Dual-Wind Discontinuous Galerkin Method Implementation Details *†

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Abstract

This is my Dual-Wind Discontinuous Galerkin Method Implementation note for [2]. You may get the details of the implementation and the integral checking boards for monomial basis function on the reference elements. Please be aware, however, that the note contains typos as well as incorrect or inaccurate solutions .

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1 The affine mapping between physical elements and the reference elements

1.1 Interval Element

Let $\hat{K}$ spanned by $\hat{A}_1 = 0$, $\hat{A}_2 = 1$ be the reference triangle (Figure. 1).

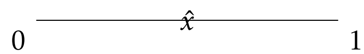

Figure 1: The interval reference element.

For a given physical element $K \in \mathcal{T}_h$, we treat it as the image of $\hat{K}$ under the affine map (Figure.2):

$$F : \hat{K} \to K.$$

If K has vertices x_i, x_{i+1}, then the map F can be defined by

$$\mathbf{x} = F(\hat{\mathbf{x}}) = h_i \hat{\mathbf{x}} + \mathbf{x}_i,$$

where

$$h_i = x_{i+1} - x_i.$$

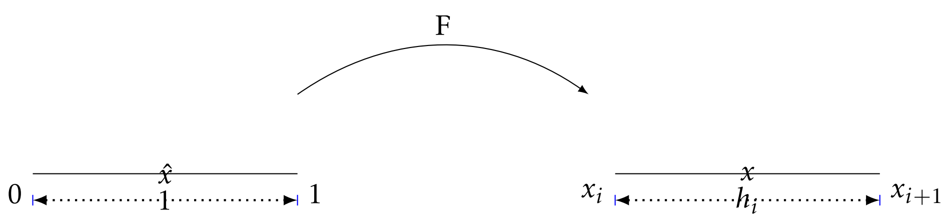

Figure 2: The affine mapping of interval element.

1.2 Triangular Element

Let $\hat{K}$ spanned by $\hat{A}_1 = (0,0)$, $\hat{A}_2 = (1,0)$ and $\hat{A}_3 = (0,1)$ be the reference triangle (Figure. 3) and $\hat{\mathbf{x}} = (\hat{x}, \hat{y})^T$.

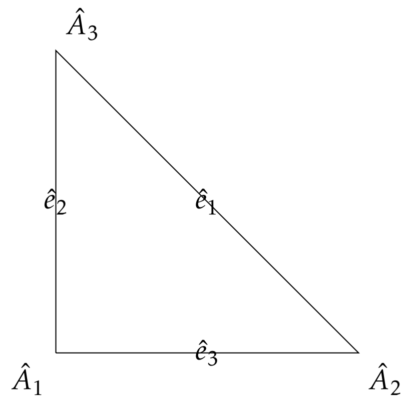

Figure 3: The triangular reference element.

For a given physical element $K \in \mathcal{T}_h$, we treat it as the image of $\hat{K}$ under the affine map (Figure.4):

$$F : \hat{K} \to K.$$

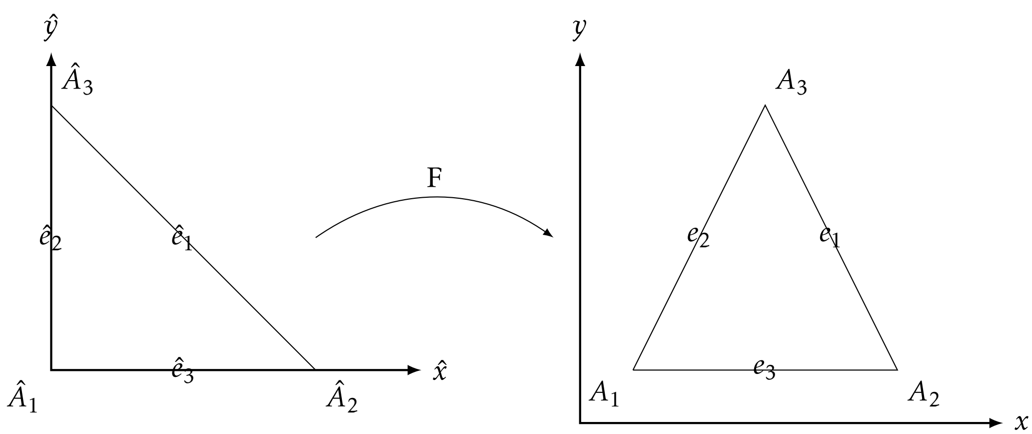

Figure 4: The affine mapping of triangular element.

If K has vertices $A_i(x_i, y_i), i = 1, 2, 3$, then the map F can be defined by

$$\mathbf{x} = F(\hat{\mathbf{x}}) = \mathbf{B}^T(\hat{\mathbf{x}}) + \mathbf{c},$$

where

$$\mathbf{B} = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

1.3 Rectangular Element

Let $\hat{K}$ spanned by $\hat{A}_1 = (-1, -1)$, $\hat{A}_2 = (1, -1)$, $\hat{A}_3 = (1, 1)$ and $\hat{A}_4 = (-1, 1)$ be the reference rectangular (Figure. 5) and $\hat{\mathbf{x}} = (\hat{x}, \hat{y})^T$.

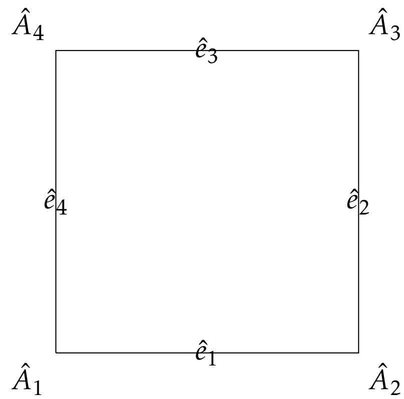

Figure 5: The rectangular reference element.

For a given physical element $K \in \mathcal{T}_h$, we treat it as the image of $\hat{K}$ under the affine map (Figure.6):

$$F : \hat{K} \to K.$$

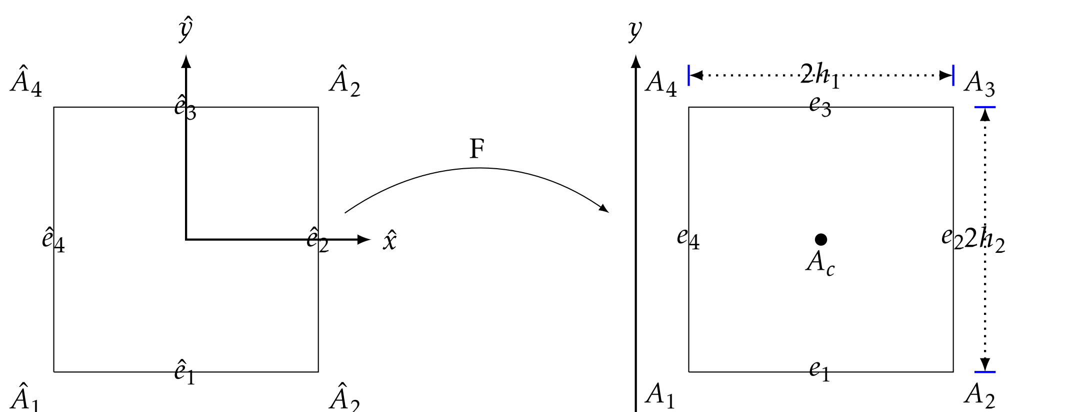

Figure 6: The affine mapping of rectangular element.

If K has vertices $A_i(x_i, y_i), i = 1, 2, 3, 4$, then the map F can be defined by

$$\mathbf{x} = F(\hat{\mathbf{x}}) = \mathbf{B}^T(\hat{\mathbf{x}}) + \mathbf{c},$$

where

$$\mathbf{B} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} x_c \\ y_c \end{pmatrix}.$$

2 The volume and face integrals on physical elements and the reference elements

From now on, we will focus on rectangular element. We define $\hat{u}(\hat{\mathbf{x}}) = u(F(\hat{\mathbf{x}})) = u(\mathbf{x})$. Then $\hat{\nabla}\hat{u} = B\nabla u$ and $dxdy = |det(B)|d\hat{x}d\hat{y}$. Let $\{\hat{\phi}_j\}_{j=1}^{n_r}$ and $\{\phi_j\}_{j=1}^{n_r}$be the basis function of $\mathbb{P}_r\hat{K}$ and $\mathbb{P}_r K$, respectively. Where $n_r = dim\mathbb{P}_r = \begin{pmatrix} d+r \\ r \end{pmatrix}$. So, we can change the computation of the integral for the basis function in K to $\hat{K}$ by

1. Volume integral $\left(\phi_j(\mathbf{x}), \phi_k(\mathbf{x})\right)_K$

$$\begin{aligned}(M_K)_{k,j} = \int_K \phi_j(\mathbf{x}) \cdot \phi_k(\mathbf{x}) dxdy &= \int_{\hat{K}} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) |det(B)| d\hat{x}d\hat{y} \\ &= |det(B)| \int_{\hat{K}} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) d\hat{x}d\hat{y}.\end{aligned}$$

2. Volume integral $\left(\phi_j(\mathbf{x}), \partial_x\phi_k(\mathbf{x})\right)_K$

$$\begin{aligned}(M_{K_x})_{k,j} = \int_K \phi_j(\mathbf{x}) \cdot \partial_x\phi_k(\mathbf{x}) dxdy &= \int_{\hat{K}} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot h_1^{-1} \hat{\partial}_{\hat{x}}\hat{\phi}_k(\hat{\mathbf{x}}) |det(B)| d\hat{x}d\hat{y} \\ &= h_2 \int_{\hat{K}} \hat{\phi}_j(\hat{\mathbf{x}}) \hat{\partial}_{\hat{x}}\hat{\phi}_k(\hat{\mathbf{x}}) d\hat{x}d\hat{y}.\end{aligned}$$

3. Volume integral $\left(\phi_j(\mathbf{x}), \partial_y\phi_k(\mathbf{x})\right)_K$

$$\begin{aligned}(M_{K_x})_{k,j} = \int_K \phi_j(\mathbf{x}) \cdot \partial_y\phi_k(\mathbf{x}) dxdy &= \int_{\hat{K}} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot h_2^{-1} \hat{\partial}_{\hat{y}}\hat{\phi}_k(\hat{\mathbf{x}}) |det(B)| d\hat{x}d\hat{y} \\ &= h_1 \int_{\hat{K}} \hat{\phi}_j(\hat{\mathbf{x}}) \hat{\partial}_{\hat{y}}\hat{\phi}_k(\hat{\mathbf{x}}) d\hat{x}d\hat{y}.\end{aligned}$$

