Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for
TMA4130 Mathematics 4N
Solution and grading manual

Academic contact during examination:

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1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

Problem 1 Interpolation [10 pts]

Find the trigonometric polynomial $t(x)$ of the form

$$t(x) = c_0 + c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x,$$

which interpolates the following values

x_i	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
y_i	2	$\sqrt{2}$	0	$\sqrt{2}$	2

.

Solution. We use the direct method. We are looking for a solution to the following linear system of equations:

$$\begin{aligned}
2 &= c_0 + 0 \cdot c_1 - c_2 - c_3 + 0 \cdot c_4 &(1)\\
\sqrt{2} &= c_0 + \frac{\sqrt{2}}{2} c_1 - \frac{\sqrt{2}}{2} c_2 + 0 \cdot c_3 - c_4 &(2)\\
0 &= c_0 + c_1 + 0 \cdot c_2 + c_3 + 0 \cdot c_4 &(3)\\
\sqrt{2} &= c_0 + \frac{\sqrt{2}}{2} c_1 + \frac{\sqrt{2}}{2} c_2 + 0 \cdot c_3 + c_4 &(4)\\
2 &= c_0 + 0 \cdot c_1 + c_2 - c_3 + 0 \cdot c_4 &(5)
\end{aligned}$$

Comparing the equations (1) and (5), we conclude that $c_2 = 0$. Then subtracting (4) from (2) we also see that $c_4 = 0$ and the system is reduced to the following system with three unknowns and three equations

$$\begin{aligned}
2 &= c_0 - c_3\\
\sqrt{2} &= c_0 + \frac{\sqrt{2}}{2} c_1\\
0 &= c_0 + c_1 + c_3
\end{aligned}$$

The solution is $c_0 = 0, c_1 = 2, c_3 = -2$. Then $t(x) = 2 \cos x - 2 \cos 2x$.

Grading manual (10p) in total.

Problem 2 Numerical integration [10 pts]

Although the 2π-periodic function $f(x) = 2^{\sin x}$ can be written as a Fourier series, most of its coefficients a_n, b_n can only be computed numerically. Using the composite Simpson's rule with two subintervals ($[-\pi, 0]$ and $[0, \pi]$), approximate a_0 and b_1.

Solution

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2^{\sin x}\,\mathrm{d}x \\
&\approx \frac{1}{2\pi}\left\{\frac{\pi}{6}\left[2^{\sin(-\pi)} + 4\times 2^{\sin(-\pi/2)} + 2^{\sin 0}\right] + \frac{\pi}{6}\left[2^{\sin 0} + 4\times 2^{\sin(\pi/2)} + 2^{\sin \pi}\right]\right\} \\
&= \frac{1}{12}\left\{\left[2^0 + 4\times 2^{-1} + 2^0\right] + \left[2^0 + 4\times 2^1 + 2^0\right]\right\} \\
&= 7/6 = 1.1666...
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} 2^{\sin x}\sin x\,\mathrm{d}x \\
&\approx \frac{1}{6}\left[2^0\sin(-\pi) + 4\times 2^{-1}\sin(-\pi/2) + 2\times 2^0\sin 0 + 4\times 2^1\sin(\pi/2) + 2^0\sin\pi\right] \\
&= \frac{1}{6}[0 - 2 + 2\times 0 + 8 + 0] \\
&= 1
\end{aligned}
$$

Grading manual (5p) for a_0 and (5p) for b_1. For each coefficient, give

- 2p for having written (stated) the integral correctly.
- 2p for stating the correct formulas for each subinterval.
- 1p for applying the formulas correctly.

Problem 3 Numerics for Nonlinear Equations [10 pts]

Consider the initial value problem

$$y'(t) = \cos(2ty)\,, \quad y(0) = 0\,.$$

Using a certain Runge–Kutta method with time-step size $h = 0.5$, the numerical approximation of $y(t)$ at time $t_{n+1} = (n+1)h$ will be

$$y_{n+1} = y_n + 0.5\cos[(n+1)y_{n+1}]\,.$$

a) Which Runge–Kutta method is being used above?

b) To compute the first time step, we have to solve a nonlinear equation:

$$y_1 = 0.5\cos y_1\,.$$

Renaming y_1 as x, the nonlinear equation can be written as

$$2x - \cos x = 0\,. \tag{6}$$

Using Newton's method to solve Eq. (6) numerically, with initial guess $x_0 = 0$, iterate until the absolute difference between two consecutive iterations is less than 0.001 (in other words, iterate until $|x_{k+1} - x_k| < 10^{-3}$).

