

# Directional Wavelet Bases Construction on Dyadic Quincunx Lattice

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## *Abstract*

We construct directional wavelet systems that will enable building efficient signal representation schemes with good direction selectivity. In particular, we focus on wavelet bases on a dyadic quincunx lattice; dilated quincunx downsampling is used to construct orthonormal and bi-orthogonal bases and standard dyadic downsampling for low-redundancy frames. We show that the supports of both orthonormal and bi-orthogonal wavelets in our framework are discontinuous in the frequency domain. However, this irregularity constraint can be avoided in frames, even with redundancy factor less than 2.

## 1 Introduction

In image compression and analysis, 2D tensor wavelet schemes are widely used. Despite the time-frequency localization inherited from 1D wavelet, 2D tensor wavelets suffer from poor orientation selectivity: only horizontal or vertical edges are well represented by tensor wavelets. To obtain better representation of 2D images, several directional wavelet schemes have been proposed and applied to image processing, such as directional wavelet filterbanks (DFB) and various extensions.

Conventional DFB [1] divides the frequency domain into eight equi-angular pairs of triangles; such schemes can be critically downsampled (maximally decimated) with perfect reconstruction (PR), but they typically do not have a multi-resolution structure. Different approaches have been proposed to generalize DFB to multi-resolution systems, including non-uniform DFB (nuDFB), contourlets, curvelets, shearlets and dual-tree wavelets. nuDFB is introduced in [2] based on multi-resolution analysis (MRA), where at each level of decomposition the square frequency domain is divided into a high frequency outer ring and a central low frequency domain. For nuDFB, the high frequency ring is primarily divided further into six equi-angular pairs of trapezoids and the central low frequency square is kept intact for division in the next level of decomposition, see Fig. 1. The nuDFB filters are solved by optimization which provides a

non-unique near orthogonal or bi-orthogonal solution depending on the initialization. Contourlets [3] combine the Laplacian pyramid scheme with DFB which has PR but with redundancy  $4/3$  inherited from the Laplacian pyramid. Shearlet [4, 5] and curvelet [6] systems construct a multi-resolution partition of the frequency domain by applying shear or rotation operators to a generator function in each level of frequency decomposition. Available shearlet and curvelet implementations have redundancy at least 4; moreover, the factor may grow with the number of directions in the decomposition level. Dual-tree wavelets [7] are linear combinations of 2D tensor wavelets (corresponding to multi-resolution systems) that constitute an approximate Hilbert transform pair, where the high frequency ring is divided into pairs of squares of different directional preference.

However, none of these multi-resolution schemes is PR, critically downsampled and regularized (localized in both time and frequency). In the framework of nuDFB ([2]), it was shown by Durand [8] that it is impossible to construct orthonormal filters localized in frequency without discontinuity in their frequency support, or – equivalently – regularized filters without aliasing. His construction of directional filters uses compositions of 2-band filters associated to quincunx lattice, similar to that of uniform DFB in [2]; as pointed out in [2] the overall composed filters are not alias-free. It is not clear whether Durand’s argument also precludes the existence of a regularized wavelet system, if one slightly weakens the set of conditions.

To study this question, we consider multi-resolution directional wavelets corresponding to the same partition of frequency domain as nuDFB and build a framework to analyze the equivalent conditions of PR for critically downsampled as well as more general redundant schemes. In our previous work [9], we show that in MRA, PR is equivalent to an identity condition and shift-cancellation condition closely related to the frequency support of filters and their downsampling scheme. Based on these two conditions, we rederived Durand’s discontinuity result of orthonormal schemes; we also show that a slight relaxation of conditions allows frames with redundancy less than 2 that circumvent the regularity limitation. Furthermore, we have an explicit approach to construct such regularized directional wavelet frames by smoothing the Fourier transform of the irregular directional wavelets. The main contribution of this paper is that we extend our previous work and show that the same obstruction to regularity as in orthonormal schemes exists in bi-orthogonal schemes. Different from our previous approach in the orthonormal case, our analysis of bi-orthogonal schemes is based on a numerical algorithm introduced by Cohen et al in [10] for constructing compactly supported symmetric wavelet bases on a hexagonal lattice. We extend and adapt this algorithm to our bi-orthogonal setting.

The paper is organized as follows. In section 2, we set up the framework of a dyadic MRA with dilated quincunx downsampling. In section 3, we review the irregularity of orthonormal schemes in [9]. In particular, we derive two conditions, *identity summation* and *shift cancellation*, equivalent to perfect reconstruction in this MRA with critical downsampling. These lead to the classification of *regular/singular* boundaries of the frequency partition and a *relaxed shift-cancellation* condition for low-redundancy MRA frame allows better reg-

ularity of the directional wavelets. In section 4, we show the irregularity for bi-orthogonal schemes. We first review the wavelet construction algorithm from [10] which solves linear systems generated from regularity constraints. Next, we extend the algorithm to our framework and show that the resulting linear system does not have feasible solution satisfying all regularity constraints, especially continuity of Fourier transforms of wavelet filters. Finally, we conclude our results and discuss future work in section 7.

## 2 Framework Setup

We summarize 2D-MRA systems and the relation between frequency domain partition and sub-lattice of  $\mathbb{Z}^2$  with critical downsampling following [9].

### 2.1 Notation

Throughout this paper, we use upper case bold font for matrices ( $\mathbf{A}, \mathbf{B}$ ), lower case bold font for vectors ( $\mathbf{a}, \mathbf{b}$ ) and upper case italics for subsets ( $C_1, C_2$ ) of the frequency domain. We denote the conjugate transpose of a matrix  $\mathbf{A}$  by  $\mathbf{A}^*$ . We use this notation for vectors: for a  $d$ -dimensional vector  $\mathbf{a}$ ,  $\mathbf{a}^*$  denotes the corresponding conjugate “row vector”. For matrices and vectors, the indexing of rows and columns starts with zero.

### 2.2 Multi-resolution analysis and sub-lattice sampling

In an MRA, given a scaling function  $\phi \in L^2(\mathbb{R}^2)$ , s.t.  $\|\phi\|_2 = 1$ , the base approximation space is defined as  $V_0 = \overline{\text{span}\{\phi_{0,\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^2}}$ , where  $\phi_{0,\mathbf{k}} = \phi(\mathbf{x} - \mathbf{k})$ . If  $\langle \phi_{0,\mathbf{k}}, \phi_{0,\mathbf{k}'} \rangle = \delta_{\mathbf{k},\mathbf{k}'}$ , then  $\{\phi_{0,\mathbf{k}}\}$  is an orthonormal basis of  $V_0$ . In addition,  $\phi$  is associated with a scaling matrix  $\mathbf{D} \in \mathbb{Z}^{2 \times 2}$ , s.t. the dilated scaling function  $\phi_1(\mathbf{x}) = |\mathbf{D}|^{-1/2} \phi(\mathbf{D}^{-1}\mathbf{x})$  is a linear combination of  $\phi_{0,\mathbf{k}}$ . Equivalently,  $\exists m_0(\boldsymbol{\omega}) = m_0(\omega_1, \omega_2)$ ,  $2\pi$ -periodic in  $\omega_1, \omega_2$ , s.t. in the frequency domain

$$\widehat{\phi}(\mathbf{D}^T \boldsymbol{\omega}) = m_0(\boldsymbol{\omega}) \widehat{\phi}(\boldsymbol{\omega}). \quad (1)$$

(1) implies that

$$\widehat{\phi}(\boldsymbol{\omega}) = (2\pi)^{-1} \prod_{k=1}^{\infty} m_0(\mathbf{D}^{-k} \boldsymbol{\omega}), \quad (2)$$

where we have implicitly assumed that  $\phi \in L^1(\mathbb{R}^2)$  and  $\int \phi dx = 1$  (which follows from the other constraints if  $\phi$  has same decay at  $\infty$ ).

Let  $\phi_{l,\mathbf{k}} = \phi(D^{-l}\mathbf{x} - \mathbf{k})$  and  $V_l = \overline{\text{span}\{\phi_{l,\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^2\}}$ ,  $l \in \mathbb{Z}$  be the nested approximation spaces. Define  $W_l$  as the orthogonal complement of  $V_l$  with respect to  $V_{l-1}$  in MRA. Suppose there are  $J$  wavelet functions  $\psi^j \in L^2(\mathbb{R}^2)$ ,  $1 \leq j \leq J$ , and  $\mathbf{Q} \in \mathbb{Z}^{2 \times 2}$ , s.t.

$$W_1 = \bigcup_{j=1}^J W_1^j = \bigcup_{j=1}^J \overline{\text{span}\{\psi_{1,\mathbf{k}}^j; \mathbf{k} \in \mathbf{Q}\mathbb{Z}^2\}} = \bigcup_{j=1}^J \overline{\text{span}\{\psi^j(\mathbf{D}^{-1}\mathbf{x} - \mathbf{k}); \mathbf{k} \in \mathbf{Q}\mathbb{Z}^2\}},$$

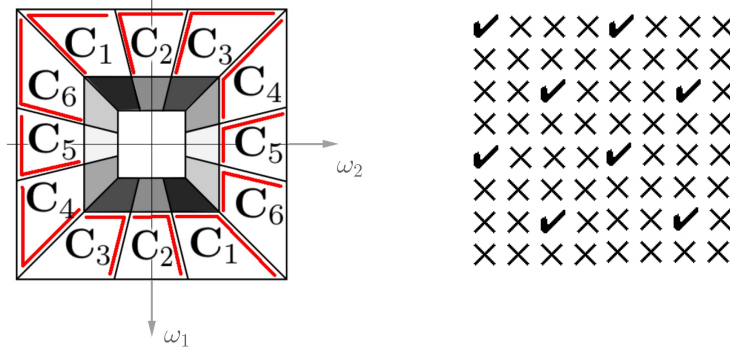


Figure 1: Left: partition of  $S_0$  and boundary assignment of  $C_j$ ,  $j = 1, \dots, 6$  (each  $C_j$  has boundaries indicated by red line segments), Right: dilated quincunx sub-lattice.

an  $L$ -level multi-resolution system with base space  $V_0$  is then spanned by

$$V_L \oplus \bigoplus_{l=1}^L \left( \bigcup_{j=1}^J W_l^j \right) = \{ \phi_{L,\mathbf{k}}, \psi_{l,\mathbf{k}'}^j, 1 \leq l \leq L, \mathbf{k} \in \mathbb{Z}^2, \mathbf{k}' \in \mathbf{Q}\mathbb{Z}^2, 1 \leq j \leq J \}. \quad (3)$$

In particular, we set  $\mathbf{D} = \mathbf{D}_2 \doteq \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  and  $\mathbf{Q} \doteq \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ . As  $W_1 \subset V_0$ , each rescaled wavelet  $\psi^j(\mathbf{D}^{-1}\cdot)$  is also a linear combination of  $\phi_{0,\mathbf{k}}$ , so that  $\exists m_j$  analogous to  $m_0$  satisfying

$$\hat{\psi}^j(\mathbf{D}^T \boldsymbol{\omega}) = m_j(\boldsymbol{\omega}) \hat{\phi}(\boldsymbol{\omega}), \quad 1 \leq j \leq J. \quad (4)$$

In this specific construction of MRA, the corresponding subsampling matrix of  $\phi_{1,\mathbf{k}}$  is  $\mathbf{D}$  and that of  $\psi_{1,\mathbf{k}}^j$  is  $\mathbf{QD}$ , the dilated quincunx subsample (see the right panel in Fig.1), as in [8].

### 2.3 Frequency domain partition and critical downsampling

Consider the canonical frequency square,  $S_0 = [-\pi, \pi) \times [-\pi, \pi)$  associated with the lattice  $\mathcal{L} = \mathbb{Z}^2$ . For  $L = 1$ , the 1-level decomposition (3) together with (1) and (4) implies that the union of the support of  $m_j$ ,  $0 \leq j \leq J$  covers  $S_0$ . Furthermore,  $\exists C_j \subset \text{supp}(m_j)$ ,  $0 \leq j \leq J$ , such that they form a partition of  $S_0$ ; we will choose  $m_j$  to be “mainly” supported on  $C_j$  (this will become more explicit in Section 5). To build an orthonormal basis with good directional selectivity, we choose the partition of  $S_0$  shown in the left of Fig.1, which is the same for Example B in [8] and the least redundant shearlet system. In this partition,  $S_0$  is divided into a central square  $C_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1} S_0$  and a ring: the ring is further cut into six pairs of directional trapezoids  $C_j$  by lines passing through the origin with slopes  $\pm 1, \pm 3$  and  $\pm \frac{1}{3}$ . The central square  $C_0$  can be further partitioned in the same way to obtain a two-level multi-resolution system, as shown in Fig.1.

Here  $J = 6$  and  $|\mathbf{D}|^{-1} + J|\mathbf{QD}|^{-1} = 1/4 + 6/(2 \cdot 4) = 1$ , hence the corresponding MRA generated by (3) achieves critical downsampling([8]). In addition,

let  $\pi_0 = (0, 0)$ ,  $\pi_1 = (\pi/2, \pi/2)$ ,  $\pi_2 = (\pi, 0)$ ,  $\pi_3 = (-\pi/2, \pi/2)$ ,  $\pi_4 = (0, \pi)$ ,  $\pi_5 = (\pi/2, -\pi/2)$ ,  $\pi_6 = (\pi, \pi)$ ,  $\pi_7 = (-\pi/2, -\pi/2)$ , then each piece  $C_j$  together with its shifts form a tiling of  $S_0$ , i.e.

$$S_0 = \bigcup_{\pi \in \Gamma_1} (C_j + \pi) = \bigcup_{\pi \in \Gamma_0} (C_0 + \pi), \quad j = 1, \dots, 6 \quad (5)$$

where  $\Gamma_0 = \{\pi_i, i = 0, 2, 4, 6\}$  and  $\Gamma_1 = \{\pi_i, i = 0, 1, \dots, 7\}$ . Alternatively, we say that  $\{C_j, j = 0, \dots, J\}$  is an *admissible* partition of  $S_0$ .

### 3 Orthonormal Bases

In this section, we discuss the conditions on  $m_j$  such that the corresponding MRA forms an orthonormal bases.

We begin with the two key conditions, i.e. *identity summation* and *shift cancellation*, on  $m_j$  such that the system (3) is perfect-reconstruction (PR) or equivalently a Parseval frame in MRA.

#### 3.1 orthonormal conditions on $m_j$

In MRA, (3) is PR if  $\forall f \in L_2(\mathbb{R}^2)$ ,

$$\sum_k \langle f, \phi_{0,k} \rangle \phi_{0,k} = \sum_k \langle f, \phi_{1,k} \rangle \phi_{1,k} + \sum_j \sum_k \langle f, \psi_{1,k}^j \rangle \psi_{1,k}^j. \quad (6)$$

Using (1) and (4) together with the admissibility of the frequency partition (5), condition (6) on  $\phi$  and  $\psi^j$  yields:

**Theorem 1.** *The perfect reconstruction condition holds for (3) iff the following two conditions hold*

$$|m_0(\omega)|^2 + \sum_{j=1}^6 |m_j(\omega)|^2 = 1 \quad (7)$$

$$\begin{cases} \sum_{j=0}^6 m_j(\omega) \overline{m_j(\omega + \pi)} = 0, & \pi \in \Gamma_0 \setminus \{0\} \\ \sum_{j=1}^6 m_j(\omega) \overline{m_j(\omega + \pi)} = 0, & \pi \in \Gamma_1 \setminus \Gamma_0 \end{cases} \quad (8)$$

Theorem 1 is a corollary of Prop. 1 and Prop. 2 in [8]. We give an alternate proof in Appendix A. In Theorem 1, Eq. (7) is the *identity summation* condition, guaranteeing conservation of  $l_2$  energy; Eq. (8) is the *shift cancellation* condition such that aliasing is canceled correctly in reconstruction from wavelet coefficients.

