Math 241 Quiz #11 version: 1

Instructor: Wenqiang Feng Name: ______________________

(1) (5 points) Evaluate $\int\int_S dS$ where the surface S is given by $z = 1 - x^2 - y^2$ $(z \geq 0)$.

① Find out the graph formula

$$\mathbf{G}(x, y) = (x, y, g(x, y)) = (x, y, 1 - x^2 - y^2)$$

② Then compute the tangent vector

$$\begin{cases} T_x = \frac{\partial \mathbf{G}}{\partial x} = \langle 1, 0, -2x \rangle, \\ T_y = \frac{\partial \mathbf{G}}{\partial y} = \langle 0, 1, -2y \rangle. \end{cases}$$

③ Compute the normal vector and its magnitude

$$\begin{aligned} \mathbf{n} &= \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} \\ &= \begin{vmatrix} 0 & -2x \\ 1 & -2y \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2x \\ 0 & -2y \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} = \langle 2x, 2y, 1 \rangle. \end{aligned}$$

So the corresponding magnitude of norm vector is as follows :

$$\|\mathbf{n}\| = \sqrt{1 + 4x^2 + 4y^2}$$

④ Plug in to the surface integral formula

$$\int\int_S dS = \int\int_D 1 \cdot \sqrt{1 + 4x^2 + 4y^2} dA$$

⑤ Determine the domain D. (Project the surface onto xy- plane, i.e. let $z = 0$ Fig.(1))

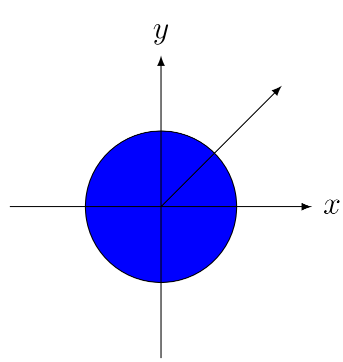

FIGURE 1. The projection of the surface onto xy-plane.

$$x^2 + y^2 \leq 1. (z \geq 0)$$

So

$$\begin{aligned}
\int\int_S dS &= \int\int_D 1 \cdot \sqrt{1+4x^2+4y^2}dA \\
&= \int_0^1 \int_0^{2\pi} \sqrt{1+4r^2} \cdot r d\theta dr \\
&= 2\pi \int_0^1 \sqrt{1+4r^2} \cdot r dr \\
&= \frac{2\pi}{8} \int_1^5 u^{1/2} du \\
&= \frac{2\pi}{8} \left(\frac{2}{3} u^{3/2} \right) \Big|_1^5 \\
&= \frac{\pi}{6} \left(5^{3/2} - 1 \right)
\end{aligned}$$

(2) (5 points) $\int\int_S \mathbf{F} d\mathbf{S}$, where $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the surface S is given by $x+y+z=1$ $(x,y,z \geq 0)$ with upward orientation.

1 Find out the graph formula

$$\mathbf{G}(x,y) = (x,y,g(x,y)) = (x,y,1-x-y)$$

2 Then compute the tangent vector

$$\begin{cases} T_x = \frac{\partial \mathbf{G}}{\partial x} = \langle 1,0,-1 \rangle, \\ T_y = \frac{\partial \mathbf{G}}{\partial y} = \langle 0,1,-1 \rangle. \end{cases}$$

3 Compute the normal vector

$$\begin{aligned}
\mathbf{n} &= \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1,1,1 \rangle.
\end{aligned}$$

4 Plug in to the surface integral formula

$$\begin{aligned}
\int\int_S \mathbf{F} d\mathbf{S} &= \int\int_D \mathbf{F} \cdot \mathbf{n} dA \\
&= \int\int_D \langle x,y,1-x-y \rangle \cdot \langle 1,1,1 \rangle dA \\
&= \int\int_D 1 dA
\end{aligned}$$

5 Determine the domain D. (Project the surface onto xy- plane, i.e. let $z=0$ Fig.(2))

$$x+y=1 (x \geq 0, y \geq 0)$$

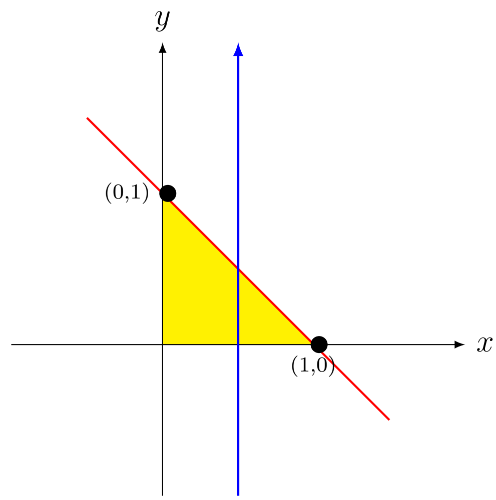

FIGURE 2. The domain of the projection region.

So,

$$
\begin{aligned}
\int\int_S \mathbf{F} d\mathbf{S} &= \int\int_D \mathbf{F}\cdot\mathbf{n} dA \\
&= \int\int_D \langle x, y, 1-x-y\rangle \cdot \langle 1,1,1\rangle\, dA \\
&= \int\int_D 1 dA \\
&= \int_0^1 \int_0^{1-x} dy dx = \frac{1}{2}.
\end{aligned}
$$

Math 241 Quiz #11 version: 2

Instructor: Wenqiang Feng Name: ____________________

(1) (5 points) Evaluate $\int\int_S zdS$ where the surface S is given by $x+y+z=1$ $(x,y,z\geq 0)$.

