MATH 572: Computational Assignment #2

Due on Thurday, April 24, 2014

TTH 12:40pm

Wenqiang Feng

Contents

The Convection Diffusion Equation

Problem 1

1. **The exact solution**

   From the problem, we know that the characteristic function is

   $$-\epsilon\lambda^2 + \lambda = 0.$$

   So, $\lambda = 0, \frac{1}{\epsilon}$. Therefore, the general solution is

   $$u = c_1 e^{0x} + c_2 e^{\frac{1}{\epsilon}x} = c_1 + c_2 e^{\frac{1}{\epsilon}x}.$$

   By using the boundary conditions, we get the solution is

   $$u(x) = 1 - \frac{1}{1 - e^{\frac{1}{\epsilon}}} + \frac{1}{1 - e^{\frac{1}{\epsilon}}} e^{\frac{1}{\epsilon}x}.$$

   And $u(x)$ is monotone.

2. **Central Finite difference scheme**

   I consider the following partition for finite difference method:

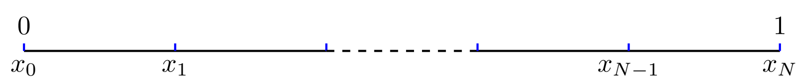

Figure 1: One dimension's uniform partition for finite difference method

   Then, the central difference scheme is as following:

   $$\begin{aligned} -\epsilon\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + \frac{U_{i+1} - U_{i-1}}{2h} &= 0, \quad i = 1, 2, \cdots, N-1. && (1) \\ U_0 = 1, U_N &= 0. && (2) \end{aligned}$$

   So

   (a) when $i = 1$, we get

   $$-\epsilon\frac{U_0 - 2U_1 + U_2}{h^2} + \frac{U_2 - U_0}{2h} = 0,$$

   i.e.

   $$-\left(\frac{\epsilon}{h^2} + \frac{1}{2h}\right)U_0 + \frac{2\epsilon}{h^2}U_1 + \left(\frac{1}{2h} - \frac{\epsilon}{h^2}\right)U_2 = 0.$$

   Since, $U_0 = 1$, so we get

   $$\frac{2\epsilon}{h^2}U_1 + \left(\frac{1}{2h} - \frac{\epsilon}{h^2}\right)U_2 = \left(\frac{\epsilon}{h^2} + \frac{1}{2h}\right). \tag{3}$$

   (b) when $i = 2, \cdots, N-2$, we get

   $$-\epsilon\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + \frac{U_{i+1} - U_{i-1}}{2h} = 0.$$

   i.e.

   $$-\left(\frac{\epsilon}{h^2} + \frac{1}{2h}\right)U_{i-1} + \frac{2\epsilon}{h^2}U_i + \left(\frac{1}{2h} - \frac{\epsilon}{h^2}\right)U_{i+1} = 0. \tag{4}$$

3. when $i = N-1$

$$-\epsilon\frac{U_{N-2}-2U_{N-1}+U_N}{h^2}+\frac{U_N-U_{N-2}}{2h}=0,$$

i.e.

$$-\left(\frac{\epsilon}{h^2}+\frac{1}{2h}\right)U_{N-2}+\frac{2\epsilon}{h^2}U_{N-1}+\left(\frac{1}{2h}-\frac{\epsilon}{h^2}\right)U_N=0.$$

Since $U_N = 0$, then,

$$-\left(\frac{\epsilon}{h^2}+\frac{1}{2h}\right)U_{N-2}+\frac{2\epsilon}{h^2}U_{N-1}=0. \tag{5}$$

From (3)-(5), we get the algebraic system is

$$AU = F,$$

where

$$A=\begin{pmatrix} \frac{2\epsilon}{h^2} & \frac{1}{2h}-\frac{\epsilon}{h^2} & & & \\ -\left(\frac{\epsilon}{h^2}+\frac{1}{2h}\right) & \frac{2\epsilon}{h^2} & \frac{1}{2h}-\frac{\epsilon}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\left(\frac{\epsilon}{h^2}+\frac{1}{2h}\right) & \frac{2\epsilon}{h^2} & \frac{1}{2h}-\frac{\epsilon}{h^2} \\ & & & -\left(\frac{\epsilon}{h^2}+\frac{1}{2h}\right) & \frac{2\epsilon}{h^2} \end{pmatrix},$$

$$U=\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-2} \\ U_{N-1}\end{pmatrix}, F=\begin{pmatrix} \frac{\epsilon}{h^2}+\frac{1}{2h} \\ \vdots \\ 0 \\ \vdots \\ \\ \end{pmatrix}.$$

4. **Numerical Results of Central Difference Method**

h	N_{nodes}	$\|u-u_h\|_{l^\infty,\epsilon=1}$	$\|u-u_h\|_{l^\infty,\epsilon=10^{-1}}$	$\|u-u_h\|_{l^\infty,\epsilon=10^{-3}}$	$\|u-u_h\|_{l^\infty,\epsilon=10^{-6}}$
0.5	3	2.540669×10^{-3}	7.566929×10^{-1}	1.245000×10^{2}	∞
0.25	5	6.175919×10^{-4}	1.933238×10^{-1}	3.050403×10^{1}	∞
0.125	9	1.563835×10^{-4}	5.570936×10^{-2}	7.449173×10^{0}	∞
0.0625	17	3.928711×10^{-5}	1.211929×10^{-2}	1.692902×10^{0}	∞
0.03125	33	9.827515×10^{-6}	3.018484×10^{-3}	2.653958×10^{-1}	∞
0.015625	65	2.457936×10^{-6}	7.484336×10^{-4}	7.515267×10^{-3}	∞
0.007812	129	6.144675×10^{-7}	1.870750×10^{-4}	2.281210×10^{-9}	∞
0.003906	257	1.536257×10^{-7}	4.674564×10^{-5}	6.661338×10^{-16}	∞

Table 1: l^∞ norms for the Central Difference Method with $\epsilon=\{1,10^{-1},10^{-3},10^{-6}\}$

From Table.1, we get that

(a) when $h<\epsilon$ the scheme is convergent with optimal convergence order (Figure.2), i.e.

$$\|u-u_h\|_{l^\infty}\approx 0.01h^{1.9992},$$

(b) when $h \approx \epsilon$ the scheme is convergent with optimal convergence order (Figure.2), i.e.

$$\|u - u_h\|_{l^\infty} \approx 3.201 h^{2.0072},$$

(c) when $h > \epsilon$ the scheme is not stable and the solution has oscillation.

