1.3 Norms.

A Norm $\|\cdot\|$ on a vector space $\mathbb{E}$ is a function $\mathbb{E} \mapsto \mathbb{R}$.

which satisfies:

1) nonnegativity: $\forall \vec{x} \in \mathbb{E}, \|\vec{x}\| \geq 0$ $\quad \|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.

2) positive homogeneity: $\forall \vec{x} \in \mathbb{E}, \forall \lambda \in \mathbb{R}, \|\lambda \vec{x}\| = |\lambda| \|\vec{x}\|$

3) triangle inequality: $\forall \vec{x}, \vec{y} \in \mathbb{E}$ $\quad \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

open ball: $B(\vec{c}, r) = \{\vec{x} \in \mathbb{E} : \|\vec{x} - \vec{c}\| < r\}$.

closed ball: $B[\vec{c}, \vec{r}] = \{\vec{x} \in \mathbb{E} : \|\vec{x} - \vec{c}\| \leq r\}$.

1.4. Inner product.

$\vec{x}, \vec{y} \in \mathbb{E} \Rightarrow$ associated with a real number. $\langle \vec{x}, \vec{y} \rangle$.

1. Commutativity: $\forall \vec{x}, \vec{y} \in \mathbb{E}$ $\quad \langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

2. linearity. $\langle \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2, \vec{y} \rangle = \alpha_1 \langle \vec{x}_1, \vec{y} \rangle + \alpha_2 \langle \vec{x}_2, \vec{y} \rangle$. $\quad \forall \vec{x}_1, \vec{x}_2, \vec{y} \in \mathbb{E}$. $\forall \alpha_1, \alpha_2 \in \mathbb{R}$

3. positive definiteness $\forall \vec{x} \in \mathbb{E}, \langle \vec{x}, \vec{x} \rangle \geq 0$ $\quad \langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$.

Inner product space: a vector space endowed (具有) with an inner product.

Underlying Spaces: a vector space endowed with $\begin{cases} \text{norm } \|\cdot\| \\ \text{inner product } \langle \cdot, \cdot \rangle \end{cases}$.

1.5 Affine Sets, Convex Sets.

Affine: For a real vector space $\mathbb{E}$.

a set $S \subseteq \mathbb{E}$ is affine if $\forall \vec{x}, \vec{y} \in S$. $\forall \lambda \in \mathbb{R}$ $\quad \lambda \vec{x} + (1-\lambda) \vec{y} \in S$

Affine hull of S : $\text{aff}(S)$

the intersection of all affine sets containing S.

$\text{aff}(S)$ is also an affine set.

$\text{aff}(S)$ is the smallest affine set containing S.

Exercise: a hyperplane: $H_{\vec{a},b} = \{\vec{x} \in \mathbb{E} : \langle \vec{a}, \vec{x} \rangle = b\}$.

prove: a hyperplane is an affine set.

proof: $\langle \vec{a}, \vec{m} \rangle = b$ $\langle \vec{a}, \vec{n} \rangle = b$ $\vec{m}, \vec{n} \in \mathbb{E}$

$\forall \lambda$ $\langle \vec{a}, \lambda\vec{m} + (1-\lambda)\vec{n} \rangle = \lambda \langle \vec{a}, \vec{m} \rangle + (1-\lambda) \langle \vec{a}, \vec{n} \rangle$

$= \lambda b + (1-\lambda) b = b.$

$\Rightarrow \lambda\vec{m} + (1-\lambda)\vec{n} \in S$ $\Rightarrow$ $H_{\vec{a},b}$ is an affine set of $\mathbb{E}$.

Convex:

A set $C \subseteq \mathbb{E}$ is convex if $\forall \vec{x}, \vec{y} \in C$, $\forall \lambda \in [0,1]$ $\lambda\vec{x} + (1-\lambda)\vec{y} \in C$

Affine sets are always convex.

Open, closed balls are convex. (regardless of the choice of norm).

proof:

$\|\vec{m} - \vec{c}\| < r$, $\|\vec{n} - \vec{c}\| < r$ $\|\lambda\vec{m} + (1-\lambda)\vec{n} - \vec{c}\|$ $\lambda \in [0,1]$

$= \|\lambda(\vec{m} - \vec{c}) + (1-\lambda)(\vec{n} - \vec{c})\|$

$\leq |\lambda| \|\vec{m} - \vec{c}\| + |1-\lambda| \|\vec{n} - \vec{c}\|$

$= \lambda \|\vec{m} - \vec{c}\| + (1-\lambda) \|\vec{n} - \vec{c}\|$

$< \lambda r + (1-\lambda) r = r.$ $\Rightarrow$ convex set.

