2.1 an "unnatural" rule: $0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$.

$f: \mathbb{E} \to [-\infty, \infty]$

(effective) domain: $\text{dom}(f) = \{x \in \mathbb{E} : f(x) < \infty\}$.

eg. Indicator functions.

$C \subseteq \mathbb{E}$ $\quad \delta_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$

$\Rightarrow \text{dom}(\delta_C) = C$

Epigraph: (for $f: \mathbb{E} \to [-\infty, \infty]$)

$\text{epi}(f) = \{(x, y) : f(x) \le y,\ x \in \mathbb{E},\ y \in \mathbb{R}\}$. 点集.

epigraph is a subset of $\mathbb{E} \times \mathbb{R}$.

$(x, y) \in \text{epi}(f) \implies x \in \text{dom}(f)$

$f: \mathbb{E} \to [-\infty, \infty]$ is $\boxed{\text{proper}}$: ① does not attain $-\infty$

② $\exists \vec{x} \in \mathbb{E}.\ f(x) < \infty$ $\quad (\text{dom}(f) \neq \phi)$

Def 2.2. Closed functions.

$f: \mathbb{E} \to [-\infty, \infty]$ is closed if its epigraph is closed.

Prop. 2.3 The indicator function $\delta_C$ is closed iff $C$ is a closed set.

proof: $epi(\delta_C) = \{(x,y) \in E \times \mathbb{R} \mid \delta_C(x) \leq y\} = C \times \mathbb{R}_+$

If $C$ is closed, then $C \times \mathbb{R}^+$ is closed,

$\delta_C$ is closed.

$\Rightarrow$ "The domain of a closed indicator function is necessarily closed"

But in general, the domain of a closed function might not be closed.

eg. $f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ \infty & \text{else} \end{cases}$ $(0, +\infty)$

---

**Def 2.5** (lower semicontinuity)

$f: \mathbb{E} \to [-\infty, \infty]$ is called lower semicontinuous at $x \in \mathbb{E}$

if $f(x) \leq \liminf_{n \to \infty} f(x_n)$ for $\boxed{\text{any}}$ seq $\{x_n\}_{n \geq 1} \in \mathbb{E}$

for which $x_n \to x$ as $n \to \infty$.

If $\forall x \in \mathbb{E}$, $f: \mathbb{E} \to [-\infty, \infty]$ is lower-semicontinuous at $x \in \mathbb{E}$, then we say $f$ is lower-semicontinuous

$$f(x) \leq \liminf_{n \to \infty} f(x_n).$$

**Def: $\alpha$-level set**

for $f: E \to [-\infty, \infty]$, its $\alpha$-level set is
$$\mathrm{Lev}(f, \alpha) = \{x \in E : f(x) \le \alpha\}.$$

Important Proposition:

Closedness $\iff$ lower semicontinuity $\iff$ Any level sets are closed.

Th 2.6 Equivalence of closedness, lower semicontinuity, closedness of level set.

Let $f: E \to [-\infty, \infty]$, the following 3 claims are equivalent:

<1> $f$ is lower semicontinuous

<2> $f$ is closed

<3> $\forall \alpha \in \mathbb{R}$, the level set $\mathrm{Lev}(f, \alpha) = \{x \in E : f(x) \le \alpha\}$ is closed

proof:

<1> $\Rightarrow$ <2> Suppose $f$ is lower semicontinuous, we show $\mathrm{epi}(f)$ is closed. $\Rightarrow$ show all limit points of $\mathrm{epi}(f)$ is inside $\mathrm{epi}(f)$.

Let seq $\{(x_n, y_n)\}_{n \ge 1} \subseteq \mathrm{epi}(f)$ as $x_n \to x^*$; $y_n \to y^*$ so, we have $f(x_n) \le y_n$.

So, by semicontinuity, $f(x^*) \le \liminf_{n\to\infty} f(x_n) \le \liminf_{n\to\infty} y_n = y^*$

which means that, $\underbrace{(x^*, y^*)}_{\text{limit points}} \in \text{epi}(f)$

In that case, all limit points are in epi(f)

$\Rightarrow$ epi(f) is closed $\Rightarrow$ $f$ is closed.

$\langle 2 \rangle \Rightarrow \langle 3 \rangle$ We show that limit points of $Lev(f, \alpha)$ are always in $Lev(f, \alpha)$.

proof: Let seq $\{(x_n, \alpha)\}$, and when $n \to \infty$, $x_n \to \bar{x}$. $\forall n$, $(x_n, \alpha) \in \text{epi}(f)$. By closedness of $f$, epi(f) is closed, then $(\bar{x}, \alpha) \in \text{epi}(f)$

that is $f(\bar{x}) \le \alpha$, $x \in E$. So $(\bar{x}, \alpha)$ is in $Lev(f, \alpha)$

Since $\{x_n\}$ are arbitarily chosed, limit points of $Lev(f, \alpha)$ are always in $Lev(f, \alpha)$.

$\langle 3 \rangle \Rightarrow \langle 1 \rangle$ Suppose: $\forall \alpha$, $Lev(f, \alpha)$ is closed. But $f$ is not lower semicontinuous.

$\Rightarrow \exists x^* \in E$ and corresponding $\{x_n\}$ $(n \to \infty,\ x_n \to x^*)$

s.t. $f(x^*) > \liminf_{n\to\infty} f(x_n)$

Take $\alpha$ s.t. $f(x^*) > \alpha > \liminf_{n\to\infty} f(x_n)$

Then there exist a subseq $\{x_{n_k}\}_{k \ge 1}$

(子列与原序列收敛[illegible])

s.t. $\forall k \geq 1$, $f(x_{n_k}) \leq \alpha$. And, as $k \to \infty$, $x_{n_k} \to x^*$

By Closedness of $Lev(f, \alpha)$, $(x^*, \alpha) \in Lev(f, \alpha)$

$\Rightarrow f(x^*) \leq \alpha$ contradicts!

