A SQUARE PEG IN A ROUND HOLE

JOHN HULL

Abstract. Today I discovered an interesting web log entry[1] on the website of Aubrey Jaffer. In Mr. Jaffer I perceive a kindred spirit, and the particular practical problem he examines at the link resonates with my predilections. I love solving constraint systems in service to the fabrication of physical artifacts. I believe that the process involved is the essence that defines the word 'design'. True, such processes are nitty gritty details, but without being informed by mastery of those details, all design becomes fatuous.

In this document I first replicate Mr. Jaffer's analysis of creating a square opening with circular tools in a way that is more understandable to me. Then I alter the optimizing constraint to equate the *area* of the corner overcuts to the central overcut, in contrast to equating the *extent* of the two overcuts, as he does. Perhaps the most useful insight extending Mr. Jaffer's effort is the discovery of the invariant $\frac{R}{r} = 3$ attached to his solution, which makes his ideas very easy to apply in a practical context, without calculation. Next, I apply that insight to *minimize* the total *area* of the overcut and *minimize* and *equate* the *uncut area* in the companion problem. Finally, I generalize to an n-gon peg in a round hole, creating a table of the associated invariants. Mostly, only an understanding of grade school algebra and trigonometry is necessary, although the minimization sections use a little elementary calculus.

1. Summary of the Problem

The problem is to use standard drills with circular profile to make a good approximation to a square opening in some sheet material. The final opening need not be further finished to be servicable, but may be 'deburred' with a file to smooth the edges of the opening. When that latter operation is applied, the resulting hole can be said to have some margin δ. In either case, we require that a square peg of the specified size will slide without interference into the finished opening. Our fabrication algorithm is:

(1) Drill four small holes near the corners of the finished square.
(2) Drill a large hole at the center of the square.
(3) Optionally, clean up the sides with a file.

Clearly, in the absence of any design constraints, there are an infinity of solutions that will work. The four small holes need not even all be the same size. Many of those solutions grossly overcut every part of the perimeter of the square target opening. There is a subset of solutions (but still infinite) that are better because all intersections of the perimeter of the large drill opening and the perimeter of each of the small drill openings lie on the perimeter of the square, and each corner of the square lies on the perimeter of a small drill opening. We wish to restrict the latter subset to a unique solution by imposing a design constraint.

The mathematical question is thus, what is the radius r of the small drill, the radius R of the large drill, and the overcut δ, when subject to some design constraint that forces those values to be unique. The constraint Mr. Jaffer chose was to equate the maximum overcut of the large drill and the small drill. The clearance is thus $2 \cdot \delta$, twice the amount of overcut. We later choose to consider the *area* of overcuts, supposing the resulting balance will yield a better fit.

In what follows, the word 'distance' always refers to a positive scalar value, where all scalars are denoted by *italic* case. The word 'position' always refers to a two dimensional vector, which may be thought of as a complex number, denoted by **bold** case. Scalar functional forms are denoted by a sans serif font. In most cases, the algebraic manipulations are belabored, as a gesture of courtesy to the math-phobic. If their abundance is intimidating, simply reading the first and last lines of each grouping and accepting that they are equivalent statements of truth suffices.

2. Equation of Overcut Extent

First of all, it is obvious all linear dimensions scale linearly with the side length s of the square. In order to avoid cluttering the equations with factors of two and thereby focus on the problem at hand, we shall use half the side length of the square hole, $h = \frac{s}{2}$ as a parameter. Then, when we are done, we will have three numbers, $\frac{r}{h}$, $\frac{R}{h}$, and $\frac{\delta}{h}$, which are factors that, when multiplied by some actual $s = 2h$, yield the ideal diameters of the small and large drill, and the total clearance (twice the overcut), repectively. As a practical matter, the ideal values will have to be rounded up, and the centers of the four small holes will have to be shifted suitably if the equality of overcut constraint is to be rigorously realized. We will not solve that secondary problem, for it is dependent on hole size and the common tooling available for execution, both of which are unknown. In practice, the shift likely would be small and neglected. To clarify the algebra, refer to figure 1 which shows the final solution, where the target opening is red.

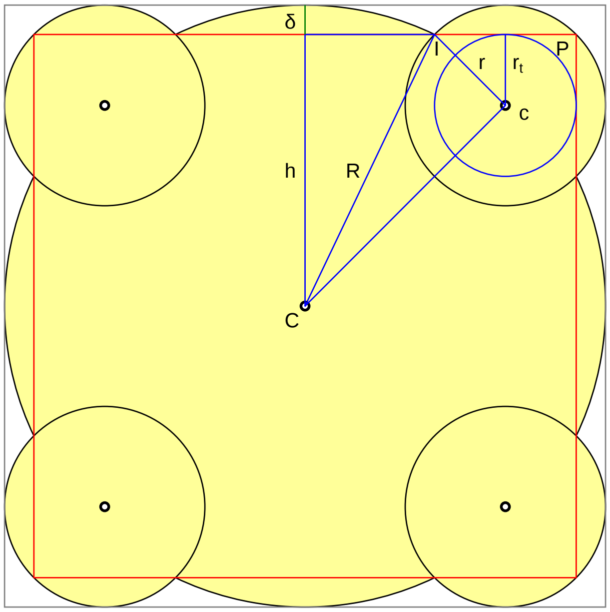

FIGURE 1.
Equal overcut extents.

Let the center of the square be at the origin: $\mathbf{C} \equiv [0\ 0]$. A corner of the square must be on the perimeter of the small drill's profile. A corner position is $\mathbf{P} = [h\ h]$. The radius of a circle tangent to both sides of the square at the corner and co-located with the small drill center is $r_t = \frac{r}{\sqrt{2}}$. The position of the corresponding small hole center is therefore:

$$\begin{aligned}\mathbf{c} &= \begin{bmatrix} h - r_t & h - r_t \end{bmatrix} \\ &= \begin{bmatrix} h - \frac{r}{\sqrt{2}} & h - \frac{r}{\sqrt{2}} \end{bmatrix}\end{aligned}$$

and the overcut is therefore $\delta_r = r - r_t = r \cdot \left(1 - \sqrt{\frac{1}{2}}\right)$. The distance from the corner of the square to the intersection of the perimeter of the small hole with the side of the square, due to the right angle at the corner, is therefore just $2 \cdot r_t = \sqrt{2} \cdot r$ and the position of the intersection as a function of r is:

$$\mathbf{I} = \begin{bmatrix} h - \sqrt{2} \cdot r & h \end{bmatrix}$$

This intersection must be the same as the position of the intersection of the perimeter of the large drill with the side of the square. The Pythagorean theorem gives us that position as a function of R:

$$\mathbf{I} = \begin{bmatrix} \sqrt{R^2 - h^2} & h \end{bmatrix}$$

The overcut of the large hole is $\delta_R = R - h$.

The equation relating r and R can now be expressed by equating x component of the two equations for $\mathbf{I}$:

$$\begin{aligned}\sqrt{R^2 - h^2} &= h - \sqrt{2} \cdot r \\ \sqrt{\left(\frac{R}{h}\right)^2 - 1} &= 1 - \sqrt{2} \cdot \frac{r}{h} \\ \left(\frac{R}{h}\right)^2 - 1 &= 1 - 2\sqrt{2} \cdot \frac{r}{h} + 2\left(\frac{r}{h}\right)^2 \\ \left(\frac{R}{h}\right)^2 &= 2\left(1 - \sqrt{2} \cdot \frac{r}{h} + \left(\frac{r}{h}\right)^2\right) = \left(\sqrt{2} - \frac{r}{h}\right)^2 + \left(\frac{r}{h}\right)^2\end{aligned}$$

Notice the last expression is just the Pythagorean theorem applied to the right triangle in figure 1 with R for hypoteneuse, r for altitude, and $\sqrt{2}h - r$ for base. This is the geometric constraint coincident intersections imply, and it holds for all values of r and R that produce mutual intersections that are also coincident with the side of the square. More will be said about this in subsequent sections.

