Math 241 Quiz #5 version: 1

Instructor: Wenqiang Feng Name: ______________________

(1) (5 points) Evaluate the double integral $\int\int_D ydA$, where D is bounded by the line $y = 1 + x$ and the parabola $x = 1 - y^2$.

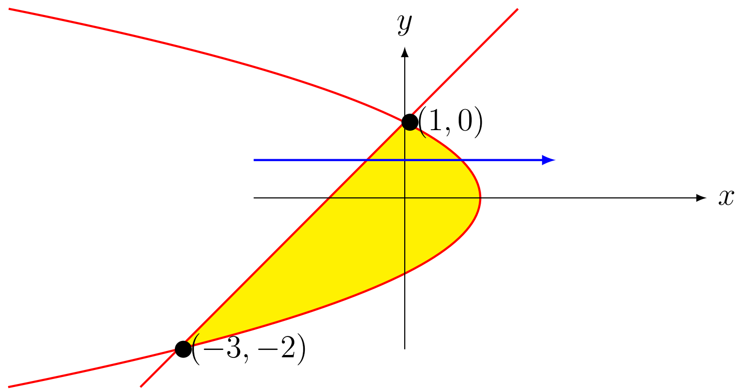

FIGURE 1. The domain of the integral.

1 Try to draw the constrains figures (Fig.1).

2 Figure out the intersection points

$$\begin{cases} y &= 1 + x \\ x &= 1 - y^2 \end{cases} \Rightarrow (0, 1),\ (-3, -2).$$

3 Determine the type of integral domain. For our this problem, we should use Horizontal domain.

4 Determine the bottom and top value of the integral domain. (According to the blue arrow in Fig.1)

$$\text{bottom function} : y = 1 + x$$
$$\text{top function} : x = 1 - y^2.$$

5 Since when we use Horizontal domain, the domain of y are fixed [-2, 1], then we need to put dy at the outer pair, i.e.

$$\int_{-2}^{1} \int f(x, y) dxdy$$

6 Next we need to figure out the domain of x (That is to say, we need to rewrite the bottom and top function as function w.r.t. y), i.e.

$$\text{bottom function} : x = y - 1$$
$$\text{top function} : x = 1 - y^2.$$

7 Set up the integral

$$\int_{-2}^{1}\int_{y-1}^{1-y^2} f(x,y)dxdy$$

8 Evaluate the double integral

$$\begin{aligned}
\int_{-2}^{1}\int_{y-1}^{1-y^2} f(x,y)dxdy &= \int_{-2}^{1}\int_{y-1}^{1-y^2} ydxdy \\
&= \int_{-2}^{1} y\int_{y-1}^{1-y^2} dxdy \\
&= \int_{-2}^{1} y\Big((1-y^2)-(y-1)\Big)dy \\
&= \int_{-2}^{1} 2y-y^2-y^3dy \\
&= \int_{-2}^{1} 2ydy-\int_{-2}^{1} y^2dy-\int_{-2}^{1} y^3dy \\
&= 2\cdot\frac{1}{2}y^2\Big|_{-2}^{1}-\frac{1}{3}y^3\Big|_{-2}^{1}-\frac{1}{4}y^4\Big|_{-2}^{1} \\
&= 1^2-(-2)^2-\frac{1}{3}1^3+\frac{1}{3}(-2)^3-\frac{1}{4}1^4+\frac{1}{4}(-2)^4 \\
&= 1-4-\frac{1}{3}-\frac{8}{3}-\frac{1}{4}+4 \\
&= -\frac{9}{4}.
\end{aligned}$$

(2) (5 points) Evaluate the triple integral $\int\int\int_E xdV$, where E is bound by the planes $x=0$, $y=1$, $z=0$ and $x+y+z=1$.

1 I prefer to project the plane to xy-plane, so we need to figure out the domain of z first,

$$\begin{aligned}
\text{bottom function} &: z=0 \\
\text{top function} &: x+y+z=1.
\end{aligned}$$

rewrite the function w.r.t x,y, i.e.

$$\begin{aligned}
\text{bottom function} &: z=0 \\
\text{top function} &: z=1-x-y.
\end{aligned}$$

Hence

$$\int\int\int_E xdV=\int\int\int_0^{1-x-y} xdzdA$$

2 Find out the region of the projection (That is to say, you need to form the system), i.e.

$$\begin{cases} x+y+z &= 1 \\ z &= 0. \end{cases} \Rightarrow x+y=1.$$

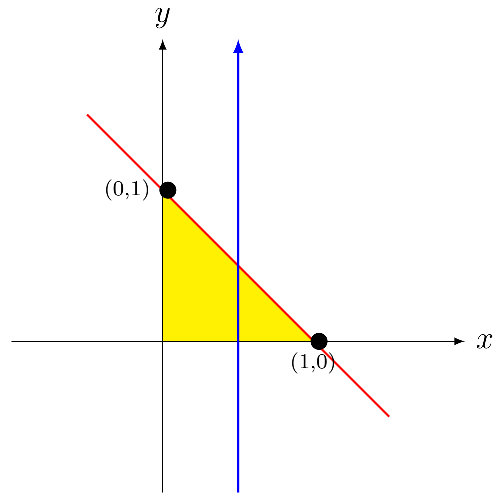

FIGURE 2. The domain of the projection region.

