Introduction to Computer Science - Homework 4

Rumen Mitov

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Problem 4.1

Definition 1:

A function, f:$X \to Y$, is injective $\iff \forall x, y \in X \mid f(x) = f(y) \implies x = y$.

Definition 2:

A function, f:$X \to Y$, is surjective $\iff \forall y \in Y, \exists x \in X \mid y = f(x)$.

Definition 3:

A function is bijective if it is injective and surjective.

a)

$$f(x) = x^3$$

Proving injectivity by contrapositive:

$$x \neq y \implies f(x) \neq f(y)$$

Since $x \neq y$, then it follows that $x^3 \neq y^3$, hence contrapositive is true.

$\therefore f(x)$ is injective. $\blacksquare$

Proving surjectivity by implications:

We know $f(x) = x^3$ is continuous in the interval $(-\infty, \infty)$.

Hence:

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} \mid y = x^3$$

$\therefore f(x)$ is surjective. $\blacksquare$

$f(x)$ is injective and surjective, hence it is bijective. $\blacksquare$

b)

$$f(x) = 2x + 1$$

Proving injectivity by contrapositive:

$$x \neq y \implies f(x) \neq f(y)$$

Since $x \neq y$, then it follows that $2x + 1 \neq 2y + 1$, hence contrapositive is true.

$\therefore f(x)$ is injective. $\blacksquare$

Proving surjectivity by implications:

We know $f(x) = 2x + 1$ is continuous in the interval $(-\infty, \infty)$.

Hence:

$$\forall y \in \mathbb{N}, \exists x \in \mathbb{N} \mid y = 2x + 1$$

$\therefore f(x)$ is surjective. $\blacksquare$

$f(x)$ is injective and surjective, hence it is bijective. $\blacksquare$

c)

$$f(x) = \sin(x)$$

Disproving injectivity by contradiction:

$$\forall x, y \in \mathbb{R} \mid f(x) = f(y) \implies x = y$$

Consider $f(x) = f(y) = 0$ which holds for $x = 0 \land y = \pi$.

$\bot x \neq y$

$\therefore f(x)$ is not injective. $\blacksquare$

Disproving surjectivity by implication:

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} \mid y = f(x)$$

Since $f(x) = \sin(x)$ is bounded by $-1 \leq x \leq 1$,

that means that $\forall y \in \mathbb{R} \mid y < -1 \lor y > 1, \nexists y = f(x) = \sin(x)$.

$\therefore f(x)$ is not surjective. $\blacksquare$

Since $f(x)$ is neither injective nor surjective, it is also not bijective. $\blacksquare$

Problem 4.2

a)

Show: If $f : X \to Y$ and $g : Y \to Z$ are injective, then $g \circ f$ is injective.

Proof by implication: We know that:

$$f(x) = f(y) \implies x = y$$

We also know that:

$$g(z) = g(w) \implies z = w$$

Hence:

$$g(f(x)) = g(f(y)) \implies f(x) = f(y) \implies x = y$$

$\therefore g \circ f$ is injective. $\blacksquare$

b)

Show: If $f : X \to Y$ and $g : Y \to Z$ are surjective, then $g \circ f$ is surjective.

Proof by implication:

We know that:

$$\forall y \in Y, \exists x \in X \mid y = f(x)$$

We also know that:

$$\forall z \in Z, \exists w \in Y \mid z = g(w)$$

Hence:

$$\forall c \in Z, \exists b \in Y \wedge \forall b \in Y, \exists a \in X \mid c = g(b), b = f(a)$$

$$\implies \forall c \in Z, \exists a \in X \mid c = g(f(a))$$

$\therefore g \circ f$ is surjective. $\blacksquare$

c)

Show: If $f : X \to Y$ and $g : Y \to Z$ are bijective, then $g \circ f$ is bijective.

Proof by implication:

Since $f(x)$ and $g(x)$ are bijective, then they are both injective and surjective.

Hence from proofs in *Problem 4.2 a)* and *Problem 4.2 b)*, we know that $(g \circ f)(x)$

will also be injective and surjective.

$\therefore g \circ f$ is bijective. ■

Problem 4.3

a)

$$\forall m \in M \mid m \text{ is a movie that is showing in the cinema}$$

$$\forall c \in C \mid c \text{ is a person paying to watch a movie in the cinema}$$

$$\forall t \in T \mid t \text{ shows that a person paid to watch a movie}$$

$$\forall x \in X \mid x \text{ is a person selling tickets to the movies}$$

$$\forall k \in K \mid k \text{ is a person checking that a ticket is valid}$$

$$\forall d \in D \mid d \text{ is a drink that is available to thirsty customers}$$

$$\forall w \in W \mid w \text{ is someone who serves drinks to customers}$$

b)

$$R \subseteq C \times D \mid (c, d) \in R, \text{ customer getting a drink}$$

$$R \subseteq W \times D \mid (w, d) \in R, \text{ waiter serving a drink}$$

$$R \subseteq C \times M \mid (c, m) \in R, \text{ customer watching a movie}$$

$$R \subseteq X \times T \mid (x, t) \in R, \text{ cashier selling a ticket}$$

$$R \subseteq K \times T \mid (k, t) \in R, \text{ ticket taker checking a ticket}$$

c)

Equivalence:

$$R \subseteq M \times M \mid (m_1, m_2) \in R, \text{ movies are directed by the same director}$$

Partial Order:

$$R \subseteq D \times D \mid (d_1, d_2) \in R, \text{ drinks have the same ingredients}$$

Strict Partial Order:

$$R \subseteq T \times T \mid (t_1, t_2) \in R, \text{ seat number is behind other seat number on tickets}$$

Other:

$$R \subseteq C \times C \mid (c_1, c_2) \in R, \text{ customers are in a monogamous relationship}$$

$$R \subseteq K \times K \mid (k_1, k_2) \in R, \text{ two ticket takers working at the same time}$$

Problem 4.4

c)

Listing 1: Haskell type of zipWith function

```
zipWith :: (a -> b -> c) -> [a] -> [b] -> [c]
```

It takes in a function (which takes in a variable of type **a** and type **b**, and returns type **c**), and two lists that hold elements of type **a** and **b**. *zipWith* returns a list of elements of type **c**.

d)

Listing 2: Haskell type of isPrefixOf function

```
isPrefixOf :: Eq a => [a] -> [a] -> Bool
```

It takes two lists, which must contain elements of variable type **a** that can be compared with the '==' operator. The function returns a boolean value.