ECE 5371: Engineering Analysis
Assignment 9 Solutions

Problem 1: An absent-minded professor has two umbrellas that she uses when commuting from home to office and back. If it rains and an umbrella is available in her location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability p each time she commutes, independently of other times. What is the steady-state probability that she gets wet on a given day?

Solution 1:

State i: i umbrellas are available in her current location, $\quad i = 0, 1, 2.$

The transition probability graph is given in Fig. 6.9, and the transition probability matrix is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

The chain has a single recurrent class that is aperiodic (assuming $0 < p < 1$), so the steady-state convergence theorem applies. The balance equations are

$$\pi_0 = (1-p)\pi_2, \quad \pi_1 = (1-p)\pi_1 + p\pi_2, \quad \pi_2 = \pi_0 + p\pi_1.$$

From the second equation, we obtain $\pi_1 = \pi_2$, which together with the first equation $\pi_0 = (1-p)\pi_2$ and the normalization equation $\pi_0 + \pi_1 + \pi_2 = 1$, yields

$$\pi_0 = \frac{1-p}{3-p}, \quad \pi_1 = \frac{1}{3-p}, \quad \pi_2 = \frac{1}{3-p}.$$

According to the steady-state convergence theorem, the steady-state probability that the professor finds herself in a place without an umbrella is π_0. The steady-state probability that she gets wet is π_0 times the probability of rain p.

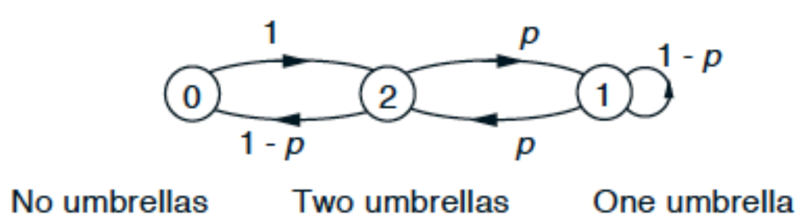

Figure 6.9: Transition probability graph for Example 6.5.

Problem 2.

The times between successive customer arrivals at a facility are independent and identically distributed random variables with the following PMF:

P(k) = 0.2, k=1

0.3, k=3

0.5, k=4

0, o.w.

Construct a four state MC model that describes the arrival process. In this model, one of the states should correspond to the times when an arrival occurs.

Problem 2 Solution

Solution to Problem 7.1. We construct a Markov chain with state space $S = \{0, 1, 2, 3\}$. We let $X_n = 0$ if an arrival occurs at time n. Also, we let $X_n = i$ if the last arrival up to time n occurred at time $n - i$, for $i = 1, 2, 3$. Given that $X_n = 0$, there is probability 0.2 that the next arrival occurs at time $n + 1$, so that $p_{00} = 0.2$, and $p_{01} = 0.8$. Given that $X_n = 1$, the last arrival occurred at time $n - 1$, and there is zero probability of an arrival at time $n + 1$, so that $p_{12} = 1$. Given that $X_n = 2$, the last arrival occurred at time $n - 2$. We then have

$$\begin{aligned} p_{20} &= \mathbf{P}(X_{n+1} = 0 \mid X_n = 2) \\ &= \mathbf{P}(T = 3 \mid T \geq 3) \\ &= \frac{\mathbf{P}(T = 3)}{\mathbf{P}(T \geq 3)} \\ &= \frac{3}{8}, \end{aligned}$$

and $p_{23} = 5/8$. Finally, given that $X_n = 3$, an arrival is guaranteed at time $n + 1$, so that $p_{40} = 1$.

Problem 3.

A mouse moves along a tiled corridor with 2m tiles, where m>1. From each tile not equal to 1,2m, it moves to either i-1 or i+1 with equal prob. From tile 1 or tile 2m, it moves to tile 2 or 2m-1 respectively with prob 1. Each time the mouse moves to a tile i>=m or i>m, an electronic device outputs a signal L or R respectively. Can the generated sequence of signals L and R be described as a Markov Chain with states L and R.

Solution Problem 3

Solution to Problem 7.2. It cannot be described as a Markov chain with states L and R, because $\mathbf{P}(X_{n+1} = L \,|\, X_n = R, X_{n-1} = L) = 1/2$, while $\mathbf{P}(X_{n+1} = L \,|\, X_n = R, X_{n-1} = R, X_{n-1} = L) = 0$.

Problem 4.

Consider a MC with states 1,2,..9 with p12=p17=1/2, pi,i+1=1 for I not equal to 1,6,9 and p61=p91=1. Is the recurrent class of the chain periodic or not?

Solution Problem 4

Solution to Problem 7.5. It is periodic with period 2. The two corresponding subsets are $\{2, 4, 6, 7, 9\}$ and $\{1, 3, 5, 8\}$.

