Introduction to Optimization - Tutorial 1

8 September 2023

Exercise 1 (Cone within a Sphere)
What is the maximal volume of a cone inscribed inside a unit sphere?

Exercise 2 (Gateaux vs Fréchet Differentiability)
This exercise is meant to distinguish Fréchet differentiability, Gateaux differentiability, continuity of partial derivatives and continuity.

1. It is known that Fréchet differentiability implies Gateaux differentiability. The converse does not hold (Else both notions would overlap). Prove that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is Gateaux differentiable, and fails to be Fréchet differentiable, at $(0,0)$.

2. Fréchet differentiability implies continuity at the given point. This is not the case for Gateaux differentiability. Prove that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2y^4}{x^4+y^8} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is Gateaux differentiable, but not continuous, at $(0,0)$.

Exercise 3 (Existence of Minimiser of Coervice Functions)
Recall a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *coercive* if $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = +\infty$. Show that a coercive function has a global minimiser, that is there exists a point $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $f(\hat{\mathbf{x}}) = \inf_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$.

Exercise 4 (Existence and Uniqueness of Projection)
Recall that for a closed and convex set $C \subset \mathbb{R}^n$, the projection map $P_C\colon \mathbb{R}^n \to C$ is defined as

$$P_C(\mathbf{x}) = \operatorname{argmin}_{\mathbf{y}\in C} \|\mathbf{x}-\mathbf{y}\|.$$

Prove that P_C is well-defined, that is that the minimum is attained and unique.