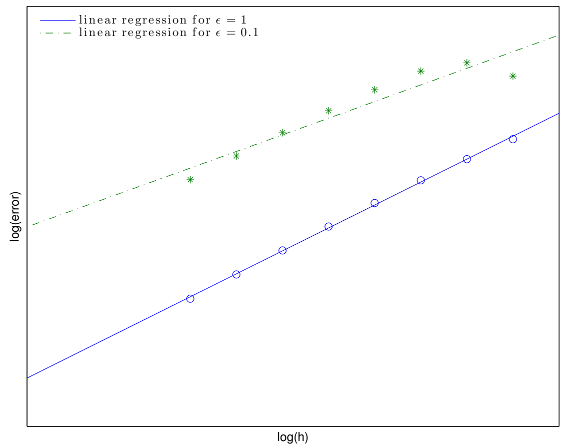

Figure 2: linear regression for l^∞ norm with $\epsilon = 1$ and $\epsilon = 0.1$

5. **Linearly Independent Solutions** $\hat{U}_i$ $\check{U}_i$

(a) Linearity

It is easy to check that

$$C_1 \hat{U}_i + C_2 \check{U}_i = 0,$$

only when $C_1 = C_2 = 0$.

(b) Solutions to (4)

Checking for $\hat{U}_i = 1$

$$-\epsilon \frac{1 - 2 * 1 + 1}{h^2} + \frac{1-1}{2h} = 0$$

Checking for $\check{U}_i = \left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^i$

$$\begin{aligned}
& - \quad \epsilon \frac{\left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^{i-1} - 2\left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^{i} + \left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^{i+1}}{h^2} + \frac{\left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^{i+1} - \left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^{i-1}}{2h} \\
&= -\left(\frac{\epsilon}{h^2} + \frac{1}{2h}\right)\left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^{i-1} + \frac{2\epsilon}{h^2}\left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^{i} + \left(\frac{1}{2h} - \frac{\epsilon}{h^2}\right)\left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^{i+1} \\
&= -\frac{2\epsilon + h}{2h^2}\left(\frac{2\epsilon+h}{2\epsilon-h}\right)^{i-1} + \frac{2\epsilon}{h^2}\left(\frac{2\epsilon+h}{2\epsilon-h}\right)^{i} + \frac{h - 2\epsilon}{2h^2}\left(\frac{2\epsilon+h}{2\epsilon-h}\right)^{i+1} \\
&= -\frac{2\epsilon - h}{2h^2}\left(\frac{2\epsilon+h}{2\epsilon-h}\right)^{i} + \frac{2\epsilon}{h^2}\left(\frac{2\epsilon+h}{2\epsilon-h}\right)^{i} - \frac{2\epsilon + h}{2h^2}\left(\frac{2\epsilon+h}{2\epsilon-h}\right)^{i} \\
&= -\frac{2\epsilon}{h^2}\left(\frac{2\epsilon+h}{2\epsilon-h}\right)^{i} + \frac{2\epsilon}{h^2}\left(\frac{2\epsilon+h}{2\epsilon-h}\right)^{i} = 0.
\end{aligned}$$

(c) The representation of $\hat{U}$ and $\check{U}$

Since $\hat{U}$ and $\check{U}$ are the solution of 1, so the linear combination is also solution to 1, i.e.

$$u = c_1\hat{U} + c_2\check{U}$$

is also solution to 1. We also need this solution to satisfy the boundary conditions, so

$$\begin{cases} u = c_1 + c_2\left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right) = 1 \\ u = c_1 + c_2\left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^N = 0. \end{cases}$$

so

$$c_1 = -\frac{(2\epsilon+h)^N}{(2\epsilon+h)(2\epsilon-h)^{N-1}-(2\epsilon+h)^N}, c_2 = \frac{(2\epsilon-h)^N}{(2\epsilon+h)(2\epsilon-h)^{N-1}-(2\epsilon+h)^N}.$$

6. **Up-wind Finite difference scheme**

By using the same partition as central difference, then the up-wind difference scheme is as following:

$$\begin{aligned} -\epsilon\frac{U_{i-1}-2U_i+U_{i+1}}{h^2} + \frac{U_i - U_{i-1}}{h} &= 0, \quad i = 1,2,\cdots,N-1. \\ U_0 = 1, U_N &= 0. \end{aligned}$$

So

(a) when $i = 1$, we get

$$-\epsilon\frac{U_0 - 2U_1 + U_2}{h^2} + \frac{U_1 - U_0}{h} = 0,$$

i.e.

$$-\left(\frac{\epsilon}{h^2}+\frac{1}{h}\right)U_0 + \left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right)U_1 - \frac{\epsilon}{h^2}U_2 = 0.$$

Since, $U_0 = 1$, so we get

$$\left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right)U_1 - \frac{\epsilon}{h^2}U_2 = \left(\frac{\epsilon}{h^2}+\frac{1}{h}\right). \tag{6}$$

(b) when $i = 2,\cdots,N-2$, we get

$$-\epsilon\frac{U_{i-1}-2U_i+U_{i+1}}{h^2} + \frac{U_i - U_{i-1}}{h} = 0.$$

i.e.

$$-\left(\frac{\epsilon}{h^2}+\frac{1}{h}\right)U_{i-1} + \left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right)U_i - \frac{\epsilon}{h^2}U_{i+1} = 0. \tag{7}$$

(c) when $i = N-1$

$$-\epsilon\frac{U_{N-2}-2U_{N-1}+U_N}{h^2} + \frac{U_{N-1}-U_{N-2}}{h} = 0,$$

i.e.

