Homework 3

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Problem 3.1

Definitions:

1. $f(n) \in \Theta(g) \iff \exists n_0, c_1, c_2 \in \mathbb{N} \| \forall n > n_0 \| c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$
2. $f(n) \in O(g) \iff \exists n_0, c \in \mathbb{N} \| \forall n > n_0 \| f(n) \leq c \cdot g(n)$
3. $f(n) \in \Omega(g) \iff \exists n_0, c \in \mathbb{N} \| \forall n > n_0 \| c \cdot g(n) \leq f(n)$
4. $f(n) \in o(g) \iff \exists n_0, c \in \mathbb{N} \| \forall n > n_0 \| f(n) \leq c \cdot g(n)$
5. $f(n) \in \omega(g) \iff \exists n_0, c \in \mathbb{N} \| \forall n > n_0 \| c \cdot g(n) \leq f(n)$

$$f(n) = 9n \wedge g(n) = 5n^3$$

Checking f(n):
$\lim n \to \infty \frac{9n}{5n^3}$
$= 0$

Checking g(n):
$\lim n \to \infty \frac{5n^3}{9n}$
$= \infty$

Hence:

1. $f(n) \notin \Theta(g)$
2. $f(n) \notin O(g)$
3. $f(n) \notin \Omega(g)$
4. $f(n) \in o(g)$
5. $f(n) \notin \omega(g)$
6. $g(n) \notin \Theta(f)$
7. $g(n) \notin O(f)$
8. $g(n) \notin \Omega(f)$
9. $g(n) \notin o(f)$
10. $g(n) \in \omega(f)$

$$f(n) = 9n^{0.8} + 2n^{0.3} + 14\log n \wedge g(n) = \sqrt{n} = n^{0.5}$$

Checking f(n):
$\lim n \to \infty \frac{9n^{0.8}+2n^{0.3}+14\log n}{n^{0.5}}$
$= \infty$

Checking g(n):
$\lim n \to \infty \frac{n^{0.5}}{9n^{0.8}+2n^{0.3}+14\log n}$
$= 0$

Hence:

1. $f(n) \notin \Theta(g)$
2. $f(n) \notin O(g)$
3. $f(n) \notin \Omega(g)$
4. $f(n) \notin o(g)$
5. $f(n) \in \omega(g)$
6. $g(n) \notin \Theta(f)$
7. $g(n) \notin O(f)$
8. $g(n) \notin \Omega(f)$
9. $g(n) \in o(f)$
10. $g(n) \notin \omega(f)$

$$f(n) = \frac{n^2}{\log n} \wedge g(n) = n \cdot \log n$$

Checking f(n):
$\lim n \to \infty \frac{\frac{n^2}{\log n}}{n \cdot \log n}$
$= \lim n \to \infty \frac{n}{\log^2 n}$ (apply L'Hopitals)
$= \lim n \to \infty \frac{n \cdot \ln 10}{2 \cdot \log n}$ (apply L'Hopitals)
$= \lim n \to \infty \frac{n \cdot \ln^2 10}{2}$
$= \infty$

Checking g(n):
$\lim n \to \infty \frac{n \cdot \log n}{\frac{n^2}{\log n}}$
$= \lim n \to \infty \frac{\log^2 n}{n}$ (apply L'Hopitals)
$= \lim n \to \infty \frac{2 \cdot \log n}{n \cdot \ln 10}$ (apply L'Hopitals)
$= \lim n \to \infty \frac{2}{n \cdot \ln^2 10}$
$= 0$

Hence:

1. $f(n) \notin \Theta(g)$
2. $f(n) \notin O(g)$
3. $f(n) \notin \Omega(g)$
4. $f(n) \notin o(g)$
5. $f(n) \in \omega(g)$
6. $g(n) \notin \Theta(f)$
7. $g(n) \notin O(f)$
8. $g(n) \notin \Omega(f)$
9. $g(n) \in o(f)$
10. $g(n) \notin \omega(f)$

$$f(n) = (\log 3n)^3 \wedge g(n) = 9 \cdot \log n$$

Checking f(n):
$\lim n \to \infty \frac{(\log 3n)^3}{9 \cdot \log n}$
$= \lim n \to \infty \frac{(\log 3n)^3}{\log n^9}$ (apply L'Hopitals)
$= \lim n \to \infty \frac{\log^2(3n)}{3}$
$= \infty$

Checking g(n):
$\lim n \to \infty \frac{9 \cdot \log n}{(\log 3n)^3}$
$= \lim n \to \infty \frac{\log n^9}{(\log 3n)^3}$ (apply L'Hopitals)
$= \lim n \to \infty \frac{3}{\log^2(3n)}$
$= 0$

Hence:

1. $f(n) \notin \Theta(g)$
2. $f(n) \notin O(g)$
3. $f(n) \notin \Omega(g)$
4. $f(n) \notin o(g)$
5. $f(n) \in \omega(g)$
6. $g(n) \notin \Theta(f)$

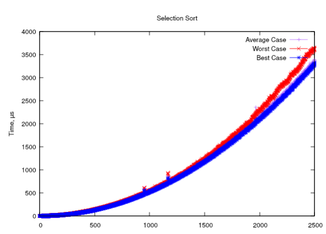
7. $g(n) \notin O(f)$
8. $g(n) \notin \Omega(f)$
9. $g(n) \in o(f)$
10. $g(n) \notin \omega(f)$

Problem 3.2

Proof that Selection Sort is Correct

Precondition: Array with n elements is unsorted ($A_{[0,n)}$).
Loop Invariant: Elements in $A_{[0,i)}$ are sorted, such that $A_i \leq A_{i+1}$.
for $i := 0$; $i < n$ {
    $A_i := min(A_{[i,n)})$
    $\forall x \in A_{[i,n)} \| A_x \geq A_i$
    $i := i + 1$
}
Postcondition: $A_{[0,n)}$ is sorted. ■

Selection Sort Runtime



Let $T(n)$ be the average time complexity.
$\exists n_0, c_1, c_2 \in \mathbb{N} \| c_1 \cdot n^2 \leq T(n) \leq c_2 \cdot n^2, \forall n > n_0$

$n_0 = 1250$

$c_1 = \frac{3292}{2500^2}$

$c_2 = \frac{3638}{2500^2}$