For (3) to be an orthonormal basis,  $\{\phi_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^2}$  need to be an orthonormal basis, which is determined by  $m_0$  in (2). In 1D MRA, Cohen's theorem in [11] provides a necessary and sufficient condition on  $m_0$  such that (3) is an orthonormal basis. This theorem generalizes to 2D as proved in e.g. [9], as follows.

**Theorem 2.** *Assume that  $m_0$  is a trigonometric polynomial with  $m_0(0) = 1$ , and define  $\hat{\phi}(\omega)$  as in (2). If  $\phi(\cdot - \mathbf{k}), \mathbf{k} \in \mathbb{Z}^2$  are orthonormal, then  $\exists K$  containing a neighborhood of 0,*



Figure 2: Boundary classification, singular (red) and regular (yellow)

*s.t.  $\forall \omega \in S_0$ ,  $\omega + 2\pi \mathbf{n} \in K$  for some  $\mathbf{n} \in \mathbb{Z}^2$ , and  $\inf_{k>0, \omega \in K} |m_0(\mathbf{D}_2^{-k} \omega)| > 0$ . Further, if  $\sum_{\pi \in \Gamma_0} |m_0(\omega + \pi)|^2 = 1$ , then the inverse is true.*

### 3.2 Design and Boundary Regularity of $m_j$

We begin with the Shannon-type wavelet construction. Let each  $m_j$  be an indicator function,  $m_j = \mathbb{1}_{C_j}$ ,  $0 \leq j \leq 6$ , and we use the boundary assignment of  $C_j$  in Fig.1, then the identity summation follows from the partition, and the shift cancellation follows from (5). Applying Theorem 2 to  $m_0$ , we check that the Shannon-type wavelets generated from these  $m$ -functions form an orthonormal basis.

Because of the discontinuity at  $\partial C_j$ , the boundaries of the  $C_j$ , the  $m_j$  are not smooth, and hence the corresponding wavelets are not spatially localized. The  $m_j$  can be regularized by smoothing at the  $\partial C_j$ . However, as shown in Proposition 3 in [8], it is not possible to smooth the behavior of the  $m_j$  at *all* the boundaries with discontinuity if the  $m_j$  have to satisfy the perfect reconstruction condition. In [9], the  $\partial C_j$  are segmented into *singular* and *regular* pieces. On regular boundaries, pairs of  $(m_j, m_{j'})$  share a boundary and can both be smoothed in a coherent way such that all the constraints in Theorem 2 remain satisfied. The singular pieces are boundaries for just one  $m_j$ , which can then not be smoothed without violating the shift cancellation condition. Fig. 2 shows the boundary classification, where the corners of  $S_0$  and  $C_0$  are singular, hence  $m_0$  and the  $m_j$ 's in two diagonal directions of an orthonormal bases are discontinuous there. A mechanism of constructing orthonormal bases by smoothing  $m_j$  on regular boundaries from indicator functions is provided in [9].

### 3.3 Extension to low-redundancy tight frame

The irregularity of orthonormal bases is overcome in the following low-redundancy tight frame construction,

$$\{\phi_{L,\mathbf{k}}, \psi_{l,\mathbf{k}'}^j, 1 \leq l \leq L, \mathbf{k}, \mathbf{k}' \in \mathbb{Z}^2, 1 \leq j \leq J\}. \quad (9)$$

where all wavelet coefficients are sub-sampled on dyadic sub-lattice and the redundancy of any  $L$ -level MRA frame doesn't exceed  $\frac{J/|D|}{1-1/|D|} = \frac{6/4}{1-1/4} = 2$ . Similar to Theorem 2, we have

**Theorem 3.** *The perfect reconstruction condition holds for (9) iff the following both hold*

$$|m_0(\boldsymbol{\omega})|^2 + \sum_{j=1}^6 |m_j(\boldsymbol{\omega})|^2 = 1 \quad (10)$$

$$\sum_{j=0}^6 m_j(\boldsymbol{\omega}) \overline{m_j}(\boldsymbol{\omega} + \boldsymbol{\pi}) = 0, \quad \boldsymbol{\pi} \in \Gamma_0 \setminus \{\mathbf{0}\} \quad (11)$$

Theorem 3 can be proved analogously to Theorem 1, but with fewer shift cancellation constraints. Following the same analysis of boundary regularity as before, we show in [9] that all boundaries are regular and can be smoothed properly. We obtained directional wavelets with much better spatial and frequency localization than those constructed by Durand in [8].

So far, we have considered two directional wavelet MRA systems (3) and (9) such that the directional wavelets characterize 2D signals in six equi-angled directions. Furthermore, these wavelets are well localized in the frequency domain such that  $\text{supp}(m_j)$  is convex and  $\exists \epsilon$  s.t.

$$\sup_{\boldsymbol{\omega}' \in \text{supp}(m_j)} \inf_{\boldsymbol{\omega} \in C_j} \|\boldsymbol{\omega}' - \boldsymbol{\omega}\| < \epsilon, \quad 0 \leq j \leq 6. \quad (12)$$

This desirable condition is hard to obtain by multi-directional filter bank assembly of several elementary filter banks.

In the next section, we analyze the more general case of directional bi-orthogonal filters constructed with respect to the same frequency partition.

## 4 Bi-orthogonal Bases

In this section, we analyze bi-orthogonal bases in the following form of MRA,

$$\{\phi_{L,\mathbf{k}}, \widetilde{\phi}_{L,\mathbf{k}}, \psi_{l,\mathbf{k}'}^j, \widetilde{\psi}_{l,\mathbf{k}'}^j, 1 \leq l \leq L, \mathbf{k} \in \mathbb{Z}^2, \mathbf{k}' \in \mathbb{Q}\mathbb{Z}^2, 1 \leq j \leq J\}, \quad (13)$$

where  $\phi$  and  $\psi^j$  satisfy (1) and (4), as well as  $\widetilde{\phi}$  and  $\widetilde{\psi}^j$ , respectively,

$$\widehat{\widetilde{\phi}}(\mathbf{D}^T \boldsymbol{\omega}) = \widetilde{m}_0(\boldsymbol{\omega}) \widehat{\phi}(\boldsymbol{\omega}), \quad \widehat{\widetilde{\psi}^j}(\mathbf{D}^T \boldsymbol{\omega}) = \widetilde{m}_j(\boldsymbol{\omega}) \widehat{\phi}(\boldsymbol{\omega}). \quad (14)$$

For such bi-orthogonal bases, we have the similar identity summation and shift cancellation condition to those in Theorem 1.

**Theorem 4.** *The perfect reconstruction iff the following two conditions hold*

$$m_0(\boldsymbol{\omega}) \widetilde{m}_0(\boldsymbol{\omega}) + \sum_{j=1}^6 m_j(\boldsymbol{\omega}) \widetilde{m}_j(\boldsymbol{\omega}) = 1 \quad (15)$$

$$\begin{cases} \sum_{j=0}^6 m_j(\boldsymbol{\omega}) \widetilde{m}_j(\boldsymbol{\omega} + \boldsymbol{\pi}) = 0, & \boldsymbol{\pi} \in \Gamma_0 \setminus \{\mathbf{0}\} \\ \sum_{j=1}^6 m_j(\boldsymbol{\omega}) \widetilde{m}_j(\boldsymbol{\omega} + \boldsymbol{\pi}) = 0, & \boldsymbol{\pi} \in \Gamma_1 \setminus \Gamma_0 \end{cases} \quad (16)$$

The conditions (15) and (16) can be combined into a linear system as follows,

$$\begin{bmatrix} \widetilde{m}_0(\omega) & \widetilde{m}_1(\omega) & \dots & \widetilde{m}_6(\omega) \\ 0 & \widetilde{m}_1(\omega + \pi_1) & \dots & \widetilde{m}_6(\omega + \pi_1) \\ \widetilde{m}_0(\omega + \pi_2) & \widetilde{m}_1(\omega + \pi_2) & \dots & \widetilde{m}_6(\omega + \pi_2) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \widetilde{m}_1(\omega + \pi_7) & \dots & \widetilde{m}_6(\omega + \pi_7) \end{bmatrix} \begin{bmatrix} m_0(\omega) \\ m_1(\omega) \\ m_2(\omega) \\ \vdots \\ m_6(\omega) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (17)$$

In addition, we have the following analogue of Theorem 2.

**Theorem 5.** Assume that  $m_0, \widetilde{m}_0$  are trigonometric polynomials with  $m_0(0) = \widetilde{m}_0(0) = 1$ , which generate  $\phi, \widetilde{\phi}$  respectively.

If  $\phi(\cdot - \mathbf{k}), \widetilde{\phi}(\cdot - \mathbf{k}), \mathbf{k} \in \mathbb{Z}^2$  are bi-orthogonal, then  $\exists K$  containing a neighborhood of 0, s.t.  $\forall \omega \in S_0, \omega + 2\pi \mathbf{n} \in K$  for some  $\mathbf{n} \in \mathbb{Z}^2$ , and  $\inf_{k>0, \omega \in K} |m_0(\mathbf{D}_2^{-k} \omega)| > 0, \inf_{k>0, \omega \in K} |\widetilde{m}_0(\mathbf{D}_2^{-k} \omega)| > 0$ . Further, if  $\sum_{\pi \in \Gamma_0} m_0(\omega + \pi) \widetilde{m}_0(\omega + \pi) = 1$ , then the inverse is true.

By Theorem 5,  $m_0$  and  $\widetilde{m}_0$  need to satisfy the following identity constraint for the MRA (13) to be bi-orthogonal,

$$m_0 \widetilde{m}_0(\omega) + m_0 \widetilde{m}_0(\omega + \pi_2) + m_0 \widetilde{m}_0(\omega + \pi_4) + m_0 \widetilde{m}_0(\omega + \pi_6) = 1. \quad (18)$$

In sum, the construction of a bi-orthogonal basis (13) is equivalent to find feasible solutions of (17) with constraint (18)<sup>1</sup>. To solve this, we use the same approach as in [10], which constructs compactly supported symmetric bi-orthogonal filters on a hexagon lattice. We next review the main scheme in [10] and adapt it to our setup of directional wavelet filter.

#### 4.1 Summary of Cohen et al's construction

We summarize the main setup and the approach in [10]. Consider a bi-orthogonal scheme consists of 3 high-pass filters  $m_1, m_2$  and  $m_3$  and a low-pass filter  $m_0$  together with their bi-orthogonal duals  $\widetilde{m}_j$ , s.t.  $m_0$  is  $\frac{2\pi}{3}$ -rotation invariant and  $m_1, m_2, m_3$  are  $\frac{2\pi}{3}$ -rotation co-variant.

This bi-orthogonal scheme satisfies the following linear system ( Lemma 2.2.2 in [10] )

$$\begin{bmatrix} \widetilde{m}_0(\omega) & \widetilde{m}_1(\omega) & \widetilde{m}_2(\omega) & \widetilde{m}_3(\omega) \\ \widetilde{m}_0(\omega + \nu_1) & \widetilde{m}_1(\omega + \nu_1) & \widetilde{m}_2(\omega + \nu_1) & \widetilde{m}_3(\omega + \nu_1) \\ \widetilde{m}_0(\omega + \nu_2) & \widetilde{m}_1(\omega + \nu_2) & \widetilde{m}_2(\omega + \nu_2) & \widetilde{m}_3(\omega + \nu_2) \\ \widetilde{m}_0(\omega + \nu_3) & \widetilde{m}_1(\omega + \nu_3) & \widetilde{m}_2(\omega + \nu_3) & \widetilde{m}_3(\omega + \nu_3) \end{bmatrix} \begin{bmatrix} m_0(\omega) \\ m_1(\omega) \\ m_2(\omega) \\ m_3(\omega) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (19)$$

where  $\nu_i = \pi_i, i = 1, 2, 3$ . Let  $\widetilde{\mathbf{M}}(\omega) \in \mathbb{C}^{4 \times 4}$  be the matrix with entries  $\widetilde{m}_j(\omega)$  and  $\mathbf{m}(\omega) \in \mathbb{C}^4$  be the vector consisting of entries  $m_j(\omega)$  in (19), then (19) can be written as

$$\widetilde{\mathbf{M}}(\omega) \mathbf{m}(\omega) = [1, 0, 0, 0]^\top.$$

<sup>1</sup>In fact, as long as (17) has a unique solution for  $m_j$  given fixed  $\widetilde{m}_j, j = 0, \dots, 6$ , (18) always holds.



Imposing that  $\widetilde{m}_1(\omega)$ ,  $\widetilde{m}_2(\omega)$  and  $\widetilde{m}_3(\omega)$  are determined by symmetry, and invoking Lemma 2.2.2 in [10] leads to

$$\begin{aligned} m_0(\omega) &= D^{-1} \begin{vmatrix} \widetilde{m}_1(\omega + \nu_1) & \widetilde{m}_2(\omega + \nu_1) & \widetilde{m}_3(\omega + \nu_1) \\ \widetilde{m}_1(\omega + \nu_2) & \widetilde{m}_2(\omega + \nu_2) & \widetilde{m}_3(\omega + \nu_2) \\ \widetilde{m}_1(\omega + \nu_3) & \widetilde{m}_2(\omega + \nu_3) & \widetilde{m}_3(\omega + \nu_3) \end{vmatrix} \\ &= D^{-1} \widetilde{\mathbf{M}}_{0,0}(\omega), \end{aligned} \quad (20)$$

where  $\widetilde{\mathbf{M}}_{0,0}(\omega)$  is the minor with respect to  $\widetilde{m}_0(\omega)$  of  $\widetilde{\mathbf{M}}(\omega)$  and  $D \equiv \det(\widetilde{\mathbf{M}}(\omega)) \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$  does not depend on  $\omega$  in [10], due to symmetry. If  $\widetilde{m}_0$  is solved, then  $m_1, m_2$  and  $m_3$  are obtained by solving the linear system (19). In particular,  $\widetilde{m}_0(\omega)$  need to satisfy

$$m_0 \widetilde{m}_0(\omega) + m_0 \widetilde{m}_0(\omega + \nu_1) + m_0 \widetilde{m}_0(\omega + \nu_2) + m_0 \widetilde{m}_0(\omega + \nu_3) = 1, \quad (21)$$

which can be obtained by expanding  $\det(\widetilde{\mathbf{M}}(\omega))$  with respect to the first column. According to Lemma 3.2.1 in [10], based on *Hilbert's Nullstellensatz*, (21) has a solution if and only if there does not exist  $(z_1, z_2) \in (\mathbb{C}^*)^2$ ,  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  s.t.  $(\pm z_1, \pm z_2)$  are all vanishing points of the  $z$ -transform of  $m_0$ .

In general, there is no efficient algorithm to solve *Hilbert's Nullstellensatz*, and how (21) is solved exactly is not mentioned in [10]. In the next section, we propose an optimization problem that solves  $\widetilde{m}_0$ , where (20)(which is the same as (18)) serves as a linear constraint and the objective function imposes regularity on  $\widetilde{m}_0$ .

## 5 Adaptation to dilated quincunx scheme

Following the same approach of Cohen et al, we first design  $\widetilde{m}_j(\omega)$ ,  $j = 1, \dots, 6$ , on the canonical frequency square  $S_0 = [-\pi, \pi) \times [-\pi, \pi)$ , then solve for  $m_0, \widetilde{m}_0$  and  $m_j$  on  $S_0$  in order with respect to (17) and (18).

