MATH 574: Finite Element Method Homework 1

Due on October 28, 2014

TTH 9:40am-10:55am

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Contents

Problem 1

Show that the function $f(x) = |x|, -1 < x < 1$ has the weak derivative

$$f'(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

Obviously f and f' blond to $L^2(-1,1)$. Thus by definition, f belongs to $H^1(-1,1)$. Also, show that $f'(x)$ does not have a weak derivative in $L^2(-1,1)$. For the latter, use the fact that $C_0^\infty(I)$ is dense in $L^2(I)$ for any interval.

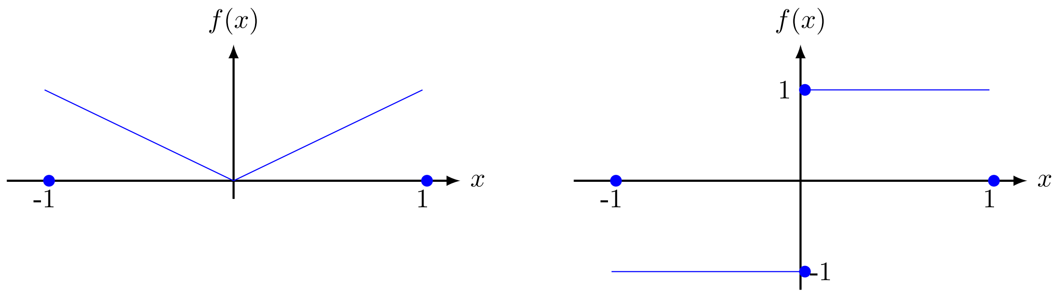

Figure 1: The function of $f(x) = |x|$ and it's weak derivative.

Proof. Let $\Omega = (-1,1)$ and

$$\mathcal{D}(\Omega) = \{u \in C^\infty : supp\ u \subset\subset \Omega\}.$$

Then integration by part yields, $\forall \phi \in \mathcal{D}(\Omega)$

$$\begin{aligned} \int_{-1}^{1} f(x)\phi'(x)dx &= \int_{-1}^{0} f(x)\phi'(x)dx + \int_{0}^{1} f(x)\phi'(x)dx \\ &= -\int_{-1}^{0} x\phi'(x)dx + \int_{0}^{1} x\phi'(x)dx \\ &= -x\phi(x)|_{-1}^{0} + \int_{-1}^{0} \phi(x)dx + x\phi(x)|_{0}^{1} - \int_{0}^{1} \phi(x)dx \\ &= \int_{-1}^{0} \phi(x)dx - \int_{0}^{1} \phi(x)dx \\ &= -\int_{-1}^{1} g(x)\phi(x)dx, \end{aligned}$$

where

$$g(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

Since $g(x) \in L^1_{\text{loc}}(\Omega)$, so $f'(x) = g(x)$.
Next, we will consider the second order distribution derivative. $\forall \phi \in \mathcal{D}(\Omega)$

$$\begin{aligned} \int_{-1}^{1} f(x)\phi''(x)dx &= -\int_{-1}^{1} g(x)\phi'(x)dx \\ &= -\int_{-1}^{0} g(x)\phi'(x)dx - \int_{0}^{1} g(x)\phi'(x)dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{-1}^{0} \phi'(x)dx - \int_{0}^{1} \phi'(x)dx \\
&= 2\phi(0) = 2\int_{-1}^{1} \delta(x)\phi(x)dx,
\end{aligned}
$$

where

$$
\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0. \end{cases} \quad \text{and} \quad \int_{-1}^{1} \delta(x)dx = 1.
$$

Since $\delta(x) \notin L^1_{\text{loc}}(\Omega)$, so $f''(x)$ does not exist. □

Problem 2

Consider the boundary value problem

$$
\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} = h, & \text{on } \Gamma_N. \end{cases}
$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ consisting of two disjoint subsets Γ_D and Γ_N. Give an appropriate weak formulation in the space $V = \{v \in H^1, v = 0 \text{ on } \Gamma_D\}$. Then give a Finite Element formula for this problem.