Solution

a) This is the implicit Euler method.

b) Newton's method for this equation will be

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k + \frac{\cos x_k - 2x_k}{\sin x_k + 2}$$

Therefore, we have

$$i)\;\; x_1 = 0 + \frac{\cos 0 - 2\times 0}{\sin 0 + 2} = 0.5$$

$$ii)\;\; x_2 = 0.5 + \frac{\cos(0.5) - 2\times 0.5}{\sin(0.5) + 2} \approx 0.4506$$

$$iii)\;\; x_3 \approx 0.4506 + \frac{\cos(0.4506) - 2\times 0.4506}{\sin(0.4506) + 2} \approx 0.4502$$

Since $|x_3 - x_2| \approx 0.4 \times 10^{-3}$, we need not iterate any further.

Grading manual (2p) for **a)**, (8p) for **b)**. Out of those 8p, give

- 5p for writing the correct expression for the iterations (with a generic x_k)
- 1p for each correct iteration.

Problem 4 Laplace transform [10 pts]

a) Solve the following ordinary differential equation using the Laplace transform:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y(t) = 2e^{2t}.$$

Show all the steps in your solution, including the application of the Laplace transform and shifting theorems (if needed), and clearly state the inverse transform to obtain the final solution for $y(t)$. Note that your solution $y(t)$ should contain the (at this point unknown) initial conditions.

b) Given the initial conditions $y(0) = 1$, $y'(0) = 3$, find the solution to the differential equation you solved in part **a)**.

Solution

a) Take the Laplace transform of both sides of the equation:

$$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = \mathcal{L}(2e^{2t}) = \frac{2}{s-2}.$$

Solving for $Y(s)$ gives:

$$Y(s)\underbrace{\left(s^2 - 4s + 4\right)}_{=(s-2)^2} - sy(0) - y'(0) + 4y(0) = \frac{2}{s-2}.$$

$$\begin{aligned}
\implies Y(s) &= \frac{(s-2)y(0) + y'(0) - 2y(0)}{(s-2)^2} + \frac{2}{(s-2)^3} \\
&= \frac{y(0)}{s-2} + \frac{y'(0) - 2y(0)}{(s-2)^2} + \frac{2}{(s-2)^3} \\
\implies y(t) &= y(0)e^{2t} + (y'(0) - 2y(0))te^{2t} + \frac{2t^2e^{2t}}{2} \\
&= (y(0) + ty'(0) - 2ty(0) + t^2)e^{2t}.
\end{aligned}$$

b) We get

$$y(t) = (y(0) + ty'(0) - 2ty(0) + t^2)e^{2t} = (1 + 3t - 2t + t^2)e^{2t} = (1 + t + t^2)e^{2t}.$$

Grading manual (9p) for **a)**, (1p) for **b)**.

Problem 5 Numerical methods for ODEs [10 pts]

```
1  import numpy as np
2
3  def F(t,y):
4      return np.array([ y[1], -y[0] ])
5
6  t0, tend = 0.0, 20.0
7  y0 = np.array([1,1])
8  h=0.01
9  N = int(np.round((tend-t0)/h))
10 h = (tend-t0)/N
11 Y = np.zeros((2,N+1))
12 Y[:,0] = y0
13 T = np.linspace(t0, tend, N+1)
14
15 for k in range(N):
16     K1 = F(T[k],Y[:,k])
17     Y[:,k+1] = Y[:,k]+ h/2*(K1 + F(T[k+1], Y[:,k] + h*K1))
```

a) The code above was written for approximating the solution to some initial value problem. Determine which initial value problem it is, and the method which is used.

b) A second order method was used to solve the problem in the code above. The step size used was $h = 0.01$ and the norm of the error at $t = 20$ turned out to be $\approx 0.47 \cdot 10^{-3}$. Give an estimate for how big the norm of the error would have been at $t = 20$ if we had chosen $h = 0.05$ instead.

Solution. As can be seen from the code lines 16–17, the numerical method is a Runge–Kutta method.

(a) The first stage is $K_1 = F(t_k, y_k)$ (line 16) and the second can be read out of the last term in the right hand side of the statement on line 17: $K_2 = F(t_{k+1}, y_k + hK_1)$. The update is $y_{k+1} = y_k + \frac{h}{2}(K_1 + K_2)$ so clearly we have found the modified Euler method which has Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

The ODE solved is given by the function `F(t,y)` where we read off the equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}$$

This is nothing but the harmonic oscillator. From lines 6–7 we read the initial values $y_1(0) = y_2(0) = 1$.

