Chapter 2: Inference and Confidence Intervals for a Single Categorical Variable

In this chapter, we will discuss methods for making decisions concerning research questions involving only a single categorical variable. Even though we'll be adding lots of terminology along the way, we will use the same logical approach to answering questions that was introduced in Chapter 1.

Formal Hypothesis Testing

In previous examples, we have used a logical process to make statistical decisions in problems involving a single categorical variable. Next, we will add more structure to these statistical investigations by introducing a procedure known as *hypothesis testing*. Before we discuss this procedure, we must introduce a few more definitions.

Population Parameters vs. Sample Statistics

In each of the previous examples, we tested a claim about a population parameter of interest.

i Definitions

- A **parameter** is a numerical descriptive measure of a population. This value is almost always unknown, and our goal is to either estimate this parameter or test claims regarding it.
- A **statistic** is a numerical descriptive measure of a sample. This value is calculated from the observed data.

Example	Statistic	Parameter
Example 1.2: Helper vs. Hinderer		
Example 1.3: Are women passed over for managerial training?		
Example 1.4: Font Preference		

Definitions

Hypothesis testing is a procedure, based on sample evidence and probability, used to test a claim regarding a population parameter. The test will measure how well our observed sample statistic agrees with some assumption about this population parameter.

Before you begin a hypothesis test, you should clearly state your research question. For instance, let's reconsider the research question from three of our previous examples.

Example	Research Question
Example 1.2: Helper vs. Hinderer	Do 10-month-old infants tend to prefer the helper toy over the hinderer toy?
Example 1.3: Are women passed over for managerial training?	Is there statistical evidence for gender discrimination against females?
Example 1.4: Font Preference	Do the majority of all consumers prefer one font over the other?

Setting up the Null and Alternative Hypothesis

Definitions

- The **null hypothesis**, H_0, is what we will assume to be true, and we will evaluate the observed data from our study against what we *expected* to see under the null hypothesis. This will always contain a statement saying that the population parameter is **equal** to some value.
- The **alternative hypothesis**, H_A, is what we are trying to show. Therefore, the research question is restated here in the alternative hypothesis. This will always contain statements of inequality, saying that the population parameter is **less than, greater than, or different from** the value in the null hypothesis.

For our three examples, the null and alternative hypotheses are shown below.

Research Question	**Hypotheses**
Do 10-month-old infants tend to prefer the helper toy over the hinderer toy?	H_0: There is no preference for one toy over the other in the population of all 10-month-olds. H_A: The majority of all 10-month-old infants prefer the helper toy.
Is there statistical evidence for gender discrimination against females?	H_0: The selection process is fair. H_A: The selection process is biased against females.
Do the majority of all consumers prefer one font over the other?	H_0: There is no preference for one font over the other in the population of all consumers. H_A: The majority of all consumers would pick one font over the other.

Note that we can also state these hypotheses in terms of the population parameter of interest:

Research Question	**Hypotheses**
Do 10-month-old infants tend to prefer the helper toy over the hinderer toy?	H_0:
	H_A:
Is there statistical evidence for gender discrimination against females?	H_0:
	H_A:
Do the majority of all consumers prefer one font over the other?	H_0:
	H_A:

Evaluating Evidence Using P-Values

In in each of our three examples, we *assumed the null hypothesis was true* when setting up our probability for our Applet simulations. Then, we used the results simulated under this scenario to help us decide whether observing results such as our sample data would be an unusual event *if the null hypothesis were true.*

Up to this point, whether an observed result was considered unusual (or extreme) has been a rather subjective decision. Now, we will discuss the guidelines used by statisticians to determine whether an observed result is extreme enough under the null hypothesis for us to conclude that the evidence supports the research question.

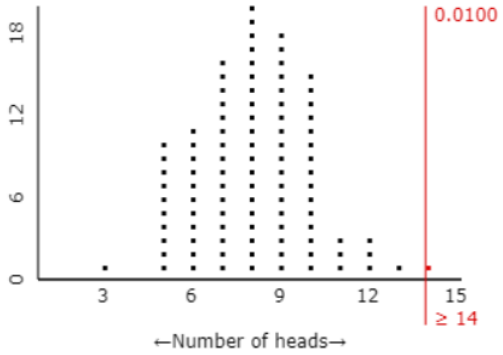
First, note that in our three examples, we examined different parts of the distribution of simulated results when deciding whether the observed data was extreme. Each of these cases is an example of a specific type of hypothesis test.

Research Question	**Hypotheses**	**Type of Test**
Do 10-month-old infants tend to prefer the helper toy over the hinderer toy?	H_0:	Upper-tailed Test
	H_A:	
Is there statistical evidence for gender gender discrimination against females?	H_0:	Lower-tailed Test
	H_A:	
Do the majority of all consumers prefer one font over the other?	H_0:	Two-tailed Test
	H_A:	

Statisticians use the following guidelines to determine whether the observed data supports the research question:

> **Note**
>
> **Upper-tailed test:** The observed result must fall in the upper 5% of the reference distribution.

