NTNU
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Department of Mathematical Sciences

Examination paper for
TMA4130 Calculus 4N
Solution and grading manual

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Originalen er:	
1-sidig ☐	**2-sidig** ☒
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skal ha flervalgskjema ☐	

Date Signature

Problem 1 Laplace transform [10 pts]

a) Consider the function $f\colon [0,\infty) \to \mathbb{R}$ given by

$$f(t) = \begin{cases} 6 \cdot t & \text{for } 0 \leq t < 1, \\ 6 & \text{for } t \geq 1. \end{cases}$$

Compute the Laplace transform $\mathcal{L}(f)(s)$ of the function f.

b) Show that for a function $y\colon [0,\infty) \to \mathbb{R}$ whose Laplace transform exists, the following identity holds true:

$$\mathcal{L}\left(\int_0^t \sin(x-t)\cdot y(x)\mathrm{d}x\right)(s) = \frac{Y(s)}{-s^2-1},$$

where $Y(s) = \mathcal{L}(y)(s)$ denotes the Laplace transform of y.

c) Use the results from **a)** and **b)** to compute the solution of the integral equation

$$y(t) + \int_0^t \sin(x-t)\cdot y(x)\mathrm{d}x = f(t)\,.$$

Solution.

a) Using the Heaviside function $u(t-a)$, f can be written as

$$f(t) = 6 \cdot t - 6 \cdot (t-1) \cdot u(t-1)\,.$$

The Laplace transform can now be calculated using 'standard techniques' to

$$\mathcal{L}(f)(s) = \frac{6}{s^2} - \frac{6 \cdot \exp(-s)}{s^2}\,.$$

b) We use that $\sin(\cdot)$ is an odd function, which allows to express the left hand side as a convolution. The convolution theorem and the Laplace transform of the sine function give the result:

$$\begin{aligned}
\mathcal{L}\left(\int_0^t \sin(x-t) \cdot y(x)\mathrm{d}x\right) &= \mathcal{L}\left(-\int_0^t \sin(t-x) \cdot y(x)\mathrm{d}x\right) \\
&= -\mathcal{L}\left((\sin(t)) * y(t)\right) = -Y(s) \cdot \mathcal{L}(\sin t) \\
&= \frac{-Y(s)}{s^2+1} = \frac{Y(s)}{-s^2-1}\,.
\end{aligned}$$

c) We apply the Laplace transform on both sides of the equation and use the above identities:

$$\begin{aligned}
\mathcal{L}\left(y(t) + \int_0^t \sin(x-t) \cdot y(x)\mathrm{d}x\right) &= \mathcal{L}(f) \\
Y(s) + \frac{Y(s)}{-s^2-1} &= \frac{6 - 6 \cdot \exp(-s)}{s^2} \\
Y(s) \cdot \frac{s^2}{s^2+1} &= \frac{6 - 6 \cdot \exp(-s)}{s^2} \\
\Rightarrow\ Y(s) &= \frac{6 \cdot (s^2+1) \cdot (1-\exp(-s))}{s^4}\,.
\end{aligned}$$

The inverse Laplace transform of the last results gives the solution $y(t)$:

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left(\frac{6 \cdot (s^2+1) \cdot (1-\exp(-s))}{s^4}\right) = \mathcal{L}^{-1}\left(\frac{6}{s^2} + \frac{6}{s^4} - \exp(-s) \cdot \left(\frac{6}{s^2} + \frac{6}{s^4}\right)\right) \\
&= 6t + t^3 - u(t-1) \cdot (6 \cdot (t-1) + (t-1)^3) \\
&= t^3 - (1-t)^3 u(t-1) + f(t)\,.
\end{aligned}$$

Grading manual

a) (2p)

b) (3p): 2 pts for rewriting as convolution and use of convolution theorem,
1 pt for final correct answer

c) (5p) 3 pts for correct Laplace transform, 2 pts for inversion.
Full points if Laplace tranform of f was miscalculated in a) but inverse Laplace transform in c) then was computed correctly.

Problem 2 Fourier series [15 pts]

a) Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function and that $a \in \mathbb{N} \setminus \{0\}$ is a constant. Decide for all the following functions whether they are also necessarily periodic. If they are, what is their (fundamental) period?

1. $g_1(x) := f(a \cdot x)$,
2. $g_2(x) := f(x + a)$,
3. $g_3(x) := f(x^a)$,
4. $g_4(x) := a + a \cdot (f(x/a + a))^a$.

b)

1. Calculate the Fourier series of the function $f(x) = |\sin x|$ defined on $[-\pi, \pi]$. Explicitly write down the first five non-vanishing terms of the Fourier sum. **Hint:** You can use the fact that the sine is an odd function, and use your knowledge about Fourier series of even functions.
2. With f from b) 1., how many terms in the partial Fourier sum $F_N(x) = a_0 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin nx)$ need to be taken into account such that the square error

$$E_N = \int_{-\pi}^{\pi} (f(x) - F_N(x))^2 \, dx,$$

is less than 0.01?

