NTNU

Game theory

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Game theory definitions

Game theory

Systematic study of strategic interactions among rational individuals

Rational individual

Has well-defined objectives and implements the best available strategy to pursue them

Applications of game theory

- Economics
- Business
- Project management
- Political science
- Defense science and technology
- Biology
- Computer science and logic
- Epidemiology
- Philosophy
- Epidemiology
- Artificial intelligence and machine learning

Wikipedia

Applications of game theory

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- Biology
- Computer science and logic
- Epidemiology
- Philosophy
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- Artificial intelligence and machine learning

Wikipedia

- Although in practice:
  - Players are not always rational
  - Might be difficult to define the utility function (the value at the end a game)

Nash equilibrium for two-player simultaneous action games

Definitions

- $\pi_p(s, a)$: strategy for player $p \in \{1, 2\}$, i.e. probability of taking action a at state s
  - In this lecture:
    - In all but the last section, the state is empty and is therefore omitted
    - Each player has only two actions, thus the probability of taking the first action is defined to be π_p and the probability of taking the second action is then $1 - \pi_p$
- $E_p(\pi_1, \pi_2)$: expected value for player p of the given players' strategies

Nash equilibrium for two-player simultaneous action games

Definitions

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The specific strategies (π_1^*, π_2^*) is a Nash equilibrium if:

$$E_1(\pi_1^*, \pi_2^*) \geq E_1(\pi_1, \pi_2^*) \text{ for all possible strategies } \pi_1$$

$$E_2(\pi_1^*, \pi_2^*) \geq E_2(\pi_1^*, \pi_2) \text{ for all possible strategies } \pi_2$$

and no player will then have an incentive to change his/her strategy

Pure vs mixed strategy Nash equilibria

- Pure strategy: selects a predetermined action, i.e. $\pi_p \in \{0, 1\}$
- Mixed strategy: selects actions according to a probability distribution, i.e. $\pi_p \in [0, 1]$
  - A pure strategy can be seen as a special case of a mixed strategy

Outline

Mixed strategies

Pure strategies

Repeated games

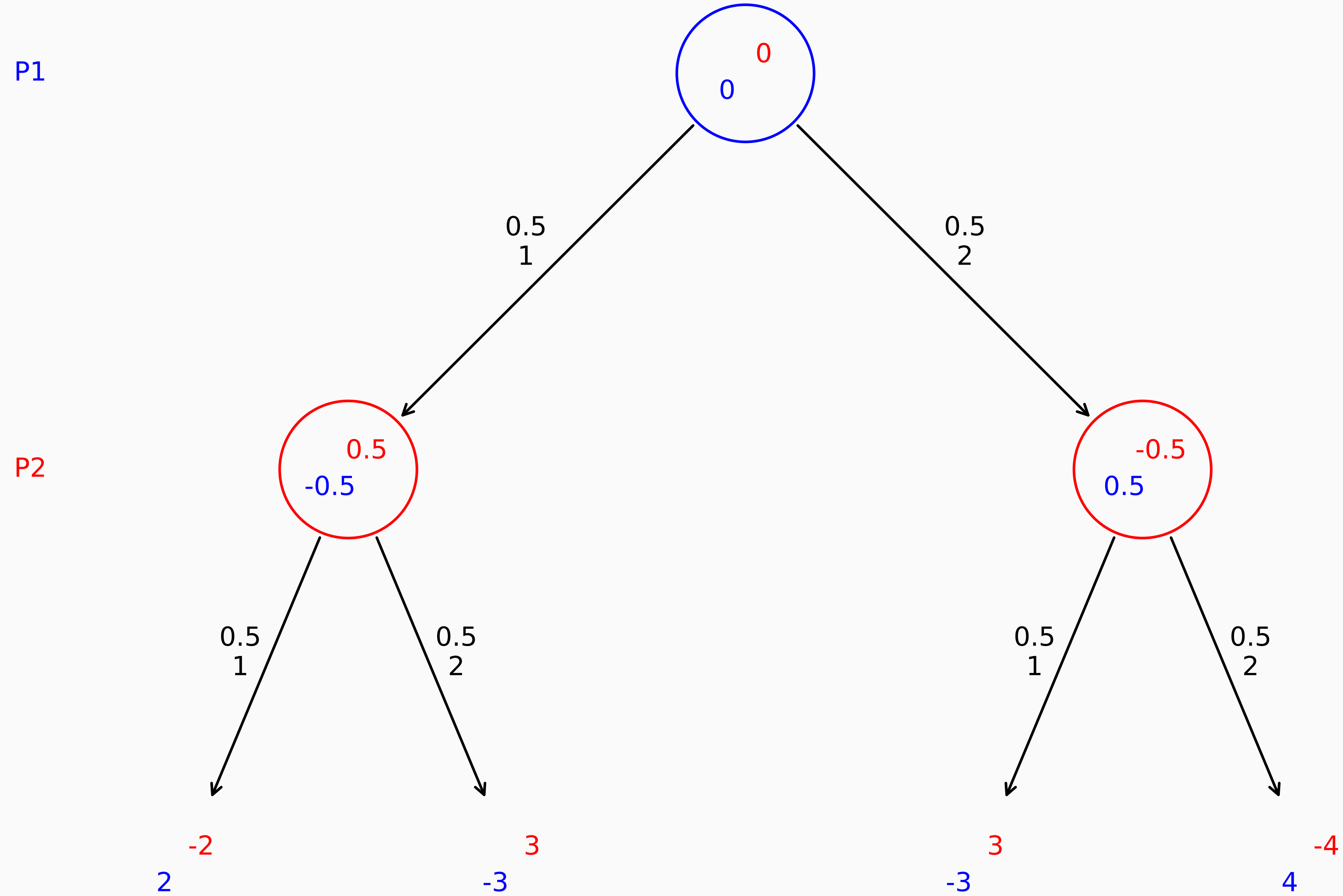
Infinite rounds

Unknown number of rounds

Two-finger Morra, payoff (utility) matrix

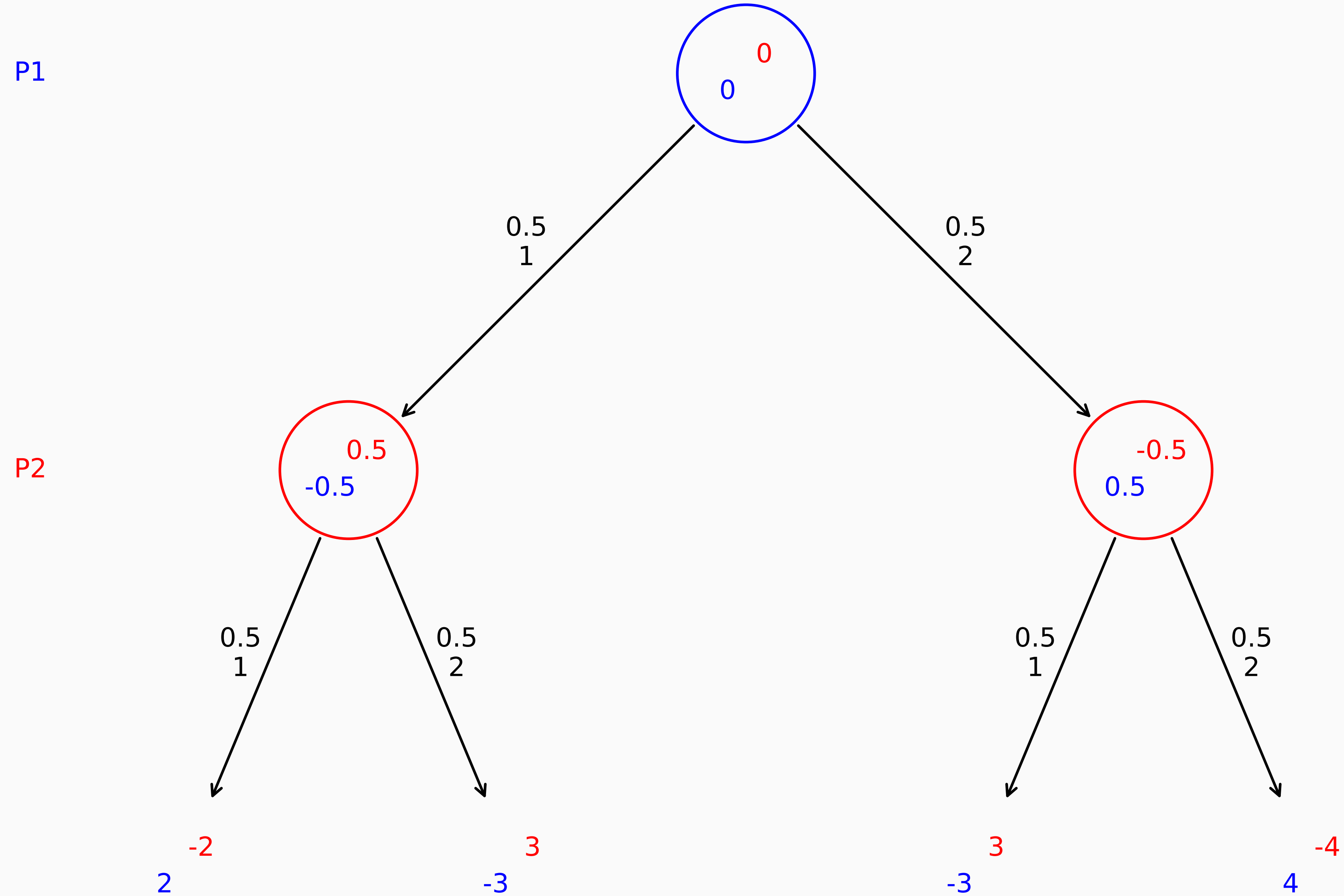
		P2	
		1	2
P1	1	2, −2	−3, 3
	2	−3, 3	4, −4

Two-finger Morra



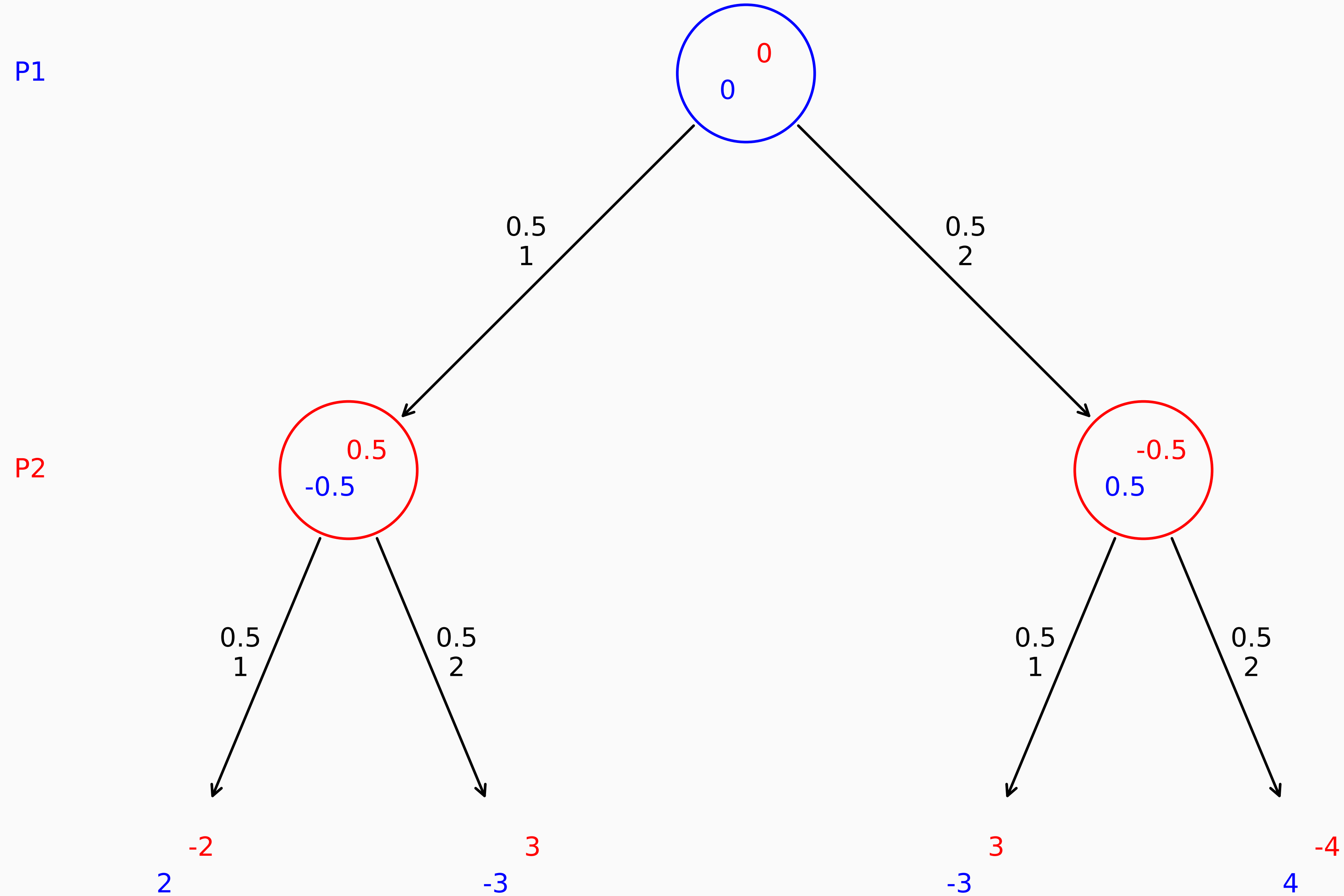
How can P1 exploit P2's strategy?

