) + \frac{1}{n})e = u(x)e \succeq x.$$

Now it remains to show that the constructed utility function $u(\cdot)$ has

1. Ability to represent the preference relation $\succeq$, and
2. Continuity.

For representation part, note that by transitivity $x \succeq y$ if and only if $u(x)e \succeq u(y)e$ (since $u(x)e \sim x \succeq y \sim u(y)e$), and by monotonicity of $\succeq$, this holds if and only if $u(x) \geq u(y)$. For continuity part, this is more subtle and not covered here.

**Remark.**

1. The construction in the proof specifies the utility of any bundle $x$ by finding the point on the 45° line on the indifference curve passing through $x$.
    - This specification is, of course, completely arbitrary; just for mathematical convenience.
    - To reflect this arbitrariness, utility representation of preferences is *ordinal*, i.e., only the induced preference ordering of choices is meaningful, instead of the exact utility numbers assigned to them.
    - In fact, if $u$ represents $\succeq$, then $U(\cdot) = v(u(\cdot))$ also represents $\succeq$ so long as

$v : \mathbb{R} \to \mathbb{R}$ is an increasing function.

2. The thought process and the introduction of the additional assumption are the most important for this part.

   - Recall that our motivation is to come up with a tractable framework to analyze the consumer's problem. One possibility is to attach a utility level to each consumption bundle (ordinal utility representation).
   - We ask whether any preference relation can be represented by a utility function. The answer is yes, if the consumption set $X$ is finite (or countable), but no if $X = \mathbb{R}^n_+$. We notice that the problem with the counter-example (e.g., the lexicographic preference relation) is that there are sudden preference reversals.
   - We then impose the restriction of "continuity" of preferences to rule out those unfavorable scenarios and show that any rational and continuous preference relation on $X = \mathbb{R}^n_+$ can be represented by a continuous utility function.

3. The continuity of preference relation and continuity of its utility representation is not interdependent. If a preference relation can be represented by a continuous utility function, then such preference relation must be continuous. On the other hand, a continuous preference relation can be represented by a discontinuous utility function, if you like. (That is not convenient for mathematical issues though.)

## 2.3 Utility Maximizing Problem

Apart from the general problem of choice theory, what makes consumer consumption decisions worthy of separate studies is its particular structure that allows us to derive economically meaningful results. The structure arises because the consumer's choice sets are assumed to be defined by certain prices and the consumer's income or wealth. With this in mind, when the preference relation can be represented by a utility function $u(\cdot)$, the Utility Maximization Problem (UMP), or Consumer Problem (CP) is reasonably defined.

**Definition 2.3.1: Utility Maximization Problem**

$$\begin{aligned} &\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \\ \text{s.t. } &\mathbf{p}\cdot\mathbf{x} \le m \\ &x_i \ge 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

The idea is that, the consumer chooses a vector of goods $\mathbf{x} = (x_1, \cdots, x_n)' \ge \mathbf{0}$ to maximize her utility, subject to a budget constraint that given a price vector $\mathbf{p} = (p_1, \cdots, p_n)' \ge \mathbf{0}, \mathbf{p} \ne \mathbf{0}$, she cannot spend more than her total wealth $m$. In other words, the CP can be formulated as:

$$C(B(\mathbf{p}, m); \succeq) = \left\{ \mathbf{x} \;\middle|\; \mathbf{x} \text{ solves } \max_{\mathbf{y}\in B(\mathbf{p},m)} u(\mathbf{y}) \right\}.$$

### 2.3.1 Existence of Optimal Choice(s)

We are then natural to seek for the existence of an optimal choice for a utility-maximizing consumer. Luckily, the answer is yes.

**Proposition 2.3.2**

Suppose that the consumer has a rational (complete and transitive) and continuous preference relation and makes rational decisions, then for $(\mathbf{p}, m) \gg \mathbf{0}$, they have (at least) one optimal choice.

***Proof for Proposition.***

- Since the consumer has a complete, transitive and continuous preference relation, then her preferences can be represented by a continuous utility function $u(\cdot)$.
- When $(\mathbf{p}, m) \gg \mathbf{0}$, the budget set $B(\mathbf{p}, m)$ is closed and bounded and hence compact.
- A continuous function on a compact set has at least one maximizer.

### 2.3.2 Further Assumptions

Now that existence of optimal choice is guaranteed (under rational and continuous preference relations, and $(\mathbf{p}, m) \gg \mathbf{0}$, we still need to answer:

- How to find the optimal choice(s)?
- In particular, what (additional) restrictions on the consumer's preference relation can simplify our analysis?

**Locally Non-Satiation** One possible simplification is to replace the inequality in the budget constraint with the equality: $\mathbf{p} \cdot \mathbf{x} = m$.

Intuitively, if a consumer thinks "more of a good is good", her preference relation should be monotone, and she would definitely exhaust all her wealth when maximizing her utility.

**Definition 2.3.3: Monotonicity**

A preference relation $\succeq$ on $X = \mathbb{R}^n_+$ is *monotone* if for any $\mathbf{x}, \mathbf{y} \in X$, we have:

- $\mathbf{x} \geq \mathbf{y} \Longrightarrow \mathbf{x} \succeq \mathbf{y}$;
- $\mathbf{x} \gg \mathbf{y} \Longrightarrow \mathbf{x} \succ \mathbf{y}$.

Indeed, the monotonicity condition proves to be a bit strong; the weaker condition of *locally non-satiated* preferences instead serves our purpose.

### Definition 2.3.4: Locally Non-Satiation

A preference relation $\succeq$ on $X = \mathbb{R}^n_+$ is *locally non-satiated* if for any $\mathbf{x} \in X$ and $\varepsilon > 0$, there exists $\mathbf{y} \in Ball(\mathbf{x}, \varepsilon) \cap X$ such that $\mathbf{y} \succ \mathbf{x}$.

Intuitively, a preference relation is locally non-satiated if there is no *bliss* point. Moreover, a monotone preference relation must be locally non-satiated.

### Proposition 2.3.5

Suppose that the consumer has a complete, transitive, continuous and locally non-satiated preference relation and makes rational decisions, then for $(\mathbf{p}, m) \gg \mathbf{0}$, the budget constraint must hold with equality at any optimal choice $\mathbf{x}^*$, i.e., $\mathbf{p} \cdot \mathbf{x}^* = m$.

***Proof for Proposition.***

Intuitively, if the consumer does not exhaust her budget, then there must be a nearby affordable bundle which is strictly more preferred, by locally non-satiation.

- Consider any bundle $\mathbf{x} \in X = \mathbb{R}^n_+$ where $\mathbf{p} \cdot \mathbf{x} < m$. Since the preference relation $\succeq$ is locally non-satiated, for any $\varepsilon > 0$, there exists $\mathbf{y} \in Ball(\mathbf{x}, \varepsilon) \cap X$ such that $\mathbf{y} \succ \mathbf{x}$.
- We claim that $\mathbf{y} \in B(\mathbf{p}, m)$, i.e., the alternative bundle falls within the consumer's budget set for $\varepsilon > 0$ small enough, so that $\mathbf{x}$ cannot be an optimal choice.

  Let $\mathbf{y} = \mathbf{x} + \mathbf{z}$. By construction, $||\mathbf{z}|| < \varepsilon$. In particular, $|z_i| < \varepsilon$, for any $i = 1, 2, \cdots, n$. It follows that $\mathbf{p} \cdot \mathbf{y} = \mathbf{p} \cdot \mathbf{x} + \mathbf{p} \cdot \mathbf{z} \leq \mathbf{p} \cdot \mathbf{x} + n \cdot \varepsilon \cdot \max_i p_i$. As $\varepsilon$ take appropriately small values, we finish our proof.

■

**Convexity** Another possible simplification is the case of *unique* optimal choice. Notice that the consumer's budget set $B(\mathbf{p}, m)$ is convex. Based on this, when the utility function $u(\cdot)$ is *strictly quasi-concave*, there is a *unique* global maximizer. This translates into the following condition of strictly convex preference relations.