Hint: The number is not very large. We recommend that you just calculate (to four digits) the error for the first few partial sums. You can use that $\int_{-\pi}^{\pi} (f(x))^2 \, dx = \pi$.

Solution

a) 1. The function is periodic with period $2\pi/a$. To see this, we calculate:

$$g_1(x + (2\pi/a)) = f(a(x + 2\pi/a)) = f(ax + 2\pi) = f(ax) = g_1(x)\,.$$

2. The function is also periodic and has period 2π. (trivial observation) $g_2(x + 2\pi) = f(x + 2\pi + a) = f(x + a) = g_2(x)$.

3. Unless $a = 1$, the function is not necessarily periodic.

4. The function is periodic and has period $2\pi a$:

$$\begin{aligned} g_4(x+2\pi a) = a+a\cdot(f((x+2\pi a)/a+a))^a &= a+a\cdot(f(x/a+2\pi+a))^a \\ &= a + a \cdot (f(x/a + a))^a = g_4(x)\,. \end{aligned}$$

b) 1. Since the sine-function is odd, the function f is even and can be expressed by cosine-terms only (i.e., $b_n = 0$ for all n). To get rid of the absolute-value, we treat the problem as an even half-range expansion of $\sin(x)$. The Fourier coefficients are now given by

$$a_0 = \frac{1}{\pi} \cdot \int_0^\pi \sin x dx = \frac{2}{\pi}\,,$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \sin(x)\cdot\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\left(\sin(x-nx)+\sin(x+nx)\right)dx \quad \sin\alpha\cos\beta = \tfrac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)) \\ &= \frac{1}{\pi}\int_0^\pi \left(-\sin((n-1)x)+\sin((n+1)x)\right)dx \\ &= \frac{1}{\pi}\left[\tfrac{1}{n-1}\cos((n-1)x) - \tfrac{1}{n+1}\cos((n+1)x)\right]_0^\pi \\ &= -\frac{2}{\pi\cdot(n-1)\cdot(n+1)}\cdot(1+(-1)^n)\,. \end{aligned}$$

So, the Fourier series is given by

$$\begin{aligned} f(x) \sim \frac{2}{\pi} - \frac{2}{\pi}\cdot\sum_{n=1}^{\infty}\frac{1+(-1)^n}{(n-1)(n+1)}\cdot\cos(nx) \\ = \frac{2}{\pi} - \frac{4}{\pi}\cdot\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)}\cdot\cos(2kx) \\ = \frac{2}{\pi} - \frac{4}{\pi}\cdot\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \frac{\cos 8x}{7\cdot 9} + \cdots\right) \end{aligned}$$

2. We know that the error can be expressed as

$$E_N = \int_{-\pi}^{\pi} (f(x))^2 dx - \pi \cdot \left(2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right) .$$

For the given function and its Fourier sum, this leads to

N	a_N	E_N
0	$2/\pi$	$\pi - 8/\pi \approx 0.5911$
1	0	
2	$\frac{-4}{3\pi}$	$\pi - 8/\pi - 16/(9\pi) \approx 0.0292$
3	0	
4	$\frac{-4}{15\pi}$	$\pi - 8/\pi - 16/(9\pi) - 16/(225\pi) \approx 0.00659 < 0.01$

Grading manual

a) (5p): 1,2,3 gives 1 pt each, 4 gives 2 pts.

b) (10p):
Subproblem 1 (7p):
2 pts identification as even function, realize that it can be written as $\cos(x)$ series/Fourier series of the even extension of $\sin(x)$,
1 pt for computation of a_0,
3 pts for computation of a_n,
1 pt for final series.
Subproblem 2 (3p) for using the error formula and finding the correct N. If coefficients in b) Subproblem 1 were miscalculated, there is no chance to answer this correctly, but give 1 pt for realizing that a mistake must have happened.
1-2 pts deduction for each minor/major miscalculations (but no minus points)

Problem 3 Heat equation [12 pts]

Consider the following heat equation

$$u_t(x,t) = c^2 u_{xx}(x,t), \quad t \geq 0, x \in [0,\pi]$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0$$

and initial condition

$$u(x,0) = \frac{\pi}{2} - \left| x - \frac{\pi}{2} \right|, \quad x \in [0,\pi].$$

a) Show that the Fourier sine series solution of this above heat equation with boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-c^2 n^2 t}$$

by using the separation of variables method.

b) Compute the Fourier sine series solution of the above heat equation with the given boundary conditions and initial condition. Write down the three first non-zero terms of the solution.

