PROBLEM SET 2 - SOLUTIONS

Problem 1. *Give an example of a ring map $A \to B$ which satisfies the infinitesimal lifting criterion for being etale, but is not flat itself.*

Proof. (cf. MathOverflow answer) Note that the group-ring $k[\mathbf{Q}_{\geq 0}]$ is the ring of Puiseux series, whose elements are of the form

$$\sum_{q \in \mathbf{Q}_{\geq 0}} a_q T^q, \quad a_q \in k.$$

We consider the projection map $k[\mathbf{Q}_{\geq 0}] \to k$. It is not flat because it is not torsion-free.

Let us prove it if formally étale. We consider a diagram

$$\begin{tikzcd} R/I & k \arrow[l] \arrow[dl, dashed] \\ R \arrow[u] & k[\mathbf{Q}_{\geq 0}] \arrow[l] \arrow[u] \end{tikzcd}$$

where I is a square-zero ideal of R. Note that $T^q \in k[\mathbf{Q}_{\geq 0}]$ maps to zero in R. Indeed, $T^{\frac{q}{2}}$ maps to a square-zero element in R by the commutation. Hence, there is the unique map $k \to R$ which is a just composition of $k \to k[\mathbf{Q}_{\geq 0}] \to R$. This proves that the projection is formally étale.

Problem 2. *Give an example of a flat ring map $A \to B$ such that $\Omega^1_{B/A} = 0$, but $A \to B$ does not satisfy the infinitesimal lifting criterion for being étale. Note that such a map cannot be finitely presented.*

Remark 1. 1. Such maps cannot be finitely presented because *étale = finite presentation + flat + Ω^1-trivial = finite presentation + formally étale*. 2. All of the properties being flat over A, admiting trivial module of Kähler differentials over A, and being formally étale over A are closed under colimit and base-change.

Proof. There is a perfection $\mathbf{F}_p[T] \to \mathbf{F}_p[T^{1/p^\infty}]$ defined by the colimit of the system

$$\mathbf{F}_p[T] \xrightarrow{T^p} \mathbf{F}_p[T] \xrightarrow{T^p} \cdots .$$

Although it gives an example (cf. [Stacks Project](https://stacks.math.columbia.edu/tag/01UA)), we give a simpler one. Let us consider

$$\mathbf{F}_p \to \mathbf{F}_p[T^{1/p^\infty}]/(T).$$

Since the base is a field, it must be flat. Its module of differentials can be computed as

$$\Omega^1_{\mathbf{F}_p \to \mathbf{F}_p[T^{1/p^\infty}]/(T)} = \varinjlim \Omega^1_{\mathbf{F}_p \to \mathbf{F}_p[T^{1/p^i}]/(T)} = 0.$$

The remainder is to show that it is not formally étale. Consider the following diagram.

$$\begin{tikzcd}
\mathbf{F}_p[T^{1/p^\infty}]/(T) \arrow[r, equal] & \mathbf{F}_p[T^{1/p^\infty}]/(T) \arrow[dl, dashed] \\
\mathbf{F}_p[T^{1/p^\infty}]/(T^p) \arrow[u] & \mathbf{F}_p \arrow[l] \arrow[u]
\end{tikzcd}$$

The existence of the dotted arrow is equivalent with the existence of a section of $\mathbf{F}[T^{1/p^\infty}]/(T^p) \to \mathbf{F}[T^{1/p^\infty}]/(T)$, but simple computation gives there is no suce map. Hence, this morphism is flat and its Ω^1 is trivial but not formally étale.

Problem 3. *Let $f : X \to S$ be a locally finitely presented unramified morphism. Show that any section of f is an open immersion. If f is additionally assumed to be separated, then show that a section has to be an isomorphism onto a connected component.*

Proof. We compute next pullback diagram.

$$\begin{tikzcd}
P' \arrow[dr, dashed] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& S \arrow[r, "s"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "(s\circ f, \mathrm{id}_X)"] \\
& X \arrow[r, "\Delta"'] & X \times_S X
\end{tikzcd}$$

Indeed, from the commutation $\Delta \circ \alpha = (s \circ f, \mathrm{id}_X) \circ \beta$, we read that $\alpha = \beta = s \circ f \circ \beta$. From this, the unique map $P' \to S$ is the composition $P' \to X \to S$.

Since f is unramified, we can conclude that Δ (thus, s) is an open immersion. If f is, moreover, separated, then Δ (thus, s) is a closed immersion.

Problem 4. *A morphism $f : X \to S$ is called a finite etale cover if f is finite, surjective, and etale. Classify all finite etale covers of:*

(1) $\mathrm{Spec}(\mathbf{R})$

(2) $\mathbf{P}^1_{\mathbf{C}}$ *and* $\mathbf{P}^1_{\overline{\mathbf{F}}_q}$

(3) $\mathbf{A}^1_{\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[T])$

(4) $\mathbf{G}_{m,\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[T^{\pm}])$

(5) $\mathrm{Spec}(\mathcal{O}_{C,x})$ *where C is a smooth projective curve over $\mathbf{C}$, and $x \in C$ is a closed point. Answer in terms of the projective geometry of C.*

(6) *An Artinian local $\mathbf{C}$-algebra*

(7) *A complete local $\mathbf{C}$-algebra*

(8) *An elliptic curve over $\mathbf{C}$*

(9) *The nodal cubic in $\mathbf{P}^2_{\mathbf{C}}$*

(10) *The cuspidal cubic in $\mathbf{P}^2_{\mathbf{C}}$. (The next exercise may be useful here.)*

(11) $\mathrm{Spec}(\mathbf{Z})$

(12) $\mathrm{Spec}(\mathbf{Z}_{(p)})$*, the local scheme of* $\mathrm{Spec}(\mathbf{Z})$ *at a prime p. Answer in terms of number fields.*

(13) $\mathrm{Spec}(\mathbf{Z}_p)$

(14) $\mathbf{A}^2_{\mathbf{C}} \smallsetminus 0$

Proof. 1. Over any field K, every étale algebra are finite product of finite separable extensions of K. Clearly, all of them defined étale covers. Hence, the answer is *finite disjoint union of $\mathrm{Spec}(\mathbf{R})$ and $\mathrm{Spec}(\mathbf{C})$*. 2.