So $f$ has to be lower semicontinuous.

## Th 2.7 operations preserving closedness.

(a) Linear Transformation.

Let $A: E \to V$ be a LT. $b \in V$. $f: V \to [-\infty, \infty]$ is an extended real-valued closed function. Then,

$g: E \to [-\infty, \infty]$ given by

$$g(x) = f(A(x) + b)$$ is closed.

proof: To show $g$ is closed, take a seq $\{(x_n, y_n)\}$ as $n \to \infty$, $(x_n, y_n) \to (x^*, y^*)$. we show $f(x^*) \leq y^*$

$\{(x_n, y_n)\} \subseteq epi(g) \Rightarrow g(x_n) = f(A(x_n) + b) \leq y_n$

By $\begin{cases} \text{continuity of LT}, & A(x_n) + b \to A(x^*) + b \text{ as } x_n \to x^* \\ \text{Closedness of } f, & (A(x^*) + b, y^*) \in epi(f) \end{cases}$

$\Rightarrow f(A(x^*) + b) \leq y^*$

$$g(x^*)$$

$\Rightarrow \operatorname{epi}(g)$ is closed

$\Rightarrow g$ is closed.

(b) (Positive weighted) Composition of closed functions

Let $f_1, f_2, \cdots f_m : \mathbb{E} \to (-\infty, \infty]$ be extended real-valued closed functions. Let $\alpha_1, \alpha_2, \cdots \alpha_m \in \mathbb{R}_+$. Then, the function $f = \sum_{i=1}^{m} \alpha_i f_i$ is closed.

proof: $\forall i$, we can find $\{(x_{in}, y_{in})\}$ $n \to \infty$, $x_{in} \to x_i^*$, $y_{in} \to y_i^*$

Since closedness $\Leftrightarrow$ semi continuity,

$$f_i(x^*) \leq \liminf_{n\to\infty} f_i(x_n)$$

$$\Rightarrow \left(\sum_{i=1}^{m} \alpha_i f_i\right)(x^*) \leq \sum_{i=1}^{m} \liminf_{n\to\infty} \alpha_i f_i(x_n) \leq \liminf_{n\to\infty} \left(\sum_{i=1}^{m} \alpha_i f_i\right)(x_n)$$

Fact: $\{a_n\}, \{b_n\}$ 2 seqs,

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty} (a_n + b_n)$$

Note: Def. of $\begin{cases} \liminf_{n\to\infty} a_n \\ \limsup_{n\to\infty} a_n \end{cases}$ Let $\{a_{n_k}\}$ be a subseq of $\{a_n\}$, each $a_{n_k} \to x$.

Let $E$ be the set of such $x$. We denotes $\liminf_{n\to\infty} a_n = \inf E$

$\limsup_{n\to\infty} a_n = \sup E$.

(c): Maximization

Let $f_i : E \to (-\infty, \infty]$, $i \in I$ be ERV closed function, where $I$ is a given index set.

Then the function $f(x) = \max_{i \in I} f_i(x)$ is closed.

proof: $\forall i \in I$, $\mathrm{epi}(f_i)$ is closed.

$\Rightarrow \mathrm{epi}(f) = \bigcap_{i=1}^{n} \mathrm{epi}(f_i)$

Intersection of closed sets is closed

$\Rightarrow \mathrm{epi}(f)$ is closed

$\Rightarrow f$ is closed.

Consider a fixed $y$:

$f_i(x) \le y$.

$\Rightarrow \max_{i \in I} f_i(x)$ has the smallest epigraph, which is a intersection of all $\mathrm{epi}(f_i)$.

## 2.2 Closedness versus Continuity.

**Thm 2.8**

{ Continuous, Closed } $\Rightarrow$ semi-continuous 充分条件.

Let $f : E \to (-\infty, \infty]$ be an extended real-valued function that ① continuous over its ② closed domain, $\mathrm{dom}(f)$, then $f$ is closed. ($f$'s epigraph is closed)

Note: Continuous: A function $g$ is continuous over its domain $\mathrm{dom}(g)$ if for any sequence $\{x_n\}_{n \ge 1} \subseteq \mathrm{dom}(g)$ satisfying $x_n \to x^*$ for some $x^* \in \mathrm{dom}(g)$, then $g(x_n) \to g(x^*)$ as $n \to \infty$.

( 连续性 强调 domain 与 codomain

超何的一致性)

proof: We show that $\operatorname{epi}(f)$ is closed.

Take a sequence $\{(x_n, y_n)\} \subset \operatorname{epi}(f)$, let $(x_n, y_n) \to (x^*, y^*)$,

we show $(x^*, y^*) \in \operatorname{epi}(f)$, that is $f(x^*) \le y^*$.

Since $\operatorname{dom}(f)$ is closed, $x^* \in \operatorname{dom}(f)$

$f(x_n) \le y_n$, $x_n \to x^* \underset{\text{(Continuity)}}{\Rightarrow} f(x_n) \to f(x^*)$. By assumption, $y_n \to y^*$

$\Rightarrow f(x^*) \le y^*$. $\Rightarrow$ $(x^*, y^*) \in \operatorname{epi}(f)$

Any limit point of $\operatorname{epi}(f)$ belongs to $\operatorname{epi}(f)$, implying $\operatorname{epi}(f)$ is closed, then $f$ is closed.

In particular, any real-valued continuous function over $E$ is closed.