$$-\left(\frac{\epsilon}{h^2}+\frac{1}{h}\right)U_{N-2} + \left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right)U_{N-1} - \frac{\epsilon}{h^2}U_N = 0.$$

Since $U_N = 0$, then,

$$-\left(\frac{\epsilon}{h^2}+\frac{1}{h}\right)U_{N-2}+\left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right)U_{N-1}=0. \tag{8}$$

From (6)-(8), we get the algebraic system is

$$AU = F,$$

where

$$A=\begin{pmatrix}
\left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right) & -\frac{\epsilon}{h^2} & & & \\
-\left(\frac{\epsilon}{h^2}+\frac{1}{h}\right) & \left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right) & -\frac{\epsilon}{h^2} & & \\
 & \ddots & \ddots & \ddots & \\
 & & -\left(\frac{\epsilon}{h^2}+\frac{1}{h}\right) & \left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right) & -\frac{\epsilon}{h^2} \\
 & & & -\left(\frac{\epsilon}{h^2}+\frac{1}{h}\right) & \left(\frac{2\epsilon}{h^2}+\frac{1}{h}\right)
\end{pmatrix},$$

$$U=\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-2} \\ U_{N-1}\end{pmatrix}, F=\begin{pmatrix} \frac{\epsilon}{h^2}+\frac{1}{h} \\ \vdots \\ 0 \\ \vdots \\ \end{pmatrix}.$$

7. **Numerical Results of Up-wind Difference Scheme**

h	N_{nodes}	$\|u-u_h\|_{l^\infty,\epsilon=1}$	$\|u-u_h\|_{l^\infty,\epsilon=10^{-1}}$	$\|u-u_h\|_{l^\infty,\epsilon=10^{-3}}$	$\|u-u_h\|_{l^\infty,\epsilon=10^{-6}}$
0.5	3	2.245933×10^{-2}	1.361643×10^{-1}	1.992032×10^{-3}	∞
0.25	5	1.270323×10^{-2}	1.988791×10^{-1}	1.587251×10^{-5}	∞
0.125	9	6.925118×10^{-3}	1.571250×10^{-1}	4.999060×10^{-7}	∞
0.0625	17	3.623644×10^{-3}	9.196290×10^{-2}	9.685710×10^{-10}	∞
0.03125	33	1.849028×10^{-3}	5.061410×10^{-2}	1.110223×10^{-15}	∞
0.015625	65	9.343457×10^{-4}	2.695432×10^{-2}	2.220446×10^{-16}	∞
0.007812	129	4.695265×10^{-4}	1.391029×10^{-2}	1.554312×10^{-15}	∞
0.003906	257	2.353710×10^{-4}	7.064951×10^{-3}	8.881784×10^{-16}	∞

Table 2: l^∞ norms for the Up-wind Difference Method with $\epsilon=\{1,10^{-1},10^{-3},10^{-6}\}$

From the Table.2 we get that

(a) when $h<\epsilon$ the scheme is convergent with optimal convergence order (Figure.2), i.e.

$$\|u-u_h\|_{l^\infty}\approx 0.0471h^{0.946},$$

(b) when $h\approx\epsilon$ the scheme is convergent, but the convergence order is not optimal (Figure.2), i.e.

$$\|u-u_h\|_{l^\infty}\approx 0.4398h^{0.6852},$$

(c) when $h>\epsilon$ the scheme is convergent, and the solution has no oscillation.

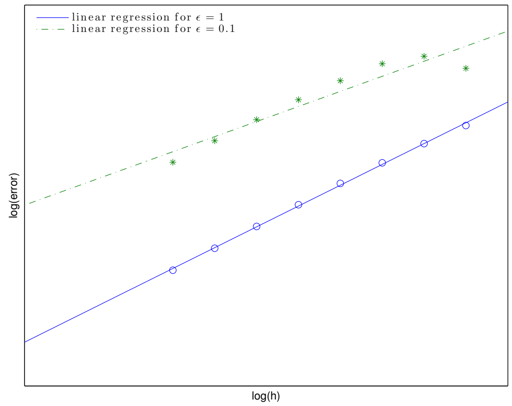

Figure 3: linear regression for l^{∞} norm with $\epsilon = 1$ and $\epsilon = 0.1$

8. **Maximum Principle of Up-wind Difference Scheme**

   Lemma 0.1 *Let $A = tridiag\{a_i, b_i, c_i\}_{i=1}^{n} \in \mathbb{R}^{n\times n}$ be a tridiagional matrix with the properties that*

   $$b_i > 0, \quad a_i, c_i \leq 0, \quad a_i + b_i + c_i = 0.$$

   Then the following maximum principle holds: If $u \in \mathbb{R}^n$ is such that $(Au)_{i=2,\cdots,n-1} \leq 0$, then $u_i \leq \max\{u_1, u_n\}$.

   From the Up-wind Difference scheme, we get that $a_1 = 0$, $a_i = -\left(\frac{\epsilon}{h^2} + \frac{1}{h}\right), i = 2, \cdots, n$, $b_i = \left(\frac{2\epsilon}{h^2} + \frac{1}{h}\right), i = 1, \cdots, n$ and $c_i = -\frac{\epsilon}{h^2}, i = 1, \cdots, n-1$, moreover $(Au)_{i=2,\cdots,n-1} = 0$. Therefore,

   $$b_i > 0, \quad a_i, c_i \leq 0, \quad a_i + b_i + c_i = 0.$$

   Since $(Au)_{i=2,\cdots,n-1} = 0$, so the corresponding matrix arising from the up-wind scheme satisfies the discrete maximum principle(Lemma 0.1).

