Math 241 Quiz #5 version: 1

Instructor: Wenqiang Feng Name: ______________________________

(1) (5 points) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, given that $f(x,y) = xe^{xy}$.

Solution. Gold Rule: Treating the rest of symbols as constants, when you compute the partial derivative.

1 Compte the $\frac{\partial f}{\partial x}$: Now, treat y as a constant

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(xe^{xy}\right) \\
&= \frac{\partial}{\partial x}(x)\,e^{xy} + x\frac{\partial}{\partial x}\left(e^{xy}\right) \text{ (product rule)} \\
&= e^{xy} + x \cdot e^{xy} y \text{ (chain rule)} \\
&= (1+xy)e^{xy}.
\end{aligned}$$

2 Compte the $\frac{\partial f}{\partial y}$: Now, treat x as a constant

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(xe^{xy}\right) \\
&= \frac{\partial}{\partial y}(x)\,e^{xy} + x\frac{\partial}{\partial y}\left(e^{xy}\right) \text{ (product rule)} \\
&= 0 + x \cdot e^{xy} x \text{ (chain rule)} \\
&= x^2 e^{xy}.
\end{aligned}$$

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(2) (5 points) Find the equations of the tangent plane to the surface $z = e^{x^2-y^2}$ at the point $(1,-1,1)$.

Solution. (a) Method 1(recommend one):

- How to drive the tangent plane?

  To see this let's start with the equation and we want to find the tangent plane to the surface given by $z = f(x,y)$ at the point (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$. In order to use the formula we need to have all the variables on one side. This is easy enough to do. All we need to do is subtract a z from both sides to get,

  $$f(x,y) - z = 0.$$

  Now, we define a new function

  $$F(x,y,z) = f(x,y) - z,$$

  we can see that the surface given by $z = f(x,y)$ is identical to the surface given by $F(x,y,z) = 0$ and this new equivalent equation is in the correct form for the equation of the tangent plane that we derived in this section. So, the first thing that we need to do is finding the gradient vector for F, i.e.

  $$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle .$$

You can take the gradient vector as the normal vector of your tangent plane. Then from the formula of the plane, we have

$$\begin{aligned} 0 &= \nabla F(x,y,z) \cdot \langle x-x_0, y-y_0, z-z_0\rangle \\ &= \langle F_x, F_y, F_z\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle . \end{aligned}$$

Thus the equation of the tangent plane is:

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - f_z(x_0,y_0)(z-z_0) = 0.$$

- what's the relationship between the tangent plane and linearization? Rewritten the tangent plane yields

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0,$$

i.e. the linearization equation

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

For our this problem:

1 Define $F(x,y,z) = e^{x^2-y^2} - z$

2 Compute the gradient

$$\nabla F(x,y,z) = \left\langle 2xe^{x^2-y^2}, -2ye^{x^2-y^2}, -1\right\rangle$$

So,

$$\nabla F(1,-1,1) = \langle 2, 2, -1\rangle .$$

3 Plugging in to the tangent plane equation

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - f_z(x_0,y_0)(z-z_0) = 0.$$

We get

$$2(x-1) + 2(y+1) - 1(z-1) = 0.$$

Thus the tangent plane is:

$$z = 1 + 2(x-1) + 2(y+1) = 2x + 2y.$$

(b) Method 2: Since $f(x,y)$ is continuous and $f_x(a,b)$ and $f_y(a,b)$ exists, then the tangent plane to the graph at $(a,b,f(a,b))$ is the plane with equation $z = L(x,y)$, where $L(x,y)$ is the linearization at (a,b), i.e.

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

For our this problem:

1 $f(x,y) = e^{x^2-y^2}$

2 compute $f(a,b), f_x(a,b), f_y(a,b)$:

$$\begin{cases} f(1,-1) &= 1, \\ f_x(x,y) &= 2xe^{x^2-y^2}, \\ f_y(x,y) &= -2ye^{x^2-y^2}. \end{cases} \Rightarrow \begin{cases} f(1,-1) &= 1, \\ f_x(1,-1) &= 2, \\ f_y(1,-1) &= 2. \end{cases}$$

3 Plugging in

$$L(x,y) = z = 1 + 2(x-1) + 2(y+1) = 1 + 2x + 2y.$$

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Math 241 Quiz #5 version: 2

Instructor: Wenqiang Feng Name: ______________________

(1) (5 points) Find $\frac{\partial^2 f}{\partial x^2}$, where $f(x, y) = \sqrt{x^2 + y^2}$.

Solution. (1) compute $1_s t$ derivative

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left((x^2+y^2)^{1/2}\right) \\
&= \frac{1}{2}\left((x^2+y^2)^{-1/2}\right)\frac{\partial}{\partial x}\left(x^2+y^2\right) \\
&= \frac{1}{2}\left((x^2+y^2)^{-1/2}\right)2x \\
&= x(x^2+y^2)^{-1/2}.
\end{aligned}$$

(2) compute $2_n d$ derivative

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(x(x^2+y^2)^{-1/2}\right) \\
&= \frac{\partial}{\partial x}(x)(x^2+y^2)^{-1/2} + x\frac{\partial}{\partial x}\left((x^2+y^2)^{-1/2}\right) \\
&= (x^2+y^2)^{-1/2} + x\left(-\frac{1}{2}(x^2+y^2)^{-3/2}\right)\frac{\partial}{\partial x}(x^2+y^2) \\
&= (x^2+y^2)^{-1/2} + x\left(-\frac{1}{2}(x^2+y^2)^{-3/2}\right)2x \\
&= (x^2+y^2)^{-1/2} - x^2(x^2+y^2)^{-3/2}
\end{aligned}$$

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(2) Find the equations of the tangent plane to the surface $z = y \ln x$ at the point $(1, 4, 0)$. For our this problem:

(1) Define $F(x, y, z) = y \ln x - z$

(2) Compute the gradient

$$\nabla F(x, y, z) = \left\langle \frac{y}{x}, \ln x, -1 \right\rangle$$

So,

$$\nabla F(1, 4, 0) = \langle 4, 0, -1 \rangle .$$

(3) Plugging in to the tangent plane equation

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - f_z(x_0, y_0)(z - z_0) = 0.$$

$$0 = 4(x - 1) + 0(y - 4) - 1(z - 0).$$

Thus the tangent plane is:

$$z = 4x - 4.$$