1 Find out the graph formula

$$\mathbf{G}(x,y)=(x,y,g(x,y))=(x,y,1-x-y)$$

2 Then compute the tangent vector

$$\begin{cases} T_x = \frac{\partial \mathbf{G}}{\partial x} = \langle 1,0,-1\rangle, \\ T_y = \frac{\partial \mathbf{G}}{\partial y} = \langle 0,1,-1\rangle. \end{cases}$$

3 Compute the normal vector and its magnitude

$$\begin{aligned} \mathbf{n} &= \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} \\ &= \mathbf{i}+\mathbf{j}+\mathbf{k} = \langle 1,1,1\rangle. \end{aligned}$$

So the corresponding magnitude of norm vector is as follows :

$$\|\mathbf{n}\| = \sqrt{1^2+1^2+1^2}=\sqrt{3}.$$

4 Plug in to the surface integral formula

$$\int\int_S dS = \int\int_D (1-x-y)\cdot\sqrt{3}dA$$

5 Determine the domain D. (Project the surface onto xy- plane, i.e. let $z=0$ Fig.(3))

$$x+y\leq 1.(x,y\geq 0)$$

So

$$\begin{aligned} \int\int_S dS &= \int\int_D (1-x-y)\cdot\sqrt{3}dA \\ &= \sqrt{3}\int_0^1\int_0^{1-x}(1-x-y)\,dydx \\ &= \sqrt{3}\int_0^1 \left((1-x)y-\frac{1}{2}y^2\right)\Big|_0^{1-x} dx \\ &= \frac{\sqrt{3}}{2}\int_0^1 (1-x)^2dx \\ &= \frac{\sqrt{3}}{2}\int_0^1 1-2x+x^2dx \\ &= \frac{\sqrt{3}}{2}\left(x-x^2+\frac{1}{3}x^3\right)\Big|_0^1 \end{aligned}$$

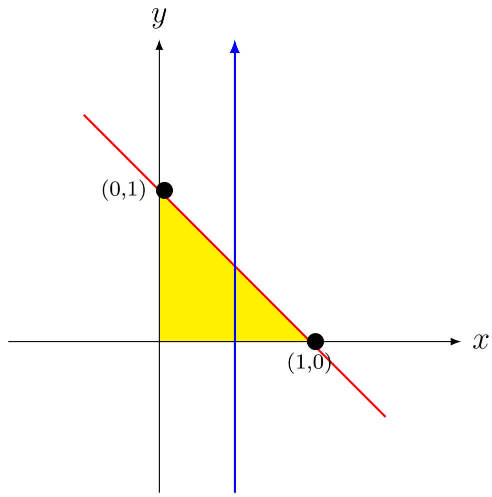

FIGURE 3. The domain of the projection region.

$$= \frac{\sqrt{3}}{2}(1 - 1 + \frac{1}{3}) = \frac{\sqrt{3}}{6}.$$

(2) (5 points) $\int\int_S \mathbf{F} d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the surface S is given by $z = 1 - x^2 - y^2$ $(z \geq 0)$ with upward orientation.

1 Find out the graph formula

$$\mathbf{G}(x, y) = (x, y, g(x, y)) = (x, y, 1 - x^2 - y^2)$$

2 Then compute the tangent vector

$$\begin{cases} T_x = \frac{\partial \mathbf{G}}{\partial x} = \langle 1, 0, -2x \rangle, \\ T_y = \frac{\partial \mathbf{G}}{\partial y} = \langle 0, 1, -2y \rangle. \end{cases}$$

3 Compute the normal vector

$$\begin{aligned} \mathbf{n} &= \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} \\ &= \begin{vmatrix} 0 & -2x \\ 1 & -2y \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2x \\ 0 & -2y \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} = \langle 2x, 2y, 1 \rangle. \end{aligned}$$

4 Plug in to the surface integral formula

$$\begin{aligned} \int\int_S \mathbf{F} d\mathbf{S} &= \int\int_D \mathbf{F} \cdot \mathbf{n} dA \\ &= \int\int_D \left\langle x, y, 1 - x^2 - y^2 \right\rangle \cdot \langle 2x, 2y, 1 \rangle \, dA \\ &= \int\int_D 1 + x^2 + y^2 dA \end{aligned}$$

5 Determine the domain D. (Project the surface onto xy- plane, i.e. let $z = 0$ Fig.(4))

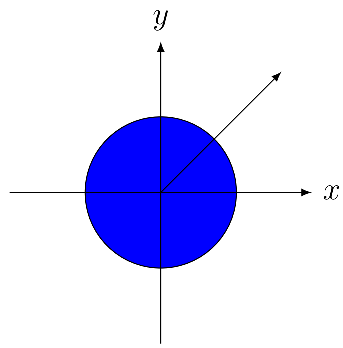

FIGURE 4. The projection of the surface onto xy-plane.

$$x^2 + y^2 \leq 1(\text{since} z \geq 0)$$

So,

$$\begin{aligned}
\int\int_S \mathbf{F} d\mathbf{S} &= \int\int_D \mathbf{F} \cdot \mathbf{n} dA \\
&= \int\int_D \left\langle x, y, 1 - x^2 - y^2 \right\rangle \cdot \langle 2x, 2y, 1 \rangle \, dA \\
&= \int\int_D 1 + x^2 + y^2 dA \\
&= \int_0^1 \int_0^{2\pi} \left(1 + r^2\right) \cdot r d\theta dr \\
&= \int_0^1 \int_0^{2\pi} r + r^3 d\theta dr \\
&= 2\pi \int_0^1 r + r^3 dr \\
&= 2\pi \int_0^1 \left(\frac{1}{2} r^2 + \frac{1}{4} r^4\right)\Big|_0^1 \\
&= 2\pi \left(\frac{1}{2} + \frac{1}{4}\right) \\
&= \frac{3\pi}{2}.
\end{aligned}$$