GK金融加速器(Bocola2016,JPE)

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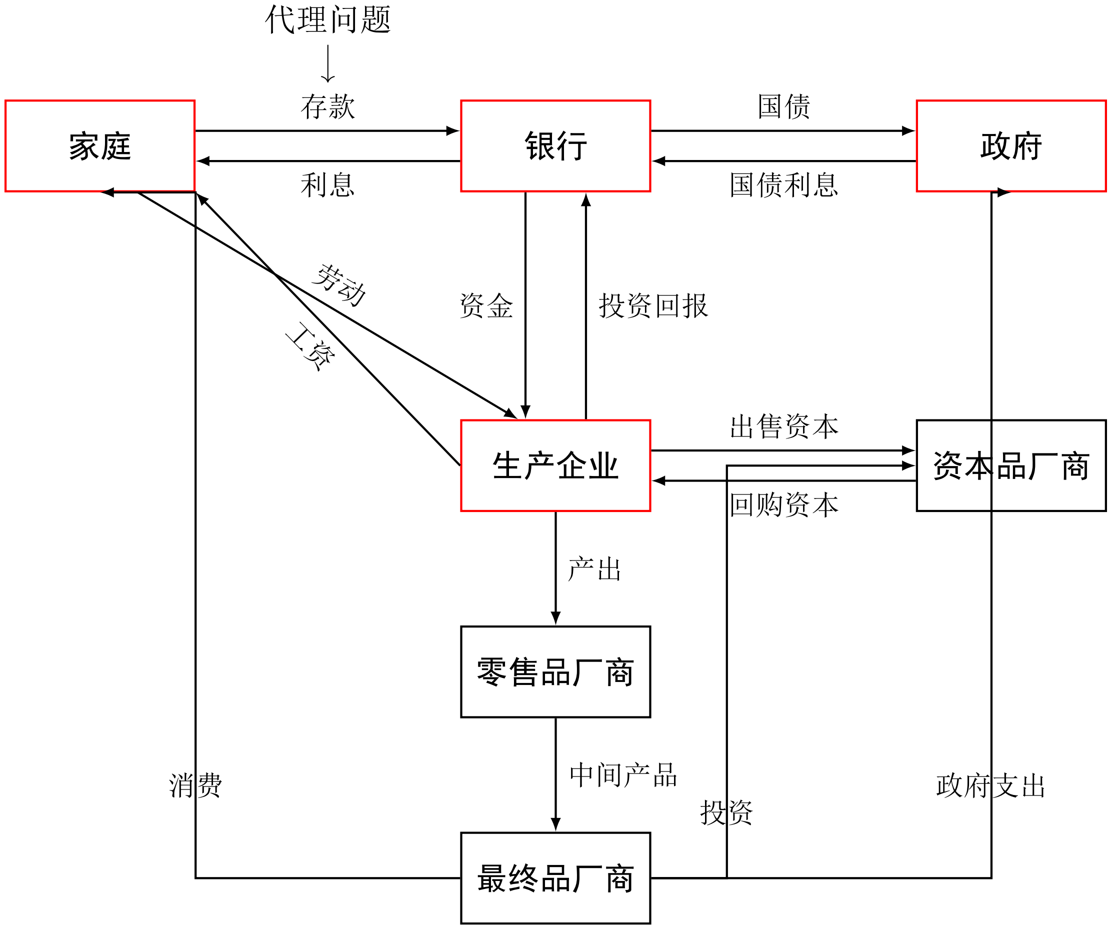
目录