Solution

a) We set
$$u(x,t) = F(x)G(t).$$
This gives
$$F(x)G'(t) = c^2F''(x)G(t).$$
Separation of variables leads to
$$\frac{G'}{c^2G} = \frac{F''}{F}.$$
As the left-hand side depends only on t and the right-hand side only on x, both fractions must be equal to a constant, say k. For $k \geq 0$ we get the trivial solution $u \equiv 0$. Therefore, $k < 0$, and we set $k = -p^2$. We get the two ODEs:
$$F'' + p^2F = 0$$
$$G' + c^2p^2G = 0.$$
The first ODE has the general solution
$$F(x) = A\cos(px) + B\sin(px).$$
Using the boundary conditions, we get
$$u(0,t) = F(0)G(t) = 0 = u(\pi,t) = F(\pi)G(t).$$
This gives $F(0) = F(\pi) = 0$, as otherwise we would get $G(t) \equiv 0$. Then $F(0) = A = 0$ and $F(\pi) = B\sin(p\pi) = 0$ with $B \neq 0$, thus $p = \frac{n\pi}{\pi} = n$, $n = 1, 2, \ldots$. We can set $B = 1$ and obtain $F_n(x) = \sin(nx)$. The second ODE has the form (with $p = n$)
$$G' + c^2p^2G = G' + (cn)^2G = 0.$$
Its general solution is
$$G_n = B_ne^{-(cn)^2t}$$
Hence, the function
$$u(x,t) = F_nG_n = B_n\sin(nx)e^{-(cn)^2t}$$
solves the heat equation with the given boundary conditions. Therefore, the series
$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n\sin(nx)e^{-c^2n^2t}$$
is also a solution of the problem.

b) The solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-c^2 n^2 t}.$$

The initial condition gives

$$u(x,0) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) = \frac{\pi}{2} - \left| x - \frac{\pi}{2} \right|$$

with

$$\begin{aligned}
B_n &= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - \left| x - \frac{\pi}{2} \right| \right) \sin(nx) dx \\
&= \frac{2}{\pi} \left(\int_0^{\pi/2} x \cdot \sin(nx) dx + \int_{\pi/2}^{\pi} (\pi - x) \cdot \sin(nx) dx \right) \\
&= \frac{2}{\pi} \cdot \left(\left[-x \cdot \frac{\cos(nx)}{n} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos(nx)}{n} dx + \left[(x-\pi) \cdot \frac{\cos(nx)}{n} \right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{\cos(nx)}{n} dx \right) \\
&= \frac{2}{\pi} \cdot \left(-\frac{\pi}{2n} \cos(n\pi/2) + \frac{1}{n^2} \left[\sin(nx) \right]_0^{\pi/2} + \frac{\pi}{2n} \cos(n\pi/2) - \frac{1}{n^2} \left[\sin(nx) \right]_{\pi/2}^{\pi} \right) \\
&= \frac{2}{\pi n^2} \cdot (\sin(n\pi/2) + \sin(n\pi/2)) \\
&= \frac{4 \sin(\frac{\pi n}{2})}{\pi n^2}
\end{aligned}$$

Therefore, we have

$$b_1 = \frac{4}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{4}{\pi 3^3}, \quad b_4 = 0, \quad b_5 = \frac{4}{\pi 5^2}, \ldots$$

So, we have

$$\begin{aligned}
u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) &= \sum_{n=1}^{\infty} \frac{4 \sin(\frac{\pi n}{2})}{\pi n^2} \sin(nx) e^{-c^2 n^2 t} \\
&= \frac{4}{\pi} \sin(x) e^{-c^2 t} - \frac{4}{\pi 3^3} \sin(3x) e^{-c^2 3^2 t} + \frac{4}{\pi 5^2} e^{-c^2 5^2 t} \ldots
\end{aligned}$$

Grading manual Generally, 1-2 pts deduction for each minor/major miscalculations (but no minus points)

a) (7p): 1 pt for ansatz/product representation $u(x,t) = F(x)G(t)$,
2 pts for inserting into PDE and deriving the 2 ODEs,
2 pts for deriving final expression for $F(x)$,
1 pt for final expression for $G(t)$,
1 pt for final series $\sum_{n=1}^{\infty} u_n(x,t)$

b) (5p): 1 pt for inserting initial condition and finding sin series for $u(x,0)$
3 pts for computing B_n
1 pt for final solution

Problem 4 Wave equation [10 pts]
Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u}{\partial x^2} = 0$$

on the real line $\mathbb{R}$. Use d'Alembert's solution formula to find the solution $u(x,t)$ satisfying initial conditions

$$u(x,0) = x,$$
$$\frac{\partial u}{\partial t}(x,0) = \cos^2(x).$$

Simplify the resulting expressions as much as possible.