Two-finger Morra



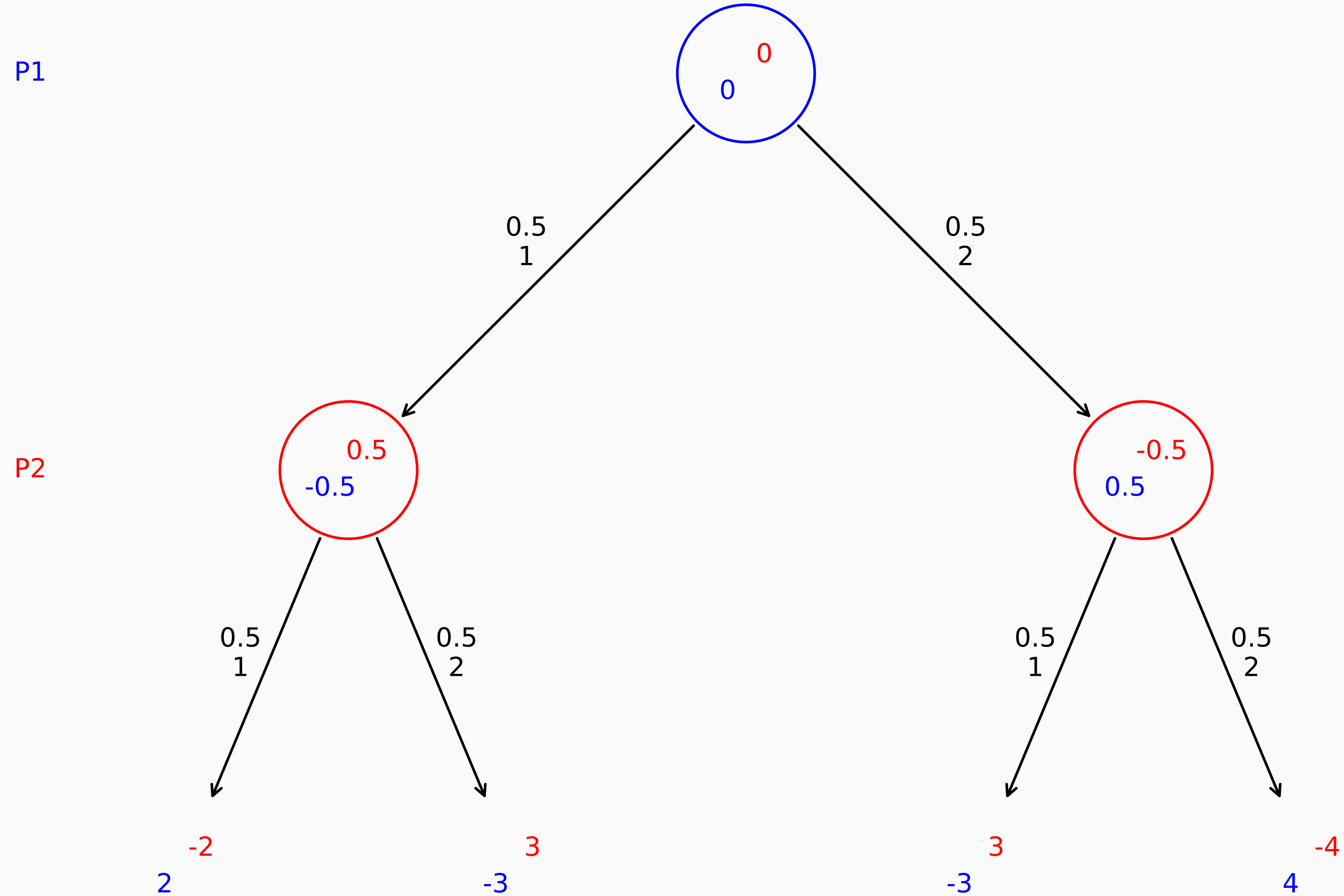
How can P1 exploit P2's strategy? Choose 2 more often than 1

Two-finger Morra



How can P2 play such that P1 cannot exploit P2?

Two-finger Morra, P2's mixed strategy

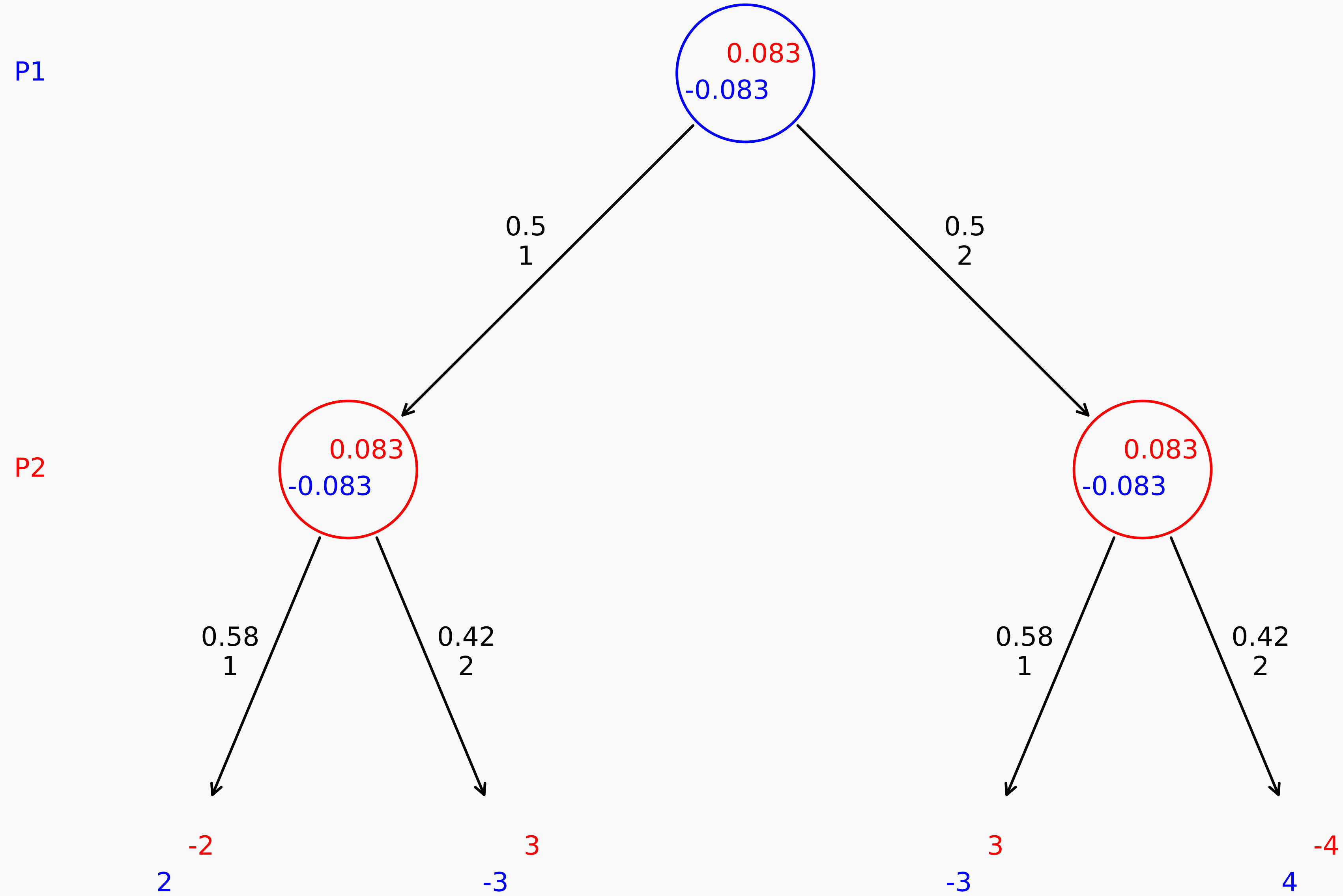


Mixed strategy for P2 with probability π_2 of choosing 1, where the expected values of P1 in the red nodes are set equal:

$$2\pi_2 - 3(1-\pi_2) = -3\pi_2 + 4(1-\pi_2)$$

$$\pi_2 = \frac{7}{12}$$

Two-finger Morra, P2's mixed strategy



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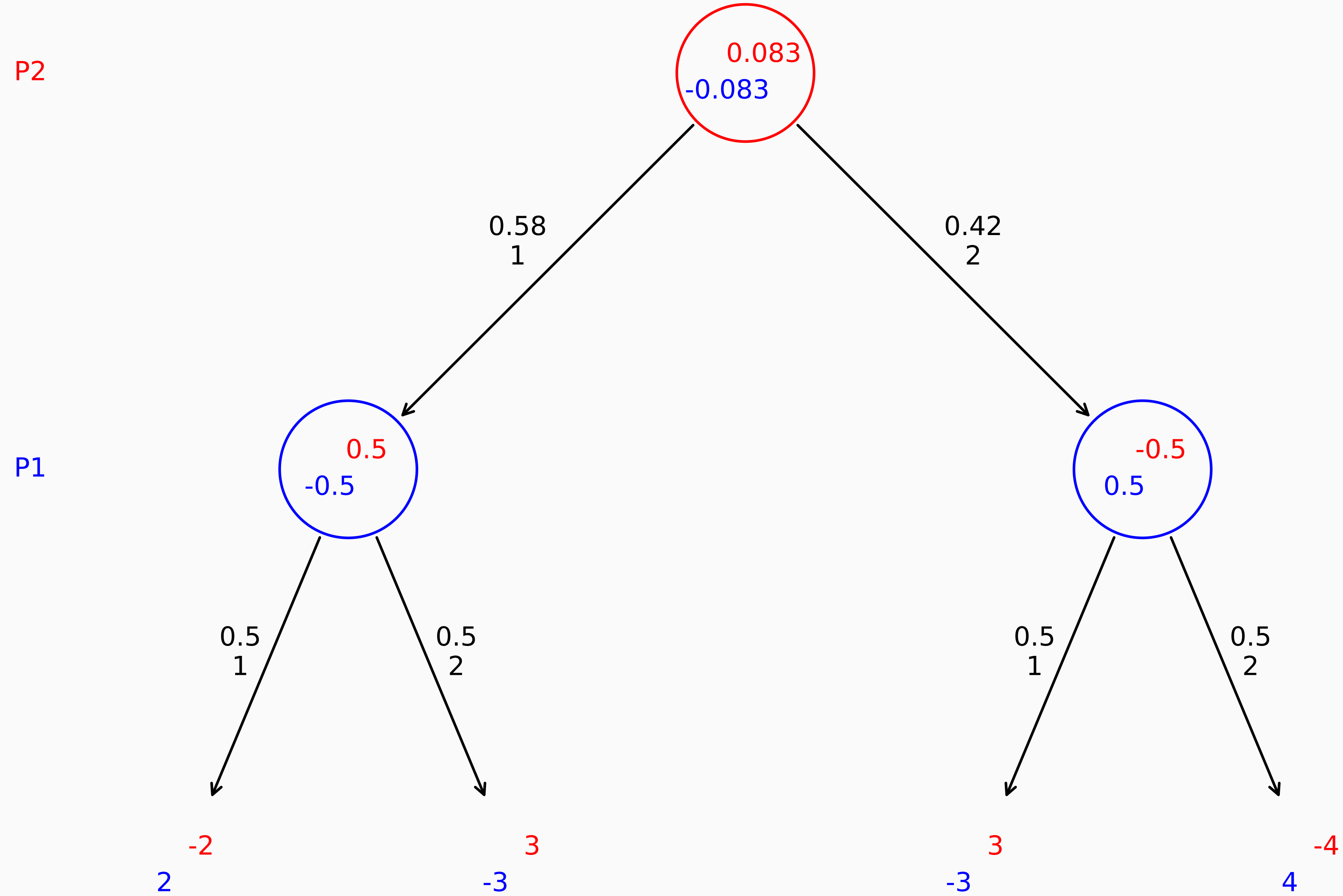
Two-finger Morra, P1's mixed strategy