1 模型结构图



2 家庭

$$
\begin{aligned}
\max_{C_t,H_t,D_{t+1}} \quad & \sum_{t=0}^{\infty}\beta^t\left[\ln(C_t-\gamma C_{t-1})+\zeta\ln(1-H_t)\right] \\
\text{s.t.} \quad & P_{t+1}D_{t+1}=R_t(P_tW_tH_t+P_tD_t+P_tJ_t-P_tC_t-P_tT_t) \\
\Rightarrow \quad & C_t+\frac{D_{t+1}}{R_t}=W_tH_t+\frac{1}{\Pi_{t+1}}D_t+J_t-T_t
\end{aligned}
$$

$$
\frac{\partial\mathcal{L}}{\partial C_t}: \quad \frac{1}{C_t-\gamma C_{t-1}}-\lambda_t-\beta\gamma\frac{1}{C_{t+1}-\gamma C_t}=0 \tag{1}
$$

$$
\frac{\partial\mathcal{L}}{\partial H_t}: \quad -\zeta\frac{1}{1-H_t}+\lambda_tW_t=0 \tag{2}
$$

$$
\frac{\partial\mathcal{L}}{\partial D_{t+1}}: \quad -\lambda_t\frac{1}{R_t}+\beta\lambda_{t+1}\frac{1}{\Pi_{t+1}}=0 \tag{3}
$$

其中D是储蓄。下一期的储蓄就是这一期存下来的钱乘上利率。

3 银行

基本设定

- 银行与家庭之间存在**信息不对称**，存在**代理人问题** 。
- 即，银行每期都面临**破产风险**，并且如果银行市值低于资产价值的一定比例，银行会宣布破产。

Balance Sheet (资产负债表):

$$\underbrace{Q_t^K K_{t+1} + Q_t^B B_{t+1}}_{\text{资产 (实物)}} = \underbrace{n_t + \frac{D_{t+1}}{R_t}}_{\text{净资产+负债}} \tag{1}$$

银行的最大化目标 (市值):

设 ψ 为银行的存活率 (Survival Rate)，且满足标准化条件 $\sum_{k=1}^{\infty} \psi^{k-1}(1-\psi) = 1$。银行最大化其预期折现红利流：

$$\begin{aligned}
V_t &= E_t\left[\sum_{k=1}^{\infty} \frac{\lambda_{t+k}}{\lambda_t}\psi^{k-1}(1-\psi)n_{t+k}\right] \\
&= E_t\left[\frac{\lambda_{t+1}}{\lambda_t}(1-\psi)n_{t+1} + \sum_{k=2}^{\infty} \frac{\lambda_{t+k}}{\lambda_t}\psi^{k-1}(1-\psi)n_{t+k}\right] \\
&\Big\downarrow \text{令 } k \equiv r+1 \\
&= E_t\left[\frac{\lambda_{t+1}}{\lambda_t}(1-\psi)n_{t+1} + \sum_{r=1}^{\infty} \frac{\lambda_{t+r+1}}{\lambda_t}\psi^{r}(1-\psi)n_{t+r+1}\right] \\
&= E_t\left[\frac{\lambda_{t+1}}{\lambda_t}(1-\psi)n_{t+1} + \psi\frac{\lambda_{t+1}}{\lambda_t}\underbrace{\sum_{r=1}^{\infty} \frac{\lambda_{t+r+1}}{\lambda_{t+1}}\psi^{r-1}(1-\psi)n_{t+r+1}}_{V_{t+1}}\right]
\end{aligned}$$

最终整理得到贝尔曼方程形式 (递归形式):

$$V_t = E_t\left\{\frac{\lambda_{t+1}}{\lambda_t}\left[(1-\psi)n_{t+1} + \psi V_{t+1}\right]\right\}$$

银行的净资产动态:

$$n_{t+1} = R_{t+1}^K Q_t^K K_{t+1} + R_{t+1}^B Q_t^B B_{t+1} - R_t \frac{D_{t+1}}{R_t} \tag{2}$$

3.1 最优化问题求解

Step 1 & 2: 猜测值函数形式(guess and verify) 由于值函数是 n_{t+1} 的线性式，我们猜测 $V_t(n_t)$ 也是线性的：

$$V_t(n_t) = A_t n_t \quad \Rightarrow \quad V_{t+1}(n_{t+1}) = A_{t+1} n_{t+1} \tag{3}$$

