Test2 Review of Calculus III

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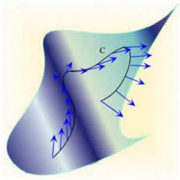
CHPATER 13: CALCULUS OD VECTOR-VALUED FUNCTIONS

Vector-Valued Functions

- The definition of Vector-Valued Functions

$$
\begin{aligned}
\mathbf{r}(t) &= \langle x(t, y(t), z(t)) \rangle \\
&= x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}
\end{aligned}
$$

- The application of the Vector-Valued Functions



- vector parametrization of the path
- trace of a space curve

Calculus of Vector-Valued Functions

- Limit

$$\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle$$

- Derivatives

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

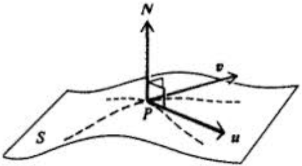
- Differentiation Rules
  - Sum:
  - Scalar multiple Rule:
  - Product Rule:
  - Quotient Rule
  - Chain Rule:

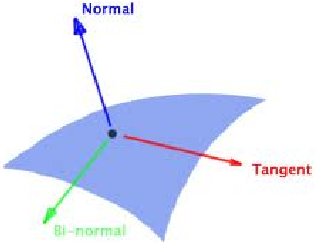
The Derivative as a Tangent Vector, Normal Vector

- Geometric property: The derivative vector $\mathbf{r}'(t)$ points in the direction tangent to the path traced by $\mathbf{r}(t)$ at $t = t_0$

$\mathbf{r}'(t_0)$: **tangent vector or velocity vector at $\mathbf{r}(t_0)$**

$\mathbf{r}''(t_0)$: **Normal vector or acceleration vector at $\mathbf{r}(t_0)$**





Application of Derivative

- The direction vector of the tangent line :

$$T(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

- United Tangent vector: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$
- Length of a Path

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

- United normal vector: $\mathbf{N}(t) = \frac{\mathbf{r}''(t)}{\|\mathbf{r}''(t)\|}$
- Curvature

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

Example: Parametric equation for the tangent line

Find parametric equation for the tangent line to the curve $\mathbf{r}(t) = \left\langle t, 2t^2, t^3 \right\rangle$ at the point (1,2,1).

Solution

1. *Find out the specific t*

$$\begin{cases} 1 = t \\ 2 = 2t^2 \\ 1 = t^3 \end{cases} \quad \Rightarrow t = 1.$$

2. *Find out the direction vector of the tangent line*
   - *direction vector for all t:*

$$\boldsymbol{r}'(t) = \frac{d}{dt}\left\langle t, 2t^2, t^3 \right\rangle = \left\langle 1, 4t, 3t^2 \right\rangle$$

   - *direction vector for specific t=1:* $\boldsymbol{r}'(1) = \langle 1, 4, 3 \rangle$.

Parametric equation for the tangent line (Con't)

3. Write out the parametric equation for the tangent line

$$\begin{aligned} T(t) &= \mathbf{r}(1) + t\mathbf{r}'(1) \\ &= \langle 1, 2, 1 \rangle + t \langle 1, 4, 3 \rangle \end{aligned}$$

Hence, the parametric equation for the tangent line is as follows

$$\begin{cases} x = 1 + t, \\ y = 2 + 4t, \\ z = 1 + 3t. \end{cases}$$

Example: Arc-length of the curve

Find the arc-length of the curve $\mathbf{r} = \langle -\cos(2t), \sin(2t), t\rangle$ over the interval $0 \leq t \leq \pi$.

Solution

1. *Find out $\boldsymbol{r}'(t)$*

$$\boldsymbol{r}'(t) = \frac{d}{dt}\langle -\cos(2t), \sin(2t), t\rangle = \langle 2\sin(2t), 2\cos(2t), 1\rangle$$

2. *Find out the length of $\boldsymbol{r}'(t)$*

$$\|\boldsymbol{r}'(t)\| \quad = \quad \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 1^2} = \sqrt{4+1} = \sqrt{5}.$$

3. *Compute the integral for $0 \leq t \leq \pi$*

$$\int_0^\pi \|\boldsymbol{r}'(t)\|\, dt = \int_0^\pi \sqrt{5}dt = \sqrt{5}\int_0^\pi 1dt = \sqrt{5}(\pi - 0) = \sqrt{5}\pi.$$

Example: Tangent Vector, Normal Vector, Curvature

Find the curvature function $\kappa(t)$ for the curve $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^2\mathbf{k}$.

