Name Solutions

Section

Instructions: read each problem carefully and provide solutionsin the space below the prompt or on the reverse side of the page. You should provide as muchjustifictation and detail in calculation as needed to clearly display yourthought process, but need not show or justify every step. If you use results fromclass in your solution, you can simply write, "by a theorem/problem/example fromclass. . . ' '. None of the problems require you to perform numerical calculations.If you are unable to obtain a complete solution, a solution sketch may receive partial creditif it reflects a clear understanding of the problem and a well-reasoned approach.Please feel free to ask any clarifying questions about the problems as they arise. Good luck!

1. Let X be distributed according to the mass function below. Determine the value of p, sketch the CDF (a log scale is acceptable), and find $\mathbb{E}X$.

x	$P(X = x)$
log(1)	0.4
log(2)	0.1
log(3)	0.2
log(4)	p
log(5)	0.2

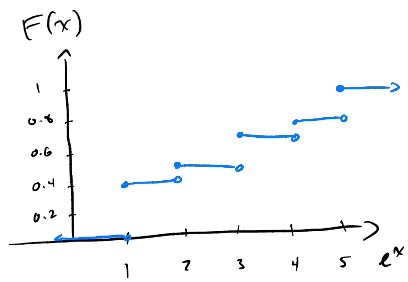
$\sum = 0.9$

(2) since $\sum_x P(X=x) = 1$ & $\sum_x P(X=x) - P(X = \log 4) = 0.9$,

$p = P(X = \log 4) = 0.1$

some reasoning for p = 0.1 is preferred, but not required; partial credit awarded if reasoning is provided but answer is wrong

(4)



CDF should:
- be a step function
- show correct values at each step
- indicate endpoints correctly
- indicate values left of log(1) and right of log(5)

partial credit awarded for sketching the mass function instead, if it is correct

expression should be specific enough to perform the calculation; writing out the exact calculation is preferred

(2) $\mathbb{E}X = \sum_x x\, P(X=x)$

$= \log 1 \cdot 0.4 + \log 2 \cdot 0.1 + \log 3 \cdot 0.2 + \log 4 \cdot 0.1 + \log 5 \cdot 0.2$

≈ 0.7496

not required

2. Consider a random variable $Z \sim f(z)$ where $f(z) = ce^{-|z|}$ where $z \in \mathbb{R}$. Determine the value of c that makes this a density and then find the moment generating function of Z.

   (*Hint*: $\int_{\mathbb{R}} g(|x|)dx = \int_{-\infty}^{0} g(-x)dx + \int_{0}^{\infty} g(x)dx$.)

(4)

$$\int_{\mathbb{R}} e^{-|z|}\,dz = \int_{-\infty}^{0} e^{z}\,dz + \int_{0}^{\infty} e^{-z}\,dz = 2\int_{0}^{\infty} e^{-z}\,dz = 2$$

$$1 = \int_{\mathbb{R}} f(z)\,dz \Rightarrow \boxed{c = \tfrac{1}{2}}$$

credit awarded for:
- seeking to find a normalizing constant by integrating exp(-|z|)
- setting up the integral correctly by applying the hint
- evaluating the integral correctly

(4)

MGF:

$$m_Z(t) = \mathbb{E}e^{tZ} = \tfrac{1}{2}\int_{\mathbb{R}} e^{tz}e^{-|z|}\,dz = \tfrac{1}{2}\int_{-\infty}^{0} e^{z+tz}\,dz + \tfrac{1}{2}\int_{0}^{\infty} e^{tz-z}\,dz \quad (2)$$

$$= \tfrac{1}{2}\int_{-\infty}^{0} e^{z(1+t)}\,dz + \tfrac{1}{2}\int_{0}^{\infty} e^{z(t-1)}\,dz$$

$$= \tfrac{1}{2}\left[\frac{1}{1+t}e^{z(1+t)}\Big|_{-\infty}^{0} + \frac{1}{t-1}e^{z(t-1)}\Big|_{0}^{\infty}\right] \quad (1)$$

$$= \tfrac{1}{2}\left[\frac{1}{1+t} - \frac{1}{t-1}\right], \quad -1 < t < 1$$

$$= \tfrac{1}{2}\left[\frac{t-1-1-t}{(t+1)(t-1)}\right], \quad |t| < 1$$

$$= \tfrac{1}{2}\left[\frac{2}{1-t^2}\right], \quad |t| < 1$$

$$\boxed{= \frac{1}{1-t^2}}, \quad \boxed{|t| < 1} \quad (1)$$

credit awarded for:
- expressing the MGF as an expectation
- setting up the integral
- applying the hint and evaluating
- identifying the constraint on t

(not part of problem)

$$\mathbb{E}Z = m_Z'(0) = 2t(1-t^2)^{-2}\Big|_{t=0} = 0$$

$$\mathbb{E}Z^2 = m_Z''(0) = 2(1-t^2)^{-2} + 4t^2 \cdot 2 \cdot (1-t^2)^{-3}\Big|_{t=0} = 2$$

$$\text{var } Z = 2$$

3. Let $U \sim \text{uniform}(0,1)$. Find the distribution of $U^{\frac{1}{k}}$ and show that $\mathbb{E}\left[U^{\frac{1}{k}}\right] \to 1$ and $\text{var}\left[U^{\frac{1}{k}}\right] \to 0$ as $k \to \infty$.