代入贝尔曼方程：

$$V_t(n_t) = A_t n_t = \frac{\lambda_{t+1}}{\lambda_t}\left[(1-\psi)n_{t+1} + \psi A_{t+1} n_{t+1}\right] = \frac{\lambda_{t+1}}{\lambda_t}(1-\psi+\psi A_{t+1})n_{t+1} \tag{4}$$

Step 3: 整合约束 由资产负债表得 $D_{t+1}/R_t = Q_t^K K_{t+1} + Q_t^B B_{t+1} - n_t$，代入净资产公式：

$$\begin{aligned} n_{t+1} &= R_{t+1}^K Q_t^K K_{t+1} + R_{t+1}^B Q_t^B B_{t+1} - R_t(Q_t^K K_{t+1} + Q_t^B B_{t+1} - n_t) \\ &= (R_{t+1}^K - R_t)Q_t^K K_{t+1} + (R_{t+1}^B - R_t)Q_t^B B_{t+1} + R_t n_t \end{aligned} \tag{5}$$

Step 4: 转化为最优化问题

$$\begin{aligned} V_t(n_t) = \max_{K_{t+1}, B_{t+1}} & E_t\left\{\frac{\lambda_{t+1}}{\lambda_t}(1-\psi+\psi A_{t+1})n_{t+1}\right\} \\ \text{s.t.} \quad & n_{t+1} = (R_{t+1}^K - R_t)Q_t^K K_{t+1} + (R_{t+1}^B - R_t)Q_t^B B_{t+1} + R_t n_t \\ & V_t(n_t) = A_t n_t \geq \lambda^K Q_t^K K_{t+1} + \lambda^B Q_t^B B_{t+1} \quad \text{(激励相容/流动性约束)} \end{aligned}$$

构造拉格朗日函数 $\mathcal{L}$：

$$\begin{aligned} \mathcal{L} \equiv & \frac{\lambda_{t+1}}{\lambda_t}(1-\psi+\psi A_{t+1})\left[(R_{t+1}^K - R_t)Q_t^K K_{t+1} + (R_{t+1}^B - R_t)Q_t^B B_{t+1} + R_t n_t\right] \\ & + \mu_t\left(A_t n_t - \lambda^K Q_t^K K_{t+1} - \lambda^B Q_t^B B_{t+1}\right) \end{aligned}$$

一阶条件 (FOCs):

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}}: \quad \frac{\lambda_{t+1}}{\lambda_t}(1-\psi+\psi A_{t+1})(R_{t+1}^K - R_t)Q_t^K = \mu_t \lambda^K Q_t^K \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}}: \quad \frac{\lambda_{t+1}}{\lambda_t}(1-\psi+\psi A_{t+1})(R_{t+1}^B - R_t)Q_t^B = \mu_t \lambda^B Q_t^B \tag{7}$$

$$\text{松弛互补}: \quad \mu_t(A_t n_t - \lambda^K Q_t^K K_{t+1} - \lambda^B Q_t^B B_{t+1}) = 0 \tag{8}$$

Step 5: 求解 A_t 与 μ_t

将 FOCs 代入目标函数，可得 A_t 的递归式：

$$V_t(n_t) = A_t n_t = \mu_t \lambda^K Q_t^K K_{t+1} + \mu_t \lambda^B Q_t^B B_{t+1} + \frac{\lambda_{t+1}}{\lambda_t}(1-\psi+\psi A_{t+1})R_t n_t$$

$$\text{由KKT} \implies A_t n_t = \mu_t A_t n_t + \frac{\lambda_{t+1}}{\lambda_t}(1-\psi+\psi A_{t+1})R_t n_t$$

$$\Rightarrow A_t = \frac{1}{1-\mu_t}\underbrace{\frac{\lambda_{t+1}}{\lambda_t}(1-\psi+\psi A_{t+1})}_{\Omega_{t+1}} R_t = \frac{1}{1-\mu_t}\Omega_{t+1} R_t \tag{9}$$