(b) The modified Euler method is a second order method, which means that the global error behaves approximately as Ch^2. We infer from the given information that $Ch^2 = C(0.01)^2 \approx 0.47 \cdot 10^{-3}$ therefore $C \approx 4.7$. Then if we use $h = 0.05$ we get the error $e \approx C(0.05)^2 \approx 0.1175 \cdot 10^{-1}$

Grading manual (5p) for **a)**, (5p) for **b)**.

Problem 6 Fourier series [10 pts]

Calculate the Fourier series of the function $f(x) = |x^3|$ defined on $[-\pi, \pi]$. Explicitly write down the first five non-vanishing terms of the Fourier sum. **Hint:** Note that $|x^3|$ is an even function. The formulas below, where $\alpha \in \mathbb{R}$, may also be useful:

$$\int x^3 \sin(\alpha x)\,\mathrm{d}x = \frac{3(\alpha^2x^2 - 2)\sin(\alpha x) - \alpha x(\alpha^2 x^2 - 6)\cos(\alpha x)}{\alpha^4} + \mathit{constant}$$

$$\int x^3 \cos(\alpha x)\,\mathrm{d}x = \frac{\alpha x(\alpha^2x^2 - 6)\sin(\alpha x) + 3(\alpha^2 x^2 - 2)\cos(\alpha x)}{\alpha^4} + \mathit{constant}$$

Solution Since x^3 is an odd function, the function f is even and can be expressed by cosine-terms only (i.e., $b_n = 0$ for all n). To get rid of the absolute-value, we treat the problem as an even half-range expansion of x^3. The Fourier coefficients are now given by

$$a_0 = \frac{1}{\pi}\int_0^\pi x^3 dx = \frac{1}{\pi}\cdot\left[\frac{1}{4}x^4\right]_0^\pi = \frac{\pi^4}{4\pi} = \frac{\pi^3}{4}\,,$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x^3 \cdot \cos(nx)dx \\
&= \frac{2}{\pi}\left(\underbrace{\left[x^3\frac{1}{n}\sin(nx)\right]_0^\pi}_{=0} - \int_0^\pi 3x^2\frac{1}{n}\sin(nx)dx\right) \\
&= -\frac{6}{\pi n}\int_0^\pi x^2\sin(nx)dx \\
&= -\frac{6}{\pi n}\left(\left[x^2\frac{1}{n}(-\cos(nx))\right]_0^\pi - \int_0^\pi 2x\frac{1}{n}(-\cos(nx))dx\right) \\
&= \frac{6\pi}{n^2}\cos(n\pi) - \frac{12}{\pi n^2}\int_0^\pi x\cos(nx)dx \\
&= \frac{6\pi}{n^2}\cos(n\pi) - \frac{12}{\pi n^2}\left(\underbrace{\left[\frac{x}{n}\sin(nx)\right]_0^\pi}_{=0} - \int_0^\pi \frac{1}{n}\sin(nx)dx\right) \\
&= \frac{6\pi}{n^2}\cos(n\pi) + \frac{12}{\pi n^3}\left[\frac{1}{n}(-\cos(nx))\right]_0^\pi \\
&= \frac{6\pi}{n^2}\cos(n\pi) - \frac{12}{\pi n^4}(\cos(n\pi) - \cos(0)) \\
&= \begin{cases} -\frac{6\pi}{n^2} - \frac{12}{\pi n^4}(-1-1) = -\frac{6\pi}{n^2} + \frac{24}{\pi n^4}, & n \text{ odd} \\ \frac{6\pi}{n^2} + \frac{12}{\pi n^4}\cdot 0 = \frac{6\pi}{n^2}, & n \text{ even} \end{cases}
\end{aligned}
$$