Since (17) takes the same form as (19), for simplicity, in the rest of this paper, we adapt the matrix and vector notations  $\widetilde{\mathbf{M}}(\omega)$ ,  $\mathbf{m}(\omega)$  that helped to simplify (19). Accordingly, we rewrite (17) as

$$\widetilde{\mathbf{M}}(\omega) \mathbf{m}(\omega) = [1, 0, 0, 0, 0, 0, 0]^\top,$$

where  $\widetilde{\mathbf{M}}(\omega) \in \mathbb{C}^{8 \times 7}$  and  $\mathbf{m}(\omega) \in \mathbb{C}^7$ . In addition, let  $\mathbf{b}_k \in \mathbb{R}^8$ ,  $0 \leq k \leq 7$ , whose only non-zero entry is  $\mathbf{b}_k[k] = 1$ , where the indexing starts with zero. Note that  $\widetilde{\mathbf{M}}(\omega) \mathbf{m}(\omega) = \mathbf{b}_0 \in \mathbb{R}^8$  is over-determined; it has a unique solution if and only if

(5.i)  $\widetilde{\mathbf{M}}(\omega)$  is full rank,

(5.ii)  $[\widetilde{\mathbf{M}}(\omega), \mathbf{b}_0]$  is singular,

where we use  $\llbracket$  to concatenate  $\widetilde{\mathbf{M}}(\omega)$  and  $\mathbf{b}_0$  into a  $8 \times 8$  matrix. The matrix  $\widetilde{\mathbf{M}}(\omega)$  is structured such that each row is associated with a shift  $\pi_i$ ,  $i = 0, \dots, 7$  and each column is associated with a dual function  $\widetilde{m}_j(\omega)$ ,  $j = 0, \dots, 7$ . In

particular,  $\widetilde{\mathbf{M}}(\boldsymbol{\omega})$  depends on the value of  $\widetilde{m}_j$  at  $\boldsymbol{\omega}$  and its shifts  $\boldsymbol{\pi}_i$  and can be considered a vector function of  $\boldsymbol{\omega}$ . We denote a sub-matrix of  $\widetilde{\mathbf{M}}$  containing all but the row associated with  $\boldsymbol{\pi}_k$  (respectively, the column associated with  $\widetilde{m}_k(\boldsymbol{\omega})$ ) as  $\widetilde{\mathbf{M}}[-k, :]$  (respectively,  $\widetilde{\mathbf{M}}[:, -k]$ ). In particular, we denote  $\widetilde{\mathbf{M}}[-0, -0]$  as  $\widetilde{\mathbf{M}}^\square$ .

We have the following observations for  $\widetilde{\mathbf{M}}(\boldsymbol{\omega})$ .

**Lemma 5.1.**  $\forall \boldsymbol{\omega} \in S_0$ , if (17) is solvable, then  $\widetilde{\mathbf{M}}[-0, :](\boldsymbol{\omega})$  is singular.

*Proof.* If (17) is solvable, then condition (ii) holds, which implies that  $\det([\widetilde{\mathbf{M}}, \mathbf{b}_0]) = 0$ . Expanding the determinant with respect to the last column  $\mathbf{b}_0$  yields  $\det(\widetilde{\mathbf{M}}[-0, :]) = 0$ .  $\square$

**Lemma 5.2.**  $\widetilde{\mathbf{M}}(\boldsymbol{\omega})$ ,  $\widetilde{\mathbf{M}}(\boldsymbol{\omega} + \boldsymbol{\pi}_2)$ ,  $\widetilde{\mathbf{M}}(\boldsymbol{\omega} + \boldsymbol{\pi}_4)$  and  $\widetilde{\mathbf{M}}(\boldsymbol{\omega} + \boldsymbol{\pi}_6)$  are the same up to row permutations. (17) holds  $\forall \boldsymbol{\omega}$  if and only if

$$\widetilde{\mathbf{M}}(\boldsymbol{\omega}) [\mathbf{m}(\boldsymbol{\omega}), \mathbf{m}(\boldsymbol{\omega} + \boldsymbol{\pi}_2), \mathbf{m}(\boldsymbol{\omega} + \boldsymbol{\pi}_4), \mathbf{m}(\boldsymbol{\omega} + \boldsymbol{\pi}_6)] = [\mathbf{b}_0, \mathbf{b}_2, \mathbf{b}_4, \mathbf{b}_6].$$

*Remark.* If we consider  $\widetilde{\mathbf{M}}(\boldsymbol{\omega})$  a vector function of  $\boldsymbol{\omega}$ , then the conditions (5.i) and (5.ii) are both pointwise, yet Lemma 5.2 shows that the set of points  $\{\boldsymbol{\omega}, \boldsymbol{\omega} + \boldsymbol{\pi}_2, \boldsymbol{\omega} + \boldsymbol{\pi}_4, \boldsymbol{\omega} + \boldsymbol{\pi}_6\}$  are linked together by the symmetry in  $\widetilde{\mathbf{M}}(\boldsymbol{\omega})$ . We exploit this symmetry in our analysis.

Due to condition (5.i),  $\forall \boldsymbol{\omega}$ ,  $\exists k_{\boldsymbol{\omega}}$  depending on  $\boldsymbol{\omega}$  such that  $\widetilde{\mathbf{M}}[-k_{\boldsymbol{\omega}}, :](\boldsymbol{\omega})$  is non-singular. On the other hand, Lemma 5.1 and Lemma 5.2 together imply that  $\widetilde{\mathbf{M}}[-k, :]$ ,  $k = 0, 2, 4, 6$  are singular. Therefore,  $k_{\boldsymbol{\omega}} \in \{1, 3, 5, 7\}$  and thus  $\widetilde{\mathbf{M}}[-k_{\boldsymbol{\omega}}, :](\boldsymbol{\omega})$  contains all rows associated with shifts  $\boldsymbol{\pi}_{2i}$ ,  $i = 0, \dots, 3$ . As in Section 4.1, we obtain the following expression of  $m_0$  by applying Cramer's rule to  $\widetilde{\mathbf{M}}[-k_{\boldsymbol{\omega}}, :](\boldsymbol{\omega})$ ,

$$m_0(\boldsymbol{\omega}) = \det(\widetilde{\mathbf{M}}^\square[-k_{\boldsymbol{\omega}}, :](\boldsymbol{\omega})) / \det(\widetilde{\mathbf{M}}[-k_{\boldsymbol{\omega}}, :](\boldsymbol{\omega})). \quad (22)$$

Moreover, based on (22), the identity condition (18) on  $m_0(\boldsymbol{\omega})$  and  $\widetilde{m}_0(\boldsymbol{\omega})$  can be derived in the same way as (21) by expanding  $\det(\widetilde{\mathbf{M}}[-k_{\boldsymbol{\omega}}, :](\boldsymbol{\omega}))$ .

In the following subsections, we first show our main result that for (17) to be solvable, the pre-designed  $\widetilde{m}_j$  are discontinuous, when the support of  $\widetilde{m}_j$  concentrates in the direction of  $C_j$  and a minimum symmetry of  $\widetilde{m}_j$  is required. We then discuss how to design  $\widetilde{m}_j$  with more symmetry and solve the corresponding system (17).

## 5.1 Discontinuity of $\widetilde{m}_j(\boldsymbol{\omega})$

In this subsection, we assume that  $|\widetilde{m}_1(\boldsymbol{\omega})|$  and  $|\widetilde{m}_6(\boldsymbol{\omega})|$  are symmetric with respect to the diagonal  $\omega_1 = \omega_2$ , i.e.

$$|\widetilde{m}_1(\boldsymbol{\omega})| = |\widetilde{m}_6(\boldsymbol{\omega}')| \quad \forall \omega_1 = \omega'_2, \omega_2 = \omega'_1,$$

and likewise for  $\widetilde{m}_3(\boldsymbol{\omega})$  and  $\widetilde{m}_4(\boldsymbol{\omega})$ ,

$$|\widetilde{m}_3(\boldsymbol{\omega})| = |\widetilde{m}_4(\boldsymbol{\omega}')| \quad \forall \omega_1 = -\omega'_2, \omega_2 = -\omega'_1.$$

Let  $\widetilde{\mathbf{m}}^i(\boldsymbol{\omega}) = [\widetilde{m}_1(\boldsymbol{\omega} + \boldsymbol{\pi}_i), \dots, \widetilde{m}_6(\boldsymbol{\omega} + \boldsymbol{\pi}_i)] \in \mathbb{C}^6$ ,  $i = 0, \dots, 7$  be the rows of  $\widetilde{\mathbf{M}}[:, -0]$ . In the following, we introduce a triangular partition of frequency

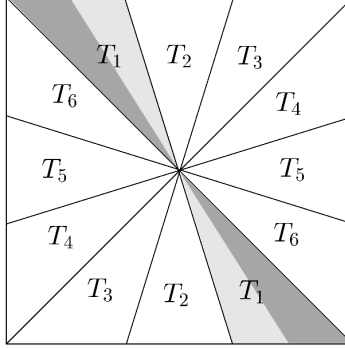


Figure 3: Partition of frequency square in six directions, where the essential support of  $\widetilde{m}_i(\omega)$  is contained in each pair of triangles  $T_i$ . The pair of dark grey triangles is  $T_1^-$  and the light grey pair is  $T_1^+$ .

square and define formally the concentration of  $\widetilde{m}_j$ 's support.

**Definition.** The *essential support*  $\Omega_i$  of a function  $\widetilde{m}_i$  is the set  $\{\omega : |\widetilde{m}_i(\omega)| > |\widetilde{m}_j(\omega)|, \forall j \neq i\}$ .

Let  $T_i$  be pairs of triangles shown in Figure 3, such that  $C_i \subset T_i$ ,  $i = 1, \dots, 6$ . Consider its decomposition,  $T_i = T_i^- \cup T_i^+$ , where  $T_i^-, T_i^+$  are halves of  $T_i$  adjacent to  $T_{i-1}$  and  $T_{i+1}$  respectively.

**Definition.**  $\widetilde{m}_i$  *concentrates* within  $T_i$  if

- (i)  $\Omega_i \subset T_i$ ;
- (ii)  $\text{supp}(\widetilde{m}_i) \subset T_{i-1}^+ \cup T_i \cup T_{i+1}^-$  and  $\int_{\Omega} |\widetilde{m}_i| > \int_{\Omega'} |\widetilde{m}_i|, \forall \Omega \subset T_i \cap \text{supp}(\widetilde{m}_i)$ , where  $\Omega' \subset T_{i-1}^+ \cup T_{i+1}^-$  is symmetric to  $\Omega$  with respect to the boundary of  $T_i$ .

*Remark.* For  $\widetilde{m}_i$  to concentrate in  $T_i$ ,  $\widetilde{m}_i$  should be “mainly” supported in  $T_i$  (condition (i)) and “decay” properly outside of  $T_i$  (condition (ii)).

Given  $\widetilde{m}_i(\omega)$  that concentrates in  $T_i$ , we study the singularity condition on  $\widetilde{\mathbf{M}}[-0, :](\omega)$  from Lemma 5.1, specifically in the domain  $S_\rho = \{(\omega_1, \omega_2) \mid \|\omega\| < \rho, \omega_1 < 0, \omega_2 < 0\}$ , see Figure 4.

**Lemma 5.3.**  $\exists \rho > 0$  s.t. if  $\widetilde{\mathbf{M}}[-0, :](\omega)$  is singular  $\forall \omega \in S_\rho$ , then  $\text{rank}(\widetilde{\mathbf{m}}^1, \widetilde{\mathbf{m}}^7) = 1$  or  $\text{rank}(\widetilde{\mathbf{m}}^3, \widetilde{\mathbf{m}}^5) = 1$ .

Lemma 5.3 can be proved by analyzing the linear dependency and independency between  $\widetilde{\mathbf{m}}^i$  on  $S_\rho$ , since  $\widetilde{\mathbf{m}}^i$  have known locations of zero entries when  $\rho$  is small due to the concentration of  $\widetilde{m}_j$ . For the full proof of Lemma 5.3, see Appendix B.

Without loss of generality, in the following analysis, we assume  $\text{rank}(\widetilde{\mathbf{m}}^1, \widetilde{\mathbf{m}}^7) = 1$  on  $S_\rho$ . Since only  $\widetilde{m}_1(\omega + \pi_1), \widetilde{m}_6(\omega + \pi_1)$  could be non-zero in  $\widetilde{\mathbf{m}}^1$  and

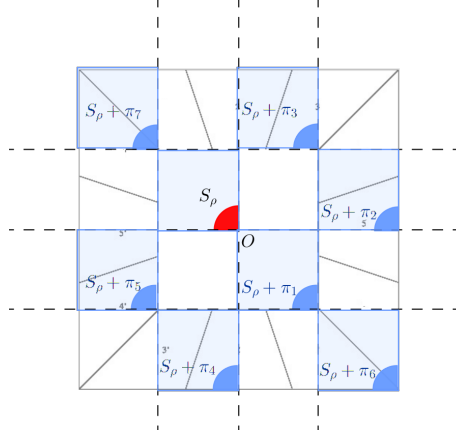


Figure 4:  $S_\rho$  and its shifts

$\widetilde{m}_1(\boldsymbol{\omega} + \boldsymbol{\pi}_7), \widetilde{m}_6(\boldsymbol{\omega} + \boldsymbol{\pi}_7)$  could be non-zero in  $\widetilde{\mathbf{m}}^7$  on  $S_\rho$ , they are linearly related. Based on this, we can show that  $\widetilde{m}_1(\boldsymbol{\omega}), \widetilde{m}_6(\boldsymbol{\omega})$  are zero almost everywhere on  $S_\rho + \boldsymbol{\pi}_1$ , see Proposition B.4 in Appendix B.1, which implies the discontinuity of  $\widetilde{m}_1(\boldsymbol{\omega})$  and  $\widetilde{m}_6(\boldsymbol{\omega})$  at  $(\frac{\pi}{2}, \frac{\pi}{2})$  or  $(-\frac{\pi}{2}, -\frac{\pi}{2})$ .

**Proposition 5.4.** *Given the same condition as in Lemma 5.3,  $\widetilde{m}_1(\boldsymbol{\omega}), \widetilde{m}_6(\boldsymbol{\omega})$  are not continuous at both  $(\frac{\pi}{2}, \frac{\pi}{2})$  and  $(-\frac{\pi}{2}, -\frac{\pi}{2})$ .*

*Proof.* If  $\widetilde{m}_1(\boldsymbol{\omega})$  is continuous at  $(\frac{\pi}{2}, \frac{\pi}{2})$ , then  $\widetilde{m}_1(\frac{\pi}{2}, \frac{\pi}{2}) = \lim_{\alpha \rightarrow 1^-} \widetilde{m}_1(\boldsymbol{\omega}(\alpha)) = 0$ , where  $\{\boldsymbol{\omega}(\alpha), 0 \leq \alpha < 1\} \subset S_\rho + \boldsymbol{\pi}_1$  and  $\boldsymbol{\omega}(1) = (\frac{\pi}{2}, \frac{\pi}{2})$ . By symmetry, we have  $\widetilde{m}_6(\frac{\pi}{2}, \frac{\pi}{2}) = 0$ . Similarly, the continuity at  $(-\frac{\pi}{2}, -\frac{\pi}{2})$  implies  $\widetilde{m}_1(-\frac{\pi}{2}, -\frac{\pi}{2}) = \widetilde{m}_6(-\frac{\pi}{2}, -\frac{\pi}{2}) = 0$ . Therefore  $\widetilde{\mathbf{m}}^1(0) = \widetilde{\mathbf{m}}^7(0) = \mathbf{0}$  which results in contradiction with Lemma 5.3.  $\square$

The following theorem summarizes our main result.