### Definition 2.3.6: Convexity

A preference relation $\succeq$ on $X = \mathbb{R}^n_+$ is *convex* if for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ such that $\mathbf{x} \succeq \mathbf{z}$ and $\mathbf{y} \succeq \mathbf{z}$, we have $t\mathbf{x} + (1-t)\mathbf{y} \succeq \mathbf{z}$, for any $t \in [0, 1]$. If $t\mathbf{x} + (1-t)\mathbf{y} \succ \mathbf{z}$ for any $\mathbf{x} \neq \mathbf{y}$ and $t \in (0, 1)$, then the preference relation is *strictly convex*.

**Remark.**

1. Convexity can be equivalently defined as: For any $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{y} \succeq \mathbf{x}$, we have $t\mathbf{y} + (1-t)\mathbf{x} \succeq \mathbf{x}$ for any $t \in [0, 1]$.
2. An equivalent way to describe convexity involves indifference curve and *upper contour set* of choice bundles, Upper Contour Set of $y = \{x \in X : x \succeq y\}$, graphically the area sitting upper-right above the indifference curve (included). Convexity of

preferences amounts to the assumption that the upper contour set of any $y \in X$, is a *convex* set.

3. Convexity is fundamental in the standard model of competitive economics. When consumer preferences are convex, market clearing prices exist; otherwise this may not exist. Convexity is also needed to be able to recover consumer preferences from choices from various budget sets. Convexity is often described as capturing the idea that the agent like diversity. However, whether convexity makes sense often depends on the interpretation of the goods space, in particular on the level of aggregation (e.g., over time or categories).

**Proposition 2.3.7**

Suppose that the consumer has a complete, transitive, continuous and strictly convex preference relation and makes rational decisions, then for $(\mathbf{p}, m) \gg \mathbf{0}$, there is exactly one optimal choice.

***Proof for Proposition.***

- The rational and continuous preference relation has guaranteed the existence of at least one optimal choice.
- Suppose to the contrary that the consumer has at least two optimal choices $\mathbf{x}^*$ and $\mathbf{y}^*$, and $\mathbf{x}^* \neq \mathbf{y}^*$, then by optimality $\mathbf{x}^* \sim \mathbf{y}^*$. Construct a bundle $\mathbf{w} = \frac{1}{2}\mathbf{x}^* + \frac{1}{2}\mathbf{y}^*$. Since the consumer's preference relation is strictly convex, $\mathbf{w} \succ \mathbf{x}^* \sim \mathbf{y}^*$. Moreover, since the budget set $B(\mathbf{p}, m)$ is convex when $(\mathbf{p}, m) \gg \mathbf{0}$, so $\mathbf{w} \in B(\mathbf{p}, m)$. So neither $\mathbf{x}^*$ nor $\mathbf{y}^*$ can be an optimal choice, which is a contradiction.

Not only do these assumptions of properties of preference relation play an important role in shaping the desired characteristics of the solutions to utility maximization problem, but each has a corresponding property in a specific preference-representing utility function $u$.

**Proposition 2.3.8**

Suppose the preference relation $\succeq$ on $X$ can be represented by $u : X \to \mathbb{R}$. Then,

1. $\succeq$ is monotone if and only if $u$ is non-decreasing.
2. $\succeq$ is locally non-satiated if and only if $u$ has no local maxima in $X$.
3. $\succeq$ is (strictly) convex if and only if $u$ is (strictly) quasi-concave.

### 2.3.3 Marshallian Demand & Indirect Utility Function

**Definition 2.3.9: Indirect Utility Function**

Given $(\mathbf{p}, m) \gg \mathbf{0}$, the *indirect utility function* is defined to give the optimal utility level:

$$v(\mathbf{p}, m) = \sup_{\mathbf{x} \in B(\mathbf{p}, m)} u(\mathbf{x}).$$

*Here we rigorously use "sup" to define $v(\boldsymbol{p}, m)$ instead of "max", because sometimes $u(\boldsymbol{x})$ does not behave well to have a maximum.*

**Definition 2.3.10: Marshallian Demand Correspondence**

The *Marshallian demand correspondence* is defined as the consumer's optimal choice(s):

$$\mathbf{x}^M(\mathbf{p}, m) = \{\mathbf{x} \in B(\mathbf{p}, m) : u(\mathbf{x}) = v(\mathbf{p}, m)\}.$$

$\mathbf{x}^M(p, m)$ is, in general, a set of utility-maximizing consumption bundles.

Assume throughout the analysis of utility maximization problem that the consumer has a rational preference relation $\succeq$ and makes rational decisions given $(\mathbf{p}, m) \gg \mathbf{0}$. Then Marshallian demand and indirect utility function must satisfy the following properties.

**Proposition 2.3.11: Properties of Marshallian Demand Correspondence and Indirect Utility Function**

- **Existence of optimal choice(s)**: If $\succeq$ is continuous, then the Marshallian demand $\mathbf{x}^M(\mathbf{p}, m) \neq \varnothing$.
- **Structure of Marshallian demand**: If $\succeq$ is convex, then $\mathbf{x}^M(\mathbf{p}, m)$ is a convex set. If $\succeq$ is strictly convex, then $\mathbf{x}^M(\mathbf{p}, m)$ Is a singleton.
- **Homogeneity**: Both $v(\mathbf{p}, m)$ and $\mathbf{x}^M(\mathbf{p}, m)$ are homogeneous of degree 0 in $(\mathbf{p}, m)$, that is, for any $t > 0$, $v(t\mathbf{p}, tm) = v(\mathbf{p}, m)$ and $\mathbf{x}^M(t\mathbf{p}, tm) = \mathbf{x}^M(\mathbf{p}, m)$.
- **Monotonicity of $v(\mathbf{p}, m)$**: $v(\mathbf{p}, m)$ is non-increasing in $\mathbf{p}$ and non-decreasing in $m$. If $\succeq$ is locally non-satiated, then $v(\mathbf{p}, m)$ is strictly increasing in $m$.
- **Walras' Law**: If $\succeq$ is locally non-satiated, then $\mathbf{p} \cdot \mathbf{x} = m$, for any $\mathbf{x} \in \mathbf{x}^M(\mathbf{p}, m)$.

### 2.3.4 Derivation of Utility Maximization Problem

The consumer's utility maximization problem is give by:

$$\begin{aligned} &\max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \\ \text{s.t. } \; &\mathbf{p} \cdot \mathbf{x} \leq m \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

For $(\mathbf{p}, m) \gg \mathbf{0}$, the constraint qualification is always satisfied, and we can apply the necessary KKT conditions when $u(\cdot)$ is continuously differentiable.
The Lagrangian is given by:

$$\mathcal{L}(\mathbf{x}, \lambda, \mu) = u(\mathbf{x}) + \lambda \left( m - \sum_{i=1}^{n} p_i x_i \right) + \sum_{i=1}^{n} \mu_i x_i,$$

where $\lambda$ is the Lagrange multiplier on the budget constraint and, for each $i$, $\mu_i$ is the multiplier on the constraint that $x_i \geq 0$. The UMP is then transformed into:

$$\max_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \lambda, \mu) = \max_{\mathbf{x}} u(\mathbf{x}) + \lambda \left( m - \sum_{i=1}^{n} p_i x_i \right) + \sum_{i=1}^{n} \mu_i x_i.$$

The first-order conditions are given by:

- w.r.t. $x_i$ : $\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i + \mu_i = 0$.
- Inequality constraints: $m - \sum_{i=1}^{n} p_i x_i \geq 0, x_i \geq 0, \lambda \geq 0, \mu_i \geq 0$.
- Complementary slackness: $\lambda \left(m - \sum_{i=1}^{n} p_i x_i\right) = 0, \mu_i x_i = 0$.

**Remark.**

- In contrast to equality constraints, the direction of each inequality constraint determines the way in which we set up the Lagrangian.

  Intuitively, we make sure each multiplier is non-negative and penalize constraint violations. For instance, when the budget constraint is violated, that is, $m - \sum_{i=1}^{n} p_i x_i < 0$, the value of the Lagrangian strictly decreases.

- Despite little economic meaning, $\lambda$ represents the *shadow price* of wealth, that is, marginal utility of an additional unit of income.

  However, note that utility only has ordinal meanings. Nothing in the consumer theory developed so far suggests any basis for using the shadow price of wealth to guide redistribution policies.