Solution. Recall that D'Alembert's formula gives a nice expression for the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

on the real line with initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, namely

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

In our case, we have $c = \dfrac{1}{2}$, $f(x) = x$ and $g(x) = \cos^2 x$,

$$u(x,t) = \frac{1}{2}[(x + \tfrac{1}{2}t) + (x - \tfrac{1}{2}t)] + \int_{x-ct}^{x+ct} \cos^2(s)ds.$$

We can use the trigonometric identities

$$\cos^2(\alpha) = \frac{1}{2}(\cos(2\alpha) + 1)$$
$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$\begin{aligned}
u(x,t) &= x + \int_{x-ct}^{x+ct} \cos^2(x)\,\mathrm{d}x \\
&= x + \int_{x-ct}^{x+ct} \frac{1+\cos(2s)}{2}\,\mathrm{d}x \\
&= x + \frac{x}{2}\Big|_{x-ct}^{x+ct} + \frac{\sin(2s)}{4}\Big|_{x-ct}^{x+ct} \\
&= x + ct + \frac{1}{4}\Big(\sin(2(x+ct) - \sin(2(x-ct))\Big) \\
&= x + ct + \frac{1}{2}\cos(2x)\sin(2ct) \\
&= x + \frac{1}{2}t + \frac{1}{2}\cos(2x)\sin(t) \quad \text{with } c = \tfrac{1}{2}.
\end{aligned}$$

Grading manual [10 pts]
Generally, 1-2 pts deduction for each minor/major miscalculations (but no minus points)
4 pts for d'Alembert's formula, e.g. 1 for general d'Alembert formula, 1 pt each for identifying c, f and g.
6 pts for computation of the integral

Problem 5 Interpolation [9 pts]
Consider the data points

x_i	-1	1	3
$f(x_i)$	1	4	3

a) Use Lagrange interpolation to find the polynomial of minimal degree interpolating these points. Express the polynomial in the form $p_n(x) = a_n x^n + \cdots + a_1 x + a_0$.

b) Determine the Newton form of the interpolating polynomial.

c) Now add the data point $(x_3, f_3) = (4, 6)$ and compute the resulting interpolation polynomial for the given 4 data points.

Solution. Lagrange and Newton polynomial for the first 3 data points are given by

$$
\begin{aligned}
L_0 &= \frac{(x-3)\,(x-1)}{8}\\
L_1 &= -\frac{(x-3)\,(x+1)}{4}\\
L_2 &= \frac{(x-1)\,(x+1)}{8}\\
p_2(x) &= \frac{(x-3)\,(x-1)}{8} - (x-3)\,(x+1) + \frac{3\,(x-1)\,(x+1)}{8}\\
p_2(x) &= -\frac{x^2}{2} + \frac{3x}{2} + 3\\
\omega_0 &= 1\\
\omega_1 &= x+1\\
\omega_2 &= (x-1)\,(x+1)
\end{aligned}
$$

```
Divided difference table =
[[-1, 1, 3], [1, 4, 3], [3/2, -1/2, None], [-1/2, None, None]]
```

Using Newton form for the 4th data point.

```
Divided difference table =
[[-1, 1, 3, 4], [1, 4, 3, 6], [3/2, -1/2, 3, None], [-1/2, 7/6, None, None],
[1/3, None, None, None]]
```

4th Newton polynomial and final interpolation polynomial.

$$
\begin{aligned}
\omega_3 &= (x-3)\,(x-1)\,(x+1)\\
p_3(x) &= \frac{x^3}{3} - \frac{3x^2}{2} + \frac{7x}{6} + 4
\end{aligned}
$$

Grading manual

a) (3p): 2 pts for interpolation polynomial in Lagrange form, 1 pt for final polynomial
no points for systematic major blunders when setting up Lagrange basis function, e.g. calculating l_0 via

$$
l_0(x) = (x-x_1)(x-x_2) \qquad \text{instead of} \qquad l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}
$$

b) (3p): 2 pts for computation of divided difference table, 1 pt for interpolation polynomial in Newton form
Only Newton form is required, no point deduction for miscalculations during additional rewriting into monomial form $p_n(x) = a_n x^n + \cdots + a_1 x + a_0$.

c) (3p): Can be computed using either the Lagrange or Newton (simpler!) form, 2 pts for computation of divided difference table or Lagrange polynomials, 1 pt for interpolation polynomial in either Newton or Lagrange form
Only correct Newton (or Lagrange) form is required, no point deduction for miscalculations during additional rewriting into monomial form $p_n(x) = a_n x^n + \cdots + a_1 x + a_0$.

