Power analyses

Post hoc power analyses and determination of sample sizes for study design

Today's agenda

1. Reading quiz [2pm section] [4pm section]
2. Type II errors: review and further exploration
3. [lecture/lab] Power analysis
   a. Sample size calculations
   b. Post-hoc power analyses

A thought experiment

Suppose that for the cloud data you'd performed a two-sided test:

$$H_0 : \mu_{\text{seeded}} = \mu_{\text{unseeded}}$$

$$H_A : \mu_{\text{seeded}} \neq \mu_{\text{unseeded}}$$

```
	Welch Two Sample t-test

data:  Rainfall by Treatment
t = 1.9982, df = 33.855, p-value = 0.05377
alternative hypothesis: true difference in means
between group Seeded and group Unseeded is not
equal to 0
95 percent confidence interval:
  -4.764295 559.556603
sample estimates:
  mean in group Seeded mean in group Unseeded
              441.9846               164.5885
```

Almost below the significance threshold but not quite.

> The data **do not provide sufficient evidence to reject** the null hypothesis that seeding has no effect relative to the alternative of an increase or decrease in mean rainfall due to seeding ($T = 1.998$ on 33.86 degrees of freedom, $p = 0.05377$).

The point estimate for the difference is 277.4 acre-feet.

- The test says this observed difference could plausibly be due to sampling variation
- But is it also plausible that our test result is wrong if the difference is real?

Type II error rates

> Recall: a type II error is failing to reject a false null hypothesis.

In the context of two-sample inference a type II error occurs when:

- the true difference is $\delta \neq 0$
- we test and fail to reject $H_0 : \delta \neq 0$

The type II error rate depends on both known and unknown factors:

- [unknown] magnitude of δ
- [unknown] population variability σ
- [known] significance level
- [known] sample sizes

What was the type II error rate for the cloud seeding test?

Simulating type II errors

Summary stats for cloud data:

Treatment	mean	sd	n
Seeded	442	650.8	26
Unseeded	164.6	278.4	26

We can approximate the type II error rate by:

1. simulating datasets with matching summary statistics
2. performing two-sided tests of no difference
3. computing the proportion of fail-to-reject decisions

```r
type2sim(delta = 277, n = 26, sd = 650, alpha = 0.05)
```

```
[1] 0.689
```

⇒ if the true difference were exactly as estimated, our test result would be incorrect nearly 70% of the time!

What would happen to the error rate if…

a. the true difference `delta` were bigger?
b. the significance level `alpha` were smaller?
c. the sample size `n` was larger?
d. the variability of rainfall `sd` were less?

Simulating type II errors

Open the lab and use the simulation function `type2sim` to fill in the table by changing arguments accordingly.

- try a few magnitudes of difference for each scenario
- repeat runs for each setting once or twice to confirm effect

Factor	Change	Effect on error rate
true difference in means	larger	
true difference in means	smaller	
population variability	larger	
population variability	smaller	
sample size	larger	
sample size	smaller	
significance level	larger	
significance level	smaller	

> Based on your explorations, do you think our original test decision was erroneous?

Statistical power

The **power** of a test refers to its true rejection rates across alternatives and is defined as:

$$\beta(\delta) = \underbrace{(1 - \text{type II error rate}_\delta)}_{\text{correct decision rate when null is false}}$$

Power is often interpreted as a detection rate for a specified alternative δ:

- high type II error $\longrightarrow$ low power $\longrightarrow$ low detection rate
- low type II error $\longrightarrow$ high power $\longrightarrow$ high detection rate

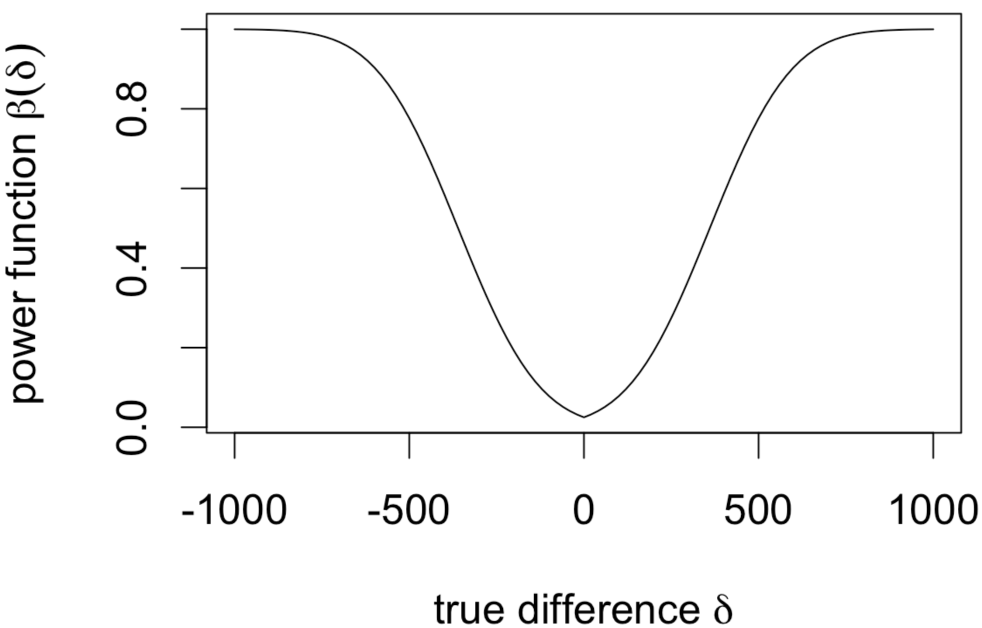
> In general tests have low power for alternatives close to the null value (where “close” is relative to sampling variability).

