Introduction to Optimization

Lecture 01: Motivation and examples. Optimization problems and their solutions.

university of groningen

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Law of Refraction

For a given pair of media, the ratio of the sines of the angles of incidence and refraction equals the ratio of the refractive indices of the two media.

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Some terminology

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where

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The minimizer is unique or strict if the corresponding inequality is strict.

Course description

Introductory course on optimization, where the fundamental concepts, techniques and tools are presented, discussed and put into practice by means of modelling and analysis.

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Assessment:

- 40% Written exam
- 15% Computational Exercises
- 45% Homework Assignments

Learning outcomes

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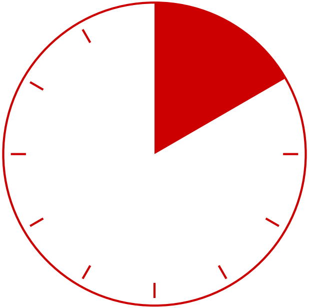
2. Characterize and calculate solutions to optimization problems by means of optimality conditions
3. Prove the convergence of optimization algorithms and establish their convergence rates
4. Implement optimization algorithms to approximate solutions numerically

Break



Motivation from data analysis

We begin with a data set

$$\mathcal{D} = \{(a_j, y_j) \in \mathbb{R}^K \times \mathbb{R}^M : j = 1, 2, \dots, J\},$$

where the a_j's are features and the y_j's are observations or labels.

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The aim is to discover or learn a function $\phi : \mathbb{R}^K \to \mathbb{R}^M$ such that

$$\phi(a_j) \simeq y_j$$

for, let us say, many j's.

The optimization point of view

Typically, ϕ belongs to a family of functions and, within that family, it is defined by some unknown parameter $x \in \mathbb{R}^N$.

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We define a loss function

$$\mathcal{L}_{\mathcal{D}}(x) = \frac{1}{J} \sum_{j=1}^{J} \ell(a_j, y_j, x),$$

in such a way that the parameters most consistent with the data set are the ones that give the lowest values of $\mathcal{L}_{\mathcal{D}}$.

Regularization

The term regularization refers loosely to changes in the problem that either simplify the resolution or induce properties in the solutions.

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Regularization also helps reduce overfitting.

Least squares

If $\phi : \mathbb{R}^N \to \mathbb{R}$ is linear, namely $y = \phi(a) = x^* a$, we find x my

$$\min_{x \in \mathbb{R}^N} \frac{1}{2J} \sum_{j=1}^{J} (x^* a_j - y_j)^2 = \min_{x \in \mathbb{R}^N} \frac{1}{2J} \|Ax - y\|^2.$$

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Regularization: Ridge, LASSO, TV...

Support vector machines

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Nonlinear transformations of the space are possible. ✍

Other examples

- Matrix factorization
- Matrix completion (a.k.a. the *Netflix Prize* Problem) ✍
- Logistic regression
- Image processing: deblurring, in-painting
  - Topic of the computational project
- Face, voice and pattern recognition
- (Deep) learning ✍

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- The functions involved are continuous. They are either smooth, or they have simple or structured forms of nonsmoothness.
- The function to be minimized is often a sum of simple functions that depend either on few data points or involve few variables.
- Usually, the nonsmooth parts can be separated from the rest (additive structure).