Solution

Since $\boldsymbol{r}(t) = t\boldsymbol{i} + 2t^2\boldsymbol{j} + t^2\boldsymbol{k}$, *then* $\boldsymbol{r}(t) = \left\langle t, 2t^2, t^2 \right\rangle$.

1. *Compte the* 1_{st} *derivative of* $\boldsymbol{r}(t)$ (*Tangent Vector*):

$$\boldsymbol{r}'(t) = \langle 1, 4t, 2t \rangle$$

2. *Compte the* 2_{nd} *derivative of* $\boldsymbol{r}(t)$ (*Normal Vector*):

$$\boldsymbol{r}''(t) = \langle 0, 4, 2 \rangle$$

3. *Compte the cross-product* $\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)$

$$\begin{aligned}\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 4t & 2t \\ 0 & 4 & 2 \end{vmatrix} \\ &= (8t - 8t)\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k} = -2\boldsymbol{j} + 4\boldsymbol{k}.\end{aligned}$$

Tangent Vector, Normal Vector, Curvature (Con't)

3. Compte the lengths:

$$\|\mathbf{r}'(t)\| = \sqrt{1^2 + (4t)^2 + (2t)^2} = \sqrt{1 + 20t^2}.$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{(-2)^2 + 4^2} = \sqrt{20} = 2\sqrt{5}.$$

4. Compute the curvature

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{2\sqrt{5}}{(1 + 20t^2)^{3/2}}.$$

Vector-Valued Integration

- Definition

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle$$

- Antiderivative function: An antiderivative of $\mathbf{r}(t)$ is a vector-valued function $\mathbf{R}(t)$ such that $\mathbf{R}'(t) = \mathbf{r}(t)$
- Fundamental Theorem

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(b) - \mathbf{R}(b)$$

Application: velocity, acceleration and path

$$\begin{cases} \mathbf{v}(t) = \mathbf{r}'(t), \\ \mathbf{a}(t) = \mathbf{r}''(t). \end{cases} \quad \Rightarrow \begin{cases} \mathbf{r}(t) = \displaystyle\int \mathbf{v}(t)dt + \mathbf{r}_0, \\ \mathbf{v}(t) = \mathbf{r}'(t) = \displaystyle\int \mathbf{a}(t)dt + \mathbf{v}_0. \end{cases}$$

Example: velocity, acceleration and path

Given the acceleration $\mathbf{a}(t) = t\mathbf{i} + t^2\mathbf{k}$, initial velocity $\mathbf{v}(0) = \mathbf{k}$ and initial position $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$ of a moving particle, find the position function $\mathbf{r}(t)$.

Solution

1. *Compte speed $\boldsymbol{v}(t)$*

$$\begin{aligned}
\boldsymbol{v}(t) &= \boldsymbol{r}'(t) = \int \boldsymbol{a}(t)dt + \boldsymbol{v}_0 = \int \left\langle t, 0, t^2 \right\rangle dt + \langle 0, 0, 1 \rangle \\
&= \left\langle \frac{1}{2}t^2, 0, \frac{1}{3}t^3 \right\rangle + \langle 0, 0, 1 \rangle = \left\langle \frac{1}{2}t^2, 0, \frac{1}{3}t^3 + 1 \right\rangle.
\end{aligned}$$

2. *Compute the path*

$$\begin{aligned}
\boldsymbol{r}(t) &= \int \boldsymbol{v}(t)dt + \boldsymbol{r}_0 = \int \left\langle \frac{1}{2}t^2, 0, \frac{1}{3}t^3 \right\rangle dt + \langle 1, 1, 0 \rangle \\
&= \left\langle \frac{1}{6}t^3, 0, \frac{1}{12}t^4 \right\rangle = \left\langle \frac{1}{6}t^3 + 1, 1, \frac{1}{12}t^4 + t \right\rangle.
\end{aligned}$$

DIFFERENTIATION IN SEVERAL VARIABLES (PARTIAL DERIVATIVES)

Partial Derivatives

- Definition: The partial derivatives are the rates of change with respect to each variable separately.
- How to compute the 1^{st} order partial derivatives ?
- How to compute the higher order partial derivatives ?