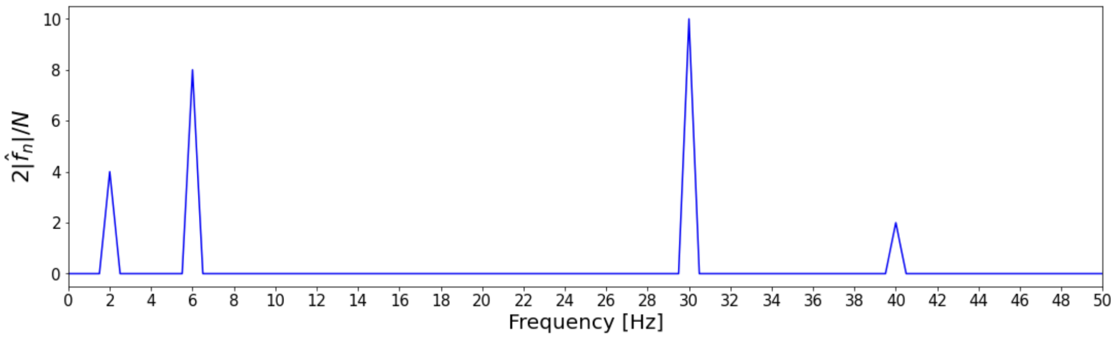
So, the Fourier series is given by

$$
\begin{aligned}
f(x) &\sim \frac{\pi^3}{4} + \sum_{n=1}^{\infty}\left(-\frac{6\pi}{n^2}\cos(n\pi) - \frac{12}{\pi n^4}(\cos(n\pi) - \cos(0))\right) \\
&= \frac{\pi^3}{4} + \underbrace{\left(-6\pi + \frac{24}{\pi}\right)\cdot\cos(x)}_{n=1} + \underbrace{\left(\frac{3\pi}{2}\right)\cdot\cos(2x)}_{n=2} + \underbrace{\left(-\frac{6\pi}{9} + \frac{24}{\pi 3^4}\right)\cdot\cos(3x)}_{n=3} \\
&\qquad + \underbrace{\left(\frac{6\pi}{16}\right)\cdot\cos(4x)}_{n=4} + \dots
\end{aligned}
$$

Grading manual (3p) for a_0, (5p) for a_n and (2p) for the formulation of the Fourier series.

Problem 7 Discrete Fourier Transform, 4D,[10 pts]

The spectrum of a certain signal f, sampled at regular time intervals, is shown below (as usual, the "mirrored" half of the spectrum is omitted from the plot):



The following Python code computes that spectrum using the `fft` function, then applies a certain type of filter to the original signal:

```
import numpy as np

N = f.size #(the signal f was pre-computed elsewhere)
dt = 0.001 #spacing between two samples, in seconds
L = N*dt #length of the sampling interval, in seconds

n = np.arange(N) #vector of indices n = 0,1,...,N-1
nShifted = N/2-np.abs(n-N/2) #vector of shifted indices
shiftedFreq=nShifted/L #shift frequencies to account for mirroring

cutoffFreq = 25 #in Hertz
DFT = np.fft.fft(f)
DFT[shiftedFreq < cutoffFreq] = 0
g = np.fft.ifft(DFT)
```

a) What kind of filter is implemented above?

b) Among the functions below, which one could best represent the *filtered* signal?

i) $g(t) = 4\cos(4\pi t) + 8\sin(12\pi t)$

ii) $g(t) = \cos(4\pi t) + \sin(12\pi t)$

iii) $g(t) = \cos(60\pi t) + \sin(80\pi t)$

iv) $g(t) = 10\cos(60\pi t) + 2\sin(80\pi t)$

Remember: if t is measured in seconds, and $\alpha > 0$, then the frequency (in Hertz) of a function such as $\cos(2\pi\alpha t)$ or $\sin(2\pi\alpha t)$ is simply α.

Solution

a) This is a **high-pass** filter, since we are setting to zero all components whose frequencies are **below** the cut-off frequency 25 Hz.

b) The first two options, *i* and *ii*, can be immediately excluded because they have low frequencies that would be filtered out. Option *iii* shows the right frequencies, but both with the same amplitude, which is clearly not the case when looking at the spectrum. That leaves us with the **correct answer** *iv*, which shows the 30 Hz coefficient fives times larger, in magnitude, than the 40 Hz coefficient, as in the spectrum.

Grading manual (5p) for **a)**, (5p) for **b)**. However,

- if the student answered "low-pass" in a) and then picked alternative *i*) in b), **with a proper justification**, give 4p.

Problem 7 **Fourier transform [10 pts], 4N**

Let

$$g(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}.$$

a) Compute the Fourier transform of g.

b) Compute the Fourier transform of $f = g * g$.