4. Face integral $\left\langle\phi_j(\mathbf{x}), \phi_k(\mathbf{x})\right\rangle_{\partial K}$

   (a) Face integral on e_1

$$\begin{aligned}(M_{K_{e_1}})_{k,j} = \int_{e_1} \phi_j(\mathbf{x}) \cdot \phi_k(\mathbf{x}) dx &= \int_{\hat{e}_1} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) h_1 d\hat{x} \\ &= h_1 \int_{\hat{e}_1} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) d\hat{x}.\end{aligned}$$

   (b) Face integral on e_2

$$\begin{aligned}(M_{K_{e_2}})_{k,j} = \int_{e_2} \phi_j(\mathbf{x}) \cdot \phi_k(\mathbf{x}) dy &= \int_{\hat{e}_2} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) h_2 d\hat{y} \\ &= h_2 \int_{\hat{e}_2} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) d\hat{y}.\end{aligned}$$

   (c) Face integral on e_3

$$\begin{aligned}(M_{K_{e_3}})_{k,j} = \int_{e_3} \phi_j(\mathbf{x}) \cdot \phi_k(\mathbf{x}) dx &= \int_{\hat{e}_3} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) h_1 d\hat{x} \\ &= h_1 \int_{\hat{e}_3} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) d\hat{x}.\end{aligned}$$

(d) Face integral on e_4

$$
\begin{aligned}
(M_{K_{e_4}})_{k,j} = \int_{e_4} \phi_j(\mathbf{x}) \cdot \phi_k(\mathbf{x}) dy &= \int_{\hat{e}_4} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) h_2 d\hat{y} \\
&= h_2 \int_{\hat{e}_4} \hat{\phi}_j(\hat{\mathbf{x}}) \cdot \hat{\phi}_k(\hat{\mathbf{x}}) d\hat{x} d\hat{y}.
\end{aligned}
$$

2.1 Integral checking boards of monomial basis function on reference elements

Table 1: Monomial reference basis function

$\mathbb{P}_0$				1				$n_r = 1$
$\mathbb{P}_1$			$\hat{x}$		$\hat{y}$			$n_r = 3$
$\mathbb{P}_2$		$\hat{x}^2$		$\hat{x}\hat{y}$		$\hat{y}^2$		$n_r = 6$
$\mathbb{P}_3$	$\hat{x}^3$		$\hat{x}^2\hat{y}$		$\hat{x}\hat{y}^2$		$\hat{y}^3$	$n_r = 10$

From appendices. A, we get

1. Checking board for volume integral $(M_{\hat{K}})_{k,j} = \left(\hat{\phi}_j(\hat{\mathbf{x}}), \hat{\phi}_k(\hat{\mathbf{x}})\right)_{\hat{K}}$

   (a) $\mathbb{P}_0$

$$M_{\hat{K}} = 4,$$

   (b) $\mathbb{P}_1$

$$M_{\hat{K}} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix},$$

   (c) $\mathbb{P}_2$

$$M_{\hat{K}} = \begin{pmatrix} 4 & 0 & 0 & \frac{4}{3} & 0 & \frac{4}{3} \\ 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 \\ \frac{4}{3} & 0 & 0 & \frac{4}{5} & 0 & \frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{4}{9} & 0 \\ \frac{4}{3} & 0 & 0 & \frac{4}{9} & 0 & \frac{4}{5} \end{pmatrix},$$

   (d) $\mathbb{P}_3$

$$M_{\hat{K}} = \begin{pmatrix} 4 & 0 & 0 & \frac{4}{3} & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{4}{9} & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & \frac{4}{9} & 0 & \frac{4}{5} \\ \frac{4}{3} & 0 & 0 & \frac{4}{5} & 0 & \frac{4}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{9} & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{3} & 0 & 0 & \frac{4}{9} & 0 & \frac{4}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{5} & 0 & 0 & 0 & 0 & \frac{4}{7} & 0 & \frac{4}{15} & 0 \\ 0 & 0 & \frac{4}{9} & 0 & 0 & 0 & 0 & \frac{4}{15} & 0 & \frac{4}{15} \\ 0 & \frac{4}{9} & 0 & 0 & 0 & 0 & \frac{4}{15} & 0 & \frac{4}{15} & 0 \\ 0 & 0 & \frac{4}{5} & 0 & 0 & 0 & 0 & \frac{4}{15} & 0 & \frac{4}{7} \end{pmatrix},$$

2. Checking board for volume integral $(M_{\hat{K}_{\hat{x}}})_{k,j} = \left(\hat{\phi}_j(\hat{\mathbf{x}}), \hat{\partial}_{\hat{x}}\hat{\phi}_k(\hat{\mathbf{x}})\right)_{\hat{K}}$

   (a) $\mathbb{P}_0$

$$M_{\hat{K}_{\hat{x}}} = 0,$$

   (b) $\mathbb{P}_1$

$$M_{\hat{K}_{\hat{x}}} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   (c) $\mathbb{P}_2$

$$M_{\hat{K}_{\hat{x}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & \frac{4}{3} & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (d) $\mathbb{P}_3$

$$M_{\hat{K}_{\hat{x}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & \frac{4}{3} & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & 0 & 0 & 0 & 0 & \frac{8}{5} & 0 & \frac{8}{9} & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & \frac{4}{9} & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & \frac{12}{5} & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{9} & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{3} & 0 & 0 & \frac{4}{9} & 0 & \frac{4}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3. Checking board for volume integral $(M_{\hat{K}_{\hat{y}}})_{k,j} = \left(\hat{\phi}_j(\hat{\mathbf{x}}), \hat{\partial}_{\hat{y}}\hat{\phi}_k(\hat{\mathbf{x}})\right)_{\hat{K}}$

   (a) $\mathbb{P}_0$

$$M_{\hat{K}_{\hat{y}}} = 0,$$

   (b) $\mathbb{P}_1$

$$M_{\hat{K}_{\hat{y}}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

   (c) $\mathbb{P}_2$

$$M_{\hat{K}_{\hat{y}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & \frac{4}{3} & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 0 & 0 & 0 \end{pmatrix}$$

(d) $\mathbb{P}_3$

$$M_{\hat{K}_{\hat{y}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & \frac{4}{3} & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{4}{9} & 0 \\ 0 & 0 & \frac{8}{3} & 0 & 0 & 0 & 0 & \frac{8}{9} & 0 & \frac{8}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{3} & 0 & 0 & \frac{4}{5} & 0 & \frac{4}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{9} & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & \frac{4}{3} & 0 & \frac{12}{5} & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. Checking board for face integral $(M_{\hat{K}_e})_{k,j} = \left\langle \hat{\phi}_j(\hat{\mathbf{x}}), \hat{\phi}_k(\hat{\mathbf{x}}) \right\rangle_{\partial \hat{K}}$
   From the appendix.B, we get
   (a) Face integral on $\hat{e}_1$
      i. $\mathbb{P}_0$

$$M_{\hat{K}_{e_1}} = 2,$$

      ii. $\mathbb{P}_1$

$$M_{\hat{K}_{e_1}} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & \frac{2}{3} & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

      iii. $\mathbb{P}_2$

$$M_{\hat{K}_{e_1}} = \begin{pmatrix} 2 & 0 & -2 & \frac{2}{3} & 0 & 2 \\ 0 & \frac{2}{3} & 0 & 0 & -\frac{2}{3} & 0 \\ -2 & 0 & 2 & -\frac{2}{3} & 0 & -2 \\ \frac{2}{3} & 0 & -\frac{2}{3} & \frac{2}{5} & 0 & \frac{2}{3} \\ 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 2 & 0 & -2 & \frac{2}{3} & 0 & 2 \end{pmatrix}$$

      iv. $\mathbb{P}_3$

$$M_{\hat{K}_{e_1}} = \begin{pmatrix} 2 & 0 & -2 & \frac{2}{3} & 0 & 2 & 0 & -\frac{2}{3} & 0 & -2 \\ 0 & \frac{2}{3} & 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{3} & 0 \\ -2 & 0 & 2 & -\frac{2}{3} & 0 & -2 & 0 & \frac{2}{3} & 0 & 2 \\ \frac{2}{3} & 0 & -\frac{2}{3} & \frac{2}{5} & 0 & \frac{2}{3} & 0 & -\frac{2}{5} & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & -\frac{2}{5} & 0 & -\frac{2}{3} & 0 \\ 2 & 0 & -2 & \frac{2}{3} & 0 & 2 & 0 & -\frac{2}{3} & 0 & -2 \\ 0 & \frac{2}{5} & 0 & 0 & -\frac{2}{5} & 0 & \frac{2}{7} & 0 & \frac{2}{5} & 0 \\ -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{2}{5} & 0 & -\frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{3} & 0 \\ -2 & 0 & 2 & -\frac{2}{3} & 0 & -2 & 0 & \frac{2}{3} & 0 & 2 \end{pmatrix}$$

   (b) Face integral on $\hat{e}_2$
      i. $\mathbb{P}_0$

$$M_{\hat{K}_{e_2}} = 2,$$

ii. $\mathbb{P}_1$

$$M_{\hat{K}_{e_2}} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$$

iii. $\mathbb{P}_2$

$$M_{\hat{K}_{e_2}} = \begin{pmatrix} 2 & 2 & 0 & 2 & 0 & \frac{2}{3} \\ 2 & 2 & 0 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 \\ 2 & 2 & 0 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}$$

iv. $\mathbb{P}_3$

$$M_{\hat{K}_{e_2}} = \begin{pmatrix} 2 & 2 & 0 & 2 & 0 & \frac{2}{3} & 2 & 0 & \frac{2}{3} & 0 \\ 2 & 2 & 0 & 2 & 0 & \frac{2}{3} & 2 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ 2 & 2 & 0 & 2 & 0 & \frac{2}{3} & 2 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{2}{5} & \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 2 & 2 & 0 & 2 & 0 & \frac{2}{3} & 2 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{2}{5} & \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \frac{2}{5} & 0 & 0 & \frac{2}{5} & 0 & \frac{2}{7} \end{pmatrix}.$$

(c) Face integral on $\hat{e}_3$

i. $\mathbb{P}_0$

$$M_{\hat{K}_{e_3}} = 2,$$

ii. $\mathbb{P}_1$

$$M_{\hat{K}_{e_3}} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & \frac{2}{3} & 0 \\ 2 & 0 & 2 \end{pmatrix},$$

iii. $\mathbb{P}_2$

$$M_{\hat{K}_{e_3}} = \begin{pmatrix} 2 & 0 & 2 & \frac{2}{3} & 0 & 2 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 2 & 0 & 2 & \frac{2}{3} & 0 & 2 \\ \frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{5} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 2 & 0 & 2 & \frac{2}{3} & 0 & 2 \end{pmatrix}$$

iv. $\mathbb{P}_3$

$$M_{\hat{K}_{e_3}} = \begin{pmatrix} 2 & 0 & 2 & \frac{2}{3} & 0 & 2 & 0 & \frac{2}{3} & 0 & 2 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{3} & 0 \\ 2 & 0 & 2 & \frac{2}{3} & 0 & 2 & 0 & \frac{2}{3} & 0 & 2 \\ \frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{5} & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{3} & 0 \\ 2 & 0 & 2 & \frac{2}{3} & 0 & 2 & 0 & \frac{2}{3} & 0 & 2 \\ 0 & \frac{2}{5} & 0 & 0 & \frac{2}{5} & 0 & \frac{2}{7} & 0 & \frac{2}{5} & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{5} & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{3} & 0 \\ 2 & 0 & 2 & \frac{2}{3} & 0 & 2 & 0 & \frac{2}{3} & 0 & 2 \end{pmatrix}$$