A Posterior Error Estimation

Problem 2

1. **Partition**

   I consider the following partition for finite element method:

Figure 4: One dimension's uniform partition for finite element method

2. **Basis Function**

I will use the linear basis function, i.e. for each element $I = [x_i, x_{i+1}]$

$$\phi_I(x) = \begin{cases} \phi_1(x) = \frac{x_{i+1}-x}{x_{i+1}-x_i} \\ \phi_2(x) = \frac{x-x_i}{x_{i+1}-x_i}. \end{cases}$$

3. **Weak Formula**

   Multiplying the testing function $v \in H_0^1$ to both side of the problem, then integrating by part we get the following weak formula

$$\int_0^1 a(x)u'v'dx = \int_0^1 fvdx.$$

4. **Approximate Problem**

   The approximate problem is to find $u_h \in H^1$, s.t

$$a(u_h, v_h) = f(v_h) \forall v \in H_0^1,$$

   where

$$a(u_h, v_h) = \int_0^1 a(x)u_h'v_h'dx \quad \text{and} \quad f(v_h) = \int_0^1 fv_h dx.$$

5. **Numerical Results of Finite Element Method for Poisson Equation**

   (a) Problem: a(x)=1, $u_e = x^3$ and $f = -6x$.

h	N_{nodes}	$\|u - u_h\|_{L^2}$	$\|u - u_h\|_{H^1}$
1/4	5	1.791646×10^{-2}	2.480392×10^{-1}
1/8	9	4.502711×10^{-3}	1.247556×10^{-1}
1/16	17	1.127148×10^{-3}	6.246947×10^{-2}
1/32	33	2.818787×10^{-4}	3.124619×10^{-2}
1/64	65	7.047542×10^{-5}	1.562452×10^{-2}
1/128	128	1.761921×10^{-5}	7.812440×10^{-3}
1/256	257	4.404826×10^{-6}	3.906243×10^{-3}

Table 3: L^2 and H^1 Errors of Finite Element Method for Poisson Equation .

Using linear regression (Figure.5), we can also see that the errors in Table.4 obey

$$\|u - u_h\|_{L^2} \approx 0.2870h^{1.9987},$$
$$\|u - u_h\|_{H^1} \approx 0.9935h^{0.9986}.$$

These linear regressions indicate that the finite element method for this problem can converge in the optimal rates, which are second order in L^2 norm and first order in H^1 norm.

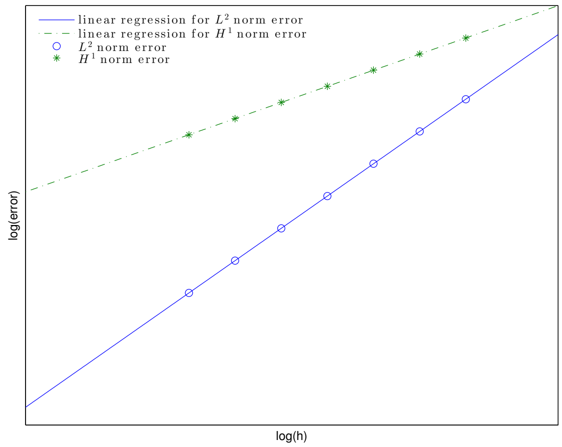

Figure 5: linear regression for L^2 and H^1 norm errors

(b) Problem: a(x)=1, $u_e = x^{\frac{3}{2}}$ and $f = -\frac{3}{4\sqrt{x}}$.

h	N_{nodes}	$\|u - u_h\|_{L^2}$	$\|u - u_h\|_{H^1}$
1/4	5	7.625472×10^{-3}	1.022294×10^{-1}
1/8	9	2.029299×10^{-3}	5.585353×10^{-2}
1/16	17	5.324774×10^{-4}	3.011300×10^{-2}
1/32	33	1.378846×10^{-4}	1.607571×10^{-2}
1/64	65	3.523180×10^{-5}	8.517032×10^{-3}
1/128	128	8.876332×10^{-6}	4.485323×10^{-3}
1/256	257	2.203920×10^{-6}	2.350599×10^{-3}

Table 4: L^2 and H^1 Errors of Finite Element Method for Poisson Equation .

Using linear regression (Figure.6), we can also see that the errors in Table.4 obey

$$\|u - u_h\|_{L^2} \approx 0.1193h^{1.9593},$$
$$\|u - u_h\|_{H^1} \approx 0.3682h^{0.9081}.$$

These linear regressions indicate that the finite element method for this problem can converge, but not in the optimal rates.

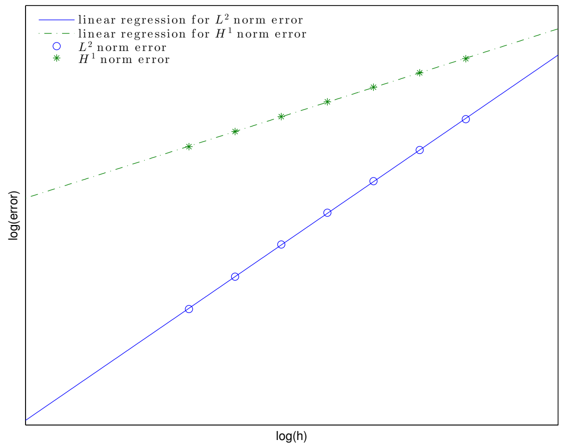

Figure 6: linear regression for L^2 and H^1 norm errors

(c) Problem: f=1,

$$a(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{\pi} \\ 2, & \frac{1}{\pi} \leq x \leq 1. \end{cases}$$

So, the exact solution should be

$$u_e = \begin{cases} -\frac{1}{2}x^2 + \frac{5\pi^2+1}{2\pi(\pi+1)}x, & 0 \leq x < \frac{1}{\pi} \\ -\frac{1}{4}x^2 + \frac{5\pi^2+1}{4\pi(\pi+1)}x + \frac{5\pi-1}{4\pi(\pi+1)}, & \frac{1}{\pi} \leq x \leq 1. \end{cases}$$

We can not use the uniform mesh to compute this problem. Since if we can use the uniform mesh, then $\frac{1}{\pi}$ should be the node point, that is to say

$$nh = n\frac{1}{N_{elem}} = \frac{1}{\pi},$$

i.e.

$$n\pi = N_{elem}, n, N_{elem} \in \mathbb{Z}.$$

This is not possible, so we can not generate such mesh.

