BA-BMECV1031U Økonometri

June 30, 2025

Copenhagen Business School
Department of Economics
Ralf A. Wilke
2025/2026, autumn semester

Problem Set 0: Revision of Fundamentals of Mathematical Statistics and Matrix Algebra

This problem set revises the content of Appendices C and D in the textbook "Introductory Econometrics" by J.Wooldridge, 8th edition, 2025. Students should ensure that they can cope with the technical level of the material.

Part 1: Fundamentals of Mathematical Statistics

1. Let Y_1, Y_2, Y_3 and Y_4 be independent, identically distributed random variables from a population with mean μ and variance σ^2. Let $\bar{Y} = \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4)$ denote the average of these four random variables.

   (a) What are the expected value and variance of $\bar{Y}$ in terms of μ and σ^2?

(b) Now, consider a different estimator of μ:

$$W = \frac{1}{8}Y_1 + \frac{1}{8}Y_2 + \frac{1}{4}Y_3 + \frac{1}{2}Y_4$$

This is an example of a weighted average of the Y_i. Show that W is also an unbiased estimator of μ. Find the variance of W.

(c) Based on your answers to parts (a) and (b), which estimator of μ do you prefer, $\bar{Y}$ or W?

2. Let Y_1, Y_2, Y_3,...,Y_n be pairwise uncorrelated random variables with common mean μ and common variance σ^2. Let $\bar{Y}$ denote the sample average.

(a) Define the class of *linear estimators* of μ by

$$W_a = a_1Y_1 + a_2Y_2 + ... + a_nY_n,$$

where the a_i are constants. What restrictions on the a_i is needed for W_a to be an unbiased estimator of μ?

(b) Find $Var(W_a)$.

(c) For any numbers $a_1, a_2, ..., a_n$, the following inequality holds: $(a_1 + a_2 + ... + a_n)^2/n \leq a_1^2 + a_2^2 + ... + a_n^2$. Use this, along with parts (a) and (b), to show that $Var(W_a) \geq Var(\bar{Y})$ whenever W_a is unbiased, so that $\bar{Y}$ is the *best linear unbiased estimator*. [Hint: What does the inequality become when a_i satisfy the restriction from part (a)?]

3. Show that when $Y = a + bX$ then $\text{Var}(Y) = b^2\text{Var}(X)$. [Hint: $\text{Var}(X) = E[X - E(X)]^2$.]

4. Let $\bar{Y}$ denote the sample average from a random sample with mean μ and variance σ^2. Consider two alternative estimators of μ: $W_1 = [(n-1)/n]\bar{Y}$ and $W_2 = \bar{Y}/2$.

(a) Show that W_1 and W_2 are both biased estimators of μ and find the biases. What happens to the biases as $n \to \infty$? Comment on any important differences in bias for the two estimators as the sample size gets large.

(b) Find the probability limit of W_1 and W_2. [Hint: Use properties PLIM.1 and PLIM.2 (as given in Wooldridge, 2020); for W_1, note that $plim[(n-1)/n] = 1$.] Which estmator is consistent?

(c) Find $Var(W_1)$ and $Var(W_2)$.

(d) Argue that W_1 is a better estimator than $\bar{Y}$ if μ is close to zero. (Consider both bias and variance.)

5. For positive random variables X and Y, suppose the expected value of Y given X is $E(Y|X) = \theta X$. The unknown parameter θ shows how the expected value of Y changes with X.

   (a) Define the random variable $Z = Y/X$. Show that $E(Z) = \theta$. [Hint: Use property CE.2 along with the law of iterated expectations, Property CE.4 (as given in Wooldridge, 2020). In particular, first show that $E(Z|X) = \theta$ and then use CE.4]

   (b) Use part (a) to prove that the estimator $W_1 = \frac{1}{n}\sum_{i=1}^{n}(Y_i/X_i)$ is unbiased for θ, where $\{(X_i, Y_i) : i = 1, 2, ..., n\}$ is a random sample.

   (c) Explain why the estimator $W_2 = \frac{\bar{Y}}{\bar{X}}$, where the overbars denote sample averages, is not the same as W_1. Nevertheless, show that W_2 is also unbiased for θ.

6. Let Y denote a Bernoulli(θ) random variable with $0 < \theta < 1$. Suppose we are interested in estimating the *odds ratio*, $\gamma = \theta/(1 - \theta)$, which is the probability of success over the probability of failure. Given a random sample $\{Y_1, ..., Y_n\}$, we know that an unbiased and consistent estimator of θ is $\bar{Y}$, the proportion of successes in n trials. A natural estimator of γ is $G = \bar{Y}/(1 - \bar{Y})$, the proportion of successes over the proportion of failures in the sample.

   (a) Why is G not an unbiased estimator of γ?

   (b) Use PLIM.2 (iii) (as given in Wooldridge, 2020) to show that G is a consistent estimator of γ.

Part 2: Matrix Algebra

7. Identify the following matrixes, i.e. are the matrixes **A**,**B** and **C** respectively a diagonal, square, scalar or identity matrix?

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. (a) Find the product $\mathbf{AB}$ using

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 7 \\ -4 & 5 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 6 \\ 1 & 8 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$$

   (b) Does $\mathbf{BA}$ exist?

9. Show that $\mathbf{AB} \neq \mathbf{BA}$

$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

10. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ diagonal matrices, show that $\mathbf{AB} = \mathbf{BA}$.

11. What is a symmetric matrix? Give an example.

12. Let $\mathbf{X}$ be any $n \times k$ matrix. Show that $\mathbf{X}'\mathbf{X}$ is a symmetric matrix.

13. (a) Use the properties of trace to argue that $\text{tr}(\mathbf{A}'\mathbf{A}) = \text{tr}(\mathbf{AA}')$ for any $n \times m$ matrix $\mathbf{A}$.

    (b) For

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \end{pmatrix},$$

    verify that $\text{tr}(\mathbf{A}'\mathbf{A}) = \text{tr}(\mathbf{AA}')$.

14. (a) Use the definition of inverse to prove the following: if $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ nonsingular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

    (b) If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are all $n \times n$ nonsingular matrices, find $(\mathbf{ABC})^{-1}$ in terms of $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, and $\mathbf{C}^{-1}$.

15. Let $\mathbf{A}$ be an $n \times n$ symmetric, positive definite matrix. Show that if $\mathbf{P}$ is any $n \times n$ nonsingular matrix, then $\mathbf{P}'\mathbf{AP}$ is positive definite.

16. Let $\mathbf{y}$ be an $n \times 1$ random vector. Prove that if $\mathbf{A}$ is an $m \times n$ nonrandom matrix and $\mathbf{b}$ is an $n \times 1$ nonrandom vector, then $\text{Var}(\mathbf{Ay} + \mathbf{b}) = \mathbf{A}[\text{Var}(\mathbf{y})]\mathbf{A}'$. <u>Hint:</u> $\text{Var}(\mathbf{y}) = E[(\mathbf{y} - \boldsymbol{\mu})(\mathbf{y} - \boldsymbol{\mu})']$ with $\boldsymbol{\mu} = E(\mathbf{y})$.

Part 3: OLS with Matrix Notation

17. Consider the linear multiple regression model as in Appendix E-1 of the textbook "Introductory Econometrics". Derive $\hat{\beta} = (X'X)^{-1}X'\mathbf{y}$. (Hint: OLS estimators minimise the sum of the squared residuals).

18. Show that $\hat{\beta}$ is an unbiased estimator for β.

Many of these problems have been taken from the textbook "Introductory Econometrics" by J.Wooldridge.