(d) Face integral on $\hat{e}_4$

i. $\mathbb{P}_0$

$$M_{\hat{K}_{e_4}} = 2,$$

ii. $\mathbb{P}_1$

$$M_{\hat{K}_{e_4}} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$$

iii. $\mathbb{P}_2$

$$M_{\hat{K}_{e_4}} = \begin{pmatrix} 2 & -2 & 0 & 2 & 0 & \frac{2}{3} \\ -2 & 2 & 0 & -2 & 0 & -\frac{2}{3} \\ 0 & 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 \\ 2 & -2 & 0 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & -\frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}$$

iv. $\mathbb{P}_3$

$$M_{\hat{K}_{e_4}} = \begin{pmatrix} 2 & -2 & 0 & 2 & 0 & \frac{2}{3} & -2 & 0 & -\frac{2}{3} & 0 \\ -2 & 2 & 0 & -2 & 0 & -\frac{2}{3} & 2 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ 2 & -2 & 0 & 2 & 0 & \frac{2}{3} & -2 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 & -\frac{2}{3} & 0 & -\frac{2}{5} \\ \frac{2}{3} & -\frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{2}{5} & -\frac{2}{3} & 0 & -\frac{2}{5} & 0 \\ -2 & 2 & 0 & -2 & 0 & -\frac{2}{3} & 2 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ -\frac{2}{3} & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & -\frac{2}{5} & \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & \frac{2}{5} & 0 & -\frac{2}{5} & 0 & 0 & \frac{2}{5} & 0 & \frac{2}{7} \end{pmatrix}$$

3 Discrete partial derivatives

Definition 3.1. *[3]Discrete partial derivatives For any $v \in \mathcal{V}_h$, we define the discrete partial derivatives $\partial^-_{h,x_i} v$, $\partial^+_{h,x_i} v$, $\partial_{h,x_i} v \in \mathbb{V}^h_r$ by*

$$\left(\partial^{\pm}_{h,x_i} v, \phi_h\right)_{\mathcal{T}_h} = \left\langle \mathbb{Q}^{\pm}_i(v) n^{(i)}, [\![\phi_h]\!]\right\rangle_{\mathcal{E}_h} - \left(v, \partial_{x_i}\phi_h\right)_{\mathcal{T}_h} + \left\langle \gamma^{\pm} [\![v]\!], [\![\phi_h]\!]\right\rangle_{\mathcal{E}^I_h}, \;\; \forall\, \phi_h \in \mathbb{V}^h_r,$$

and

$$\partial_{h,x_i} v = \frac{1}{2}\left(\partial^{+}_{h,x_i} v + \partial^{-}_{h,x_i} v\right),$$

for all $i = 1, 2, \cdots, d$.

Since $\mathbb{V}^h_r$ is a totally discontinuous piecewise polynomial space, the discrete derivatives $\partial^{\pm}_{h,x_i} v$ can also be written in their equivalent local versions:

$$\left(\partial^{\pm}_{h,x_i} v, \phi_h\right)_K = \left\langle \mathbb{Q}^{\pm}_i(v) n^{(i)}_K, \phi_h \right\rangle_{\partial K} - \left(v, \partial_{x_i} \phi_h\right)_K + \sum_{e \in \partial K \setminus \partial \Omega} \left\langle \gamma^{\pm}_{i,e} [\![v]\!], [\![\phi_h]\!] \right\rangle_e, \ \ \forall\ \phi_h \in \mathbb{V}^h_r,$$

for $i = 1, 2, \cdots, d$ and $K \in \mathcal{T}_h$. Here $\gamma^{\pm}_{i,e} = \gamma^{\pm}_i|_e$.

From now on, we only consider the penalty parameters $\gamma^{\pm}_{i,e}$ are zeros. Since for any $v \in \mathcal{V}_h$, $\partial^{-}_{h,x_i} v$, $\partial^{+}_{h,x_i} v$, $\partial_{h,x_i} v \in \mathbb{V}^h_r$, so $\partial^{\pm}_{h,x_i} v|_K$ can also be represented by the basis $\{\phi_j\}_{j=1}^{n_r}$, i.e.

$$\partial^{\pm}_{h,x_i} v|_K = \sum_{j=1}^{n_r} \alpha^{\pm}_{i,j} \phi_j.$$

Now, we need to compute the weak derivatives of the basis function along each dimension, i.e. to determinate the coefficient $\alpha^{\pm}_{i,j}$. For fixed basis ϕ_ℓ,

$$\partial^{\pm}_{h,x_i} \phi_\ell|_K = \sum_{j=1}^{n_r} \alpha^{\pm \ell}_{i,j} \phi_j.$$

Therefore, for a fixed i,

$$\left(\sum_{j=1}^{n_r} \alpha^{\pm \ell}_{i,j} \phi_j, \phi_k\right)_K = \left\langle \mathbb{Q}^{\pm}_i(\phi_\ell) n^{(i)}_K, \phi_k \right\rangle_{\partial K} - \left(\phi_\ell, \partial_{x_i} \phi_k\right)_K, \ \ k = 1, 2, \cdots, n_r.$$

Let

$$b^{\pm \ell}_{i,k} = \left\langle \mathbb{Q}^{\pm}_i(\phi_\ell) n^{(i)}_K, \phi_k \right\rangle_{\partial K} - \left(\phi_\ell, \partial_{x_i} \phi_k\right)_K$$

then we have

$$A \vec{\alpha}^{\pm \ell}_i = \vec{b}^{\ell}_i,$$

where,

$$A = \begin{pmatrix} (\phi_1, \phi_1)_K & (\phi_2, \phi_1)_K & \cdots & (\phi_{n_r}, \phi_1)_K \\ (\phi_1, \phi_2)_K & (\phi_2, \phi_2)_K & \cdots & (\phi_{n_r}, \phi_2)_K \\ \vdots & \vdots & \ddots & \vdots \\ (\phi_1, \phi_{n_r})_K & (\phi_2, \phi_{n_r})_K & \cdots & (\phi_{n_r}, \phi_{n_r})_K \end{pmatrix}$$

and

$$\vec{\alpha}^{\pm m}_i = \begin{pmatrix} \alpha^{\pm \ell}_{i,1} \\ \alpha^{\pm \ell}_{i,2} \\ \vdots \\ \alpha^{\pm \ell}_{i,n_r} \end{pmatrix}, \vec{b}^{\pm \ell}_i = \begin{pmatrix} b^{\pm \ell}_{i,1} \\ b^{\pm \ell}_{i,2} \\ \vdots \\ b^{\pm \ell}_{i,n_r} \end{pmatrix}.$$

3.1 The value of trace operators

We have two ways to define or understand the trace operators. Generally, one way is to define the upwinding and downwinding trace operators according to the DG edge unit normal, the other way is according to the elements' unit outer normal vector.

3.1.1 upwinding and downwinding operators according to DG edge unit normal

Let $K^+, K^- \in \mathcal{T}_h$ and $e = \partial K^+ \cap \partial K^-$. Without loss of generality, we assume that the global labeling number of K^+ is smaller than that of K^-. We then introduce the following standard jump and average notations for the face/edge e:

1. For interior edge $e \in \mathcal{E}_h^I$,

$$[\![v]\!] = v|_{K^+} - v|_{K^-}, \quad \{v\} = \frac{1}{2}\left(v|_{K^+} + v|_{K^-}\right),$$

2. For boundary edge $e \in \mathcal{E}_h^B$,

$$[\![v]\!] = v, \quad \{v\} = v.$$

We also define $n_e = n_K|_e$ as the unit normal on e(Figure. 7).

Definition 3.2. *[3]Trace operators for interior edge $\mathcal{E}_h^I$ we define the trace operator for interior edge $e \in \mathcal{E}_h^I$ as*

$$\mathbb{Q}_i^-(v)(x) := \begin{cases} \lim\limits_{\substack{y \in K^- \\ y \to x}} v(y) & \text{if } n_e^{(i)} < 0, \\ \lim\limits_{\substack{y \in K^+ \\ y \to x}} v(y) & \text{if } n_e^{(i)} \geq 0, \end{cases}$$

$$\mathbb{Q}_i^+(v)(x) := \begin{cases} \lim\limits_{\substack{y \in K^+ \\ y \to x}} v(y) & \text{if } n_e^{(i)} < 0, \\ \lim\limits_{\substack{y \in K^- \\ y \to x}} v(y) & \text{if } n_e^{(i)} \geq 0, \end{cases}$$

and

$$\mathbb{Q}_i(v)(x) := \frac{1}{2}\Big(\mathbb{Q}_i^+(v)(x) + \mathbb{Q}_i^-(v)(x)\Big),$$

for any $x \in e$ and $i = 1, 2, 3, \cdots, n$.

Definition 3.3. *[3]Trace operators for boundary edge $\mathcal{E}_h^B$ we define the trace operator for boundary edge $e \in \mathcal{E}_h^B$ as*

$$\mathbb{Q}_i^+(v)(x) = \mathbb{Q}_i^-(v)(x) = \mathbb{Q}_i(v)(x) := \lim_{\substack{y \in \Omega \\ y \to x}} v(y).$$

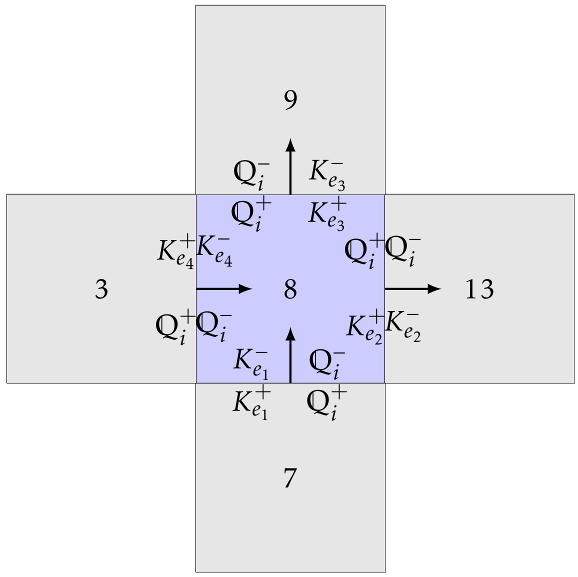

Figure 7: The value of trace operators according to DG edge unit normal.