Using our design constraint that equates overcuts, we also have:

$$\begin{aligned}R - h &= r \cdot \left(1 - \sqrt{\frac{1}{2}}\right) \\ \frac{R}{h} - 1 &= \frac{r}{h} \cdot \left(1 - \sqrt{\frac{1}{2}}\right) \\ \frac{R}{h} &= \frac{r}{h} \cdot \left(1 - \sqrt{\frac{1}{2}}\right) + 1 \qquad (2.1)\end{aligned}$$

yielding the simultaneous system of equations:

$$\left.\begin{array}{rcl}\frac{R}{h} & = & \frac{r}{h}\cdot\left(1-\sqrt{\frac{1}{2}}\right)+1 \\ \left(\frac{R}{h}\right)^2 & = & 2\left(1-\sqrt{2}\cdot\frac{r}{h}+\left(\frac{r}{h}\right)^2\right)\end{array}\right\}$$

We solve first for $\frac{r}{h}$:

$$\begin{aligned}\left(\frac{r}{h}\cdot\left(1-\sqrt{\frac{1}{2}}\right)+1\right)^2 &= 2\left(1-\sqrt{2}\cdot\frac{r}{h}+\left(\frac{r}{h}\right)^2\right) \\ \left(\frac{r}{h}\cdot\left(1-\sqrt{\frac{1}{2}}\right)\right)^2+2\frac{r}{h}\cdot\left(1-\sqrt{\frac{1}{2}}\right)+1 &= 2-2\sqrt{2}\cdot\frac{r}{h}+2\left(\frac{r}{h}\right)^2 \\ \left(\frac{r}{h}\right)^2\cdot\left(\frac{3}{2}-\sqrt{2}\right)+\frac{r}{h}\cdot\left(2-\sqrt{2}\right) &= 1-2\sqrt{2}\cdot\frac{r}{h}+2\left(\frac{r}{h}\right)^2 \\ 0 &= 1-\left(2+\sqrt{2}\right)\frac{r}{h}+\left(\frac{1}{2}+\sqrt{2}\right)\left(\frac{r}{h}\right)^2\end{aligned}$$

The quadratic formula tells us:

$$\begin{aligned}\frac{r}{h} &= \frac{2+\sqrt{2}}{2\left(\frac{1}{2}+\sqrt{2}\right)}\pm\sqrt{\left(\frac{2+\sqrt{2}}{2\left(\frac{1}{2}+\sqrt{2}\right)}\right)^2-\frac{1}{\frac{1}{2}+\sqrt{2}}} \\ &= \frac{2+\sqrt{2}}{1+2\sqrt{2}}\pm\sqrt{\left(\frac{2+\sqrt{2}}{1+2\sqrt{2}}\right)^2-\frac{2}{1+2\sqrt{2}}} \\ &= \frac{1}{1+2\sqrt{2}}\left(2+\sqrt{2}\pm\sqrt{\left(2+\sqrt{2}\right)^2-2\left(1+2\sqrt{2}\right)}\right) \\ &= \frac{1}{1+2\sqrt{2}}\left(2+\sqrt{2}\pm\sqrt{4+4\sqrt{2}-4\sqrt{2}}\right) \\ &= \frac{\left(2+\sqrt{2}\right)\pm 2}{1+2\sqrt{2}}\to\sqrt{2},\ \frac{2}{4+\sqrt{2}}\simeq 0.3693980625181293\end{aligned}$$

where we choose the smaller value, which gives a small drill. The large solution would solve the problem by drilling a single hole the diameter of the square's diagonal at the center of the square, eliminating the need for the large drill. That solution has excessive overcut, and, obviously, produces a round hole. Notice the small solution expresses the harmonic mean of $\frac{1}{4}$and $\frac{1}{\sqrt{2}}$. It is not immediately clear, however, that there is any useful geometric interpretation.

The solution we calculated is twice that found by Mr. Jaffer because h is half of s, and he used s in his calculations. Like him, we also seek a diameter, which is the dimension that drills are indexed by, so this value is doubled again when we multiply by $s = 2h$ to get the diameter. Using the overcut constraint now yields:

$$\begin{aligned}\frac{R}{h} &= \frac{r}{h}\cdot\left(1-\sqrt{\frac{1}{2}}\right)+1 \\ &= \frac{2}{4+\sqrt{2}}\cdot\left(1-\sqrt{\frac{1}{2}}\right)+1 \\ &= \frac{2-\sqrt{2}}{4+\sqrt{2}}+1 \\ &= \frac{\sqrt{2}-1}{2\sqrt{2}+1}+1=\frac{3\sqrt{2}}{2\sqrt{2}+1}\simeq 1.108194187554388\end{aligned}$$

where again our value is twice that of Mr. Jaffer's because h is half of s. When multiplied by $s = 2h$, using his exemplar of $s = 1.125$, we get the same drill diameters as he does. In contrast, he multiplies his numbers by $2s = 4h$ to get the same diameters. Notice that the proportion $\frac{R}{h} : \frac{r}{h} = 3 = \frac{R}{r}$ is invariant.

Now δ is the overcut, and Mr. Jaffer does not calculate it. It does, however, tell us how good or bad this approximation is. According to our equations $\frac{\delta}{h} = \frac{\delta_r}{h} = \frac{r}{h}\cdot\left(1-\sqrt{\frac{1}{2}}\right) = 0.108\cdots = \frac{R}{h} - 1 = \frac{R-h}{h} = \frac{\delta_R}{h}$, which is about 5.5% of the *total* side length of the square. This yields a clearance, $2\cdot\delta$, of 22% of *half* the side length or

11% of the *total* side length. This is actually a pretty rough hole, but when you are using it to mount some panel gadget with a bezel, it is entirely appropriate.

If you seek a tight fit, then the companion problem that requires the drill perimeters to be inside the square is more suitable, and then final finishing with a file is no longer optional. We can solve that companion problem easily from what we have already done. Let H be half the side length of the circumscribing square. All drill perimeters must be tangent to that square. Therefore, $H = R$ where R is the solution we just calculated for the associated h which is not known. The proportion $\frac{r}{h} : \frac{R}{h}$ is invariant, that is $\frac{r}{R} = \frac{1}{3}$ for the design constraint that forces equal overcuts, so $r = \frac{r}{R} \cdot R = \frac{R}{3}$. From the geometry, we have $H = \left(h - \frac{r}{\sqrt{2}}\right) + r$, so $h = H - \frac{R}{3}\left(1 - \frac{1}{\sqrt{2}}\right) = H \cdot \frac{4-\sqrt{2}}{6}$. The canonical solution to the companion problem is therefore obtained by finding the canonical solution of the original problem with $h = 1$ and scaling all values by $\frac{h}{H} = \frac{4-\sqrt{2}}{6}$. The solution for a specific H is then obtained by multiplying all values by H. The solution to this companion problem is also depicted in figure 1, where the target opening is gray.

It is interesting that the companion problem has no immediately obvious natural design constraint to make the solution unique, for merely requiring tangency does not restrict the solution set. Solving the overcut problem and then mapping the solution to the undercut problem via scaling is a good way to impose a natural constraint that yields a unique solution. This works because of the invariant. Therefore, probably the most important discovery here is the invariant. It is a very easy to remember practical rule of thumb.

So, for example, if you wish to make a square hole in a 2x4 as a socket for a 2x2, you would first drill four half inch holes tangent to the corners of a 1.5'' square, since that is the nominal dimension of a 2x2. Then you would drill a 1.5'' hole in the center and clean up the edges with a wood chisel. The paradigm is a versatile means to make sturdy structures from inexpensive materials.