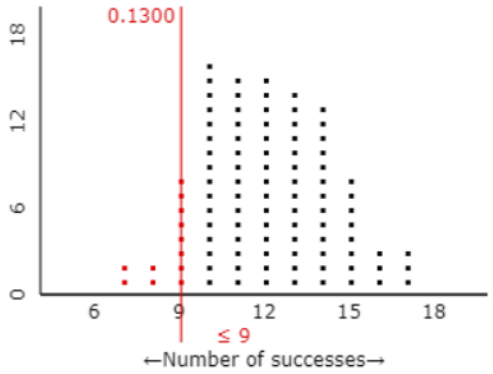
Do 10-month-old infants tend to prefer the helper toy over the hinderer toy?



Statisticians use what is called a ***p-value*** to quantify the amount of evidence that an observed result from a set of data provides for a research question.

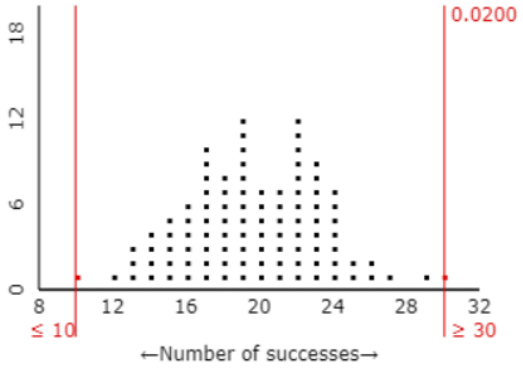
> **Note**
>
> **Lower-tailed test:** The observed result must fall in the lower 5% of the reference distribution.

Is there statistical evidence for gender discrimination against females?



> **Note**
>
> **Two-tailed test:** The observed result must fall in either the upper 2.5% or the lower 2.5% of the reference distribution.

Do the majority of all consumers prefer one font over the other?



> **Tip**
>
> In practice, two-tail tests are typically conducted over an upper- or lower-tailed test.

Statisticians use what is called a **p-value** to quantify the amount of evidence that an observed result from a set of data provides for a research question.

> **Definition**
>
> **P-value**: The probability of observing an outcome as extreme (or even more extreme) than the observed study result, assuming the null hypothesis is true.

Note that in each of the above examples, we obtained the simulation results assuming the null hypothesis was true. Therefore, to estimate the p-value, we simply determine how often outcomes as extreme (or more extreme) than the observed study results appeared in our simulation study.

Example	**Estimate of p-value**
Helper vs. Hinderer?	
Are Women Passed Over for Managerial Training?	
Font Preference?	

Making a Decision with P-Values

If the p-value is **less than** 0.05 (5%), then the data provide enough statistical evidence to support the research question.
If the p-value is not less than 0.05 (5%), then the data do not provide enough statistical evidence to support the research question.

Why does this decision rule work? Consider the "Helper vs. Hinderer" example. Because the p-value falls below 5%, the observed result *must* have been in the upper 5% of the reference distribution. As stated earlier, this implies that the observed study result is very unlikely to happen by chance under the null hypothesis, which supports the research question.

On the other hand, consider the "Are Women Passed Over for Managerial Training" example. Because the p-value was larger than 5%, the observed result *can't* have been in the lower 5% of the reference distribution. This implies that the observed study result is not all that unusual and could have easily happened by chance under the null hypothesis. Therefore, the null hypothesis *could* be true, and we have no evidence to support the research question.

This decision rule is widely accepted for determining whether study results are statistically significant; however, some researchers do advocate using a more flexible rule similar to the following:

> **Note**
>
> - If the p-value falls below 0.05, we have strong evidence to support the alternative hypothesis (i.e., the research question).
> - If the p-value falls below 0.10 but above 0.05, we have "marginal" evidence to support the alternative hypothesis (i.e., the research question).
> - If the p-value is above 0.10, we have no evidence to support the research question

Next, we will review the steps involved in a formal hypothesis test for each of our three examples. Note that our conclusions are written in the context of the problem. Moreover, even a person with no statistical background should be able to understand these conclusions (i.e., a conclusion should NOT say something like "We reject the null hypothesis.")

Helper vs. Hinderer

Research Question	Do 10-month-old infants tend to prefer the helper toy over the hinderer toy?
Hypotheses	H_0: There is no preference for one toy over the other in the population of all 10-month-olds; that is, the probability a 10-month-old selects the helper toy is 50%.
	H_A: The majority of all 10-month-old infants prefer the helper toy; that is, the probability a 10-month-old selects the helper toy is greater than 50%.
p-value estimated from simulation	
Conclusion	

Are Women Passed Over for Managerial Training?

Research Question	Is there statistical evidence of gender discrimination against females?
Hypotheses	H_0: The selection process is fair; that is, the probability a woman is selected is 60%.
	H_A: The selection process is biased against females; that is, the probability a woman is selected is less than 60%.
p-value estimated from simulation	
Conclusion	

Font Preference

Research Question	Do the majority of all consumers prefer one font over the other?
Hypotheses	H_0: There is no preference for one font over the other in the population of all consumers; that is, the probability of a consumer selecting the Signet font is 50%.
	H_A: The majority of all consumers would pick one font over the other; that is, the probability of a consumer selecting the Signet font is different from 50%.
p-value estimated from simulation	
Conclusion	

Next, let's carry out a formal hypothesis test for a few new examples.

