Math 241 Quiz #5 version: 1

Instructor: Wenqiang Feng Name: ______________________

(1) (5 points) Find a potential function of the field $\mathbf{F} = \langle y, x\rangle$ and evaluate the work done by $\mathbf{F}$ in sending a partial from $(0,0)$ to $(1,2)$.

1 We know that the potential function satisfies

$$\mathbf{F} = \nabla v = \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = \langle y, x\rangle$$

then we have

$$\begin{cases} \frac{\partial v}{\partial x} = y, \\ \frac{\partial v}{\partial y} = x. \end{cases} \Rightarrow \begin{cases} v = xy + \mathbf{C}(y), \\ v = xy + \mathbf{C}(x) \end{cases} \Rightarrow v = xy.$$

2 Since

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} = 1,$$

then this field is conservative field. By fundamental theorem for the conservative vector fields, we have

$$\int_C \mathbf{F} ds = V(Q) - V(P) = 1 \cdot 2 - 0 = 2.$$

(2) (5 points) Evaluate the line integral

$$\oint_C (2y + e^{x^2} dx + (x + \sin(y^2)))dy,$$

where C is the circle $x^2 + y^2 = 1$ with the counter-clockwise orientation.

1 From the problem, we know that

$$\begin{cases} F_1 &= 2y + e^{x^2}, \\ F_2 &= x + \sin(y^2). \end{cases}$$

Hence

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - 2 = -1.$$

2 Then by green theorem, we have

$$\begin{aligned} \oint_C F_1 dx + F_2 dy &= \int\int_C \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA \\ &= -\int\int_C dA \\ &= -\int_0^1 \int_0^{2\pi} r d\theta dr \\ &= -\pi. \end{aligned}$$

Math 241 Quiz #5 version: 2

Instructor: Wenqiang Feng Name: ____________________

(1) (5 points) Find a potential function of the field $\mathbf{F} = \langle x, y\rangle$ and evaluate the work done by $\mathbf{F}$ in sending a partial from $(0,0)$ to $(1,1)$.

1 We know that the potential function satisfies

$$\mathbf{F} = \nabla v = \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = \langle y, x\rangle$$

then we have

$$\begin{cases} \frac{\partial v}{\partial x} = x, \\ \frac{\partial v}{\partial y} = y. \end{cases} \Rightarrow \begin{cases} v = \frac{1}{2}x^2 + \mathbf{C}(y), \\ v = \frac{1}{2}y^2 + \mathbf{C}(x) \end{cases} \Rightarrow v = \frac{x^2+y^2}{2}.$$

2 Since

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} = 0,$$

then this field is conservative field. By fundamental theorem for the conservative vector fields, we have

$$\int_C \mathbf{F} ds = V(Q) - V(P) = 1 \cdot 1 - 0 = 1.$$

(2) (5 points) Evaluate the line integral

$$\oint_C (3y + e^{\sin(x)} dx + (7x + \sqrt{y^4+1}) dy,$$

where C is the circle $x^2 + y^2 = 9$ with the counter-clockwise orientation.

1 From the problem, we know that

$$\begin{cases} F_1 &= 3y + e^{\sin(x)}, \\ F_2 &= 7x + \sqrt{y^4+1}. \end{cases}$$

Hence

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 7 - 3 = 4.$$

2 Then by green theorem, we have

$$\begin{aligned} \oint_C F_1 dx + F_2 dy &= \int\int_C \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA \\ &= 4 \int\int_C dA \\ &= 4 \int_0^3 \int_0^{2\pi} r d\theta dr \\ &= 36\pi. \end{aligned}$$