<u>Collary 2.9</u> Let $f: E \to \mathbb{R}$ be continuous. Then $f$ is closed.

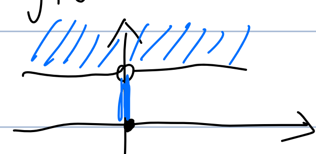
---

Note: Closedness $\not\Rightarrow$ Continuity.
$\Leftarrow$

eg 2-10 Consider $f_\alpha(x) = \begin{cases} \alpha & x = 0 \\ x & 0 \le x \le 1 \\ \infty & \text{else.} \end{cases}$ $\alpha \le 0 \Rightarrow$ closed
$\alpha = 0 \Rightarrow$ continuous.

eg 2-11 $l_0$-norm, $f: \mathbb{R}^n \to \mathbb{R}$. $f(x) = \|x\|_0 = \#\{i : x_i \ne 0\}$.

Actually $f = \sum_{i=1}^{n} I(x_i)$, where $I(y) = \begin{cases} 0 & y = 0 \\ 1. & y \ne 0 \end{cases}$ (Indicator)

All $I(x_i)$ $(\mathbb{R} \to \mathbb{R})$ are closed 

since epi(I) is closed

By Thm 2.7(b), $f$ is closed.

blue: epi(I).

## Thm 2-12 (Weierstrass theorem for closed functions)

Let $f: E \to (-\infty, \infty]$ be a proper ? closed function and assume that $C$ is a compact set with $C \cap \text{dom}(f) \neq \phi$. Then

(1) $f$ is bounded below over $C$

(2) $f$ attains its minimal value over $C$.

(a) Proof by contradiction.

Suppose by contradiction, $f$ is not bounded below over $C$.

Then $\exists$ a seq $\{x_n\}_{n\geq1} \subseteq C$ s.t. $\lim_{n\to\infty} f(x_n) = -\infty$.

By BW (Bolzano-Weierstrass) Thm, since $C$ is compact, $\exists \{x_{n_k}\}_{k\geq1}$, $x_{n_k} \to \bar{x}$, $\bar{x} \in C$.

By Thm 2.6, $f$ is lower semicontinuous,

($C \cap \text{dom}(f)$ 保证 $\bar{x}$ 在 dom($f$) 中)

$f(\bar{x}) \leq \liminf_{k\to\infty} f(x_{n_k})$. Contradicts with $\lim_{n\to\infty} f(x_n) = -\infty$.

Bolzano-Weierstrass:

Every infinite subset of compact set has a limit point in this compact set.

$\Downarrow$

If $C$ is compact, $\{x_n\}_{n\geq1} \subseteq K$, then

(b). Denote $f_{opt}$ the minimal value of $f$ over $C$. $\Rightarrow \exists$ a seq $\{x_n\}_{n\geq1}$, $f(x_n) \to f_{opt}$ as $n \to \infty$.

Similar to (a), take a subseq $\{x_{n_k}\}_{k\geq 1}$ s.t. $x_{n_k} \to \bar{x} \in C$ as $k \to \infty$.

$\exists \{x_{n_k}\}_{k\geq 1}$, if $x_{n_k} \to \bar{x}$ then $\bar{x} \in C$

紧致空间中的序列一定有收敛子列

then $f(\bar{x}) \leq \liminf_{k\to\infty} f(x_{n_k}) \leq \lim_{k\to\infty} f(x_{n_k})$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\quad f_{opt}$

与 $f(x_n)$ 同收敛至 $f_{opt}$.

$\Rightarrow \bar{x}$ is a minimizer of $f$ over $C$.

Note: Without $\boxed{\text{Compactness of } C}$, the existence of minimizer can not be ensured.

But, if the function has a property called "coerciveness", then "compactness" can be replaced by "closedness".

**Def 2.13** $\boxed{\text{Coerciveness.}}$ A proper function $f: E \to (-\infty, \infty]$ is called coercive if $\lim_{\|x\|\to\infty} f(x) = \infty$

**Thm 2.14** (Attainment (of minimizer) under coerciveness). Let $f: E \to (-\infty, \infty]$ be a proper ① closed and ② coercive function, $S \subseteq E$ is a nonempty ③ closed set s.t. $S \cap \mathrm{dom}(f) \neq \emptyset$. Then $f$ attains its minimal value over $S$.

proof: Let $x_0 \in S \cap \mathrm{dom}(f)$, an arbitrary point.

By coerciveness of $f$, $\exists M>0$ s.t.

$f(x) > f(x_0)$ for any $x$ s.t. $\|x\| > M$.

For any minimizer $x^*$ of $f$ over $S$, $f(x^*) \le f(x_0)$

$\Downarrow$

The set of all minimizes of $f$ over $S$

is the same as the set of all minimizers of $f$ over $S \cap B_{\|\cdot\|}[0, M]$,

which is compact ( $B_{\|\cdot\|}[0, M]$ is compact, $S$ is closed,

$B_{\|\cdot\|}[0,M] \cap S$ is a subset of $B_{\|\cdot\|}[0,M]$ )

By Thm 2.12, we can attain the minimizer of $S$ over $S$

---

## 2.3 Convex Functions.

Like closedness, the definition of convexity for ERV function can be written in terms of the epigraph.

**Def 2.15** $\boxed{\text{Convex Functions}}$

A extended real-valued function : $f: E \to [-\infty, \infty]$ is <u>convex</u> if epi$(f)$ is a convex set.

$$p, q \in \text{dom}(f)$$

$$\text{then } \lambda p + (1-\lambda) q \in \text{dom}(f), \quad \forall \lambda \in [0, 1].$$

For an extended real-valued function $f: E \to [-\infty, \infty]$,

$f$ is convex $\Longleftrightarrow$ $f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y)$,

标准定义 $\quad \forall x,y\in E, \lambda\in[0,1]$

$\Longleftrightarrow$ $\mathrm{dom}(f)$ is convex,

$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y)$,

$\forall x,y\in \mathrm{dom}(f)$, $\lambda\in[0,1]$.