Example: 1^{st} order partial derivatives

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, given that $f(x, y) = xe^{xy}$. Gold Rule: Treating the rest of symbols as constants, when you compute the partial derivative.

1. Compte the $\frac{\partial f}{\partial x}$: Now, treat y as a constant

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(xe^{xy}\right) = \frac{\partial}{\partial x}(x)\, e^{xy} + x\frac{\partial}{\partial x}\left(e^{xy}\right) \quad \text{(product rule)} \\ &= e^{xy} + x \cdot e^{xy} y \quad \text{(chain rule)} \\ &= (1 + xy)e^{xy}. \end{aligned}$$

2. Compte the $\frac{\partial f}{\partial y}$: Now, treat x as a constant

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(xe^{xy}\right) = \frac{\partial}{\partial y}(x)\, e^{xy} + x\frac{\partial}{\partial y}\left(e^{xy}\right) \quad \text{(product rule)} \\ &= 0 + x \cdot e^{xy} x \quad \text{(chain rule)} = x^2 e^{xy}. \end{aligned}$$

Example: Higher order partial derivatives

Find $\frac{\partial^2 f}{\partial x^2}$, where $f(x, y) = \sqrt{x^2 + y^2}$.

1. compute 1^{st} derivative

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left((x^2 + y^2)^{1/2}\right) \\
&= \frac{1}{2}\left((x^2 + y^2)^{-1/2}\right)\frac{\partial}{\partial x}\left(x^2 + y^2\right) \\
&= \frac{1}{2}\left((x^2 + y^2)^{-1/2}\right)2x \\
&= x(x^2 + y^2)^{-1/2}.
\end{aligned}$$

Example: Higher order partial derivatives (Con't)

2. compute 2^{nd} derivative

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(x(x^2+y^2)^{-1/2}\right) \\
&= \frac{\partial}{\partial x}(x)(x^2+y^2)^{-1/2} + x\frac{\partial}{\partial x}\left((x^2+y^2)^{-1/2}\right) \\
&= (x^2+y^2)^{-1/2} + x\left(-\frac{1}{2}(x^2+y^2)^{-3/2}\right)\frac{\partial}{\partial x}(x^2+y^2) \\
&= (x^2+y^2)^{-1/2} + x\left(-\frac{1}{2}(x^2+y^2)^{-3/2}\right)2x \\
&= (x^2+y^2)^{-1/2} - x^2(x^2+y^2)^{-3/2}
\end{aligned}
$$

Application of partial derivatives

- Linearization
- Tangent plane
- Linear Approximation
- Gradient of a function and its application
- Directional Derivative
- Optimization
- Lagrange multipliers

Linearization

- Definition: Linearization at (a, b), defined by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Tangent plane to the graph at (a, b, f (a, b))

$$z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Linear Approximation: $f(x, y) \approx L(x, y)$ for (x,y) near (a,b) where

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

2D: $f(a + h, b + k) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)k$
3D: $f(a + h, b + k, c + m) \approx$
$f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)k + f_z(a, b, c)(z - c)$

Example: Linearization tangent plane

Find the equations of the tangent plane to the surface $z = e^{x^2-y^2}$ at the point $(1, -1, 1)$

