Introduction to Optimization

Lecture 09: Subgradient descent.
The proximal-gradient algorithm.

university of groningen

Reminder

A vector $v \in \mathbb{R}^N$ is a subgradient of f a convex function $f : \mathrm{dom}(f) \subset \mathbb{R}^N \to \mathbb{R}$ at the point x if

$$f(y) \geq f(x) + \langle v, y - x \rangle$$

for all $y \in \mathbb{R}^N$.

Proposition

If $f : \mathrm{dom}(f) \subset \mathbb{R}^N \to \mathbb{R}$ is convex, for each $x \in \mathrm{int}(\mathrm{dom}(f))$, there exist $L_x, r_x > 0$ such that

$$|f(z) - f(y)| \leq L_x \|z - y\|$$

for all $z, y \in B(x, r_x)$. Moreover, $\emptyset \neq \partial f(x) \subseteq \bar{B}(0, L_x)$.

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Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex and Lipschitz-continuous with constant M ($|f(x) - f(y)| \leq M\|x - y\|$) with minimizers, and let (x_n) be defined by the subgradient method. Set $\bar{x}_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k$. Then,

$$\min_{k=1,\ldots,n} \left(f(x_k) - \min(f)\right) \leq f(\bar{x}_n) - \min(f) \leq \frac{\alpha M^2}{2} + \frac{dist(x_0, S)^2}{2\alpha(n+1)}.$$

The subgradient method

$x_{n+1} = x_n - \alpha v_n$ with $v_n \in \partial f(x_n)$

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Question

Given $\varepsilon > 0$, after how many iterations can we be sure to have found a point $\hat{x}$ such that $f(\hat{x}) - \min(f) \leq \varepsilon$?

Extended real-valued functions, I

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$$\begin{aligned} \mathrm{dom}(f) &= \{x \in \mathbb{R}^N \ : \ f(x) < +\infty\} \\ \mathrm{epi}(f) &= \{(x, z) \in \mathbb{R}^{N+1} \ : \ f(x) \le z\}, \end{aligned}$$

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Notice that $\mathrm{dom}(\lambda f + g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

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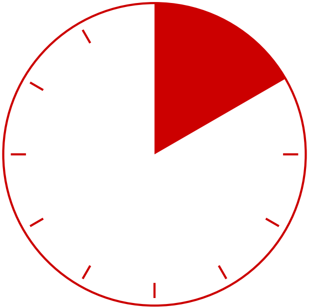
If $f : \mathbb{R}^N \to \mathbb{R}$, $\min\{f(x) : x \in C\} = \min\left\{f(x) + \iota_C(x) : x \in \mathbb{R}^N\right\}$.

Break



Closedness and proximity operator

If $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is closed and convex, and $y \in \mathbb{R}^N$, the function

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$$\partial f_y(x) = \partial f(x) + x - y,$$

for each $x \in \mathrm{dom}(f)$.

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for each $x \in \mathrm{dom}(f)$. The unique minimizer of f_y is denoted by $\mathrm{prox}_f(y)$, and is characterized by

$$y - \mathrm{prox}_f(y) \in \partial f\big(\mathrm{prox}_f(y)\big).$$

The proximal method

If $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is closed and convex, and fix $\alpha > 0$. From an initial point $x_0 \in \mathbb{R}^N$, define a sequence inductively by

$$x_{n+1} = \text{prox}_{\alpha f}(x_n).$$

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Exercise

If $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is closed and convex, and $S \neq \emptyset$, then x_n converges to a point in S. Moreover,

$$f(x_n) - \min(f) \leq \frac{\mathrm{dist}(x_0, S)^2}{2\alpha n}, \qquad n \geq 1.$$

Proximal-gradient algorithm

Suppose we want to find the minima of $f = g + h$, where $g : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is convex and lower-semicontinuous, and $h : \mathbb{R}^N \to \mathbb{R}$ is convex and L-smooth.

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Example

A typical example in image and signal processing, statistics, ML, is

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \rho\|x\|_1$$

for $x \in \mathbb{R}^N$.

Proximal-gradient algorithm

The proximal-gradient method consists in applying proximal iterations while linearizing the smooth function:

$$x_{n+1} = \operatorname{Argmin}\left\{g(x) + h(x_n) + \langle \nabla h(x_n), x - x_n \rangle + \frac{1}{2\gamma}\|x - x_n\|^2\right\}$$

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This subproblem has a unique solution characterized by

$$0 \in \partial g(x_{n+1}) + \nabla h(x_n) + \frac{1}{\gamma}(x_{n+1} - x_n).$$

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In turn, this can be rewritten as

$$x_{n+1} = (I + \gamma \partial g)^{-1}(I - \gamma \nabla h)x_n,$$

where we identify a gradient subiteration with respect to h, and then a proximal subiteration with respect to g.

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Convergence of proximal-gradient sequences

Theorem

Let $f = g + h$, *where* $g : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ *is convex and lower-semicontinuous, and* $h : \mathbb{R}^N \to \mathbb{R}$ *is convex and L-smooth. Take* $\gamma \in (0, 2/L)$ *and define* (x_n) *by*

$$x_{n+1} = \big(I + \gamma \partial g\big)^{-1}(I - \gamma \nabla h)x_n, \qquad n \geq 0.$$

Then, x_n *converges to a minimizer of* f, *and there is* $C > 0$ *such that*

$$f(x_n) - \min(f) \leq \frac{dist(x_0, \operatorname{Argmin}(f))^2}{2\gamma n}, \qquad n \geq 1.$$

Moreover, $\lim\limits_{n \to \infty} n\big(f(x_n) - \min(f)\big) = 0.$