(4)

let $Y = U^{1/k}$. support $(Y) = (0,1)$.

$$P(Y \le y) = P(U^{1/k} \le y) = P(U \le y^k) = \begin{cases} 0 & y^k \le 0 \\ y^k & 0 < y^k < 1 \\ 1 & y^k \ge 1 \end{cases}$$ (4)

credit awarded for finding the CDF; partial credit awarded for identifying starting points (CDF or other characterization of a U(0, 1) r.v.)

$$\frac{d}{dy} P(Y \le y) = k y^{k-1}, \quad y \in (0,1)$$ (PDF only needed for calculating expectations)

(2) $\mathbb{E} Y = \int_0^1 y \cdot k y^{k-1} \, dy = \int_0^1 k y^k \, dy = \frac{k}{k+1} \to 1$ as $k \to \infty$

(2) $\mathbb{E} Y^2 = \int_0^1 y^2 \cdot k y^{k-1} \, dy = \int_0^1 k y^{k+1} \, dy = \frac{k}{k+2} \to 1$ as $k \to \infty$

(1) as $k \to \infty$, $\mathbb{E} Y \to 1$ AND $\mathbb{E} Y^2 \to 1$, so

$\text{var } Y = \mathbb{E} Y^2 - (\mathbb{E} Y)^2 \to 0$

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alternate solution: $f(u) = 1, \; u \in (0,1)$.

(2) $\mathbb{E}[U^{1/k}] = \int_0^1 u^{1/k} \, du = \frac{1}{1 + 1/k} \to 1$ as $k \to \infty$

(2) $\mathbb{E}[(U^{1/k})^2] = \int_0^1 u^{2/k} \, du = \frac{1}{1 + 2/k} \to 1$ as $k \to \infty$

(1) $\text{var}(U^{1/k}) = \underbrace{\frac{1}{1 + 2/k}}_{\to 1} - \Big[\underbrace{\frac{1}{1 + 1/k}}_{\to 1}\Big]^2 \to 0$ as $k \to \infty$

credit awarded for:
- setting up appropriate expressions for the first and second moments
- applying the variance formula (or other method)
- evaluating limits

partial credit allowable for other approaches. lim[E(X^{1/k})] = E[lim(X^{1/k})] may be used if some justification is provided (e.g., writing as integral)

Name	PDF/PMF	Support	Parameters	Mean	Variance	MGF
Discrete uniform	$\frac{1}{n}$	$\{a_1, \ldots, a_n\}$	none	$\bar{a}$	$\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2$	$\frac{1}{n}\sum_{i=1}^{n} e^{ta_i}$
Bernoulli	$p^x(1-p)^{1-x}$	$\{0, 1\}$	$p \in (0, 1)$	p	$p(1-p)$	$1 - p + pe^t$
Geometric	$(1-p)^x p$	$\mathbb{N}$	$p \in (0, 1)$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$ $t < -\log(1-p)$
Binomial	$\binom{n}{p} p^x (1-p)^{n-x}$	$\{0, 1, \ldots, n\}$	$n \in \mathbb{Z}^+$ $p \in (0, 1)$	np	$np(1-p)$	$\left(1 - p + pe^t\right)^n$
Negative binomial	$\binom{r+x-1}{x} p^r (1-p)^x$	$\mathbb{N}$	$r \in \mathbb{Z}^+$ $p \in (0, 1)$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$ $t < -\log(1-p)$
Poisson	$\frac{\lambda^x e^{-\lambda}}{x!}$	$\mathbb{N}$	$\lambda > 0$	λ	λ	$\exp\left\{-\lambda(1-e^t)\right\}$
Continuous uniform	$\frac{1}{b-a}$	(a, b)	$-\infty < a <$ $b < \infty$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\begin{cases} \frac{e^{ta} - e^{tb}}{t(b-a)} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$
Gaussian	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$	$\mathbb{R}$	$\mu \in \mathbb{R}, \sigma > 0$	μ	σ^2	$\exp\left\{\mu t + \frac{1}{2}t^2\sigma^2\right\}$
Gamma	$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$	$x > 0$	$\alpha > 0, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$