1. $f(x, y) = e^{x^2-y^2}$
2. compute $f(a, b), f_x(a, b), f_y(a, b)$:

$$\begin{cases} f(1,-1) & = 1, \\ f_x(x,y) & = 2xe^{x^2-y^2}, \\ f_y(x,y) & = -2ye^{x^2-y^2}. \end{cases} \Rightarrow \begin{cases} f(1,-1) & = 1, \\ f_x(1,-1) & = 2, \\ f_y(1,-1) & = 2. \end{cases}$$

3. Plugging in
   $z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

$$z = L(x, y) = z = 1 + 2(x - 1) + 2(y + 1) = 1 + 2x + 2y.$$

Example: Linearization approximation

Compute $e^{0.01^2-0.02}$ by hand

1. Set up the math model $f(x,y)=e^{x^2-y}$
2. $e^{0.01^2-0.02}=f(0.01,0.02)$
3. Linearization approximation

$$\begin{aligned} f(0.01,0.02) &= f(0+0.01,0+0.02) \\ &\approx f(0,0)+f_x(0,0)*0.01+f_y(0,0)0.02 \end{aligned}$$

$$\begin{cases} f(x,y) &= e^{x^2-y}, \\ f_x(x,y) &= 2xe^{x^2-y}, \\ f_y(x,y) &= -e^{x^2-y}. \end{cases} \Rightarrow \begin{cases} f(0,0) &= 1, \\ f_x(0,0) &= 0, \\ f_y(0,0) &= -1. \end{cases}$$

4. Plugging in

$$f(0.01,0.02)\approx 1+0*0.01-1*0.02=0.98$$

Gradient

- Gradient definition

$$2D : \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \quad 3D : \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

- Directional Derivatives: directional derivative in the direction of a unit vector $\mathbf{u} = \langle h, k \rangle$ is the limit

$$D_{\mathbf{u}} f(a, b) = \lim_{t \to 0} \frac{f(a + th, b + tk) - f(a, b)}{t}$$

- Computing the Directional Derivative

$$D_{\mathbf{u}} f(a, b) = \nabla f_{(a,b)} \cdot \frac{\mathbf{v}}{||\mathbf{v}||}, \quad \mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

Example: Tangent plane by using gradient

Find the equations of the tangent plane to the surface $z = e^{x^2 - y^2}$ at the point $(1, -1, 1)$

1. Define $F(x, y, z) = e^{x^2 - y^2} - z$
2. Compute the gradient

$$\nabla F(x, y, z) = \left\langle 2xe^{x^2-y^2}, -2ye^{x^2-y^2}, -1 \right\rangle$$

   So,

$$\nabla F(1, -1, 1) = \langle 2, 2, -1 \rangle .$$

3. Plugging in to the tangent plane equation

$$\nabla F \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

   We get

$$2(x - 1) + 2(y + 1) - 1(z - 1) = 0.$$

   Thus the tangent plane is:

$$z = 1 + 2(x - 1) + 2(y + 1) = 1 + 2x + 2y.$$

Example: Directional Derivative

Find the change rate at the specific direction:

Example

Find the rate of change of pressure at the point $Q = (1, 2, 1)$ in the direction if $\mathbf{v} = \langle 0, 1, 1 \rangle$, assuming that the pressure is given by

$$f(x, y, z) = 1000 + 0.01(yz^2 + x^2z - xy^2)$$

1. First compute the gradient at $Q = (1, 2, 1)$:
2. compute the derivative with respect to **v**
3. The rate of change per kilometer is the directional derivative

OPTIMIZATION

Critical Point

- Definition: A point $P = (a, b)$ in the domain of $f(x, y)$ is called a critical point if:
  1. $f_x(a, b) = 0$ or $f_x(a, b)$ does not exist, and
  2. $f_y(a, b) = 0$ or $f_y(a, b)$ does not exist
- Fermat's Theorem : If $f(x, y)$ has a local minimum or maximum at $P = (a, b)$, then (a, b) is a critical point of $f(x, y)$.
- Hessian determinant:

$$D = D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$$

- Second Derivative Test:
  1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
  2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum
  3. If $D < 0$, then f has a saddle point at (a,b)
  4. If $D = 0$, the test inconclusive

Example: Critical Point

Find the critical points of

$$f(x,y) = (x^2+y^2)e^{-x}$$

and analyze them using the Second Derivative Test.

