Lecture 9: Public Key Cryptography and RSA

TTM4135

Relates to Stallings Chapter 9

Spring Semester, 2025

Motivation

- Public key cryptography (PKC) provides some features which cannot be achieved with symmetric key cryptography
- PKC is widely applied for key management in protocols such as TLS and IPSec
- RSA is probably the best known public key cryptosystem, widely deployed in many kinds of applications

Outline

- Public Key Cryptography
- RSA algorithms
- Implementing RSA
- Security of RSA
  - Factorisation
  - Side channel attacks

One-way functions

- A function f is said to be a *one-way function* if it is easy to compute $f(x)$ given x, but is computationally hard to compute $f^{-1}(y) = x$ given y
- It is an open problem in computer science whether any one-way functions formally exist
- Two examples of functions believed to be one-way are:
  1. Multiplication of large primes: the inverse function is integer factorisation
  2. Exponentiation: the inverse function is taking discrete logarithms

Trapdoor one-way functions

- A *trapdoor one-way function* f is a one-way function such that given additional information (the trapdoor) it is easy to compute f^{-1}
- An example of a trapdoor one-way function is *modular squaring*
- Let $n = pq$ be the product of two large prime numbers p and q and define $f(x) = x^2 \bmod n$
- If there is an algorithm to take square roots (compute f^{-1}) then this algorithm can be used to factorise n
- The trapdoor is the factorisation of n – knowledge of p and q gives an efficient algorithm to find square roots (exercise)

Ciphers based on computationally hard problems

- In 1976 Diffie and Hellman published their famous paper *New Directions in Cryptography*
- They suggested that computational complexity be applied in the design of encryption algorithms
- A public key cryptosystem can be designed by using a trapdoor one-way function
- The trapdoor will become the decryption key

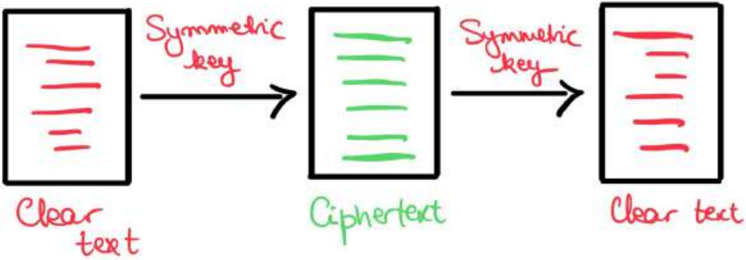
Prior claims

- In 1997 is was revealed that researchers at UK's intelligence agency (GCHQ) had previously invented public key cryptography in the early 1970s
- James H. Ellis, Clifford Cocks, and Malcolm Williamson invented what is now known as Diffie-Hellman key exchange and also a special case of RSA
- The GCHQ cryptographers used the name *non-secret encryption*

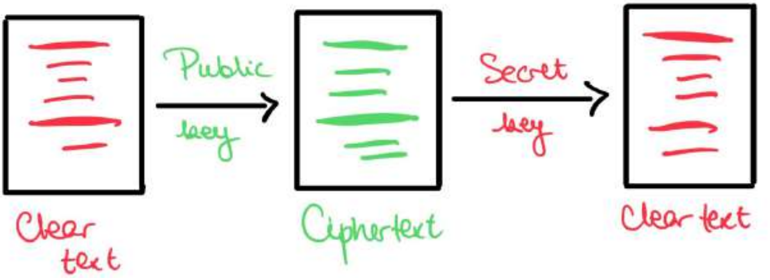
Public and private keys

- Public key cryptography is another name for *asymmetric cryptography*
- The encryption and decryption keys are different
- The encryption key is a *public key* which can be known to anybody
- The decryption key is a *private key* which should be known only to the owner of the key
- Finding the private key from knowledge of the public key must be a hard computational problem

Symmetric-key Cryptography – reminder



Public-key Cryptography – reminder



Why public key cryptography?