- If the preference relation is well-behaved (i.e., locally non-satiated and strictly convex) and the non-negativity constraints are not binding, then $\frac{\partial u}{\partial x_i} = \lambda p_i$, and we are back to the familiar "tangency conditions", that is, for all $i, j$:

$$MRS_{ij} = \frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} = \frac{p_i}{p_j}.$$

  The "tangency conditions" say that at the consumer's maximum, relative marginal utility of any two choices equals their relative price, that is, all possible inner "gains from trade" has been realized (thus no more inner "gain from trade").

To facilitate the derivation of the consumer's problem, we should **first check whether the preference relation is locally non-satiated and strictly convex**.

- Case 1: If *locally non-satiated* and *strictly convex*, then we proceed in two steps
  1. Directly apply the "tangency conditions": $\frac{MU_i}{p_i} = \frac{MU_j}{p_j} = \lambda$, for any $i, j$.
  2. Check the *non-negativity constraints* and apply the complementary slackness condition(s) if necessary.

     *Note that in the previous step we first assume that the non-negativity holds and that we have interior solution.*
- Case 2: If neither locally non-satiated nor strictly convex, then use logic or economic intuition to tackle the problem.

**Cobb-Douglas Utility Example.**

Suppose that the consumer's preference relation can be represented by the following utility function:

$$u(x_1, x_2, x_3) = (x_1 + a)(x_2 + b)(x_3 + c)$$

where $a, b, c \geq 0$ are non-negative constants. Moreover, the consumer faces constant prices $p_1, p_2, p_3\mathbf{0}$ and has income $m \geq 0$.

1. First suppose that $a = b = c = 0$. Solve for the consumer's Marshallian demand correspondence $\mathbf{x}^M(x_1, x_2, x_3)$ and indirect utility function $v(p_1, p_2, p_3, m)$.
2. Next suppose that $a, b, c > 0$. Solve for the consumer's Marshallian demand correspondence $\mathbf{x}^M(p_1, p_2, p_3, m)$ and indirect utility function $v(p_1, p_2, p_3, m)$.

## Claim: Cobb-Douglas Utility

Cobb-Douglas utility representation for a preference relation $\succeq$ can be written as

$$u(x_1, x_2, \cdots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Then the consumer would spend her income on each good according to its share,

$$x_i = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} \cdot \frac{m}{p_i} \iff p_i x_i = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} \cdot m$$

**Solution.**

1. To solve the first question, we need to rigorously follow the steps:
   - Step 1:Apply the positive monotonic transformation for computation-convenience: $v = \ln u$.
   - Step 2: Check the locally non-satiation and convexity of preference relation for simplification of optimal solution.
     - Check that the preference relation is monotonic and hence locally non-satiated.
     - Check that the preference relation is strictly convex, that is, the utility function is strictly quasi-concave.

- Step 3: Now that both non-satiation and convexity are satisfied, apply the "tangency conditions":

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \frac{MU_3}{p_3}$$

- Step 4: Together with the budget constraint, so we have the Marshallian demand correspondence $\mathbf{x}^M(\mathbf{p}, m)$.

$$\mathbf{x}^M(p_1, p_2, p_3, m) = \left(\frac{m}{3p_1}, \frac{m}{3p_2}, \frac{m}{3p_3}\right) \geq \mathbf{0}.$$

2. The first three steps are quite similar to the previous part and thus omitted. In step 4, this time we have

$$\mathbf{x}_1^M(p_1, p_2, p_3, m) = \frac{m + p_1 a + p_2 b + p_3 c}{3p_1} - a$$
$$\mathbf{x}_2^M(p_1, p_2, p_3, m) = \frac{m + p_1 a + p_2 b + p_3 c}{3p_2} - b$$
$$\mathbf{x}_3^M(p_1, p_2, p_3, m) = \frac{m + p_1 a + p_2 b + p_3 c}{3p_3} - c$$

Note that depending on the parameter values, we may or may not have $\mathbf{x}^M(\mathbf{p}, m) \geq \mathbf{0}$. In other words, the non-negativity constraints may be binding and are for us to check. For simplicity, suppose that

$$2p_1 a - p_2 b - p_3 c \geq 2p_2 b - p_1 a - p_3 c \geq 2p_3 c - p_1 a - p_2 b$$

The other five symmetric cases are similar. In our assumption, $p_1 a \geq p_2 b \geq p_3 c$.

(a) $m \geq 2p_1 a - p_2 b - p_3 c$, then the non-negativity constraints are not binding, and we have the interior solution described above.

(b) $2p_1 a - p_2 b - p_3 c \geq m$, then at optimum, $x_1^* = 0$. The constraint $x_1^M \geq 0$ is binding, so the consumer only purchases goods 2 and 3, and we have

$$\mathbf{x}^M(p_1, p_2, p_3, m) = \left(0, \frac{m + p_2 b + p_3 c}{2p_2} - b, \frac{m + p_2 b + p_3 c}{2p_3} - c\right)$$

i. $m \geq p_2 b - p_3 c$, then $\mathbf{x}^M$ is as above.

ii. $p_2 b - p_3 c > m$, then the two constraints $x_1^M \geq 0$ and $x_2^M \geq 0$ are both binding, so the consumer only purchases good 3, and

$$\mathbf{x}^M(p_1, p_2, p_3, m) = \left(0, 0, \frac{m}{p_3} - c\right)$$

**Inter-Temporal Choice Example.**

You have a saving $s > 0$ to spend for this year and next year. Since you are now in graduate school, you will not earn any additional income over the two years. Suppose your utility is time-separable and is given by $v(c_1, c_2) = u(c_1) + \beta u(c_2)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable function with $u'(c) > 0$ and $u''(c) < 0$ for any $c \in \mathbb{R}_+$

and $0 < \beta < 1$ is the discount factor. Further suppose that the going interest rate is $0 < r < 1$, which will remain constant. Let $c_1^*$ and $c_2^*$ be your optimal consumption choices for this year and next year.

1. Give a (necessary and sufficient) condition for $(c_1^*, c_2^*) \gg \mathbf{0}$.
2. Suppose that $(c_1^*, c_2^*) \gg \mathbf{0}$, compare $c_1^*$ with $c_2^*$.

**Solution.**

1. The utility maximization problem is given by:

$$\begin{aligned} &\max_{c_1, c_2} v(c_1, c_2) = u(c_1) + \beta u(c_2) \\ \text{s.t. } &c_1 + \frac{c_2}{1+r} \leq s \\ &c_1, c_2 \geq 0 \end{aligned}$$

Since $u(\cdot)$ is strictly increasing and strictly concave, $v(\cdot)$ is also strictly increasing and strictly concave (in particular, strictly quasi-concave), and we can directly apply the "tangency condition":

$$\frac{u'(c_1)}{1} = \frac{\beta u'(c_2)}{\frac{1}{1+r}}$$

By strictly monotonicity, $c_1 + \frac{c_2}{1+r} = s$. For $(c_1^*, c_2^*) \gg \mathbf{0}$, we need:

$$\frac{u'(s)}{u'(0)} < \beta(1+r) < \frac{u'(0)}{u'\left((1+r)s\right)}$$

2. At the optimum,

$$\frac{u'(c_1)}{u'(c_2)} = \beta(1+r)$$

- If $\beta(1+r) > 1$, then $\frac{u'(c_1)}{u'(c_2)} > 1$ and $c_1^* < c_2^*$.
- If $0 < \beta(1+r) < 1$, then $\frac{u'(c_1)}{u'(c_2)} < 1$ and $c_1^* > c_2^*$.
- If $\beta(1+r) = 1$, then $\frac{u'(c_1)}{u'(c_2)} = 1$ and $c_1^* = c_2^*$.

Intuitively, $\beta(1+r)$ represents how your tradeoff between next year and this year compares with that of the market when $c_1 = c_2$.

**Utility Representation Example.**

Let $X = \mathbb{R}_+ \times \mathbb{N}$, where $(x, t)$ is interpreted as receiving $x$ yuan at time $t$. Consider the following six properties of preference relations on $X$:

- Rationality (completeness and transitivity).
- Continuity.
- There is indifference between receiving 0 yuan at time 0 and receiving 0 yuan at any

other time.

- It is (strictly) better to receive any positive amount of money as soon as possible.
- Money is always desirable.
- The preference between $(x, t)$ and $(y, t+1)$ is independent of $t$.

