Math 241 Quiz #5 version: 1

Instructor: Wenqiang Feng Name: ______________________________

(1) (5 points) Evaluare the triple integral $\int\int\int_E \frac{1}{\sqrt{x^2+y^2+z^2}} dV$, where

$$E = \left\{(x, y, z)|x^2 + y^2 + z^2 \leq 4 \text{ and } x, y \geq 0\right\}$$

.

1. For ρ,

$$0 \leq \rho^2 = x^2 + y^2 + z^2 \leq 4 \Rightarrow 0 \leq \rho \leq 2.$$

2. For θ

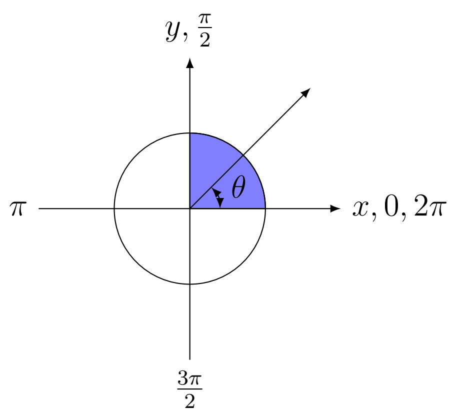

FIGURE 1. The projection region of sphere onto xy-plane.

$$x, y \geq 0 \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}.$$

3. For ϕ

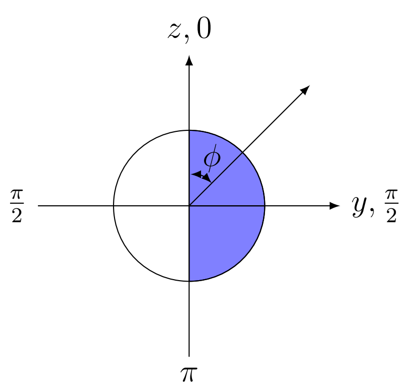

FIGURE 2. The projection region of sphere onto yz-plane.

$$y \geq 0 \Rightarrow 0 \leq \phi \leq \pi.$$

(4) Change coordinate

$$
\begin{aligned}
\int\int\int_E \frac{1}{\sqrt{x^2+y^2+z^2}} dV &= \int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{2} \frac{1}{\rho}\rho^2 \sin(\phi) d\rho d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{2} \rho \sin(\phi) d\rho d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\pi} \sin(\phi) \int_0^{2} \rho d\rho d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\pi} \sin(\phi) \frac{\rho^2}{2}\Big|_0^2 d\phi d\theta \\
&= 2\int_0^{\frac{\pi}{2}}\int_0^{\pi} \sin(\phi) d\phi d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (-\cos(\phi))\Big|_0^{\pi} d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (-\cos(\pi)+\cos(0))\, d\theta \\
&= 4\int_0^{\frac{\pi}{2}} d\theta \\
&= 2\pi.
\end{aligned}
$$

(2) (5 points) Find the total mass and center of the mass of the semi-desk

$$D = \left\{(x,y)|x^2+y^2 \le 1, x \ge 0\right\},$$

given the density function $\rho(x,y) = \sqrt{x^2+y^2}$.

(1) For r.

$$0 \le r^2 = x^2+y^2 \le 1 \Rightarrow 0 \le r \le 1.$$

(2) For θ. Since θ only involves in xy-plane, so we project the sphere to xy-plane.

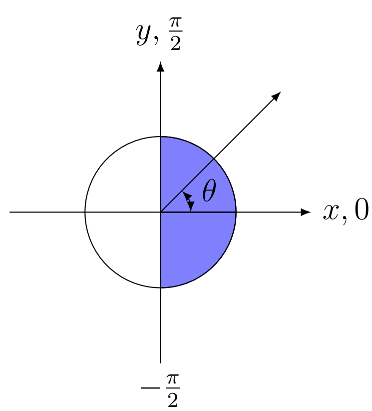

FIGURE 3. The projection region of sphere onto xy-plane.

$$x \ge 0 \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

3 Change coordinate

$$
\begin{aligned}
M &= \int\int \rho(x,y)dA \\
&= \int\int \sqrt{x^2+y^2}dA \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r\cdot r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}r^3\Big|_0^1 dr d\theta \\
&= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \\
&= \frac{\pi}{3}.
\end{aligned}
$$

$$
\begin{aligned}
M_x &= \int\int y\rho(x,y)dA \\
&= \int\int y\sqrt{x^2+y^2}dA \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r\sin(\theta)r\cdot r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\theta)\int_0^1 rr\cdot r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\theta)\frac{1}{4}r^4\Big|_0^1 dr d\theta \\
&= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\theta)d\theta \\
&= \frac{1}{4}\left(-\cos(\theta)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \int\int x\rho(x,y)dA \\
&= \int\int x\sqrt{x^2+y^2}dA \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r\sin(\theta)r\cdot r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\int_0^1 rr\cdot r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\frac{1}{4}r^4\Big|_0^1 dr d\theta \\
&= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)d\theta
\end{aligned}
$$

$$= \quad \frac{1}{4}\left(\sin(\theta)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{4}\cdot 2 = \frac{1}{2}.$$

4

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M},\frac{M_x}{M}\right) = \left(\frac{3}{2\pi},0\right).$$

Math 241 Quiz #5 version: 2

Instructor: Wenqiang Feng Name: ____________________

(1) (5 points) Evaluare the triple integral $\int\int\int_E \sqrt{x^2+y^2+z^2}dV$, where

$$E = \left\{(x,y,z)|x^2+y^2+z^2 \leq 1 \text{ and } z \geq 0\right\}$$

.