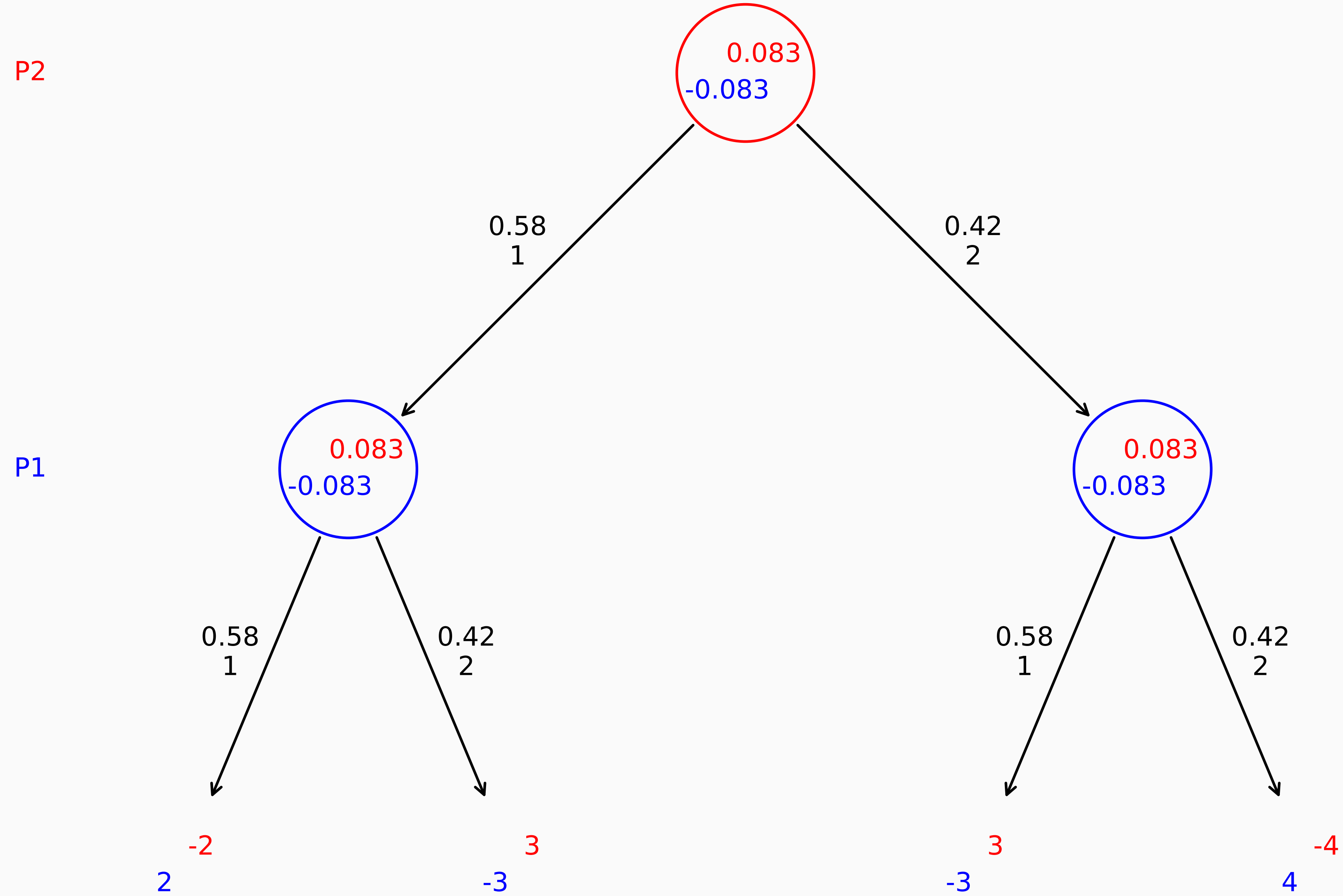


Mixed strategy for P1 with probability π_1 of choosing 1, where the expected values of P2 in the blue nodes are set equal:

$$-2\pi_1 + 3(1-\pi_1) = 3\pi_1 - 4(1-\pi_1)$$

$$\pi_1 = \frac{7}{12}$$

Two-finger Morra, P1's mixed strategy



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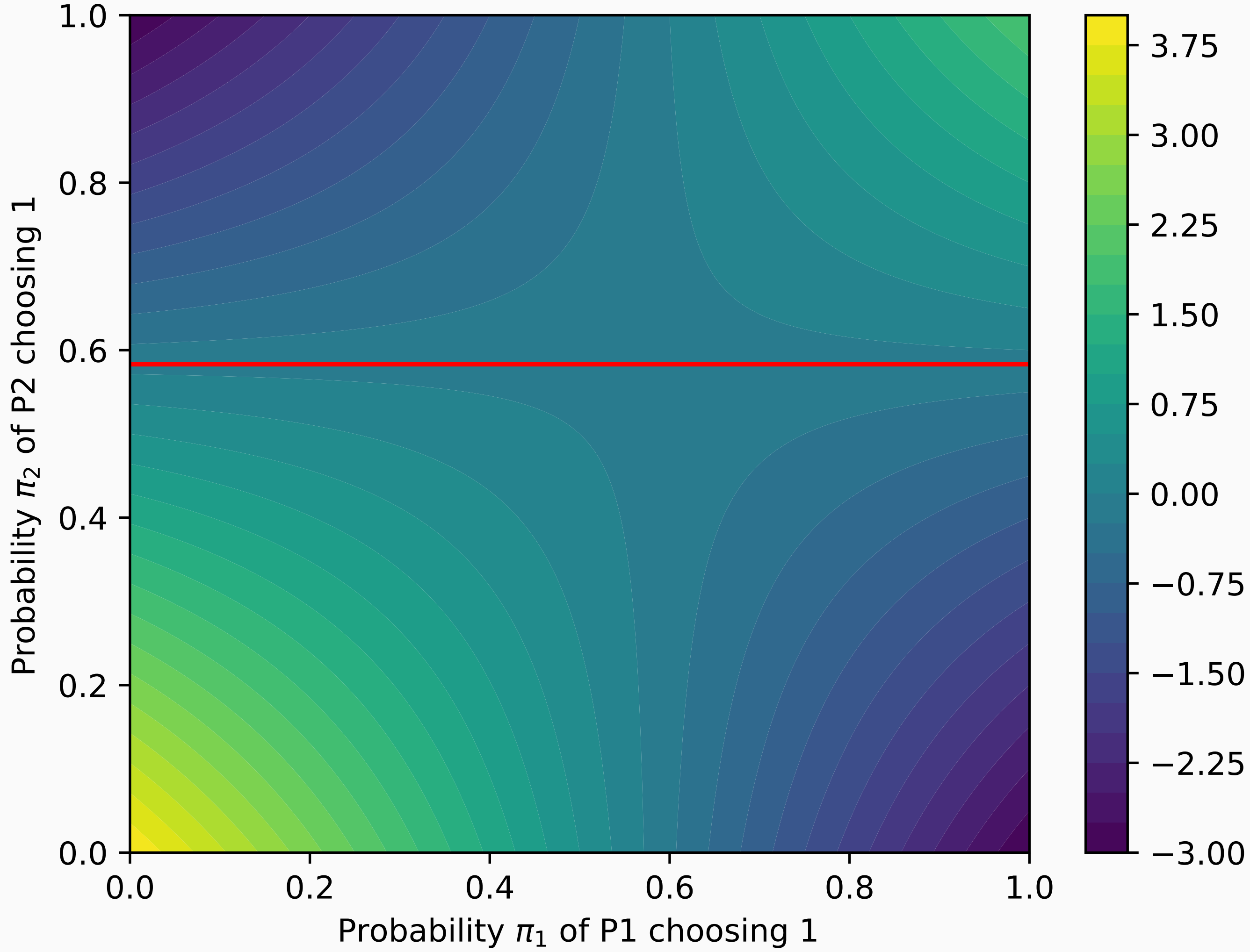
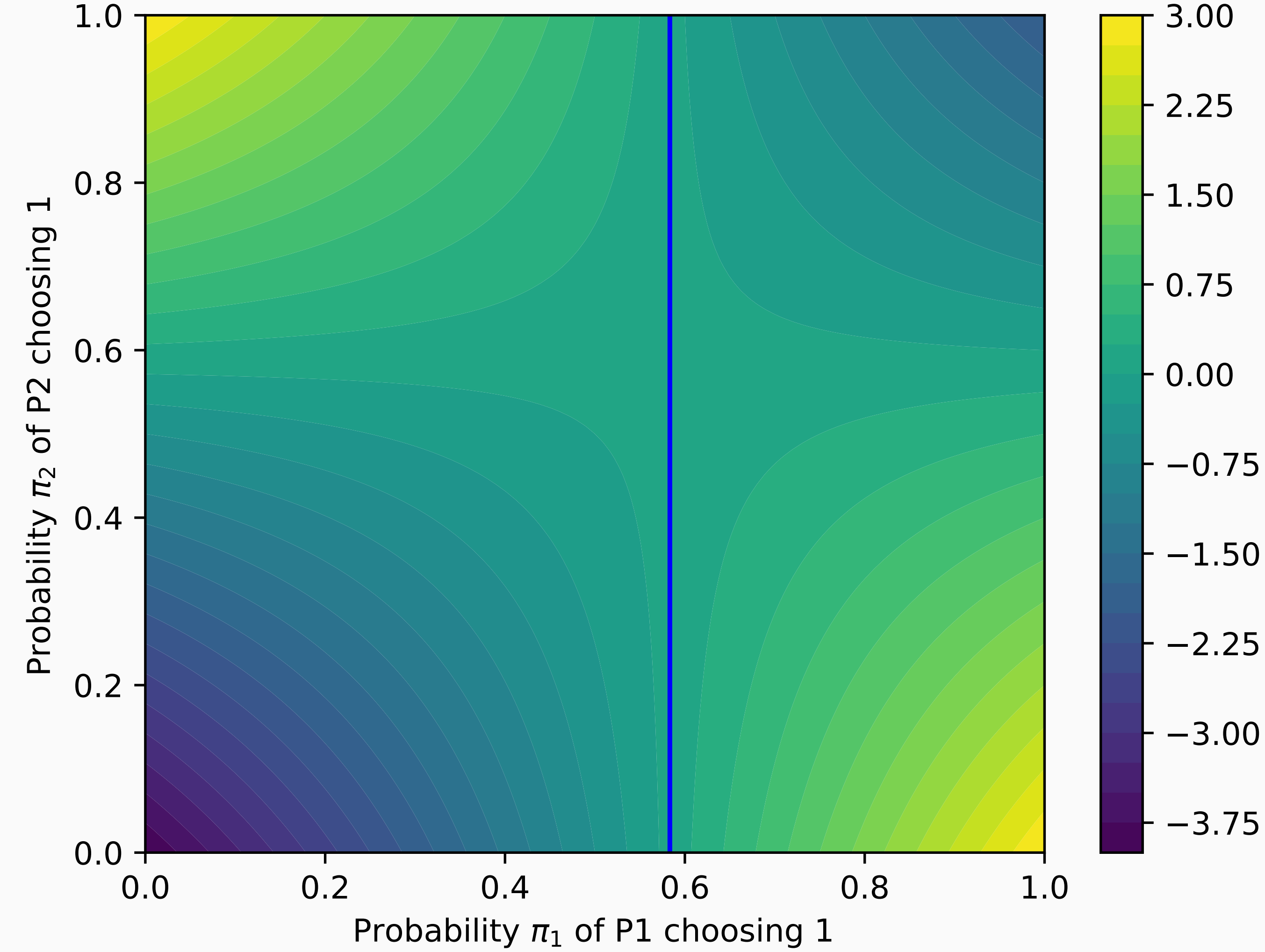
Two-finger Morra, expected values of every π_1 and π_2



