# Introduction to Optimization - Tutorial 7

19th of October 2023

**Exercise 1 (Proximal-Gradient Method)**
In this exercise we shall prove the convergence of the Proximal-Gradient method. We consider two functions $f, g\colon \mathbb{R}^d \to \mathbb{R}$ which are closed, proper, convex and lower-semicontinuous. We additionally assume that $f$ is $L$-smooth. Out aim is to find a minimiser of $f+g$ over $\mathbb{R}^d$. As usually, we assume $f+g$ has a minimiser, and we call it $\hat{x}$.

To this extend, we iteratively apply the *Proximal-Gradient method*, which is given, for an initial guess $x_0 \in \mathbb{R}^d$, by

$$x_{k+1} = \mathrm{prox}_{\alpha g}(x_n - \alpha \nabla f(x_n)).$$

To guarantee convergence, we assume $\alpha \in (0, 1/L)$.

1. Start by showing that
$$0 \in \partial g(x_{n+1}) + \nabla f(x_n) + \frac{1}{\alpha}(x_{n+1} - x_n).$$
2. Next show that
$$g(x_{n+1}) \leq g(x_n) + \frac{1}{\alpha}(x_n - x_{n+1} - \alpha \nabla f(x_n)) \cdot (x_{n+1} - x_n).$$
3. Conclude, using the Descent Lemma on $f$, that $f(x_n) + g(x_n)$ is nonincreasing.
4. Show that
$$\|x_{n+1} - x_n - \alpha(\nabla f(x_{n+1}) - \nabla f(x_n))\| \leq \|x_{n+1} - x_n\|$$
5. Show that
$$f(x_n) \leq f(\hat{x}) + \nabla f(x_n) \cdot (x_n - \hat{x}),$$
and
$$g(x_{n+1}) \leq g(\hat{x}) + \frac{1}{\alpha}(x_n - x_{n+1} - \alpha \nabla f(x_n)) \cdot (x_{n+1} - \hat{x}).$$
6. Conclude that $f(x_n) + g(x_n) - \min(f+g)$ converges with rate $C/n$, for some constant $C > 0$ not depending on $n$.

**Exercise 2 (Super-Additivity of Subdifferential Operator)**
Let $f, g\colon \mathbb{R}^d \to \mathbb{R}$ be closed, convex and lower-semicontinuous. Show that, for all $x \in \mathbb{R}^d$, it holds that

$$\partial f(x) + \partial g(x) \subset \partial(f+g)(x).$$