Problem 6 Quadrature rules [15 pts]

a) It is known that the quadrature rule $MR[f](a,b)$ defined by the midpoint rule satisfies the error estimate

$$|MR[f](a,b) - \int_a^b f(x)dx| \leqslant \frac{7M}{24}(b-a)^3,$$

where $M = \max_{x\in[a,b]} |f''(x)|$. Which degree of exactness has this quadrature rule and why?

b) Show that the corresponding *composite midpoint rule* $CMR[f](a,b,m)$ defined on m equally spaced subintervals satisfies an estimate for the quadrature error of the form

$$|CMR[f](a,b,m) - \int_a^b f(x)dx| \leqslant M(b-a)\frac{7h^2}{24},$$

where $h = \frac{b-a}{m}$ and M is defined as in **b)**.

c) Consider the the integral $\int_0^1 \cos(x)\,\mathrm{d}x$. Find the number of intervals m which guarantees that the quadrature error for the composite midpoint rule is below 10^{-3}.

d) Write down a `Python` code snippet, which for given function f, interval endpoints a, b and number of intervals m uses the composite midpoint rule to compute the integral $\int_a^b f(x)dx$ numerically.

Solution.

a) A quadrature rule has a degree of exactness equal to p if every polynomial up to order p is integrated exactly by the quadrature rule (i.e quadrature error is 0). The the error estimate involves the second derivative $f''(x)$. Thus the quadrature rule has a degree of exactness of 1 since the second derivative vanishes for all polynomials up to order 1.

b) Define $h = (b-a)/m$ and m subintervals $[x_{i-1}, x_i]$ for $i = 1, \dots m$, with $x_i = a + ih$ for $i = 0, \dots m$. Then

$$\begin{aligned}
|CMR[f](a,b,m) - \int_a^b f(x)dx| &= \left| \sum_{i=1}^m MR[f](x_{i-1}, x_i) - \sum_{i=1}^m \int_{x_{i-1}}^{x_i} f(x)dx \right| \\
&\leqslant \sum_{i=1}^m \left| MR[f](x_{i-1}, x_i) - \int_{x_{i-1}}^{x_i} f(x)dx \right| \\
&\leqslant \sum_{i=1}^m \max_{x\in[x_{i-1},x_i]} |f''(x)| \frac{7}{24} h^3 \\
&\leqslant \underbrace{\max_{x\in[a,b]} |f''(x)|}_{=:M} \frac{7}{24} h^3 m \\
&= M \frac{7}{24} h^3 (b-a)/h = M \frac{7}{24} h^2 (b-a).
\end{aligned}$$

c) For the given integrand $\cos(x)$ we observe that

$$M = \max_{x\in[0,1]} |\cos''(x)| \max_{x\in[0,1]} |\cos(x)| = 1$$

With $h = (b-a)/m$, the error estimate can be rewritten as follows

$$\begin{aligned}
|CMR[f](a,b,m) - \int_a^b f(x)dx| &\leqslant M(b-a)\frac{7h^2}{24} \\
&= M\frac{(b-a)^3}{m^2}\frac{7}{24}
\end{aligned}$$

With $M = 1$ and $(b-a) = 1$, we thus want m to be large enough to satisfy

$$10^{-3} \geqslant M\frac{(b-a)^3}{m^2}\frac{7}{24} = \frac{1}{m^2}\frac{7}{24} \Rightarrow m \geqslant \sqrt{\frac{7000}{24}} \approx 17.078,$$

so for $m = 18$, we can guarantee that the quadrature error is less than 10^{-3}.

d) Alternative 1 using the `np.sum` function:

```python
def CMR(f, a, b, N):
  h = (b-a)/N
  xi = np.linspace(a+h/2,b-h/2, N)
  return h*np.sum(f(xi))
```

Alternative 2 using a simple `for` loop to compute the sum of function evaluations at the interval midpoints:

```python
def CMR(f, a, b, N):
    h = (b-a)/N
    f_sum = 0
    for xi in np.linspace(a+h/2,b-h/2, N):
        f_sum += f(xi)
    return h*f_sum
```

Grading manual

a) (3p) 1pt for reviewing degree of exactness,
2 pts for determining degree of exactness and arguing via second order derivative in error representation

b) (4p) 2 pts for bounding quadrature error of the composite rule by a sum of the quadrature errors on each subinterval
2 pts for using the error estimate for the simple midpoint rule from a) and the final error estimate.

c) (4p) 1 pt for finding M
2 pts for relating quadrature error estimate to given tolerance
1 pt for finding correct m.

d) (4p) for correct code, -1 pt for each "bug/error" in the code (at most 4 minus point).

Problem 7 Nonlinear equations [8 pts]

Let r be a solution of the following equation

$$x + \sin(x-2) = 0, \quad 0 \le x \le 2$$

Show that the solution is unique by using the intermediate value theorem.
Starting from

$$x_0 = 0.5,$$

perform two iterations of the Newton method.