Example 2.1 Claims of Numbness After Automobile Accident

A 28-year-old developed pain involving the spine and the left side of her body after an automobile collision. They were actively involved in a personal litigation against the company that owned the other vehicle, and they reported constant pain and numbness in the left arm. To test their claims, researchers touched their left arm with either 1 finger or 2 fingers simultaneously while their eyes were closed. The word "touch" was said simultaneously with the presentation of the tactile stimulus so that the subject knew when to respond. She then had to indicate whether she felt 1 single touch or 2 simultaneous touches (with the double-touch stimulus, the fingertips were always spaced 2 inches apart). The subject received 100 stimuli overall; they were correct on 30 of them. Is there statistical evidence that they are intentionally answering incorrectly?

1. Identify both the population and sample of interest.

2. Identify the single categorical variable of interest.

3. Identify both the parameter and statistic of interest.

4. Carry out the formal hypothesis test to address the research question.

- **Research Question** Is there statistical evidence that she is intentionally answering incorrectly?

- Hypotheses:

- Estimate p-value (Carry out the simulation study to investigate this p-value, sketch or paste your simulation results here)

Use the simulation results to estimate the p-value: _____

- Conclusion:

Example 2.2: Effectiveness of an Experimental Drug

Suppose a commonly prescribed drug for relieving nervous tension is believed to be only 70% effective. Experimental results with a *new* drug administered to a random sample of 20 adults who were suffering from nervous tension show that 18 received relief. Is there statistical evidence that the new experimental drug is more than 70% effective?

1. Identify both the population and sample of interest.

2. Identify the single categorical variable of interest.

3. Identify both the parameter and statistic of interest.

4. Carry out the formal hypothesis test to address the research question.

- **Research Question** Is there statistical evidence that the new drug is more than 70% effective?

- Hypotheses:

- Estimate p-value (Carry out the simulation study to investigate this p-value, sketch or paste your simulation results here)

Use the simulation results to estimate the p-value: _____

- Conclusion:

Theoretical Approach to P-values

There is one caveat regarding our current approach to obtaining a p-value. Certainly, different simulations will produce slightly different reference distributions. The general pattern will be the same, but variations do exist. For example, consider the Helper vs. Hinderer study.

Helper vs. Hinderer

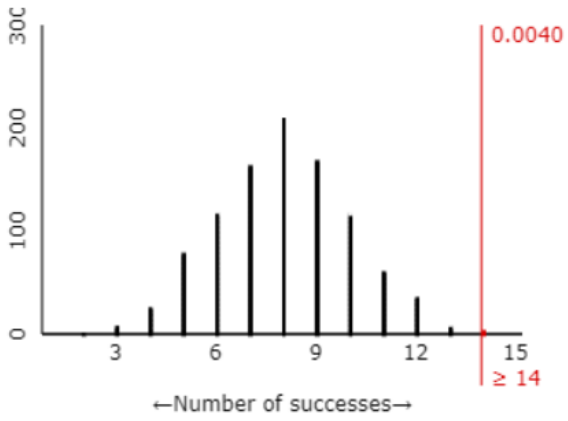
- Research Question: Do 10-month-olds tend to prefer the helper toy over the hinderer toy?
- Hypotheses
  - H_0: There is no preference for one toy over the other in the population of all 10-month-olds; that is, the probability a randomly selected 10-month-old selects the helper toy is 50%

  - H_A: The majority of all 10-month-old infants prefer the helper toy; that is, the probability a randomly selected 10-month-old selects the helper toy is greater than 50%.

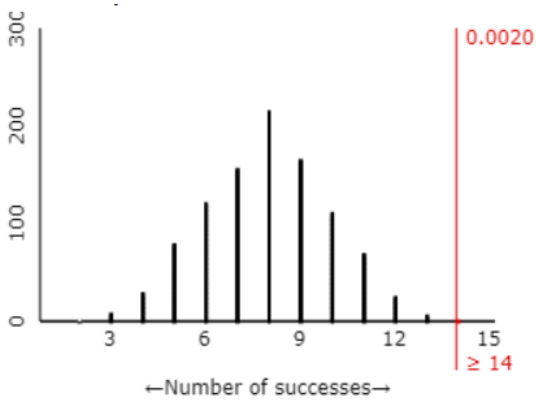
The *observed* result was as follows: 14 out of 16 infants chose the helper toy. What if two different researchers each carried out their own simulation study with 1,000 repetitions to estimate the p-value?

- p-value

  Simulation #1: p-value: _____



  Simulation #2: p-value: _____



- Conclusion: We do have evidence that 10-month-old infants prefer the helper toy.

Fortunately, regardless of which simulation study we use in the previous example, the final conclusion is the same and the discrepancy between the two estimated p-values is minimal.

Note that as the number of repetitions in our simulation study increases, we would expect less discrepancy between these two estimates. So, instead of using a simulation study with only 100 (or 1000) repetitions to *estimate* the p-value, we would ideally like to simulate this experiment over and over again, say an *infinite* number of times. This would provide us with the theoretical probabilities of interest so that can get exact p-values instead of an estimate of the p-value.