3.1.2 upwinding and downwinding operators according to elements' unit outer normal vector

Next we introduce some standard notation for DG methods. Let $K^+, K^- \in \mathcal{T}_h$ and $e = \partial K^- \cap \partial K^+$. Without a loss of generality, we assume the global labeling number of K^+ is smaller than that of K^-. We then define the sided-flux values for v as

$$v^+\big|_e \equiv v\big|_{e\cap\partial K^+}, \qquad v^-\big|_e \equiv v\big|_{e\cap\partial K^-},$$

where $v|_{\partial K}$ is understood to be the trace of v defined on $\overline{K}$. Suppose K is a boundary simplex. We extend the sided-flux definitions to the boundary of Ω by

$$v^\pm\big|_{\partial K\cap\partial\Omega} \equiv v\big|_{\partial K\cap\partial\Omega}.$$

We also define jump and average operators on $\mathcal{E}_h$ by

$$[\![v]\!] \equiv \begin{cases} v^- - v^+ & \text{on } \mathcal{E}_h^I \\ v & \text{on } \mathcal{E}_h^B \end{cases}, \qquad \{v\} \equiv \frac{v^- + v^+}{2},$$

with the convention that the outward normal vector on e, denoted by $\mathbf{n}$, is always given by the outward normal vector for K^-.

Definition 3.4. *[2] Choose $K \in \mathcal{T}_h$. Let $\vec{n}$ denote the normal vector to ∂K, and let $K' \in \mathcal{T}_h$ such that $\emptyset \neq \partial K \cap \partial K' \equiv e \in \mathcal{E}_h^I$. We define the upwinding and downwinding trace operators $\mathbf{T}^+, \mathbf{T}^- : H^1(\mathcal{T}_h) \to \mathbf{L}^2(\mathcal{E}_h)$ by*

$$T_j^+(v)\big|_e \equiv \begin{cases} v\big|_{K'} & \text{if } \mathrm{n}_j > 0, \\ v\big|_K & \text{if } \mathrm{n}_j < 0, \\ \{v\} & \text{if } \mathrm{n}_j = 0, \end{cases} \qquad T_j^-(v)\big|_e \equiv \begin{cases} v\big|_K & \text{if } \mathrm{n}_j > 0, \\ v\big|_{K'} & \text{if } \mathrm{n}_j < 0, \\ \{v\} & \text{if } \mathrm{n}_j = 0, \end{cases}$$

and

$$T_j^\pm(v)\big|_{\partial K\cap\partial\Omega} \equiv v\big|_K$$

for all $j = 1,2,\ldots,d$. We define the central trace operator $\overline{\mathbf{T}} : H^1(\mathcal{T}_h) \to \mathbf{L}^2(\mathcal{E}_h)$ by $\overline{\mathbf{T}} \equiv \frac{1}{2}(\mathbf{T}^+ + \mathbf{T}^-)$. If $\vec{v} \in \mathbf{H}^1(\mathcal{T}_h)$, we let $T_j(\vec{\varphi}) \equiv T_j(v_j)$ for $j = 1,2,\ldots,d$ for $\mathbf{T} = \mathbf{T}^+, \mathbf{T}^-, \overline{\mathbf{T}}$.

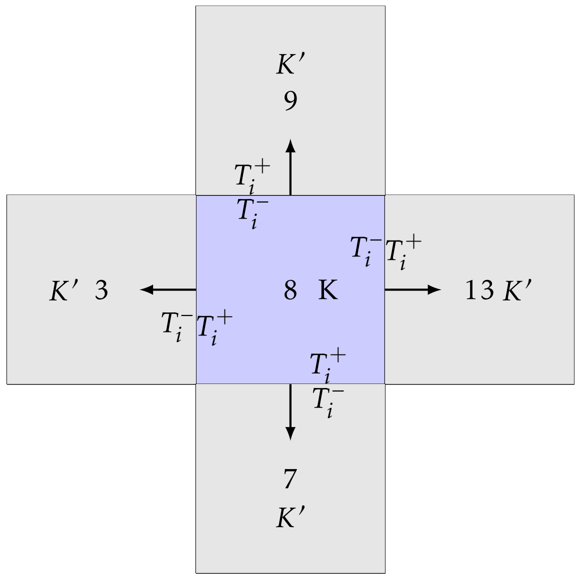

Figure 8: The value of trace operators according to elements' unit outer normal vector.

An example of how the trace operators are defined can be found in Figure 8. Observe, on interior faces/edges, T_j^+ corresponds to trace values from the positive x_i (Cartesian) direction and T_j^- corresponds to the trace values from the negative x_i (Cartesian) direction.

3.2 The value of $b_{i,k}^{\pm\ell}$ according to the DG edge normal

From the definition of the trace operator and the DG normal, for each edge of a fixed element K, we get that

$$\mathrm{Q}_i^+(\phi_\ell) = \begin{cases} \phi_\ell|_{K_e} & \text{if } \#K < \#K^n \text{ or } e \in \mathcal{E}_h^B \\ \phi_\ell|_{K_e^n} & \text{if } \#K > \#K^n. \end{cases}$$

$$\mathrm{Q}_i^-(\phi_\ell) = \begin{cases} \phi_\ell|_{K_e} & \text{if } \#K > \#K^n \text{ or } e \in \mathcal{E}_h^B \\ \phi_\ell|_{K_e^n} & \text{if } \#K < \#K^n. \end{cases}$$

Where, K^n is the neighbor element of K at edge e and # is the global labeling number. For instance, the $b_{i,k}^{\pm\ell}$ of element 8 in Figure.(7) is as follows:

$$b_{i,k}^{\pm\ell} = (M_E^{\pm})_{(k,\ell)} - (M_{K_{x_i}})_{(k,\ell)},$$

where,

$$(M_{K_{x_i}})_{(k,\ell)} = \left(\phi_\ell, \partial_{x_i}\phi_k\right)_K$$

and

$$
\begin{aligned}
(M_E^+)_{(k,\ell)} &= \left\langle \mathbb{Q}_i^+(\phi_\ell) n_K^{(i)}, \phi_k \right\rangle_{\partial K} \\
&= \left\langle \mathbb{Q}_i^+(\phi_\ell) {n_{e_1}}^{(i)}, \phi_k \right\rangle_{e_1} + \left\langle \mathbb{Q}_i^+(\phi_\ell) {n_{e_2}}^{(i)}, \phi_k \right\rangle_{e_2} + \left\langle \mathbb{Q}_i^+(\phi_\ell) {n_{e_3}}^{(i)}, \phi_k \right\rangle_{e_3} + \left\langle \mathbb{Q}_i^+(\phi_\ell) {n_{e_4}}^{(i)}, \phi_k \right\rangle_{e_4} \\
&= {n_{e_1}}^{(i)} \int_{e_1} \phi_\ell|_{K_7} \phi_k|_{K_8} + {n_{e_2}}^{(i)} \int_{e_2} \phi_\ell|_{K_8} \phi_k|_{K_8} + {n_{e_3}}^{(i)} \int_{e_3} \phi_\ell|_{K_8} \phi_k|_{K_8} + {n_{e_4}}^{(i)} \int_{e_4} \phi_\ell|_{K_3} \phi_k|_{K_8},
\end{aligned}
$$

$$
\begin{aligned}
(M_E^-)_{(k,\ell)} &= \left\langle \mathbb{Q}_i^-(\phi_\ell) n_K^{(i)}, \phi_k \right\rangle_{\partial K} \\
&= \left\langle \mathbb{Q}_i^-(\phi_\ell) {n_{e_1}}^{(i)}, \phi_k \right\rangle_{e_1} + \left\langle \mathbb{Q}_i^-(\phi_\ell) {n_{e_2}}^{(i)}, \phi_k \right\rangle_{e_2} + \left\langle \mathbb{Q}_i^-(\phi_\ell) {n_{e_3}}^{(i)}, \phi_k \right\rangle_{e_3} + \left\langle \mathbb{Q}_i^-(\phi_\ell) {n_{e_4}}^{(i)}, \phi_k \right\rangle_{e_4} \\
&= {n_{e_1}}^{(i)} \int_{e_1} \phi_\ell|_{K_8} \phi_k|_{K_8} + {n_{e_2}}^{(i)} \int_{e_2} \phi_\ell|_{K_{13}} \phi_k|_{K_8} + {n_{e_3}}^{(i)} \int_{e_3} \phi_\ell|_{K_9} \phi_k|_{K_8} + {n_{e_4}}^{(i)} \int_{e_4} \phi_\ell|_{K_8} \phi_k|_{K_8}.
\end{aligned}
$$

From $(M_E^+)_{(k,\ell)}$ and $(M_E^+)_{(k,\ell)}$, we get that we need to compute the self-inner product for all basis on K and the interacting-inner product between K and its edge neighbor.

4 Dual-Wind Discontinuous Galerkin Scheme

We consider the following scheme :

$$
-\frac{\mathrm{div}_{h,g}^+ \nabla_h^- u_h + \mathrm{div}_{h,g}^- \nabla_h^+ u_h}{2} + \mathcal{P}_h(c^2 u_h) + j_{h,g}(u_h) = \mathcal{P}_h f. \tag{1}
$$

where $\mathcal{P}_h$ is the L^2 projection operator onto $\mathbb{V}_r^h$, and $j_{h,g} : \mathcal{V}_h \to \mathbb{V}_r^h$ is the penalty operator satisfying

$$
\left(j_{h,g}(v), \phi_h \right)_{\mathcal{T}_h} = \langle \gamma [\![v]\!], [\![\phi_h]\!] \rangle_{\mathcal{E}_h^I}, \forall \phi_h \in \mathbb{V}_r^h,
$$

where γ is a piecewise penalty function satisfying $\gamma|_e = \gamma_e h_e^{-1}$ for all $e \in \mathcal{E}_h$.

From the definition of [4], we get that

$$
\begin{aligned}
-\left(\mathrm{div}_{h,g}^\pm \nabla_h^\mp u_h, \phi_h\right)_{\mathcal{T}_h} &= -\left(\mathrm{div}_h^\pm \nabla_h^\mp u_h, \phi_h\right)_{\mathcal{T}_h} - \left\langle (g - \nabla_h^\mp u_h) \cdot \mathbf{n}, \phi_h \right\rangle_{\mathcal{E}_h^B} \\
&= \left(\nabla_h^\mp u_h, \nabla_h^\mp \phi_h\right)_{\mathcal{T}_h} - \left\langle \nabla_h^\mp u_h \cdot \mathbf{n}, \phi_h \right\rangle_{\mathcal{E}_h^B} - \left\langle g - \nabla_h^\mp u_h \cdot \mathbf{n}, \phi_h \right\rangle_{\mathcal{E}_h^B} \\
&= \left(\nabla_h^\mp u_h, \nabla_h^\mp \phi_h\right)_{\mathcal{T}_h} - \langle g, \phi_h \rangle_{\mathcal{E}_h^B}.
\end{aligned}
$$

Then we have the scheme is defined by: Find $u_h \in \mathbb{V}_r^h$ such that

$$
\frac{1}{2}\left(\left(\nabla_h^+ u_h, \nabla_h^+ \phi_h\right)_{\mathcal{T}_h} + \left(\nabla_h^- u_h, \nabla_h^- \phi_h\right)_{\mathcal{T}_h} \right) + \left(c^2 u_h, \phi_h\right)_{\mathcal{T}_h} = (f, \phi_h)_{\mathcal{T}_h} + \langle g, \phi_h \rangle_{\mathcal{E}_h^B}, \tag{2}
$$

for all $\phi_h \in \mathbb{V}_r^h$.