Actually a special case of Jensen's inequality

Any $x_1, x_2, \cdots x_k \in E$ and $\lambda\in\Delta_k$, if $f$ is convex,

then:

simplex $\begin{cases} \forall i, \lambda_i\in[0,1]. \\ \sum_{i=1}^{k}\lambda_i=1 \end{cases}$

$$f\left(\sum_{i=1}^{k}\lambda_i x_i\right) \le \sum_{i=1}^{k}\lambda_i f(x_i)$$

---

**Thm. 2.16** (Operations Preserving Convexity)

Note: The proof below mainly combines 2 facts:

(1) Same properties holds for real-valued convex functions defined on a given convex domain.

(2) A proper extended real-valued function is convex iff its domain is convex & its restriction to its domain is a real-valued convex function.

(1) Linear Transformation.

Let $A: E\to V$ be a linear transformation (2 underlying spaces) and $b\in V$. Let $f: V\to[-\infty,\infty]$ be an extended real-valued convex function.

Then the extended real-valued function $g: E \to (-\infty, \infty]$ given by $g(x) = f(A(x)+b)$ is convex.

(2) Weighted Composition.

Let $f_1, \dots, f_m : E \to (-\infty, \infty]$ be extended real-valued functions and let $\alpha_1, \dots \alpha_m \in \mathbb{R}_+$. Then $\sum_{i=1}^{m} f_i$ is convex.

(3) Maximization.

Let $f_i : E \to (-\infty, \infty]$, $i \in I$, be extended real-valued convex functions, where $I$ is a given index set.

Then the function $f(x) = \max_i f_i(x)$ is convex.

---

Eg. Suppose the underlying space $E$ is Euclidean $(\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle})$

Let $C \subset E$, $C \neq \emptyset$.

Define the distance function $d_C(x) = \min_{y \in C} \|x - y\|$.

Consider $\varphi_C(x) = \frac{1}{2}(\|x\|^2 - d_C^2(x))$. We show $\varphi(x)$ is convex.

proof: $d_C^2(x) = \min_{y \in C} \|x-y\|^2 = \|x\|^2 - \max_{y \in C} [2\langle y, x \rangle - \|y\|^2]$

Hence $\varphi_C(x) = \frac{1}{2} \max_{y \in C} [2\langle y, x \rangle - \|y\|^2]$.

Since $\langle y, x \rangle$, $\|y\|^2$ are both convex, $\varphi_C(x)$ is convex.

Another operation: Partial minimization of jointly convex functions, also preserve convexity.

Thm 2.18

Let $f: E \times V \to [-\infty, \infty]$ be a convex function satisfying the following property:

$\forall x \in E, \exists y \in V$ for which $f(x, y) < \infty$,

let $g: E \to [-\infty, \infty)$ be defined by $g(x) = \min_{y \in V} f(x, y)$.

Then $g$ is convex.

proof: Let $x_1, x_2 \in E$, $\lambda \in [0, 1]$. To show the convexity of $g$, we will prove that:

$$g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda g(x_1) + (1-\lambda) g(x_2)$$

$\lambda = 0, 1 \Rightarrow$ Obviously holds.

Therefore we assume $\lambda \in (0, 1)$.

Case 1. $g(x_1), g(x_2) > -\infty$. Take $\varepsilon > 0 \Rightarrow \exists y_1, y_2 \in V$ s.t.

$$\begin{cases} f(x_1, y_1) \leq g(x_1) + \varepsilon, \\ f(x_2, y_2) \leq g(x_2) + \varepsilon. \end{cases}$$

By convexity of $f$,

$$g(\lambda x_1 + (1-\lambda)x_2) \leq f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \leq \lambda f(x_1, y_1) + (1-\lambda) f(x_2, y_2)$$

$$\leq \lambda (g(x_1) + \varepsilon) + (1-\lambda)(g(x_2) + \varepsilon)$$

$$= \lambda g(x_1) + (1-\lambda) g(x_2) + \varepsilon.$$

Done.

Case 2. At least one of $g(x_1), g(x_2)$ equals to $-\infty$.

Assume $g(x_1) = -\infty$. Then we only need to prove $g(\lambda x_1 + (1-\lambda) x_2) = -\infty$.

$g(x_1) = -\infty \Rightarrow \forall M \in \mathbb{R}, \exists y_1 \in V$ s.t. $f(x_1, y_1) \le M$.

By assumption ☆, for $x_2$, $\exists y_2 \in V$ s.t. $f(x_2, y_2) < \infty$.

By convexity of $f$,

$$g(\lambda x_1 + (1-\lambda) x_2) \le f(\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \le \lambda f(x_1, y_1) + (1-\lambda) f(x_2, y_2)$$

$$\le \lambda M + (1-\lambda) f(x_2, y_2)$$

$$\Rightarrow g(\lambda x_1 + (1-\lambda) x_2) = -\infty.$$

## 2.32. The Infimal Convolution 卷积下确界

Def. Let $h_1, h_2 : E \to [-\infty, \infty]$ be 2 proper functions. The infimal convolution of $h_1, h_2$ is:

$$(h_1 \square h_2)(x) \coloneqq \min_{u \in E} \{ h_1(u) + h_2(x-u) \}.$$

A direct result of Thm 2.18

Thm 2.19 Convexity of infimal convolution.