The following graphic shows what the distribution would look like if we kept repeating the simulation study over and over again, each time counting and plotting the number of infants that chose the helper toy (assuming there was no real preference in the population of all infants). This is known as the **binomial distribution**.

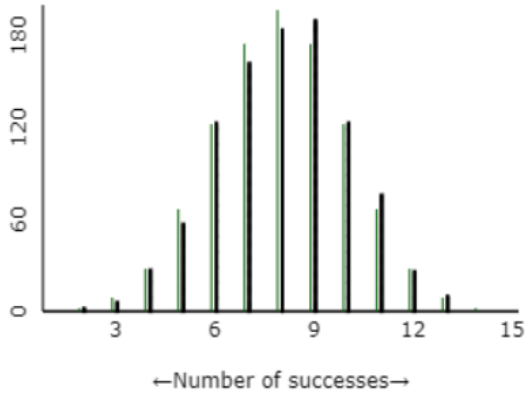


Figure 1: The black dots/bars show results from 1000 simulations; the green bars show the results from the binomial distribution. Notice they follow a similar pattern.

Statisticians often use the binomial distribution to calculate p-values when testing claims about a population proportion. However, before using the binomial distribution, we should check to make sure the following conditions are met.

The Binomial Distribution - When can we use it?

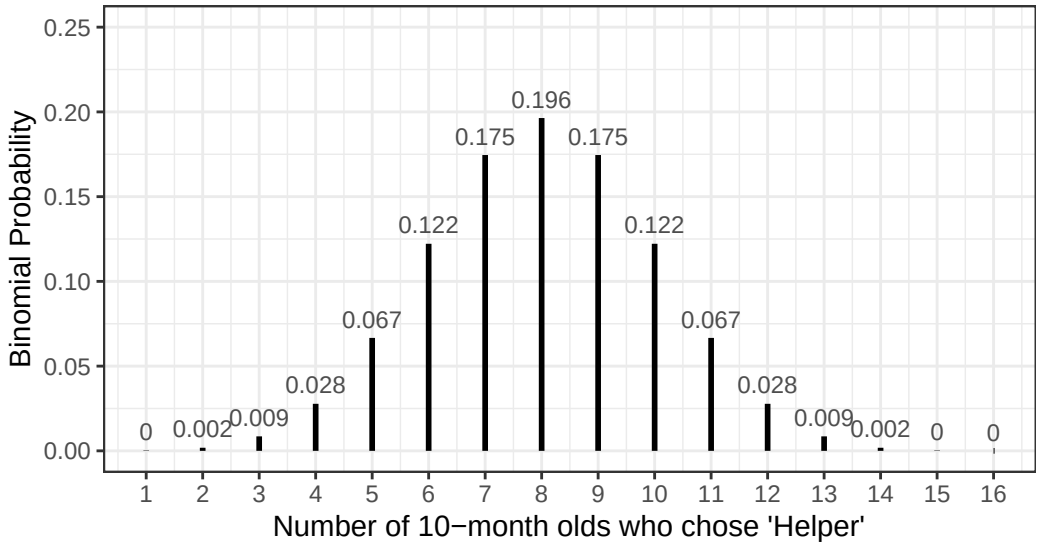
This distribution can be used whenever the following assumptions are met:

- There exist a fixed number of trials, n.
- There are only two possible outcomes on each trial ("success" or "failure").
- The probability of "success" π remains constant from trial to trial.
- The n trials are independent.

1. Check whether these assumptions seem reasonable for the Helper vs. Hinderer study.

- There exist a fixed number of trials, n.
- There are only two possible outcomes on each trial ("success" or "failure").
- The probability of success (π) remains constant from trial to trial.
- The n trials are independent.

Once the conditions for the binomial distribution have been met, we can calculate binomial probabilities of interest. While there is a mathematical formula for calculating these probabilities, we will not go into that in this class. Instead, I have provided you the calculated probabilities for the Helper vs. Hinder example in the graph below:



1. Find the probability of observing *exactly* 14 of the 16 10-month-old infants choosing the helper toy, assuming that the population of all 10-month-olds has no preference.

2. What is the exact p-value found using the binomial probabilities?

Alternatively, there is a handy function, `binom.test()` that helps us calculate our p-value from binomial probabilities and takes the following arguments:

- `x`: the number of successes in our study
- `n`: the sample size for our study
- `p`: the proportion of successes we would expect to see under the null
- `alternative`: "two-sided", "greater", "less"

Below are the results from the Helper vs. Hinder example.

```
binom.test(x = 14, n = 16, p = 0.5, alternative = "greater")
```

```
    Exact binomial test

data:  14 and 16
number of successes = 14, number of trials = 16, p-value = 0.00209
alternative hypothesis: true probability of success is greater than 0.5
95 percent confidence interval:
 0.656175 1.000000
sample estimates:
probability of success
                 0.875
```

Confidence Intervals for a Single Population Proportion

When carrying out hypothesis tests, we are testing claims about population parameters. Sometimes, however, our goal is simply to *estimate* a parameter of interest. Statisticians typically do this with a **confidence interval**

> **i** Definition
>
> **Confidence interval:** a range of plausible values for the parameter of interest.

