# MATH 574: Finite Element Method Homework 2

Due on December 12, 2014

*TTH 9:40am-10:55am*

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# Problem 1

Show that the Gram matrix $G_{ij} = (\phi_j, \phi_i)\,,\ i, j = 1, \cdots, N$ is symmetric positive definite.

*Proof.* 1. Symmtric-ness: $G_{ij} = (\phi_j, \phi_i)$ is obviously symmetric by the construction. Moreover, $G_{ii} > 0$, since $\phi_i$ are Finite Element Basis functions and $(\phi_i, \phi_i) > 0$ for all $i = 1, \cdots, N$.

2. positive-ness: Since G is symmetric, then it has a spectral decomposition:

$$G = \sum_{i}^{N} \lambda_i u_i u_i^T = U\Lambda U^T, \quad \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N) \tag{1}$$

where the matrix $U = [u_1, u_2, \cdots, u_N]$ is orthogonal, and contanis the eigenvector of G, while the diagonal matrix $\Lambda$ contains the eigenvalues of G. Since $G_{ii}$ are positive, so $\lambda_i$ should be positive from (1). Moreover

$$\begin{aligned}
\mathbf{x}^T G \mathbf{x} &= \mathbf{x}^T U \Lambda U^T \mathbf{x} \\
&= \left(U^T \mathbf{x}\right)^T \Lambda \left(U^T \mathbf{x}\right) \\
&= \mathbf{y}^T \Lambda \mathbf{y} \\
&= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_N y_N^2.
\end{aligned}$$

Since $\lambda_i > 0$ and $y_i$ are not all zeros, so the resulting diagonal quadratic form is positive definite.

□

# Problem 2

Consider the matrix $\tilde{S} = G^{-1/2} S G^{-1/2}$ where $G$ is the Gram matrix and $S$ is the stiffness matrix. Show that the eigenvalues of $\tilde{S}$ are real and positive.

*Proof.* 1.

**Lemma 0.1.** *If $A \in \mathbb{C}^{n\times n}_{sym}$, then the eigenvalue of $A$ are real.*

*Proof.* Let $\lambda$ be arbitrary eigenvalue of A, then

$$(Ax, x) = (\lambda x, x) = \lambda(x, x),$$
$$(Ax, x) = (x, A^* x) = (x, Ax)(x, \lambda x) = \overline{\lambda}(x, x),$$

and then $\lambda = \overline{\lambda}$, so $\lambda$ is real. □

Since the stiffness matrix $S$ and Gram matrix $G$ are SPD. So,

$$\begin{aligned}
\tilde{S}^T &= \left(G^{-1/2} S G^{-1/2}\right)^T \\
&= \left(G^{-1/2}\right)^T S^T \left(G^{-1/2}\right)^T \\
&= \left(G^T\right)^{-1/2} S^T \left(G^T\right)^{-1/2} \\
&= G^{-1/2} S G^{-1/2} = \tilde{S}.
\end{aligned}$$

Hence, $\tilde{S}$ is symmetric. Therefore, the eigenvalues of $\tilde{S}$ are real by Lemma.0.1.

2. Since $S$ is SPD, then S has the following decomposition:

$$S = U\Lambda U^T, \ \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_N), \ \lambda_i > 0, \forall i.$$

$$\begin{aligned}
\mathbf{x}^T \tilde{S}\mathbf{x} &= \mathbf{x}^T G^{-1/2} U\Lambda U^T G^{-1/2}\mathbf{x} \\
&= \left(U^T G^{-1/2}\mathbf{x}\right)^T \Lambda \left(U^T G^{-1/2}\mathbf{x}\right) \\
&= \mathbf{y}^T \Lambda \mathbf{y} \\
&= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_N y_N^2.
\end{aligned}$$

Since $\lambda_i > 0$ and $y_i$ are not all zeros, so the resulting diagonal quadratic form is positive definite.

□

# Problem 3

Let the function $u_h$ in $V_h$ be given by $u_h = \sum_{j=1}^N \xi_j \phi_j(x)$. Suppose we know that $\left\|G^{1/2}\xi\right\| \leq 1$. Show that $\|u_h\| \leq 1$.