The approximation solution (or Dual-Wind solution) is the linear combination of the basis functions, i.e. $u_h \in \mathbb{V}_r^h$ and

$$
u_h = \sum_{m=1}^{N} c^m \phi_m,
$$

where $N = N_{elem} * n_r$. Then

$$\begin{aligned}\left(\nabla_h^{\pm} u_h, \nabla_h^{\pm}\phi_h\right)_{\mathcal{T}_h} &= \left(\nabla_h^{\pm}\left(\sum_{m=1}^{N} c^m \phi_m\right), \nabla_h^{\pm}\phi_h\right)_{\mathcal{T}_h} \\ &= \sum_{m=1}^{N} c^m \left(\nabla_h^{\pm}\phi_m, \nabla_h^{\pm}\phi_h\right)_{\mathcal{T}_h}, \forall \phi_h \in \mathbb{V}_r^h.\end{aligned}$$

Since $\nabla_h^{\pm} v = \left(\partial_{h,x_1}^{\pm} v, \partial_{h,x_2}^{\pm} v, \cdots, \partial_{h,x_d}^{\pm}\right)^T$, so

$$\left(\nabla_h^{\pm} u_h, \nabla_h^{\pm}\phi_h\right)_{\mathcal{T}_h} = \sum_{i=1}^{d}\sum_{m=1}^{N} c^m \left(\partial_{h,x_i}^{\pm}\phi_m, \partial_{h,x_i}^{\pm}\phi_h\right)_{\mathcal{T}_h}, \forall \phi_h \in \mathbb{V}_r^h.$$

From the definition (3.1) of the discrete partial derivatives, we know

$$\left(\partial_{h,x_i}^{\pm} v, \phi_h\right)_{\mathcal{T}_h} = \left\langle \mathbb{Q}_i^{\pm}(v) n^{(i)}, [\![\phi_h]\!]\right\rangle_{\mathcal{E}_h} - \left(v, \partial_{x_i}\phi_h\right)_{\mathcal{T}_h}, \quad \forall\ \phi_h \in \mathbb{V}_r^h.$$

4.1 Method 1

So, for all $\partial_{h,x_i}^{\pm}\phi_h \in \mathbb{V}_r^h$,

$$\left(\partial_{h,x_i}^{\pm}\phi_m, \partial_{h,x_i}^{\pm} w_h\right)_{\mathcal{T}_h} = \left\langle \mathbb{Q}_i^{\pm}(\phi_m) n^{(i)}, [\![\partial_{h,x_i}^{\pm} w_h]\!]\right\rangle_{\mathcal{E}_h} - \left(v, \partial_{x_i}\left(\partial_{h,x_i}^{\pm} w_h\right)\right)_{\mathcal{T}_h}$$

Now we choose w_h as $\phi_1, \phi_2, \cdots, \phi_N$ respectively to get the algebraic system. For a fixed i, we get

$$\begin{aligned}\left(\partial_{h,x_i}^{\pm}\phi_m, \partial_{h,x_i}^{\pm}\phi_\ell\right)_{\mathcal{T}_h} &= \left\langle \mathbb{Q}_i^{\pm}(\phi_m) n^{(i)}, [\![\partial_{h,x_i}^{\pm}\phi_\ell]\!]\right\rangle_{\mathcal{E}_h} - \left(v, \partial_{x_i}\left(\partial_{h,x_i}^{\pm}\phi_\ell\right)\right)_{\mathcal{T}_h} \\ &= \left\langle \mathbb{Q}_i^{\pm}(\phi_m) n^{(i)}, \partial_{h,x_i}^{\pm}\phi_\ell|_{K^+}\right\rangle_{\mathcal{E}_h} - \left\langle \mathbb{Q}_i^{\pm}(\phi_m) n^{(i)}, \partial_{h,x_i}^{\pm}\phi_\ell|_{K^-}\right\rangle_{\mathcal{E}_h} - \left(v, \partial_{x_i}\left(\partial_{h,x_i}^{\pm}\phi_\ell\right)\right)_{\mathcal{T}_h}\end{aligned}$$

for all $m = 1, 2, \cdots, N, \ell = 1, 2, \cdots, n_r$. Since,

$$\partial_{h,x_i}^{\pm}\phi_\ell|_K = \sum_{j=1}^{n_r} \alpha_{i,j}^{\pm\ell}\phi_j.$$

Therefore,

$$\begin{aligned}\left(\partial_{h,x_i}^{\pm}\phi_m, \partial_{h,x_i}^{\pm}\phi_\ell\right)_{\mathcal{T}_h} &= \left\langle \mathbb{Q}_i^{\pm}(\phi_m) n^{(i)}, \left(\sum_{j=1}^{n_r}\alpha_{i,j}^{\pm\ell}\phi_j\right)|_{K^+}\right\rangle_{\mathcal{E}_h} - \left\langle \mathbb{Q}_i^{\pm}(\phi_m) n^{(i)}, \left(\sum_{j=1}^{n_r}\alpha_{i,j}^{\pm\ell}\phi_j\right)|_{K^-}\right\rangle_{\mathcal{E}_h} \\ &\quad - \left(\phi_m, \partial_{x_i}\left(\sum_{j=1}^{n_r}\alpha_{i,j}^{\pm\ell}\phi_j\right)\right)_{\mathcal{T}_h}.\end{aligned}$$

So,

$$\begin{aligned}\frac{1}{2}\left(\left(\nabla_h^{+} u_h, \nabla_h^{+}\phi_\ell\right)_{\mathcal{T}_h} + \left(\nabla_h^{-} u_h, \nabla_h^{-}\phi_\ell\right)_{\mathcal{T}_h}\right) &= \frac{1}{2}\sum_{m=1}^{N} c^m \left(\partial_{h,x_1}^{+}\phi_m, \partial_{h,x_1}^{+}\phi_\ell\right)_{\mathcal{T}_h} + \frac{1}{2}\sum_{m=1}^{N} c^m \left(\partial_{h,x_2}^{+}\phi_m, \partial_{h,x_2}^{+}\phi_\ell\right)_{\mathcal{T}_h} \\ &\quad + \frac{1}{2}\sum_{m=1}^{N} c^m \left(\partial_{h,x_1}^{-}\phi_m, \partial_{h,x_1}^{-}\phi_\ell\right)_{\mathcal{T}_h} + \frac{1}{2}\sum_{m=1}^{N} c^m \left(\partial_{h,x_2}^{-}\phi_m, \partial_{h,x_2}^{-}\phi_\ell\right)_{\mathcal{T}_h},\end{aligned}$$

where,

$$\left(\partial^+_{h,x_1}\phi_m,\partial^+_{h,x_1}\phi_\ell\right)_{\mathcal{T}_h} = \left\langle \mathbb{Q}^+_i(\phi_m)n^{(1)},\left(\sum_{j=1}^{n_r}\alpha^{+\ell}_{1,j}\phi_j\right)|_{K^+}\right\rangle_{\mathcal{E}_h} - \left\langle \mathbb{Q}^+_i(\phi_m)n^{(1)},\left(\sum_{j=1}^{n_r}\alpha^{+\ell}_{1,j}\phi_j\right)|_{K^-}\right\rangle_{\mathcal{E}_h}$$
$$-\left(\phi_m,\partial_{x_1}\left(\sum_{j=1}^{n_r}\alpha^{+\ell}_{1,j}\phi_j\right)\right)_{\mathcal{T}_h}$$
$$= n^{(1)}\sum_{j=1}^{n_r}\alpha^{+\ell}_{1,j}|_{K^+}\left\langle\phi_m|_{K^+},\phi_j|_{K^+}\right\rangle_{\mathcal{E}_h} - n^{(1)}\sum_{j=1}^{n_r}\alpha^{+\ell}_{1,j}|_{K^-}\left\langle\phi_m|_{K^+},\phi_j|_{K^-}\right\rangle_{\mathcal{E}_h}$$
$$-\sum_{j=1}^{n_r}\alpha^{+\ell}_{1,j}\left(\phi_m,\partial_{x_1}\phi_j\right)_{\mathcal{T}_h}.$$

Similarly,

$$\left(\partial^+_{h,x_2}\phi_m,\partial^+_{h,x_2}\phi_\ell\right)_{\mathcal{T}_h} = n^{(2)}\sum_{j=1}^{n_r}\alpha^{+\ell}_{2,j}|_{K^+}\left\langle\phi_m|_{K^+},\phi_j|_{K^+}\right\rangle_{\mathcal{E}_h} - n^{(2)}\sum_{j=1}^{n_r}\alpha^{+\ell}_{2,j}|_{K^-}\left\langle\phi_m|_{K^+},\phi_j|_{K^-}\right\rangle_{\mathcal{E}_h}$$
$$-\sum_{j=1}^{n_r}\alpha^{+\ell}_{2,j}\left(\phi_m,\partial_{x_2}\phi_j\right)_{\mathcal{T}_h}.$$

$$\left(\partial^-_{h,x_1}\phi_m,\partial^-_{h,x_1}\phi_\ell\right)_{\mathcal{T}_h} = \left\langle \mathbb{Q}^-_i(\phi_m)n^{(1)},\left(\sum_{j=1}^{n_r}\alpha^{-\ell}_{1,j}\phi_j\right)|_{K^+}\right\rangle_{\mathcal{E}_h} - \left\langle \mathbb{Q}^-_i(\phi_m)n^{(1)},\left(\sum_{j=1}^{n_r}\alpha^{-\ell}_{1,j}\phi_j\right)|_{K^-}\right\rangle_{\mathcal{E}_h}$$
$$-\left(\phi_m,\partial_{x_1}\left(\sum_{j=1}^{n_r}\alpha^{-\ell}_{1,j}\phi_j\right)\right)_{\mathcal{T}_h}$$
$$= n^{(1)}\sum_{j=1}^{n_r}\alpha^{+\ell}_{1,j}|_{K^+}\left\langle\phi_m|_{K^-},\phi_j|_{K^+}\right\rangle_{\mathcal{E}_h} - n^{(1)}\sum_{j=1}^{n_r}\alpha^{-\ell}_{1,j}|_{K^-}\left\langle\phi_m|_{K^-},\phi_j|_{K^-}\right\rangle_{\mathcal{E}_h}$$
$$-\sum_{j=1}^{n_r}\alpha^{-\ell}_{1,j}\left(\phi_m,\partial_{x_1}\phi_j\right)_{\mathcal{T}_h}.$$

$$\left(\partial^-_{h,x_2}\phi_m,\partial^-_{h,x_2}\phi_\ell\right)_{\mathcal{T}_h} = n^{(2)}\sum_{j=1}^{n_r}\alpha^{+\ell}_{2,j}|_{K^+}\left\langle\phi_m|_{K^-},\phi_j|_{K^+}\right\rangle_{\mathcal{E}_h} - n^{(2)}\sum_{j=1}^{n_r}\alpha^{-\ell}_{1,j}|_{K^-}\left\langle\phi_m|_{K^-},\phi_j|_{K^-}\right\rangle_{\mathcal{E}_h}$$
$$-\sum_{j=1}^{n_r}\alpha^{-\ell}_{2,j}\left(\phi_m,\partial_{x_2}\phi_j\right)_{\mathcal{T}_h}.$$