Theory allows a direct calculation of power, given sample size, significance level, population standard deviation, and population difference in means.

Power curves

> Power is usually construed as a *curve* depending on the true difference.

Power curve for the test exactly as performed with the cloud seeding data:



All other attributes of the test are fixed to approximate the test performed:

- sample size $n = 26$
- significance level $\alpha = 0.05$
- population standard deviation $\sigma = 650$ (larger of two group estimates)

Factors affecting power

> Power depends on all the same factors as type II error rates

Factor	Change	Effect on error rate	Effect on power
true difference in means	larger		
true difference in means	smaller		
population variability	larger		
population variability	smaller		
sample size	larger		
sample size	smaller		
significance level	larger		
significance level	smaller		

Two common power analyses

Post hoc analysis: how much power does the test I conducted have if the true difference is exactly equal to my estimate?

Helps to interpret negative results:

- low power → failure to reject was likely
- high power → failure to reject was not likely

Sample size determination: how much data do I need to collect to detect a difference of δ using a particular test?

Helps avoid two potential issues:

- too little data → study not likely to yield significant results
- too much data → study is too likely to yield significant results

Don't over-interpret post-hoc analyses

Failure to reject using a well-powered test *does not confirm the null hypothesis*.

Post-hoc analysis

> Can we estimate the power of a test we already performed?

Feasible if we assume (a) a population standard deviation and (b) test conditions are met.

For the cloud seeding test:

```
power.t.test(delta = 250, # magnitude of difference
             sd = 650, # largest population SD
             n = 26, # smallest sample size
             sig.level = 0.05,
             type = 'two.sample',
             alternative = 'two.sided')
```

```

     Two-sample t test power calculation 

              n = 26
          delta = 250
             sd = 650
      sig.level = 0.05
          power = 0.2743235
    alternative = two.sided

NOTE: n is number in *each* group
```

For a conservative estimate, use:

- *smallest* of the two sample sizes
- *largest* of the two standard deviations
- *smaller* difference than observed

> $\Longrightarrow$ our test would only reject in favor of a difference of the observed magnitude about 27% of the time

Failure to reject doesn’t strongly rule out the alternative.

Your turn: post-hoc analysis

Consider testing whether body temperature differs by sex.

Summary stats and test result:

sex	mean	sd	n
female	98.66	0.9929	19
male	98.17	0.7876	20

```r
t.test(body.temp ~ sex, data = temps)
```

```
	Welch Two Sample t-test

data:  body.temp by sex
t = 1.7118, df = 34.329, p-value = 0.09595
alternative hypothesis: true difference in means between group female and group male is not equal to 0
95 percent confidence interval:
 -0.09204497  1.07783444
sample estimates:
mean in group female   mean in group male 
            98.65789             98.16500 
```

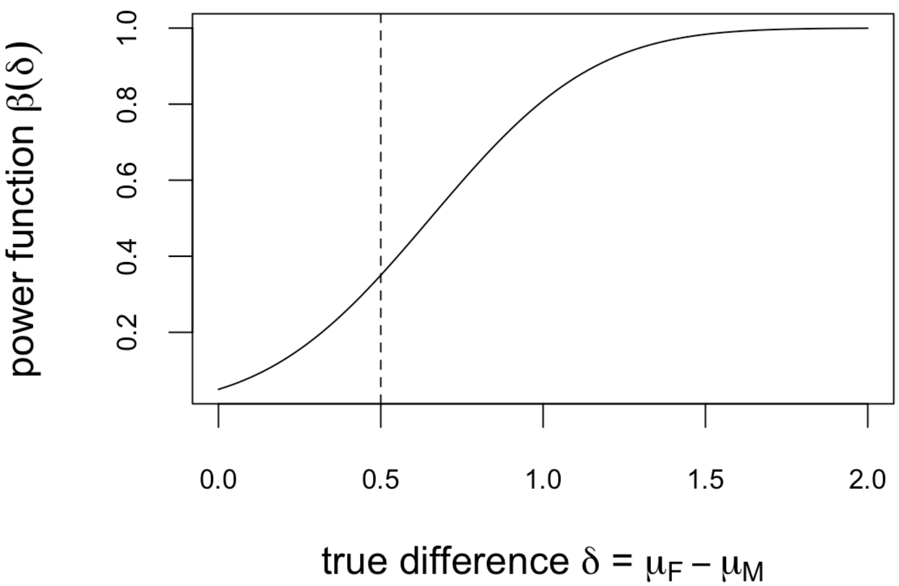
Assume the true difference is actually 0.5 °F. Determine the power of the test above when:

1. Population SD is the smaller of the two groups
2. Population SD is the larger of the two groups
3. A one-sided test is used instead

> Based on your answers, do you think the negative test result rules out the alternative?