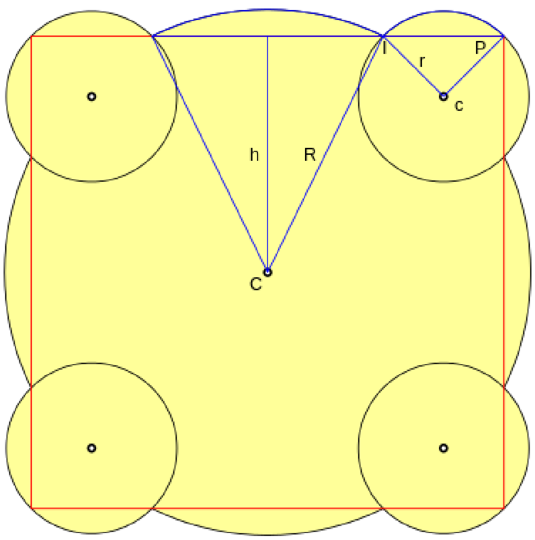

Figure 2.
Equal overcut areas.

3. Equation of Overcut Areas

Looking at the image of the previous solution, it seems that the corners overcut disproportionately, that is, there is a sense that the square has ‘ears’. The perception suggests the constraint of the previous section, equating overcut, errs on the side of making the large hole too small, regarding only aesthetic considerations. What we desire is to have the contributions from each kind of overcut to be equal so that the defects in the opening are visually balanced.

If the small holes were very small, the large hole would protrude excessively and the imbalance would be visually very obvious. If the large hole were nearly tangent, the corner holes would form significant ‘ears’ and the imbalance would be equally obvious. This qualitative analysis makes it clear some median balance point does exist. If we use a design constraint that equates overcut areas for the two different drills, there will be less overcut in the corners and more in the center of an edge. In consequence, if the final shape is smoothed with a file, the sides will bulge a bit, but the closeness of fit in the corners should be better.

Once more, to clarify the algebra, refer to the final solution in figure 2. We begin by revisiting the idea of an invariant from the perspective of our geometric constraint that requires coincidence of the intersections of the small and large drill openings with the side of the square. The lowest right triangle in figure 1 nicely expresses this constraint:

$$\begin{aligned} R^2 &= r^2 + \left(\sqrt{2} \cdot h - r\right)^2 \\ 1 - \left(\frac{r}{R}\right)^2 &= \left(\sqrt{2} \cdot \frac{h}{R} - \frac{r}{R}\right)^2 \\ \frac{1}{\sqrt{2}}\left(\frac{r}{R} + \sqrt{1 - \left(\frac{r}{R}\right)^2}\right) &= \frac{h}{R} \end{aligned} \tag{3.1}$$

We have rearranged the relationship to express $\frac{h}{R}$ as a function of the invariant $\frac{r}{R}$. The solution for a different design constraint will have a different invariant, but it will still satisfy this relationship. The invariant expresses an infinity of possible design constraints as a simple numerical constant with $0 < \frac{r}{R} < \frac{1}{\sqrt{2}}$. In the previous section, the equal overcut constraint formula 2.1 can be rewritten as $\frac{h}{R} = 1 - \frac{r}{R} \cdot \left(1 - \sqrt{\frac{1}{2}}\right)$, and that, combined with the universal geometric contraint 3.1 for this problem, yields a relationship that may be solved for the invariant:

$$\begin{aligned}
\frac{1}{\sqrt{2}}\left(\frac{r}{R} + \sqrt{1 - \left(\frac{r}{R}\right)^2}\right) &= 1 - \frac{r}{R} \cdot \left(1 - \sqrt{\frac{1}{2}}\right) \\
\sqrt{1 - \left(\frac{r}{R}\right)^2} &= \sqrt{2} - \frac{r}{R} \cdot \sqrt{2} \\
0 &= 3\left(\frac{r}{R}\right)^2 - 4\frac{r}{R} \cdot +1
\end{aligned}$$

where the reader may easily check that the solutions are $\frac{1}{3}$ and 1 with only $\frac{1}{3}$ applicable. Feeding that back into either the equal overcut constraint or the geometric constraint, with $h = 1$, gives $R = 1.108\cdots$ and $r = \frac{R}{3} = .369\cdots$ as before. In the sequel, we will find a new relationship for $\frac{h}{R}$ and use it with the universal geometric constraint 3.1 to solve for $\frac{r}{R}$.

As a preliminary, we find the formula for the area between the perimeter of a circle and a chord. For convenience, we will call that a 'lune' although it is really only half of a proper lune. Let α be the angle that the chord subtends, in radian measure, and r be the radius of the circle. The area of the lune is the difference between the area of the circular sector and the triangle that the chord cuts off from that sector. This is given by:

$$\begin{aligned}
\mathsf{A}_l(r, \alpha) &= \frac{\alpha}{2\pi} \cdot \pi r^2 - \frac{1}{2} r \cdot \cos\left(\frac{\alpha}{2}\right) \cdot \left(2r \cdot \sin\left(\frac{\alpha}{2}\right)\right) \\
&= \frac{\alpha}{2} \cdot r^2 - r^2 \cdot \cos\left(\frac{\alpha}{2}\right) \cdot \sin\left(\frac{\alpha}{2}\right) \\
&= \frac{r^2}{2}(\alpha - \sin(\alpha)) = \frac{r^2}{2}\left(\alpha - \sqrt{1 - \cos^2(\alpha)}\right) = \frac{r^2}{2}\left(\alpha - 2 \cdot \cos\left(\frac{\alpha}{2}\right) \cdot \sin\left(\frac{\alpha}{2}\right)\right)
\end{aligned}$$

The total area of the corner overcuts is 8 times this, and for the corner hole, $\alpha = \frac{\pi}{2}$, therefore $\mathsf{A}_r = 8 \cdot \mathsf{A}_l\left(r, \frac{\pi}{2}\right) = 8\left(\frac{\pi}{4} - \frac{1}{2}\right) \cdot r^2 = 2(\pi - 2) \cdot r^2$.

Let α be the angle subtended by the chord of the large drill opening. We will use the half angle version of the formula for the lune area because we have the cosine and sine readily available in the geometry. The total area of the overcuts of the large drill is $\mathsf{A}_R = 4 \cdot \mathsf{A}_l(R, \alpha)$. The relationship equating areas is thus:

$$\begin{aligned}
\mathsf{A}_r &= \mathsf{A}_R \\
2(\pi - 2) \cdot r^2 &= 2R^2\left(\alpha - 2\cos\left(\frac{\alpha}{2}\right) \cdot \sin\left(\frac{\alpha}{2}\right)\right) \\
(\pi - 2)\left(\frac{r}{R}\right)^2 &= 2\arccos\left(\frac{h}{R}\right) - 2 \cdot \frac{h}{R} \cdot \frac{h - \sqrt{2}r}{R} \\
2\arccos\left(\frac{h}{R}\right) &= (\pi - 2)\left(\frac{r}{R}\right)^2 + 2\left(\frac{h}{R}\right)^2 - 2\sqrt{2} \cdot \frac{h}{R} \cdot \frac{r}{R} \qquad (3.2) \\
\frac{h}{R} &= \cos\left(\frac{1}{2}(\pi - 2)\left(\frac{r}{R}\right)^2 + \left(\frac{h}{R}\right)^2 - \sqrt{2} \cdot \frac{h}{R} \cdot \frac{r}{R}\right)
\end{aligned}$$