The big difference between **hypothesis testing** and **confidence intervals** is as follows: the construction of a confidence interval does NOT require any hypotheses concerning our population parameter of interest. The goal of a **hypothesis test** is to draw a conclusion about a claim whereas the goal of a **confidence interval** is to make an estimate for the parameter.

For example, consider the following scenario.

Example 2.3: Perceived Effectivenss of Acupuncture

In 2008, researchers from the University of Pennsylvania used data from the 2002 National Health Interview Survey (NHIS) to determine the proportion of survey respondents who reported perceived effectiveness of acupuncture treatment for specific conditions. The NHIS is a nationally representative household interview survey of the U.S. civilian non-institutionalized population aged 18 years or older.

In one part of this study, the researchers identified 89 survey respondents who reported seeing a practitioner of acupuncture to treat back pain. Of these 89 respondents, 77 reported that the acupuncture helped improve their symptoms.

Source: 2008. Patrick J. LaRiccia et al. "Perceived Effectiveness of Acupuncture: Findings from the National Health Interview Survey." *Medical Acupuncture,* Volume 20, Number 4.

1. Identify the population of interest.

2. Identify the sample

3. Identify the parameter of interest.

4. Identify the observed sample statistic.

Note that the goal of this study was clearly identified in the abstract for this manuscript:

ABSTRACT

Background: Knowledge of perceived benefit from acupuncture treatment is important to predict who is using acupuncture, to inform physicians of the possible benefits of acupuncture, to determine where rigorous research should be focused, and to help policy makers predict future demand.
Objectives: To determine the proportions of survey respondents who reported perceived effectiveness of acupuncture treatment for specific conditions;

Recall that for the specific condition of back pain, 77 respondents reported that the acupuncture helped improve their symptoms.

The objectives don't mention anything about a specific research hypothesis (for example, they're not conducting this study in order to show that "the majority of patients with back pain get relief from acupuncture"). So, we have no null or alternative hypothesis of interest. Instead, the goal is simply to estimate , the population parameter of interest. What is our best guess for π based on the observed data from the sample?

$\hat{\pi} =$

This is called a **point estimate.** However, we know that if we obtained a different random sample of 89 survey respondents, this point estimate would most likely change. The purpose of using an **interval estimate** (i.e., a confidence interval) instead of this point estimate is to account for the inherent sample-to-sample variation that exists in a sample statistic. Once we determine how much variation is expected in the statistic, we can obtain a range of likely values for the population parameter (called the confidence interval). So, our next goal is to determine how much variation we should expect under repeated sampling.

Approach 1: Simulation

Describe process:

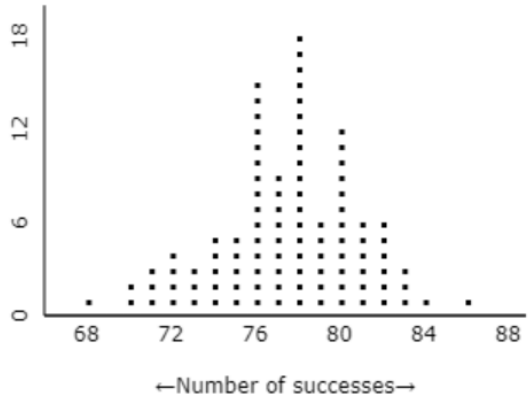
Probability of success (π):	0.87
Sample size (*n*):	89
Number of samples:	100

5. Why is the Sample size (n) set to 89?

6. Where did the 0.87 value come from?

7. Why did we use 0.87 for the probability of success?

8. Once we obtain the results of several runs of the simulation and plot the outcomes, on what value do you expect this distribution to be centered? Why?



A formal approach to calculating a confidence interval involves identifying the middle 95% of this distribution. That is, we need to find a lower endpoint that separates the bottom 2.5% of the distribution and an upper endpoint that separates the top 2.5% of the distribution. Let's zoom in on this distribution so that we can find these endpoints:

9. Identify the lower and upper endpoints that separate close to the bottom 2.5% and the top 2.5% of the outcomes.

10. What information does this range of values really provide us?

Note that instead of considering the *number* that get relief from acupuncture, we could have equivalently considered the *proportion* that get relief. This would simply change the values on the x-axis on the above plot.

11. Fill in the appropriate proportion that correspond to the following points of interest, and write each value in its position on the x-axis above.