Solution The function $f(x) = x + \sin(x-2)$ is continuous on the interval $[0,2]$. We have $f(0) = \sin(-2) \approx -0.909$ and $f(2) = 2$. Therefore, by the intermediate value theorem, there exists a zero in $[0,2]$. This solution is unique as

$$f'(x) = 1 + \cos(x-2) > 0$$

for all $x \in [0,2]$. Applying Newton's method, we get

$$\begin{aligned} x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} &= 0.5 - \frac{0.5 + \sin(-1.5)}{1 + \cos(-1.5)} = 0.964 \\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} &= 0.964 - \frac{0.964 + \sin(0.964 - 2)}{1 + \cos(0.964 - 2)} = 0.895. \end{aligned}$$

Grading manual [8 pts]:
2 pts for application of intermediate value theorem,
2 pts for uniqueness
2pts for computation of x_1
2pts for computation of x_2

Problem 8 Numerical methods for ODE [12 pts]

Consider the following implementation of a 3-stage Runge-Kutta method.

```
def rkm(y0, t0, T, f, Nmax):
    ts = [t0]
    ys = [y0]
    dt = (T-t0)/Nmax

    while (ts[-1] < T):
        t, y  = ts[-1], ys[-1]

        k1 = f(t,y)
        k2 = f(t+2/3*dt,y+2/3*dt*k1)
        k3 = f(t+dt,y+dt/2*(k1+k2))

        ys.append(y + dt/4*(k1+3*k2))
        ts.append(t + dt)

    return np.array(ts), np.array(ys)
```

a) Extract the Butcher table from the given implementation. Can you simplify the Butcher table and/or implementation code?

b) Determine the consistency order of the Runge-Kutta method implemented in **a)**.

c) Now imagine you have run a convergence rate study for three different Runge-Kutta methods, one of which was the method implemented in the code snippet above. You obtained the following tables which tabulate the number of used, equidistant time-steps N against the resulting error.

What are the experimentally observed orders of convergence for each method and which table was likely produced by the method implemented above? Justify your answers!

	N	Error
0	4	0.221199
1	8	0.096199
2	16	0.044258
3	32	0.021231
4	64	0.010403
5	128	0.005141

Table 1

	N	Error
0	4	3.1795e-02
1	8	3.0213e-03
2	16	3.1609e-04
3	32	3.5879e-05
4	64	4.2818e-06
5	128	5.2306e-07

Table 2

	N	Error
0	4	0.071203
1	8	0.010207
2	16	0.001986
3	32	0.000446
4	64	0.000106
5	128	0.000026

Table 3

Solution. **a)**

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 2/3 & 2/3 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 \\ \hline & 1/4 & 3/4 & 0 \end{array}.$$

Since the computation of the next step

$$y_{n+1} = y_n + \frac{\tau}{4}(k_1 + 3k_2)$$

does not use k_3, we could savely skip the computation of k_3, that is simply delete line 11 in the given code.

The presented method is in fact a 2 stage Runge-Kutta method represented by the Butcher table

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 3/4 \end{array}$$

b) Note again that $b_3 = 0$ and we have in fact a two stage method. Then the order conditions are (omitting all terms involving $b_3 = 0$):

$p = 1$ $$b_1 + b_2 = \frac{1}{4} + \frac{3}{4} = 1$$ OK

$p = 2$ $$b_1c_1 + b_2c_2 = \frac{1}{4} \cdot 0 + \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$ OK

$p = 3$ $$b_1c_1^2 + b_2c_2^2 = \frac{1}{4} \cdot 0^2 + \frac{3}{4} \cdot \left(\frac{2}{3}\right)^2 = \frac{1}{3}$$ OK

$$b_1(a_{11}c_1 + a_{12}c_2) + b_2(a_{21}c_1 + a_{22}c_2) = \frac{1}{4}(0 \cdot 0 + 0 \cdot \frac{2}{3}) + \frac{3}{4}(\frac{2}{3} \cdot 0 + 0 \cdot \frac{2}{3})$$
$$= 0 \neq \frac{1}{6}$$ Not satisfied

Thus the given Runge-Kutta method has consistency order 2.

c) For a Runge-Kutta methods which has consistency order p, we expect the method to have convergence order 2 as well for reasonable right-hand side functions in an ODE problem. Thus, whenever the number of subintervals is doubled, the error should be reduced by a roughly a factor of $\frac{1}{2^p}$, that is $err_k/err_{k+1} \approx 2^p$. Taking the log gives

$$\frac{\log(err_k/err_{k+1}))}{\log 2} \approx p. \tag{1}$$

Table 1: For each doubling of the number of time-steps, the error is roughly reduced by a factor $\frac{1}{2} = \frac{1}{2^1}$ so the experimentally observed convergence rate EOC is 1. We can confirm that using (1), e.g. for the first doubling we get $\log(0.221199/0.096199)/\log(2) \approx 1.20$ and for the last two doublings, we get

$$\log(0.021231/0.010403)/\log(2) \approx 1.029$$
$$\log(0.010403/0.005141)/\log(2) \approx 1.017$$

so the EOC is around 1.