Consider the following questions:

1. Use precise mathematical language to formally define the six properties.
2. Suppose a preference relation on $X$ can be represented by the utility function $v(x,t) = u(x)\beta^t$, where $0 < \beta < 1$ and $u(\cdot)$ is continuous, strictly increasing and $u(0) = 0$. Check whether this preference relation satisfies each of the six properties.
3. Suppose a preference relation on $X$ satisfies all of the six properties. Show that this preference relation must admit a utility representation.
4. Use precise mathematical language to formalize the idea that "one preference is more patient than another".
5. Based on your definition in part (4), prove or disprove the following statement: A preference relation represented by $v_1(x,t) = u_1(x)\beta_1^t$ is more patient than another preference relation represented by $v_2(x,t) = u_2(x)\beta_2^t$ if $0 < \beta_2 < \beta_1$ (where $u_1(\cdot)$ and $u_2(\cdot)$ are both continuous, strictly increasing and $u_1(0) = u_2(0) = 0$).

**Solution.**

1. Mind yourself that the time $t$ is not continuous here, so take caution when you try to define a "limit" with regard to $t$.
   - Continuity: For any $t, t' \in \mathbb{N}$, and any pair of sequences $\{x(n)\}_{n=1}^{\infty}$ and $\{y(n)\}_{n=1}^{\infty}$ from $\mathbb{R}^+$ with $x(n) \to x^*$, $y(n) \to y^*$. If $(x(n), t) \succeq (y(n), t')$ for all $n$, we have $(x^*, t) \succeq (y^*, t')$.
   - The preference between $(x, t)$ and $(y, t+1)$ is independent of $t$: For any $x, y \in \mathbb{R}^+$, and $t, t' \in \mathbb{N}$, we have $(x, t) \succeq (y, t+1) \iff (x, t') \succeq (y, t'+1)$.
2. A continuous preference relation can be represented by a discontinuous function. However, if a preference relation can be represented by a continuous function, then the preference relation must be continuous.
3. Recall the proof in our lecture of existence of utility representation for any rational and continuous preference relation.
   - Claim 1: For any pair $(x, t)$, there is a unique number $u(x,t) \in \mathbb{R}^+$ such that $(x, t) \sim (u(x,t), 0)$.

     *Proof:* First, by indifference when receiving nothing, we have $(0, t) \sim (0, 0)$. For any pair $(x, t)$, we have $(x, t) \succeq (0, 0)$ and $(x+1, t) \succeq (x, t)$. Then by continuity, there is $y$ for which $(x, t) \sim (y, 0)$, and we then define $u(x, t) := y$.

- Claim 2: The preference relation is represented by $u(x,t)$.

  By claim 1 and property that money is more desirable:

$$u(x,t) \geq u(y,t') \iff (u(x,t),0) \succeq (u(y,t'),0) \iff (x,t) \succeq (y,t')$$

4. The definition should be clearly based on preference relations and try to be somewhat math-irrelevant.

   $\succeq_1$ is more patient than $\succeq_2$ if for any $(x,t)$ and any $(y,t')$ with $t' > t$, $y > x$:

$$(y,t') \succeq_2 (x,t) \Longrightarrow (y,t') \succeq_1 (x,t)$$

   The definition means that if I prefer to wait from $t$ to $t'$ under $\succeq_1$, then I will also prefer to wait from $t$ to $t'$ when I'm more patient (say under $\succeq_2$).

5. Intuitively, if the two preference relations value money differently at the baseline level (i.e., simply in terms of money), they would generate different preference over combinations of money and receiving time.

**Example.**

Let $\succeq$ be a rational (complete and transitive) preference relation on $X = \mathbb{R}^2_+$. Consider the following three properties:

- Additivity: If $(x_1, x_2) \succeq (y_1, y_2)$, then for any $t, s$ such that $(x_1 + t, x_2 + s), (y_1 + t, y_2 + s) \in \mathbb{R}^2_+$, $(x_1 + t, x_2 + s) \succeq (y_1 + t, y_2 + s)$.
- Strong monotonicity: If $x_1 \geq y_1$ and $x_2 \geq y_2$, then $(x_1, x_2) \succeq (y_1, y_2)$. If in addition, $x_1 > y_1$ or $x_2 > y_2$, then $(x_1, x_2) \succ (y_1, y_2)$.
- Standard continuity: For any two sequences $\{\mathbf{x}_n\}_{n=1}^\infty$ and $\{\mathbf{y}_n\}_{n=1}^\infty$, if $\mathbf{x}_n \succeq \mathbf{y}_n$ for any $n$, and $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}^*$ and $\lim_{n\to\infty} \mathbf{y}_n = \mathbf{y}^*$, then $\mathbf{x}^* \succeq \mathbf{y}^*$.

Consider the following questions:

1. Show that if $\succeq$ has a linear utility representation, i.e., $u(x_1, x_2) = ax_1 + bx_2$, for some $a, b > 0$, then this preference relation satisfies the above three properties.
2. Show that these three properties are necessary for the preference relation $\succeq$ to have a linear utility representation, i.e., show that for any pair of the three properties, there is a preference relation that does not satisfy the third property.
3. Show that if $\succeq$ satisfies the three properties, then this preference relation admits a linear utility representation, i.e., there exists $a, b > 0$ such that $u(x_1, x_2) = ax_1 + bx_2$, for any $(x_1, x_2) \in \mathbb{R}^2_+$. (Hint: Think about the indifference curves/sets of this preference relation.)

**Solution.**

1. Easy to verify. Notice that if a preference relation can be represented by a continuous function, then the preference relation must be continuous.

2. This question means the three properties are "parallel" from a preference relation to have a linear utility representation.

   - (i)(ii)$\overset{\times}{\Longrightarrow}$(iii): The lexicographic preference: $(x_1, x_2) \succeq (y_1, y_2)$ if either $x_1 > x_2$ or $x_1 = x_2$ and $y_1 > y_2$.
   - (i)(iIi)$\overset{\times}{\Longrightarrow}$(ii): $u(x_1, x_2) = x_1 - x_2$; $u(x_1, x_2) = x_1 - \frac{1}{x_2}$.
   - (iI)(iIi)$\overset{\times}{\Longrightarrow}$(i): $u(x_1, x_2) = x_1^2 + x_2^2$; $u(x_1, x_2) = x_1$.

3. Starting from possible intuitions from linear utility representation, we need to establish the following two properties of the indifference curve:

   - Property 1: The indifference curves are linear.
   - Property 2: The indifference curves are parallel, downward sloping and not thick.

   In order to establish the two properties, we first prove the following two lemmas:

   - Lemma 1: For $\mathbf{x} \neq \mathbf{y}$, if $\mathbf{x} \sim \mathbf{y}$, then for $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ and $\mathbf{z}' = 2\mathbf{y} - \mathbf{x}$, we have $\mathbf{x} \sim \mathbf{y} \sim \mathbf{z} \sim \mathbf{z}'$.
   - Lemma 2: For $\mathbf{x} \neq \mathbf{y}$, if $\mathbf{x} \sim \mathbf{y}$, then for any $\boldsymbol{w} = t\mathbf{x} + (1-t)\mathbf{y}, 0 \leq t \leq 1$, we have $\mathbf{x} \sim \mathbf{y} \sim \boldsymbol{w}$.

   Here we omit the technical proofs and move on with establishment of property 1. Pick any point on the horizontal axis $\mathbf{x} = (x, 0), x > 0$. By the proof of the utility representation in lecture and strong monotonicity, $\exists 0 < w < x$ such that $\mathbf{x} \sim \boldsymbol{w} = w\boldsymbol{e} = (w, w)$. Connect $\mathbf{x}$ and $\boldsymbol{w}$ and extend it to the vertical axis. Denote the intersection of the ray $\mathbf{x}\boldsymbol{w}$ with the vertical axis as $\mathbf{y} = (0, y)$. Jointly from Lemma 1 and 2 we can say the points on the line $\mathbf{xy}$ are indifferent to each other. Finally, by strong monotonicity, the indifference curves must be downward sloping and for any $x \neq x'$, we cannot have $(x, 0) \sim (x', 0)$. Moreover, if $(x, 0) \sim (w, w)$, then for any $t \geq -w$, we have $(x + t, 0) \sim (w + t, w)$, so the indifference curves are parallel.

## 2.4 Expenditure Minimization Problem

In order to disentangle the price effect and the income effect, we introduce the following expenditure minimization problem, holding the consumer's utility at or above a certain level.