Expected values for P1, red line shows mixed strategy for P2



Expected values for P2, blue line shows mixed strategy for P1

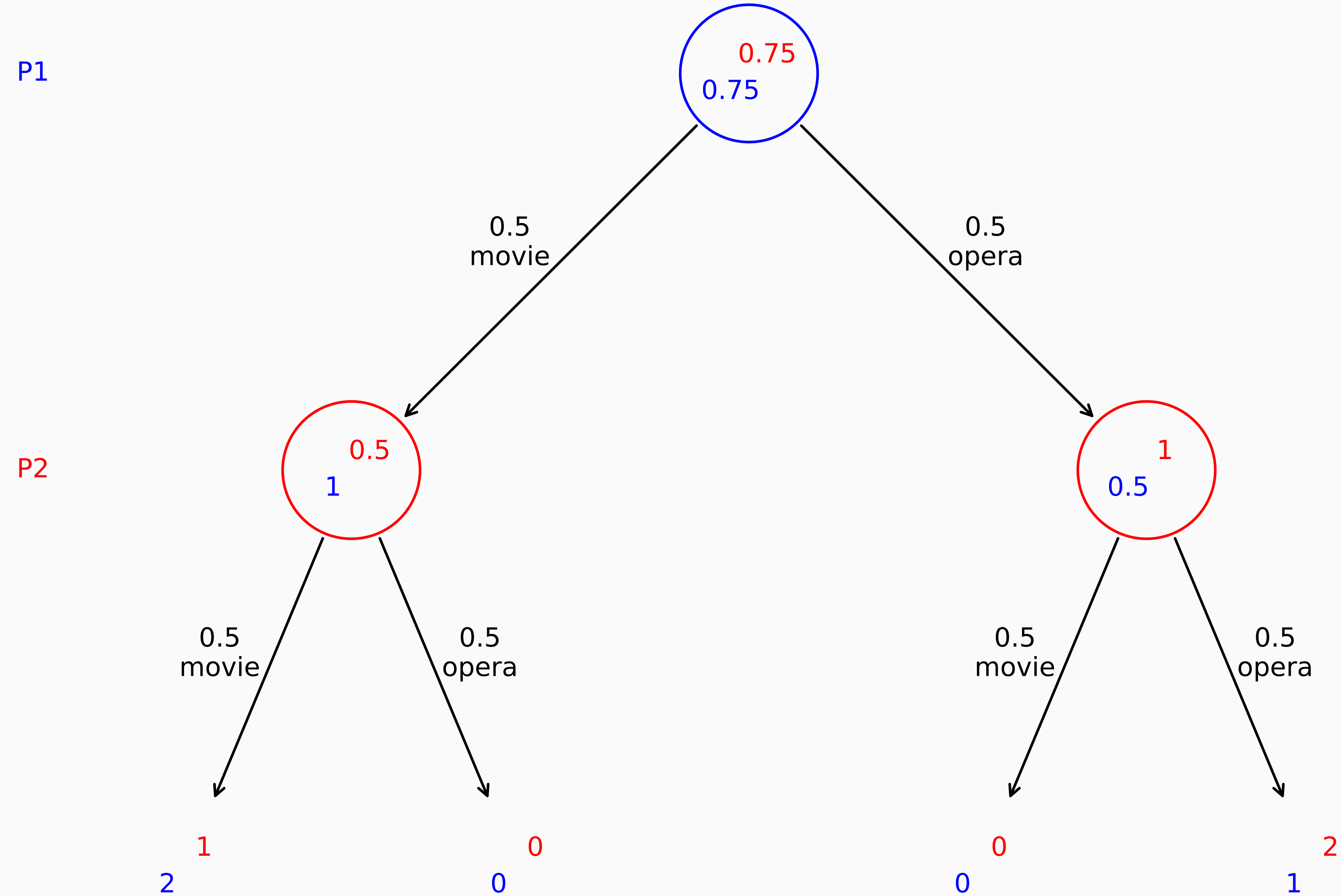
Battle of the Sexes

		P2	
		movie	opera
P1	movie	$\underline{2}$, $\underline{1}$	0, 0
	opera	0, 0	$\underline{1}$, $\underline{2}$

Not a zero-sum game

Two pure strategy Nash equilibria: P1 and P2 choose movie, and P1 and P2 choose opera

Battle of the Sexes, P2's mixed strategy

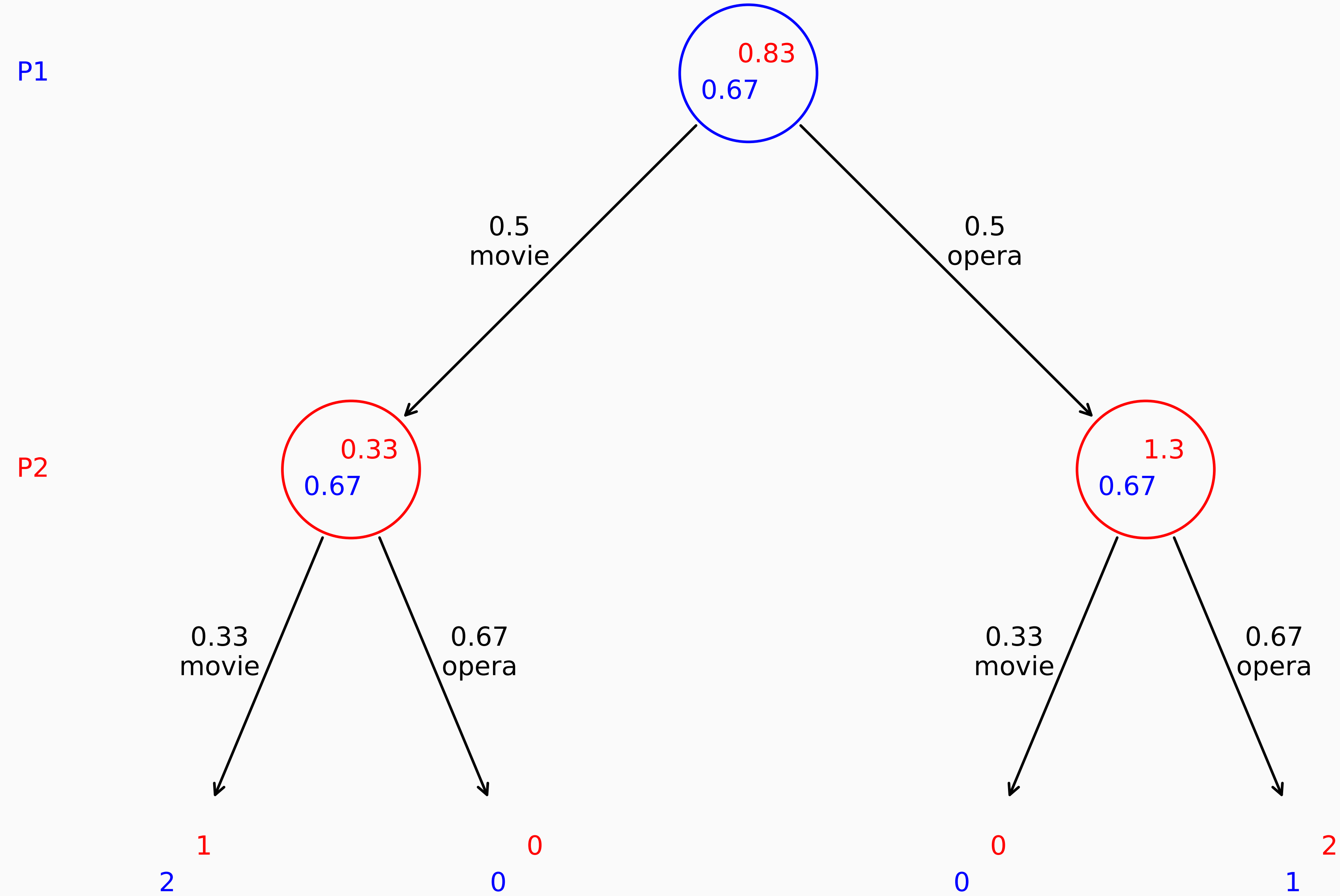


Mixed strategy for P2 with probability π_2 of choosing movie, where the expected values of P1 in the red nodes are set equal:

$$2\pi_2 + 0(1 - \pi_2) = 0\pi_2 + 1(1 - \pi_2)$$

$$\pi_2 = \frac{1}{3}$$

Battle of the Sexes, P2's mixed strategy



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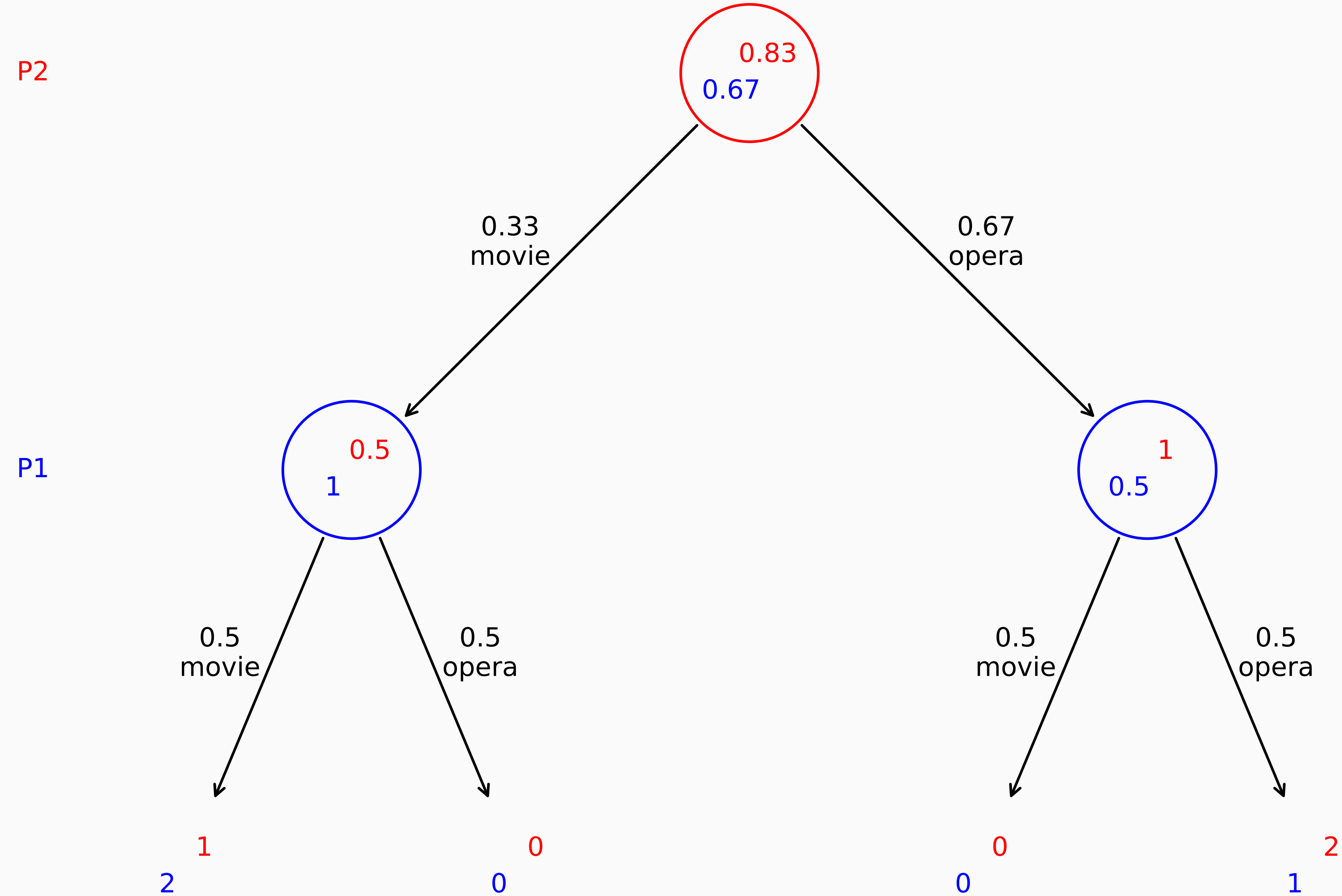
Battle of the Sexes, P1's mixed strategy