Same for closed balls.

examples for convex sets.

Closed line segment : $[\vec{x}, \vec{y}] = \{\alpha \vec{x} + (1-\alpha)\vec{y} : \alpha \in [0,1]\}$.

Open line segment : $(\vec{x}, \vec{y}) = \{\alpha \vec{x} + (1-\alpha)\vec{y} : \alpha \in (0,1)\}$.

half-spaces : $H^-_{\vec{a},b} = \{\vec{x} \in \mathbb{E} : \langle \vec{a}, \vec{x} \rangle \le b\}$.

1.6 Euclidean Spaces.

A finite dimensional real vector space is Euclidean space if

① equipped with an inner product $\langle \cdot, \cdot \rangle$.

② endowed with the norm $\|\vec{x}\| = \sqrt{\langle \cdot, \cdot \rangle}$ Euclidean norm.

1.7 $\mathbb{R}^n$.

Inner product in $\mathbb{R}^n$: $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i$

(Unless otherwise stated, inner product in $\mathbb{R}^n$ is dot product).

Q-inner product : $\langle \vec{x}, \vec{y} \rangle_Q = \vec{x}^T Q \vec{y}$, Q positive $n \times n$.

$Q = I_n \Rightarrow \langle \vec{x}, \vec{y} \rangle_Q = \langle \vec{x}, \vec{y} \rangle$.

Then $\|\vec{x}\|_2 = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$

$\|\vec{x}\|_Q = \sqrt{\vec{x}^T Q \vec{x}}$

$p \ge 1$, ℓ_p-norm : $\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$.

ℓ_∞ - norm on $\mathbb{R}^n$: $\|\vec{x}\|_\infty = \max_{i=1,\cdots n} |x_i|$.

1.7.) Subset of $\mathbb{R}^n$.

nonnegative orthant : $\mathbb{R}^n_+ = \{(x_1, \cdots x_n)^T : x_1, \cdots x_n \geq 0\}$.

positive orthant : $\mathbb{R}^n_{++} = \{(x_1, \cdots x_n)^T : x_1, \cdots x_n > 0\}$.

unit simplex : $\Delta_n = \{\vec{x} \in \mathbb{R}^n : x \geq \vec{0}, \ \vec{e}^T\vec{x} = 1\}$.

$\mathrm{Box}[\vec{l}, \vec{u}] = \{\vec{x} \in \mathbb{R}^n : \vec{l} \leq \vec{x} \leq \vec{u}\}$ eg. $\mathrm{Box}[-\vec{e}, \vec{e}] = [-1, 1]^n$.

1.72 Operations on vectors in $\mathbb{R}^n$.

$[\vec{x}]_+ = (\max\{x_i, 0\})_{i=1}^n$

$|\vec{x}| = (|x_i|)_{i=1}^n$

$$\mathrm{sgn}(\vec{x})_i = \begin{cases} 1 & x_i \geq 0 \\ -1 & x_i < 0 \end{cases}$$

Hadamard product : $\vec{a} \odot \vec{b} = (a_i b_i)_{i=1}^n$

(component-wise product)

1.8 $\mathbb{R}^{m\times n}$ ... The set of all real-valued $m\times n$ matrices.

dot product in $\mathbb{R}^{m\times n}$: $\langle A, B\rangle = \mathrm{Tr}(A^T B) = \sum_{i=1}^m \sum_{j=1}^n A_{ij} B_{ij}$

(unless otherwise stated, is inner product in $\mathbb{R}^{m\times n}$).

1.8.1 Subset of $\mathbb{R}^{n\times n}$.

$S^n = \{A \in \mathbb{R}^{n\times n} : A = A^T\}$

$S^n_+ = \{A \in \mathbb{R}^{n\times n} : A \succeq 0\}$. $\Rightarrow S^n_{++} \subseteq S^n_+ \subseteq S^n$

$S^n_{++} = \{A \in \mathbb{R}^{n\times n} : A \succ 0\}$.

Similarly, $S^n_- = \{A \in \mathbb{R}^{n\times n} : A \preceq 0\}$

$S^n_{--} = \{A \in \mathbb{R}^{n\times n} : A \prec 0\}$.