**Definition 2.4.1: Expenditure Minimization Problem**

$$\begin{aligned} &\min_{\mathbf{x}} \mathbf{p} \cdot \mathbf{x} \\ \text{s.t. } &u(\mathbf{x}) \geq u \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

**Definition 2.4.2: Expenditure Function; Hicksian Demand Correspondence**

Let $F(\mathbf{p}, u) = \{\mathbf{x} \in \mathbb{R}^n_+ : u(\mathbf{x}) \geq u\}$ be the feasible set. Define the optimal (minimal) value function as the *expenditure function*:

$$e(\mathbf{p}, u) = \inf_{\mathbf{x} \in F(\mathbf{p}, u)} \mathbf{p} \cdot \mathbf{x},$$

and the consumer's optimal choice(s) as the *Hicksian demand correspondence*:

$$\mathbf{x}^H(\mathbf{p}, u) = \{\mathbf{x} \in F(\mathbf{p}, u) : \mathbf{p} \cdot \mathbf{x} = e(\mathbf{p}, u)\}.$$

*Since the consumer's utility is held at or above a certain level, when there is a price change,* ***the change in the Hicksian demand only captures the substitution effect.***

Similar to the analysis of the consumer's utility maximization problem, before seeking the optimal solution(s), we approach the expenditure minimization problem from two aspects:

- When can we simplify the problem (e.g., existence of solution(s), binding utility level and uniqueness of solution)?
- Properties of expenditure function and Hicksian demand correspondence.

### 2.4.1 Existence of Solution

**Proposition 2.4.3**

Suppose $u(\cdot)$ represents a continuous preference relation and that $\mathbf{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$, then the expenditure minimization problem has at least one minimizer, i.e.,

$$\mathbf{x}^H(\mathbf{p}, u) \neq \varnothing.$$

***Proof for Proposition.***

- Pick any $\mathbf{x}_0 \in F(u)$ and consider the alternative feasible set:

$$\tilde{F} = \{\mathbf{x} \in \mathbb{R}^n_+ : u(\mathbf{x}) \geq u \text{ and } \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{x}_0\}.$$

  It is easy to see that the expenditure minimization problem with the feasible set $\tilde{F}$ has the same solution as the original problem. Moreover, the key motivation for this construction is that $\tilde{F}$ is closed and bounded, and hence compact.

  - Boundedness is crafted by picking $\mathbf{x}_0$ purposefully but without loss of generality.
  - Closedness is guaranteed by continuity of $u(\cdot)$.
    * Note that $u(\cdot)$ need not be continuous, because we can always find an alternative continuous function $v(\cdot)$ that represents the same preference relation regardless of the continuity of $u(\cdot)$. Thus, $\tilde{F}$ is closed.
    * Note that the continuity of preference relation has no direct relation to the continuity of its utility representation.

- The objective function $\mathbf{p} \cdot \mathbf{x}$ is continuous. We know that a continuous function on a compact set has at least one minimizer, which ends our proof.

### 2.4.2 Biding Utility Level

**Proposition 2.4.4**

Suppose $u(\cdot)$ represents a continuous preference relation and that $\mathbf{p} \gg \mathbf{0}$, $u \geq u(\mathbf{0})$ and $F(u) \neq \varnothing$, then at any minimizer $\mathbf{x}^*$,

$$u(\mathbf{x}^*) = u.$$

***Proof for Proposition.***

- By the argument in the previous proposition, we can assume without loss that $u(\cdot)$ is continuous. Suppose to the contrary that a minimizer $\mathbf{x}^*$, we have $u(\mathbf{x}^*) > u$. Since $u \geq u(\mathbf{0})$, $\mathbf{x}^* \neq \mathbf{0}$.
- By the continuity of $u(\cdot)$, we know for some $\varepsilon > 0$. $u\left((1-\varepsilon)\mathbf{x}^*\right) > u$. It follows that $(1-\varepsilon)\mathbf{x}^* \in F(u)$ and that $\mathbf{p} \cdot (1-\varepsilon)\mathbf{x}^* < \mathbf{p} \cdot \mathbf{x}^*$, which is a contradiction to the optimality of $\mathbf{x}^*$.

**Remark.**

The binding condition here in EMP is much weaker than that in UMP, where we do not put "any" additional condition on preference relation. One can understand this as the objective function in EMP is itself locally non-satiated.

### 2.4.3 Unique Minimizer

**Proposition 2.4.5**

Suppose $u(\cdot)$ represents a continuous and strictly convex preference relation and that $\mathbf{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$. Then the expenditure minimization problem has exactly one minimizer, i.e., $\mathbf{x}^H(\mathbf{p}, u)$ is a singleton.

***Proof for Proposition.***

- By the preceding proposition, *at least one minimizer exists.*
- Suppose to the contrary that there are two minimizers $\mathbf{x}^* \neq \mathbf{y}^*$. Then by feasibility, $u(\mathbf{x}^*) \geq u$ and $u(\mathbf{y}^*) \geq u$. Since the preference relation is strictly convex, $u(\cdot)$ is strictly quasi-concave, so for the bundle $\mathbf{w} = \frac{1}{2}\mathbf{x}^* + \frac{1}{2}\mathbf{y}^* \neq \mathbf{0}$, we have $u(\mathbf{w}) > \min\{u(\mathbf{x}^*), u(\mathbf{y}^*)\} \geq u$. By continuity, for some small $\varepsilon > 0$, $u((1-\varepsilon)\mathbf{w}) \geq u$. Moreover, $\mathbf{p} \cdot (1-\varepsilon)\mathbf{w} = \frac{1}{2}(1-\varepsilon)\mathbf{p} \cdot \mathbf{x}^* + \frac{1}{2}(1-\varepsilon)\mathbf{p} \cdot \mathbf{y}^* < \mathbf{p} \cdot \mathbf{x}^* = \mathbf{p} \cdot \mathbf{y}^*$, which is a contradiction.

### 2.4.4 Summary of Properties

**Proposition 2.4.6: Properties of Expenditure Function and Hicksian Demand Correspondence**

Suppose $\mathbf{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$.

- **Existence of minimizer**: If $u(\cdot)$ represents a continuous preference relation and that, then the Hicksian demand $\mathbf{x}^H(\mathbf{p}, u) \neq \varnothing$.
- **Structure of Hicksian demand**: If $u(\cdot)$ represents a convex preference relation, then $\mathbf{x}^H(\mathbf{p}, u)$ is a convex set. If $u(\cdot)$ represents a continuous and strictly convex preference relation, then $\mathbf{x}^H(\mathbf{p}, u)$ is a singleton.
- **Homogeneity**: $e(\mathbf{p}, u)$ is homogeneous of degree 1 in $\mathbf{p}$, that is, for any $t > 0$, $e(t\mathbf{p}, u) = te(\mathbf{p}, u)$.
- **Monotonicity of $e(\mathbf{p}, u)$**: $e(\mathbf{p}, u)$ is non-decreasing in $\mathbf{p}$ and $u$. If $u(\cdot)$ represents a continuous preference relation, then $e(\mathbf{p}, u)$ is strictly increasing in $u$ when $u \geq u(\mathbf{0})$.
- **Binding utility level**: Suppose $u(\cdot)$ represents a continuous preference relation and $u \geq u(\mathbf{0})$. Then at any minimizer $\mathbf{x}^*$, $u(\mathbf{x}^*) = u$.

Since $\min \mathbf{p} \cdot \mathbf{x}$ is equivalent to $\max -\mathbf{p} \cdot \mathbf{x}$, EMP can be solved in an analogous manner to UMP. EMP and UMP share the same "tangency condition" for interior solutions:

$$\frac{MU_i}{MU_j} = \frac{p_i}{p_j}$$

**Example.**

Suppose a consumer's preference relation can be represented by the following utility function:

$$u(x_1, x_2) = \ln x_1 + x_2.$$

Moreover, the consumer faces constant prices $(p_1, p_2) \gg \mathbf{0}$ and has income $m > 0$.

1. Solve the consumer's utility maximization problem to derive the Marshallian de-

mand $\mathbf{x}^M(p_1, p_2, m)$ and indirect utility function $v(p_1, p_2, m)$.

2. Solve the consumer's expenditure minimization problem to derive the Hicksian demand $\mathbf{x}^H(p_1, p_2, u)$ and expenditure function $e(p_1, p_2, u)$.