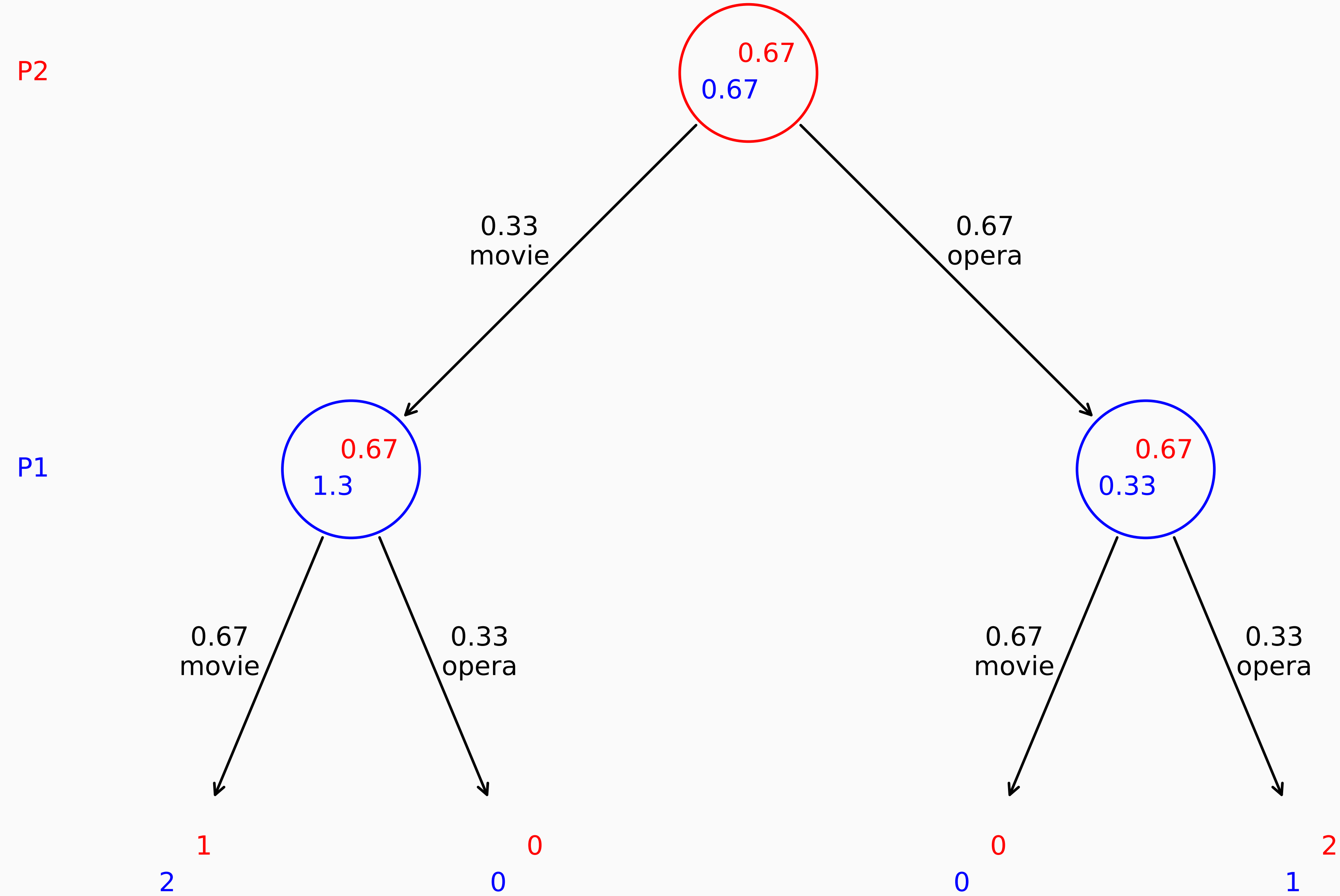


Mixed strategy for P1 with probability π_1 of choosing movie, where the expected values of P2 in the blue nodes are set equal:

$$1\pi_1 + 0(1-\pi_1) = 0\pi_1 + 2(1-\pi_1)$$

$$\pi_1 = \frac{2}{3}$$

Battle of the Sexes, P1’s mixed strategy



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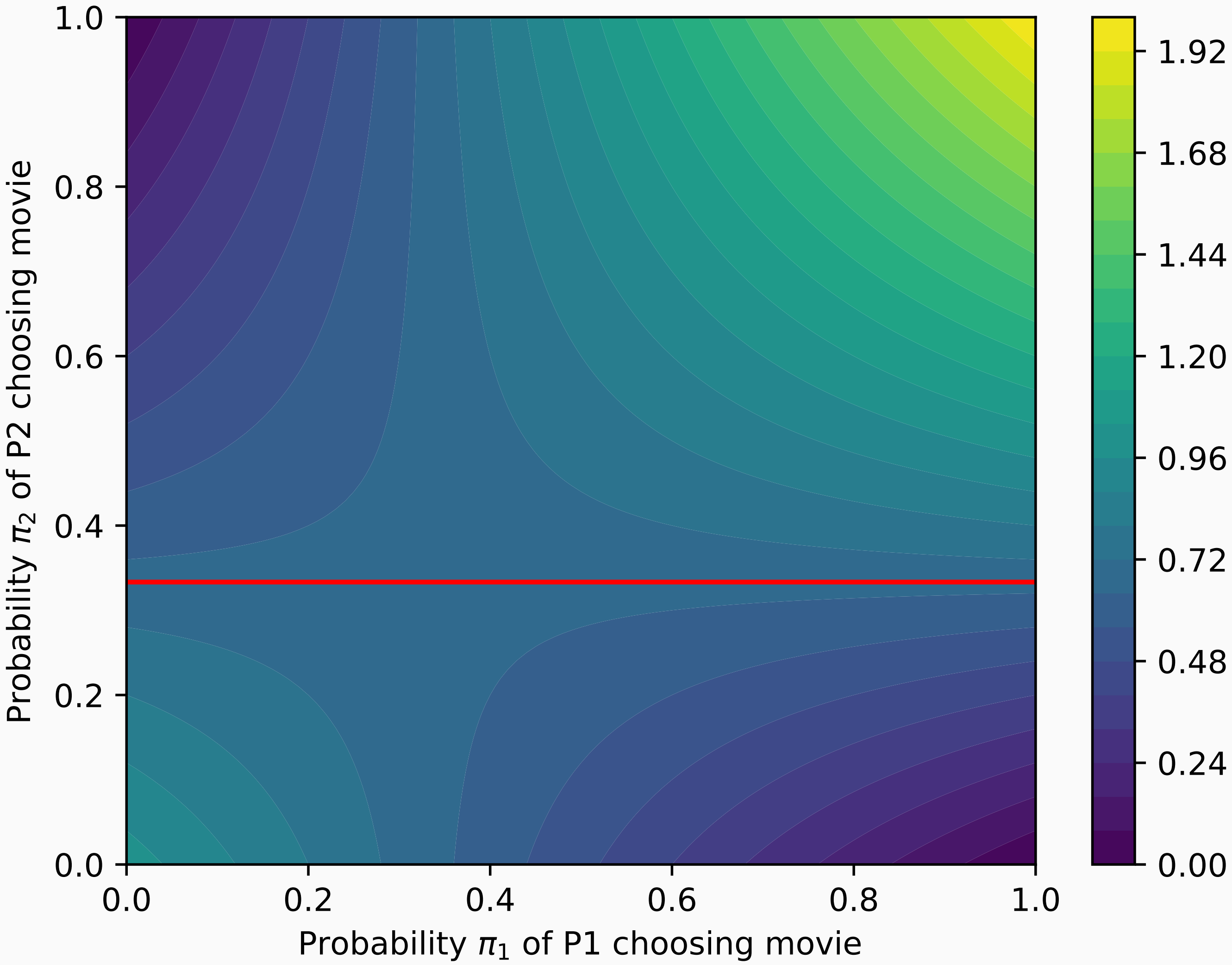
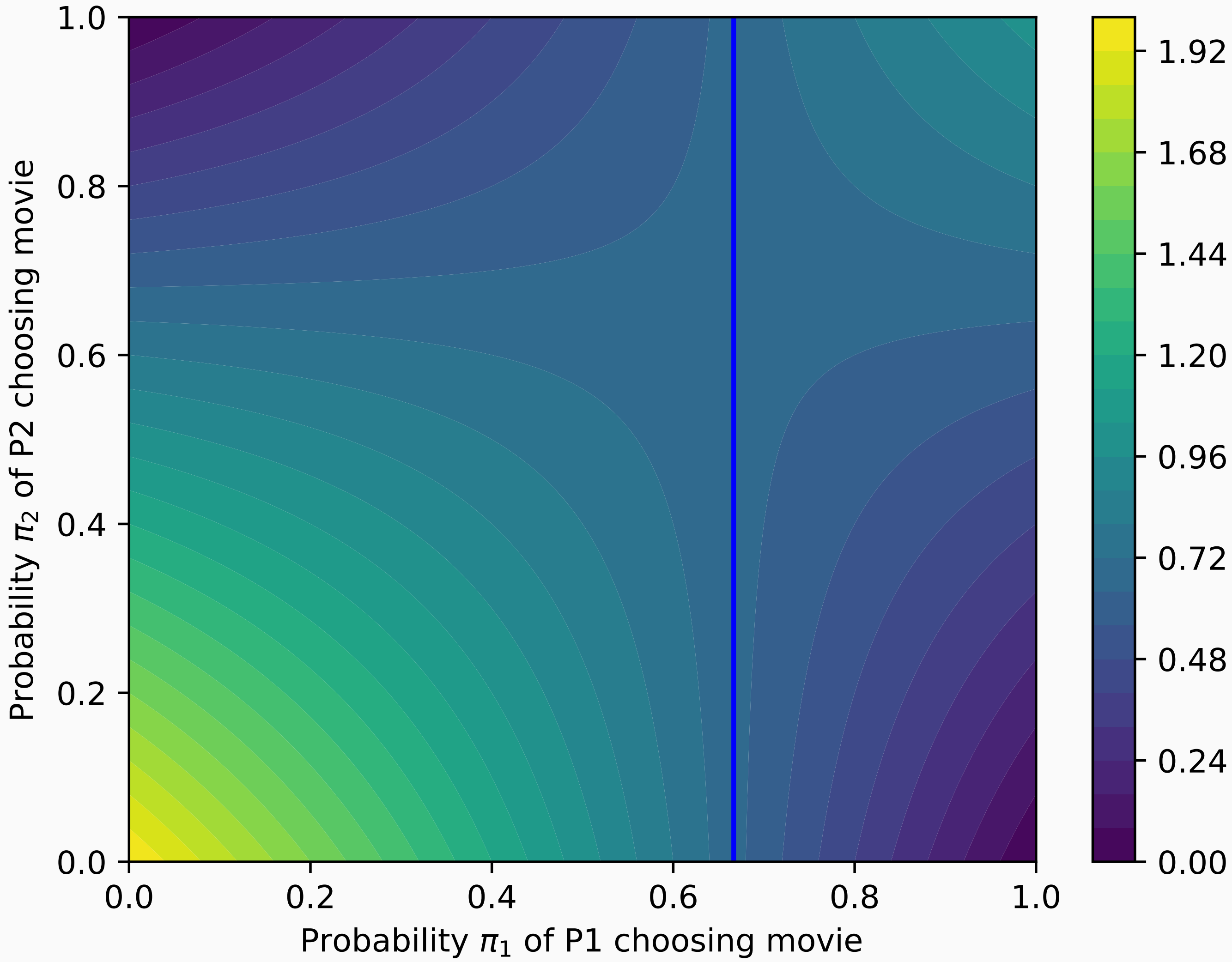
Battle of the Sexes, expected values of every π_1 and π_2



Expected values for P1, red line shows mixed strategy for P2



Expected values for P2, blue line shows mixed strategy for P1

Outline

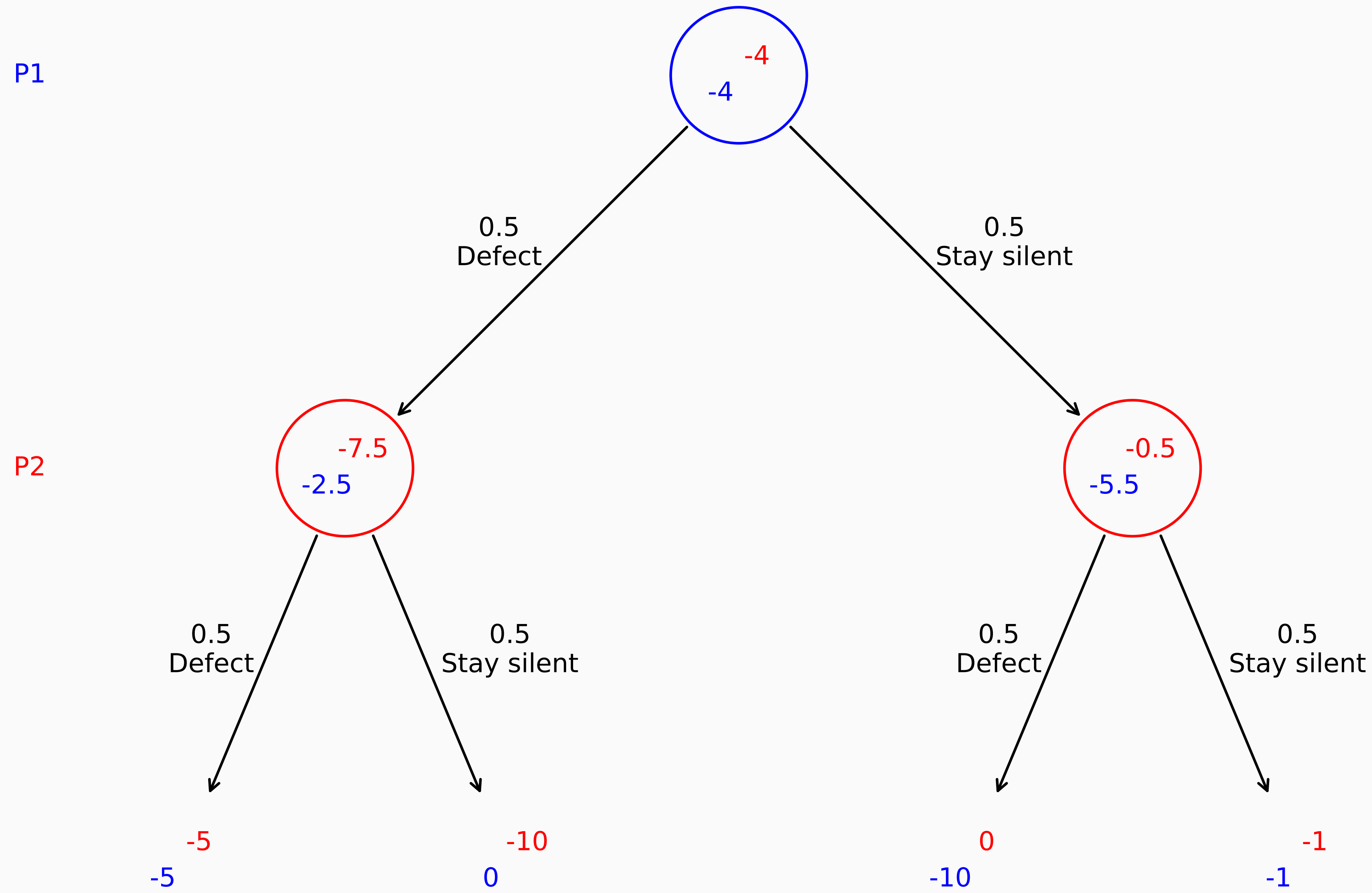
Prisoner's dilemma

P2

		Defect	Stay silent
P1	Defect	-5 (P1), -5 (P2)	0 (P1), -10 (P2)
	Stay silent	-10 (P1), 0 (P2)	-1 (P1), -1 (P2)