4.2 Method 2

Since $\mathbb{V}^h_r$ is a totally discontinuous piecewise polynomial space, the discrete derivatives $\partial^\pm_{h,x_i}v$ can also be written in their equivalent local versions:

$$\left(\partial^\pm_{h,x_i}v,\phi_h\right)_K = \left\langle \mathbb{Q}^\pm_i(v)n^{(i)}_K,\phi_h\right\rangle_{\partial K} - \left(v,\partial_{x_i}\phi_h\right)_K, \quad \forall\ \phi_h\in\mathbb{P}_r(K),$$

for $i = 1, 2, \cdots, d$ and $K \in \mathcal{T}_h$.

So, for all $\partial^{\pm}_{h,x_i}\phi_h \in \mathbb{V}^h_r$,

$$\begin{aligned}\left(\partial^{\pm}_{h,x_i}\phi_m, \partial^{\pm}_{h,x_i}w_h\right)_{\mathcal{T}_h} &= \sum_{K\in\mathcal{T}_h}\left(\partial^{\pm}_{h,x_i}\phi_m, \partial^{\pm}_{h,x_i}\phi_h\right)_K \\ &= \sum_{K\in\mathcal{T}_h}\left\langle \mathbb{Q}^{\pm}_i(\phi_m)n^{(i)}_K, \partial^{\pm}_{h,x_i}\phi_h\right\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\left(\phi_m, \partial_{x_i}\left(\partial^{\pm}_{h,x_i}\phi_h\right)\right)_K\end{aligned}$$

Now we choose w_h as $\phi_1, \phi_2, \cdots, \phi_N$ respectively to get the algebraic system. For a fixed i, we get

$$\left(\partial^{\pm}_{h,x_i}\phi_m, \partial^{\pm}_{h,x_i}\phi_\ell\right)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}\left\langle \mathbb{Q}^{\pm}_i(\phi_m)n^{(i)}_K, \partial^{\pm}_{h,x_i}\phi_\ell\right\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\left(\phi_m, \partial_{x_i}\left(\partial^{\pm}_{h,x_i}\phi_\ell\right)\right)_K,$$

for all $m = 1, 2, \cdots, N, \ell = 1, 2, \cdots, n_r$. Since,

$$\partial^{\pm}_{h,x_i}\phi_\ell|_K = \sum_{j=1}^{n_r}\alpha^{\pm\ell}_{i,j}\phi_j.$$

Then,

$$\begin{aligned}\left(\partial^{\pm}_{h,x_i}\phi_m, \partial^{\pm}_{h,x_i}\phi_\ell\right)_{\mathcal{T}_h} &= \sum_{K\in\mathcal{T}_h}\left\langle \mathbb{Q}^{\pm}_i(\phi_m)n^{(i)}_K, \partial^{\pm}_{h,x_i}\phi_\ell\right\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\left(\phi_m, \partial_{x_i}\left(\partial^{\pm}_{h,x_i}\phi_\ell\right)\right)_K \\ &= \sum_{K\in\mathcal{T}_h}\left\langle \mathbb{Q}^{\pm}_i(\phi_m)n^{(i)}_K, \sum_{j=1}^{n_r}\alpha^{\pm\ell}_{i,j}\phi_j\right\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\left(\phi_m, \partial_{x_i}\left(\sum_{j=1}^{n_r}\alpha^{\pm\ell}_{i,j}\phi_j\right)\right)_K \\ &= \sum_{K\in\mathcal{T}_h}n^{(i)}_K\sum_{j=1}^{n_r}\alpha^{\pm\ell}_{i,j}\left\langle \mathbb{Q}^{\pm}_i(\phi_m), \phi_j\right\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\sum_{j=1}^{n_r}\alpha^{\pm\ell}_{i,j}\left(\phi_m, \partial_{x_i}\phi_j\right)_K.\end{aligned}$$

5 Dual-Wind Discontinuous Galerkin in Matrix format

This will provide a way to code DWDG in Matrix form which is very efficient in MATLAB. Let N denote the number of basis functions. Let $M, S^{I\pm}_k, S^B_k, B \in \mathbb{R}^{N\times N}$ for all $k = 1, 2, \cdots, d$ be defined by

$$\begin{aligned}M_{j,i} &= \left(\phi_i, \phi_j\right)_{\mathcal{T}_h} \\ S^{I\pm}_{k;j,i} &= \left\langle \mathbb{Q}^{\pm}_i(\phi_i)n^{(i)}, [\![\phi_j]\!]\right\rangle_{\mathcal{E}_h} \\ S^B_{k;j,i} &= \left\langle \phi_i n^{(i)}_K, \phi_j\right\rangle_{\mathcal{E}^B_h} \\ B_{j,i} &= \left(v, \partial_{x_k}\phi_h\right)_{\mathcal{T}_h}\end{aligned}$$

Appendices

A Volume integrals of reference basis on rectangular element

$$
\begin{aligned}
(M_{\hat{K}})_{1,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot 1 \; d\hat{x}d\hat{y} = 4,\\
(M_{\hat{K}})_{2,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{x} \; d\hat{x}d\hat{y} = 0,\\
(M_{\hat{K}})_{3,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{y} \; d\hat{x}d\hat{y} = 0,\\
(M_{\hat{K}})_{4,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{x}^2 \; d\hat{x}d\hat{y} = \frac{4}{3},\\
(M_{\hat{K}})_{5,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{x}\hat{y} \; d\hat{x}d\hat{y} = 0,\\
(M_{\hat{K}})_{6,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{y}^2 \; d\hat{x}d\hat{y} = \frac{4}{3},\\
(M_{\hat{K}})_{7,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{x}^3 \; d\hat{x}d\hat{y} = 0,\\
(M_{\hat{K}})_{8,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{x}^2\hat{y} \; d\hat{x}d\hat{y} = 0,\\
(M_{\hat{K}})_{9,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{x}\hat{y}^2 \; d\hat{x}d\hat{y} = 0,\\
(M_{\hat{K}})_{10,1} &= \left(\hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} 1 \cdot \hat{y}^3 \; d\hat{x}d\hat{y} = 0.
\end{aligned}
$$

$$(M_{\hat{K}})_{2,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{x}\ d\hat{x}d\hat{y} = \frac{4}{3},$$

$$(M_{\hat{K}})_{3,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{y}\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{4,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{x}^2\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{5,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{x}\hat{y}\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{6,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{y}^2\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{7,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{x}^3\ d\hat{x}d\hat{y} = \frac{4}{5},$$

$$(M_{\hat{K}})_{8,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{x}^2\hat{y}\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{9,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{x}\hat{y}^2\ d\hat{x}d\hat{y} = \frac{4}{9},$$

$$(M_{\hat{K}})_{10,2} = \left(\hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\cdot\hat{y}^3\ d\hat{x}d\hat{y} = 0.$$

$$(M_{\hat{K}})_{3,3} = \left(\hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}\cdot\hat{y}\ d\hat{x}d\hat{y} = \frac{4}{3},$$

$$(M_{\hat{K}})_{4,3} = \left(\hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}\cdot\hat{x}^2\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{5,3} = \left(\hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}\cdot\hat{x}\hat{y}\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{6,3} = \left(\hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}\cdot\hat{y}^2\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{7,3} = \left(\hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}\cdot\hat{x}^3\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{8,3} = \left(\hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}\cdot\hat{x}^2\hat{y}\ d\hat{x}d\hat{y} = \frac{4}{9},$$

$$(M_{\hat{K}})_{9,3} = \left(\hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}\cdot\hat{x}\hat{y}^2\ d\hat{x}d\hat{y} = 0,$$

$$(M_{\hat{K}})_{10,3} = \left(\hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}\cdot\hat{y}^3\ d\hat{x}d\hat{y} = \frac{4}{5}.$$

$$(M_{\hat{K}})_{4,4} = \left(\hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2 \; d\hat{x}d\hat{y} = \frac{4}{5},$$
$$(M_{\hat{K}})_{5,4} = \left(\hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y} \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{6,4} = \left(\hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^2 \; d\hat{x}d\hat{y} = \frac{4}{9},$$
$$(M_{\hat{K}})_{7,4} = \left(\hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^3 \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{8,4} = \left(\hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2\hat{y} \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{9,4} = \left(\hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{10,4} = \left(\hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^3 \; d\hat{x}d\hat{y} = 0.$$

$$(M_{\hat{K}})_{5,5} = \left(\hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y} \; d\hat{x}d\hat{y} = \frac{4}{9},$$
$$(M_{\hat{K}})_{6,5} = \left(\hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^2 \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{7,5} = \left(\hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^3 \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{8,5} = \left(\hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{9,5} = \left(\hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{10,5} = \left(\hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^3 \; d\hat{x}d\hat{y} = 0.$$

$$(M_{\hat{K}})_{6,6} = \left(\hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^2 \; d\hat{x}d\hat{y} = \frac{4}{5},$$
$$(M_{\hat{K}})_{7,6} = \left(\hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^3 \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{8,6} = \left(\hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2\hat{y} \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{9,6} = \left(\hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{10,6} = \left(\hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^3 \; d\hat{x}d\hat{y} = 0.$$

$$(M_{\hat{K}})_{7,7} = \left(\hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^3 \; d\hat{x}d\hat{y} = \frac{4}{7},$$
$$(M_{\hat{K}})_{8,7} = \left(\hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2\hat{y} \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{9,7} = \left(\hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{x}d\hat{y} = \frac{4}{15},$$
$$(M_{\hat{K}})_{10,7} = \left(\hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^3 \; d\hat{x}d\hat{y} = 0.$$

$$(M_{\hat{K}})_{8,8} = \left(\hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{x}d\hat{y} = \frac{4}{15},$$
$$(M_{\hat{K}})_{9,8} = \left(\hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{x}d\hat{y} = 0,$$
$$(M_{\hat{K}})_{10,8} = \left(\hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^3 \; d\hat{x}d\hat{y} = \frac{4}{15}.$$

$$(M_{\hat{K}})_{9,9} = \left(\hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x}d\hat{y} = \frac{4}{15},$$
$$(M_{\hat{K}})_{10,9} = \left(\hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^3 \; d\hat{x}d\hat{y} = 0.$$

$$(M_{\hat{K}})_{10,10} = \left(\hat{\phi}_1 0(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}})\right)_{\hat{K}} = \int_{-1}^{1}\int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^3 \; d\hat{x}d\hat{y} = \frac{4}{7}.$$