**Solution.**

1. It is easy to check that the preference relation is monotonic and strictly convex. The utility miximization can then be simplified as:

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) = \ln x_1 + x_2 \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = m.$$

   The "tangency condition" is: $|MRS| = \frac{MU_1}{MU_2} = \frac{1}{x_1} = \frac{p_1}{p_2}$.

   Together with the binding budget constraint, we have $x_1^* = \frac{p_2}{p_1}$ and $x_2^* = \frac{m}{p_2} - 1$.

   Note that the constraint $x_2 \geq 0$ may be binding. Marshallian demand is given by

$$\mathbf{x}^M(p_1, p_2, m) = \begin{cases} \left(\frac{p_2}{p_1}, \frac{m}{p_2} - 1\right), & \text{if } m \geq p_2 \\ \left(\frac{m}{p_1}, 0\right), & \text{if } 0 < m < p_2 \end{cases}$$

2. Similar to the previous part, the expenditure minimization problem can be simplified as:

$$\min_{x_1, x_2 \geq 0} p_1 x_1 + p_2 x_2 \quad \text{s.t. } u(x_1, x_2) = \ln x_1 + x_2 = u.$$

   The "tangency condition" is $|MRS| = \frac{MU_1}{MU_2} = \frac{1}{x_1} = \frac{p_1}{p_2}$.

   Together with binding utility level, we have $x_1^* = \frac{p_2}{p_1}$ and $x_2^* = u - \ln \frac{p_2}{p_1}$.

   Notice the constraint $x_2 \geq 0$ may be binding, so the Hicksian demand is given by:

$$\mathbf{x}^H(p_1, p_2, m) = \begin{cases} \left(\frac{p_2}{p_1}, u - \ln \frac{p_2}{p_1}\right), & \text{if } u \geq \ln \frac{p_2}{p_1} \\ (e^u, 0), & \text{if } u < \ln \frac{p_2}{p_1} \end{cases}$$

*Remember to check if the utility function is monotone and quasi-concave beforehand!*

## 2.5 Duality and Comparative Statics

### 2.5.1 Duality between UMP and EMP

Recall consumer's utility maximization problem (UMP):

$$\begin{aligned} &\max_{\mathbf{x}} u(\mathbf{x}) \\ \text{s.t. } &\mathbf{p} \cdot \mathbf{x} \leq m \\ &x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

Consumer's expenditure minimization problem (EMP):

$$\begin{aligned} &\min_{\mathbf{x}} \mathbf{p}\cdot\mathbf{x} \\ \text{s.t. } & u(\mathbf{x}) \geq u \\ & x_i \geq 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

Notice that the objective function in the UMP is precisely the constraint in the EMP and vice versa. In mathematics, the two optimization problems are called *dual problems*. By duality, Marshallian and Hicksian demand have a special relationship.

**Proposition 2.5.1**

Suppose $u(\cdot)$ is a utility function that represents a continuous and locally non-satiated preference relation on $X = \mathbb{R}^n_+$, then for any $\mathbf{p} \gg \mathbf{0}$, we have:

1. For any $m \geq 0$, $\mathbf{x}^M(\mathbf{p}, m) = \mathbf{x}^H(\mathbf{p}, v(\mathbf{p}, m))$ and $e(\mathbf{p}, v(\mathbf{p}, m)) = m$.
2. For any $u \geq u(\mathbf{0})$, $\mathbf{x}^H(\mathbf{p}, u) = \mathbf{x}^M(\mathbf{p}, e(\mathbf{p}, u))$ and $v(\mathbf{p}, e(\mathbf{p}, u)) = u$.

***Proof for Proposition.***

- Fix $m > 0$ and any $\mathbf{x}_0^M \in \mathbf{x}^M(\mathbf{p}, m)$, we have:

  $$e(\mathbf{p}, v(\mathbf{p}, m)) \leq \mathbf{p}\cdot\mathbf{x}_0^M = m.$$

  Fix any $u \geq u(\mathbf{0})$ and any $\mathbf{x}_0^H \in \mathbf{x}^H(\mathbf{p}, u)$, we have

  $$v(\mathbf{p}, e(\mathbf{p}, u)) \geq u(\mathbf{x}_0^H) = u.$$

- Applying the first inequality to the wealth level $m = e(\mathbf{p}, u)$, we have:

  $$e(\mathbf{p}, v(\mathbf{p}, e(\mathbf{p}, u))) \leq e(\mathbf{p}, u).$$

  On the other hand, since the preference relation is continuous, $e(\mathbf{p}, u)$ is strictly increasing in $u$, so from the second inequality, we have

  $$e(\mathbf{p}, v(\mathbf{p}, e(\mathbf{p}, u))) \geq e(\mathbf{p}, u).$$

- Again by strict monotonicity of $e(\mathbf{p}, u)$ in $u$, we have $v(\mathbf{p}, e(\mathbf{p}, u)) = u$. Similarly, $e(\mathbf{p}, v(\mathbf{p}, m)) = m$.

- Finally, since the preference relation is continuous and locally non-satiated, the budget constraint must bind at the UMP and the utility level must bind at the EMP. Correspondingly,

  $$\begin{aligned} \mathbf{x}^M(\mathbf{p}, m) &= \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p}\cdot\mathbf{x} = m, u(\mathbf{x}) = v(\mathbf{p}, m)\} = \mathbf{x}^H(\mathbf{p}, v(\mathbf{p}, m)) \\ \mathbf{x}^H(\mathbf{p}, u) &= \{\mathbf{x} \in \mathbb{R}^n_+ : u(\mathbf{x}) = u, \mathbf{p}\cdot\mathbf{x} = e(\mathbf{p}, u)\} = \mathbf{x}^M(\mathbf{p}, e(\mathbf{p}, u)) \end{aligned}$$

Intuitively, their relationship implies that, at the "right" level of wealth and utility, the Marshallian demand correspondence is identical to the Hicksian demand correspondence.

### 2.5.2 Envelope Theorem

**Theorem 2.5.2: Envelope Theorem for Unconstrained Optimization**

Let $f : X \times [\underline{\theta}, \bar{\theta}] \to \mathbb{R}$ and $V(\theta) = \sup_{\mathbf{x} \in X} f(\mathbf{x}, \theta)$. Suppose $f(\mathbf{x}, \cdot)$ is differentiable in $\theta$ for all $\mathbf{x} \in X$. Moreover, there exists an integrable function $b : [\underline{\theta}, \bar{\theta}] \to \mathbb{R}_+$ such that $|f_\theta(\mathbf{x}, \theta)| \leq b(\theta)$ for all $\mathbf{x} \in X$ and almost all $\theta \in [\underline{\theta}, \bar{\theta}]$. Then $V(\cdot)$ is absolutely continuous and hence differentiable almost everywhere. In addition, for any $\mathbf{x}^* \in \arg\max_{\mathbf{x} \in X} f(\mathbf{x}, \theta)$,

$$V'(\theta) = f_2(\mathbf{x}^*(\theta), \theta).$$

**Remark.**

- The intuition is that, only the direct effect matters; the indirect effect of $\mathbf{x}^*(\theta)$ on $V(\theta)$ can be ignored, since $f(\cdot)$ does not change with $\mathbf{x}$ at the optimum.
- May get some inspiration from the simplest version.
  - Suppose $x$ is one-dimensional and that the optimizer is unique and differentiable in $\theta$. Combined with F.O.C., we would have:

$$V'(\theta) = f_1(x^*(\theta), \theta) \cdot (x^*)'(\theta) + f_2(x^*(\theta), \theta) = f_2(x^*(\theta), \theta).$$

  - For the envelope theorem to hold, $\mathbf{x}$ need not be one-dimensional, $f(\cdot, \theta)$ need not be differentiable in $\mathbf{x}$, and the optimal $\mathbf{x}^*(\theta)$ need not be unique or differentiable in $\theta$.