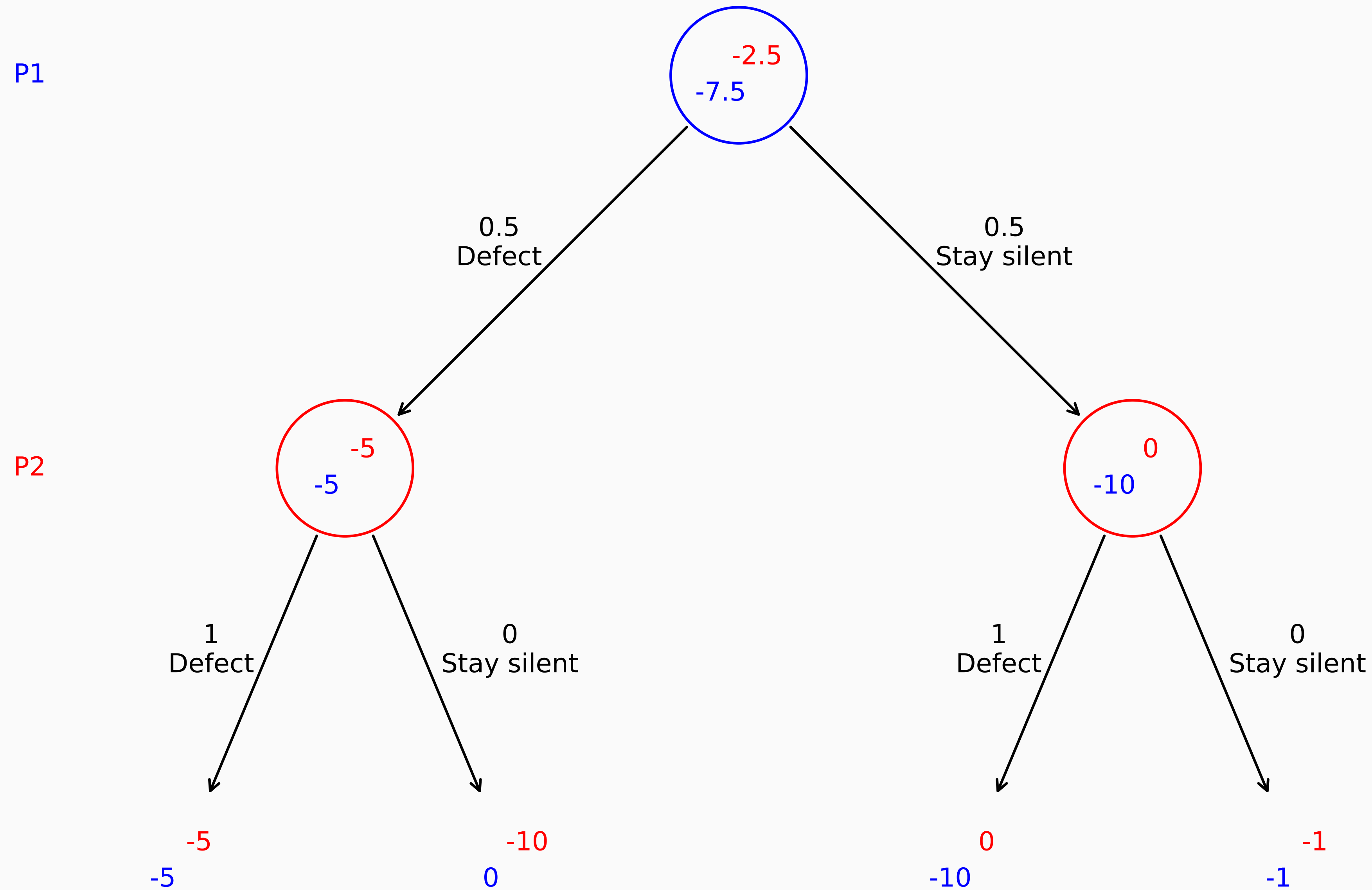
Has no mixed strategy (shown on page 18)

Prisoner's dilemma



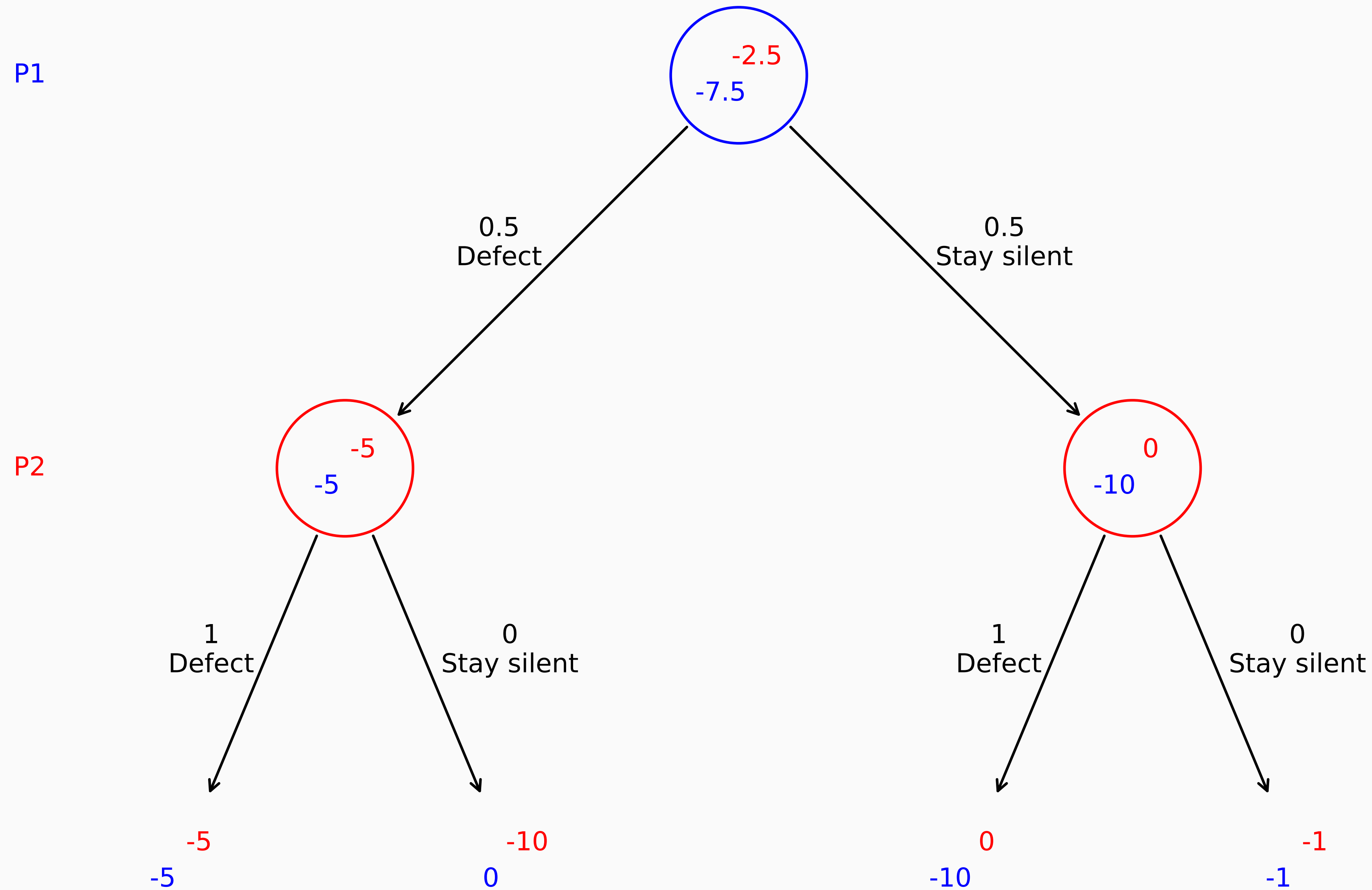
What is P2's best strategy?

Prisoner's dilemma, P2's pure strategy



What is P2's best strategy? Defect

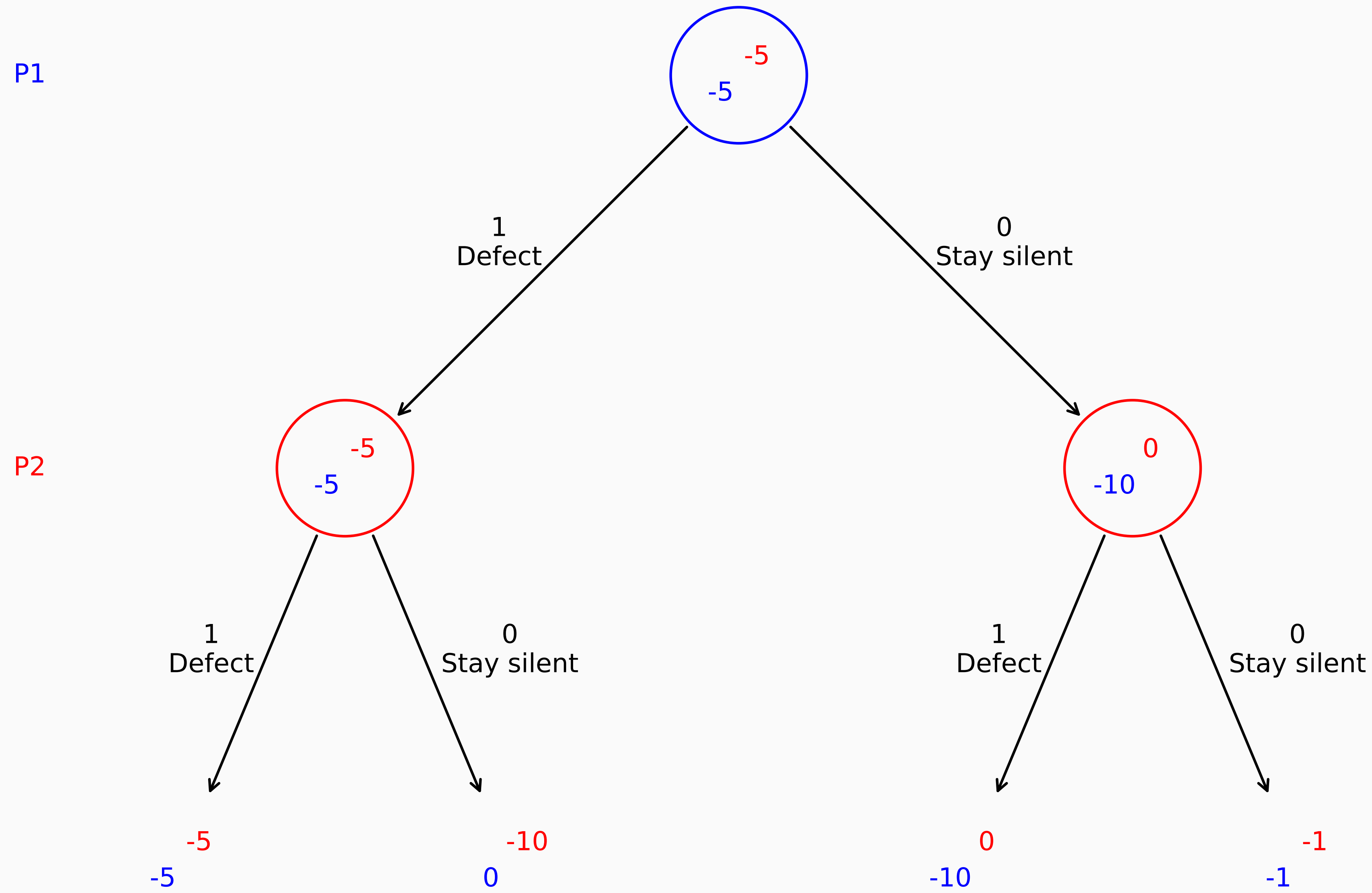
Prisoner's dilemma, P2's pure strategy



What is P2's best strategy? Defect

What is P1's best strategy?

