Introduction to Optimization

Lecture 12: Duality and algorithms.

The Fenchel conjugate

The Fenchel conjugate of a closed convex function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is the closed convex function $f^* : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in \mathbb{R}^N} \{x^* \cdot x - f(x)\}.$$

Fenchel-Young Inequality: $f(x) + f^*(x^*) \geq x^* \cdot x$.

There is equality if, and only if, $x^* \in \partial f(x)$.

Legendre-Fenchel Reciprocity Formula: $x^* \in \partial f(x) \iff x \in \partial f^*(x^*)$.

Fenchel-Rockafellar duality

Let $P \in \mathbb{R}^{M \times N}$, and let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ be closed and convex.

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The dual problem is $\inf_{y \in \mathbb{R}^M} \big\{ f^*(-P^T y) + g^*(y) \big\}$, with optimal value $v^* \in \mathbb{R}$, and set of dual solutions $S^* \subset \mathbb{R}^M$.

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Proposition

The duality gap $v + v^*$ is nonnegative.

Characterization of the primal-dual solutions

Theorem

The following statements concerning $\hat{x} \in \mathbb{R}^N$ and $\hat{y} \in \mathbb{R}^M$ are equivalent:

i) $-P^T\hat{y} \in \partial f(\hat{x})$ *and* $\hat{y} \in \partial g(P\hat{x})$*;*

ii) $f(\hat{x}) + f^*(-P^T\hat{y}) = \langle -P^T\hat{y}, \hat{x}\rangle$ *and* $g(P\hat{x}) + g^*(\hat{y}) = \langle \hat{y}, P\hat{x}\rangle$*;*

iii) $f(\hat{x}) + g(P\hat{x}) + f^*(-P^T\hat{y}) + g^*(\hat{y}) = 0$*; and*

iv) $\hat{x} \in S$ *and* $\hat{y} \in S^*$ *and* $v + v^* = 0$*.*

Moreover, if $\hat{x} \in S$ *and* g *is continuous, there exists* $\hat{y} \in \mathbb{R}^M$ *such that all four statements hold.*

Structured optimization problem

We consider the problem

$$\min \{f(x) + g(Px) + h(x)\},$$

where

- $P \in \mathbb{R}^{M\times N}$;
- $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ are closed and convex; and
- $h : \mathbb{R}^N \to \mathbb{R}$ is ℓ-smooth and convex.

Primal-dual algorithm

Chambolle-Pock (2011), Condat-Vũ, (2013):

$$\left\{\begin{array}{rcl} x_{k+1} & = & \operatorname{prox}_{\tau f}\left(x_k - \tau\nabla h(x_k) - \tau P^T y_k\right) \\ y_{k+1} & = & \operatorname{prox}_{\sigma g^*}\left(y_k + \sigma P(2x_{k+1} - x_k)\right), \end{array}\right.$$

with $\tau\sigma\|P\|^2 + \frac{\tau\ell}{2} \leq 1$.

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Limit points are solutions of the problem.

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Implementation trick: Moreau's Identity

$$\mathrm{prox}_{\sigma g^*}(y) = y - \sigma\, \mathrm{prox}_{\sigma^{-1} g}\left(\sigma^{-1} y\right).$$

TV Regularization

The Total Variation Regularization Problem is

$$\min_{x \in \mathbb{R}^{N_1 \times N_2}} \left\{ \frac{1}{2} \|Fx - b\|^2 + \rho \|Dx\|_1 \right\},$$

where F models or approximates the process by which an image x has been modified (usually deteriorated) to produce b, and D is the discrete gradient.

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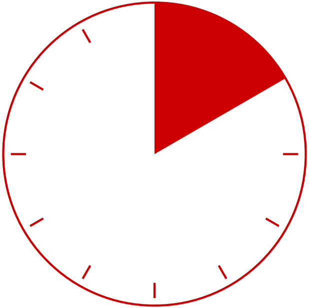
Question

Can we apply the primal-dual algorithm to this problem?

Break



Linear programming

The linear programming problem is

$$(LP) \qquad \min_{x \in \mathbb{R}^N} \{ c \cdot x \ : \ Ax \leq b \},$$

where $c \in \mathbb{R}^N$, A is a matrix of size $M \times N$, and $b \in \mathbb{R}^M$.

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Exercise

Compute the dual of the dual.

Slack variables and optimality conditions

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The primal problem can be rewritten, using a slack variable $s \in \mathbb{R}^M$, as

$$(LP) \qquad \min_{x\in\mathbb{R}^N} \{\, c\cdot x \,:\, Ax + s = b, \text{ and } s \ge 0 \,\}.$$

Primal-dual optimality conditions

Primal feasibility: $Ax + s = b$ and $s \ge 0$.
Dual feasibility: $A^T y + c = 0$ and $y \ge 0$.
Complementarity: $y_i s_i = 0$ for $i = 1, \ldots, M$.