Introduction to Computer Science - Homework 2

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September 20, 2023

Problem 2.1

Show: $\forall a \in \mathbb{Z} \mid a^{32}$ is odd $\implies a^4$ is odd.

Proof by contrapositive:

Consider that a^4 is even.

We know that $\forall n \in \mathbb{Z} \mid$ n is even $\implies n^2$ is even.

Hence:

${a^4}^{2^{2^2}} = a^{32}$ is even.

$\therefore$ Contrapositive is true $\implies$ original statement is true.

a^{32} is odd $\implies a^4$ is odd. $\blacksquare$

Problem 2.2

Show: $\forall n \in \mathbb{N} \mid n \geq 1 \mid n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Proof by induction:

Base case:

$$(1)^3 + (1+1)^3 + (1+2)^3$$

$$= 1 + 8 + 9$$

$$= 18 \land 18 \equiv 0 \pmod 9$$

Induction hypothesis:

$$n^3 + (n+1)^3 + (n+2)^3$$

$$= 3n^3 + 9n^2 + 15n + 9$$

Induction step: $n \to n+1$

Substituting the *induction step* into the *induction hypothesis* results in:

$$((n+1)+1)^3 + ((n+1)+2)^3 + ((n+1)+2)^3$$

$$= (n+1)^3 + (n+2)^3 + (n+3)^3$$

$$= 3n^3 + 18n^2 + 42n + 36$$

Now we check if the difference between the $n+1$ -th term and the n-term is divisible by 9:

$$3n^3 + 18n^2 + 42n + 36$$

$$-3n^3 + 9n^2 + 15n + 9$$

$$= 9n^2 + 27n + 27$$

Notice that $\forall x \in \{9n^2, 27n, 27\} \mid x \equiv 0 \pmod 9$.

$\therefore$ Since the difference between our *induction hypothesis* and its subsequent term are divisible by 9, then $\forall n \in \mathbb{N} \mid n \geq 1$ the *induction hypothesis* is true. $\blacksquare$

Problem 2.3

Listing 1: Haskell code of divisor and sigma functions

```
-- Return the list of positive divisors of an integer n.
divisors :: Int -> [Int]
divisors n = [ x | x <- [1..n], mod n x == 0 ]

-- Return the sum of divisors of n taken to the power of z
sigma :: Int -> Int -> Int
sigma z n = sum [x ^ z | x <- divisors n]
```