**Theorem 5.5.** *If  $\widetilde{m}_i(\boldsymbol{\omega})$  concentrates in  $T_i$  and  $\widetilde{m}_1(\boldsymbol{\omega}), \widetilde{m}_6(\boldsymbol{\omega})$  are symmetric with respect to the diagonal  $\omega_1 = \omega_2$  to each other, then (17) doesn't have feasible solution given continuous  $\widetilde{m}_1(\boldsymbol{\omega})$  and  $\widetilde{m}_6(\boldsymbol{\omega})$ .*

## 5.2 Design of input $\widetilde{m}_j(\boldsymbol{\omega})$

In this sub-section, we construct  $\widetilde{m}_j(\boldsymbol{\omega})$ ,  $j = 1, \dots, 6$ , which concentrate in  $T_i$ . Specifically, following the orthonormal construction in [9], we consider  $\widetilde{m}_j(\boldsymbol{\omega})$  in the form

$$\widetilde{m}_j(\boldsymbol{\omega}) = e^{-i\boldsymbol{\eta}_j^\top \boldsymbol{\omega}} |\widetilde{m}_j(\boldsymbol{\omega})|, \quad j = 1, \dots, 6, \quad (23)$$

where  $\boldsymbol{\eta}_j \in \mathbb{Z}^2$  is the phase constant of  $\widetilde{m}_j$ . In addition to the symmetry of pairs  $(|\widetilde{m}_1|, |\widetilde{m}_6|)$  and  $(|\widetilde{m}_3|, |\widetilde{m}_4|)$  assumed in Section 5.1, we further request that  $|\widetilde{m}_2|$  and  $|\widetilde{m}_5|$  are symmetric with respect to the  $\omega_1$ -axis and  $\omega_2$ -axis accordingly. Figure 5 shows such a design of  $|\widetilde{m}_j(\boldsymbol{\omega})|$  that has the imposed strong symmetries.

Based on (23) and the symmetries of  $|\widetilde{m}_j(\omega)|$ , for  $|m_0(\omega)| > 0, \forall |\omega| < \rho$ ,  $\eta_j$  have to satisfy certain constraints.

**Lemma 5.6.** *If  $\exists \omega \in D_1 := \{\omega_1 = \omega_2, \omega_1 \in (-\frac{\pi}{2}, 0)\}$ , s.t.  $|m_0(\omega)| \neq 0$ , then  $(\eta_1 - \eta_6)^\top (\pi_6 - \pi_7) \neq 0 \pmod{2\pi}$ .*

Because  $m_0(\omega)$  can be expressed as in (22),  $|m_0(\omega)| \neq 0$  is equivalent to  $\det(\widetilde{\mathbf{M}}^\square[-k_\omega, :](\omega)) \neq 0$ , i.e.  $\widetilde{\mathbf{M}}^\square(\omega)$  is full rank. The constraint on  $\eta_1$  and  $\eta_6$  then follows from substituting non-zero entries of  $\widetilde{\mathbf{M}}^\square(\omega)$  by (23) and consider the linear dependency of the columns in  $\widetilde{\mathbf{M}}^\square(\omega)$ . For the full proof of Lemma 5.6, see Appendix B.2.

Similarly, if  $\exists \omega \in \{\omega_1 = \omega_2, \omega_1 \in (0, \frac{\pi}{2})\}$ , s.t.  $|m_0(\omega)| \neq 0$ , then  $(\eta_1 - \eta_6)^\top (\pi_6 - \pi_1) \neq 0 \pmod{2\pi}$ . These two conditions are equivalent to

$$(\eta_1 - \eta_6)^\top (\pi/2, \pi/2) \neq 0 \pmod{2\pi} \quad (\mathbf{c1.1})$$

since  $\eta_1, \eta_6 \in \mathbb{Z}^2$ . Considering the other diagonal segment  $\{\omega_2 = -\omega_1, |\omega_1| < \frac{\pi}{2}\}$ , we also have

$$(\eta_3 - \eta_4)^\top (-\pi/2, \pi/2) \neq 0 \pmod{2\pi} \quad (\mathbf{c1.2})$$

Next, we consider  $\widetilde{m}_0(\mathbf{0})$  and investigate  $\widetilde{\mathbf{M}}^\square(\omega)$  at the origin.

**Proposition 5.7.** *If  $|\widetilde{m}_0(\mathbf{0})| \neq 0$ , then  $\pi_1^\top (\eta_1 - \eta_6) \neq \pi \pmod{2\pi}$  or  $\pi_3^\top (\eta_3 - \eta_4) \neq \pi \pmod{2\pi}$ .*

*Remark.* The proof of Proposition 5.7 is similar to that of Lemma 5.6 but more involved. See Appendix B.2 for the full proof.

We propose the following set of phases such that (c1.1) and (c1.2) as well as the necessary condition from Proposition 5.7 are all satisfied,

$$\begin{aligned} \eta_1 &= (0, 0), \quad \eta_2 = (-1, 1), \quad \eta_3 = (0, 2), \\ \eta_4 &= (1, 0), \quad \eta_5 = (0, -1), \quad \eta_6 = (0, 1). \end{aligned} \quad (24)$$

### 5.3 Solving (17) and (18) for $m_0, \widetilde{m}_0$ and $m_j$

Once  $\widetilde{m}_j(\omega), j = 1, \dots, 6$  are fixed on  $S_0$ , (17) can be reformulated as follows,

$$\widetilde{\mathbf{M}}[:, -0](\omega) \begin{bmatrix} m_1(\omega) \\ m_2(\omega) \\ m_3(\omega) \\ m_4(\omega) \\ m_5(\omega) \\ m_6(\omega) \end{bmatrix} = \mathbf{b}_0 - m_0(\omega) \begin{bmatrix} \widetilde{m}_0(\omega) \\ 0 \\ \widetilde{m}_0(\omega + \pi_2) \\ 0 \\ \widetilde{m}_0(\omega + \pi_4) \\ 0 \\ \widetilde{m}_0(\omega + \pi_6) \\ 0 \end{bmatrix} \doteq \mathbf{b}'_0(\omega), \quad (25)$$

where  $\widetilde{\mathbf{M}}[:, -0](\omega)$  is completely determined by  $\widetilde{m}_j(\omega), j = 1, \dots, 6$  and  $m_j, j = 1, \dots, 6$  can be uniquely solved on  $S_0$  if and only if  $\forall \omega \in S_0$

(5.3.i)  $\widetilde{\mathbf{M}}[:, -0](\omega)$  is full rank,

(5.3.ii)  $\mathbf{b}'_0(\omega)$  is in  $\text{col}(\widetilde{\mathbf{M}}[:, -0](\omega))$ , the column space of  $\widetilde{\mathbf{M}}[:, -0](\omega)$ .

Next, we show that (5.3.ii) breaks down to constraints on two submatrices of  $\widetilde{\mathbf{M}}[:, -0](\omega)$  and quadruples  $(m_0(\omega), m_0(\omega + \pi_2), m_0(\omega + \pi_4), m_0(\omega + \pi_6)), (m_0(\omega + \pi_1), m_0(\omega + \pi_3), m_0(\omega + \pi_5), m_0(\omega + \pi_7))$ .

**Proposition 5.8.** *Let  $\widetilde{\mathbf{M}}[\text{odd}, -0](\omega), \widetilde{\mathbf{M}}[\text{even}, -0](\omega) \in \mathbb{C}^{4 \times 6}$  be the sub-matrices of  $\widetilde{\mathbf{M}}[:, -0](\omega)$  consisting of odd and even indexed rows respectively.  $\forall \omega \in S_0$ , suppose (5.3.i) holds, then (5.3.ii) holds if and only if  $\text{rank}(\widetilde{\mathbf{M}}[\text{odd}, -0](\omega)) = \text{rank}(\widetilde{\mathbf{M}}[\text{even}, -0](\omega)) = 3$  and*

$$[m_0(\omega), m_0(\omega + \pi_2), m_0(\omega + \pi_4), m_0(\omega + \pi_6)] \widetilde{\mathbf{M}}[\text{even}, -0](\omega) = \mathbf{0}, \quad (26)$$

$$[m_0(\omega + \pi_1), m_0(\omega + \pi_3), m_0(\omega + \pi_5), m_0(\omega + \pi_7)] \widetilde{\mathbf{M}}[\text{odd}, -0](\omega) = \mathbf{0}. \quad (27)$$

For the proof of Proposition 5.8, see Appendix B.3.

*Remark.* Note that the submatrices  $\widetilde{\mathbf{M}}[\text{odd}, -0](\omega)$  and  $\widetilde{\mathbf{M}}[\text{even}, -0](\omega)$  are dual to each other under the shift of variable  $\omega \mapsto \omega + \pi_i$ , when  $i$  is odd. Therefore, the constraints  $\text{rank}(\widetilde{\mathbf{M}}[\text{even}, -0](\omega)) = 3$  and (26) from Proposition 5.8 are sufficient for (5.3.ii) to hold on  $S_0$ . Furthermore, because  $\widetilde{\mathbf{M}}[\text{even}, -0](\omega)$  and  $(\omega, \omega + \pi_2, \omega + \pi_4, \omega + \pi_6)$  are invariant to the shift of variable  $\omega \mapsto \omega + \pi_i$  when  $i$  is even, we only need to consider the constraints above on the subset  $[-\pi, 0) \times [-\pi, 0)$  of  $S_0$ .

In sum,  $\widetilde{\mathbf{M}}[:, -0](\omega)$  (or equivalently  $\widetilde{m}_j$ ) has to satisfy the following rank constraints on  $[-\pi, 0) \times [-\pi, 0)$  for (25) to be uniquely solvable on  $S_0$ ,

$$\text{rank}(\widetilde{\mathbf{M}}[:, -0](\omega)) = 6, \text{rank}(\widetilde{\mathbf{M}}[\text{even}, -0](\omega)) = 3. \quad (28)$$

As the rank constraints are hard to impose while designing  $\widetilde{m}_j$ , in our numerical experiments, we check if these rank constraints are satisfied afterwards, see step 1. in Algorithm 1.

Suppose (28) hold, the quadruple  $(m_0(\omega), m_0(\omega + \pi_2), m_0(\omega + \pi_4), m_0(\omega + \pi_6))$  can then be solved by (26) as the vector that is orthogonal to the column space of  $\widetilde{\mathbf{M}}[\text{even}, -0](\omega)$  of co-dimension 1. Hence the quadruple is uniquely determined up to a constant. In particular, we obtain  $m_0(\omega)$  on  $S_0$  by solving (26) independently at each  $\omega$  on  $[-\pi, 0) \times [-\pi, 0)$ , see step 2. in Algorithm 1. Since the constant can change drastically as  $\omega$  changes, there is potential lack of regularity of  $m_0(\omega)$  as an artifact of the algorithm. Fortunately, this artificial irregularity can be removed as suggested by the following proposition.

**Proposition 5.9.** *If  $\widetilde{m}_j(\omega), m_j(\omega), j = 0, 1, \dots, 6$  satisfy (17) and (18), then  $m'_0(\omega) \doteq m_0(\omega)c(\omega), \widetilde{m}'_0(\omega) \doteq \widetilde{m}_0(\omega)\bar{c}(\omega)^{-1}$  together with the same  $m_j(\omega), \widetilde{m}_j(\omega), j = 1, \dots, 6$  satisfy (17) and (18) if  $c(\omega) = c(\omega + \pi_2) = c(\omega + \pi_4) = c(\omega + \pi_6) \neq 0$ , i.e.  $c(\omega)$  is  $\pi$ -periodic in both  $\omega_1$  and  $\omega_2$ .*

*Proof.* It suffices to show that  $m'_0(\omega)\widetilde{m}'_0(\omega + \pi_i) = m_0(\omega)\widetilde{m}_0(\omega + \pi_i)$ , when  $i$  is even.  $\square$

*Remark.* Proposition 5.9 suggests that we can design  $c(\omega)$  such that  $(m'_0, \widetilde{m}'_0)$  have desired properties, e.g. continuity, smoothness or fast decay rate. In practice, we first solve  $\widetilde{m}_0(\omega)$ , see step 3. in Algorithm 1, then compute  $m_0(\omega)\widetilde{m}_0(\omega)$  and re-decompose the product into  $m'_0(\omega)$  and  $\widetilde{m}'_0(\omega)$ .

To obtain  $\widetilde{m}_0(\boldsymbol{\omega})$  on  $S_0$ , we solve the identity condition (18) on  $[-\pi, 0) \times [-\pi, 0)$  for the quadruple  $(\widetilde{m}_0(\boldsymbol{\omega}), \widetilde{m}_0(\boldsymbol{\omega} + \boldsymbol{\pi}_2), \widetilde{m}_0(\boldsymbol{\omega} + \boldsymbol{\pi}_4), \widetilde{m}_0(\boldsymbol{\omega} + \boldsymbol{\pi}_6))$ . As mentioned at the end of Section 4.1, we are not aware of existing efficient algorithms that solve (18). Hence we reformulate it as an optimization problem where (18) serves as a linear constraint. In particular, on a  $2N \times 2N$  regular grid  $\mathcal{G} = \{\boldsymbol{\omega}_i\}_{i=1}^{4N^2}$  of  $[-\pi, \pi) \times [-\pi, \pi)$ , (18) can be rewrite as

$$\mathbf{A} \widetilde{\mathbf{m}}_0 = \mathbf{1}_{4N^2}, \quad (29)$$

where  $\widetilde{\mathbf{m}}_0 = [\widetilde{m}_0(\boldsymbol{\omega}_i)]_{i=1}^{4N^2}$  and  $\mathbf{A} \in \mathbb{C}^{N^2 \times 4N^2}$  is a sparse matrix with entries

$$A_{i,j} = m_0(\boldsymbol{\omega}_j) \sum_{k=0}^3 \delta(\boldsymbol{\omega}_j - \boldsymbol{\omega}_i - \boldsymbol{\pi}_{2k}), \quad \boldsymbol{\omega}_j \in [-\pi, 0) \times [-\pi, 0).$$

For the objective function to be minimized, we choose the total variation of  $\widetilde{m}_0(\boldsymbol{\omega})m_0(\boldsymbol{\omega})$ , as the product will be later re-decomposed and its regularity controls the regularity of  $(m'_0(\boldsymbol{\omega}), \widetilde{m}'_0(\boldsymbol{\omega}))$  that can be achieved.

In sum, we solve the following quadratic minimization problem with linear constraint,

$$\min_{\mathbf{x}} \|\mathbf{D}(\mathbf{m}_0 \circ \mathbf{x})\|^2, \quad s.t. \mathbf{A}\mathbf{x} = \mathbf{1}, \quad (30)$$

where  $\mathbf{D}$  is the gradient operator,  $\circ$  is Hadamard product and  $\mathbf{A}$  is the linear operator from (18).

Supplementary numerical results on solving  $\widetilde{m}_0(\boldsymbol{\omega})$  by optimization are provided in Appendix C, where we test this optimization method on known bi-orthogonal filters  $m_0$  and  $\widetilde{m}_0$  and compare the solution from the optimization with the ground truth.

Finally, we plug  $m_0(\boldsymbol{\omega})$  and  $\widetilde{m}_0(\boldsymbol{\omega})$  into  $\mathbf{b}'_0(\boldsymbol{\omega})$  on the right of (25) and solve the linear system, which has a guaranteed unique solution.

To sum up, we propose the following algorithm for bi-orthogonal directional filter construction with dilated quincunx downsampling scheme:

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**Algorithm 1. Construction of  $m_0, \widetilde{m}_0$  and  $\widetilde{m}_j$  in bi-orthogonal basis**

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Input:  $\widetilde{m}_j(\boldsymbol{\omega})$ ,  $j = 1, \dots, 6$

step 1. construct  $\widetilde{\mathbf{M}}[:, -0](\boldsymbol{\omega})$  on sub-grid  $[-\pi, 0) \times [-\pi, 0)$  and check rank constraints (28),

step 2. solve quadruple  $(m_0(\boldsymbol{\omega}), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_2), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_4), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_6))$  using (26) on  $[-\pi, 0) \times [-\pi, 0)$ ,

step 3. solve the optimization (30) for  $\widetilde{m}_0(\boldsymbol{\omega})$  on grid  $[-\pi, \pi) \times [-\pi, \pi)$ ,

step 4. re-decompose  $m_0(\boldsymbol{\omega})\widetilde{m}_0(\boldsymbol{\omega})$  into  $m'_0(\boldsymbol{\omega})$  and  $\widetilde{m}'_0(\boldsymbol{\omega})$ ,

step 5. solve the reduced linear system (25) for  $m_j(\boldsymbol{\omega})$ ,  $j = 1, \dots, 6$ .