(3) Draw the figure of the projection region (Fig.2) So, we get

$$\text{bottom function}: y = 0$$
$$\text{top function}: x + y = 1.$$

(4) Next, we need to use the method which was used in 2D to figure out the domain of x and y.

- (1) Determine the type of integral domain. I will use vertical domain to solve this problem.
- (2) Since when we use vertical domain, the domain of x are fixed [0, 1] (Fig.2), then we need to put dx at the outer pair, i.e.

$$\int\int\int_E x dV = \int_0^1 \int \int_0^{1-x-y} x dz dy dx$$

- (3) Next we need to figure out the domain of y (That is to say, we need to rewrite the bottom and top function as function w.r.t. x), i.e.

$$\text{bottom function}: y = 0$$
$$\text{top function}: y = 1 - x.$$

- (4) Set up the integral

$$\boxed{\int\int\int_E x dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz dy dx}$$

- (5) Evaluate the double integral

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz dy dx &= \int_0^1 x \int_0^{1-x} \int_0^{1-x-y} dz dy dx \\
&= \int_0^1 x \int_0^{1-x} (1 - x - y) - 0 dy dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 x\left(\int_0^{1-x} dy - \int_0^{1-x} xdy - \int_0^{1-x} ydy\right) dx \\
&= \int_0^1 x\left((1-x) - x(1-x) - \frac{1}{2}y^2\bigg|_0^{1-x}\right) dx \\
&= \int_0^1 x(1-x) - x^2(1-x) - \frac{1}{2}(1-x)^2 dx \\
&= \int_0^1 \frac{1}{2}x - x^2 + \frac{1}{2}x^3 \\
&= \left(\frac{1}{2}\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{2}\frac{1}{4}x^4\right)\bigg|_0^1 \\
&= \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{1}{24}.
\end{aligned}
$$

Math 241 Quiz #5 version: 2

Instructor: Wenqiang Feng Name: ______________________________

(1) (5 points) Evaluate the double integral $\int\int_D y dA$, where D is bounded by the curves $y = x$ and the parabola $y = x^2$.

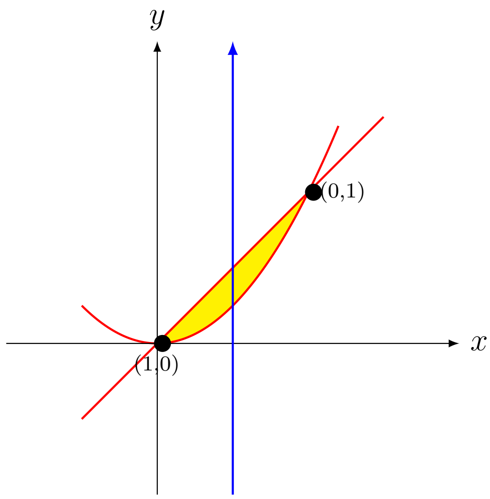

FIGURE 3. The domain of the integral.

1 Try to draw the constrains figures (Fig.3).

2 Figure out the intersection points

$$\begin{cases} y &= x \\ x &= x^2 \end{cases} \Rightarrow (0,0),\ (1,1).$$

3 Determine the type of integral domain. For our this problem, I will use vertical domain.

4 Determine the bottom and top value of the integral domain. (According to the blue arrow in Fig.3)

$$\text{bottom function} : y = x^2$$
$$\text{top function} : y = x.$$

5 Since when we use vertical domain, the domain of x are fixed [0, 1], then we need to put dx at the outer pair, i.e.

$$\int_0^1 \int f(x, y) dy dx$$

6 Next we need to figure out the domain of y (That is to say, we need to rewrite the bottom and top function as function w.r.t. x), i.e.

$$\text{bottom function} : y = x^2$$
$$\text{top function} : y = x.$$

(7) Set up the integral

$$\boxed{\int_0^1 \int_{x^2}^{x} f(x,y)\,dy\,dx}$$

(8) Evaluate the double integral

$$\begin{aligned}
\int_0^1 \int_{x^2}^{x} f(x,y)dydx &= \int_0^1 \int_{x^2}^{x} ydydx \\
&= \int_0^1 \left(\frac{1}{2}y^2\right)\Big|_{x^2}^{x} dx \\
&= \int_0^1 \frac{1}{2}x^2 - \frac{1}{2}x^4 dx \\
&= \frac{1}{2}\frac{1}{3}x^3\Big|_0^1 - \frac{1}{2}\frac{1}{5}x^5\Big|_0^1 \\
&= \frac{1}{6} - \frac{1}{10} = \frac{1}{15}.
\end{aligned}$$

(2) (5 points) Evaluate the triple integral $\int\int\int_E xdV$, where E is bound by the planes $x = 0$, $y = 1$, $z = 0$ and $x + 2y + 3z = 1$.