6. **Adaptive Finite Element Method for Possion Equation**

I will follow the standard local mesh refinement loops :

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.$$

(a) Problem: a(x)=1, $u_e = x^{\frac{3}{2}}$ and $f = -\frac{3}{4\sqrt{x}}$.

$Iter$	N_{elem}	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1	32	2.797720×10^{-5}	1.378846×10^{-4}	1.607571×10^{-2}
2	47	1.022508×10^{-5}	6.093669×10^{-5}	9.927148×10^{-3}
3	75	3.674022×10^{-6}	2.038303×10^{-5}	5.935496×10^{-3}
4	102	1.313414×10^{-6}	1.400631×10^{-5}	4.239849×10^{-3}
5	145	4.663453×10^{-7}	6.119733×10^{-6}	2.933869×10^{-3}
6	171	1.654010×10^{-7}	4.589394×10^{-6}	2.432512×10^{-3}
7	192	5.970786×10^{-8}	4.010660×10^{-6}	2.185324×10^{-3}
8	208	5.956431×10^{-8}	3.587483×10^{-6}	2.034418×10^{-3}
9	219	5.957050×10^{-8}	3.297922×10^{-6}	1.942123×10^{-3}
10	229	5.976916×10^{-8}	3.076573×10^{-6}	1.864147×10^{-3}

Table 5: L^2 and H^1 Errors of Finite Element Method for Poisson Equation .

Using linear regression, we can also see that the errors (Figure.7) in Table.5 obey

$$\|u-u_h\|_{H^1} \approx 0.6454 N_{elem}^{-1.0798}.$$

These linear regressions indicate that the adaptive finite element method for this problem can converge in the optimal rates, which is first order in H^1 norm.

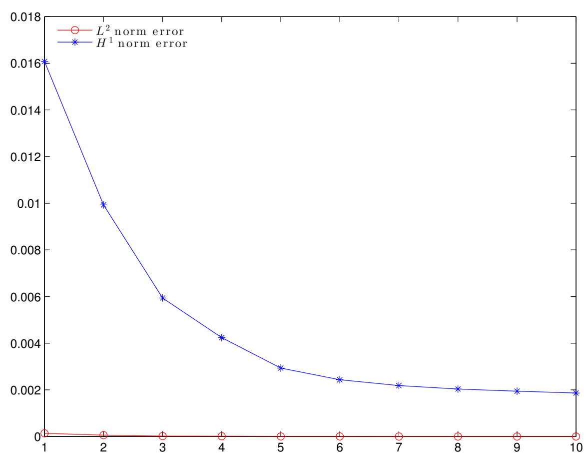

Figure 7: L^2 and H^1 norm errors for each iteration

(b) Problem: f=1,

$$a(x) = \begin{cases} 1, & 0 \le x < \frac{1}{\pi} \\ 2, & \frac{1}{\pi} \le x \le 1. \end{cases}$$

So, the exact solution should be

$$u_e = \begin{cases} -\frac{1}{2}x^2 + \frac{5\pi^2+1}{2\pi(\pi+1)}x, & 0 \le x < \frac{1}{\pi} \\ -\frac{1}{4}x^2 + \frac{5\pi^2+1}{4\pi(\pi+1)}x + \frac{5\pi-1}{4\pi(\pi+1)}, & \frac{1}{\pi} \le x \le 1. \end{cases}$$

$Iter$	N_{elem}	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1	2	5.652041×10^{-1}	4.506966×10^{-1}	9.637043×10^{-2}
2	4	5.652041×10^{-1}	4.626630×10^{-1}	4.818522×10^{-2}
3	8	5.652041×10^{-1}	4.656590×10^{-1}	2.409261×10^{-2}
4	16	5.652041×10^{-1}	4.664083×10^{-1}	1.204630×10^{-2}
5	32	5.652041×10^{-1}	4.665956×10^{-1}	6.023152×10^{-3}
6	48	5.652041×10^{-1}	4.666425×10^{-1}	4.116248×10^{-3}
7	96	5.652041×10^{-1}	4.666542×10^{-1}	2.058124×10^{-3}
8	160	5.652041×10^{-1}	4.666571×10^{-1}	1.739956×10^{-3}
9	192	5.652041×10^{-1}	4.666571×10^{-1}	1.029062×10^{-3}
10	192	5.652041×10^{-1}	4.666571×10^{-1}	1.029062×10^{-3}

Table 6: L^2 and H^1 Errors of Finite Element Method for Interface Problems .

Using linear regression, we can also see that the errors (Figure.8) in Table.6 obey

$$\|u-u_h\|_{H^1} \quad \approx 0.1825 N_{elem}^{-0.9706}.$$

These linear regressions indicate that the adaptive finite element method for this problem can converge in the optimal rates, which is first order in H^1 norm.

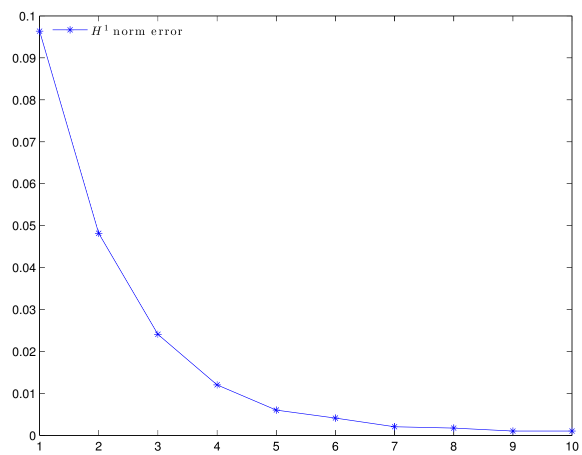

Figure 8: L^2 and H^1 norm errors for each iteration

Heat Equation

Problem 3

1. **Partition**

   I consider the following partition for finite element method:

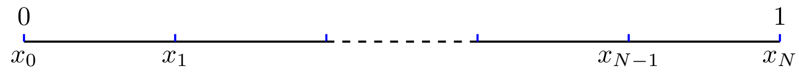

Figure 9: One dimension's uniform partition for finite element method

2. **The corresponding value of f and u_0**

   I choose the following corresponding value of f and u_0:

$$u_0 = \sin(3\pi x), f(t,x) = -2\sin(3\pi x)e^{-2t} + 9\pi^2 \sin(3\pi x)e^{-2t}.$$

3. **θ Method Scheme** The θ Method Discretization Scheme of this problem is as following

$$\frac{U_i^{k+1}-U_i^k}{\tau} - \theta\frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2} - (1-\theta)\frac{U_{i-1}^{k+1} - 2U_i^{k+1} + U_{i+1}^{k+1}}{h^2} = \theta f_i^k + (1-\theta) f_i^{k+1}. \tag{9}$$

   Let $\mu = \frac{\tau}{h^2}$, then the scheme (9) can be rewritten as

$$U_i^{k+1} - U_i^k - \theta\mu(U_{i-1}^k - 2U_i^k + U_{i+1}^k) - (1-\theta)\mu(U_{i-1}^{k+1} - 2U_i^{k+1} + U_{i+1}^{k+1}) = \theta\tau f_i^k + (1-\theta)\tau f_i^{k+1}.$$

   Combining of similar terms, we get

$$\begin{aligned} & -(1-\theta)\mu U_{i-1}^{k+1} + (2(1-\theta)\mu + 1)U_i^{k+1} - (1-\theta)\mu U_{i+1}^{k+1} \\ = \; & \theta\mu U_{i-1}^k - (2\theta\mu - 1)U_i^k + \theta\mu U_{i+1}^k + \theta\tau f_i^k + (1-\theta)\tau f_i^{k+1}. \end{aligned}$$

   Since $U(0) = U(1) = 0$So, the θ-scheme can be written as the following matrix form

$$AU^{k+1} = BU^k + F,$$

   where

$$A = \begin{pmatrix} 2(1-\theta)\mu+1 & -(1-\theta)\mu & & & \\ -(1-\theta)\mu & 2(1-\theta)\mu+1 & -(1-\theta)\mu & & \\ & \ddots & \ddots & \ddots & \\ & & -(1-\theta)\mu & 2(1-\theta)\mu+1 & -(1-\theta)\mu \\ & & & -(1-\theta)\mu & 2(1-\theta)\mu+1 \end{pmatrix},$$

$$B = \begin{pmatrix} -(2\theta\mu-1) & \theta\mu & & & \\ \theta\mu & -(2\theta\mu-1) & \theta\mu & & \\ & \ddots & \ddots & \ddots & \\ & & \theta\mu & -(2\theta\mu-1) & \theta\mu \\ & & & \theta\mu & -(2\theta\mu-1) \end{pmatrix},$$

$$U^{k+1} = \begin{pmatrix} U^{k+1}(x_1) \\ U^{k+1}(x_2) \\ \vdots \\ U^{k+1}(x_{N-2}) \\ U^{k+1}(x_{N-1}) \end{pmatrix}, U^k = \begin{pmatrix} U^k(x_1) \\ U^k(x_2) \\ \vdots \\ U^k(x_{N-2}) \\ U^k(x_{N-1}) \end{pmatrix},$$

$$F = \theta\tau \begin{pmatrix} f^k(x_1) \\ \vdots \\ f^k(x_i) \\ \vdots \\ f^k(x_{N-1}) \end{pmatrix} + (1-\theta)\tau \begin{pmatrix} f^{k+1}(x_1) \\ \vdots \\ f^{k+1}(x_i) \\ \vdots \\ f^{k+1}(x_{N-1}) \end{pmatrix}.$$

4. **Numerical Results of Finite difference Method (θ Method) for Heat Equation**

(a) Numerical results for θ-Method for fixed $\tau = 1 \times 10^{-5}$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty,\theta=0}$	$\|u-u_h\|_{L^\infty,\theta=1}$	$\|u-u_h\|_{L^\infty,\theta=\frac{1}{2}}$
1/4	5	0.00016	8.794539×10^{-2}	8.794522×10^{-2}	8.794531×10^{-2}
1/8	9	0.00064	1.723827×10^{-2}	1.723819×10^{-2}	1.723823×10^{-2}
1/16	17	0.00256	4.076556×10^{-3}	4.076490×10^{-3}	4.076523×10^{-3}
1/32	33	0.01024	1.005390×10^{-3}	1.005327×10^{-3}	1.005359×10^{-3}
1/64	65	0.04096	2.505219×10^{-4}	2.504594×10^{-4}	2.532024×10^{-4}
1/128	129	0.16384	6.260098×10^{-5}	6.253858×10^{-5}	6.256978×10^{-5}

Table 7: L^∞ norms for the θ-Method for fixed $\tau = 1 \times 10^{-5}$

h	N_{nodes}	μ	$\|u-u_h\|_{L^2,\theta=0}$	$\|u-u_h\|_{L^2,\theta=1}$	$\|u-u_h\|_{L^2,\theta=\frac{1}{2}}$
1/4	5	0.00016	6.218678×10^{-2}	6.218666×10^{-2}	6.218672×10^{-2}
1/8	9	0.00064	1.218929×10^{-2}	1.218924×10^{-2}	1.218927×10^{-2}
1/16	17	0.00256	2.882561×10^{-3}	2.882514×10^{-3}	2.882537×10^{-3}
1/32	33	0.01024	7.109183×10^{-4}	7.108736×10^{-4}	7.108959×10^{-4}
1/64	65	0.04096	1.771458×10^{-4}	1.771015×10^{-4}	1.771236×10^{-4}
1/128	129	0.16384	4.426558×10^{-5}	4.422145×10^{-5}	4.424352×10^{-5}