**Theorem 2.5.3: Envelope Theorem for Constrained Optimization**

Suppose $X$ is compact and convex. Let $f, g : X \times \left[\underline{\theta}, \bar{\theta}\right] \to \mathbb{R}$ and $V(\theta) = \sup_{\mathbf{x} \in X : g(\mathbf{x}, \theta) \geq 0} f(\mathbf{x}, \theta)$. The Lagrangian is given by $\mathcal{L}(\mathbf{x}, \theta; \lambda) = f(\mathbf{x}, \theta) + \lambda g(\mathbf{x}, \theta)$. Suppose $f$ and $g$ are continuous and concave in $\mathbf{x}$, $f_2(\mathbf{x}, \theta)$ and $g_2(\mathbf{x}, \theta)$ are continuous in $(\mathbf{x}, \theta)$, and there exists $\mathbf{x}_0 \in X$ such that $g(\mathbf{x}_0, \theta) > 0$ for all $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$. Then $V(\cdot)$ is absolutely continuous and hence differentiable (a.e.). In addition, for any $\mathbf{x}^*(\theta) \in \arg\max_{\mathbf{x} \in X : g(\mathbf{x}, \theta) \geq 0} f(\mathbf{x}, \theta)$,

$$V'(\theta) = \mathcal{L}_2(\mathbf{x}^*(\theta), \theta; \lambda^*).$$

Roy's identity and Shepard's lemma are two direct applications of envelope theorem.

**Corollary 2.5.4: Roy's Identity**

Suppose $u(\cdot)$ represents a locally non-satiated and strictly convex preference relation on $X = \mathbb{R}^n_+$. Then, for any $(\mathbf{p}, m) \gg \mathbf{0}$, the Marshallian demand for good $i$, $x_i^M(\mathbf{p}, m)$, is given by:

$$x_i^M(\mathbf{p}, m) = -\frac{\partial v(\mathbf{p}, m)/\partial p_i}{\partial v(\mathbf{p}, m)/\partial m}.$$

***Proof for Corollary.***

The Lagrangian of the utility maximization problem is given by:

$$\mathcal{L}(\mathbf{x}, \lambda, \mu; \mathbf{p}, m) = u(x_1, x_2, \cdots, x_n) + \lambda\left(m - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} \mu_i x_i$$

By the envelope theorem, we have:

$$\begin{cases} \frac{\partial v(\mathbf{p},m)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}|_{(\mathbf{x}^*, \lambda^*, \mu^*)} = -\lambda^* x_i^M(\mathbf{p}, m) \\ \frac{\partial v(\mathbf{p},m)}{\partial m} = \frac{\partial \mathcal{L}}{\partial m}|_{(\mathbf{x}^*, \lambda^*, \mu^*)} = \lambda^* \end{cases} \Longrightarrow x_i^M(\mathbf{p}, m) = -\frac{\partial v(\mathbf{p}, m)/\partial p_i}{\partial v(\mathbf{p}, m)/\partial m}$$

**Corollary 2.5.5: Shepard's Lemma**

Suppose $u(\cdot)$ represents a locally non-satiated and strictly convex preference relation on $X = \mathbb{R}^n_+$. Then, for any $\mathbf{p} \gg \mathbf{0}$ and $F(u) \neq \varnothing$, the Hicksian demand for good $i$, $x_i^H(\mathbf{p}, u)$, is given by:

$$x_i^H(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}.$$

***Proof for Corollary.***

The Lagrangian of the expenditure minimization problem is given by:

$$\mathcal{L}(\mathbf{x}, \lambda, \mu; \mathbf{p}, u) = \sum_{i=1}^{n} p_i x_i - \lambda(u(\mathbf{x}) - u) - \sum_{i=1}^{n} \mu_i x_i.$$

By the envelope theorem, we have:

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}|_{(\mathbf{x}^*, \lambda^*, \mu^*)} = x_i^H(\mathbf{p}, u).$$

*By strict convexity, both Marshallian demand and Hicksian demand are single-valued.*

### 2.5.3 Slutsky Equation

Slutsky equation mathematically describes how Marshallian demand reacts to a price change, decomposing into income effect and substitution effect.

### Theorem 2.5.6: Slutsky Equation

Suppose $u(\cdot)$ represents a continuous, locally non-satiated and strictly convex preference relation $\succeq$ on $X = \mathbb{R}^n_+$ and that $\mathbf{x}^M(\mathbf{p}, m)$ and $\mathbf{x}^H(p, u)$ are both differentiable and single-valued. Then

$$\frac{\partial x_i^M(\mathbf{p}, m)}{\partial p_j} = \frac{\partial x_i^H(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_j} - \frac{\partial x_i^M(\mathbf{p}, m)}{\partial m} x_j^M(\mathbf{p}, m).$$

***Proof for Theorem***

The proof proceeds with duality of Marshallian and Hicksian demand throughout.

$$x_i^H(\mathbf{p}, u) = x_i^M(\mathbf{p}, e(\mathbf{p}, u)), \text{ as long as } u \geq u(\mathbf{0})$$

Take partial derivatives with respect to $p_j$:

$$\frac{\partial x_i^H(\mathbf{p}, u)}{\partial p_j} = \frac{x_i^M(\mathbf{p}, e(\mathbf{p}, u))}{\partial p_j} + \frac{\partial x_i^M(\mathbf{p}, e(\mathbf{p}, u))}{\partial m} \cdot \frac{\partial e(\mathbf{p}, u)}{\partial p_j}$$

By Shepard's lemma:

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_j} = x_j^H(\mathbf{p}, u)$$

By duality,

$$\begin{cases} x_j^H(\mathbf{p}, v(\mathbf{p}, m)) = x_j^M(\mathbf{p}, m) \\ e(\mathbf{p}, v(\mathbf{p}, m)) = m \end{cases}$$

Evaluating the partial derivative equation at $u = v(\mathbf{p}, m)$, we have:

$$\frac{\partial x_i^H(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_j} = \frac{x_i^M(\mathbf{p}, e(\mathbf{p}, v(\mathbf{p}, m)))}{\partial p_j} + \frac{\partial x_i^M(\mathbf{p}, m)}{\partial m} \cdot x_j^M(\mathbf{p}, m)$$

Rearrange the terms to finish the proof of Slutsky Equation. ■

The interpretation for Slutsky Equation is that, the LHS represents how the Marshallian demand changes with respect to a price change, and that the first term on the RHS represents the substitution effect and the second term reveals the income effect.

Slutsky Equation allows us to develop a body of comparative statics based on it. The following definitions related to comparative statics are quite useful.

For comparative statics with regard to income $m$, we define normal good and inferior good. Intuitively, a good is normal if it loses its charm when its price increases.

### Definition 2.5.7: Normal Good; Inferior Good

Good $i$ is a *normal good* if $x_i(\mathbf{p}, m)$ is increasing in $m$, an *inferior good* if $x_i(\mathbf{p}, m)$ is decreasing in $m$.

*Inferior good is purely a definition for goods whose demand is negatively related with income level, but not informative of its quality.*

Regular good and Giffen good are defined upon the comparative statics with regard to the good's own price.

**Definition 2.5.8: Regular Good; Giffen Good**

Good $i$ is a *regular good* if $x_i(\mathbf{p}, m)$ is decreasing in $p_i$, a *Giffen good* if $x_i(\mathbf{p}, m)$ is increasing in $p_i$.

*A Giffen good has to be an inferior good first, because the substitution effect of a price increase will never favor the demand of that good; the income effect has to be not only positive but positive enough to counteract the negative impact of the substitution effect.*

Another aspect of comparative statics grows in the view of analyzing a good's demand change given the price change of another good.

**Definition 2.5.9: Substitute; Complement**

Good $i$ is a *substitute* for good $j$ if $x_i^H(\mathbf{p}, u)$ is increasing in $p_j$, a *complement* for good $j$ if $x_i^H(\mathbf{p}, u)$ is decreasing in $p_j$.

**Definition 2.5.10: Gross Substitute; Gross Complement**

Good $i$ is a *gross substitute* for good $j$ if $x_i^M(\mathbf{p}, m)$ is increasing in $p_j$, a *gross complement* for good $j$ if $x_i^M(\mathbf{p}, m)$ is decreasing in $p_j$.

*Good $i$ being a complement for good $j$ means that, any increase in $p_j$ would shift part of the original share of consumption onto alternative goods other than good $i$.*

## 2.6 Consumer Welfare

Consumer surplus is a basic quantitative measure of consumer welfare. But here are some issues with it:

- What if more than one price would change at the same time?
- No equivalent or immediate interpretation in utility theory.

To address the two shortcomings, we introduce two additional measures to quantify changes in consumer welfare. That is, compensating variation and equivalent variation, on measures of "conceived wealth".