1 For ρ,

$$0 \leq \rho^2 = x^2+y^2+z^2 \leq 1 \Rightarrow 0 \leq \rho \leq 1.$$

2 For θ

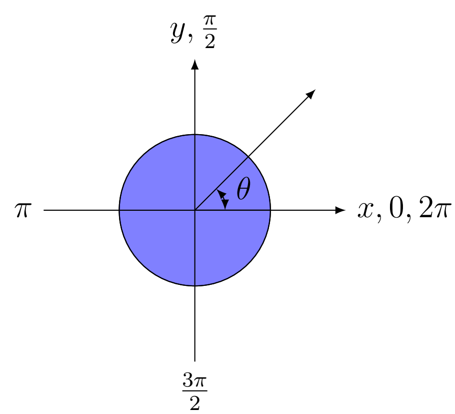

FIGURE 4. The projection region of sphere onto xy-plane.

$$\text{no constrains for } x, y \Rightarrow 0 \leq \theta \leq 2\pi.$$

3 For ϕ

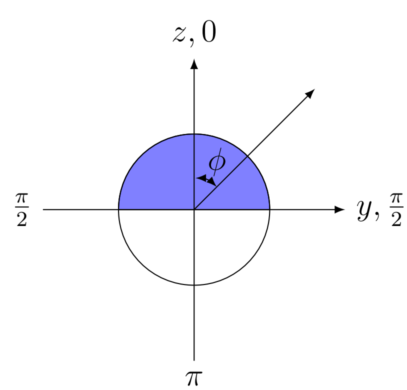

FIGURE 5. The projection region of sphere onto yz-plane.

$$z \geq 0 \Rightarrow 0 \leq \phi \leq \frac{\pi}{2}.$$

(4) Change coordinate

$$\int\int\int_E \sqrt{x^2+y^2+z^2}dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho\rho^2\sin(\phi)d\rho d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3\sin(\phi)d\rho d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin(\phi)\int_0^1 \rho d\rho^3 d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin(\phi)\frac{1}{4}\rho^4\Big|_0^1 d\phi d\theta$$
$$= \frac{1}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin(\phi)d\phi d\theta$$
$$= \frac{1}{4}\int_0^{2\pi}(-\cos(\phi))\Big|_0^{\frac{\pi}{2}} d\theta$$
$$= \frac{1}{4}\int_0^{2\pi}\Big(-\cos(\frac{\pi}{2})+\cos(0)\Big)d\theta$$
$$= \frac{1}{4}\int_0^{2\pi}d\theta$$
$$= \frac{\pi}{2}.$$

(2) (5 points) Find the total mass and center of the mass of the semi-desk

$$D = \Big\{(x,y)|x^2+y^2\le 1, y\ge 0\Big\},$$

given the density function $\rho(x,y) = \sqrt{x^2+y^2}$.

(1) For r.

$$0 \le r^2 = x^2+y^2 \le 1 \Rightarrow 0 \le r \le 1.$$

(2) For θ. Since θ only involves in xy-plane, so we project the sphere to xy-plane.

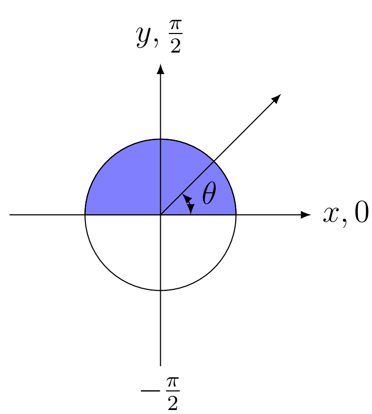

FIGURE 6. The projection region of sphere onto xy-plane.

$$y \ge 0 \Rightarrow 0 \le \theta \le \pi.$$

3 Change coordinate

$$\begin{aligned}
M &= \int\int \rho(x,y)dA \\
&= \int\int \sqrt{x^2+y^2}dA \\
&= \int_0^{\pi}\int_0^1 r\cdot rdrd\theta \\
&= \int_0^{\pi} \frac{1}{3}r^3\Big|_0^1 drd\theta \\
&= \frac{1}{3}\int_0^{\pi} d\theta \\
&= \frac{\pi}{3}.
\end{aligned}$$

$$\begin{aligned}
M_x &= \int\int y\rho(x,y)dA \\
&= \int\int y\sqrt{x^2+y^2}dA \\
&= \int_0^{\pi}\int_0^1 r\sin(\theta)r\cdot rdrd\theta \\
&= \int_0^{\pi}\sin(\theta)\int_0^1 rr\cdot rdrd\theta \\
&= \int_0^{\pi}\sin(\theta)\frac{1}{4}r^4\Big|_0^1 drd\theta \\
&= \frac{1}{4}\int_0^{\pi}\sin(\theta)d\theta \\
&= \frac{1}{4}\left(-\cos(\theta)\right)\Big|_0^{\pi} = \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
M_y &= \int\int x\rho(x,y)dA \\
&= \int\int x\sqrt{x^2+y^2}dA \\
&= \int_0^{\pi}\int_0^1 r\sin(\theta)r\cdot rdrd\theta \\
&= \int_0^{\pi}\cos(\theta)\int_0^1 rr\cdot rdrd\theta \\
&= \int_0^{\pi}\cos(\theta)\frac{1}{4}r^4\Big|_0^1 drd\theta \\
&= \frac{1}{4}\int_0^{\pi}\cos(\theta)d\theta \\
&= \frac{1}{4}\left(\sin(\theta)\right)\Big|_0^{\pi} = \frac{1}{4}\cdot 0 = 0.
\end{aligned}$$

4

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(0, \frac{3}{2\pi}\right).$$