We may gain some geometric insight into the nature of the solutions satisfying this relationship by performing two substitutions which yield the equivalent relations:

$$\begin{aligned}
u &= \cos\left(\left(\frac{\pi}{2}-1\right)v^2+u^2-\sqrt{2}\cdot u\cdot v\right) \\
&\quad \mid u,\, v=\frac{h}{R},\,\frac{r}{R} \\
\frac{1}{\sqrt{2}}\left(v+\sqrt{1-v^2}\right) &= \cos\left(\left(\frac{\pi}{2}-1\right)v^2+\frac{1}{2}\left(v+\sqrt{1-v^2}\right)^2-\left(v+\sqrt{1-v^2}\right)\cdot v\right) \\
&\quad \mid u=\frac{1}{\sqrt{2}}\left(v+\sqrt{1-v^2}\right) \\
\frac{1}{\sqrt{2}}\left(v+\sqrt{1-v^2}\right) &= \cos\left(\left(\frac{\pi}{2}-1\right)v^2+\frac{1}{2}\left(1+2v\sqrt{1-v^2}\right)-\left(v^2+v\sqrt{1-v^2}\right)\right) \\
\frac{1}{\sqrt{2}}\left(v+\sqrt{1-v^2}\right) &= \cos\left(\left(\frac{\pi}{2}-2\right)v^2+\frac{1}{2}\right)
\end{aligned}$$

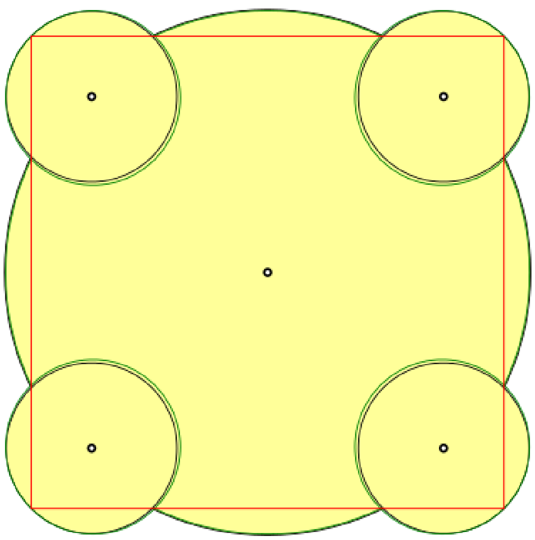

FIGURE 3.
Comparison of solutions.

The last equation expresses the intersection of the cosine function using a parabolic domain mapping with a skewed square root function when plotted in a Cartesian coordinate system. In arriving at this formula, we performed two substitutions. The first was simply to clarify the algebra. The second expresses the universal geometric constraint given by equation 3.1 relating the first substitution variables. This final relationship yields values for the invariant under the design constraint that equates overcut areas. There are two solutions: $v \simeq 0.32464171480530516$ or $v \simeq 0.83421552624000018$ which may be found using numerical methods, for example, Newton's method. Only the small value is in the domain of $\frac{r}{R}$ that produces geometrically meaningful results. Feeding that value back into the geometric constraint, with $h = 1$, we find $R = 1.113134341796548$ and $r = R \cdot \frac{r}{R} = 0.361369841529506$. The invariant $v \equiv \frac{r}{R}$ is no longer an integer, or even a nice number.

From a practical perspective, this solution is identical to the first solution. Working out the solution was, however, a useful exercise as a prelude to minimizing the total area of all overcuts. Moreover, the images could not have been made without solving the problem. Figure 3 shows a comparison of the two solutions. The green lines show the outline of the previous solution. The increase in large radius is negligable, despite the visibly distinct difference in the small radii. Nevertheless, the difference between overcuts is also negligable. Thus, ultimately we find that Mr. Jaffer's original solution is a superior practical solution.

4. Minimizing Total Overcut

Another approach that comes to mind is summing the overcuts and minimizing the total area. This solution has the practical value of minimizing the material that needs to be removed to form the hole, which is a quantitative metric for closeness of fit. The first step is to rearrange equation 3.2 so that it expresses a function giving the total area. Notice signs change, since this is now a *sum* of areas, not an *equation* of areas:

$$\begin{aligned}
\mathsf{A}_t\left(\frac{h}{R},\,\frac{r}{R}\right) &= 2\arccos\left(\frac{h}{R}\right)-2\left(\frac{h}{R}\right)^2+2\sqrt{2}\cdot\frac{h}{R}\cdot\frac{r}{R}+(\pi-2)\left(\frac{r}{R}\right)^2 \\
\mathsf{A}_t\left(u,\,v\right) &= 2\arccos\left(u\right)-2u^2+2\sqrt{2}\cdot u\cdot v+(\pi-2)\,v^2 \\
&\quad \mid u,\, v=\frac{h}{R},\,\frac{r}{R}
\end{aligned}$$

As before, some substitutions will clarify and simplify:

$$\begin{aligned}
\mathsf{A}_t(v) &= 2\arccos\left(\frac{1}{\sqrt{2}}\left(v+\sqrt{1-v^2}\right)\right)-\left(v+\sqrt{1-v^2}\right)^2+2\left(v+\sqrt{1-v^2}\right)\cdot v+(\pi-2)\,v^2\\
&\quad \mid u=\frac{1}{\sqrt{2}}\left(v+\sqrt{1-v^2}\right)\\
\mathsf{A}_t(w) &= 2\arccos\left(\frac{w}{\sqrt{2}}\right)-w^2+w\cdot\left(w+\sqrt{2-w^2}\right)+\left(\frac{\pi}{4}-\frac{1}{2}\right)\left(w+\sqrt{2-w^2}\right)^2\\
&\quad \mid w=v+\sqrt{1-v^2}\ \Rightarrow\ v=\frac{1}{2}\left(w\pm\sqrt{2-w^2}\right)\\
&= 2\arccos\left(\frac{w}{\sqrt{2}}\right)+w\cdot\sqrt{2-w^2}+\left(\frac{\pi}{4}-\frac{1}{2}\right)\left(w+\sqrt{2-w^2}\right)^2
\end{aligned}$$

The formula $v=\frac{1}{2}\left(w\pm\sqrt{2-w^2}\right)$ may be derived from the second substitution by rearrangement and solving for v using the quadratic formula (see Appendix A). Since v is positive, we use the positive root. These substitutions greatly simplify computing the derivative:

$$\begin{aligned}
\mathsf{A}_t'(w) &= -\frac{2}{\sqrt{2-w^2}}+\sqrt{2-w^2}-\frac{w^2}{\sqrt{2-w^2}}+\left(\frac{\pi}{2}-1\right)\left(w+\sqrt{2-w^2}\right)\left(1-\frac{w}{\sqrt{2-w^2}}\right)\\
&= -\frac{2}{\sqrt{2-w^2}}+\sqrt{2-w^2}-\frac{w^2}{\sqrt{2-w^2}}+\left(\frac{\pi}{2}-1\right)\left(\sqrt{2-w^2}-\frac{w^2}{\sqrt{2-w^2}}\right)\\
&= -\frac{2}{\sqrt{2-w^2}}+\frac{\pi}{2}\left(\sqrt{2-w^2}-\frac{w^2}{\sqrt{2-w^2}}\right)\\
&= -\frac{2}{\sqrt{2-w^2}}+\frac{\pi}{2}\left(\frac{2-w^2-w^2}{\sqrt{2-w^2}}\right)\\
&= \frac{\pi\left(1-w^2\right)-2}{\sqrt{2-w^2}}
\end{aligned}$$

The zeros of this derivative mark extrema of the area function. They may be found by solving $0=-\pi w^2+(\pi-2)$, yielding two zeros: $w=\pm\sqrt{\frac{\pi-2}{\pi}}\simeq\pm0.6028102749890869$. I looked up this sequence of digits in the Online Encylopedia of Integer Sequences[3], but there are no matching entries. Since we have a symbolic form, we will follow it through the inverse sequence of substitutions to obtain a symbolic expression for the invariant. From our definition of w, Appendix A shows:

$$\begin{aligned}
v\equiv\frac{r}{R} &= \frac{1}{2}\left(w\pm\sqrt{2-w^2}\right)\\
&= \frac{1}{2}\left(\pm\sqrt{\frac{\pi-2}{\pi}}\pm\sqrt{2-\left(\pm\sqrt{\frac{\pi-2}{\pi}}\right)^2}\right)\\
&= \frac{1}{2}\left(\pm\sqrt{\frac{\pi-2}{\pi}}\pm\sqrt{2-\frac{\pi-2}{\pi}}\right)\\
&= \frac{1}{2\sqrt{\pi}}\left(\pm\sqrt{\pi-2}\pm\sqrt{\pi+2}\right)
\end{aligned}$$

The permutations of sign yield $v\simeq\pm0.33824706729482124$ and $v\simeq\pm0.9410573422839081$. Only the positive small value is consistent with the geometry. Thus, $R=1.1054550830781478$, and $r=0.37391693987733643$. Unexpectedly, this has a larger small drill than the previous solutions. Figure 4 shows what it looks like in comparison to the equal overcut extent solution.