- Public key cryptography has two main advantages in comparison with shared key (symmetric key) cryptography

1. The key management is simplified: keys do not need to be transported confidentially
2. Digital signatures can be obtained. We look at digital signatures in a later lecture

Using public key encryption

- In a public key encryption scheme the receiver's key is made public
- Suppose that user A stores her public key, PK_A, in a public directory
- Anyone can obtain this public key and use it to encrypt a message M for A: $C = E(M, PK_A)$
- Since only A has the private key, SK_A, only A can decrypt and recover the message: $M = D(C, SK_A)$

Hybrid encryption

- Public key cryptography is usually computationally much more expensive than symmetric-key cryptography
- A typical usage of public key cryptography is to:
  - encrypt a random key for a symmetric-key encryption algorithm
  - encrypt the message M using the symmetric-key algorithm

  1. B chooses a random symmetric key k, finds A's public key PK_A and computes $C_1 = E(k, PK_A)$
  2. B computes $C_2 = E_s(M, k)$ where E_s is encryption with a symmetric-key algorithm, such as AES in CTR mode
  3. B sends (C_1, C_2) to A
- On receipt of (C_1, C_2), A recovers $k = D(C_1, SK_A)$ and then $M = D_s(C_2, k)$

Introduction to RSA

- Rivest-Shamir-Adleman, MIT, 1977
- Public-key cryptosystem and digital signature scheme
- Based on integer factorisation problem
- RSA patent expired in 2000

RSA Key Generation

1. Let p, q be distinct prime numbers, randomly chosen from the set of all prime numbers of a certain size
2. Compute $n = pq$
3. Select e randomly with $\gcd(e, \phi(n)) = 1$
4. Compute $d = e^{-1} \bmod \phi(n)$
5. The public key is the pair n and e
6. The private key consists of the values p, q and d

RSA operations

Encryption The public key for encryption is $K_E = (n, e)$

1. Input is any value M where $0 < M < n$
2. Compute $C = E(M, K_E) = M^e \bmod n$

Decryption The private key for decryption is $K_D = d$ (we will see later how to use values p and q)

1. Compute $D(C, K_D) = C^d \bmod n = M$

Note that any message needs to be pre-processed to become the input M: this includes coding as a number and adding randomness (details later)

Numerical example

- ***Key Generation***:
  - Suppose $p = 43, q = 59$ then $n = pq = 2537$ and $\phi(n) = (p-1)(q-1) = 2436$
  - Choose $e = 5$ then

$$d = e^{-1} \bmod \phi(n) = 5^{-1} \bmod 2436 = 1949$$

- ***Encryption***:
  Let $M = 50 \implies C = M^5 \bmod 2537 = 2488$
- ***Decryption***:
  $M = C^{1949} \bmod 2537 = 50$

Correctness of RSA Encryption (blackbaord)

We need to know that encryption followed by decryption gets back where we started from:

$$(M^e)^d \bmod n = M$$

Since $d = e^{-1} \bmod \phi(n)$ we know that $ed \bmod \phi(n) = 1$ and so $ed = 1 + k\phi(n)$ for some integer k. Therefore:

$$\begin{aligned}(M^e)^d \bmod n &= M^{ed} \bmod n \\ &= M^{1+k\phi(n)} \bmod n\end{aligned}$$

To complete the proof we need to show

$$M^{1+k\phi(n)} \bmod n = M \tag{1}$$

Proving equation 1: Case 1

There are two cases. We first assume $\gcd(M, n) = 1$
We can apply Euler's theorem directly to get

$$M^{\phi(n)} \bmod n = 1$$

Therefore

$$\begin{aligned} M^{1+k\phi(n)} \bmod n &= M \times (M^{\phi(n)})^k \bmod n \\ &= M \times (1)^k \bmod n \\ &= M \end{aligned}$$

Proving equation 1: Case 2

- If $\gcd(M, n) \neq 1$ then it must be the case that either $\gcd(M, p) = 1$ or $\gcd(M, q) = 1$
- Suppose that $\gcd(M, p) = 1$ (the other case is similar) Then $\gcd(M, q) = q$ so $M = lq$ for some integer l
- Applying Fermat's theorem we obtain $(M^{\phi(n)})^k \bmod p = (M^{p-1})^{(q-1)k} \bmod p = 1$ Therefore
$$M^{1+k\phi(n)} \bmod p = M \bmod p \tag{2}$$
- Since $M = lq$ it follows that
$$M^{1+k\phi(n)} \bmod q = 0 \tag{3}$$

Case 2 continued

- Finally the Chinese Remainder Theorem tells us that there is a unique solution $x \bmod n$ to the two equations (2) and (3) where $x = M^{1+k\phi(n)}$
- The solution $x = M$ satisfies both equations (2) and (3) and therefore this is the unique solution for $M^{1+k\phi(n)} \bmod n$
- Thus equation (1) is satisfied in this case too

RSA applications

The RSA operations can be used in a variety of applications.