Proof. Take $V = \{v \in H^1, v = 0 \text{ on } \Gamma_D\}$, then $\forall v \in V$,

$$
\begin{aligned}
\int_{\Omega} fvdx &= \int_{\Omega} -\Delta uvdx \\
&= \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot vds \\
&= \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\Gamma_N} h \cdot vds
\end{aligned}
$$

The weak form is to find $u \in V$ s.t

$$
a(u, v) = f(u, v), \forall v \in V,
$$

where

$$
\begin{aligned}
a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla vdx, \\
(f, v) &= \int_{\Omega} f \cdot vdx - \int_{\Gamma_N} h \cdot vds.
\end{aligned}
$$

□

Problem 3

Let $(K, P_K, \sum_K)$ be finite element with $\sum_K = \{\ell_1, \ell_2, \cdots, \ell_N\}$. We say that the set $\sum_K$ is P_K-unisolvent if given any set of N real numbers $\alpha_1, \alpha_2, \cdots, \alpha_N$, there exists a unique $v \in P_K$ such that $\ell_i(v) = \alpha_i, i = 1, 2, \cdots, N$. Let $\{\phi_i\}_{i=1}^N$ be the uniquely defined function in P_K satisfying $\ell_i(\phi_j) = \delta_{ij}, i, j = 1, 2, \cdots, N$.

Problem 3 continued on next page. . .

1. Show that the function $\{\phi_i\}_{i=1}^N$ are linearly independent.
2. Show that any v in P_K can be expressed as $v = \sum_{i=1}^N \ell_i(v)\phi_i$

Thus the set $\{\phi_i\}_{i=1}^N$ is a basis for P_K.

Proof. 1. Assume that $\{\phi_i\}_{i=1}^N$ are linearly dependent. There is to say, there exists a $\beta_j \neq 0$, such that

$$\phi_j = \frac{\beta_1}{\beta_j}\phi_1 + \frac{\beta_2}{\beta_j}\phi_2 + \cdots + \frac{\beta_N}{\beta_j}\phi_N.$$

Then, we have

$$\begin{aligned} \ell_j(\phi_j) &= \ell_j\left(\frac{\beta_1}{\beta_j}\phi_1 + \frac{\beta_2}{\beta_j}\phi_2 + \cdots + \frac{\beta_N}{\beta_j}\phi_N\right) \\ &= \frac{\beta_1}{\beta_j}\ell_j(\phi_1) + \frac{\beta_2}{\beta_j}\ell_j(\phi_2) + \cdots + \frac{\beta_N}{\beta_j}\ell_j(\phi_N) \\ &= 0 \neq 1. \end{aligned}$$

So, we get the contradiction. Therefore, the function $\{\phi_i\}_{i=1}^N$ are linearly independent.

2. Since , the function $\{\phi_i\}_{i=1}^N$ are linearly independent, so $\{\phi_i\}_{i=1}^N$ is a basis for P_K, i.e.

$$v = \beta_1\phi_1 + \beta_2\phi_2 + \cdots + \beta_N\phi_N.$$

Then,

$$\begin{aligned} \ell_i(v) &= \ell_i\left(\beta_1\phi_1 + \beta_2\phi_2 + \cdots + \beta_N\phi_N\right) \\ &= \beta_1\ell_i(\phi_1) + \beta_2\ell_i(\phi_2) + \cdots + \beta_N\ell_i(\phi_N) \\ &= \beta_i, \end{aligned}$$

for all $i = 1, 2, \cdots, N$. Hence, any v in P_K can be expressed as

$$v = \sum_{i=1}^N \ell_i(v)\phi_i.$$

□

Problem 4

Let $\mathcal{T}_h$ be a regular partition of Ω. Show that if v is such that $v \in H^1(K), \forall K \in \mathcal{T}_h$ and $v \in C^0(\Omega)$, then v belongs to $H^1(\Omega)$. Similarly, it can be shown that if v is such that $v \in H^2(K), \forall K \in \mathcal{T}_h$ and $v \in C^1(\Omega)$, then v belongs to $H^2(\Omega)$