5. The Companion Problem Revisited

So we went through a lot of work here due to some nebulous insult to our sense of aesthetics, only to discover that the only other obvious alternatives for a design constraint produce essentially indistinguishable results. What is intriguing is the stability of the solution under really quite varied heuristic conceptions of optimization. Nevertheless, something still appears aesthetically wrong with the images, irrespective of the original utilitarian motive that inspired Mr. Jaffer. Looking at figure 1, this time focussing on the gray target square of the companion problem, we can see that the large drill size is fixed by the size of the square in that companion problem. It is therefore a new invariant and may replace the universal geometric constraint.

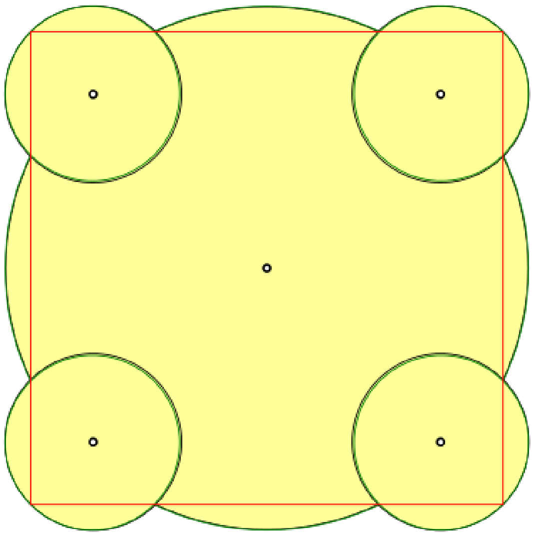

Figure 4.
Minimal overcut area.

What can vary is the small drill size, subject to the constraint that it is tangent to both sides of the square near the corner. Depending on the size we choose, the red square's corner will either be in the uncut material at the corner, on the perimeter of the small drill opening, or somewhere interior to the small drill opening. If we abandon the universal geometric constraint that requires the intersection between drill openings to lie on the red square, which is now optional instead of given, we need some new constraint to force a unique solution. There are two obvious ones that are distinct from the previous analyses:

(1) Equate the uncut area at the corners to the uncut area between the points of tangency.
(2) Minimize the total uncut area.

In order to calculate these areas, we need to know the two points of intersection of the two drill openings so we may calculate the area of the overlapping lunes. Then the total uncut area is just the area of the square, minus the area of the large drill, minus four times the area of the small drill, plus the area of the four lunes. The corner uncut area may be split out by computing rectangular areas bounded by the tangencies and small drill centers, then adding the area of one small drill opening, which is equal to the area of the four sectors missing from the rectangular sum. The corner area is then the total area of the square minus this latter sum. More simply, the corner uncut area is just the area of the square circumscribing the small drill opening minus the area of the small drill.

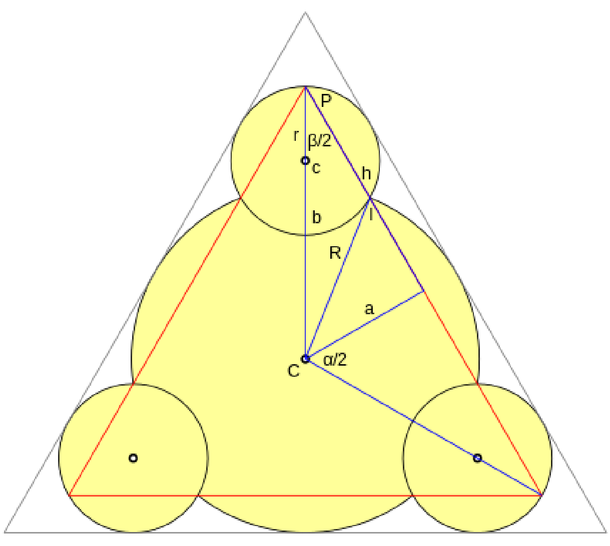

Figure 5.
n-gon peg: $n = 3$.

Looking at the figure again, what offends the aesthetic sensibility is now apparent: the uncut area at the corner is too small in comparison to the flanking uncut areas between tangent points. The small drill needs to be made *larger*, while maintaining tangency to the gray square. When this is done, the corners of the red square will lie in uncut material, interfering with the square peg. When used as a solution to Mr. Jaffer's original problem, finishing the corners with a few strokes of a square file is necessary, but the result is a tight fit in the corners and smaller clearance everywhere, providing a mechanically more robust construction.

Since, very likely, the result of these calculations will also be indistinguishable, I will defer them to some later time (possibly never) and proceed with the more interesting generalization.

6. An n-gon Peg in a Round Hole

The stability of the solutions suggests that Mr. Jaffer's original heuristic is the heuristic of choice for generalization to an n sided polygon. It certainly is the simplest to analytically apply. It is clear that as n increases the approximation will improve, however, if one wishes to insert a peg with more sides than say Gauss' beloved 17-gon, the simplest approach is probably to just use a single round hole. In fact, the essence of this observation is the basis of Archimedes method for estimating the value of π. Polygonal pegs with eight sides or less are the most likely targets for this generalization. The equilateral triangular hole captures the essential details for calculating all the rest, and so we will illustrate with a triangle. Figure 5 shows the final solution for the case $n = 3$.

Let $\alpha = \frac{2\pi}{n}$ radians be the central angle subtended by one side of the given n-gon. Let β be the angle between two adjacent sides. Then $\beta = \pi - \alpha = \pi\frac{n-2}{n}$. As before, let s be the given side length and h be half of that so that $s \equiv 2h$. Let the distance from the centroid to the center of a side be a and the distance from the centroid to a vertex be b. From the geometric definition of a cosine and a sine, $b = \frac{h}{\cos\left(\frac{\beta}{2}\right)}$ and $a = b \cdot \sin\left(\frac{\beta}{2}\right) = h \cdot \tan\left(\frac{\beta}{2}\right)$. As before, we will find the formulas for R, the radius of the large hole that overcuts the edge, and r, the radius of the small hole that overcuts the edge. As before, we will require the intersection of the two openings to lie on the n-gon perimeter, providing a geometric constraint. As before, we require the polygonal vertices to lie on the perimeter of a small drill opening. As before, we will equate the extents of the two overcuts as a design constraint. The universal geometric constraint for an n-gon thus becomes:

$$\begin{aligned}
R^2 &= a^2 + \left(h - 2r \cdot \cos\left(\frac{\beta}{2}\right)\right)^2 \\
R^2 &= \left(h \cdot \tan\left(\frac{\beta}{2}\right)\right)^2 + \left(h - 2r \cdot \cos\left(\frac{\beta}{2}\right)\right)^2 \\
0 &= \left(\frac{h}{R} \cdot \tan\left(\frac{\beta}{2}\right)\right)^2 + \left(\frac{h}{R} - 2\frac{r}{R} \cdot \cos\left(\frac{\beta}{2}\right)\right)^2 - 1 \\
0 &= \left(\frac{h}{R}\right)^2 \left(\tan^2\left(\frac{\beta}{2}\right) + 1\right) - 4\frac{h}{R} \cdot \frac{r}{R} \cdot \cos\left(\frac{\beta}{2}\right) + \left(4\left(\frac{r}{R}\right)^2 \cdot \cos^2\left(\frac{\beta}{2}\right) - 1\right)
\end{aligned}$$