*Proof.*

$$\begin{aligned}
\|u_h\| &= (u_h, u_h)^{1/2} \\
&= \left(\sum_{j=1}^N \xi_j \phi_j(x), \sum_{i=1}^N \xi_i \phi_i(x)\right)^{1/2} \\
&= \left(\sum_{i,j=1}^N \xi_j \left(\phi_j(x), \phi_i(x)\right) \xi_i\right)^{1/2} \\
&= \left(\xi^T G \xi\right)^{1/2} \\
&= \left(\xi^T G^{1/2} G^{1/2} \xi\right)^{1/2} \\
&= \left(\left(G^{1/2}\xi\right)^T \left(G^{1/2}\xi\right)\right)^{1/2} \\
&= \left\|G^{1/2}\xi\right\| \leq 1.
\end{aligned}$$

□

# Problem 4

The *transition* matrix $T$ for the trapezoidal method for the parabolic problem is given by

$$T = \left(G + \frac{1}{2}\tau S\right)^{-1} \left(G - \frac{1}{2}\tau S\right).$$

As was done for the Backward Euler method, use the spectral approach to show the bound $\|T\| \leq 1$. Use this to show the stability of the Trapezoidal method for the parabolic problem with $f = 0$ by showing that $\left\|u_h^{n+1}\right\| \leq \|u_h^n\|$.

*Proof.* Since

$$\left(G + \frac{1}{2}\tau S\right)^{-1} \left(G - \frac{1}{2}\tau S\right) = \left(G + \frac{1}{2}\tau S\right)^{-1} G G^{-1} \left(G - \frac{1}{2}\tau S\right)$$

$$
\begin{aligned}
&= \left(G^{-1}\left(G+\frac{1}{2}\tau S\right)\right)^{-1}\left(G^{-1}\left(G-\frac{1}{2}\tau S\right)\right)\\
&= \left(I+\frac{1}{2}\tau G^{-1}S\right)\left(I-\frac{1}{2}\tau G^{-1}S\right),
\end{aligned}
$$

Then, for any $\tau > 0$,

$$
\begin{aligned}
\rho(T) &= \rho\left(\left(G+\frac{1}{2}\tau S\right)^{-1}\left(G-\frac{1}{2}\tau S\right)\right)\\
&= \rho\left(\left(I+\frac{1}{2}\tau G^{-1}S\right)\left(I-\frac{1}{2}\tau G^{-1}S\right)\right)\\
&= \max_{1\le k\le N}\frac{\left|1-\frac{1}{2}\tau\lambda_k(G^{-1}S)\right|}{1+\frac{1}{2}\tau\lambda_k(G^{-1}S)} \le 1.
\end{aligned}
$$

Hence, $\|T\| \le 1$. The Trapezoidal method (Crank-Nicolson method) for homogeneous heat equation can be read as

$$u_h^{n+1} = T u_h^n,$$

so,

$$\left\|u_h^{n+1}\right\|^2 = \left(u_h^{n+1}, u_h^{n+1}\right) \le \left(u_h^n, u_h^{n+1}\right) \le \left\|u_h^n\right\|\left\|u_h^{n+1}\right\|.$$

Hence

$$\left\|u_h^{n+1}\right\| \le \left\|u_h^n\right\|.$$

□

# Problem 5

As an alternative to Problem.4 above, use the energy method to show the stability of the Trapezoidal method for the parabolic problem with $f = 0$. Indeed, this is much simper once you know how to choose the test function $v_h$.

*Proof.* The Trapezoidal method (Crank-Nicolson method) for heat equation can be read as follows:

$$\left(\frac{u_h^{n+1}-u_h^n}{\tau_n}, v_h\right) + \frac{1}{2}a\left(u_h^{n+1}+u_h^n, v_h\right) = \frac{1}{2}\left(f(t^{n+1})+f(t^n), v_h\right).$$

For homogeneous case, we get

$$\left(u_h^{n+1}-u_h^n, v_h\right) + \frac{1}{2}\tau\left(\nabla u_h^{n+1}+\nabla u_h^n, \nabla v_h\right) = 0.$$

Taking $v_h = \frac{u_h^{n+1}+u_h^n}{2}$ yields

$$\left(u_h^{n+1}-u_h^n, \frac{u_h^{n+1}+u_h^n}{2}\right) + \frac{1}{2}\tau\left(\nabla u_h^{n+1}+\nabla u_h^n, \frac{\nabla u_h^{n+1}+\nabla u_h^n}{2}\right) = 0.$$

Hence,

$$\frac{1}{2}\left\|u_h^{n+1}\right\|^2 - \frac{1}{2}\left\|u_h^n\right\|^2 + \frac{1}{4}\tau\left\|\nabla u_h^{n+1}+\nabla u_h^n\right\|^2 = 0.$$

Therefore

$$\left\|u_h^{n+1}\right\|^2 \le \left\|u_h^n\right\|^2.$$

Hence,

$$\left\|u_h^{n+1}\right\| \le \left\|u_h^n\right\|.$$

□