Proof. Since $v \in H^1(K)$ and each element K has a Lipschitz-continuous boundary ∂K, applying the Green's formula yields: For each $K \in \mathcal{T}_h$,

$$\int_K \partial_i(v|_K)\phi dx = \int_{\partial K} v|_K \phi \cdot \mathbf{n}_i^K ds - \int_K v|_K \partial_i(\phi)dx,$$

for any $\phi \in \mathcal{D}(\Omega)$, where $\mathbf{n}_i^K$ is the i-th component of the unit outer normal vector along ∂K. By summing over all of the elements, we have

$$\int_\Omega \partial_i(v|_K)\phi dx = \sum_{K\in\mathcal{T}_h} \int_{\partial K} v|_K \phi \cdot \mathbf{n}_i^K ds - \int_\Omega v|_K \partial_i(\phi)dx.$$

From the observation, we have that $\sum_{K\in\mathcal{T}_h}\int_{\partial K} v|_K\phi\cdot\mathbf{n}_i^K ds$ vanishes on the interior edges when $v\in C^0$. Moreover, since $\phi\in\mathcal{D}(\Omega)$, so $\sum_{K\in\mathcal{T}_h}\int_{\partial K} v|_K\phi\cdot\mathbf{n}_i^K ds$ vanishes on the boundary edges. Therefore,

$$\int_\Omega \partial_i(v|_K)\phi dx = -\int_\Omega v|_K\partial_i(\phi)dx.$$

Hence, $v\in H^1(\Omega)$. $\square$

Problem 5

Show that the Argyris triangle, when assembled, leads to a (global) finite element space V_h which is a subspace of $C^1(\Omega)$, Thus, according to the preceding problem, V_h is a subspace of $H^2(\Omega)$.

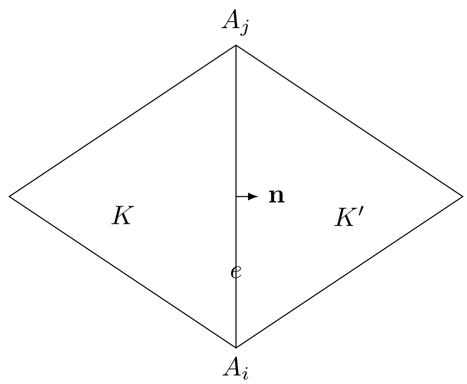

Figure 2: The adjacent side of triangle.

Proof. Let

$$v_h^K = v_h|_K,\quad v_h^{K'} = v_h|_{K'},\quad \mathbf{n}=n^T.$$

If we want to show that Argyris element is C^1, we only need to show that the Argyris element v_h satisfy

$$\begin{cases} v_h^K|_e = v_h^{K'}|_e \\ \partial_n v_h^K|_e = \partial_n v_h^{K'}|_e. \end{cases}$$

on the interior edges (Fig.2)of two arbitrary two element. Let

$$q_1(t) = v_h^K|_e,\quad q_2(t) = v_h^{K'}|_e,$$

be the Argyris elements on edge e. From the class, we have known that

$$q_1, q_2 \in \mathbb{P}_5,$$

so

$$q(t) = q_1(t) - q_2(t) \in \mathbb{P}_5.$$

Moreover, we have

$$\begin{cases} q(A_i) = q(A_j) = 0, \\ q_t'(A_i) = q_t'(A_j) = 0, \\ q_{tt}''(A_i) = q_{tt}''(A_j) = 0. \end{cases}$$

Thus $v_h^K|_e = v_h^{K'}|_e$ and $\partial_\tau v_h^K|_e = \partial_\tau v_h^{K'}|_e$. Denote

$$r_1(t) = \partial_n v_h^K|_e, \quad r_2(t) = \partial_n v_h^{K'}|_e,$$

then

$$r(t) = r_1(t) - r_2(t) \in \mathbb{P}_4(e).$$

Thus

$$\begin{cases} r(A_i) = r(A_j) = 0, \\ r_t'(A_i) = \partial_{nt} v_h^K|_e(A_i) - \partial_{nt} v_h^{K'}|_e(A_i) = 0, \\ r_t'(A_j) = \partial_{nt} v_h^K|_e(A_j) - \partial_{nt} v_h^{K'}|_e(A_j) = 0. \end{cases}$$