Table 8: L^2 norms for the θ-Method for fixed $\tau = 1 \times 10^{-5}$

h	N_{nodes}	μ	$\|u-u_h\|_{H^1,\theta=0}$	$\|u-u_h\|_{H^1,\theta=1}$	$\|u-u_h\|_{H^1,\theta=\frac{1}{2}}$
1/4	5	0.00016	1.838499×10^{-0}	1.838496×10^{-0}	1.838497×10^{-0}
1/8	9	0.00064	8.668172×10^{-1}	8.668132×10^{-1}	8.668152×10^{-1}
1/16	17	0.00256	4.284228×10^{-1}	4.284158×10^{-1}	4.284193×10^{-1}
1/32	33	0.01024	2.136338×10^{-1}	2.136204×10^{-1}	2.136271×10^{-1}
1/64	65	0.04096	1.067553×10^{-1}	1.067286×10^{-1}	1.067419×10^{-1}
1/128	129	0.16384	5.338867×10^{-2}	5.333545×10^{-2}	5.336206×10^{-2}

Table 9: H^1 norms for the θ-Method for fixed $\tau = 1 \times 10^{-5}$

From the Table(7)-(9), we can conclude that when $\mu < 0.5$, Implicit Euler method, Explicit Euler method and Crank-Nicolson method are convergent with optimal order in spacial, which are second order in L^∞, L^2 norm and first order in H^1 norm.

(b) Numerical results for θ-Method for $\tau = \sqrt{h}$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	8.00	9.334285×10^{-2}	6.600336×10^{-2}	1.951333×10^{0}
1/8	9	22.63	1.418498×10^{-1}	1.003029×10^{-1}	7.132843×10^{0}
1/16	17	64.00	5.067314×10^{-3}	3.583132×10^{-3}	5.325457×10^{-1}
1/32	33	181.02	3.744691×10^{-2}	2.647897×10^{-2}	7.957035×10^{0}
1/64	65	512	6.776843×10^{-4}	4.791952×10^{-4}	2.887826×10^{-1}
1/128	129	1228.15	8.093502×10^{-3}	5.722970×10^{-3}	6.902469×10^{0}
1/256	257	4096	2.192061×10^{-4}	1.550021×10^{-4}	3.739592×10^{-2}

Table 10: Error norms for the Implicit Euler method with $\tau = \sqrt{h}$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	8.00	4.341161×10^{2}	3.069664×10^{2}	9.075199×10^{3}
1/8	9	22.63	8.631363×10^{1}	6.103296×10^{1}	4.340236×10^{3}
1/16	17	64.00	4.466761×10^{3}	3.158477×10^{3}	4.694310×10^{5}
1/32	33	181.02	2.482730×10^{3}	1.755559×10^{3}	5.275526×10^{5}
1/64	65	512	5.556307×10^{10}	2.439517×10^{10}	1.962496×10^{14}
1/128	129	1228.15	4.383362×10^{25}	1.193837×10^{25}	3.823127×10^{29}
1/256	257	4096	3.530479×10^{51}	1.095038×10^{51}	1.420743×10^{56}

Table 11: Error norms for the Explicit Euler method with $\tau = \sqrt{h}$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	8.00	3.937504×10^{-1}	2.784236×10^{-1}	8.231355×10^{0}
1/8	9	22.63	4.372744×10^{-2}	3.091997×10^{-2}	2.198812×10^{0}
1/16	17	64.00	1.007102×10^{-2}	7.121285×10^{-3}	1.058406×10^{0}
1/32	33	181.02	3.858423×10^{-2}	2.728317×10^{-2}	8.198702×10^{0}
1/64	65	512	1.408511×10^{-4}	9.959676×10^{-5}	6.002108×10^{-2}
1/128	129	1228.15	7.776086×10^{-3}	5.498523×10^{-3}	6.631764×10^{0}
1/256	257	4096	1.158509×10^{-5}	8.191894×10^{-6}	1.976382×10^{-2}

Table 12: Error norms for the Crank-Nicolson method with $\tau = \sqrt{h}$

From the Table(10)-(12), we can conclude that Implicit Euler method and Crank-Nicolson method are unconditional stable, while when $\mu > \frac{1}{2}$ Explicit Euler method is not stable.

(c) Numerical results for θ-Method for $\tau = h$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	4	9.048357×10^{-2}	6.398155×10^{-2}	1.891560×10^{0}
1/8	9	8	1.777939×10^{-2}	1.257192×10^{-2}	8.940271×10^{-1}
1/16	17	16	4.292498×10^{-3}	3.035255×10^{-3}	4.511170×10^{-1}
1/32	33	32	1.106397×10^{-3}	7.823405×10^{-4}	2.350965×10^{-1}
1/64	65	64	2.999114×10^{-4}	2.120694×10^{-4}	1.278017×10^{-1}
1/128	129	128	8.707869×10^{-5}	6.157393×10^{-5}	7.426427×10^{-2}
1/256	257	256	2.785209×10^{-5}	1.969440×10^{-5}	4.751484×10^{-2}

Table 13: Error norms for the Implicit Euler method with $\tau = h$

Problem 3 continued on next page. . .

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	4	1.633634×10^{4}	1.155154×10^{4}	3.415113×10^{5}
1/8	9	8	4.782087×10^{6}	3.381446×10^{6}	2.404647×10^{8}
1/16	17	16	3.367080×10^{12}	2.023268×10^{12}	1.028718×10^{15}
1/32	33	32	1.762004×10^{51}	8.628878×10^{50}	1.756719×10^{54}
1/64	65	64	5.115840×10^{137}	2.577582×10^{137}	2.101478×10^{141}
1/128	129	128	4.972138×10^{-17}	∞	∞
1/256	257	256	4.972138×10^{-17}	∞	∞

Table 14: Error norms for the Explicit Euler method with $\tau = h$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	4	1.115040×10^{-1}	7.884526×10^{-2}	2.330993×10^{0}
1/8	9	8	1.245553×10^{-2}	8.807388×10^{-3}	6.263197×10^{-1}
1/16	17	16	4.072106×10^{-3}	2.879414×10^{-3}	4.279551×10^{-1}
1/32	33	32	1.004329×10^{-3}	7.101680×10^{-4}	2.134083×10^{-1}
1/64	65	64	2.502360×10^{-4}	1.769436×10^{-4}	1.066335×10^{-1}
1/128	129	128	6.250630×10^{-5}	4.419863×10^{-5}	5.330792×10^{-2}
1/256	257	256	1.562328×10^{-5}	1.104733×10^{-5}	2.665286×10^{-2}

Table 15: Error norms for the Crank-Nicolson method with $\tau = h$

From the Table(13)-(15), we can conclude that Implicit Euler method and Crank-Nicolson method are unconditional stable, while when $\mu > \frac{1}{2}$ Explicit Euler method is not stable.