B Face integrals of reference basis on rectangular element

B.1 Face integrals of reference basis on $\hat{e}_1$

$$
\begin{aligned}
(M_{K_{\hat{e}_1}})_{1,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot 1 \; d\hat{x} = 2, \\
(M_{K_{\hat{e}_1}})_{2,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{x} \; d\hat{x} = 0, \\
(M_{K_{\hat{e}_1}})_{3,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{y} \; d\hat{x} = -2, \\
(M_{K_{\hat{e}_1}})_{4,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{3}, \\
(M_{K_{\hat{e}_1}})_{5,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{x}\hat{y} \; d\hat{x} = 0, \\
(M_{K_{\hat{e}_1}})_{6,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{y}^2 \; d\hat{x} = 2, \\
(M_{K_{\hat{e}_1}})_{7,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{x}^3 \; d\hat{x} = 0, \\
(M_{K_{\hat{e}_1}})_{8,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{x}^2\hat{y} \; d\hat{x} = -\frac{2}{3}, \\
(M_{K_{\hat{e}_1}})_{9,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0, \\
(M_{K_{\hat{e}_1}})_{10,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} 1 \cdot \hat{y}^3 \; d\hat{x} = -2.
\end{aligned}
$$

$$(M_{K_{\hat{e}_1}})_{1,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot 1 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{2,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{x} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{3,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{4,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{5,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{x}\hat{y} \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{6,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{7,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^3 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{8,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^2\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{9,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{x}\hat{y}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{10,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x} \cdot \hat{y}^3 \; d\hat{x} = 0.$$

$$(M_{K_{\hat{e}_1}})_{1,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot 1 \; d\hat{x} = -2,$$

$$(M_{K_{\hat{e}_1}})_{2,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{3,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{y} \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_1}})_{4,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^2 \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{5,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{6,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{y}^2 \; d\hat{x} = -2,$$

$$(M_{K_{\hat{e}_1}})_{7,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{8,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{9,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{10,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y} \cdot \hat{y}^3 \; d\hat{x} = 2.$$

$$(M_{K_{\hat{e}_1}})_{1,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot 1 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{2,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{3,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y} \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{4,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{5,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{6,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{7,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{8,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2\hat{y} \; d\hat{x} = -\frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{9,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{10,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^3 \; d\hat{x} = -\frac{2}{3}.$$

$$(M_{K_{\hat{e}_1}})_{1,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot 1 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{2,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x} \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{3,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{4,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{5,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{6,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{7,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^3 \; d\hat{x} = -\frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{8,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{9,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{10,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^3 \; d\hat{x} = 0.$$

$$(M_{K_{\hat{e}_1}})_{1,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot 1 \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_1}})_{2,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{3,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y} \; d\hat{x} = -2,$$

$$(M_{K_{\hat{e}_1}})_{4,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{5,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{6,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^2 \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_1}})_{7,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{8,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2\hat{y} \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{9,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{10,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^3 \; d\hat{x} = -2.$$

$$(M_{K_{\hat{e}_1}})_{1,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot 1 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{2,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x} \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{3,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{4,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{5,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y} \; d\hat{x} = -\frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{6,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{7,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^3 \; d\hat{x} = \frac{2}{7},$$

$$(M_{K_{\hat{e}_1}})_{8,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{9,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{10,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^3 \; d\hat{x} = 0.$$

$$(M_{K_{\hat{e}_1}})_{1,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot 1 \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{2,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{3,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{4,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2 \; d\hat{x} = -\frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{5,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{6,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^2 \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{7,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{8,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{9,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{10,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^3 \; d\hat{x} = \frac{2}{3}.$$

$$(M_{K_{\hat{e}_1}})_{1,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot 1 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{2,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{3,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{4,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{5,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y} \; d\hat{x} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{6,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{7,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^3 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_1}})_{8,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^2\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_1}})_{9,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_1}})_{10,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^3 \; d\hat{x} = 0.$$

$$
\begin{aligned}
(M_{K_{\hat{e}_1}})_{1,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot 1 \; d\hat{x} = -2, \\
(M_{K_{\hat{e}_1}})_{2,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x} \; d\hat{x} = 0, \\
(M_{K_{\hat{e}_1}})_{3,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y} \; d\hat{x} = 2, \\
(M_{K_{\hat{e}_1}})_{4,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^2 \; d\hat{x} = -\frac{2}{3}, \\
(M_{K_{\hat{e}_1}})_{5,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}\hat{y} \; d\hat{x} = 0, \\
(M_{K_{\hat{e}_1}})_{6,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^2 \; d\hat{x} = -2, \\
(M_{K_{\hat{e}_1}})_{7,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^3 \; d\hat{x} = 0, \\
(M_{K_{\hat{e}_1}})_{8,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{3}, \\
(M_{K_{\hat{e}_1}})_{9,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0, \\
(M_{K_{\hat{e}_1}})_{10,10} &= \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_1} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^3 \; d\hat{x} = 2.
\end{aligned}
$$

B.2 Face integrals of reference basis on $\hat{e}_2$

$$
\begin{aligned}
(M_{K_{\hat{e}_2}})_{1,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot 1 \; d\hat{y} = 2, \\
(M_{K_{\hat{e}_2}})_{2,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{x} \; d\hat{y} = 2, \\
(M_{K_{\hat{e}_1 2}})_{3,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{y} \; d\hat{y} = 0, \\
(M_{K_{\hat{e}_2}})_{4,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{x}^2 \; d\hat{y} = 2, \\
(M_{K_{\hat{e}_2}})_{5,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{x}\hat{y} \; d\hat{y} = 0, \\
(M_{K_{\hat{e}_2}})_{6,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{y}^2 \; d\hat{y} = \frac{2}{3}, \\
(M_{K_{\hat{e}_2}})_{7,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{x}^3 \; d\hat{y} = 2, \\
(M_{K_{\hat{e}_2}})_{8,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0, \\
(M_{K_{\hat{e}_2}})_{9,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = \frac{2}{3}, \\
(M_{K_{\hat{e}_2}})_{10,1} &= \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} 1 \cdot \hat{y}^3 \; d\hat{y} = 0.
\end{aligned}
$$

$$
\begin{aligned}
(M_{K_{\hat{e}_2}})_{1,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot 1 \; d\hat{y} = 2, \\
(M_{K_{\hat{e}_2}})_{2,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{x} \; d\hat{y} = 2, \\
(M_{K_{\hat{e}_2}})_{3,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{y} \; d\hat{y} = 0, \\
(M_{K_{\hat{e}_2}})_{4,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^2 \; d\hat{y} = 2, \\
(M_{K_{\hat{e}_2}})_{5,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{x}\hat{y} \; d\hat{y} = 0, \\
(M_{K_{\hat{e}_2}})_{6,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{y}^2 \; d\hat{y} = \frac{2}{3}, \\
(M_{K_{\hat{e}_2}})_{7,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^3 \; d\hat{y} = 2, \\
(M_{K_{\hat{e}_2}})_{8,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0, \\
(M_{K_{\hat{e}_2}})_{9,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{x}\hat{y}^2 \; d\hat{y} = \frac{2}{3}, \\
(M_{K_{\hat{e}_2}})_{10,2} &= \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x} \cdot \hat{y}^3 \; d\hat{y} = 0.
\end{aligned}
$$

$$(M_{K_{\hat{e}_2}})_{1,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot 1 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{2,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{x} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{3,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{y} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{4,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{5,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{x}\hat{y} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{6,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{y}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{7,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^3 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{8,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{9,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{10,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y} \cdot \hat{y}^3 \; d\hat{y} = \frac{2}{5}.$$

$$(M_{K_{\hat{e}_2}})_{1,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot 1 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_2}})_{2,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x} \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_2}})_{3,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{4,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_2}})_{5,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{6,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^2 \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{7,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^3 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_2}})_{8,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{9,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{10,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^3 \; d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_2}})_{1,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot 1 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{2,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x} \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{3,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y} \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{4,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{5,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y} \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{6,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^2 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{7,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^3 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{8,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2\hat{y} \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{9,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y}^2 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{10,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^3 \, d\hat{y} = \frac{2}{5}.$$

$$(M_{K_{\hat{e}_2}})_{1,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot 1 \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{2,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x} \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{3,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y} \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{4,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2 \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{5,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y} \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{6,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^2 \, d\hat{y} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_2}})_{7,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^3 \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{8,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2\hat{y} \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{9,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y}^2 \, d\hat{y} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_2}})_{10,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^3 \, d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_2}})_{1,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot 1 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_2}})_{2,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x} \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_2}})_{3,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{4,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_2}})_{5,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{6,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^2 \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{7,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^3 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_2}})_{8,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{9,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{10,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^3 \; d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_2}})_{1,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot 1 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{2,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{3,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{4,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{5,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{6,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{7,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^3 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{8,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_2}})_{9,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_2}})_{10,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^3 \; d\hat{y} = \frac{2}{5}.$$

$$(M_{K_{\hat{e}_2}})_{1,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot 1 \; d\hat{y} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_2}})_{2,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x} \; d\hat{y} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_2}})_{3,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y} \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_2}})_{4,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^2 \; d\hat{y} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_2}})_{5,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y} \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_2}})_{6,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^2 \; d\hat{y} = \frac{2}{5},$$
$$(M_{K_{\hat{e}_2}})_{7,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^3 \; d\hat{y} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_2}})_{8,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_2}})_{9,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = \frac{2}{5},$$
$$(M_{K_{\hat{e}_2}})_{10,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^3 \; d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_2}})_{1,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot 1 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_2}})_{2,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x} \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_2}})_{3,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y} \; d\hat{y} = \frac{2}{5},$$
$$(M_{K_{\hat{e}_2}})_{4,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^2 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_2}})_{5,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}\hat{y} \; d\hat{y} = \frac{2}{5},$$
$$(M_{K_{\hat{e}_2}})_{6,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^2 \; d\hat{y} =, 0$$
$$(M_{K_{\hat{e}_2}})_{7,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^3 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_2}})_{8,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^2\hat{y} \; d\hat{y} = \frac{2}{5},$$
$$(M_{K_{\hat{e}_2}})_{9,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_2}})_{10,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_2} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^3 \; d\hat{y} = \frac{2}{7}.$$