1. Find the critical point

$$\begin{cases} f_x(x,y) & = 2xe^{-x} - (x^2+y^2)e^{-x} = 0 \\ f_y(x,y) & = 2ye^{-x} = 0. \end{cases} \Rightarrow \begin{cases} x = 0, 2 \\ y = 0. \end{cases}$$

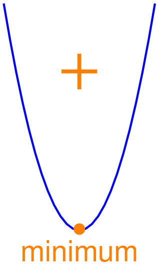
2. Compute the second-order partials:

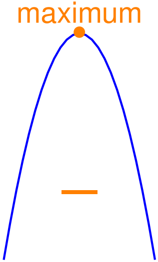
$$\begin{cases} f_{xx}(x,y) & = \frac{\partial}{\partial x}\left(2xe^{-x} - (x^2+y^2)e^{-x}\right) = (2-4x+x^2+y^2)e^{-x} \\ f_{yy}(x,y) & = \frac{\partial}{\partial y}\left(2ye^{-x}\right) = 2e^{-x} \\ f_{xy}(x,y) & = \frac{\partial}{\partial x}\left(2ye^{-x}\right) = -2ye^{-x}. \end{cases}$$

Example: Critical Point (Con't)

3. Apply the Second Derivative Test.

Critical Point	f_{xx}	f_{yy}	f_{xy}	D	Type
(0,0)	2	2	0	$= 4$	Local min
(2,0)	$-2e^{-2}$	$2e^{-2}$	0	$-4e^{-4}$	Saddle





Lagrange multipliers

- Constraint optimization problem

$$\begin{array}{rl} \min & f(x,y) \\ \text{subject to} & g(x,y) \end{array}$$

- Unconstraint auxiliary optimization problem

$$\begin{aligned} F(x,y) &= \text{original fun} - \lambda\text{constraint} \\ &= f(x,y) - \lambda g(x,y) \end{aligned}$$

Assume that $f(x,y)$ and $g(x,y)$ are differentiable functions. If $f(x,y)$ has a local minimum or a local maximum on the constraint curve $g(x,y)=0$ at $P=(a,b)$, and then there is a scalar λ such that

$$\nabla F(x,y) = \nabla f(x,y) - \lambda \nabla g(x,y) = 0$$

Example: Lagrange multipliers

Find the extreme values of $f(x, y) = xy$ under the constraint $4x^2 + 9y^2 = 32$

1. Write out the Lagrange Equation.

$$\begin{aligned} F(x, y) &= f(x, y) - \lambda g(x, y) \\ &= xy - \lambda(4x^2 + 9y^2 - 32) \end{aligned}$$

$$\begin{cases} F_x(x, y) & = y - \lambda(8x) = 0 \\ F_y(x, y) & = x - \lambda(18y) = 0 \end{cases} \Rightarrow \begin{cases} x = \lambda(18y) \\ y = \lambda(8x). \end{cases}$$

$$\begin{cases} x = \lambda(18y) \\ y = \lambda(8x) \\ 4x^2 + 9y^2 = 32 \end{cases} \Rightarrow \begin{cases} \lambda = ? \\ x = \pm 2 \\ y = \pm \frac{4}{3}. \end{cases}$$

Example: Lagrange multipliers (Con't)

2. We obtain the following critical points:
   $(-2,-\frac{4}{3})$,$(-2,\frac{4}{3})$,$(2,-\frac{4}{3})$,$(2,\frac{4}{3})$
3. Calculate f at the critical points: Since the extreme values of $f(x,y)=xy$ attain at the critical points

$$\begin{cases} f(-2,-\frac{4}{3}) = f(2,\frac{4}{3}) = \frac{8}{3} \\ f(-2,\frac{4}{3}) = f(2,-\frac{4}{3}) = -\frac{8}{3} \end{cases} \Rightarrow \begin{cases} \min = -\frac{8}{3} \\ \max = \frac{8}{3} \end{cases}$$

Thank you!