- In this lecture we consider only message encryption.
- We look at RSA digital signatures in a later lecture.
- RSA is often used to distribute a key for symmetric-key encryption (often known as *hybrid encryption*).
- RSA can be used for user authentication by proving knowledge of the private key corresponding to an authenticated public key.

Implementation issues

Optimisations in the implementation of RSA have been widely studied. We examine some of the most important issues:

- key generation
  - choice of e
  - generating large primes
- encryption and decryption algorithms
  - fast exponentiation
  - using CRT for decryption
- formatting data (padding)

Generating p and q

- The primes p and q should be random of a chosen length. Today this length is usually recommended to be at least 1024 bits.
- A simple method of selecting a random prime is given by the following algorithm:
  1. Select a random odd number r of the required length.
  2. Check whether r is prime
  3. - If so, output r and halt
     - If not, increment r by 2 and go to the previous step.
  4. We require a fast way to check for primality such as the Miller–Rabin test.

Are there enough prime numbers?

- The *prime number theorem* tells us that the primes thin out as the numbers get larger.
- Let $\pi(x)$ denote the number of prime numbers less than x. The prime number theorem say that the ratio of $\pi(x)$ and $\frac{x}{\ln(x)}$ tends to 1 as x gets large.
- We can use the prime number theorem to give a rule of thumb that the proportion of prime numbers up to size x is $\ln(x)$.
- Since $\ln(2^{1024}) = 710$ we can estimate that one in every 710 numbers of size 1024 bits is a prime number. Therefore there are well over 2^{1000} 1024-bit primes.
- Thus brute-force searching for randomly chosen primes is completely infeasible.

Selecting e

- The public exponent e should be chosen at random for best security
- A small value of e is often used in practice since it can have a large effect on efficiency.
  - $e = 3$ is the smallest possible value and is sometimes used. However, there are possibly security problems when encrypting small messages.
  - $e = 2^{16} + 1$ is a popular choice. More exponentiations, but reduces the constraints on p and q, and avoids aforementioned attacks.
- A smaller than average d value is also possible. However, to avoid known attacks d should be at least $\sqrt{n}$
  - A low value of d implies a total break, since one can just brute-force all possible values.

Fast exponentiation

- To compute the RSA encryption and decryption functions we use the *square-and-multiply* modular exponentiation algorithm.
- We write e in binary representation.

$$e = e_0 2^0 + e_1 2^1 + \cdots + e_k 2^k$$

  where e_i are bits.
- The basic idea behind fast exponentiation is the *square and multiply* algorithm.
- There are many variants and optimisations of the basic idea.

Square and multiply algorithm

$$m^e = m^{e_0}(m^2)^{e_1}(m^4)^{e_2}\dots(m^{2^k})^{e_k}$$

Data: $m, n, e = e_k e_{k-1} \dots e_1 e_0$
Result: $m^e \bmod n$

```
z ← 1;
for i = 0 to k do
    if e_i = 1 then
        z ← z * m mod n;
    end
    if i < k then
        m ← m^2 mod n;
    end
end
return z
```

Algorithm 1: Square and multiply algorithm

Cost of square and multiply

- If $2^k \leq e < 2^{k+1}$ then the algorithm uses k squarings. If b of the e_i bits are 1 then the algorithm uses $b - 1$ multiplications. Note that the first computation $z \to z * m$ is not counted because then $z = 1$.
- Suppose that n is a 2048-bit RSA modulus. The public exponent e is length at most 2048 bits. To compute $M^e \bmod n$ requires at most:
  - 2048 modular squarings; and
  - 2048 modular multiplications.
- On average only half of the bits of e are '1' bits and so only 1024 multiplications are needed.
- Remember that we can reduce modulo n after every operation.