Notice this is more complex than the square case, where the right angles in the geometry caused a to equal h. It is, nevertheless, a quadratic equation in $\frac{h}{R}$, so the quadratic formula gives us $\frac{h}{R}$ as a function of the invariant $\frac{r}{R}$:

$$\begin{aligned}
\frac{h}{R} &= 2\frac{r}{R} \cdot \frac{\cos\left(\frac{\beta}{2}\right)}{\tan^2\left(\frac{\beta}{2}\right) + 1} \pm \sqrt{\left(2\frac{r}{R} \cdot \frac{\cos\left(\frac{\beta}{2}\right)}{\tan^2\left(\frac{\beta}{2}\right) + 1}\right)^2 - \frac{4\left(\frac{r}{R}\right)^2 \cdot \cos^2\left(\frac{\beta}{2}\right) - 1}{\tan^2\left(\frac{\beta}{2}\right) + 1}} \\
&= 2\frac{r}{R} \cdot \cos^3\left(\frac{\beta}{2}\right) \pm \sqrt{\left(2\frac{r}{R} \cdot \cos^3\left(\frac{\beta}{2}\right)\right)^2 - 4\left(\frac{r}{R}\right)^2 \cdot \cos^4\left(\frac{\beta}{2}\right) + \cos^2\left(\frac{\beta}{2}\right)} \\
&= 2\frac{r}{R} \cdot \cos\left(\frac{\beta}{2}\right)\left(\cos^2\left(\frac{\beta}{2}\right) \pm \sqrt{\cos^4\left(\frac{\beta}{2}\right) - \cos^2\left(\frac{\beta}{2}\right) + \left(2\frac{r}{R}\right)^{-2}}\right) \\
&= 2\frac{r}{R} \cdot \cos\left(\frac{\beta}{2}\right)\left(\cos^2\left(\frac{\beta}{2}\right) \pm \sqrt{\left(2\frac{r}{R}\right)^{-2} - \cos^2\left(\frac{\beta}{2}\right) \cdot \sin^2\left(\frac{\beta}{2}\right)}\right) \\
&= 2\frac{r}{R} \cdot \cos\left(\frac{\beta}{2}\right)\left(\cos^2\left(\frac{\beta}{2}\right) \pm \sqrt{\frac{1}{4}\left(\frac{r}{R}\right)^{-2} - \frac{1}{4}\sin^2(\beta)}\right) \\
&= \cos\left(\frac{\beta}{2}\right)\left(2\frac{r}{R} \cdot \cos^2\left(\frac{\beta}{2}\right) \pm \sqrt{1 - \left(\frac{r}{R}\right)^2 \cdot \sin^2(\beta)}\right)
\end{aligned}$$

If you use $\beta = \frac{\pi}{2}$ for the square, you will find that this gives exactly the same relationship as equation 3.1. The design constraint equating the extent of the overcuts becomes:

$$\begin{aligned}
R - a &= r - r \cdot \sin\left(\frac{\beta}{2}\right) \\
R - h \cdot \tan\left(\frac{\beta}{2}\right) &= r\left(1 - \sin\left(\frac{\beta}{2}\right)\right) \\
\cos\left(\frac{\beta}{2}\right) - \frac{h}{R} \cdot \sin\left(\frac{\beta}{2}\right) &= \frac{r}{R}\left(\cos\left(\frac{\beta}{2}\right) - \cos\left(\frac{\beta}{2}\right) \cdot \sin\left(\frac{\beta}{2}\right)\right) \\
\frac{h}{R} \cdot \sin\left(\frac{\beta}{2}\right) &= \cos\left(\frac{\beta}{2}\right) - \frac{r}{R}\left(\cos\left(\frac{\beta}{2}\right) - \cos\left(\frac{\beta}{2}\right) \cdot \sin\left(\frac{\beta}{2}\right)\right) \\
\frac{h}{R} &= \cot\left(\frac{\beta}{2}\right) - \frac{r}{R}\left(\cot\left(\frac{\beta}{2}\right) - \cos\left(\frac{\beta}{2}\right)\right)
\end{aligned}$$

It is a little easier this time to see that $\beta = \frac{\pi}{2}$ gives the same relationship as before for the case of a square. Thus the equation defining the invariant $\frac{r}{R}$ for the n-gon is:

$$
\begin{aligned}
\cos\left(\frac{\beta}{2}\right)\left(2\frac{r}{R}\cdot\cos^2\left(\frac{\beta}{2}\right)\pm\sqrt{1-\left(\frac{r}{R}\right)^2\cdot\sin^2(\beta)}\right) &= \cot\left(\frac{\beta}{2}\right)-\frac{r}{R}\left(\cot\left(\frac{\beta}{2}\right)-\cos\left(\frac{\beta}{2}\right)\right)\\
\pm\cos\left(\frac{\beta}{2}\right)\cdot\sqrt{1-\left(\frac{r}{R}\right)^2\cdot\sin^2(\beta)} &= \cot\left(\frac{\beta}{2}\right)-\frac{r}{R}\left(\cot\left(\frac{\beta}{2}\right)-\cos\left(\frac{\beta}{2}\right)\cdot\left(1-2\cdot\cos^2\left(\frac{\beta}{2}\right)\right)\right)\\
\cos^2\left(\frac{\beta}{2}\right)\cdot\left(1-\left(\frac{r}{R}\right)^2\cdot\sin^2(\beta)\right) &= \left(\cot\left(\frac{\beta}{2}\right)-\frac{r}{R}\left(\cot\left(\frac{\beta}{2}\right)-\cos\left(\frac{\beta}{2}\right)\cdot\left(\sin^2\left(\frac{\beta}{2}\right)-\cos^2\left(\frac{\beta}{2}\right)\right)\right)\right)^2\\
\cos^2\left(\frac{\beta}{2}\right)\cdot\sin^2\left(\frac{\beta}{2}\right)\cdot\left(1-\left(\frac{r}{R}\right)^2\cdot\sin^2(\beta)\right) &= \left(\cos\left(\frac{\beta}{2}\right)-\frac{r}{R}\left(\cos\left(\frac{\beta}{2}\right)+\cos\left(\frac{\beta}{2}\right)\cdot\sin\left(\frac{\beta}{2}\right)\cdot\cos(\beta)\right)\right)^2\\
\frac{1}{4}\sin^2(\beta)\cdot\left(1-\left(\frac{r}{R}\right)^2\cdot\sin^2(\beta)\right) &= \left(\cos\left(\frac{\beta}{2}\right)-\frac{r}{R}\left(\cos\left(\frac{\beta}{2}\right)+\frac{1}{2}\sin(\beta)\cdot\cos(\beta)\right)\right)^2\\
\sin^2(\beta)\cdot\left(1-\left(\frac{r}{R}\right)^2\cdot\sin^2(\beta)\right) &= \left(2\cos\left(\frac{\beta}{2}\right)-\frac{r}{R}\left(2\cos\left(\frac{\beta}{2}\right)+\frac{1}{2}\sin(2\beta)\right)\right)^2\\
\sin^2(\beta)\cdot\left(1-\left(\frac{r}{R}\right)^2\cdot\sin^2(\beta)\right) &= 4\cos^2\left(\frac{\beta}{2}\right)+\left(\frac{r}{R}\right)^2\left(2\cos\left(\frac{\beta}{2}\right)+\frac{1}{2}\sin(2\beta)\right)^2\\
&\quad -2\frac{r}{R}\left(4\cos^2\left(\frac{\beta}{2}\right)+\cos\left(\frac{\beta}{2}\right)\cdot\sin(2\beta)\right)\\
-\left(\frac{r}{R}\right)^2\left(\left(2\cos\left(\frac{\beta}{2}\right)+\frac{1}{2}\sin(2\beta)\right)^2+\sin^4(\beta)\right) &= -\frac{r}{R}\left(8\cos^2\left(\frac{\beta}{2}\right)+2\cos\left(\frac{\beta}{2}\right)\cdot\sin(2\beta)\right)+4\cos^2\left(\frac{\beta}{2}\right)-\sin^2(\beta)\\
\left(\frac{r}{R}\right)^2\left(\left(2\cos\left(\frac{\beta}{2}\right)+\frac{1}{2}\sin(2\beta)\right)^2+\sin^4(\beta)\right) &= \frac{r}{R}\left(8\cos^2\left(\frac{\beta}{2}\right)+2\cos\left(\frac{\beta}{2}\right)\cdot\sin(2\beta)\right)-4\cos^2\left(\frac{\beta}{2}\right)+\sin^2(\beta)\\
\left(\frac{r}{R}\right)^2\left(\left(2\cos\left(\frac{\beta}{2}\right)+\frac{1}{2}\sin(2\beta)\right)^2+\sin^4(\beta)\right) &= \frac{r}{R}\left(2\cos\left(\frac{\beta}{2}\right)\cdot\left(4\cos\left(\frac{\beta}{2}\right)+\sin(2\beta)\right)\right)-4\cos^4\left(\frac{\beta}{2}\right)
\end{aligned}
$$