Therefore,

$$\partial_n v_h^K|_e = \partial_n v_h^{K'}|_e.$$

□

Problem 6

Let K be an triangle and let $\lambda_1, \lambda_2, \lambda_3$ be the barycentric coordinates of a point $x \in K$. Let $(\alpha_1, \alpha_2, \alpha_3)$ be any multiinteger. Show that

$$\int_K \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} dx = |K| \frac{\alpha_1! \alpha_2! \alpha_3! 2!}{(\alpha_1 + \alpha_2 + \alpha_3 + 2)!}.$$

Proof. Let

$$\xi_i = x_i - x_k, \quad \eta_i = y_j - y_k, \quad \omega_i = x_j y_k - x_k y_j,$$

where i, j, k is in anticolckwise order (Fig.3)

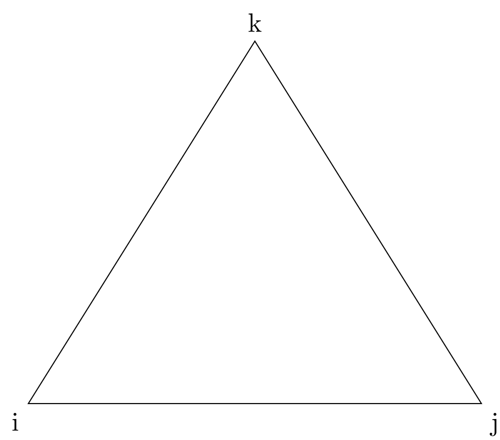

Figure 3: The barycentric coordinates of triangle.

Then we have

$$|K| = \frac{1}{2} \det(D) = \omega_1 \omega_2 \omega_3,$$

where

$$D = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

And

$$\lambda_i(x,y) = \frac{1}{2\,|K|}\left(\eta_i x - \xi_i y + \omega_i\right), \quad i = 1,2,3.$$

Moreover, we have

$$\begin{cases} \sum_{i=1}^{3} \lambda_i(x,y) = 1 \\ \sum_{i=1}^{3} \lambda_i(x,y) x_i = x, \\ \sum_{i=1}^{3} \lambda_i(x,y) y_i = y. \end{cases}.$$

on K.

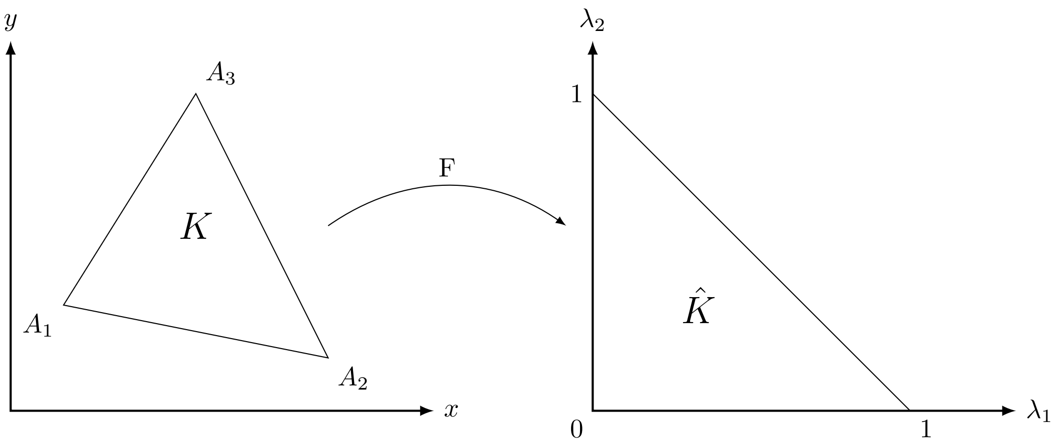

Figure 4: The affine mapping of triangular element.