Solution a) We use the definition to compute the Fourier transform of g:

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) e^{-iwx} dx.$$

We note that g is an even function and

$$\hat{g}(w) = \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x) \cos wx dx = \sqrt{\frac{2}{\pi}} \int_0^1 \cos wx dx - \sqrt{\frac{2}{\pi}} \int_0^1 x \cos wx dx.$$

We compute the second integral integrating by parts:

$$\int_0^1 x \cos wx dx = x \frac{\sin wx}{w} |_0^1 - \int_0^1 \frac{\sin wx}{w} dx = \frac{\sin w}{w} + \frac{\cos wx}{w^2} |_0^1 = \frac{\sin w}{w} - \frac{1 - \cos w}{w^2}.$$

Then

$$\hat{g}(w) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin wx}{w} |_0^1 - \frac{\sin w}{w} + \frac{1 - \cos w}{w^2} \right) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos w}{w^2}.$$

b) We know that $\hat{f} = \mathcal{F}(g * g) = \sqrt{2\pi} \hat{f}^2$. Therefore

$$\hat{g}(w) = \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{(1 - \cos w)^2}{w^4}.$$

Grading manual (8p) for **a)**, (2p) for **b)**.

Problem 8 Wave Equation [10 pts]

Use the d'Alembert formula to find the solution to the wave equation $u_{tt} = 9u_{xx}$ with initial conditions $u(0,x) = \sin x$, $u_t(0,x) = \cos x$

Solution The d'Alembert formula gives

$$u(t,x) = \frac{1}{2}(\sin(x+3t) + \sin(x-3t)) + \frac{1}{6}\int_{x-3t}^{x+3t} \cos y dy.$$

We compute the integral and obtain

$$\begin{aligned} u(t,x) = \frac{1}{2}(\sin(x+3t) + \sin(x-3t)) + \frac{1}{6}(\sin(x+3t) - \sin(x-3t) \\ = \frac{2}{3}\sin(x+3t) + \frac{1}{3}\sin(x-3t). \end{aligned}$$

Grading manual (3p) for d'Alembert's formula, (7p) for computing the integral and u.

Problem 9 Heat Equation [10 pts]

For the one-dimensional heat equation $u_t = u_{xx}$ on $[0, \infty) \times [0, 2]$, use the method of separation of variables to find the solution satisfying the Neumann boundary conditions $u_x(t, 0) = u_x(t, 2) = 0$ and the initial condition $u(0, x) = 1 - \cos 3\pi x$.

Solution First we look for solutions of the form $u(t, x) = G(t)F(x)$. We obtain two separate ODEs, $F'' = -kF$ and $G' = -kG$.

The boundary conditions imply that we are looking for a solution with $F'(0) = F'(2) = 0$. When $k = 0$ we get a constant solution $F_0 = C_0$ and $G_0 = 1$, and $u_0(t, x) = C_0$. When $k = w^2 > 0$ we obtain $F(x) = C_1 \cos wx + C_2 \sin wx$ and $F'(x) = -wC_1 \sin wx + C_2 w \cos wx$. The condition $F'(0) = F'(2) = 0$ implies that $C_2 = 0$ and if $F \neq 0$ then $w = n\pi/2$ for some $n = 1, 2, \ldots$. We have $w_n = n\pi/2$, $F_n = C \cos n\pi x/2$. For $k = -\kappa^2 < 0$ we have $F(x) = C_1 e^{\kappa x} + C_2 e^{-\kappa x}$, $F'(x) = C_1 \kappa e^{\kappa x} - C_2 \kappa e^{-\kappa x}$ and the condition $F'(0) = F'(2) = 0$ implies that $F' = 0$. We obtain only the trivial solution.

For the values $w_n = n\pi/2$, we solve the second equation $G' = -w_n^2 G$ and get $G_n(t) = \exp(-w_n^2 t)$. Then solutions of the form $u(t, x) = G(t)F(x)$ are given by

$$u_n(t, x) = \cos(n\pi x/2) \exp(-n^2\pi^2 t/4).$$

The superposition principles implies that linear combinations of such solutions is also a solution

$$u(t, x) = c_0 + \sum_n c_n \cos(n\pi x/2) \exp(-n^2\pi^2 t/4).$$

Our initial condition $u(0, x) = 1 - \cos 3\pi x$ is satisfied if we choose $c_0 = 1$ and $c_6 = -1$ while all other coefficients are zeros. We obtain the following solution

$$u(t, x) = 1 - \cos(3\pi x) \exp(-9\pi^2 t).$$

Grading manual (10p).

Overall grading scheme:

A	80-90
B	68–79
C	56–67
D	44–55
E	33–43
F	0–32