1. (Normal-$\chi^2$ relationship) Show that if $Z \sim N(0,1)$ then $Z^2 \sim \chi^2_1$.

$Z \sim N(0,1)$ : PDF is $\varphi(z) = (2\pi)^{-1/2} \exp\{-\frac{1}{2} z^2\}$ ; CDF is $\Phi(z)$

if $Y = Z^2$, the CDF is

$$P(Y \le y) = P(Z^2 \le y) = P(-\sqrt{y} \le Z \le \sqrt{y}) = P(-\sqrt{y} < Z \le \sqrt{y})$$
$$= P(Z \le \sqrt{y}) - P(Z \le -\sqrt{y})$$
$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

then the PDF is:

$$f(y) = \frac{d}{dy} P(Y \le y) = \frac{1}{2\sqrt{y}} \varphi(\sqrt{y}) + \frac{1}{2\sqrt{y}} \varphi(-\sqrt{y})$$
$$= \frac{1}{2\sqrt{y}} \left[ \varphi(\sqrt{y}) + \varphi(\sqrt{y}) \right] \qquad \varphi(z) = \varphi(-z)$$
$$= \frac{1}{\sqrt{y}} \varphi(\sqrt{y})$$
$$= y^{-1/2} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} y\}$$
$$= \underbrace{\frac{1}{\Gamma(1/2)\, 2^{1/2}} y^{1/2 - 1} e^{-y/2}}_{\Gamma(1/2,\,2) \text{ PDF}}, \quad y > 0$$

so $Y \sim \Gamma(1/2, 2)$, i.e., $Y \sim \chi^2_1$

score out of 4

(1pt) identify or use the distribution of Z correctly

(1pt) express CDF of Z^2 in terms of CDF of Z (note: students may also use symmetry properties of the standard normal and may not obtain exactly the expression above)

(1pt) differentiate to obtain PDF

(1pt) express in the form of a gamma/chisq

2. (Stochastic ordering) Two random variables $X$ and $Y$ are stochastically ordered if either $F_X(x) \leq F_Y(x)$ or the reverse inequality is true for every $x \in \mathbb{R}$. We say that:

$$X \geq_{st} Y \quad \text{if} \quad F_X(x) \leq F_Y(x) \quad \text{for every} \quad x \in \mathbb{R}$$

Show that the exponential distribution is stochastically ordered in its parameter: that is, if $X \sim$ exponential($\alpha$) and $Y \sim$ exponential($\alpha + c$) where $c > 0$, then $X \geq_{st} Y$. (Use the 'rate' parametrization: $f(x) = \alpha e^{-\alpha x}, x > 0, \alpha > 0$).

the CDFs are $\begin{cases} F_X(x) = 1 - e^{-\alpha x}, & x > 0 \\ F_Y(y) = 1 - e^{-(\alpha+c)y}, & y > 0 \end{cases}$

note for $c > 0$, $\alpha > 0$, and any $x > 0$

$$e^{-\alpha x} > e^{-(\alpha+c)x}$$

$$1 - e^{-\alpha x} < 1 - e^{-(\alpha+c)x}$$

$$F_X(x) < F_Y(x)$$

so since $F_X(x) < F_Y(x)$ for every $x > 0$ and $F_X(x) = F_Y(y) = 0$ for $x \leq 0$,

$$F_X(x) \leq F_Y(x) \quad \forall x \in \mathbb{R}$$

i.e., $X \geq_{ST} Y$

3. Let $X$ be a random variable with moment generating function $m_X(t)$. Define $s_X(t) = \log(m_X(t))$. Show that $s'_X(0) = \mathbb{E}X$ and $s''_X(0) = \text{var}(X)$. Then use this approach to find the mean and variance of a random variable $X$ when:

   i. $X \sim N(\mu, \sigma^2)$
   ii. $X \sim \Gamma(\alpha, \beta)$

note that $m_X(0) = \mathbb{E}e^{0X} = \mathbb{E}(1) = 1$. then:

$$s'_X(0) = \frac{d}{dt} m_X(t)\Big|_{t=0} = \frac{m'_X(0)}{m_X(0)} = m'_X(0) = \mathbb{E}X$$

$$s''_X(0) = \frac{d^2}{d^2t} m_X(t)\Big|_{t=0} = \frac{m''_X(0)}{m_X(0)} - m'_X(0) \cdot \frac{m'_X(0)}{[m_X(0)]^2}$$

score out of 6

(2pt) express derivatives of s_x in terms of derivatives of m_x, or obtain equivalent solution

(1pt) obtain s_x for the gaussian
(1pt) differentiate to obtain mean and variance

(1pt) obtain s_x for the gamma
(1pt) differentiate to obtain mean and variance

$$= m''_X(0) - [m'_X(0)]^2$$

$$= \mathbb{E}X^2 - [\mathbb{E}X]^2$$

$$= \text{var } X$$

i. $X \sim N(\mu, \sigma^2)$ : $m_X(t) = \exp\{\mu t + \frac{1}{2}t^2\sigma^2\}$ so $s_X(t) = \mu t + \frac{1}{2}t^2\sigma^2$

$$s'_X(0) = [\mu + t\sigma^2]_{t=0} = \mu$$

$$s''_X(0) = [\sigma^2]_{t=0} = \sigma^2$$

ii. $X \sim \Gamma(\alpha, \beta)$ : $m_X(t) = (1-\beta t)^{-\alpha}$ so $s_X(t) = -\alpha \log(1-\beta t)$

$$s'_X(0) = \alpha\beta(1-\beta t)^{-1}\Big|_{t=0} = \alpha\beta$$

$$s''_X(0) = \alpha\beta^2(1-\beta t)^{-2}\Big|_{t=0} = \alpha\beta^2$$