Table 2: With the same rationale as for the first table, we see that the EOC for the first doubling is $\log(3.1795e - 2/3.0213e - 3)/\log(2) \approx 3.396$ while for the last two doubling we see that

$$\log(3.5879e - 5/4.2818e - 6)/\log(2) \approx 3.067$$
$$\log(4.2818e - 6/5.2306e - 7)/\log(2) \approx 3.033$$

so the EOC is very close to 3.

Table 3: With the same rationale as for the first table, we see that the EOC for the first doubling is $\log(0.071203/0.010207)/\log(2) \approx 2.8023791301743497$ while for the last two doubling we see that the EOC approaches 2,

$$\log(0.000446/0.000106)/\log(2) \approx 2.073$$
$$\log(0.000106/0.000026)/\log(2) \approx 2.0275$$

In problem part **b)**, we saw that the given Runge-Kutta method has consistency order 2, we thus expect the method to have convergence order 2 as well for reasonable right-hand side functions in an ODE problem, so Table 3 is most likely produced by the code given above.

Grading manual

a) (4p) 2 pts for correct Butcher table,
1 pt for realizing that k_3 is not used and that line 11 can be deleted
1 pt for reduced/simplified Butcher table

b) (4p) 1 pt for each correctly checked order condition

c) (4p) 1 pt for each table/ correctly identified EOC, short rationale is ok (e.g. "the error roughly is reduced by a factor 4 for each doubling of the number of time steps which corresponds to an EOC of around 2.")
1 pt for determining right table produced by code from a).

Problem 9 Numerical Methods for Partial Differential Equations [9 pts]

a) Let $u : [a, b] \to \mathbb{R}$ be a 4 times differentiable function and assume that all derivatives are continuous on $[a, b]$.

Show that the central difference operator

$$\partial^+\partial^- u(x) := \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

satisfies

$$u''(x) - \partial^+\partial^- u(x) = \mathcal{O}(h^2) \quad h \to 0.$$

b) On the 2-dimensional unit square $\Omega = (0, 1)^2 \subset \mathbb{R}^2$, consider the two-dimensional Laplace equation

$$\Delta u(x, y) := \partial_{xx} u(x, y) + \partial_{yy} u(x, y) = 0. \tag{2}$$

Assume that $(N + 1)^2$ grid points $\{(x_i, y_j)\}_{i,j}^N$ are defined by uniformly subdividing each axis into N subintervals; that is, for a given double index (i, j), a grid point is given by $(x_i, y_j) = (ih, jh)$ where $h = 1/N$.

Write down the definition of the 5-point stencil used in the finite-difference based discretization of the two-dimensional Laplace operator Δ.

c) On the three-dimensional unit cube $\Omega = (0, 1)^3 \subset \mathbb{R}^3$ consider the three-dimensional Laplace equation

$$\Delta u(x, y, z) := \partial_{xx} u(x, y, z) + \partial_{yy} u(x, y, z) + \partial_{zz} u(x, y, z) = 0. \tag{3}$$

Assume that $(N + 1)^3$ grid points $\{(x_i, y_j, z_k)\}_{i,j,k}^N$ are defined by uniformly subdividing each axis into N subintervals; that is, for a given triple index (i, j, z), a grid point is given by $(x_i, y_j, z_k) = (ih, jh, kh)$ where $h = 1/N$.

Find the the corresponding stencil to discretize the three-dimensional Laplace operator Δ using the finite difference method. Give a *short* rationale of how you arrived at the formula.

Solution

a) Taylor-expand the terms $u(x+h)$ and $u(x-h)$ around x

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)}{2}h^2 + \frac{u^{(3)}}{6!}h^3 + \frac{u^{(4)}(x)}{4!}h^4 \tag{4}$$

$$u(x-h) = u(x) - u'(x)h + \frac{u''(x)}{2}h^2 - \frac{u^{(3)}}{6!}h^3 + \frac{u^{(4)}(x)}{4!}h^4 \tag{5}$$