Let $h_1 : E \to [-\infty, \infty]$ be a proper convex function,

$h_2 : E \to \mathbb{R}$, a real-valued convex function.

Then $h_1 \square h_2$ is convex.

Proof: Define $f(x,y) \equiv h_1(y) + h_2(x-y)$

$h_1, h_2$ is convex $\Rightarrow$ $f$ is convex.

$\forall x \in E$, $\exists y \in \mathrm{dom}(h_1)$ s.t. $f(x,y) < \infty$

$\Rightarrow$ $h_1 \square h_2$ is a partial minimization function of $f(\cdot,\cdot)$

So $h_1 \square h_2$ is convex by Thm 2.18.

Ex 2.20 (Convexity of distance function.)

Let $C \subseteq E$, $C \neq \phi$. The distance function can be written as the infimal convolution:

$$d_C(x) = \min_y \{ \|x-y\| : y \in C \} = \min_{y \in E} \{ \delta_C(y) + \|x-y\| \} = (\delta_C \square h_1)(x)$$

$\begin{cases} h_1 = \|\cdot\| & \cdots \text{ real-valued convex.} \\ \delta_C & \cdots \text{ proper \& convex} \end{cases}$ $\xRightarrow{\text{Thm 2.19}}$ $d_C$ is convex.

$\delta_C$ → indicator function $\delta_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$

## 2.3.3 Continuity of Convex Functions

Thm 2.21 Local Lipschitz continuity of convex functions

Let $f : E \to (-\infty, \infty]$ be convex. Let $x_0 \in \mathrm{int}(\mathrm{dom}(f))$.

interior points in dom(f)

Then $\exists\, \varepsilon > 0$, $L > 0$ s.t. $B[x_0, \varepsilon]$ and

$$|f(x) - f(x_0)| \le L \|x - x_0\|.$$

for all $x \in B[x_0, \varepsilon]$.     $L$: lipschitz constant.

Note:
Convex function $f$ is continuous at points in $\text{int}(\text{dom}(f))$
local Lipschitz continuous at nbhds of points in $\text{int}(\text{dom}(f))$
闭邻域

For univariate functions, when the function is closed and convex, the continuity can be guaranteed.

Thm 2.22    Continuity of closed convex univariate functions.
Let $f: \mathbb{R} \to (-\infty, \infty]$ be a proper closed and convex function, then $f$ is continuous over $\text{dom}(f)$.
Proof: Let $I = \text{dom}(f)$.
1° $\text{int}(I) = \emptyset \Rightarrow I$ is a singleton $\Rightarrow$ Obviously $f$ continuous over $I$.
2° $\text{int}(I) \neq \emptyset$   $f$ is continuous over $\text{int}(I)$, by Thm 2.21.
3° We only need to show the continuity of $f$ at the endpoints of $I$ (if it exists).
Suppose $f$ has a left endpoint $a$, we first prove $\lim_{t \to a^+} f(t)$ exists.

Let $c \in I$, $c < a$.  $g(t) = \dfrac{f(c-t) - f(c)}{t}$, $t \in [0, c-a]$.

Easy to find $g(t) \uparrow$ by convexity of $f$.

$\Rightarrow g(t) \le g(c-a)$, $\forall t \in (0, c-a]$.

$\Rightarrow \lim\limits_{t \to (c-a)^-} g(t)$ exists. Let $\lim\limits_{t \to (c-a)^-} g(t) = \ell$.

$\Rightarrow f(c-t) = t g(t) + f(c) \to f(c) + (c-a)\ell$. as $t \to (c-a)^-$

$\Rightarrow \lim\limits_{t \to a^+} f(t) \to f(c) + (c-a)\ell$. exist.

$$f(c-t) = t g(t) + f(c) \le f(c) + (c-a) g(c-a)$$
$$= f(c) + f(a) - f(c) = f(a)$$

$\Rightarrow \lim\limits_{t \to (c-a)^-} f(c-t) = \lim\limits_{t \to a^+} f(t) \le f(a)$

On the other hand. $f$ is closed, then $f$ is lower-semicontinuous $\Rightarrow f(a) \le \liminf\limits_{t \to a^+} f(t) \le \lim\limits_{t \to a^+} f(t)$

$\Rightarrow \lim\limits_{t \to a^+} f(t) = f(a)$, proving the right continuity of $f$ at $a$.

---

## 2.4 Support Functions.

**Def. Support Functions.**

Let $C \subset E$ be a nonempty set, the support function of $C$ is the function $\sigma_C : E^* \to (-\infty, \infty]$ given by

$$\sigma_C(y) = \max \langle y, x \rangle.$$

recall: $E^*$, dual space

$x \in C$ of $E$.

Lemma 2.23. Closedness & Convexity of support functions.

Let $C \subseteq E$ be a nonempty set, then $\sigma_C$ is <u>closed</u> and <u>convex</u>.

Proof: The linear function $y \mapsto \langle y, x \rangle$ is obviously closed and convex, by Thm 2.7(c), 2.16(c), support functions are always closed and convex (Regardless of whether $C$ is closed and/or convex)

note: Minkowski sum:

$$A+B = \{a+b : a \in A, b \in B\}.$$

for a scalar $\alpha \in \mathbb{R}$, a set $A \subseteq E$,

$$\alpha A = \{\alpha a : a \in A\}.$$

Lemma 2.24.