Faster decryption with the CRT

- We can use the Chinese Remainder Theorem to decrypt ciphertext C faster with regard to p and q separately.
- First compute:

$$\begin{aligned} M_p &= C^{d \bmod p-1} \bmod p \\ M_q &= C^{d \bmod q-1} \bmod q \end{aligned}$$

- Solve for $M \bmod n$ using the Chinese remainder theorem.

$$\begin{aligned} M &\equiv M_p \pmod{p} \\ M &\equiv M_q \pmod{q} \end{aligned}$$

$$M = q \times (q^{-1} \bmod p) \times M_p + p \times (p^{-1} \bmod q) \times M_q \bmod n$$

Why it works (blackbaord)

Note that $d = d \bmod (p-1) + k(p-1)$ for some k.

$$\begin{aligned} M \bmod p &= (C^d \bmod n) \bmod p \\ &= C^d \bmod p \\ &= C^{d \bmod p-1} C^{k(p-1)} \bmod p \\ &= C^{d \bmod p-1} \\ &= M_p \end{aligned}$$

- Similarly $M \bmod q = M_q$
- Therefore $M \bmod n$ is the unique solution to the above two equations.

Example

- Same example as before: $n = 43 \times 59$. Ciphertext is $C = 2488$. Decryption exponent is $d = 1949$.
- $d \bmod p - 1 = 1949 \bmod 42 = 17$
  $d \bmod q - 1 = 1949 \bmod 58 = 35$
- $$\begin{aligned} M_p &\equiv 2488^{17} \pmod{43} = 37^{17} \pmod{43} = 7 \\ M_q &\equiv 2488^{35} \pmod{59} = 16^{35} \pmod{59} = 50 \end{aligned}$$
- Using CRT solution is $M = 50$.

How much faster is decryption with the CRT?

- Note that the exponents $(d \bmod p-1)$ and $(d \bmod q-1)$ are about half the length of d.
- Since the complexity of exponentiation (square and multiply) increases with the cube of the input length, computing M_p and M_q each use 1/8 the computation of computing $M = C^d \bmod n$.
- Overall there is about 4 times less computation. If M_p and M_q can be computed in parallel the time can be up to 8 times faster.
- This is a good reason to store p and q with the private exponent d.

RSA Padding

- Using the RSA encryption function directly on messages encoded as numbers is a weak cryptosystem. It is vulnerable to attacks such as:
  - building up a dictionary of known plaintexts
  - guessing the plaintext and checking to see if it encrypts to the ciphertext
  - Håstad's attack (next slide)
- Therefore padding mechanisms must be used to prepare messages for encryption. These mechanisms must include redundancy and randomness.

Håstad's Attack

- Suppose that the *same message* is encrypted without padding to three different recipients.
- Suppose that public exponent $e = 3$ is used by all recipients
- Then the cryptanalyst has three ciphertexts:

$$\begin{aligned} c_1 &= m^3 \bmod n_1 \\ c_2 &= m^3 \bmod n_2 \\ c_3 &= m^3 \bmod n_3 \end{aligned}$$

- These equations can be solved by the Chinese Remainder Theorem to obtain m^3 in the ordinary (non-modular) integers. Then m can be found by taking a cube root.

Types of padding

- PKCS #1: simple, ad-hoc design for encryption and signatures
- Optimal Asymmetric Encryption Padding (OAEP) designed by Bellare and Rogaway in 1994.
  - Has a security proof in a suitable model
  - Standardised in IEEE P1363: Standard Specifications for Public Key Cryptography

Example RSA block format: PKCS Number 1

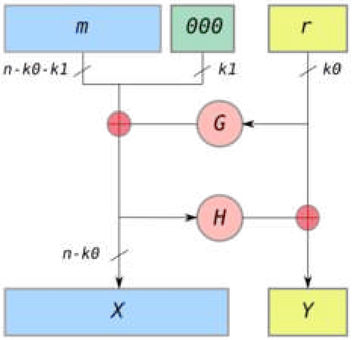
Encryption block format is:

00	02	*PS*	00	*D*

where 00 and 02 are bytes, *PS* is a pseudo-random string of nonzero bytes, and *D* is the data to be encrypted.

- The length of the block is the same as the length of the modulus.
- PS is a minimum of 8 bytes,
- The byte 02 and padding ensure that even short messages result in a large integer value for encryption.