---

We may also manipulate pairs of  $(m_j, \widetilde{m}_j)$  in a similar way due to the following generalization of Proposition 5.9.

**Proposition 5.10.** *If  $\widetilde{m}_j(\omega), m_j(\omega), j = 0, 1, \dots, 6$  satisfy (17) and (18),  $m_j^c(\omega) \doteq m_j(\omega)c_j(\omega), \widetilde{m}_j^c(\omega) \doteq \widetilde{m}_j(\omega)\overline{c_j(\omega)}^{-1} j = 0, \dots, 6$  satisfy (17) and (18) if  $c_0(\omega) = c_0(\omega + \pi_{2k}), \forall k = 0, \dots, 3$  and  $c_j(\omega) = c_j(\omega + \pi_k), \forall k = 0, \dots, 7, j = 1, \dots, 6$ .*

## 6 Numerical Experiments

In this section, we show an example of the numerical construction of biorthogonal directional wavelets on a quincunx lattice using our proposed algorithm in Matlab.

We design  $\widetilde{m}_j$  in the form of (23), using phases in (24) and amplitudes  $|\widetilde{m}_j|$  shown in Figure 5. For the construction of  $|\widetilde{m}_j|$ , we start with  $|\widetilde{m}_2|$ , then compute  $|\widetilde{m}_1|$  and  $|\widetilde{m}_3|$  by shearing  $|\widetilde{m}_2|$  counter-clockwise and clockwise respectively.  $|\widetilde{m}_4|, |\widetilde{m}_5|$  and  $|\widetilde{m}_6|$  are obtained by symmetry along the diagonal. This is the same approach used in the shearlet construction [12]. Furthermore,  $\widetilde{m}_j(\omega) = 0, \forall \omega \in C_0 = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ . In particular, by Proposition 5.4, we enforce  $|\widetilde{m}_1(\frac{\pi}{2}, \frac{\pi}{2})| \neq 0$  and  $|\widetilde{m}_6(\frac{\pi}{2}, \frac{\pi}{2})| \neq 0$ .

Using these  $\widetilde{m}_j(\omega), j = 1, \dots, 6$  as input, the main algorithm in Section 5.3 yields  $m_0, \widetilde{m}_0$  and  $m_j, j = 1, \dots, 6$ . As the first step, we numerically check that this particular design of  $\widetilde{m}_j$  satisfies the rank constraints (28). As shown in Figure 6  $m_0(\omega)$  obtained in step 2 is supported on the central square  $C_0$ , where  $|m_0(\omega)| = 1$ . Moreover, Figure 7 shows that  $\widetilde{m}_0(\omega) = \overline{m_0(\omega)}^{-1} = m_0(\omega)$  on  $C_0$ , i.e.  $\widetilde{m}_0(\omega)$  solved in step 3 is numerically the same as  $m_0(\omega)$ . Figure 8 and Figure 9 show  $|\widetilde{m}_j(\omega)|$  and  $|m_j\widetilde{m}_j(\omega)|$  for  $\widetilde{m}_j(\omega)$  obtained by solving (25). We also numerically check that  $m_j(\omega)$  and  $\widetilde{m}_j(\omega)$  have the same phase, i.e.  $m_j\widetilde{m}_j(\omega) \in \mathbb{R}$ .

So far, we construct a set of  $(m_j, \widetilde{m}_j)_{j=0, \dots, 6}$  that satisfies (17) and (18), thus it can be used to construct biorthogonal wavelets based on (4) and (14). However,  $m_0$  and  $\widetilde{m}_0$  here are not continuous. To improve the regularity of the corresponding wavelets, by Proposition 5.9 and the observation  $m_0\widetilde{m}_0(\omega) = \mathbf{1}_{C_0}$ , we replace  $(m_0, \widetilde{m}_0)$  by  $(m'_0, \widetilde{m}'_0)$ , where  $\widetilde{m}'_0$  is a smooth real function supported on  $C_0$  and  $m'_0(\omega) = \mathbf{1}_{C_0}(\widetilde{m}'_0)^{-1}(\omega)$ . Figure 10 shows such a pair of  $(m'_0, \widetilde{m}'_0)$ . Finally, using pre-designed  $\widetilde{m}_j(\omega)$  and post-designed  $\widetilde{m}'_0$ , we construct the dual wavelets  $\widetilde{\psi}^j$  in (13) using (14). As shown in Figure 11, the dual wavelets are spatially localized and have good direction selection. The wavelets and scaling functions in (13) can be constructed using (14) similarly.

## 7 Conclusion and future work

In this paper, we consider directional wavelet schemes on dilated quincunx sublattice and analyze their regularity. We show that filters in bi-orthogonal bases have the same discontinuity in the frequency domain as that of the orthonormal bases at the corners of  $S_1 = [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ .



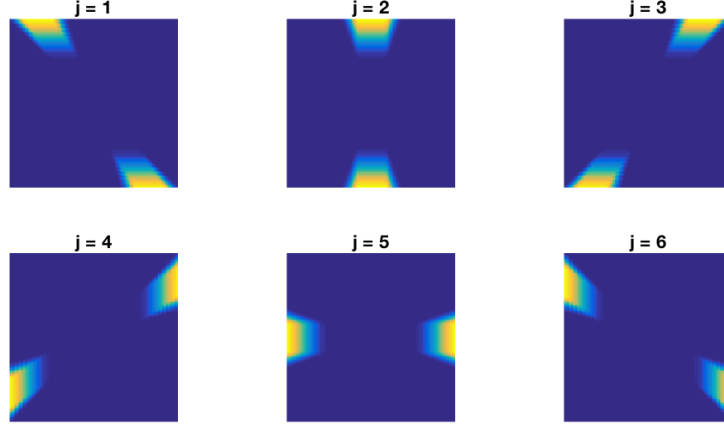


Figure 5: Input  $|\widetilde{m}_j(\omega)|$  constructed in the same way as shearlets.

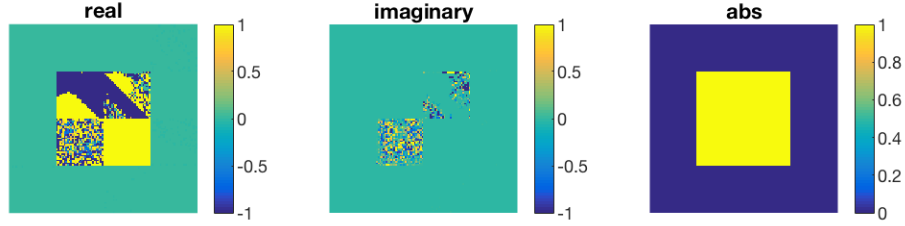


Figure 6:  $m_0(\omega)$  constructed from  $\widetilde{m}_j$ . Left to right:  $Re(m_0(\omega))$ ,  $Im(m_0(\omega))$  and  $|m_0(\omega)|$ .

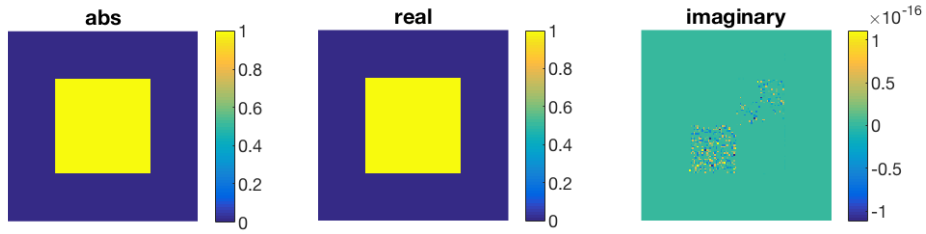


Figure 7:  $|\widetilde{m}_0(\omega)|$  and  $m_0 \widetilde{m}_0(\omega)$ , where  $\widetilde{m}_0(\omega)$  is solved by optimization (30), given  $\widetilde{m}_j$  in Figure 5 and  $m_0$  in Figure 6. Left to right:  $|\widetilde{m}_0(\omega)|$ ,  $Re(m_0 \widetilde{m}_0(\omega))$  and  $Im(m_0 \widetilde{m}_0(\omega))$ .

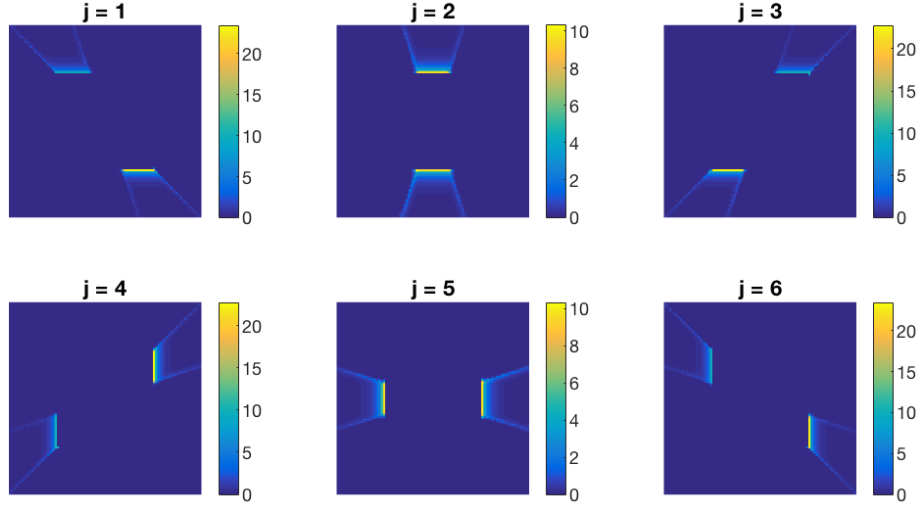


Figure 8:  $|m_j(\omega)|$ , where  $m_j(\omega)$  is solved from (25) given  $\widetilde{m}_j$  in Figure 5,  $m_0$  in Figure 6 and  $\widetilde{m}_0$  in Figure 7.

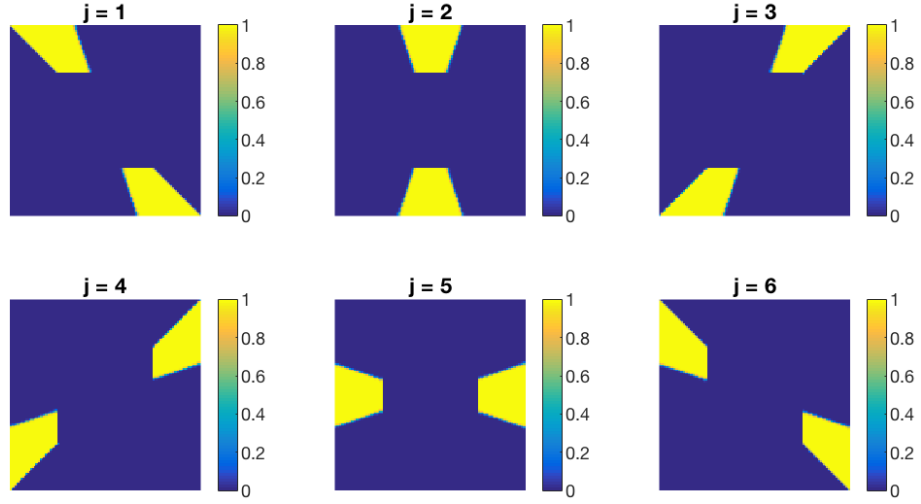


Figure 9:  $|m_j(\omega)\widetilde{m}_j(\omega)|$

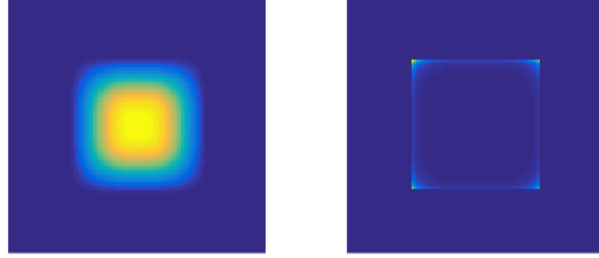


Figure 10: Left:  $\widetilde{m}_0'$ , designed smooth function mainly supported on the central square  $C_0$ , right:  $m_0'$ , s.t.  $m_0' \widetilde{m}_0'(\omega) = m_0 \widetilde{m}_0(\omega) = \mathbf{1}_{C_0}(\omega)$ .

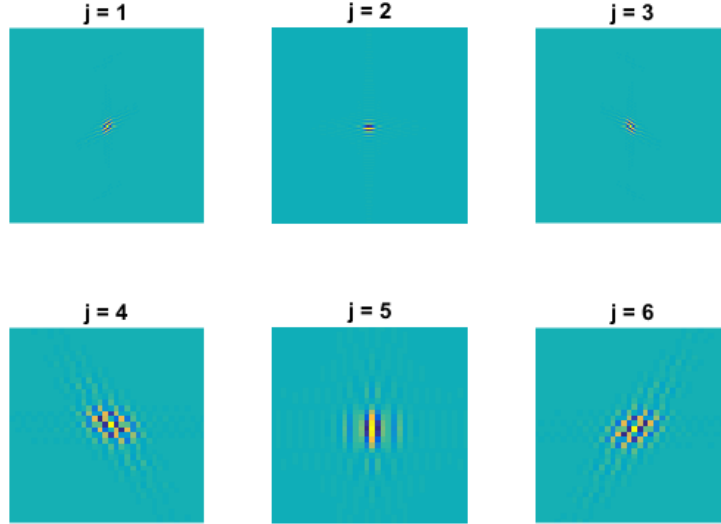


Figure 11: Real part of  $\psi^j$  constructed from  $\widetilde{m}_0'$  in Figure 10 and  $\widetilde{m}_j$ ,  $j = 1, \dots, 6$  in Figure 5 using (14). Top:  $\widetilde{\psi}^j$  without scaling, bottom:  $\widetilde{\psi}^j$  with eight time zoom-in

We propose a different approach to construct biorthogonal wavelets from our previous approach for the orthonormal bases construction [9]. The directional dual filters are first designed such that they can be extended to a bi-orthogonal frame and the remaining filters are obtained by solving linear systems and a constrained quadratic optimization derived from the identity summation and shift cancellation conditions for a biorthogonal MRA. We show numerically, that regularized dual wavelets  $\widetilde{\psi}^j$  can be constructed, yet their corresponding wavelets  $\psi^j$  are discontinuous in frequency domain, which is unavoidable according to our analysis.

The extension of the biorthogonal bases to low-redundancy dual frame construction is not studied here and will be our future focus.