Power curve for body temps

Assuming we underestimated the population standard deviation a bit, the power curve for a one-sided test would look like this:



Assumptions:

- n = 19 per group
- $\sigma = 1.2$ per group
- significance level $\alpha = 0.05$
- one-sided test

> Fairly low power for alternatives near the estimated difference (dashed line), so failure to reject doesn't strongly rule out the alternative.

The equal-variance t-test

If it is reasonable to assume the (population) standard deviations are the same in each group, one can gain a bit of power by using a different standard error:

$$SE_{\text{pooled}}(\bar{x} - \bar{y}) = \sqrt{\frac{s_p^2}{n_x} + \frac{s_p^2}{n_y}} \quad \text{where} \quad s_p = \underbrace{\sqrt{\frac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}}}_{\text{weighted average of } s_x^2 \ \& \ s_y^2}$$

In the case of the body temperature data, $s_p = 0.8934$. Check:

- How much power do we gain if we assume a common SD of 0.89?
- Does it change the outcome of the test (add `var.equal = T`)?

> Produces minimal gains and inflates type I error if not warranted, so better avoided unless you have a small sample size

Sample size calculation

> If you were (re)designing the study, how much data should you collect to detect a specified effect size?

To detect a difference of 250 or more due to cloud seeding with power 0.9:

```
power.t.test(power = 0.9, # target power level
             delta = 250, # smallest difference
             sd = 650, # largest population SD
             sig.level = 0.05,
             type = 'two.sample',
             alternative = 'two.sided')
```

```
     Two-sample t test power calculation 

              n = 143.0276
          delta = 250
             sd = 650
      sig.level = 0.05
          power = 0.9
    alternative = two.sided

NOTE: n is number in *each* group
```

For a conservative estimate, use:

- *overestimate* of the larger of the two standard deviations
- *minimum* difference of interest

> $\Longrightarrow$ we need at least 144 observations in each group to detect a difference of 250 or more at least 90% of the time

Your turn: sample size calculation

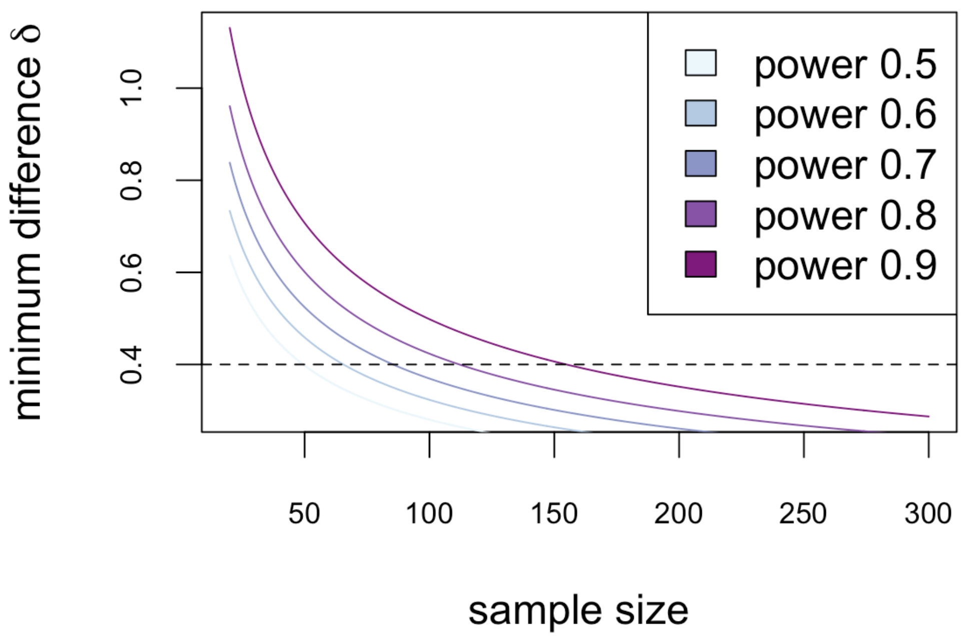
Suppose you are designing a follow-up study and wish to detect a difference of 0.4 °F at least 70% of the time. You know women have slightly higher body temperatures than men on average.

Known direction?	Population SD	Minimum n
No	larger of prior estimates	
No	1.2 times larger than larger of prior estimates	
Yes	larger of prior estimates	
Yes	1.2 times larger than larger of prior estimates	

> If it costs $10 per participant to run the study, what's the best power achievable within a $2K budget for the target detection magnitude?

Power vs. sample size curves

Minimum detectable difference at 5 levels of power as a function of sample size for a one-sided test:



Assumes $\sigma = 1.2$ for a conservative estimate.

The best power achievable within budget for the target detection range is 0.7593159.

- increasing power to 0.8 will require *n* = 112 per group
  - $240 over budget
- increasing power to 0.9 will require *n* = 155 per group
  - $1050 over budget