(a) (positive homogeneity) $\forall C \neq \emptyset$, $C \subseteq E$, $y \in E^*$, $\alpha \geq 0$

$$\sigma_C(\alpha y) = \alpha \sigma_C(y)$$

Proof: $\sigma_C(\alpha y) = \max_{x \in C} \langle \alpha y, x \rangle = \alpha \max_{x \in C} \langle y, x \rangle = \alpha \sigma_C(y)$

(b). (subadditivity) $\forall C \neq \emptyset$, $C \subseteq E$, $y_1, y_2 \in E^*$,

$$\sigma_C(y_1+y_2) \leq \sigma_C(y_1) + \sigma_C(y_2)$$

Proof: $\sigma_C(y_1+y_2) = \max_{x \in C} \langle y_1+y_2, x \rangle = \max_{x \in C} [\langle y_1, x \rangle + \langle y_2, x \rangle]$

$\leq \max \langle y_1, x \rangle + \max \langle y_2, x \rangle = \sigma_C(y_1) + \sigma_C(y_2)$

$x\in C$ $\quad$ $x\in C$

equality holds when $y_1 = y_2$

(c) $\forall C \neq \phi,\ C \subseteq E,\ y \in E^*,\ \alpha \geq 0$

$$\sigma_{\alpha C}(y) = \alpha\,\sigma_C(y).$$

proof: $\sigma_{\alpha C}(y) = \max\limits_{x\in\alpha C} \langle y, x\rangle = \max\limits_{x\in C} \langle y, \alpha x\rangle$

$= \alpha \max\limits_{x\in C} \langle y, x\rangle = \alpha\,\sigma_C(y).$

(d) $\forall A, B \neq \phi,\ A, B \subseteq E,\ y \in E^*$

proof: $\sigma_{A+B}(y) = \max\limits_{x\in A+B} \langle y, x\rangle = \max\limits_{x_1\in A, x_2\in B} \langle y, x_1 + x_2\rangle$

$= \max\limits_{x_1\in A} \langle y, x_1\rangle + \max\limits_{x_2\in B} \langle y, x_2\rangle = \sigma_A(y) + \sigma_B(y).$

Examples of support functions:

eg 2.25. Support function on finite set:

$$C = \{b_1, b_2, \dots b_m\}, \quad b_1 \sim b_m \in E.$$

$$\Rightarrow \sigma_C(y) = \max\{\langle b_1, y\rangle, \langle b_2, y\rangle, \dots \langle b_m, y\rangle\}.$$

eg 2.26. support functions of cones.

Let $K \subseteq E$ be a cone.

$S \subseteq E$, if $\forall x \in S, \lambda \geq 0$, $\lambda x \in S$, then $S$ is "cone"

Define polar cone of $K$ as:

$$K^\circ = \{ y \in E^* : \langle y, x \rangle \leq 0 \quad \text{for all } x \in K \}.$$

We show that: $\sigma_K(y) = \delta_{K^\circ}(y)$.

proof: $1^\circ$ $y \in K^\circ \Rightarrow \langle y, x \rangle \leq 0, \ \forall x \in K$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\delta_{K^\circ}(y) = 0 \qquad \sigma_K(y) = \max_{x \in K} \langle y, x \rangle = 0$

$\Rightarrow \sigma_K(y) = \delta_{K^\circ}(y)$.

$2^\circ$ $y \notin K^\circ \Rightarrow \exists \tilde{x} \in K, \ \langle y, \tilde{x} \rangle > 0 \qquad \lambda \tilde{x} \in K, \lambda \geq 0$ since $K$ is cone.

$\Downarrow$

$\delta_{K^\circ}(y) = \infty \qquad \langle y, \lambda \tilde{x} \rangle = \lambda \langle y, \tilde{x} \rangle > 0$

$\Rightarrow \sigma_K(y) \geq \langle y, \lambda \tilde{x} \rangle = \lambda \langle y, \tilde{x} \rangle, \ \forall \lambda \geq 0$

Take $\lambda \to \infty \Rightarrow \sigma_K(y) = \infty$ for $y \notin K^\circ$

$\Rightarrow \sigma_K(y) = \delta_{K^\circ}(y)$.

eg 2.27. support function of the $\boxed{\text{nonnagative orthant}}$

Consider $E = \mathbb{R}^n$. As a special case of eg 2.26,

since $\underline{(\mathbb{R}_+^n)^\circ = \mathbb{R}_-^n}$ (obviously)

Then $\sigma_{\mathbb{R}_+^n}(y) = \delta_{\mathbb{R}_-^n}(y)$

Before ex 2.29, we introduce a lemma:

Lemma 2.28 Farka's lemma – second formulation.

Let $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$. Then, the following ①, ② is equivalent

① If $Ax \le 0$ then $c^T x \le 0$

② $\exists\, y \in \mathbb{R}_+^m$ s.t. $A^T y = c$.

(A kind of cone)

eg 2.29. support functions of convex polyhedral cones.

Let $E = \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$.

Define $S = \{x \in \mathbb{R}^n : Ax \le 0\}$. (P27, def of cone) Obviously, $S$ is a cone.

Therefore $\sigma_S(y) = \delta_{S^\circ}(y)$

$y \in S^\circ$ iff $\forall x$ s.t. $Ax \le 0$ we have $\langle y, x \rangle \le 0$. ($\forall x \in S$)

By Farka's Lemma,

The claim ☆ $\Longleftrightarrow$ $\exists\, \lambda \in \mathbb{R}_+^m$ s.t. $A^T \lambda = y$

$\Rightarrow S^\circ = \{A^T \lambda : \lambda \in \mathbb{R}_+^m\}$.

To conclude: $\sigma_S(y) = \delta_{\{A^T\lambda : \lambda \in \mathbb{R}_+^m\}}(y)$.

example of polyhedral cone:

$C = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : 2x_1 - x_2 = 0,\ x_1 + 3x_2 \le 0 \right\}$.

$$\begin{cases} 2x_1 - x_2 \le 0 \\ \text{and} \\ -2x_1 + x_2 \le 0 \end{cases} \quad A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \quad Ax \le 0$$

eg 2.30 support functions of $\boxed{\text{affine sets}}$.