B.3 Face integrals of reference basis on $\hat{e}_3$

$$(M_{K_{\hat{e}_3}})_{1,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot 1 \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{2,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{3,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{y} \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{4,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{5,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{6,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{y}^2 \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{7,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{8,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{9,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{10,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} 1 \cdot \hat{y}^3 \; d\hat{x} = 2.$$

$$(M_{K_{\hat{e}_3}})_{1,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot 1 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{2,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{x} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{3,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{4,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{5,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{x}\hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{6,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{7,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^3 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{8,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^2\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{9,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{x}\hat{y}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{10,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x} \cdot \hat{y}^3 \; d\hat{x} = 0.$$

$$(M_{K_{\hat{e}_3}})_{1,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot 1 \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{2,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{3,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{y} \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{4,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{5,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{6,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{y}^2 \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{7,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{8,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{9,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{10,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y} \cdot \hat{y}^3 \; d\hat{x} = 2.$$

$$(M_{K_{\hat{e}_3}})_{1,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot 1 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{2,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{3,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{4,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{5,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{6,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{7,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{8,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{9,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{10,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^3 \; d\hat{x} = \frac{2}{3}.$$

$$(M_{K_{\hat{e}_3}})_{1,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot 1 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{2,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{3,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{4,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{5,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{6,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{7,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^3 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{8,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{9,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{10,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^3 \; d\hat{x} = 0.$$

$$(M_{K_{\hat{e}_3}})_{1,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot 1 \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{2,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{3,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y} \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{4,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{5,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{6,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^2 \; d\hat{x} = 2,$$

$$(M_{K_{\hat{e}_3}})_{7,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{8,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{9,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{10,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^3 \; d\hat{x} = 2.$$

$$(M_{K_{\hat{e}_3}})_{1,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot 1 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{2,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x} \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{3,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{4,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{5,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y} \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{6,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{7,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^3 \; d\hat{x} = \frac{2}{7},$$

$$(M_{K_{\hat{e}_3}})_{8,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{9,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{10,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^3 \; d\hat{x} = 0.$$

$$(M_{K_{\hat{e}_3}})_{1,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot 1 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{2,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{3,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y} \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{4,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{5,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{6,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^2 \; d\hat{x} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_3}})_{7,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^3 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{8,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_3}})_{9,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{x} = 0,$$

$$(M_{K_{\hat{e}_3}})_{10,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^3 \; d\hat{x} = \frac{2}{3}.$$

$$(M_{K_{\hat{e}_3}})_{1,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot 1 \; d\hat{x} = 0,$$
$$(M_{K_{\hat{e}_3}})_{2,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x} \; d\hat{x} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_3}})_{3,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y} \; d\hat{x} = 0,$$
$$(M_{K_{\hat{e}_3}})_{4,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^2 \; d\hat{x} = 0,$$
$$(M_{K_{\hat{e}_3}})_{5,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y} \; d\hat{x} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_3}})_{6,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^2 \; d\hat{x} = 0,$$
$$(M_{K_{\hat{e}_3}})_{7,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^3 \; d\hat{x} = \frac{2}{5},$$
$$(M_{K_{\hat{e}_3}})_{8,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^2\hat{y} \; d\hat{x} = 0,$$
$$(M_{K_{\hat{e}_3}})_{9,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{x} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_3}})_{10,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^3 \; d\hat{x} = 0.$$

$$(M_{K_{\hat{e}_3}})_{1,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot 1 \; d\hat{x} = 2,$$
$$(M_{K_{\hat{e}_3}})_{2,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x} \; d\hat{x} = 0,$$
$$(M_{K_{\hat{e}_3}})_{3,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y} \; d\hat{x} = 2,$$
$$(M_{K_{\hat{e}_3}})_{4,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^2 \; d\hat{x} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_3}})_{5,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}\hat{y} \; d\hat{x} = 0,$$
$$(M_{K_{\hat{e}_3}})_{6,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^2 \; d\hat{x} = 2,$$
$$(M_{K_{\hat{e}_3}})_{7,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^3 \; d\hat{x} = 0,$$
$$(M_{K_{\hat{e}_3}})_{8,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^2\hat{y} \; d\hat{x} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_3}})_{9,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{x} =, 0$$
$$(M_{K_{\hat{e}_3}})_{10,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_3} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^3 \; d\hat{x} = 2.$$

B.4 Face integrals of reference basis on $\hat{e}_4$

$$(M_{K_{\hat{e}_4}})_{1,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot 1 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_4}})_{2,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{x} \; d\hat{y} = -2,$$

$$(M_{K_{\hat{e}_{1}2}})_{3,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{4,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{x}^2 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_4}})_{5,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{x}\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{6,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{y}^2 \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{7,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{x}^3 \; d\hat{y} = -2,$$

$$(M_{K_{\hat{e}_4}})_{8,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{9,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{10,1} = \left\langle \hat{\phi}_1(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} 1 \cdot \hat{y}^3 \; d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_4}})_{1,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot 1 \; d\hat{y} = -2,$$

$$(M_{K_{\hat{e}_4}})_{2,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{x} \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_4}})_{3,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{4,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^2 \; d\hat{y} = -2,$$

$$(M_{K_{\hat{e}_4}})_{5,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{x}\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{6,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{y}^2 \; d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{7,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^3 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_4}})_{8,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{9,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{x}\hat{y}^2 \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{10,2} = \left\langle \hat{\phi}_2(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x} \cdot \hat{y}^3 \; d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_4}})_{1,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot 1 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{2,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{x} \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{3,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{y} \; d\hat{y} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_4}})_{4,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^2 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{5,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{x}\hat{y} \; d\hat{y} = -\frac{2}{3},$$
$$(M_{K_{\hat{e}_4}})_{6,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{y}^2 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{7,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^3 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{8,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{y} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_4}})_{9,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{10,3} = \left\langle \hat{\phi}_3(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y} \cdot \hat{y}^3 \; d\hat{y} = \frac{2}{5}.$$

$$(M_{K_{\hat{e}_4}})_{1,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot 1 \; d\hat{y} = 2,$$
$$(M_{K_{\hat{e}_4}})_{2,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x} \; d\hat{y} = -2,$$
$$(M_{K_{\hat{e}_4}})_{3,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y} \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{4,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2 \; d\hat{y} = 2,$$
$$(M_{K_{\hat{e}_4}})_{5,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y} \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{6,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^2 \; d\hat{y} = \frac{2}{3},$$
$$(M_{K_{\hat{e}_4}})_{7,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^3 \; d\hat{y} = -2,$$
$$(M_{K_{\hat{e}_4}})_{8,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0,$$
$$(M_{K_{\hat{e}_4}})_{9,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = -\frac{2}{3},$$
$$(M_{K_{\hat{e}_4}})_{10,4} = \left\langle \hat{\phi}_4(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2 \cdot \hat{y}^3 \; d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_4}})_{1,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot 1 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{2,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x} \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{3,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y} \, d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{4,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{5,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y} \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{6,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^2 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{7,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^3 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{8,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}^2\hat{y} \, d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{9,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{x}\hat{y}^2 \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{10,5} = \left\langle \hat{\phi}_5(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y} \cdot \hat{y}^3 \, d\hat{y} = -\frac{2}{5}.$$

$$(M_{K_{\hat{e}_4}})_{1,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot 1 \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{2,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x} \, d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{3,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y} \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{4,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2 \, d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{5,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y} \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{6,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^2 \, d\hat{y} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_4}})_{7,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^3 \, d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{8,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}^2\hat{y} \, d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{9,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{x}\hat{y}^2 \, d\hat{y} = -\frac{2}{5},$$

$$(M_{K_{\hat{e}_4}})_{10,6} = \left\langle \hat{\phi}_6(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^2 \cdot \hat{y}^3 \, d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_4}})_{1,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot 1 \; d\hat{y} = -2,$$

$$(M_{K_{\hat{e}_4}})_{2,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x} \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_4}})_{3,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{4,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2 \; d\hat{y} = -2,$$

$$(M_{K_{\hat{e}_4}})_{5,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{6,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^2 \; d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{7,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^3 \; d\hat{y} = 2,$$

$$(M_{K_{\hat{e}_4}})_{8,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{9,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{10,7} = \left\langle \hat{\phi}_7(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^3 \cdot \hat{y}^3 \; d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_4}})_{1,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot 1 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{2,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{3,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{4,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{5,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y} \; d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{6,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{7,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^3 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{8,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}^2\hat{y} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{9,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{x}\hat{y}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{10,8} = \left\langle \hat{\phi}_8(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}^2\hat{y} \cdot \hat{y}^3 \; d\hat{y} = \frac{2}{5}.$$

$$(M_{K_{\hat{e}_4}})_{1,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot 1 \; d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{2,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x} \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{3,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{4,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^2 \; d\hat{y} = -\frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{5,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{6,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^2 \; d\hat{y} = -\frac{2}{5},$$

$$(M_{K_{\hat{e}_4}})_{7,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^3 \; d\hat{y} = \frac{2}{3},$$

$$(M_{K_{\hat{e}_4}})_{8,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}^2\hat{y} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{9,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_4}})_{10,9} = \left\langle \hat{\phi}_9(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{x}\hat{y}^2 \cdot \hat{y}^3 \; d\hat{y} = 0.$$

$$(M_{K_{\hat{e}_4}})_{1,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_1(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot 1 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{2,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_2(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x} \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{3,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_3(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y} \; d\hat{y} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_4}})_{4,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_4(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{5,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_5(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}\hat{y} \; d\hat{y} = -\frac{2}{5},$$

$$(M_{K_{\hat{e}_4}})_{6,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_6(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{7,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_7(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^3 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{8,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_8(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}^2\hat{y} \; d\hat{y} = \frac{2}{5},$$

$$(M_{K_{\hat{e}_4}})_{9,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_9(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{x}\hat{y}^2 \; d\hat{y} = 0,$$

$$(M_{K_{\hat{e}_4}})_{10,10} = \left\langle \hat{\phi}_{10}(\hat{\mathbf{x}}), \hat{\phi}_{10}(\hat{\mathbf{x}}) \right\rangle_{\hat{e}_4} = \int_{-1}^{1} \hat{y}^3 \cdot \hat{y}^3 \; d\hat{y} = \frac{2}{7}.$$

References

[1] W. Feng, T. L. Lewis, and S. Wise, *Discontinuous galerkin derivative operators with applications to second-order elliptic problems and stability*, Mathematical Methods in the Applied Sciences, 38 (2015), pp. 5160–5182. 1, 13

[2] X. Feng, T. Lewis, and M. Neilan, *Discontinuous Galerkin finite element differential calculus and applications to numerical solutions of linear and nonlinear partial differential equations*, ArXiv e-prints, (2013). 10, 12

[3] T. Lewis and M. Neilan, *Convergence analysis of a symmetric dual-wind discontinuous galerkin method*, Journal of Scientific Computing, (2013), pp. 1–24. 15