Problem 5

A professor gives tests that are hard, medium or easy. If he gives a hard test, the next test will either be M or E with equal prob. If medium or easy test, then there is a .5 probability that the next test will be of same difficulty , and a .25 probability for each of the other two levels of difficulty. Construct an appropriate MC and find the steady state probabilities.

Solution problem 5

Solution to Problem 7.11. We use a Markov chain model with 3 states, H, M, and E, where the state reflects the difficulty of the most recent exam. We are given the transition probabilities

$$\begin{bmatrix} r_{HH} & r_{HM} & r_{HE} \\ r_{MH} & r_{MM} & r_{ME} \\ r_{EH} & r_{EM} & r_{EE} \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}.$$

It is easy to see that our Markov chain has a single recurrent class, which is aperiodic. The balance equations take the form

$$\pi_1 = \frac{1}{4}(\pi_2 + \pi_3),$$
$$\pi_2 = \frac{1}{2}(\pi_1 + \pi_2) + \frac{1}{4}\pi_3,$$
$$\pi_3 = \frac{1}{2}(\pi_1 + \pi_3) + \frac{1}{4}\pi_2,$$

and solving these with the constraint $\sum_i \pi_i = 1$ gives

$$\pi_1 = \frac{1}{5}, \qquad \pi_2 = \pi_3 = \frac{2}{5}.$$

Problem 6 A superstitious professor works in a circular building with m doors, where m is odd, and never uses the same door twice in a row. Instead he uses with probability p (or probability $1-p$) the door that is adjacent in the clockwise direction (or the counterclockwise direction, respectively) to the door he used last. What is the probability that a given door will be used on some particular day far into the future?

Solution problem 6

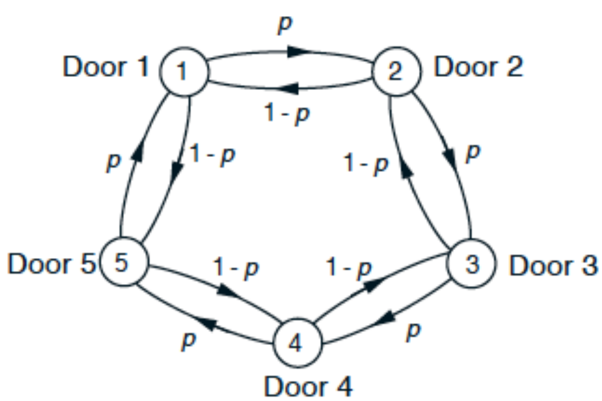

Figure 6.10: Transition probability graph in Example 6.6, for the case of $m = 5$ doors.

We introduce a Markov chain with the following m states:

State i: Last door used is door i, $\quad i = 1, \ldots, m$.

The transition probability graph of the chain is given in Fig. 6.10, for the case $m = 5$. The transition probability matrix is

$$\begin{bmatrix} 0 & p & 0 & 0 & \dots & 0 & 1-p \\ 1-p & 0 & p & 0 & \dots & 0 & 0 \\ 0 & 1-p & 0 & p & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p & 0 & 0 & 0 & \dots & 1-p & 0 \end{bmatrix}.$$

Assuming that $0 < p < 1$, the chain has a single recurrent class that is aperiodic. [To verify aperiodicity, argue by contradiction: if the class were periodic, there could be only two subsets of states such that transitions from one subset lead to the other, since it is possible to return to the starting state in two transitions. Thus, it cannot be possible to reach a state i from a state j in both an odd and an even number of transitions. However, if m is odd, this is true for states 1 and m – a contradiction (for example, see the case where $m = 5$ in Fig. 6.10, doors 1 and 5 can be reached from each other in 1 transition and also in 4 transitions).] The balance equations are

$$\begin{aligned} \pi_1 &= (1-p)\pi_2 + p\pi_m, \\ \pi_i &= p\pi_{i-1} + (1-p)\pi_{i+1}, \qquad i = 2, \dots, m-1, \\ \pi_m &= (1-p)\pi_1 + p\pi_{m-1}. \end{aligned}$$

These equations are easily solved once we observe that by symmetry, all doors should have the same steady-state probability. This suggests the solution

$$\pi_j = \frac{1}{m}, \qquad j = 1, \dots, m.$$

Indeed, we see that these π_j satisfy the balance equations as well as the normalization equation, so they must be the desired steady-state probabilities (by the uniquenes part of the steady-state convergence theorem).

Note that if either $p = 0$ or $p = 1$, the chain still has a single recurrent class but is periodic. In this case, the n-step transition probabilities $r_{ij}(n)$ do not converge to a limit, because the doors are used in a cyclic order. Similarly, if m is even, the recurrent class of the chain is periodic, since the states can be grouped into two subsets, the even and the odd numbered states, such that from each subset one can only go to the other subset.