The set of all orthogonal matrices $O^n = \{A \in \mathbb{R}^{n\times n} : AA^T = A^TA = I\}$.

1.8.2 Norms in $\mathbb{R}^{m\times n}$.

Frobenius norm: $\|A\|_F = \sqrt{Tr(A^TA)} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2}$, $A \in \mathbb{R}^{m\times n}$

Induced norm $\|A\|_{a,b} = \max_{\vec{x}} \{\|A\vec{x}\|_b : \|\vec{x}\|_a \le 1\}$.

Inequality: $\|A\vec{x}\|_b \le \|A\|_{a,b}\|\vec{x}\|_a$. $\|\cdot\|_{a,b}$ (a,b) norm.

when $a=b \Rightarrow$ a-norm.

eg. 1.1 spectral norm.

if $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|^2$ $\Rightarrow$ the induced norm of $A^{m\times n}$ is the maximum singular value of A.

$\|A\|_2 = \|A\|_{2,2} = \sqrt{\lambda_{max}(A^TA)} \equiv \sigma_{max}(A)$.

1-norm. If $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_1$ $\|A\|_1 = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |A_{ij}|$

∞-norm　If $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_\infty$　$\|A\|_\infty = \max_{i=1,2\cdots m} \sum_{j=1}^{n} |A_{ij}|$

1.9. Cartesian Product of Vector Spaces.

1.10. Linear Transformation　$\mathbb{R}^n \to \mathbb{R}^m$　$\mathcal{A}(\vec{x}) = A\vec{x}$.

$\mathbb{R}^{m\times n} \to \mathbb{R}^k$　$\mathcal{A}(X) = \begin{pmatrix} Tr(A_1^T X) \\ Tr(A_2^T X) \\ \vdots \\ Tr(A_k^T X) \end{pmatrix}$

$X, A_1 \sim A_k \in \mathbb{R}^{m\times n}$

1.11 Dual Space.

linear functional on $\mathbb{E}$: a linear transformation from E to $\mathbb{R}$.

E^* : dual space --- a set of all linear functionals on $\mathbb{E}$.

For inner products space, $\forall f \in E^*$, $\exists \vec{v} \in E$ s.t. $f(\vec{x}) = \langle \vec{v}, \vec{x} \rangle$.

Correspondance between $\begin{cases} \text{linear functionals} \\ \text{elements in } \mathbb{E}. \end{cases}$

Elements in E are exactly the same as elements in E^*?

Only difference : norm.

E is endowed with a norm $\|\cdot\|$,

then dual norm (the norm of dual space)　$\|y\|_* = \max_x \{\langle y, x\rangle : \|x\| \le 1\}$, $y \in E^*$.

actually, $\|y\|_* = \max_x \{\langle y, x\rangle : \|x\| = 1\}$, $y \in E^*$

is also valid.

$\|f\|_* = \sup\left\{\frac{\|f(x)\|}{\|x\|}, \|x\|\neq 0\right\}$

Lemma 1.4. generalized Cauchy–Schwarz inequality

$\mathbb{E}$: inner product vector space endowed with norm $\|\cdot\|$.

Then $\|\langle y,x\rangle\| \le \|y\|_* \|x\|$ for $\forall y\in\mathbb{E}^*, x\in\mathbb{E}$.

proof: $\|x\|=0$ trivially correct.

Let $\boxed{\tilde{x} = \frac{x}{\|x\|}}$ $\Rightarrow \|\tilde{x}\| = 1$.

$\Rightarrow \forall x\in\mathbb{E}$, $\|y\|_* \ge \langle y, \tilde{x}\rangle = \langle y, \frac{x}{\|x\|}\rangle = \frac{1}{\|x\|}\langle y,x\rangle$

$\Rightarrow \langle y,x\rangle \le \|y\|_* \|x\|$ ①

Also we have $\|y\|_* \ge \langle y, -\tilde{x}\rangle = -\frac{1}{\|x\|}\langle y,x\rangle$

$\langle y,x\rangle \ge -\|y\|_* \|x\|$ ②

①, ② $\Rightarrow$ $\|\langle y,x\rangle\| \le \|y\|_* \|x\|$.

Euclidean norms are self-dual. $\|\cdot\| = \|\cdot\|_*$

for euclidean space, $\mathbb{E} = \mathbb{E}^*$. (disregarding the fact that the members of $\mathbb{E}^*$ are actually linear functionals on $\mathbb{E}$)

Confused: Adjoint Transformation. 1.13