The final equation is a quadratic equation in the invariant $\frac{r}{R}$ with coefficients that depend on $\beta = \pi - \alpha = \pi\frac{n-2}{n}$. Therefore, given n, that is, given a regular polygon with a specified number of sides, the coefficients are numerically determined, and in consequence of the quadratic relation, the invariant is determined. I ran the coefficients of the penultimate equation through Wolfram Alpha[2], asking for simplification. The final simple form of the constant term was returned, while the linear term final simplification is obvious, and is the only useful improvement offered by Wolfram Alpha. Wolfram Alpha does not provide any useful simplification of the quadratic term.

To summarize, the algorithm for calculating a solution for an n-gon with side length $s = 2h$ is therefore:

(1) Calculate $\beta = \pi\frac{n-2}{n}$, the angle between adjacent sides in radians.
(2) Calculate the coefficients of the quadratic form:
  (a) $\mathsf{a}(\beta) = \left(2\cos\left(\frac{\beta}{2}\right)+\frac{1}{2}\sin(2\beta)\right)^2+\sin^4(\beta)$
  (b) $\mathsf{b}(\beta) = 2\cos\left(\frac{\beta}{2}\right)\cdot\left(4\cos\left(\frac{\beta}{2}\right)+\sin(2\beta)\right)$
  (c) $\mathsf{c}(\beta) = 4\cos^4\left(\frac{\beta}{2}\right)$
(3) Find the small solution of the quadratic form $0 = \mathsf{a}(\beta)\cdot\left(\frac{r}{R}\right)^2 - \mathsf{b}(\beta)\cdot\left(\frac{r}{R}\right) + \mathsf{c}(\beta)$:

$$\frac{r}{R} = \frac{\mathsf{b}(\beta)}{2\mathsf{a}(\beta)} - \sqrt{\left(\frac{\mathsf{b}(\beta)}{2\mathsf{a}(\beta)}\right)^2 - \frac{\mathsf{c}(\beta)}{\mathsf{a}(\beta)}}$$

(4) Calculate $\frac{h}{R} = \cos\left(\frac{\beta}{2}\right)\left(2\frac{r}{R}\cdot\cos^2\left(\frac{\beta}{2}\right)+\sqrt{1-\left(\frac{r}{R}\right)^2\cdot\sin^2(\beta)}\right)$, which is the large solution of the universal geometric constraint $R^2 = \left(h\cdot\tan\left(\frac{\beta}{2}\right)\right)^2 + \left(h - 2r\cdot\cos\left(\frac{\beta}{2}\right)\right)^2$ implied by the geometry, the Pythagorean theorem, and the quadratic formula.
(5) Calculate $R = \frac{R}{h}\cdot h = \frac{R}{h}\cdot\frac{s}{2}$ and $r = \frac{r}{R}\cdot R$.
(6) Calculate the distance of the small drill center from the centroid $b - r = \frac{h}{\cos\left(\frac{\beta}{2}\right)} - r = \frac{s}{2\cos\left(\frac{\beta}{2}\right)} - r$.

A short table of values follows, assuming $h = 1$. I have included degenerate cases, just for fun. The calculations were performed by a short Python script. Quadratic form coefficients are written as the formula produces them, that is, they have not been simplified to integral surds or smaller integers. The quadratic form, as well as a and b, are omitted when the coefficients do not have a simple symbolic form. It is interesting that the second solution for $\frac{r}{R}$ is unity in all cases. The second solution implies drilling just a single circular clearance hole.

n	α	β	b	a	quadratic form	$\frac{r}{R}$	R	r
0	$\frac{2}{0}\pi$	$\frac{-2}{0}\pi$	-	-	-	-	-	-
1	$\frac{2}{1}\pi$	$\frac{-1}{1}\pi$	$\frac{1}{0}$	$\frac{1}{0}$	$0 = 0$	-	-	-
2	$\frac{2}{2}\pi$	$\frac{0}{2}\pi$	1	0	$4\left(\frac{r}{R}\right)^2 - 8\frac{r}{R} + 4 = 0$	1, 1	1	1
3	$\frac{2}{3}\pi$	$\frac{1}{3}\pi$	$\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{21}{4}\left(\frac{r}{R}\right)^2 - \frac{15}{2}\frac{r}{R} + \frac{9}{4} = 0$	1, $\frac{3}{7}$	0.734809	0.314918
4	$\frac{1}{2}\pi$	$\frac{1}{2}\pi$	$\frac{\sqrt{2}}{1}$	$\frac{1}{1}$	$3\left(\frac{r}{R}\right)^2 - 4\frac{r}{R} + 1 = 0$	1, $\frac{1}{3}$	1.108194	0.369398
5	$\frac{2}{5}\pi$	$\frac{3}{5}\pi$	$\sqrt{\frac{8}{5-\sqrt{5}}}$	$\sqrt{1+\frac{2}{\sqrt{5}}}$	$\frac{15-\sqrt{5}}{8}\left(\frac{r}{R}\right)^2 - \frac{15-3\sqrt{5}}{4}\frac{r}{R} + \frac{15-5\sqrt{5}}{8} = 0$	1, 0.299254	1.459814	0.436855
6	$\frac{1}{3}\pi$	$\frac{2}{3}\pi$	2	$\sqrt{3}$	$\frac{7-2\sqrt{3}}{4}\left(\frac{r}{R}\right)^2 - \frac{1-\sqrt{3}}{2}\frac{r}{R} + \frac{1}{4} = 0$	1, 0.282814	1.800262	0.509139
7	$\frac{2}{7}\pi$	$\frac{5}{7}\pi$				1, 0.273525	2.134335	0.583794
8	$\frac{\pi}{4}$	$\frac{3}{4}\pi$	$\sqrt{\frac{4}{2-\sqrt{2}}}$	$1+\sqrt{2}$		1, 0.267732	2.464438	0.659808
9	$\frac{2}{9}\pi$	$\frac{7}{9}\pi$				1, 0.263862	2.791904	0.736677
10	$\frac{1}{5}\pi$	$\frac{4}{5}\pi$	$\frac{4}{\sqrt{5}-1}$	$\sqrt{5+2\sqrt{5}}$		1, 0.261144	3.117530	0.814123
11	$\frac{2}{11}\pi$	$\frac{9}{11}\pi$				1, 0.259158	3.441819	0.891976
12	$\frac{1}{6}\pi$	$\frac{5}{6}\pi$	$\sqrt{\frac{4}{2-\sqrt{3}}}$	$2+\sqrt{3}$		1, 0.257663	3.765107	0.970129
13	$\frac{2}{13}\pi$	$\frac{11}{13}\pi$				1, 0.256508	4.087627	1.048510
14	$\frac{1}{7}\pi$	$\frac{6}{7}\pi$				1, 0.255597	4.409544	1.127067
15	$\frac{2}{15}\pi$	$\frac{13}{15}\pi$				1, 0.254866	4.730979	1.205764
16	$\frac{1}{8}\pi$	$\frac{7}{8}\pi$	$\sqrt{\frac{4}{\sqrt{2-\sqrt{2+\sqrt{2}}}}}$	$\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}}$		1, 0.254269	5.052022	1.284573
17	$\frac{2}{17}\pi$	$\frac{15}{17}\pi$				1, 0.253776	5.372743	1.363475

