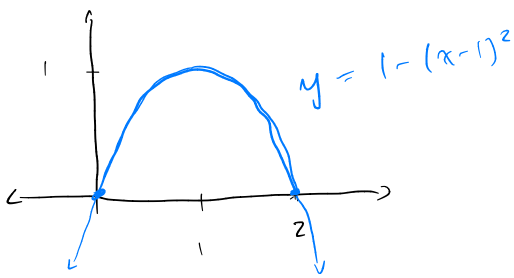
This is a practice test – we will work through it as an in-class activity in small groups.

Name ______________________________

Section ______________________________

Instructions: read each problem carefully and provide solutions in the space below the prompt or on the reverse side of the page. You should provide as much justifictation and detail in calculation as needed to clearly display your thought process, but need not show or justify every step. If you use results from class in your solution, you can simply write, "by a theorem/problem/example from class…". None of the problems require you to perform numerical calculations.If you are unable to obtain a complete solution, a solution sketch may receive partial creditif it reflects a clear understanding of the problem and a well-reasoned approach.Please feel free to ask any clarifying questions about the problems as they arise. Good luck!

1. Let $f(x) = c\left[1-(x-1)^2\right]$. Find the value of c and support set for which f is a PDF.



f is a PDF if:

- $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
- $\int_{\mathbb{R}} f(x)dx = 1$

while there are multiple possible solutions, the most natural is:

choose support to be $(0,2) \Rightarrow f(x) \geq 0$

set $c = \left[\int_0^2 (1-(1-x)^2)\,dx\right]^{-1} \Rightarrow \int f(x)dx = 1$

$$\int_0^2 \left(1-(1-x)^2\right)dx = \int_0^2 dx - \int_0^2 (1-x)^2 dx$$

$$= 2 - \left[-\tfrac{1}{3}(1-x)^3\right]_0^2$$

$$= 2 + \tfrac{1}{3}\left[-1-1\right]$$

$$= 2 - \tfrac{2}{3}$$

$$= \tfrac{4}{3}$$

so $c = \frac{3}{4}$, $S = (0,2)$ ensures a PDF:

$$f(x) = \tfrac{3}{4}\left[1-(1-x)^2\right], \quad 0 < x < 2$$

2. Intuitively, a log transformation should reduce the skewness of a distribution. The skewness of a random variable is defined as the third standardized moment, i.e., the expectation $\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3 - 3\mu_1\sigma^2 - \mu_1^3}{\sigma^3}$, where μ_k denotes the kth moment and σ denotes the square root of the variance, i.e., $\sigma = \sqrt{\mu_2 - \mu_1^2}$. Let $\log(X) \sim N(0, 1)$; show that the skewness of X is larger than the skewness of $\log(X)$.

need μ_1, μ_2, μ_3 for both X, $\log(X)$.

but since for $\log X$, $\mu_1 = \mu_3 = 0$, skewness $= 0$.

so need to find μ_1, μ_2, μ_3 for X and show

$$\mu_3 - 3\mu_1(\mu_2 - \mu_1^2) - \mu_1^3 > 0.$$

since $\log X \sim N(0,1)$, $m_{\log X}(t) = e^{t^2/2}$, but

$$e^{t^2/2} = m_{\log X}(t) = \mathbb{E}\, e^{t \log X} = \mathbb{E}\, X^t$$

so:
$$\mu_1 = e^{1/2}$$
$$\mu_2 = e^{2}$$
$$\mu_3 = e^{9/2}$$

so: $\mu_3 - 3\mu_1(\mu_2 - \mu_1^2) - \mu_1^3 = e^{9/2} - 3e^{5/2} + 3e^{3/2} - e^{3/2} > 0$

3. Let $X \sim \text{exponential}(\beta)$ and $c > 0$ and define

$$Y = \begin{cases} 1 & , X > c \\ 0 & , X \leq c \end{cases}$$

Find the distribution of Y and its mean and variance.

since Y only takes two values, we can find the probabilities directly as:

$$P(Y=1) = P(X > c) = 1 - P(X \leq c) = 1 - (1 - e^{-\beta c}) = e^{-\beta c}$$

$$P(Y=0) = P(X \leq c) = 1 - e^{-\beta c}$$

so $Y \sim \text{bernoulli}(p = e^{-\beta c})$

thus: $\mathbb{E}Y = p = e^{-\beta c}$

$\text{var}\, Y = p(1-p) = e^{-\beta c}(1 - e^{-\beta c})$

Name	PDF/PMF	Support	Parameters	Mean	Variance	MGF
Discrete uniform	$\frac{1}{n}$	$\{a_1, \dots, a_n\}$	none	$\bar{a}$	$\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2$	$\frac{1}{n}\sum_{i=1}^{n} e^{ta_i}$
Bernoulli	$p^x(1-p)^{1-x}$	$\{0, 1\}$	$p \in (0, 1)$	p	$p(1-p)$	$1 - p + pe^t$
Geometric	$(1-p)^x p$	$\mathbb{N}$	$p \in (0, 1)$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$, $t < -\log(1-p)$
Binomial	$\binom{n}{p}p^x(1-p)^{n-x}$	$\{0, 1, \dots, n\}$	$n \in \mathbb{Z}^+$ $p \in (0, 1)$	np	$np(1-p)$	$(1 - p + pe^t)^n$
Negative binomial	$\binom{r+x-1}{x}p^r(1-p)^x$	$\mathbb{N}$	$r \in \mathbb{Z}^+$ $p \in (0, 1)$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$, $t < -\log(1-p)$
Poisson	$\frac{\lambda^x e^{-\lambda}}{x!}$	$\mathbb{N}$	$\lambda > 0$	λ	λ	$\exp\{-\lambda(1-e^t)\}$
Continuous uniform	$\frac{1}{b-a}$	(a, b)	$-\infty < a < b < \infty$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\begin{cases} \frac{e^{ta}-e^{tb}}{t(b-a)} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$
Gaussian	$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$	$\mathbb{R}$	$\mu \in \mathbb{R}, \sigma > 0$	μ	σ^2	$\exp\left\{\mu t + \frac{1}{2}t^2\sigma^2\right\}$
Gamma	$\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}$	$x > 0$	$\alpha > 0, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$	$(1-\beta t)^{-\alpha}$, $t < \frac{1}{\beta}$