(d) Numerical results for θ-Method for $\tau = h^2$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	1	8.849982×10^{-2}	6.257882×10^{-2}	1.850089×10^{0}
1/8	9	1	1.730081×10^{-2}	1.223352×10^{-2}	8.699621×10^{-1}
1/16	17	1	4.089480×10^{-3}	2.891699×10^{-3}	4.297810×10^{-1}
1/32	33	1	1.008450×10^{-3}	7.130822×10^{-4}	2.142840×10^{-1}
1/64	65	1	2.512547×10^{-4}	1.776639×10^{-4}	1.070675×10^{-1}
1/128	129	1	6.276023×10^{-5}	4.437819×10^{-5}	5.352449×10^{-2}
1/256	257	1	1.568672×10^{-5}	1.109219×10^{-5}	2.676109×10^{-2}

Table 16: Error norms for the Implicit Euler method with $\tau = h^2$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	1	8.603950×10^{5}	6.083912×10^{5}	1.798656×10^{7}
1/8	9	1	8.967110×10^{12}	6.340704×10^{12}	7.960153×10^{14}
1/16	17	1	3.903063×10^{104}	2.759883×10^{104}	1.406256×10^{107}
1/32	33	1	4.972138×10^{-17}	∞	∞
1/64	65	1	4.972138×10^{-17}	∞	∞
1/128	129	1	4.972138×10^{-17}	∞	∞
1/256	257	1	4.972138×10^{-17}	∞	∞

Table 17: Error norms for the Explicit Euler method with $\tau = h^2$

h	N_{nodes}	μ	$\|u-u_h\|_{L^\infty}$	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{H^1}$
1/4	5	1	8.793428×10^{-2}	6.217892×10^{-2}	1.838267×10^{0}
1/8	9	1	1.723790×10^{-2}	1.218904×10^{-2}	8.667990×10^{-1}
1/16	17	1	4.076506×10^{-3}	2.882525×10^{-3}	4.284175×10^{-1}
1/32	33	1	1.005358×10^{-3}	7.108952×10^{-4}	2.136269×10^{-1}
1/64	65	1	2.504906×10^{-4}	1.771236×10^{-4}	1.067419×10^{-1}
1/128	129	1	6.256978×10^{-5}	4.424351×10^{-5}	5.336206×10^{-2}
1/256	257	1	1.563914×10^{-5}	1.105854×10^{-5}	2.667992×10^{-2}

Table 18: Error norms for the Crank-Nicolson method with $\tau = h^2$

From the Table(16)-(18), we can conclude that Implicit Euler method and Crank-Nicolson method are unconditional stable, while when $\mu > \frac{1}{2}$ Explicit Euler method is not stable. Moreover, by using linear regression (Figure.10) for Implicit Euler method errors, we can see that the errors in Table.16 obey

$$\|u-u_h\|_{L^2} \approx 0.9435h^{2.0580},$$
$$\|u-u_h\|_{H^1} \approx 7.2858h^{1.0137}.$$

These linear regressions indicate that the finite element method for this problem can converge in the optimal rates, which are second order in L^2 norm and first order in H^1 norm.

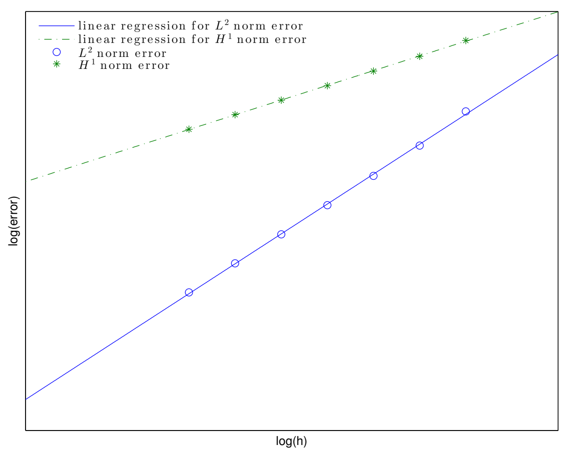

Figure 10: linear regression for L^2 and H^1 norm errors of Implicit Euler method with $\tau = h^2$

Similarly, by using linear regression (Figure.11) for Crank-Nicolson Method, we can also see that the errors in Table.18 obey

$$\|u-u_h\|_{L^2} \approx 0.9382h^{2.0574},$$
$$\|u-u_h\|_{H^1} \approx 7.2445h^{1.0131}.$$

These linear regressions indicate that the finite element method for this problem can converge in the optimal rates, which are second order in L^2 norm and first order in H^1 norm.

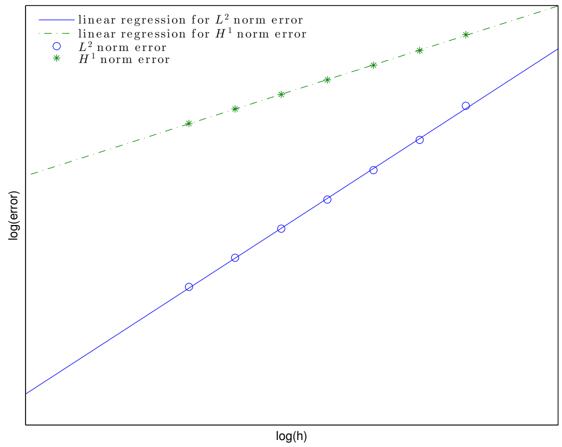

Figure 11: linear regression for L^2 and H^1 norm errors of Crank-Nicolson method with $\tau = h^2$