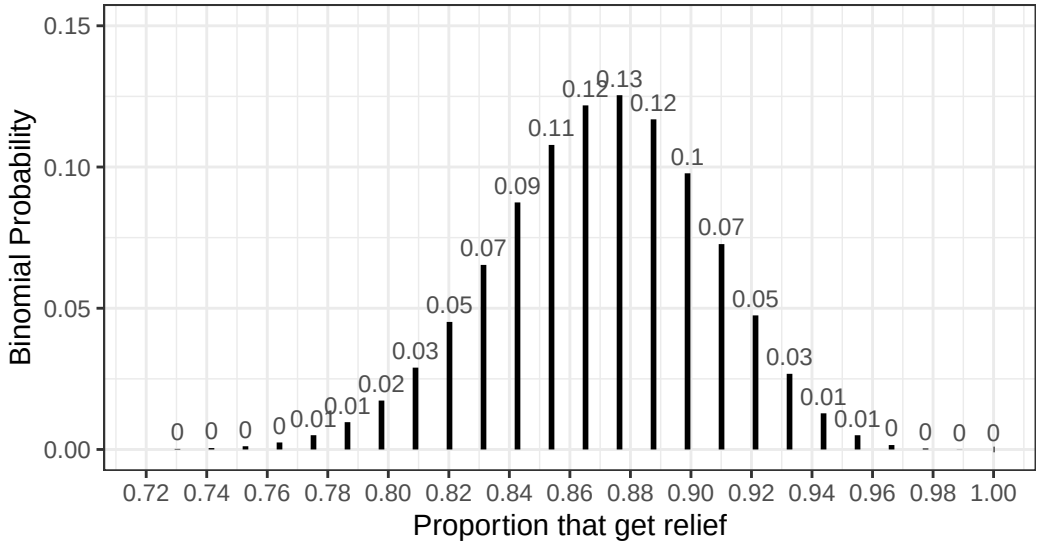
- A value of 76 in the above plot becomes _____ on the proportion scale
- Our lower endpoint of _____ becomes _____ on the proportion scale
- Our upper endpoint of _____ becomes _____ on the proportion scale

12. How do we interpret the meaning of the lower and upper endpoints when the outcomes are measured as proportions instead of as counts?

Approach 2: Binomial

Next, recall that we previously discussed the fact that running a simulation study enables us to estimate probabilities that we actually could obtain using the binomial distribution.

The distribution appears as follows:



13. Identify the lower and upper endpoints that separate close to the bottom 2.5% and the top 2.5% of the outcomes.

Approach 3: Normal Approximation

With the previous two approaches, we were simply trying to understand the basic idea behind a confidence interval. A statistician would probably not use Simulation or the binomial distribution in the same way that we have just illustrated to calculate the endpoints for this confidence interval (think about some reasons why not).

Instead, to calculate a confidence interval for a proportion, a statistician could make use of the normal curve (i.e., the bell-shaped curve). Note that the normal curve approximates the binomial distribution with a sample size of 89 and $\pi = 0.87$ fairly well:

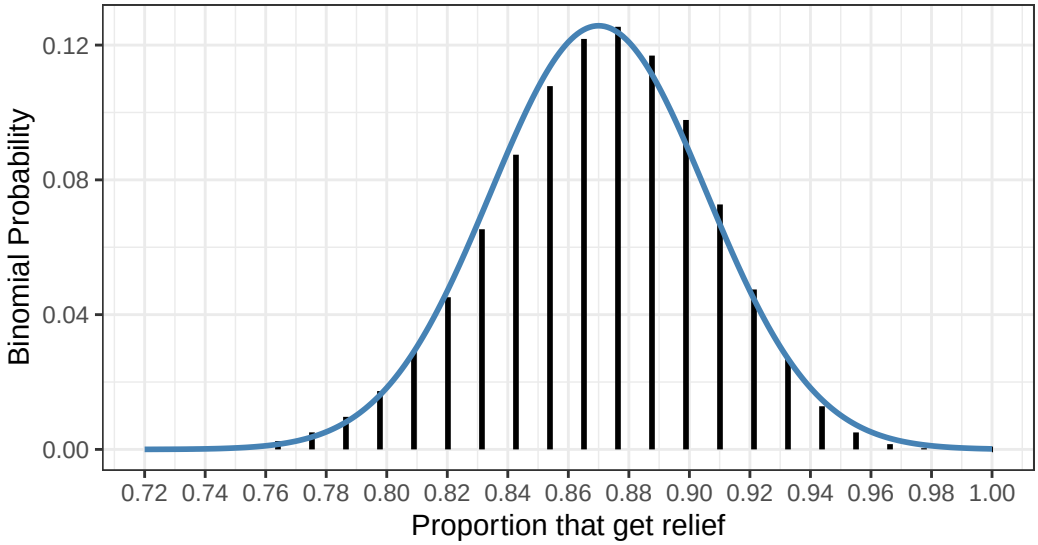


Figure 2: The normal curve superimposed on binomial probabilities.

To calculate the endpoints for a 95% confidence interval constructed using normal theory methods, we find the values on the x-axis of the above graph that separate the middle 95% from the rest. This is done using the following steps:

14. Start with the **point estimate**, $\hat{\pi}$:

15. Calculate the **standard error** associated with this point estimate:

$$\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} =$$

16. Calculate the **margin of error**: This is defined as 1.96 standard errors for a 95% confidence interval. *(This is discussed more on the next page).*

$$1.96 \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} =$$

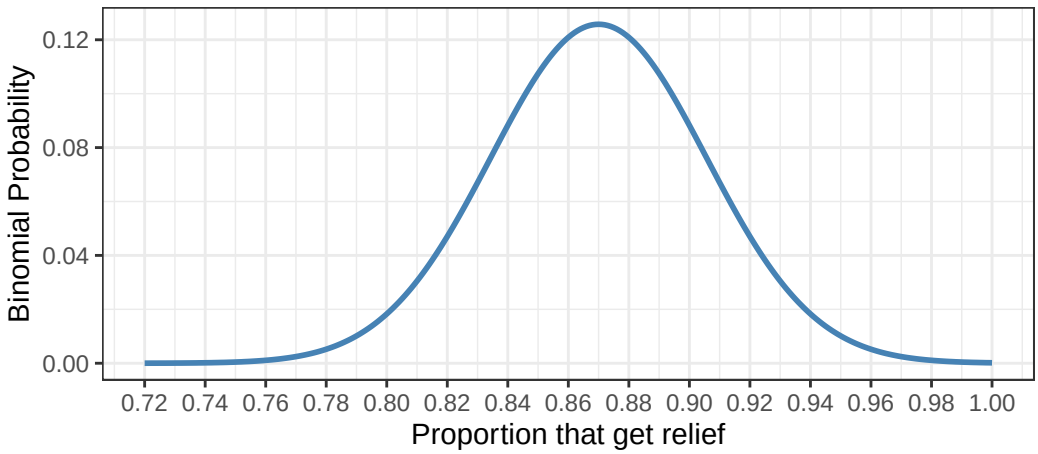
17. Find the **endpoints** of the confidence interval:

Lower endpoint $= \hat{\pi} - 1.96 \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} =$

Upper endpoint $= \hat{\pi} + 1.96 \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} =$

Sketch the following values for our example on the normal curve shown below:

- the point estimate
- the lower and upper endpoints
- the margin of error.

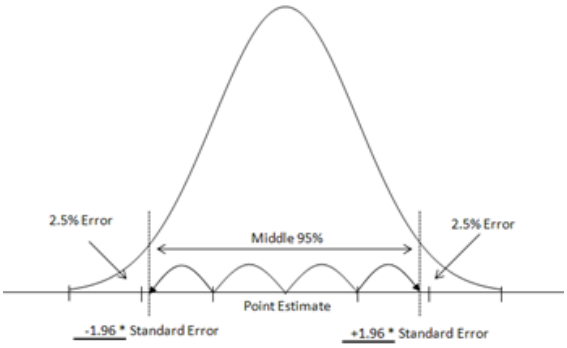


18. What is the 95% confidence interval based on the normal theory approach for this problem?

19. What is the interpretation of this confidence interval?

More on the Margin of Error

Question: Why is 1.96 used in the formula for a 95% confidence interval?
Answer: Because with the normal distribution, using a margin of error of 1.96 × standard error gives us an error rate of 2.5% in each tail, resulting in a total error rate of 5%.

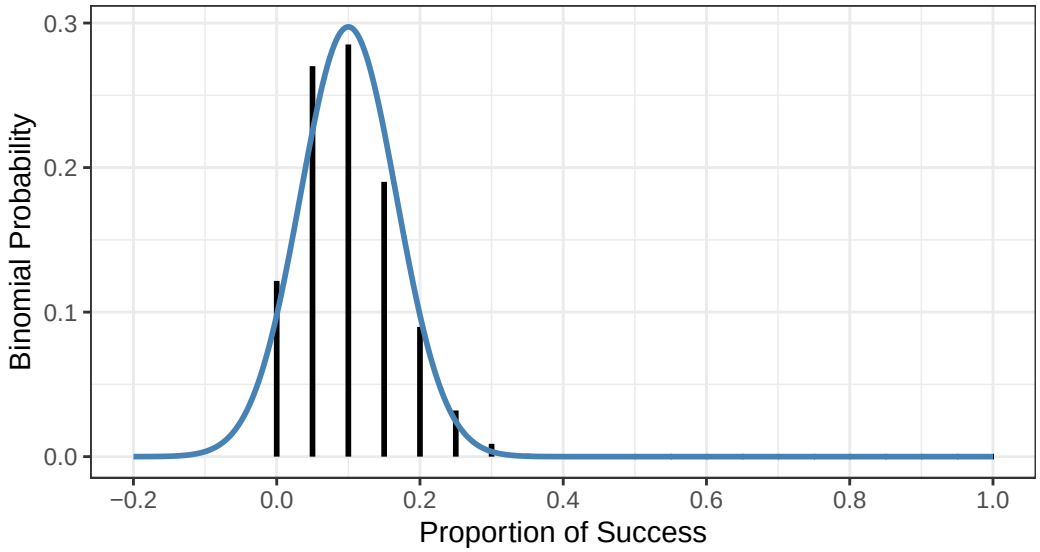


- If you are constructing a 95% confidence interval, then margin of error = 1.96 * standard error
- Similarly, if you are constructing a 90% confidence interval, it can be shown that margin of error = 1.645 * standard error
- If you are constructing a 99% confidence interval, it can be shown that margin of error = 2.575 * standard error

Warning

Several methods exist for constructing a confidence interval for a binomial proportion. We have considered a method known as the "Wald" interval (normal approximation). **Note that this method does not typically perform well for very small sample sizes or when the point estimate is very close to either zero or one.**

This is because the normal curve does not approximate the binomial distribution very well in these situations. For example, consider the binomial distribution with $\pi = 0.10$ and $n = 20$:



This distribution is not approximated well by a bell-shaped curve. If we encountered a study with $n = 20$ and $\pi = 0.10$, then the normal approxmiation method would not provide us with a very reliable interval estimate of the true population proportion.