## Appendices

### A Proof of Theorem 1

Take the Fourier transform of both sides of (6), we have

$$\begin{aligned} \sum_{\mathbf{k}} \langle f, \phi_{\mathbf{k}} \rangle \hat{\phi}(\boldsymbol{\omega}) e^{-i\boldsymbol{\omega}^T \mathbf{k}} &= \sum_{\mathbf{k}} \langle f, \phi_{1,\mathbf{k}} \rangle e^{-i\boldsymbol{\omega}^T \mathbf{D}_2 \mathbf{k}} |\mathbf{D}_2|^{1/2} \hat{\phi}(\mathbf{D}_2^T \boldsymbol{\omega}) \\ &\quad + \sum_{j=1}^J \sum_{\mathbf{k}} \langle f, \psi_{1,\mathbf{k}}^j \rangle e^{-i\boldsymbol{\omega}^T \mathbf{D} \mathbf{k}} |\mathbf{D}|^{1/2} \hat{\phi}(\mathbf{D}^T \boldsymbol{\omega}) \end{aligned}$$

Suppose  $m_j$  are trigonometric series

$$m_0(\boldsymbol{\omega}) = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{-i\boldsymbol{\omega}^T \mathbf{k}} \quad m_j(\boldsymbol{\omega}) = \sum_{\mathbf{k}} g_{\mathbf{k}} e^{-i\boldsymbol{\omega}^T \mathbf{k}}, \quad j = 1, \dots, J \quad (31)$$

The first term on the right hand side can be represented by  $\hat{\phi}(\boldsymbol{\omega})$  and  $\langle f, \phi_{\mathbf{k}} \rangle$  using (1) and (31).

$$\begin{aligned} \text{the first term on R.H.S.} &= \sum_{\mathbf{k}} \langle f, \phi_{1,\mathbf{k}} \rangle e^{-i\boldsymbol{\omega}^T \mathbf{D}_2 \mathbf{k}} |\mathbf{D}_2|^{1/2} m_0(\boldsymbol{\omega}) \hat{\phi}(\boldsymbol{\omega}) \\ &= \sum_{\mathbf{k}} \left( \sum_{\mathbf{k}'} \langle f, \phi_{\mathbf{k}'} \rangle \overline{c_{\mathbf{k}' - \mathbf{D}_2 \mathbf{k}}} |\mathbf{D}_2|^{1/2} \right) e^{-i\boldsymbol{\omega}^T \mathbf{D}_2 \mathbf{k}} |\mathbf{D}_2|^{1/2} m_0(\boldsymbol{\omega}) \hat{\phi}(\boldsymbol{\omega}) \\ &= \sum_{\mathbf{k}'} \langle f, \phi_{\mathbf{k}'} \rangle \left( |\mathbf{D}_2| \sum_{\mathbf{k}} \overline{c_{\mathbf{k}' - \mathbf{D}_2 \mathbf{k}}} e^{i\boldsymbol{\omega}^T (\mathbf{k}' - \mathbf{D}_2 \mathbf{k})} \right) e^{-i\boldsymbol{\omega}^T \mathbf{k}'} m_0(\boldsymbol{\omega}) \hat{\phi}(\boldsymbol{\omega}). \end{aligned}$$

*Remark.* If we have a shift  $\mathbf{k}_0$  in the down-sample scheme, i.e.  $\mathbf{D}_2 \mathbb{Z}^2 - \mathbf{k}_0$  instead of  $\mathbf{D}_2 \mathbb{Z}^2$ , so that we obtain coefficient of  $\tilde{\phi}_{1,\mathbf{k}} = \phi_{1,\mathbf{k}+\mathbf{k}_0}$  instead of  $\phi_{1,\mathbf{k}}$ , and  $\tilde{\phi}_1(\mathbf{x}) = \phi_1(\mathbf{x} - \mathbf{k}_0) = |\mathbf{D}_2|^{1/2} \sum_{\mathbf{k}} c_{\mathbf{k}} \phi(\mathbf{x} - \mathbf{k} - \mathbf{k}_0) = |\mathbf{D}_2|^{1/2} \sum_{\mathbf{k}} c_{\mathbf{k} - \mathbf{k}_0} \phi(\mathbf{x} - \mathbf{k})$ . This change of down-sample scheme results in an extra phase term  $e^{-i\boldsymbol{\omega}^T \mathbf{k}_0}$  in  $m_0$ . Here, we use the down-sample scheme without translation.

Since  $\bigcup_{\beta \in B} \{\beta\} := \bigcup_{\beta \in B} (\mathbf{D}_2 \mathbb{Z}^2 + \beta) = \mathbb{Z}^2$ , where  $B = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ ,

the summation over  $\mathbf{k}' \in \mathbb{Z}^2$  can be written as a double sum  $\sum_{\beta \in B} \sum_{\mathbf{k}' \in \{\beta\}}$ ,

$$\begin{aligned} & \sum_{\beta \in B} \sum_{\mathbf{k}' \in \{\beta\}} \langle f, \phi_{\mathbf{k}'} \rangle \sum_{\mathbf{k}} \overline{c_{\mathbf{k}' - \mathbf{D}_2 \mathbf{k}}} e^{i\omega^T (\mathbf{k}' - \mathbf{D}_2 \mathbf{k})} e^{-i\omega^T \mathbf{k}'} |\mathbf{D}_2| m_0(\omega) \hat{\phi}(\omega) \\ &= \sum_{\beta \in B} \sum_{\mathbf{k}' \in \{\beta\}} \langle f, \phi_{\mathbf{k}'} \rangle \sum_{\mathbf{k} \in \{\beta\}} \overline{c_{\mathbf{k}}} e^{i\omega^T \mathbf{k}} e^{-i\omega^T \mathbf{k}'} |\mathbf{D}_2| m_0(\omega) \hat{\phi}(\omega) \end{aligned}$$

The summation over  $\mathbf{k}$  in the middle is similar to the trigonometric form of  $m_0$  in (31), but  $\mathbf{k}$  takes value on the shifted sub-lattice  $\{\beta\}$  instead of  $\mathbb{Z}^2$ . Therefore, the summation equals to instead a linear combination of  $m_0$  with shifts  $\Gamma_0$ ,

$$\sum_{\pi \in \Gamma_0} m_0(\omega + \pi) e^{i\beta^T \pi} = \sum_{\mathbf{k} \in \{\beta\}} c_{\mathbf{k}} e^{-i\omega^T \mathbf{k}} \quad (32)$$

Substitute (32) into the previous expression,

$$\sum_{\beta \in B} \sum_{\mathbf{k}' \in \{\beta\}} \langle f, \phi_{\mathbf{k}'} \rangle \sum_{\pi \in \Gamma_0} \overline{m_0(\omega + \pi)} e^{-i\beta^T \pi} e^{-i\omega^T \mathbf{k}'} m_0(\omega) \hat{\phi}(\omega)$$

Since  $e^{i\pi^T \beta} = e^{i\pi^T \mathbf{k}'}$ ,  $\forall \mathbf{k}' \in \{\beta\}$ , after rewriting the double sum over  $\mathbf{k}'$  back to a unit sum on  $\mathbb{Z}^2$ , we get

$$\sum_{\mathbf{k}'} \langle f, \phi_{\mathbf{k}'} \rangle e^{-i\omega^T \mathbf{k}'} \hat{\phi}(\omega) \left( \sum_{\pi \in \Gamma_0} \overline{m_0(\omega + \pi)} m_0(\omega) e^{-i\pi^T \mathbf{k}'} \right)$$

Similarly, the second term on the R.H.S. of (6) equals to

$$\sum_{j=1}^J \sum_{\mathbf{k}'} \langle f, \phi_{\mathbf{k}'} \rangle e^{-i\omega^T \mathbf{k}'} \hat{\phi}(\omega) \left( \sum_{\pi \in \Gamma_1} \overline{m_j(\omega + \pi)} m_j(\omega) e^{-i\pi^T \mathbf{k}'} \right)$$

(For Theorem 3 on frame construction, the summation of shifts  $\pi$  is over  $\Gamma_0$  instead of  $\Gamma_1$ .) Combining the two terms on the R.H.S. of (6), and compare the coefficients of  $\langle f, \phi_{\mathbf{k}'} \rangle e^{-i\omega^T \mathbf{k}'} \hat{\phi}(\omega)$  on both sides, the perfect reconstruction condition is then equivalent to  $\forall \mathbf{k}'$ ,

$$\sum_{\pi \in \Gamma_0} e^{-i\pi^T \mathbf{k}'} \overline{m_0(\omega + \pi)} m_0(\omega) + \sum_j \sum_{\pi \in \Gamma_1} e^{-i\pi^T \mathbf{k}'} \overline{m_j(\omega + \pi)} m_j(\omega) = 1.$$

This is equivalent to

$$|m_0(\omega)|^2 + \sum_j |m_j(\omega)|^2 = 1$$

and

$$\begin{aligned} \sum_{j=0}^J \overline{m_j(\omega + \pi)} m_j(\omega) &= 0, \quad \pi \in \Gamma_0 \setminus \{\mathbf{0}\} \\ \sum_{j=1}^J \overline{m_j(\omega + \pi)} m_j(\omega) &= 0, \quad \pi \in \Gamma_1 \setminus \Gamma_0 \end{aligned}$$

*Remark.* Because each  $m_j$  is  $(2\pi, 2\pi)$  periodic, we only need to check the above equality  $\forall \omega \in S_0$ . If we downsample  $\psi_1^j$  on a shifted sub-lattice  $\mathbf{D}\mathbb{Z}^2 - \mathbf{k}_j$ , we then have an extra phase  $e^{i\pi^T \mathbf{k}_j}$  before  $\overline{m_j(\omega + \pi)} m_j(\omega)$  in shift cancellation condition. This provides additional freedom in the construction yet it is not substantial.

## B Proof of lemmas and propositions

### B.1 Discontinuity of $\widetilde{m}_j(\omega)$

**Lemma B.1.** Define  $d_{i,j}(\omega) = \det([\widetilde{\mathbf{m}}^{k_1}(\omega)^\top, \dots, \widetilde{\mathbf{m}}^{k_6}(\omega)^\top])$ , where  $0 \leq k_1 < \dots < k_6 \leq 7$ , s.t.  $k_l \neq i, j$ . (17) is solvable  $\forall \omega$  if and only if

$$\mathfrak{D}(\omega) \begin{bmatrix} \widetilde{m}_0(\omega) \\ \widetilde{m}_0(\omega + \pi_2) \\ \widetilde{m}_0(\omega + \pi_4) \\ \widetilde{m}_0(\omega + \pi_6) \end{bmatrix} \doteq \begin{bmatrix} 0 & d_{0,2} & d_{0,4} & d_{0,6} \\ -d_{0,2} & 0 & d_{2,4} & d_{2,6} \\ -d_{0,4} & -d_{2,4} & 0 & d_{4,6} \\ -d_{0,6} & -d_{2,6} & -d_{4,6} & 0 \end{bmatrix} \begin{bmatrix} \widetilde{m}_0(\omega) \\ \widetilde{m}_0(\omega + \pi_2) \\ \widetilde{m}_0(\omega + \pi_4) \\ \widetilde{m}_0(\omega + \pi_6) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (33)$$

*Proof.* By Lemma 5.1 and Lemma 5.2,  $\widetilde{\mathbf{M}}[-k, :]$ ,  $k = 0, 2, 4, 6$  are singular, The singularity condition on  $\widetilde{\mathbf{M}}[-0, :](\omega)$  can be rewritten as follows,

$$\begin{aligned} 0 &= \det(\widetilde{\mathbf{M}}[-0, :]) \\ &= \widetilde{m}_0(\omega + \pi_2) \cdot \det(\widetilde{\mathbf{M}}^\square[-2, :]) \\ &\quad + \widetilde{m}_0(\omega + \pi_4) \cdot \det(\widetilde{\mathbf{M}}^\square[-4, :]) + \widetilde{m}_0(\omega + \pi_6) \cdot \det(\widetilde{\mathbf{M}}^\square[-6, :]) \\ &= 0 \cdot \widetilde{m}_0(\omega) + d_{0,2} \cdot \widetilde{m}_0(\omega + \pi_2) \\ &\quad + d_{0,4} \cdot \widetilde{m}_0(\omega + \pi_4) + d_{0,6} \cdot \widetilde{m}_0(\omega + \pi_6) \end{aligned} \quad (34)$$

Similarly, the second to fourth equations can be obtained by rewriting the singularity condition on  $\widetilde{\mathbf{M}}[-2, :]$ ,  $\widetilde{\mathbf{M}}[-4, :]$  and  $\widetilde{\mathbf{M}}[-6, :]$  respectively.  $\square$

The identity constraint (18) on  $m_0$  and the singularity condition (33) together imply the following proposition,

**Proposition B.2.** Given  $\widetilde{m}_i, i = 1, \dots, 6$ , (17) has no solution for  $\widetilde{m}_0$ , if  $\exists \omega$ , s.t.  $[m_0(\omega), m_0(\omega + \pi_2), m_0(\omega + \pi_4), m_0(\omega + \pi_6)]$  is a linear combination of the rows of  $\mathfrak{D}(\omega)$ .

**Proof of Lemma 5.3:**

**Lemma 5.3.**  $\exists \rho > 0$  s.t. if  $\widetilde{\mathbf{M}}[-0, :](\omega)$  is singular  $\forall \omega \in S_\rho$ , then  $\text{rank}(\widetilde{\mathbf{m}}^1, \widetilde{\mathbf{m}}^7) = 1$  or  $\text{rank}(\widetilde{\mathbf{m}}^3, \widetilde{\mathbf{m}}^5) = 1$ .

*Proof.* When  $\rho$  is small enough, due to the concentration property,  $\widetilde{m}_i(\omega)$  is zero on all but a few sets  $S_\rho + \pi_j$  (see Fig.4 for reference of  $S_\rho$  and its shifts), thus  $\widetilde{\mathbf{m}}^i(\omega)$  is sparse on  $S_\rho$  and  $\widetilde{\mathbf{M}}[:, -0]$  takes the following form

$$\widetilde{\mathbf{M}}[:, -0](\omega) = \begin{bmatrix} \widetilde{\mathbf{m}}^0 \\ \widetilde{\mathbf{m}}^1 \\ \widetilde{\mathbf{m}}^2 \\ \widetilde{\mathbf{m}}^3 \\ \widetilde{\mathbf{m}}^4 \\ \widetilde{\mathbf{m}}^5 \\ \widetilde{\mathbf{m}}^6 \\ \widetilde{\mathbf{m}}^7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ * & 0 & * & * & 0 & * \\ * & 0 & 0 & 0 & 0 & * \end{bmatrix} \quad (35)$$

where \*s denote possible non-zero entries. We make the following observation of  $\tilde{\mathbf{m}}^i$ :

- (i)  $\tilde{\mathbf{m}}^0$  is a zero vector
- (ii)  $\tilde{\mathbf{m}}^2$  and  $\tilde{\mathbf{m}}^4$  are linearly independent of each other and the rest of  $\tilde{\mathbf{m}}^i$
- (iii)  $\text{span}\{\tilde{\mathbf{m}}^1, \tilde{\mathbf{m}}^7\} \perp \text{span}\{\tilde{\mathbf{m}}^3, \tilde{\mathbf{m}}^5\}$  and  $\text{rank}(\tilde{\mathbf{m}}^1, \tilde{\mathbf{m}}^7) \leq 2$ ,  
 $\text{rank}(\tilde{\mathbf{m}}^3, \tilde{\mathbf{m}}^5) \leq 2$
- (iv)  $\text{span}\{\tilde{\mathbf{m}}^1, \tilde{\mathbf{m}}^7, \tilde{\mathbf{m}}^3, \tilde{\mathbf{m}}^5, \tilde{\mathbf{m}}^6\} \leq 4$

Since  $S_\rho$  is in the low frequency domain,  $m_0(\omega) \neq 0$ . (22) then implies that  $\det(\tilde{\mathbf{M}}^\square[-k_\omega, :]) \neq 0$  hence  $\tilde{\mathbf{M}}^\square$  is full rank, or equivalently,  $\text{rank}(\tilde{\mathbf{M}}[:, -0]) = 6$ . It follows from (ii) and (iv) that  $\text{rank}(\tilde{\mathbf{m}}^1, \tilde{\mathbf{m}}^6, \tilde{\mathbf{m}}^7, \tilde{\mathbf{m}}^3, \tilde{\mathbf{m}}^5) = 4$ .