Let underlying space be $E = \mathbb{R}^n$ and let $B = \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Define the affine set:

$$C = \{x \in \mathbb{R}^n : Bx = b\}. \qquad \sigma_C(y) = \max_{x \in C} \langle y, x \rangle$$

We assume $C \neq \phi \Rightarrow \exists x_0 \in \mathbb{R}^n$ s.t. $Bx_0 = b$.

Then $\sigma_C(y) = \max_x \{\langle y, x\rangle : Bx = b\}$, 换元, 转换约束

make the change of variable: $x = z + x_0 \quad \Rightarrow Bz = Bx - Bx_0 = 0$

$$\sigma_C(y) = \max_z \{\langle y, z\rangle + \langle y, x_0\rangle : Bz = 0\}$$

$$= \langle y, x_0\rangle + \max_z \{\langle y, z\rangle : Bz = 0\}$$

$$= \langle y, x_0\rangle + \sigma_{\tilde{C}}(y)$$

where $\tilde{C} = \{x \in \mathbb{R}^n : Bx = 0\}$.

$= \{x \in \mathbb{R}^n : Ax \le 0\} \qquad A = \begin{pmatrix} -B \\ B \end{pmatrix}$

a convex polyhedral cone.

it follows that $\sigma_{\tilde{C}} = \delta_{\tilde{C}^\circ}$,

where $\tilde{C}^\circ$ is a polar cone of $C^\circ$

$$\tilde{C}^\circ = \{B'\lambda_1 - B\lambda_2 : \lambda_1, \lambda_2 \in \mathbb{R}^m_+\}.$$

We will show that $\tilde{C}^{\circ} = \text{Range}(B^T)$.

($\Rightarrow$) $C^{\circ} \subset \text{Range}(B^T)$.

If $x \in C^{\circ}$, then $\exists \lambda_1, \lambda_2 \in \mathbb{R}_+^m$ s.t. $x = B^T(\lambda_1 - \lambda_2) \in \text{Range}(B^T)$

($\Leftarrow$) $\text{Range}(B^T) \subset C^{\circ}$.

If $x \in \text{Range}(B^T)$, then $\exists \lambda \in \mathbb{R}^m$ s.t. $x = B^T \lambda$

Define $\lambda_1 = [\lambda]_+$, $\lambda_2 = [-\lambda]_+$, we obtain that $\lambda = \lambda_1 - \lambda_2$

with $\lambda_1, \lambda_2 \in \mathbb{R}_+^m$. Here $x = B^T(\lambda_1 - \lambda_2) \in C$

$$\Rightarrow \sigma_C(y) = \langle y, x_0 \rangle + \delta_{\text{Range}(B^T)}(y).$$

$$([v]_+)_i = \begin{cases} v_i & v_i \geq 0 \\ 0 & v_i < 0 \end{cases}$$

eg 2.32. support functions of $\boxed{\text{unit balls.}}$

Suppose $E$ is the underlying space endowed with a norm $\|\cdot\|$.

Consider the unit ball given by $B_{\|\cdot\|}[0,1] = \{x \in E : \|x\| \leq 1\}$.

$$\Rightarrow \sigma_{B_{\|\cdot\|}[0,1]}(y) = \max_{\|x\| \leq 1} \langle y, x \rangle = \|y\|_*$$

dual norm.

For example, for space $\mathbb{R}^n$ we have

$$\sigma_{B_{\|\cdot\|_p}[0,1]}(y) = \|y\|_q \qquad (1 \leq p \leq \infty,\ \frac{1}{p} + \frac{1}{q} = 1)$$

$$\sigma_{\|\cdot\|_Q}[0,1](y) = \|y\|_{Q^{-1}} \qquad (Q \in S_{++}^n).$$

eg. 2.32.

Consider $C \subset \mathbb{R}^2$, $C = \{(x_1, x_2)^T : x_1 + \frac{x_2^2}{2} \le 0\}$.

$$\sigma_C(y) = \max_{x_1, x_2} \{x_1 y_1 + x_2 y_2 : x_1 + \frac{x_2^2}{2} \le 0\}.$$

$\sigma_C(0) = 0$.

we only consider $\boxed{y \neq 0}$ The maximum is attained at the boundary of C.

suppose we attain the maximum at an interior point of C. $\frac{\partial}{\partial x}\langle x, y\rangle = \frac{\partial}{\partial x} y^T x = y = 0$

contradicts with $y \neq 0$.

$$\Rightarrow \sigma_C(y) = \max_{x_1, x_2} \{x_1 y_1 + x_2 y_2 : x_1 + \frac{x_2^2}{2} = 0\}$$

$$= \max_{x_2} \{-\frac{y_1}{2} x_2^2 + y_2 x_2\}$$

$1^\circ$ $y_1 < 0 \Rightarrow \sigma_C(y) = \infty$.

$2^\circ$ $y_1 = 0$, $y_2 \neq 0 \Rightarrow \sigma_C(y) = \infty$

$3^\circ$ $y_1 > 0$ when $x_2 = -\frac{y_2}{y_1}$, $\sigma_C(y) = \frac{y_2^2}{2y_1}$

$$\Rightarrow \sigma_C(y) = \begin{cases} \frac{y_2^2}{2y_1} & y_1 > 0 \\ 0 & y_1 = y_2 = 0 \end{cases}$$

$\quad\quad \infty \quad$ else.

By Thm 2.23, $\sigma_C$ is convex and closed.