(1) I prefer to project the plane to xy-plane, so we need to figure out the domain of z first,

$$\begin{aligned}
\text{bottom function} &: z = 0 \\
\text{top function} &: x + 2y + 3z = 1.
\end{aligned}$$

rewrite the function w.r.t x, y, i.e.

$$\begin{aligned}
\text{bottom function} &: z = 0 \\
\text{top function} &: z = \frac{1 - x - 2y}{3}.
\end{aligned}$$

Hence

$$\int\int\int_E xdV = \int\int\int_0^{\frac{1-x-2y}{3}} xdzdA$$

(2) Find out the region of the projection (That is to say, you need to form the system), i.e.

$$\begin{cases} x + 2y + 3z &= 1 \\ z &= 0. \end{cases} \Rightarrow x + 2y = 1.$$

(3) Draw the figure of the projection region (Fig.4) So, we get

$$\begin{aligned}
\text{bottom function} &: y = 0 \\
\text{top function} &: x + 2y = 1.
\end{aligned}$$

(4) Next, we need to use the method which was used in 2D to figure out the domain of x and y.

(1) Determine the type of integral domain. I will use vertical domain to solve this problem.

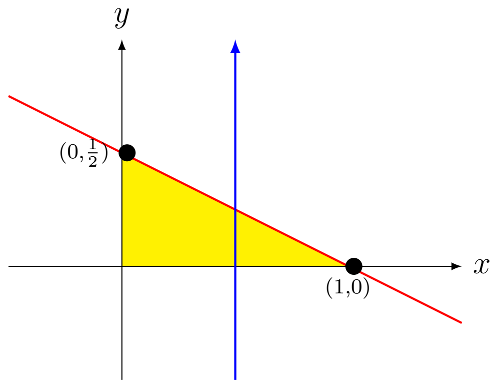

FIGURE 4. The domain of the projection region.

(2) Since when we use vertical domain, the domain of x are fixed [0, 1] (Fig.4), then we need to put dx at the outer pair, i.e.

$$\int\int\int_E xdV = \int_0^1 \int \int_0^{\frac{1-x-2y}{3}} xdzdydx$$

(3) Next we need to figure out the domain of y (That is to say, we need to rewrite the bottom and top function as function w.r.t. x), i.e.

$$\begin{aligned} \text{bottom function} : y &= 0 \\ \text{top function} : y &= \frac{1-x}{2}. \end{aligned}$$

(4) Set up the integral

$$\boxed{\int\int\int_E xdV = \int_0^1 \int_0^{\frac{1-x}{2}} \int_0^{\frac{1-x-2y}{3}} xdzdydx}$$

(5) Evaluate the double integral

$$\begin{aligned} \int_0^1 \int_0^{\frac{1-x}{2}} \int_0^{1-x-y} xdzdydx &= \int_0^1 x \int_0^{\frac{1-x}{2}} \int_0^{1-x-y} dzdydx \\ &= \frac{1}{3} \int_0^1 x \int_0^{\frac{1-x}{2}} 1-x-2ydydx \\ &= \frac{1}{3} \int_0^1 x \left(\int_0^{\frac{1-x}{2}} 1dy - \int_0^{\frac{1-x}{2}} xdy - \int_0^{\frac{1-x}{2}} 2ydy \right) dx \\ &= \frac{1}{3} \int_0^1 x \left(\frac{1-x}{2} - x\frac{1-x}{2} - 2\frac{1}{2}y^2 \Big|_0^{\frac{1-x}{2}} \right) dx \\ &= \frac{1}{6} \int_0^1 \left(x(1-x) - x^2(1-x) - x\frac{1-2x+x^2}{2} \right) dx \\ &= \frac{1}{6} \int_0^1 \frac{1}{2}x - x^2 + \frac{1}{2}x^3 dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{12}\int_0^1 x dx - \frac{1}{6}\int_0^1 x^2 + \frac{1}{12}\int_0^1 x^3 dx \\
&= \frac{1}{12}\frac{1}{2}x^2\Big|_0^1 - \frac{1}{6}\frac{1}{3}x^3\Big|_0^1 + \frac{1}{12}\frac{1}{4}x^4 dx \\
&= \frac{1}{24} - \frac{1}{18} + \frac{1}{48} = \frac{1}{144}.
\end{aligned}
$$