Math 241 Quiz #3 version: 1

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(1) (5 points) Find the curvature function $\kappa(t)$ for the curve $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^2\mathbf{k}$.

Solution. Since $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^2\mathbf{k}$, then $\mathbf{r}(t) = \langle t, 2t^2, t^2\rangle$.

1 Compte the 1_{st} derivative of $\mathbf{r}(t)$

$$\mathbf{r}'(t) = \langle 1, 4t, 2t\rangle$$

2 Compte the 2_{nd} derivative of $\mathbf{r}(t)$

$$\mathbf{r}''(t) = \langle 0, 4, 2\rangle$$

3 Compte the cross-product $\mathbf{r}'(t) \times \mathbf{r}''(t)$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4t & 2t \\ 0 & 4 & 2 \end{vmatrix} \\ &= (8t - 8t)\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \\ &= -2\mathbf{j} + 4\mathbf{k}.\end{aligned}$$

4 Compte the lengths:

$$\|\mathbf{r}'(t)\| = \sqrt{1^2 + (4t)^2 + (2t)^2} = \sqrt{1 + 20t^2}.$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{(-2)^2 + 4^2} = \sqrt{20} = 2\sqrt{5}.$$

5 Compute the curvature

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{2\sqrt{5}}{(1 + 20t^2)^{3/2}}.$$

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(2) (5 points) Given the acceleration $\mathbf{a}(t) = t\mathbf{i} + t^2\mathbf{k}$, initial velocity $\mathbf{v}(0) = \mathbf{k}$ and initial position $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$ of a moving particle, find the position function $\mathbf{r}(t)$.

Solution. From the physical meaning, we know that

$$\begin{cases} \mathbf{v}(t) = \mathbf{r}'(t), \\ \mathbf{a}(t) = \mathbf{r}''(t). \end{cases} \Rightarrow \begin{cases} \mathbf{r}(t) = \int \mathbf{v}(t)dt + \mathbf{r}_0, \\ \mathbf{v}(t) = \mathbf{r}'(t) = \int \mathbf{a}(t)dt + \mathbf{v}_0. \end{cases}$$

1 Compte speed $\mathbf{v}(t)$

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = \int \mathbf{a}(t)dt + \mathbf{v}_0 \\ &= \int t\mathbf{i} + t^2\mathbf{k}dt + \mathbf{v}_0 \\ &= \frac{1}{2}t^2\mathbf{i} + \frac{1}{3}t^3\mathbf{k} + \mathbf{k}.\end{aligned}$$

2 Compute the path

$$\begin{aligned}
\mathbf{r}(t) &= \int \mathbf{v}(t) dt + \mathbf{r}_0 \\
&= \int \frac{1}{2}t^2\mathbf{i} + \frac{1}{3}t^3\mathbf{k} + \mathbf{k} dt + \mathbf{r}_0 \\
&= \frac{1}{6}t^3\mathbf{i} + \frac{1}{12}t^4\mathbf{k} + t\mathbf{k} + \mathbf{i} + \mathbf{j} \\
&= \left(\frac{1}{6}t^3 + 1\right)\mathbf{i} + \mathbf{j} + \left(\frac{1}{12}t^4 + t\right)\mathbf{k}.
\end{aligned}$$

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