(注: 若 $\mu_t < 1$，则 $A_t > 0, V_t > 0$)

关于乘子 μ_t 的求解：

1. 若 $\mu_t > 0$，则约束绑定，$A_t n_t = \lambda^K Q_t^K K_{t+1} + \lambda^B Q_t^B B_{t+1}$。代入 A_t 可解得：

$$\mu_t = 1 - \frac{\Omega_{t+1} R_t n_t}{\lambda^K Q_t^K K_{t+1} + \lambda^B Q_t^B B_{t+1}}$$

2. 若约束不绑定，则 $\mu_t = 0$。

综上所述：

$$\mu_t = \max\left(1 - \frac{\Omega_{t+1} R_t n_t}{\lambda^K Q_t^K K_{t+1} + \lambda^B Q_t^B B_{t+1}},\quad 0\right) < 1 \tag{10}$$

4 生产企业

从家庭雇佣劳动，以 $P_{m,t}$ 价格生产商品。每期生产结束向银行支付上期贷款本息。生产结束后将减耗的资本品卖给资本品厂商。

最大化目标:

$$\begin{aligned}
&\max_{H_t} \quad P_{m,t} Y_t^e - W_t H_t - R_t^K Q_{t-1} K_t + Q_t^K (1-\delta) K_t \\
&\text{s.t.} \quad Y_t^e = Z_t K_t^\alpha H_t^{1-\alpha}
\end{aligned} \tag{14}$$

FOC:

$$\begin{aligned}
W_t &= P_{m,t} Z_t K_t^\alpha (1-\alpha) H_t^{-\alpha} \\
W_t H_t &= (1-\alpha) P_{m,t} Y_t^e
\end{aligned} \tag{15}$$

假设生产企业完全竞争，zero-profit condition:

$$\begin{aligned}
&0 = P_{m,t} Y_t^e - (1-\alpha) P_{m,t} Y_t^e - R_t^K Q_{t-1}^K K_t + Q_t^K (1-\delta) K_t \\
\Rightarrow \quad &R_t^K = \frac{\alpha P_{m,t} Y_t^e}{Q_{t-1}^K K_t} + \frac{Q_t^K (1-\delta)}{Q_{t-1}^K}
\end{aligned} \tag{16}$$

4. 最终产品企业

Max:

$$\begin{aligned}
&\max_{Y_t^i} \quad P_t Y_t - \int_0^1 P_t^i Y_t^i di \\
&\text{s.t.} \quad Y_t = \left[\int_0^1 (Y_t^i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}
\end{aligned}$$

FOC (Demand Function):

$$Y_t^i = \left(\frac{P_t^i}{P_t}\right)^{-\varepsilon} Y_t$$

Zero-profit condition (Price Index):

$$P_t = \left(\int_0^1 P_t^{i^{1-\varepsilon}} di \right)^{\frac{1}{1-\varepsilon}}$$

5 零售商 (Retailers)

$$\begin{aligned}
\max_{P_t^{i*}} \quad & E_t \sum_{k=0}^{\infty} (\theta\beta)^k \frac{\lambda_{t+k}}{\lambda_t} \left(\frac{P_t^{i*}}{P_{t+k}} - P_{m,t+k} \right) Y_{t+k}^{i*} \\
\text{s.t.} \quad & Y_{t+k}^{i*} = \left(\frac{P_t^{i*}}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k}
\end{aligned}$$

最优定价与递归形式:

$$\Pi_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{X_t^1}{X_t^2} \tag{17}$$

$$X_t^1 = \lambda_t P_{m,t} Y_t + \theta\beta \Pi_{t+1}^{\varepsilon} X_{t+1}^1 \tag{18}$$

$$X_t^2 = \lambda_t Y_t + \theta\beta \Pi_{t+1}^{\varepsilon-1} X_{t+1}^2 \tag{19}$$

价格演变:

$$1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta)(\Pi_t^*)^{1-\varepsilon} \tag{20}$$