Adding these two equations gives

$$u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + 2\frac{u^{(4)}(x)}{4!}h^4 \tag{6}$$

$$= 2u(x) + u''(x)h^2 + \mathcal{O}(h^4) \tag{7}$$

Rearranging the last equation yields

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + \mathcal{O}(h^2). \tag{8}$$

b) 5-point stencil is given by

$$\begin{aligned}\Delta u(x_i, y_j) &\approx \partial_x^+\partial_x^- u(x_i, y_j) + \partial_y^+\partial_y^- u(x_i, y_j) \\ &= \frac{u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j-1}) + u(x_i, y_{j+1}) - 4u(x_i, y_j)}{h^2}\end{aligned}$$

c) To obtain the right stencil for the Laplace operator in 3D, we use the same idea as before/in the course when we passed from 1D to 2D: One simply applies a 2nd order central finite difference operator along each coordinate axis to approximate $\partial_{xx}, \partial_{yy}$, and ∂_{zz}, which results in a 7 point stencil

$$\begin{aligned}\Delta u(x_i, y_j, z_k) &\approx \partial_x^+\partial_x^- u(x_i, y_j, z_k + \partial_y^+\partial_y^- u(x_i, y_j, z_k) + \partial_z^+\partial_z^- u(x_i, y_j, z_k) \\ &= \frac{1}{h^2}\Big(u(x_{i+1}, y_j, z_k) + u(x_{i-1}, y_j, z_k) - 2u(x_i, y_j) \\ &\qquad + u(x_i, y_{j-1}, z_k) + u(x_i, y_{j+1}, z_k) - 2u(x_i, y_j) \\ &\qquad + u(x_i, y_j, z_{k-1}) + u(x_i, y_j, z_{k+1}) - 2u(x_i, y_j)\Big) \\ &= \frac{1}{h^2}\Big(u(x_{i+1}, y_j, z_k) + u(x_{i-1}, y_j, z_k) \\ &\qquad + u(x_i, y_{j-1}, z_k) + u(x_i, y_{j+1}, z_k) \\ &\qquad + u(x_i, y_j, z_{k-1}) + u(x_i, y_j, z_{k+1}) - 6u(x_i, y_j, z_k)\Big)\end{aligned}$$

Grading manual

a) (4p):
1 pt for each Taylor expansion, ‘
2 pts for final estimate via inserting and rearranging
-2 pts if Taylor expansions were too short, e.g only $\mathcal{O}(h^3)$ remainder term instead of $\mathcal{O}(h^4)$ term.

b) (2p) Only definition of 5 point stencil is asked for.

c) (3p): 1 pts for short reasoning/rationale (apply 1D FD in each direction) 2 pts for final 7-point stencil

Grading scale and distribution

The exam contained 9 problems in total, the maximum score for each problem is documented in the solution manual above, with a total score of 100 points.

Problem set	1	2	3	4	5	6	7	8	9
Average score	5.7	5.8	7.5	6.2	7.9	4.6	5.5	5.2	2.2
Max points	10	15	12	10	9	15	8	12	9

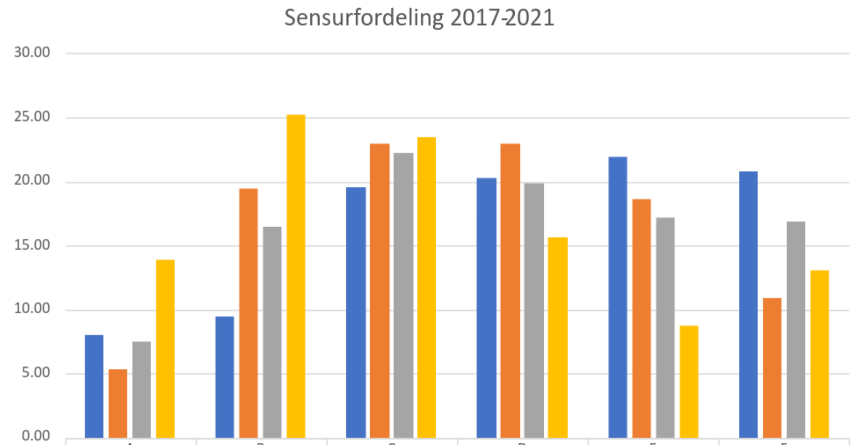
The grading scale used for the assessment of this exam corresponds to the NTNU standard grading scale describe at

`https://i.ntnu.no/wiki/-/wiki/Norsk/Prosentvurderingsmetoden`

but scaled with a factor 0.9:

- A: $\geqslant 80$
- B: $\geqslant 69$
- C: $\geqslant 58$
- D: $\geqslant 47$
- E: $\geqslant 36$
- F: $0 < 36$

The resulting grade distribution is shown below, together with the grade distributions for the autumn semesters 2017-2019. Note that in autumn 2020, a digital home exam with pass/not pass grade scale was conducted and thus the results are not included.



	A	B	C	D	E	F
H2021	8.06	9.43	19.55	20.24	21.96	20.75
H2019	5.34	19.45	22.89	22.89	18.59	10.84
H2018	7.48	16.45	22.24	19.81	17.20	16.82
H2017	13.88	25.20	23.44	15.63	8.77	13.08

H2021 H2019 H2018 H2017

Figure 1: Grade distribution for TMA4130 exam results from autumn 2021 and autumn 2017-2019.