Conditions for One Proportion Normal Approximation

One rule of thumb is that as long as there are 10 "successes" and 10 "failures", the normal approximation should work reasonably well.

20. Check this condition for the Acupuncture example. Does the Wald interval seem to be appropriate here?

What Does 95% Confidence Really Mean?

Once again, consider Example 2.3 regarding acupuncture treatment for back pain. Note that if we were to obtain another random sample of 89 individuals, the sample proportion from this new study will probably not be equal to $77/89 = 86.5\%$. Thus, the confidence interval we calculate based on this statistic will also change.

Also, note that the population parameter (π = the proportion of the population that get relief) is equal to one true value, which happens to be unknown. In a single study conducted in the real world, we have no way of knowing for sure whether or not the confidence interval obtained from this study has captured this one true value.

We can, however, conduct a simulation study to see how well our methods for constructing a confidence interval work. In this simulation study, we will set the population proportion to a value that is known, construct confidence intervals from several random samples, and see how often the resulting interval actually captures the true value of the population proportion.

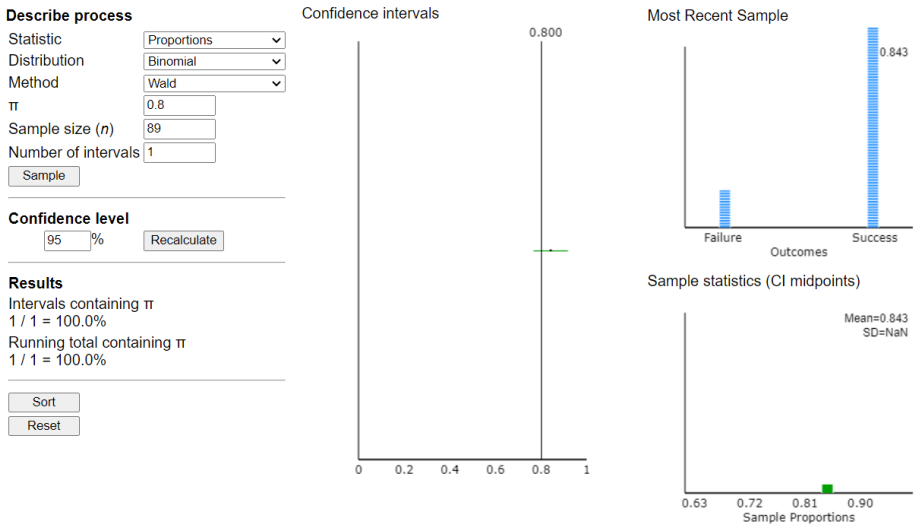
This process is easy to do using an applet available on the web. `Online Simulation Applets > Simulating Confidence Intervals for Population Parameter`

Link: http://www.rossmanchance.com/applets/2021/confsim/ConfSim.html

Carry out the following steps.

1. Let's assume that the true population proportion for our acupuncture example is $\pi = 0.80$. In the applet, set the value of π to be 0.80 and n to be 89. Also, make sure that the method is set to "Wald" for "Proportions."

2. Click on "Sample." Then, click on the interval to see the sample proportion that was obtained. You should see something similar to the following.



In the example shown above, the first random sample resulted in $= 69/89 = .775$. The confidence interval constructed using the Wald method was $0.689 \leq \pi \leq 0.862$. Note that this interval did in fact capture the true population proportion, $\pi = 0.80$.

Identify the following from your first random sample in the simulation study.

- Sample proportion :

- 95% confidence interval:

- Did your confidence interval capture the true value of $\pi = 0.80$?

3. Click "Sample" again. Did you get the same interval? Does this new interval capture the true value of $\pi = 0.80$?

4. To investigate what happens in the long run, we can use the applet to take many more random samples and construct a confidence interval for each. Change the number of "Intervals" from 1 to 198 and click "Sample."

   What percentage of the 200 intervals captures the true value of the population proportion, $\pi = 0.80$? Note that this information is given in the "Running Total."

5. Change the number of "Intervals" from 198 to 200 and click "Sample" over and over again until you've created 1,000 confidence intervals. What percentage of these 1000 intervals captures the true value of the population proportion, $\pi = .80$? This is often referred to as the *coverage rate*.

6. Predict how the coverage rate will change if you were to construct 90% confidence intervals instead of 95% confidence intervals. Then, in the applet, change the confidence level to 90% and click "Recalculate." Continue to press "Sample" until you have constructed 1,000 intervals. What percentage of these 1000 intervals captures the true value of the population proportion, $\pi = 0.80$?

7. Now, recall that in the actual study, you get only one sample and thus one confidence interval. Do you know for sure whether this one interval captures the truth?

Meaning of 95% Confidence

These simulations show us that if take repeated samples from a population and construct 95% confidence intervals each time, in the long run approximately 95% of our intervals will succeed in capturing the true value of the population proportion, π. So, when we collect data and calculate a single 95% confidence interval, we say we are 95% confident (or certain) that this interval captures the truth.