On the other hand, (ii) and (iv) imply that

$$\text{rank}(\tilde{\mathbf{M}}^\square(\omega + \pi_2)) = \text{rank}(\tilde{\mathbf{m}}^0, \tilde{\mathbf{m}}^4, \tilde{\mathbf{m}}^6, \tilde{\mathbf{m}}^1, \tilde{\mathbf{m}}^3, \tilde{\mathbf{m}}^5, \tilde{\mathbf{m}}^7) = 5$$

and likewise

$$\text{rank}(\tilde{\mathbf{M}}^\square(\omega + \pi_4)) = \text{rank}(\tilde{\mathbf{m}}^0, \tilde{\mathbf{m}}^2, \tilde{\mathbf{m}}^6, \tilde{\mathbf{m}}^1, \tilde{\mathbf{m}}^3, \tilde{\mathbf{m}}^5, \tilde{\mathbf{m}}^7) = 5.$$

Therefore,  $\det(\tilde{\mathbf{M}}^\square(\omega + \pi_2)) = \det(\tilde{\mathbf{M}}^\square(\omega + \pi_4)) = 0$  and (22) implies  $m_0(\omega + \pi_2) = m_0(\omega + \pi_4) = 0$ .

If  $\tilde{\mathbf{m}}^1$  and  $\tilde{\mathbf{m}}^7$  are linearly independent and so are  $\tilde{\mathbf{m}}^3$  and  $\tilde{\mathbf{m}}^5$ , then

$$\text{rank}(\tilde{\mathbf{M}}^\square(\omega + \pi_6)) = \text{rank}(\tilde{\mathbf{m}}^2, \tilde{\mathbf{m}}^4, \tilde{\mathbf{m}}^1, \tilde{\mathbf{m}}^3, \tilde{\mathbf{m}}^5, \tilde{\mathbf{m}}^7) = 6,$$

hence  $m_0(\omega + \pi_6) \neq 0$ . Therefore,

$$[m_0(\omega), m_0(\omega + \pi_2), m_0(\omega + \pi_4), m_0(\omega + \pi_6)] = [*, 0, 0, *].$$

In addition,  $d_{i,j} = 0$ ,  $\forall (i, j)$  except  $(0, 6)$ , so in (33)

$$\mathfrak{D}(\omega) = [d_{0,6}, 0, 0, 0]^\top [0, 0, 0, 1] + [0, 0, 0, d_{0,6}]^\top [-1, 0, 0, 0].$$

By Proposition B.2, the linear system (17) has no solution  $\tilde{m}_0$  and this proves the lemma.  $\square$

**Lemma B.3.** *If  $\tilde{m}_1(\omega)(\tilde{m}_6(\omega))$  concentrates in  $T_1(T_6)$ , then  $|\tilde{m}_6(\omega)| > |\tilde{m}_1(\omega)|$  a.e. on  $T_6 \cap \text{supp}(\tilde{m}_6)$  ( $|\tilde{m}_1(\omega)| > |\tilde{m}_6(\omega)|$  a.e. on  $T_1 \cap \text{supp}(\tilde{m}_1)$ ).*

*Proof* Let  $B_6 = \{\omega : |\tilde{m}_6(\omega)| \leq |\tilde{m}_1(\omega)|\} \cap T_6 \cap \text{supp}(\tilde{m}_1)$  and  $B_1$  be its mirror set with respect to  $\omega_1 = \omega_2$  and suppose  $|B_6| > 0$ , then  $\int_{B_6} |\tilde{m}_6(\omega)| \leq \int_{B_6} |\tilde{m}_1(\omega)|$ . On the other hand, since  $\tilde{m}_1(\omega)$  concentrates in  $T_1$ , we know  $\int_{B_1} |\tilde{m}_1(\omega)| > \int_{B_6} |\tilde{m}_1(\omega)|$ . Moreover, due to the symmetry of  $\tilde{m}_1(\omega), \tilde{m}_6(\omega)$  and  $B_1, B_6$ ,  $\int_{B_1} |\tilde{m}_1(\omega)| = \int_{B_6} |\tilde{m}_6(\omega)|$ , hence  $\int_{B_6} |\tilde{m}_1(\omega)| \geq \int_{B_6} |\tilde{m}_6(\omega)| = \int_{B_1} |\tilde{m}_1(\omega)|$  which results in contradiction.  $\square$

**Proposition B.4.** *If  $m_1(m_6)$  concentrates within  $T_1(T_6)$ , then  $\tilde{m}_1(\omega) = \tilde{m}_6(\omega) = 0$ , a.e. on  $S_\rho + \pi_1$ .*

*Proof.* Consider frequency domain  $S'_\rho = S_\rho \cap \{\omega_1 < \omega_2\}$ . By Lemma 5.3,  $\exists \alpha_\omega \in \mathbb{C}$ , s.t.  $\tilde{\mathbf{m}}^1(\omega) = \alpha_\omega \tilde{\mathbf{m}}^7(\omega)$ ,  $\forall \omega \in S'_\rho$ , i.e.  $\tilde{m}_1(\omega + \pi_1) = \alpha_\omega \cdot \tilde{m}_1(\omega + \pi_7)$  and  $\tilde{m}_6(\omega + \pi_1) = \alpha_\omega \cdot \tilde{m}_6(\omega + \pi_7)$ . On the other hand, Lemma B.3 implies

that  $|\widetilde{m}_1(\boldsymbol{\omega} + \boldsymbol{\pi}_7)| \geq |\widetilde{m}_6(\boldsymbol{\omega} + \boldsymbol{\pi}_7)|$ , hence  $|\widetilde{m}_1(\boldsymbol{\omega} + \boldsymbol{\pi}_1)| \geq |\widetilde{m}_6(\boldsymbol{\omega} + \boldsymbol{\pi}_1)|$ . Let  $\Omega'_6 := (S_\rho + \boldsymbol{\pi}_1) \cap T_6$ , then  $\int_{\Omega'_6} |\widetilde{m}_1(\boldsymbol{\omega})| \geq \int_{\Omega'_6} |\widetilde{m}_6(\boldsymbol{\omega})|$ , which will contradict Lemma B.3 unless  $|\Omega'_6 \cap \text{supp}(\widetilde{m}_6)| = 0$ , or equivalently  $\alpha_\omega = 0$  and so  $\widetilde{m}_6(\boldsymbol{\omega}) = \widetilde{m}_1(\boldsymbol{\omega}) = 0$ , *a.e.* on  $\Omega'_6$ . By symmetry,  $\widetilde{m}_6(\boldsymbol{\omega}) = \widetilde{m}_1(\boldsymbol{\omega}) = 0$ , *a.e.* on  $(S_\rho + \boldsymbol{\pi}_1) \setminus \Omega'_6$  as well.  $\square$

## B.2 Design of input $\widetilde{m}_j(\boldsymbol{\omega})$

**Proof of Lemma 5.6:**

**Lemma 5.6.** *If  $\exists \boldsymbol{\omega} \in D_1 := \{\omega_1 = \omega_2, \omega_1 \in (-\frac{\pi}{2}, 0)\}$ , s.t.  $m_0(\boldsymbol{\omega}) > 0$ , then  $(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_6)^\top (\boldsymbol{\pi}_6 - \boldsymbol{\pi}_7) \neq 0 \pmod{2\pi}$ .*

*Proof.* As  $\widetilde{m}_1(\boldsymbol{\omega})$  and  $\widetilde{m}_6(\boldsymbol{\omega})$  concentrate in  $T_1$  and  $T_6$  respectively,  $\widetilde{m}_1(\boldsymbol{\omega} + \boldsymbol{\pi}_i) = 0$  and  $\widetilde{m}_6(\boldsymbol{\omega} + \boldsymbol{\pi}_i) = 0$ ,  $i = 1, \dots, 5$ . Due to symmetry,  $|\widetilde{m}_1(\boldsymbol{\omega})| = |\widetilde{m}_6(\boldsymbol{\omega})|$  on  $\{\omega_1 = \omega_2\}$ . Let  $A = |\widetilde{m}_1(\boldsymbol{\omega} + \boldsymbol{\pi}_7)| = |\widetilde{m}_6(\boldsymbol{\omega} + \boldsymbol{\pi}_7)|$  and  $B = |\widetilde{m}_1(\boldsymbol{\omega} + \boldsymbol{\pi}_6)| = |\widetilde{m}_6(\boldsymbol{\omega} + \boldsymbol{\pi}_6)|$ , then the first and the last columns of  $\widetilde{\mathbf{M}}^\square$  are

$$\widetilde{\mathbf{M}}^\square[:, 1] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ Ae^{i\boldsymbol{\eta}_1^\top (\boldsymbol{\omega} + \boldsymbol{\pi}_6)} \\ Be^{i\boldsymbol{\eta}_1^\top (\boldsymbol{\omega} + \boldsymbol{\pi}_7)} \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{M}}^\square[:, 6] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ Ae^{i\boldsymbol{\eta}_6^\top (\boldsymbol{\omega} + \boldsymbol{\pi}_6)} \\ Be^{i\boldsymbol{\eta}_6^\top (\boldsymbol{\omega} + \boldsymbol{\pi}_7)} \end{bmatrix}.$$

By (22), if  $m_0(\boldsymbol{\omega}) > 0$ ,  $\boldsymbol{\omega} \in D_1$  then  $\widetilde{\mathbf{M}}^\square(\boldsymbol{\omega})$  is full rank, hence its columns are linearly independent. In particular,  $\widetilde{\mathbf{M}}^\square[:, 1]$  and  $\widetilde{\mathbf{M}}^\square[:, 6]$  are linearly independent, which implies that  $e^{i(\boldsymbol{\eta}_1^\top \boldsymbol{\pi}_6 + \boldsymbol{\eta}_6^\top \boldsymbol{\pi}_7)} \neq e^{i(\boldsymbol{\eta}_6^\top \boldsymbol{\pi}_6 + \boldsymbol{\eta}_1^\top \boldsymbol{\pi}_7)}$  or equivalently  $(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_6)^\top (\boldsymbol{\pi}_6 - \boldsymbol{\pi}_7) \neq 0 \pmod{2\pi}$ .  $\square$

**Proof of Proposition 5.7**

**Proposition 5.7.** *If  $\widetilde{m}_0(\mathbf{0}) \neq 0$ , then  $\boldsymbol{\pi}_1^\top (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_6) \neq \pi \pmod{2\pi}$  or  $\boldsymbol{\pi}_3^\top (\boldsymbol{\eta}_3 - \boldsymbol{\eta}_4) \neq \pi \pmod{2\pi}$ .* *Proof.* Since  $\widetilde{m}_0(\mathbf{0}) \neq 0$ , as shown in Lemma 5.3, at  $\boldsymbol{\omega} = \mathbf{0}$   $\text{rank}(\widetilde{\mathbf{m}}^1, \widetilde{\mathbf{m}}^6, \widetilde{\mathbf{m}}^7, \widetilde{\mathbf{m}}^3, \widetilde{\mathbf{m}}^5) = 4$ . This is equivalent to the matrix  $\mathbf{A}$  defined in (36) to be full rank.

$$\mathbf{A} = \begin{bmatrix} \widetilde{m}_1(\boldsymbol{\pi}_6) & \widetilde{m}_6(\boldsymbol{\pi}_6) & \widetilde{m}_3(\boldsymbol{\pi}_6) & \widetilde{m}_4(\boldsymbol{\pi}_6) \\ \widetilde{m}_1(\boldsymbol{\pi}_1) & \widetilde{m}_6(\boldsymbol{\pi}_1) & 0 & 0 \\ \widetilde{m}_1(\boldsymbol{\pi}_7) & \widetilde{m}_6(\boldsymbol{\pi}_7) & 0 & 0 \\ 0 & 0 & \widetilde{m}_3(\boldsymbol{\pi}_3) & \widetilde{m}_4(\boldsymbol{\pi}_3) \\ 0 & 0 & \widetilde{m}_3(\boldsymbol{\pi}_5) & \widetilde{m}_4(\boldsymbol{\pi}_5) \end{bmatrix} \quad (36)$$

Let  $|\widetilde{m}_1(\boldsymbol{\pi}_1)| = a$ ,  $|\widetilde{m}_1(\boldsymbol{\pi}_6)| = b$ . Due to the symmetry of  $\widetilde{m}_j(\boldsymbol{\omega})$ ,  $|\widetilde{m}_1(\boldsymbol{\pi}_1)| = |\widetilde{m}_1(\boldsymbol{\pi}_7)| = |\widetilde{m}_6(\boldsymbol{\pi}_1)| = |\widetilde{m}_6(\boldsymbol{\pi}_7)| = |\widetilde{m}_3(\boldsymbol{\pi}_3)| = |\widetilde{m}_3(\boldsymbol{\pi}_5)| = |\widetilde{m}_4(\boldsymbol{\pi}_3)| = |\widetilde{m}_4(\boldsymbol{\pi}_5)|$  and  $|\widetilde{m}_1(\boldsymbol{\pi}_6)| = |\widetilde{m}_6(\boldsymbol{\pi}_6)| = |\widetilde{m}_3(\boldsymbol{\pi}_6)| = |\widetilde{m}_4(\boldsymbol{\pi}_6)|$ . Rewrite  $\mathbf{A}$  as



follows,

$$\mathbf{A} = \begin{bmatrix} be^{-i\pi_6^\top \eta_1} & be^{-i\pi_6^\top \eta_6} & be^{-i\pi_6^\top \eta_3} & be^{-i\pi_6^\top \eta_4} \\ ae^{-i\pi_1^\top \eta_1} & ae^{-i\pi_1^\top \eta_6} & 0 & 0 \\ ae^{i\pi_1^\top \eta_1} & ae^{i\pi_1^\top \eta_6} & 0 & 0 \\ 0 & 0 & ae^{-i\pi_3^\top \eta_3} & ae^{-i\pi_3^\top \eta_4} \\ 0 & 0 & ae^{i\pi_3^\top \eta_3} & ae^{i\pi_3^\top \eta_4} \end{bmatrix}$$

The product of singular values of  $\mathbf{A}$  is

$$\sqrt{\det(\mathbf{A}^* \mathbf{A})} = 4a^3 \sqrt{a^2 K_1^2 K_2^2 + b^2(Q_1 K_2^2 + Q_2 K_1^2)}, \quad (37)$$

where  $Q_1 = 1 - \cos(\pi_6^\top(\eta_1 - \eta_6)) \cos(\pi_1^\top(\eta_1 - \eta_6))$ ,  $Q_2 = 1 - \cos(\pi_6^\top(\eta_3 - \eta_4)) \cos(\pi_3^\top(\eta_3 - \eta_4))$ ,  $K_1 = \sin(\pi_1^\top(\eta_1 - \eta_6))$ ,  $K_2 = \sin(\pi_3^\top(\eta_3 - \eta_4))$ . If  $\pi_1^\top(\eta_1 - \eta_6) = \pi_3^\top(\eta_3 - \eta_4) = \pi \pmod{2\pi}$ , then  $K_1 = K_2 = 0$  and  $\mathbf{A}$  becomes singular.  $\square$

### B.3 Solving (17) and (18) for $m_0, \widetilde{m}_0$ and $m_j$

**Lemma B.5.** Let  $\mathbf{P} \in \mathbb{C}^{n \times n}$  be a projection matrix of rank 2 and  $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in \mathbb{C}^n$ , s.t.  $\mathbf{a}^* \mathbf{b} = (\mathbf{a}')^* \mathbf{b}' = 1$ ,  $\mathbf{a}'^* \mathbf{b} = \mathbf{a}^* \mathbf{b}' = \mathbf{b}^* \mathbf{b}' = 0$ . If  $\mathbf{P}(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}') = \mathbf{0}$ , then  $\mathbf{P}$  is the projection of  $\text{span}\{\mathbf{b}, \mathbf{b}'\}$ .

