Notes on Discrete Time Signal Processing 3

Rom Parnichkun

November 17, 2022

1 Discrete Time Signals

$$x[n] = x(nT) \tag{1}$$

A discrete signal x sampled at period T.

1.1 Basic sequences

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases} \tag{2}$$

This is the **unit sample** (impulse) function.

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[k-n]x[k] \tag{3}$$

This derivation seems quite redundant, however it will be useful later on when analyzing linear time invariant systems.

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases} = \sum_{k=-\infty}^{\infty} \delta = \sum_{k=0}^{\infty} \delta[n-k] \tag{4}$$

This is the **unit step** function.

$$x[n] = A\alpha^n \tag{5}$$

General exponential sequence.

$$\begin{aligned} x[n] &= |A||\alpha|^n e^{(\omega_0 n + \phi)j} && (6) \\ &= |A||\alpha|^n \left[\cos(\omega_0 n + \phi)\sin(\omega_0 n + \phi)\right] && (7) \end{aligned}$$

Setting $A = |A|e^{j\phi}$ and $\alpha = |\alpha|e^{j\omega_0}$.

$$
\begin{aligned}
x[n] &= |A||\alpha|^n e^{(\omega_1 n+\phi)j} && (8)\\
&= |A||\alpha|^n e^{((2\pi+\omega_0)n+\phi)j} && (9)\\
&= |A||\alpha|^n e^{(\omega_0 n+\phi)j} e^{2\pi nj} && (10)\\
&= |A||\alpha|^n e^{(\omega_0 n+\phi)j} 1 && (11)
\end{aligned}
$$

It could be noted that the effective frequency range in discrete systems is 2π because n is an integer. As an example, here we set $\omega_1 = 2\pi + \omega_0$.

---

2 Discrete time systems

$$y[n] = T\{x[n]\} \tag{12}$$

General representation of discrete time systems. Note that the output sequence at each value of the index n may depend on input samples $x[n]$ of all n.

$$
\begin{aligned}
T\{x_1[n] + x_2[n]\} &= T\{x_1[n]\} + T\{x_2[n]\}\\
&= y_1[n] + y_2[n] && (13)
\end{aligned}
$$

$$T\{ax[n]\} = aT\{x[n]\} = ay[n] \tag{14}$$

Linear systems are defined by the principle of superposition. Equation 13 follows the property of *additivity*, and Equation 14 follows the property of *scaling* or *homogeneity.*

$$
\begin{aligned}
y[n] &= T\{\sum_{-\infty}^{\infty} x[k]\delta[n-k]\} && (15)\\
&= \sum_{-\infty}^{\infty} x[k]T\{\delta[n-k]\} && (16)\\
&= \sum_{-\infty}^{\infty} x[k]h_k[n] && (17)\\
&= \sum_{-\infty}^{\infty} x[k]h[n-k] && (18)
\end{aligned}
$$

Linear time-invariant systems (LTI). Because of the time-invariant property, $h_k[n]$ in Equation 17 can be used for all values of k.

$$y[n] = x[n] * h[n] \tag{19}$$

As a consequence of Equation 18, LTI systems can be represented as a convolutional operator on the input data.

$$x[n] * h[n] = h[n] * x[n] \tag{20}$$

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n] \tag{21}$$

$$y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]) \tag{22}$$

Properties of LTI systems. LTI are commutative (Equation 20), distributive (Equation 21), associative (Equation 22). Therefore, multiple LTI systems can be combined into one.

$h[n] = 0, \quad n < 0$	(23)	Causal LTI system only have response for $n >= 0$.