Since the table assumes a fixed side length, the area of the polygon increases with n, and so do R and r. If you want to get a feel for how they change relatively, then the values need to be divided by the semi-perimeter which is just n. Alternatively, just look at how the invariant $\frac{r}{R}$ changes.

Although the geometric meaning restricts the solutions to integral n, the invariant equation is a continuous function in β and $\frac{r}{R}$, and so the general solution space is the intersection of a surface in three dimensions with the $\beta, \frac{r}{R}$ plane. This is called a 'level set'. The geometrically meaningful solutions are those where $\beta = \pi\frac{n-2}{n}$. Figure 6 shows the solution for a pentagonal peg.

The invariant $\frac{r}{R}$ is assymptotic toward zero. Therefore, the level set coordinate $\frac{r}{R}$ will pass through all of the harmonic fractions as n increases, supposing for the moment that n is now continuous, not discrete. We see so far that only 1 and $\frac{1}{3}$ are included among the geometrically meaningful solutions, that is, solutions that are harmonic fractions and n is also integral. The fraction $\frac{1}{2}$ is missing. This raises an interesting question: are any other harmonic fraction solutions geometrically meaningful? If so, what is the sequence of values for n that produce them? This is what mathematics is all about - wonder. Calculation and problem solving are simply means to explore our wonder. Mastery of those skills is necessary, but ultimately, they are important only because there are interesting questions to wonder about.

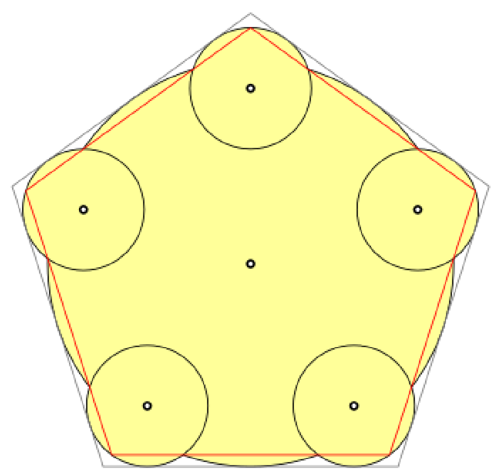

FIGURE 6.
Pentagonal peg in a round hole.

7. Epilogue

Only two values of n in the table produce a rational value of $\frac{r}{R}$, specifically $n = 3$ and $n = 4$. Whether there are more beyond the table is unknown. In all cases, the geometry implies another circle, one tangent to the small drill openings and concentric with the large drill opening. That raised the interesting question of whether any of those inscibed circles are in rational proportion. Of the values in the table, only $n = 3$ yields such a rational proportion and it is $\frac{5}{3}r$. That immediately suggested to me the possibility of using the ratios as gear ratios. Figure 7 depicts the resulting planetary gear train. The red outline is the solution to the triangular peg problem, and it coincides with the pitch circles of the ring gear and the darker blue planet gears. The pitch circle of the copper color sun gear is tangent to the pitch circles of the darker blue planet gears. Gear clearance was chosen to be 5% in order to accomodate printing registration variance. The undercut on small pinions is calculated using the machinist's 'cheat', instead of the mathematically 'correct' trochoidal curve.

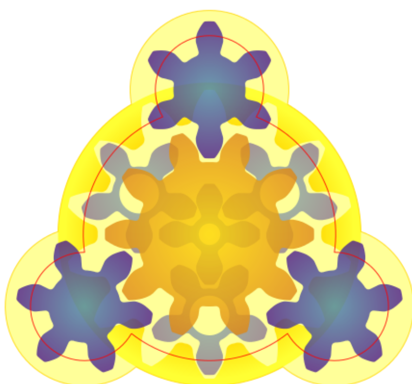

FIGURE 7.
Planetary gears.

Not shown is a spider coupling the lighter blue planet gears to the copper colored sun gear. The casing outline is *not* a solution to the triangular peg problem, since all solutions to the triangular peg problem are uniform scalings of one another. The casing outline is the result of adding a uniform margin to the pitch circle line. Using a solution for the casing outline either creates a casing that interferes with the darker blue planet gears, or is grossly oversized.

As it happens, I'd been looking for some inspiration for creating an iconic logo for my document license. Although it is a bit busy, this image seemed like it might make a good starting point. I decluttered it a bit by removing some transparency, reducing other `transparencies`, and subduing the colors. Finally, I embossed a copyright symbol on the principal gears. Figure 8 shows the final result. The busyness now has more of the character of a watermark.

In addition to files related to producing this document, the repository[4] contains the file '`polygonal_pegs.py`'. That file is self contained and holds a class definition that encapsulates the calculations in this document as class properties. It was instrumental in producing the table of values for small polygons. The file's text may be copied and pasted into the Python interpreter command line, facilitating easy calculation of specific cases that may interest the reader. So, for example, if you are the reincarnation of Gauss, the first thing you would do is type '`p=PolygonalPeg(17)`', and then you would type '`p.r`', '`p.R`', '`p.intersection`', '`p.r_R`' (calculates $\frac{r}{R}$), et cetera.

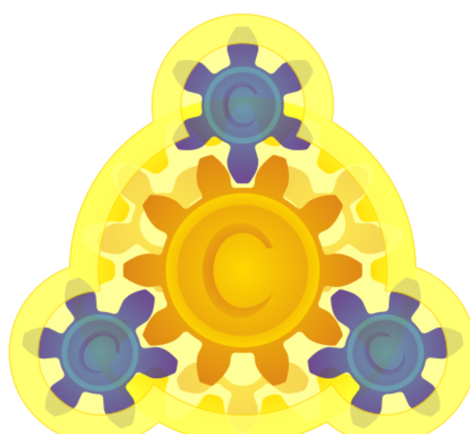

FIGURE 8. ASSAIL license logo.

Appendix A

$$\begin{aligned}
w &= v + \sqrt{1 - v^2} \\
(w - v)^2 &= 1 - v^2 \\
2v^2 - 2wv + w^2 - 1 &= 0 \qquad \Rightarrow \\
v &= \frac{w}{2} \pm \sqrt{\left(\frac{w}{2}\right)^2 - \frac{w^2 - 1}{2}} \\
v &= \frac{1}{2}\left(w \pm \sqrt{w^2 - 2w^2 + 2}\right) \\
v &= \frac{1}{2}\left(w \pm \sqrt{2 - w^2}\right)
\end{aligned}$$

Appendix B: License

References

[1] http://voluntocracy.blogspot.ca/2016/03/make-square-opening-by-drilling-five.html
[2] https://www.wolframalpha.com
[3] https://oeis.org
[4] https://github.com/ruminations/Essays/tree/master/Polygonal_Pegs