*Proof.* Since

$$\text{rank}(\mathbf{I}_n) \leq \text{rank}(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}') + \text{rank}(\mathbf{a} \otimes \mathbf{b}) + \text{rank}(\mathbf{a}' \otimes \mathbf{b}'),$$

it follows that  $\text{rank}(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}') \geq n - 2$ . On the other hand, because  $\text{rank}(\mathbf{P}) = 2$ ,  $\mathbf{P}(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}') = \mathbf{0}$  implies that  $\text{rank}(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}') \leq n - 2$ . Hence  $\text{rank}(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}') = n - 2$  and  $\mathbf{P}$  is the projection of  $\text{col}(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}')^\perp$ . On the other hand,

$$\begin{aligned} \mathbf{b}^*(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}') &= \mathbf{b}^* - (\mathbf{b}^* \mathbf{a}) \mathbf{b}^* - (\mathbf{b}^* \mathbf{a}') (\mathbf{b}')^* \\ &= \mathbf{b}^* - \mathbf{b}^* - 0 \cdot (\mathbf{b}')^* = \mathbf{0}^* \end{aligned}$$

Therefore,  $\mathbf{P} \mathbf{b} = \mathbf{b}$ . Similarly,  $(\mathbf{b}')^*(\mathbf{I}_n - \mathbf{a} \otimes \mathbf{b} - \mathbf{a}' \otimes \mathbf{b}') = \mathbf{0}^*$  and  $\mathbf{P} \mathbf{b}' = \mathbf{b}'$ . Moreover, as  $\mathbf{b}^* \mathbf{b}' = 0$  and  $\text{rank}(\mathbf{P}) = 2$ ,  $\mathbf{P} = \|\mathbf{b}\|^{-2} \cdot \mathbf{b} \otimes \mathbf{b} + \|\mathbf{b}'\|^{-2} \cdot \mathbf{b}' \otimes \mathbf{b}'$ .  $\square$

**Lemma B.6.** Given  $\widetilde{\mathbf{M}}[:, -0](\omega)$  is full rank  $\forall \omega$ ,  $\widetilde{\mathbf{M}}[-0, :](\omega)$  is singular if (18) holds.

*Proof.* If (18) holds, then by Lemma B.5,  $\mathbf{m}_0^\mathcal{E}, \mathbf{m}_0^\mathcal{O}$  are orthogonal to  $\text{col}(\widetilde{\mathbf{M}}[:, -0])$ , therefore  $[\mathbf{m}_0^\mathcal{O}, \mathbf{m}_0^\mathcal{E}, \widetilde{\mathbf{M}}[:, -0]] \in \mathbb{C}^{8 \times 8}$  is full rank. Due to (18),  $\mathbf{m}_0^\mathcal{E}$  and  $\widetilde{\mathbf{m}}_0^\mathcal{E}$  are not orthogonal to each other, hence  $[\mathbf{m}_0^\mathcal{O}, \widetilde{\mathbf{m}}_0^\mathcal{E}, \widetilde{\mathbf{M}}[:, -0]] = [\mathbf{m}_0^\mathcal{O}, \widetilde{\mathbf{M}}]$  is full rank as well. Because  $(\mathbf{m}_0^\mathcal{O})^* \widetilde{\mathbf{M}}[:, i] = 0, i = 0, \dots, 7$  and  $\mathbf{m}_0^\mathcal{O}[-0]^* \widetilde{\mathbf{M}}[-0, i] = (\mathbf{m}_0^\mathcal{O})^* \widetilde{\mathbf{M}}[:, i]$ ,  $\mathbf{m}_0^\mathcal{O}[-0]$  is orthogonal to  $\text{col}(\widetilde{\mathbf{M}}[-0, :])$ . Since  $[\mathbf{m}_0^\mathcal{O}[-0], \widetilde{\mathbf{M}}[-0, :]] \in \mathbb{C}^{7 \times 8}$  is full rank,  $\widetilde{\mathbf{M}}[-0, :]$  must be singular.  $\square$

**Proof of Proposition 5.8:**

**Proposition 5.8.** Let  $\widetilde{\mathbf{M}}[\text{odd}, -0], \widetilde{\mathbf{M}}[\text{even}, -0] \in \mathbb{C}^{4 \times 6}$  be the sub-matrices of

$\widetilde{\mathbf{M}}[:, -0]$  consisting of odd and even rows respectively. Suppose (5.3.i) holds, then (5.3.ii) holds if and only if  $\text{rank}(\widetilde{\mathbf{M}}[\text{odd}, -0]) = \text{rank}(\widetilde{\mathbf{M}}[\text{even}, -0]) = 3$  and

$$[m_0(\boldsymbol{\omega}), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_2), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_4), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_6)] \widetilde{\mathbf{M}}[\text{even}, -0](\boldsymbol{\omega}) = \mathbf{0}, \quad (26)$$

$$[m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_1), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_3), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_5), m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_7)] \widetilde{\mathbf{M}}[\text{odd}, -0](\boldsymbol{\omega}) = \mathbf{0}. \quad (27)$$

*Proof.* Note that  $\widetilde{\mathbf{M}}[:, -0]$  have the same rows at  $\boldsymbol{\omega} + \boldsymbol{\pi}_i$ ,  $i = 0, \dots, 7$ , we define row permutation matrix  $\mathbf{P}_i$ , s.t.  $\mathbf{P}_i(\widetilde{\mathbf{M}}[:, -0](\boldsymbol{\omega} + \boldsymbol{\pi}_i)) = \widetilde{\mathbf{M}}[:, -0](\boldsymbol{\omega})$ . Let  $\mathbf{P}_{\widetilde{\mathbf{M}}}(\boldsymbol{\omega})$  be the projection matrix of the  $\text{col}(\widetilde{\mathbf{M}}[:, -0](\boldsymbol{\omega}))^\perp = \text{null}(\widetilde{\mathbf{M}}[:, -0]^*)$ , then (5.3.ii) is equivalent to  $\mathbf{P}_{\widetilde{\mathbf{M}}} \mathbf{b}'_0(\boldsymbol{\omega}) = \mathbf{0}$ . Group this equality at  $\boldsymbol{\omega} + \boldsymbol{\pi}_i$ , we have

$$\begin{aligned} \mathbf{0} &= [\mathbf{P}_i \mathbf{P}_{\widetilde{\mathbf{M}}} \mathbf{b}'_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)]_{i=0, \dots, 7} \\ &= [\mathbf{P}_i \mathbf{P}_{\widetilde{\mathbf{M}}}(\boldsymbol{\omega} + \boldsymbol{\pi}_i) \mathbf{P}_i^2 \mathbf{b}'_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)]_{i=0, \dots, 7} \\ &= [\mathbf{P}_{\widetilde{\mathbf{M}}}(\boldsymbol{\omega}) \mathbf{P}_i \mathbf{b}'_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)]_{i=0, \dots, 7} \\ &= \mathbf{P}_{\widetilde{\mathbf{M}}}(\boldsymbol{\omega}) [\mathbf{P}_i \mathbf{b}'_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)]_{i=0, \dots, 7} \end{aligned} \quad (38)$$

Let

$$\begin{aligned} \widetilde{\mathbf{m}}_0^\mathcal{E} &= [(1 + i \bmod 2) \cdot \widetilde{m}_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)]_{i=0, \dots, 7}^\top = \widetilde{\mathbf{M}}[:, 0](\boldsymbol{\omega}), \\ \widetilde{\mathbf{m}}_0^\mathcal{O} &= [(i \bmod 2) \cdot \widetilde{m}_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)]_{i=0, \dots, 7}^\top, \\ \mathbf{m}_0^\mathcal{E} &= [(1 + i \bmod 2) \cdot m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)]_{i=0, \dots, 7}^\top, \\ \mathbf{m}_0^\mathcal{O} &= [(i \bmod 2) \cdot m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)]_{i=0, \dots, 7}^\top. \end{aligned}$$

The identity constraint (18) thus can be written as  $(\widetilde{\mathbf{m}}_0^\mathcal{E})^* \widetilde{\mathbf{m}}_0^\mathcal{E} = 1$  and  $(\widetilde{\mathbf{m}}_0^\mathcal{O})^* \widetilde{\mathbf{m}}_0^\mathcal{O} = 1$ . By definition,

$$\mathbf{P}_i \mathbf{b}'_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i) = \mathbf{P}_i (\mathbf{b}_0 - m_0 \widetilde{\mathbf{M}}[:, 0](\boldsymbol{\omega} + \boldsymbol{\pi}_i)) = \mathbf{b}_i - m_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i) \mathbf{P}_i (\widetilde{\mathbf{M}}[:, 0](\boldsymbol{\omega} + \boldsymbol{\pi}_i))$$

and

$$\mathbf{P}_i (\widetilde{\mathbf{M}}[:, 0](\boldsymbol{\omega} + \boldsymbol{\pi}_i)) = \begin{cases} \widetilde{\mathbf{M}}[:, 0] = \widetilde{\mathbf{m}}_0^\mathcal{E}, & i \text{ is even} \\ \widetilde{\mathbf{m}}_0^\mathcal{O}, & i \text{ is odd} \end{cases}$$

Substitute the above expression of  $\mathbf{P}_i \mathbf{b}'_0(\boldsymbol{\omega} + \boldsymbol{\pi}_i)$  in (38) and we have

$$\mathbf{0} = \mathbf{P}_{\widetilde{\mathbf{M}}} (\mathbf{I}_8 - \widetilde{\mathbf{m}}_0^\mathcal{E} \otimes \overline{\mathbf{m}}_0^\mathcal{E} - \widetilde{\mathbf{m}}_0^\mathcal{O} \otimes \overline{\mathbf{m}}_0^\mathcal{O}) \quad (39)$$

Therefore, by Lemma B.5,  $\mathbf{P}_{\widetilde{\mathbf{M}}}$  is the projection of  $\text{span}\{\overline{\mathbf{m}}_0^\mathcal{O}, \overline{\mathbf{m}}_0^\mathcal{E}\}$ . This is equivalent to (26) and (27). Finally, since

$$6 = \text{rank}(\widetilde{\mathbf{M}}[:, -0]) \leq \text{rank}(\widetilde{\mathbf{M}}[\text{odd}, -0]) + \text{rank}(\widetilde{\mathbf{M}}[\text{even}, -0]) \leq (4-1) + (4-1),$$

$$\text{rank}(\widetilde{\mathbf{M}}[\text{odd}, -0]) = \text{rank}(\widetilde{\mathbf{M}}[\text{even}, -0]) = 3. \quad \square$$

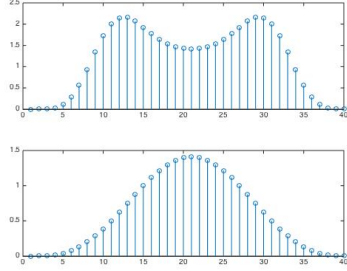


Figure 13:  $m_0(\omega)$  and  $\widetilde{m}_0(\omega)$

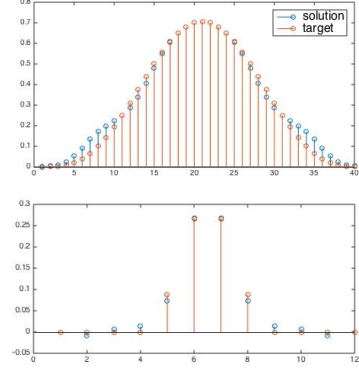


Figure 14:  $\widehat{m}_0$  vs.  $\widetilde{m}_0$ , top: frequency domain, bottom: time domain

## C Supplementary Numerical Results

### C.1 Numerical optimization of $\widetilde{m}_0(\omega)$ in 1D

To test whether numerical optimization is a practical way to solve (18) and (21), we experiment on  $m_0$  and  $\widetilde{m}_0$  of existing bi-orthogonal wavelets. We consider a pair of low frequency filters corresponding to bi-orthogonal scaling functions  $\phi, \tilde{\phi}$  with vanishing moments 3 and 5 respectively.

The 1D filters are shown in Fig.12. Suppose we know the decomposition filter, and we want to find the reconstruction filter by solving (21), such that the filter has support as concentrated as possible. Without loss of generality, (21) can be solved assuming that  $m_0$  is real. It is not necessary that the corresponding  $\widetilde{m}_0$  is also real, but in this testing case,  $m_0$  and  $\widetilde{m}_0$  are both real. Fig.13 shows the ground truth  $m_0$  and  $\widetilde{m}_0$  considered in this simulation.

Let  $\widehat{m}_0(\omega)$  be the approximation of  $\widetilde{m}_0(\omega)$ , which is solution of the following optimization problem

$$\min_{\mathbf{x}} \|\mathbf{D}\mathbf{x}\|^2 + \|\mathbf{x}\|^2, \quad s.t. \mathbf{A}\mathbf{x} = \mathbf{1} \quad (40)$$

where  $\mathbf{A}$  in the constraint is the matrix generated from (18) (in 1D, only a single shift of  $\pi$  appears in the condition, so each row of  $\mathbf{A}$  has two non-zero entries). Figure 14 compares the solution of (40) and the ground truth. The support of the solution is slightly more spread out than the ground truth.

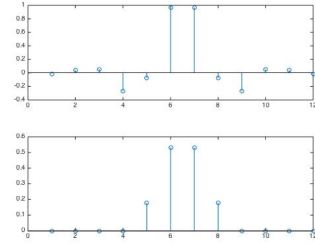


Figure 12: 1D filters, up: LoD, down: LoR

## C.2 Numerical optimization of $\widetilde{m}_0(\omega)$ in 2D

In the 2D case, we use the pair of bi-orthogonal low-pass filters that are the tensor products of the 1D filters in Section C.1 as ground truth. We solve the 2D version of the optimization problem (40). Figure 15 shows the solution and compares it with the ground truth.

To make the support of  $\widetilde{m}_0(\omega)$  better concentrate within the low frequency domain, we change the squared  $\ell_2$ -norm penalty in (40) to a weighted version (corresponding to Modulation space) as follows,

$$\min_{\mathbf{x}} \|\mathbf{D}\mathbf{x}\|^2 + \lambda \|\mathbf{w} \circ \mathbf{x}\|^2, \quad s.t. \mathbf{A}\mathbf{x} = \mathbf{1} \quad (41)$$

where  $\circ$  is Hadamard product and  $\mathbf{w}$  is a weight vector. In particular, we choose  $\forall \omega$ ,  $\mathbf{w}(\omega) = \|\omega\|$ . Fig.16 shows the solution of (41) with  $\lambda = 600$ .

Compared to (30) proposed to solve  $\widetilde{m}_0(\omega)$ , both optimization problems (40) and (41) in this simulation minimize total variation(TV) of  $\widetilde{m}_0$  instead of  $\widetilde{m}_0 \cdot m_0$  and have an extra (weighted)  $\ell_2$  regularization term. Although (40) and (41) work better than (30) for tensor wavelet construction here, they do not provide better solutions in the construction of directional wavelets and cost more computation time.

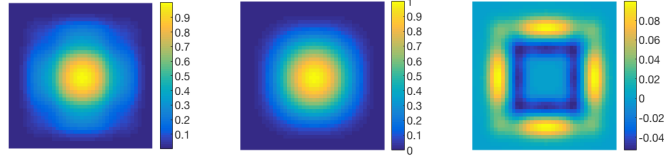


Figure 15: Left to right: solution of (40) in 2D, ground truth and their difference

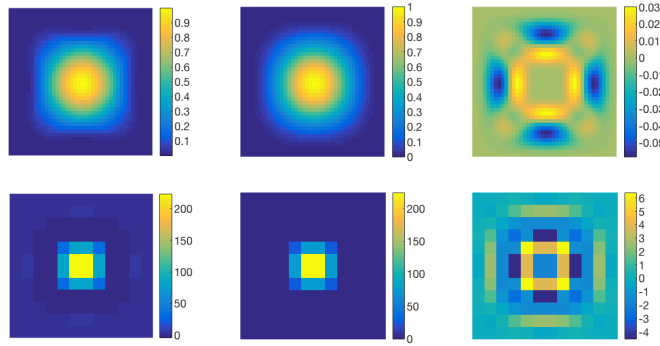


Figure 16: Left to right: solution of (41) ( $\lambda = 600$ ), ground truth and their difference; Top: frequency domain, bottom: time domain.

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