6 政府部门

每期有 δ 比例的国债到期，剩余的国债支付 l 的利息。

6.1 无风险情况

$$\begin{aligned}
R_{t+1}^B Q_t^B B_{t+1} &= \delta B_{t+1} + (1-\delta)(l + Q_{t+1}^B) B_{t+1} \\
\Rightarrow \quad R_{t+1}^B &= \frac{\delta + (1-\delta)(1 + Q_{t+1}^B)}{Q_t^B}
\end{aligned}$$

6.2 假设国债有违约的风险

假设违约后进行清算，清算成本为 P。

破产概率：

$$PD_{t+1} = P(e_{t+1} = 1) = \frac{\exp(M_{t+1})}{1 + \exp(M_{t+1})}$$

其中过程为：

$$M_{t+1} = (1-\rho_s)\overline{M} + \rho_s M_t + \varepsilon^s_{t+1}$$

变量替换：$S_t \equiv \exp(M_t) \Leftrightarrow M_t = \ln(S_t)$

那么：

$$PD_{t+1} = \frac{S_{t+1}}{1+S_{t+1}} \tag{21}$$

$$\ln(S_{t+1}) = (1-\rho_s)\ln(\overline{S}) + \rho_s \ln(S_t) + \varepsilon^S_{t+1} \tag{22}$$

有风险情况下，国债回报率成为预期值:

$$\begin{aligned} R^B_{t+1} &= (1-PD_{t+1})\left(\frac{\delta+(1-\delta)(1+Q^B_{t+1})}{Q^B_t}\right) \\ &\quad + PD_{t+1}(1-P)\left(\frac{\delta+(1-\delta)(1+Q^B_{t+1})}{Q^B_t}\right) \\ &= [(1-P)PD_{t+1} + (1-PD_{t+1})]\left(\frac{\delta+(1-\delta)(1+Q^B_{t+1})}{Q^B_t}\right) \end{aligned} \tag{23}$$

财政规则:

$$T_t = \gamma_\tau B_{t-1} \tag{24}$$

这个是政府债务的一个比例。保证债务收敛。 **政府预算:**

$$Q^B_t B_{t+1} = [\delta + (1-\delta)(1+Q^B_t)]B_t + G_t - T_t \tag{25}$$

政府支出:

$$\ln(G_t) = \rho_g \ln(G_{t-1}) + (1-\rho_g)\ln(\overline{G}) + \varepsilon_t \tag{26}$$

7 资本品生产企业

资本积累方程:

$$K_{t+1} = (1-\delta)K_t + I_t - \frac{\chi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t \tag{27}$$

最优化问题:

$$\max_{I_t} \quad Q^K_t \left[I_t - \frac{\chi}{2}\left(\frac{I_t}{K_t}-\delta\right)^2 K_t\right] - I_t$$

FOC:

$$Q^K_t = \left[1 - \chi\left(\frac{I_t}{K_t} - \delta\right)\right]^{-1} \tag{28}$$

8 央行

$$i_t = \frac{R_t}{\Pi_{t+1}} \tag{29}$$

泰勒规则:

$$\ln(i_t) = \phi_\pi \ln\left(\frac{\pi_t}{\overline{\pi}}\right) + \varepsilon_t^i \tag{30}$$

9 市场出清和加总

生产加总:

$$Y_t^e = \int_0^1 Y_t^i di = \int_0^1 \left(\frac{P_t^i}{P_t}\right)^{-\varepsilon} diY_t = d_t Y_t \tag{31}$$

价格离散度 :

$$d_t = \theta \Pi_t^\varepsilon d_{t-1} + (1-\theta)(\Pi_t^*)^{-\varepsilon} \tag{32}$$

技术冲击:

$$\ln(Z_t) = \rho_z \ln(Z_{t-1}) + \varepsilon_t^z \tag{33}$$

资源约束:

$$Y_t = C_t + I_t + G_t + \frac{\chi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t \tag{34}$$