Now, we will transform (x,y) plane to (λ_1,λ_2) plane (Fig.4), then we get the determine of the Jacobi matrix is as follow

$$\begin{cases} \det\left(\frac{\partial(\lambda_1,\lambda_2)}{\partial(x,y)}\right) = \begin{bmatrix} \frac{\partial\lambda_1}{\partial x} & \frac{\partial\lambda_1}{\partial y} \\ \frac{\partial\lambda_2}{\partial x} & \frac{\partial\lambda_2}{\partial y} \end{bmatrix} = \frac{1}{2|K|}, \\ \det\left(\frac{\partial(x,y)}{\partial(\lambda_1,\lambda_2)}\right) = \det\left(\frac{\partial(\lambda_1,\lambda_2)}{\partial(x,y)}\right)^{-1} = 2\,|K|\,. \end{cases}$$

$$\begin{aligned} \int_K \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}dx &= \int\int_K \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}dxdy \\ &= \int\int_{\hat{K}} \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\left(1-\lambda_1-\lambda_2\right)^{\alpha_3}\det\left(\frac{\partial(x,y)}{\partial(\lambda_1,\lambda_2)}\right)d\lambda_1 d\lambda_2 \\ &= 2\,|K|\int\int_{\hat{K}} \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\left(1-\lambda_1-\lambda_2\right)^{\alpha_3}d\lambda_1 d\lambda_2 \\ &= 2\,|K|\int_0^1\int_0^{1-\lambda_2} \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\left(1-\lambda_1-\lambda_2\right)^{\alpha_3}d\lambda_1 d\lambda_2. \end{aligned}$$

Changing of variables $t = \frac{\lambda_1}{1-\lambda_2}$, $\left(1 - t = \frac{1-\lambda_1-\lambda_2}{1-\lambda_2}\right)$ yields

$$\begin{aligned}
\int_K \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}dx &= 2|K| \int_0^1\int_0^1 (t(1-\lambda_2))^{\alpha_1}\lambda_2^{\alpha_2}((1-t)(1-\lambda_2))^{\alpha_3}(1-\lambda_2)dtd\lambda_2 \\
&= 2|K| \int_0^1\int_0^1 t^{\alpha_1}(1-\lambda_2)^{\alpha_1}\lambda_2^{\alpha_2}((1-t)(1-\lambda_2))^{\alpha_3}(1-\lambda_2)dtd\lambda_2 \\
&= 2|K| \int_0^1 (1-\lambda_2)^{\alpha_1+\alpha_3+1}\lambda_2^{\alpha_2}d\lambda_2 \int_0^1 t^{\alpha_1}(1-t)^{\alpha_3}\,dt.
\end{aligned}$$

By apply the Euler integral formula

$$\int_0^1 s^\alpha(1-s)^\beta ds = \frac{\alpha!\beta!}{(\alpha+\beta+1)!},$$

we have

$$\begin{aligned}
\int_K \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}dx &= 2|K| \frac{\alpha_2!\,(\alpha_1+\alpha_3+1)!}{(\alpha_2+\alpha_1+\alpha_3+2)!}\frac{\alpha_1!\alpha_3!}{(\alpha_1+\alpha_3+1)!} \\
&= |K| \frac{2\alpha_1!\alpha_2!\alpha_3!}{(\alpha_2+\alpha_1+\alpha_3+2)!}.
\end{aligned}$$

□

Problem 7

Using the preceding problem show that the quadrature rule

$$\int_K f(x)dx \approx \frac{|K|}{60}\left(3\sum_{i=1}^3 f(a_i) + 8\sum_{1\le i<j\le 3} f(a_{ij}) + 27f(a_{123})\right)$$

is exact on $\mathbb{P}_3(K)$.