**Definition 2.6.1: Compensating Variation; Equivalent Variation**

Suppose the initial price is $\mathbf{p}^0$ and $u^0 = v(\mathbf{p}, m)$, and that the final price is $\mathbf{p}'$ and $u' = v(\mathbf{p}', m)$. Compensating variation and equivalent variation are defined as:

1. *Compensating variation*: $CV = e(\mathbf{p}^0, u^0) - e(\mathbf{p}', u^0)$.
2. *Equivalent variation*: $EV = e(\mathbf{p}^0, u') - e(\mathbf{p}', u')$.

Notice that $CV$ and $EV$ have the same sign, and is positive for a price drop and negative for a price increase (though the two cases are not exhaustive).

Mathematically, by duality,

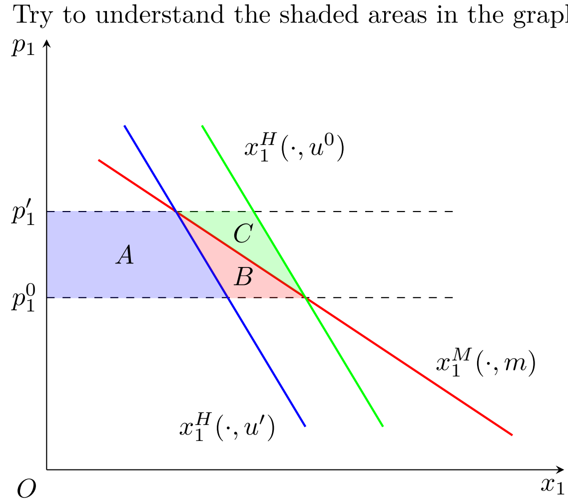
$$\begin{aligned} CV &= e(\mathbf{p}^0, u^0) - e(\mathbf{p}', u^0) = m - e(\mathbf{p}', u^0) \\ EV &= e(\mathbf{p}^0, u') - e(\mathbf{p}', u') = e(\mathbf{p}^0, u') - m \end{aligned}$$

Intuitively, $-CV$ measures how much we need to *compensate* the consumer for them to achieve the original level of utility at the new price vector, while $EV$ measures what is the equivalent amount of money that the consumer values this price change if the price vector were fixed at the original level.

Suppose the price of a single good $i$ changes from $p_i^0$ to $p_i'$, then

$$\begin{aligned} CV &= \int_{p_i'}^{p_i^0} \frac{\partial e(\mathbf{p}, u^0)}{\partial p_i} \mathrm{d}p_i = \int_{p_i'}^{p_i^0} x_i^H(\mathbf{p}, u^0) \mathrm{d}p_i \\ EV &= \int_{p_i'}^{p_i^0} \frac{\partial e(\mathbf{p}, u')}{\partial p_i} \mathrm{d}p_i = \int_{p_i'}^{p_i^0} x_i^H(\mathbf{p}, u') \mathrm{d}p_i \\ \Delta CS &= \int_{p_i'}^{p_i^0} x_i^M(\mathbf{p}, m) \mathrm{d}p_i \end{aligned}$$

Suppose the price of good 1 increases from $p_1^0$ to $p_1'$. The change is depicted in the following graph, with Marshallian demand, Hicksian demands and the change of $CV$, $EV$ and $\Delta CS$. Try to understand the shaded areas in the graph.




$$\begin{cases} -CV = A + B + C \\ -EV = A \\ -\Delta CS = A + B \end{cases}$$

**Remark.**

1. If the Marshallian demand curve is steeper than the Hicksian demand curve, it implies that the good is an inferior good. In the graph, the represented good is a normal good.

2. On any range where the good in question is either normal or inferior, then:

$$\min\{CV, EV\} \leq \Delta CS \leq \max\{CV, EV\}.$$

   Notice that the two cases are not exhaustive. For example, a good may be normal good at some lower ranger of price, but reversed to inferior good at higher range of price.

**Example.**

Suppose a consumer has a locally non-satiated and strictly convex preference relation on $\mathbb{R}^2_+$ that can be represented by a twice continuously differentiable utility function $u(x_1, x_2) \geq 0$. Moreover, for $(p_1, p_2) \gg \mathbf{0}$ and $u \geq 0$, the expenditure function is given by:

$$e(p_1, p_2, u) = \frac{p_1 p_2 u^2}{p_1 + p_2}$$

1. For $(p_1, p_2) \gg \mathbf{0}$ and $u > 0$, derive the Hicksian demand $\mathbf{x}^H(p_1, p_2, u)$.

2. For $(p_1, p_2, m) \gg \mathbf{0}$, derive the Marshallian demand $\mathbf{x}^M(p_1, p_2, m)$.

3. Now suppose $p_2 = 1$ and $m = 2$. Consider a price drop from $p_1^0 = 2$ to $p_1' = 1$. Calculate the compensating variation ($CV$), the equivalent variation ($EV$), and the change in consumer surplus ($\Delta CS$) of this price change.

**Solution.**

1. By Shepard's Lemma,

$$x_1^H(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{p_2^2 u^2}{(p_1 + p_2)^2}$$
$$x_2^H(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_2} = \frac{p_1^2 u^2}{(p_1 + p_2)^2}$$

   It follows that $\mathbf{x}^H(p_1, p_2, u) = \left(\frac{p_2^2 u^2}{(p_1+p_2)^2}, \frac{p_1^2 u^2}{(p_1+p_2)^2}\right)$.

2. By duality,

$$e(p_1, p_2, v(\mathbf{p}, m)) = \frac{p_1 p_2 v(\mathbf{p}, m)^2}{p_1 + p_2} = m.$$

   We then have $v(\mathbf{p}, m)^2 = \frac{p_1+p_2}{p_1 p_2} m$.

Again by duality,

$$\mathbf{x}^M(p_1, p_2, m) = \mathbf{x}^H(p_1, p_2, v(\mathbf{p}, m)) = \left(\frac{p_2 m}{p_1(p_1+p_2)}, \frac{p_1 m}{p_2(p_1+p_2)}\right).$$

3. Apply the definition of $CV$, $EV$ and $\Delta CS$:

$$\begin{cases} CV = e(p_1^0, p_2, u^0) - e(p_1', p_2, u') = m - e(p_1', p_2, u^0) \\ EV = e(p_1^0, p_2, u') - e(p_1^0, p_2, u') = e(p_1^0, p_2, u') - m \\ \Delta CS = \int_1^2 x_1^M(p_1, p_2, m)\mathrm{d}p_1 = 2\ln\frac{4}{3} = 0.58 \end{cases}$$

Notice that $u^0 = v(p_1^0, p_2, m) = \sqrt{3}$ and $u' = v(p_1', p_2, m) = 2$. Then the results are pinned down to be:

$$\begin{cases} CV = 2 \\ EV = \frac{2}{3} \approx 0.5 \\ \Delta CS = 2\ln\frac{4}{3} \approx 0.58 \end{cases}$$

There is one thing noteworthy about aggregation. While we have laid the foundation for individually analyzing consumer's utility maximization problem, it may fail if we directly make the aggregation.

**Example.**

Consider two consumers 1 and 2, whose preferences can be represented by the following utility functions:

$$u^1(x_1, x_2) = \begin{cases} x_1 x_2^3 & \text{if } 0 \le x_2 \le 7.7 \\ (7.7)^3 x_2 & \text{if } x_2 \ge 7.7 \end{cases}$$

$$u^2(x_1, x_2) = \begin{cases} x_1^3 x_2 & \text{if } x_1 \ge 3x_2 \\ \frac{1}{3}x_1^4 & \text{if } 0 \le x_1 \le 3x_2 \end{cases}$$

Consider the following budget sets:

- Budget set $A$: $p_1 = p_2 = 2$, $m = 20$.
- Budget set $B$: $p_1 = 3$, $p_2 = 1$, $m = 20$.

Intuitively, the failure of aggregation is due to *diverse income effects*. For instance, in the example above, the price change has a positive income effect on consumer 1, but a negative income effect on consumer 2. Aggregation is possible in the special case where all consumers have the same wealth effect, that is, $\frac{\partial \mathbf{x}^i}{\partial m^i} = \frac{\partial \mathbf{x}^j}{\partial m^j}$, for every two consumers $i, j$ and $p, m^i, m^j$.