Prisoner's dilemma, P1's and P2's pure strategies



What is P2's best strategy? Defect

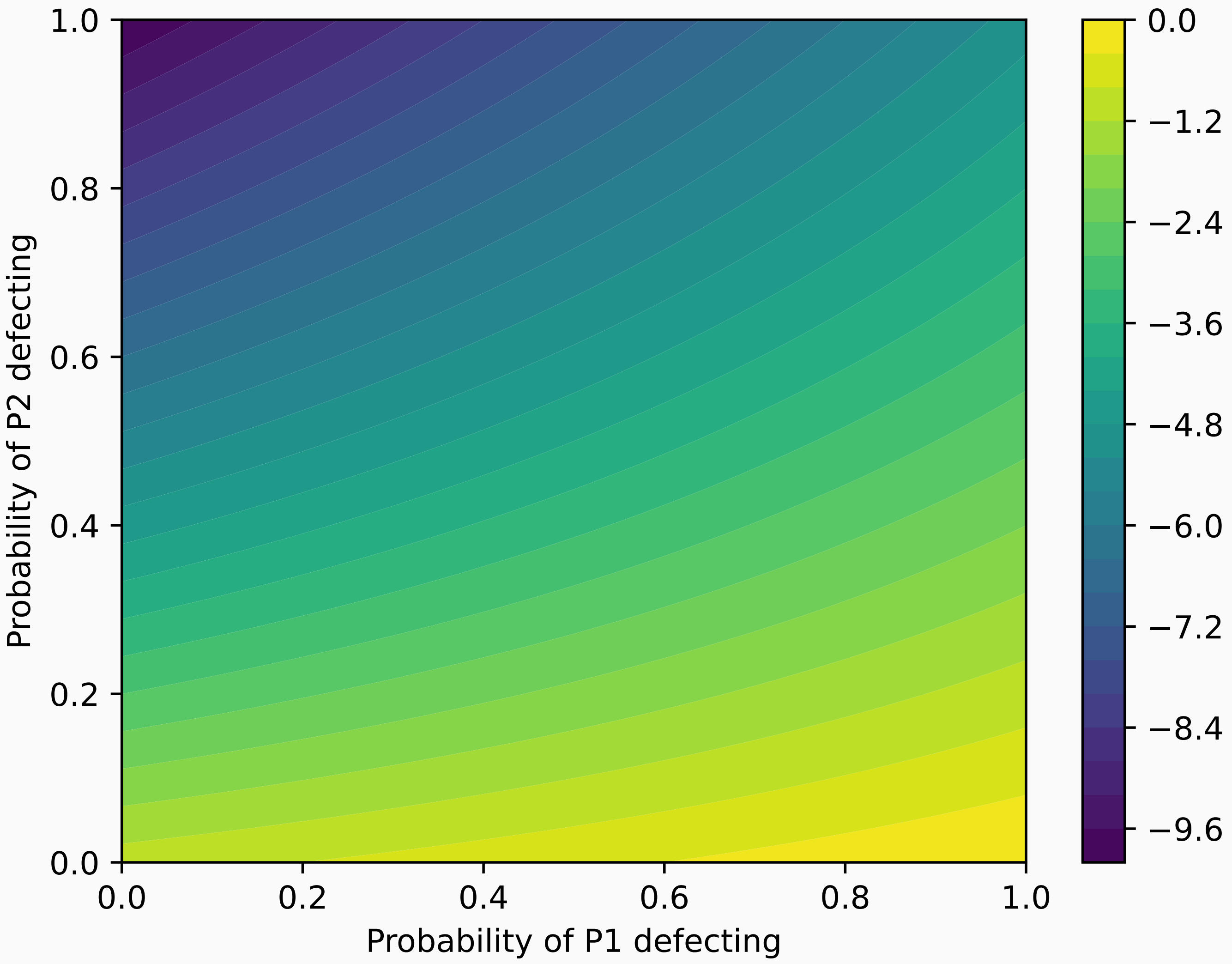
What is P1's best strategy? Defect

Prisoner's dilemma

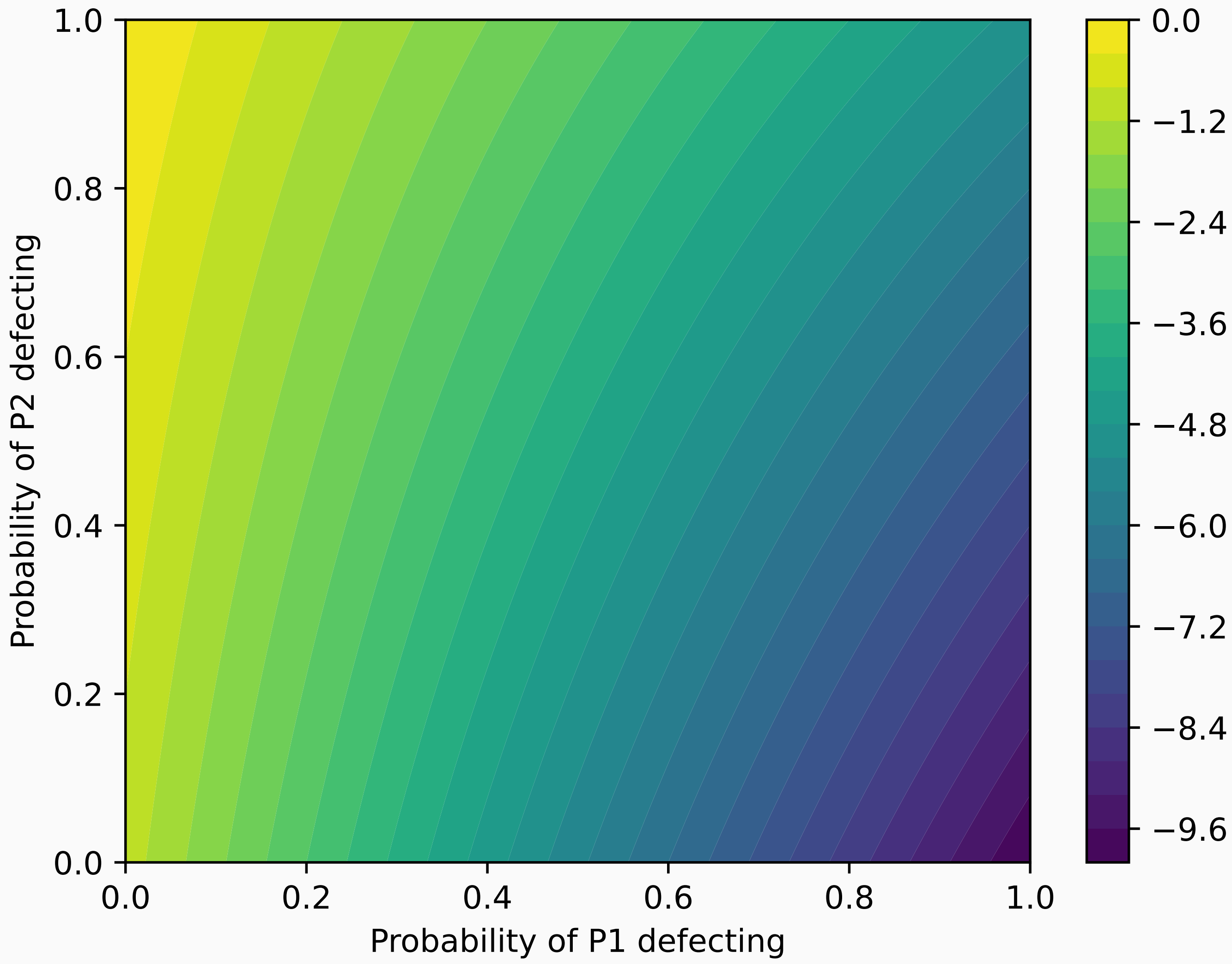
		P2	
		Defect	Stay silent
P1	Defect	$\underline{-5}$, $\underline{-5}$	0, -10
	Stay silent	-10, 0	-1, -1

Defecting is a pure strategy Nash equilibrium

Prisoner's dilemma, expected values of every π_1 and π_2



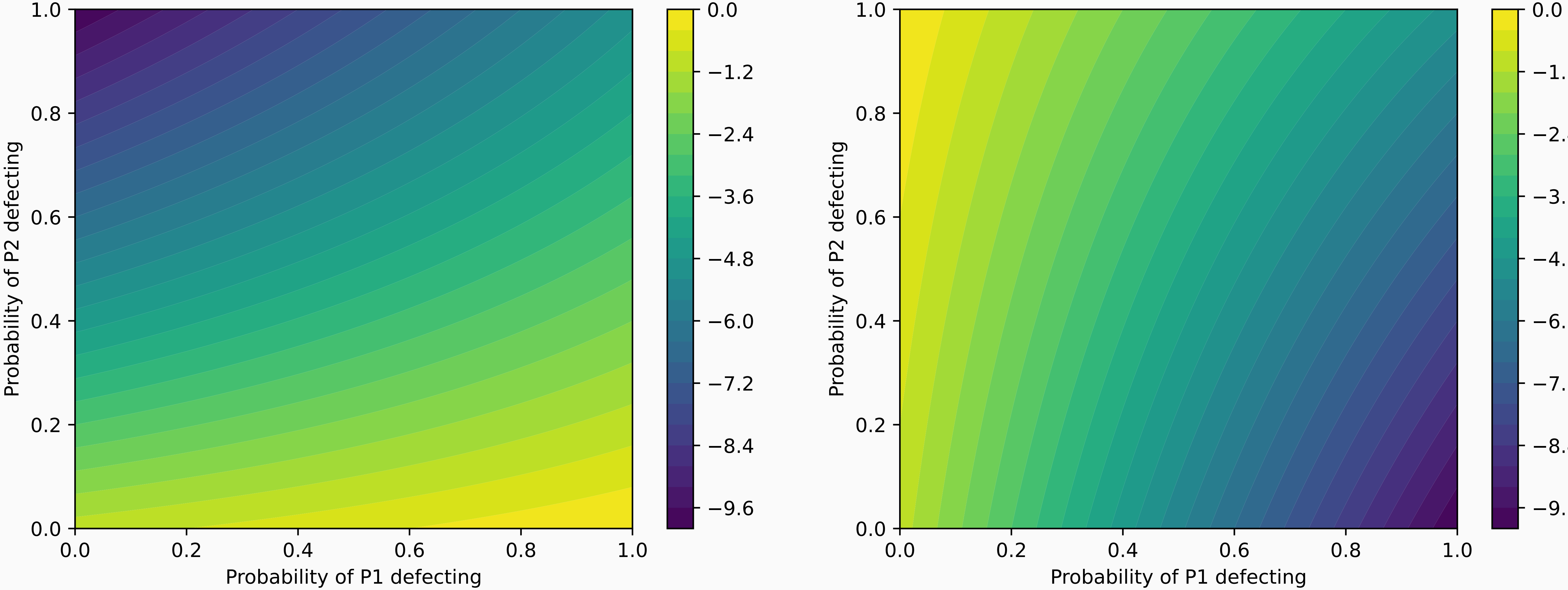
Expected values for P1



Expected values for P2

No mixed strategies. Why?

Prisoner's dilemma, expected values of every π_1 and π_2



Expected values for P1 — Expected values for P2

No mixed strategies. Why? Expected values are not constant on any horizontal or vertical line over $\pi_1 \in [0, 1]$ and $\pi_2 \in [0, 1]$

Outline

Backward induction: when the number of rounds is fixed, finite and known

- Players will play 10 rounds of prisoner's dilemma
  - What will the players do?

		P2: Defect	P2: Stay silent
P1	Defect	-5, -5	0, -10
	Stay silent	-10, 0	-1, -1

Backward induction: when the number of rounds is fixed, finite and known

- Players will play 10 rounds of prisoner's dilemma
  - What will the players do?
  - What will the players do at the tenth round?

		P2: Defect	P2: Stay silent
P1	Defect	-5, -5	0, -10
P1	Stay silent	-10, 0	-1, -1

Backward induction: when the number of rounds is fixed, finite and known

- Players will play 10 rounds of prisoner's dilemma
  - What will the players do?
  - What will the players do at the tenth round?
    - Defect

P2

		P2: Defect	P2: Stay silent
P1	Defect	-5, -5	0, -10
	Stay silent	-10, 0	-1, -1

Backward induction: when the number of rounds is fixed, finite and known

- Players will play 10 rounds of prisoner's dilemma
  - What will the players do?
  - What will the players do at the tenth round?
    - Defect
  - What will the players do at the ninth round?