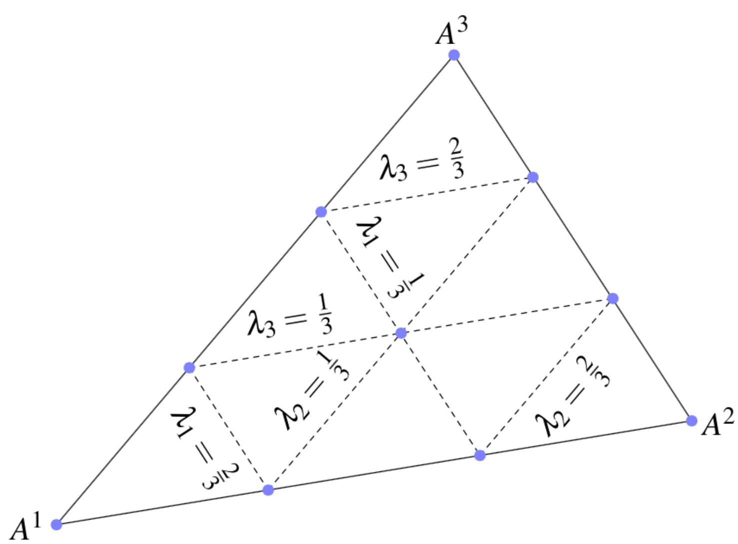

Figure 5: The 10 nodes of a $\mathbb{P}_3(K)$ Lagrange triangle.

Proof. We need to check that the basis functions of $\mathbb{P}_3(K)$ are exact for the quadrature rule. From Fig.5, we can get the 10 basis function are as follows:

$$\begin{cases} \text{1 center:} & \begin{cases}\phi_1 = 27\lambda_1\lambda_2\lambda_3\end{cases} \\ \text{3 vertices:} & \begin{cases}\phi_2 = \frac{1}{2}\lambda_1(3\lambda_1-1)(3\lambda_1-2)\\ \phi_3 = \frac{1}{2}\lambda_2(3\lambda_2-1)(3\lambda_2-2)\\ \phi_4 = \frac{1}{2}\lambda_3(3\lambda_3-1)(3\lambda_3-2)\end{cases} \\ \text{6 edge nodes:} & \begin{cases}\phi_5 = \frac{9}{2}\lambda_1\lambda_2(3\lambda_1-1)\\ \phi_6 = \frac{9}{2}\lambda_1\lambda_2(3\lambda_1-2)\\ \phi_7 = \frac{9}{2}\lambda_2\lambda_3(3\lambda_2-1)\\ \phi_8 = \frac{9}{2}\lambda_2\lambda_3(3\lambda_2-2)\\ \phi_9 = \frac{9}{2}\lambda_1\lambda_3(3\lambda_3-1)\\ \phi_{10} = \frac{9}{2}\lambda_1\lambda_3(3\lambda_3-2).\end{cases} \end{cases}$$

For $f(x) = \phi_1$, we have

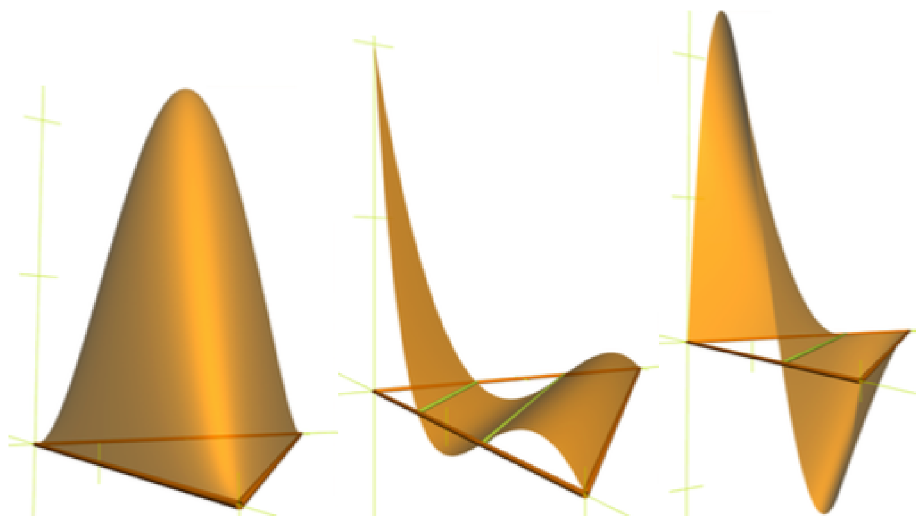

Figure 6: The three different kinds of $\mathbb{P}_3(K)$ basis polynomials..