Optimal Asymmetric Encryption Padding (OAEP)



Picture from Wikipedia

- The OAEP scheme includes k_0 bits of randomness and k_1 bits of redundancy into the message before encryption.
- Reasonable values of k_0 and k_1 are 128.
- Two random hash functions G and H are used
- Note that OAEP is an encoding algorithm - it can be easily inverted without any secret.

Outline

- Public Key Cryptography
- RSA algorithms
- Implementing RSA
- Security of RSA
  - Factorisation
  - Side channel attacks

Factorising the RSA modulus

- If an adversary can factorise the modulus n into its prime factors p and q then the adversary can easily recover the private key d and reveal all messages. Thus breaking RSA is not harder than the factorisation problem.
- Using a formal definition of encryption security it can be shown that breaking RSA is as hard as the so-called *RSA problem*.
- It is unknown whether the RSA problem is as hard as the factorisation problem. Remember that it is also unknown whether factorisation is really computationally hard.
- One positive security result is that finding the private key from the public key is as hard as factorising the modulus.

Equivalence with factorisation problem

- Is it possible to find the private key without factorising the modulus? No!

Theorem (Miller)

Determining d from e and n is as hard as factorising n.

- To show this, suppose that a cryptanalyst can find d from e and n.
- Then cryptanalyst could factorise n using Miller's algorithm (next slide).
- Algorithm uses same ideas as Miller–Rabin test for primality.

Miller's Algorithm

- Define u, v such that $ed - 1 = 2^v u$, where u is odd
- Consider the sequence $a^u, a^{2u}, \ldots, a^{2^{v-1}u}, a^{2^v u} \pmod{n}$, where a is random with $0 < a < n$.
- Notice that $a^{2^v u} \equiv a^{ed-1} \equiv a^{ed} a^{-1} \equiv aa^{-1} \equiv 1 \pmod{n}$. Therefore there is a square root of 1 somewhere in this sequence.
- With probability at least $\frac{1}{2}$ the sequence contains a non-trivial square root of 1 modulo n, thereby revealing the factors of n.
- If not, choose a new a and repeat.

Quantum computers

- Quantum computers do not exist yet (commercially at least).
- Shor's algorithm can factorise in polynomial time on a quantum computer.
- NIST is currently running an open competition to standardise signature schemes against quantum computers, and the standardisation process for signature schemes was (partly) finalised in 2023.

https://csrc.nist.gov/projects/pqc-dig-sig/round-1-additional-signatures
https://csrc.nist.gov/projects/post-quantum-cryptography/post-quantum-cryptography-standardization

Outline

Public Key Cryptography

RSA algorithms

Implementing RSA

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- Factorisation
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Side Channel Attacks

- First made public in 1996 by Paul Kocher
- Many different kinds of side-channels are now known including:

Timing attacks
: Uses timing of the private key operations to obtain information about the private key.

Power analysis
: Uses power usage profile of the private key operations to obtain information about the private key.

Fault analysis
: Measures the effect of interfering with the private key operations to obtain information about the private key.

Timing attacks

- Recall that the square-and-multiply algorithm performs either a squaring or a squaring and a multiplication in each step
- The multiplication step is included exactly when each exponent bit $e_i = 1$
- Thus step i takes around twice as long when $e_i = 1$ as when $e_i = 0$

Demonstrated practically.

Some side channel countermeasures

- computing in constant time - run a “dummy” multiplication when $e_i = 0$
- Montgomery ladder - makes every operation depend on the key to avoid some fault attacks
- randomising the RSA message - mitigates “differential” attacks by preventing multiple timings on the same operation

Practical problems with RSA key generation

- In 2008 it was discovered that the implementation of OpenSSL used in Debian-based linux system used massively reduced randomness for RSA key generation.
- In 2012 a group of researchers led by Arjen Lenstra published a study of over 6 million RSA keys deployed on the Internet (many have expired).
  - 270 000 keys (about 4%) were identical, causing potential problems for those that share keys.
  - 12934 (about 0.2% of keys examined) provide no security because they share one prime factor with each other.
  - These problems are almost certainly due to poor random number generation.

Summary of RSA encryption security

- Standardised padding should always be used before encryption.
- Factorisation of the modulus is the best known attack against RSA in the case that standardised padding is used.
- Finding the private key from the public key is as hard as factorising the modulus.
- It is an open problem whether there is any way of breaking RSA encryption without factorising the modulus.
- Side channels