P2

P1		Defect	Stay silent
	Defect	-5, -5	0, -10
	Stay silent	-10, 0	-1, -1

Backward induction: when the number of rounds is fixed, finite and known

- Players will play 10 rounds of prisoner's dilemma
  - What will the players do?
  - What will the players do at the tenth round?
    - Defect
  - What will the players do at the ninth round?
    - Defect

P2

P1		Defect	Stay silent
	Defect	-5 (P1), -5 (P2)	0 (P1), -10 (P2)
	Stay silent	-10 (P1), 0 (P2)	-1 (P1), -1 (P2)

Backward induction: when the number of rounds is fixed, finite and known

- Players will play 10 rounds of prisoner's dilemma
  - What will the players do?
  - What will the players do at the tenth round?
    - Defect
  - What will the players do at the ninth round?
    - Defect
  - ...
  - What will the players do at the first round?
    - Defect

P2

P1		Defect	Stay silent
	Defect	P1: -5, P2: -5	P1: 0, P2: -10
	Stay silent	P1: -10, P2: 0	P1: -1, P2: -1

Outline

When the number of rounds is infinite

We cannot use the sum of the expected values of each game:

$$\sum_{t=0}^{\infty} E_p(\pi_1(s_t), \pi_2(s_t))$$

where state s_t contains all the previous actions prior to time t

For example, both the strategies

$\pi_1(s_t) = \pi_2(s_t) = 1 \; \forall s_t$ (always defect) and

$\pi_1(s_t) = \pi_2(s_t) = 0 \; \forall s_t$ (always stay silent) would

then get the sum $-\infty$

		P2	
		Defect	Stay silent
P1	Defect	-5 (P2), -5 (P1)	-10 (P2), 0 (P1)
	Stay silent	0 (P2), -10 (P1)	-1 (P2), -1 (P1)

When the number of rounds is infinite

However, we can use a discount factor $\delta \in (0, 1)$:

$$\sum_{t=0}^{\infty} \delta^t E_p(\pi_1(s_t), \pi_2(s_t))$$

where a small δ will lead to discounting all but the first few games

P2

		Defect	Stay silent
P1	Defect	-5 (P2), -5 (P1)	-10 (P2), 0 (P1)
	Stay silent	0 (P2), -10 (P1)	-1 (P2), -1 (P1)

Grim trigger strategy

Let the state s_t be defined as all the previous actions:

$$s_t = ((a_{1,0}, a_{2,0}), (a_{1,1}, a_{2,1}), \dots, (a_{1,t-1}, a_{2,t-1}))$$

where $a_{1,0}, a_{2,0}$ are player 1's and player 2's actions (1: defect, 0: stay silent) at the first round

The grim trigger strategy for example for P1 is then:

$$\pi_1(s_t) = \begin{cases} 1, & \text{if } t > 0 \wedge \sum_{i=0}^{t-1} a_{2,i} > 0 \\ 0, & \text{otherwise} \end{cases}$$

i.e. defect if the opponent has defected previously, otherwise stay silent

		P2	
		Defect	Stay silent
P1	Defect	-5 -5	-10 0
	Stay silent	0 -10	-1 -1

Grim trigger strategy: Nash equilibrium

If both players follow the grim trigger strategies, the discount factor determines if for example player 1 has no incentive to change his/her strategy:

$$\sum_{t=0}^{\infty} \delta^t E_1(0,0) \geq \delta^0 E_1(1,0) + \sum_{t=1}^{\infty} \delta^t E_1(1,1)$$

$$-1\delta^0 - 1\delta^1 - 1\delta^2 - \ldots \geq 0\delta^0 - 5\delta^1 - 5\delta^2 - 5\delta^3 - \ldots$$

$$-\frac{1}{1-\delta} \geq 5 - \frac{5}{1-\delta}$$

$$\delta \geq \frac{1}{5}$$

i.e. if both players follow the grim strategy and use discount factor $\delta \geq \frac{1}{5}$, they are in a Nash equilibrium

		P2: Defect	P2: Stay silent
P1	Defect	P1: -5, P2: -5	P1: 0, P2: -10
P1	Stay silent	P1: -10, P2: 0	P1: -1, P2: -1

Outline

Mixed strategies

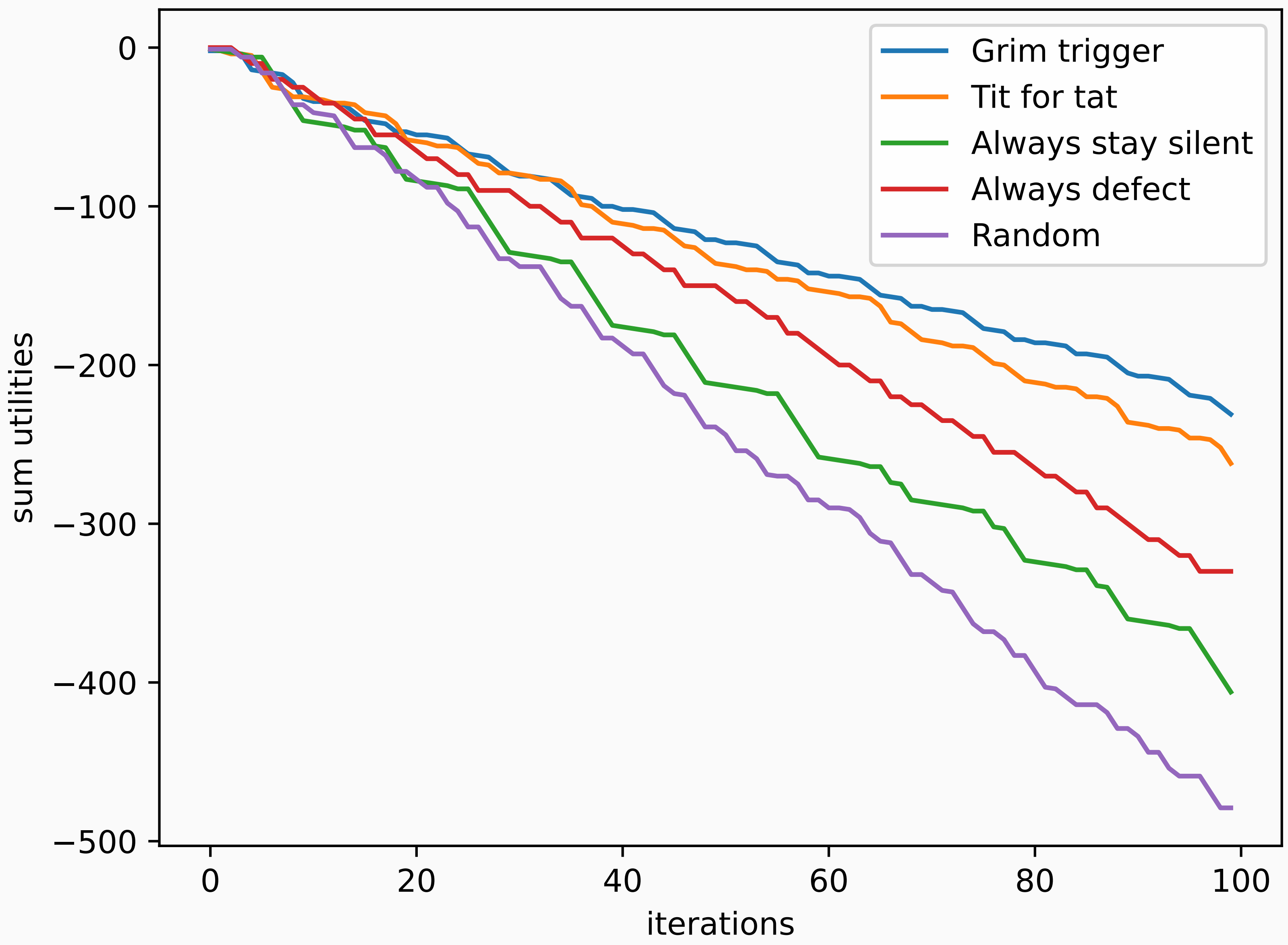
Pure strategies

Repeated games

Infinite rounds

Unknown number of rounds

Strategy competition: search for optimal strategy, unknown number of rounds



The tit for tat strategy is to repeat the opponent's last action, for example for P1:

$$\pi_1(s_t) = \begin{cases} 0, & \text{if } t = 0 \\ a_{2,t-1}, & \text{otherwise} \end{cases}$$

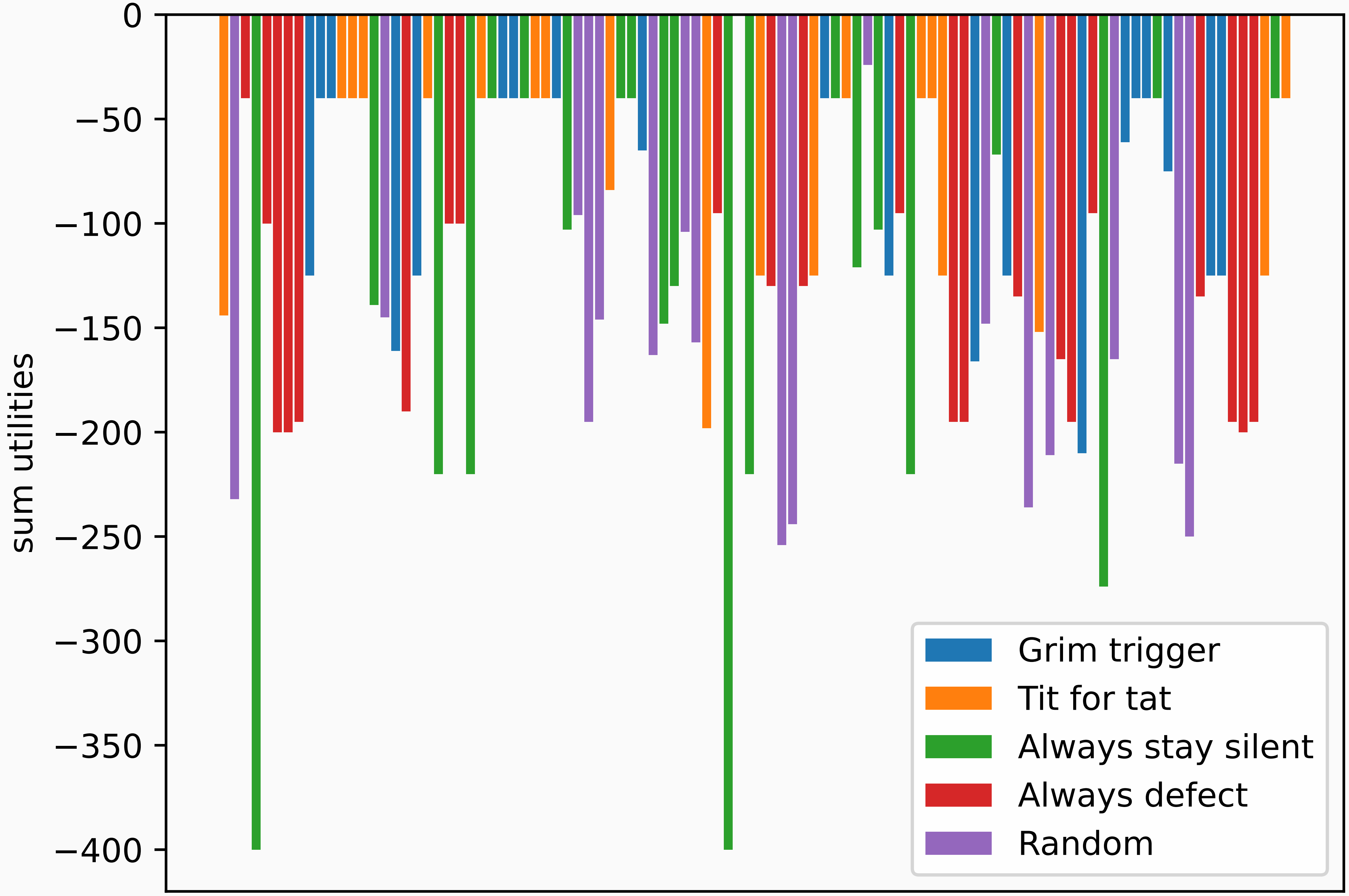
Axelrod’s Tournament

“In 1980, Robert Axelrod, professor of political science at the University of Michigan, held a tournament of various strategies for the prisoner’s dilemma. He invited a number of well-known game theorists to submit strategies to be run by computers. In the tournament, programs played games against each other and themselves repeatedly. Each strategy specified whether to cooperate or defect based on the previous moves of both the strategy and its opponent.”

“The winner of Axelrod’s tournament was the TIT FOR TAT strategy.”

https://cs.stanford.edu/people/eroberts/courses/soco/projects/1998-99/game-theory/axelrod.html

Strategy competition: search for optimal strategy, unknown number of rounds



Randomly placed agents are only playing against their immediate neighbors