$$\begin{aligned} LHS &= \int_K 27\lambda_1\lambda_2\lambda_3 dx \\ &= 27\int_K \lambda_1\lambda_2\lambda_3 dx \\ &= 27\,|K|\,\frac{2}{(1+1+1+2)!} \\ &= \frac{9\,|K|}{20}. \end{aligned}$$

$$RHS = \frac{|K|}{60}\left(0+0+27\cdot 27\cdot \frac{1}{3}\frac{1}{3}\frac{1}{3}\right) = \frac{9\,|K|}{20}.$$

For $f(x) = \phi_2$, we have

$$LHS = \int_K \frac{1}{2}\lambda_1(3\lambda_1-1)(3\lambda_1-2)dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_K \lambda_1(9\lambda_1^2 - 9\lambda_1 + 2)dx \\
&= \frac{9}{2}\int_K \lambda_1^3 dx - \frac{9}{2}\int_K \lambda_1^2 dx + \int_K \lambda_1 dx \\
&= \frac{9}{2}|K|\frac{2\cdot 3!}{(3+0+0+2)!} - \frac{9}{2}|K|\frac{2\cdot 2!}{(2+0+0+2)!} + |K|\frac{2\cdot 1!}{(1+0+0+2)!} \\
&= \frac{9}{20}|K| - \frac{3}{4}|K| + \frac{1}{3}|K| \\
&= \frac{1}{30}|K|. \\
RHS &= \frac{|K|}{20}(1+0+0) \\
&+ \frac{2|K|}{15}\left(\frac{1}{2}\cdot\frac{1}{2}(\frac{3}{2}-1)(\frac{3}{2}-2)+0+0\right) \\
&+ \frac{9|K|}{20}(0) \\
&= \frac{|K|}{30}.
\end{aligned}
$$

Similarly, we can get that basis functions of $\phi_3, \cdots, \phi_{10}$ are exact for the quadrature rule. Therefore, the quadrature rule is exact on $\mathbb{P}_3(K)$. □

Problem 8

Use the definition to show that the $n\times n$ tridiagonal matrix $A = [-1\ \ 2\ \ -1]$ is symmetric, positive definite.

Proof. Since

$$
A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n\times n},
$$

obviously, A is symmetric. Next, we will show that A is positive definite. First, I would like to proof the following two important lemma

Lemma 0.1. *(Eigenvalues of Tridiagonal Matrices) If $A = diag(b, a, b) \in \mathcal{M}_n$ is an tridiagonal matrix, then the eigenvaules of A are*

$$
\lambda_k = a + 2b\cos(\theta_k), \quad k = 1, 2, 3, \cdots, N
$$

and its corresponding eigenvector are

$$
\vec{\xi}_k = \sqrt{2}\,(\sin(1\theta_k), \sin(2\theta_k), \cdots, \sin(N\theta_k))
$$

where

$$
\theta_k = k\theta = k\pi h = \frac{k\pi}{N+1}.
$$

Proof. It can be easily verified by the trigonometric identities

$$\sin(2\theta_k) = 2\sin(\theta_k)\cos(\theta_k),$$

and

$$2\sin(ki\theta_k)\cos(k\theta_k) = \sin(k(i-1)\theta_k) + \sin(k(i+1)\theta_k).$$

□

Lemma 0.2. *Suppose $A \in \mathbb{C}_{her}^{n\times n}$, and $\rho(A) \subset (0,\infty)$. Prove that A is Hermitian Positive Definite.*

Proof. Since $A \in \mathbb{C}_{her}^{n\times n}$, then the eigenvalue of A are real. Let λ be arbitrary eigenvalue of A, then

$$(Ax,x) = (\lambda x, x) = \lambda(x,x),$$
$$(Ax,x) = (x, A^* x) = (x, Ax)(x, \lambda x) = \overline{\lambda}(x,x),$$

and then $\lambda = \overline{\lambda}$, so λ is real. Moreover, since $\rho(A) \subset (0,\infty)$, then we have λ is positive.

$$x^* Ax = x^* \lambda x = \lambda x^* x = \lambda(x_1^2 + x_2^2 + \cdots + x_n^2) > 0.$$

for all $x \neq 0$. Hence, A is Hermitian Positive Definite. □

For our this problem, Form Lemma.0.1, we know that $\rho(A) \subset (0,\infty)$. Then by using Lemma.0.2, we prove that A is Hermitian Positive Definite. □