(but not continuous on $(0,0)$)

(remember for 1-dim function, convexity and closedness imply continuity)

take $\alpha > 0$, $\begin{cases} y_1(t) = \frac{t^2}{2\alpha} \\ y_2(t) = t \end{cases}$ $(t > 0)$ $\Rightarrow \sigma_C(y_1(t), y_2(t)) = \alpha$.

$\lim_{t \to 0^+} \sigma_C(y_1(t), y_2(t)) = \alpha \neq 0 \Rightarrow$ discontinuity on $(0,0)$

Important property of support functions:

$\Rightarrow$ Support functions are completely determined by their underlying sets as long as their sets are closed and convex

(We prove it by using Thm 2.33)

Thm 2.33 Strictly seperation theorem. Let $C \subseteq E$ be a nonempty closed and convex set, let $y \notin C$. Then $\exists p \in E^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ s.t.

$$\begin{cases} \langle p, y\rangle > \alpha \\ \langle p, x\rangle \le \alpha, \ \forall x \in C. \end{cases}$$

$y \notin$ 闭凸集的
等价表述

Proof: By second projection thm ([10, Thm 9.8])

证明R难点 part.

the vector $\bar{x} = P_C(y) \in C$ satisfies

$$(y-\bar{x})^T(x-\bar{x}) \le 0, \ \forall x \in C.$$

$$\Longleftrightarrow \ (y-\bar{x})^T x \le (y-\bar{x})^T \bar{x}, \ \forall x \in C.$$

denote $\begin{cases} p = y-\bar{x} \neq 0 \ (\text{since } y \notin C) \\ \alpha = (y-\bar{x})^T \bar{x} \end{cases}$

so we have $\langle p, x\rangle \le \alpha, \ \forall x \in C.$

For $y \notin C$, $\langle p, y\rangle = (y-\bar{x})^T y$

$$= (y-\bar{x})^T(y-\bar{x}) + (y-\bar{x})^T \bar{x}$$

$$= \|y-\bar{x}\|^2 + (y-\bar{x})^T \bar{x} > (y-\bar{x})^T \bar{x} = \alpha.$$

$(\text{since } y \notin C, \ \|y-\bar{x}\| > 0)$

Then we are done.

**Lemma 2.34** Let $A, B \subseteq E$ be nonempty closed and convex sets. Then $A = B$ iff $\sigma_A = \sigma_B$.

Proof: $(\Rightarrow)$ $A = B \Rightarrow \sigma_A = \sigma_B$ is obvious.

($\Leftarrow$) Suppose $\sigma_A = \sigma_B$, we show that $A = B$.

Assume for contradition $A \neq B$.

$\Rightarrow \exists\, y \in A$ s.t. $y \notin B$.

$B$ is nonempty closed and convex, by Thm 2.33,

$\exists\ p \in E^* \setminus \{0\}$, $\alpha > 0$

s.t. $\langle p, x \rangle \le \alpha < \langle p, y \rangle$, $\forall x \in B$.

Taking the maximum over $x \in B$.

$$\sigma_B(p) \le \alpha < \langle p, y \rangle \le \sigma_A(p).$$

then $\sigma_A \neq \sigma_B$, contradicts! $\Rightarrow A = B$.

Lemma 2.35. Let $A \subseteq E$ be nonempty. Then.

(a) $\sigma_A = \sigma_{cl(A)}$

(b) $\sigma_A = \sigma_{conv(A)}$

The support function stays the same under the operations of $\begin{cases} \text{closure 闭包} \\ \text{convex hull. 凸包} \end{cases}$

proof:

(a) Since $A \subseteq cl(A)$, $\sigma_A(y) \le \sigma_{cl(A)}(y)$, $\forall y \in E^*$.

$\Rightarrow \sigma_A \le \sigma_{cl(A)}$.

We will show the reverse inequality.

Note: Def of convex hull of set $E$

... The smallest convex polygon that contains all the points in the set.

类似闭包，指包含所有$E$的点的极小凸集.

Let $y \in E^*$, by def of support functions, $\exists$ a seq $\{x_k\}_{k \ge 1} \subseteq cl(A)$ s.t. $\langle y, x_k \rangle \to \sigma_{cl(A)}(y)$ as $k \to \infty$.

By def of closure, $\exists \{z_k\}_{k\geq 1} \subseteq A$ s.t. $z_k \to x_k$ as $k\to\infty$.
Since $z_k \in A$, $\sigma_A(y) \geq \langle y, z_k\rangle = \langle y, x_k\rangle + \langle y, z_k - x_k\rangle$
take $k\to\infty$ $\Rightarrow \sigma_A(y) \geq \sigma_{cl(A)}(y) + 0 = \sigma_{cl(A)}(y)$.

$\Rightarrow \sigma_A = \sigma_{cl(A)}$.

(b). Since $A \subseteq conv(A)$, $\sigma_A(y) \leq \sigma_{conv(A)}(y)$, $\forall y \in E^*$.
We will show the reverse inequality.
Similarly to A, $\exists \{x_k\}_{k\geq 1} \subseteq conv(A)$ s.t. $\langle y, x_k\rangle \to \sigma_{conv(A)}(y)$ as $k\to\infty$.

$x_k \in conv(A)$, by def of convex hull, for $\forall k$,
$\exists x_{k_1}, x_{k_2}, \dots x_{k_n} \in A$ and $\lambda_k \in \Delta_{nk}$ s.t.

$$x_k = \sum_{i=1}^n \lambda_{ki} z_{ki}$$

$$\Rightarrow \langle y, x_k\rangle = \langle y, \sum_{i=1}^n \lambda_{ki} z_{ki}\rangle = \sum_{i=1}^n \lambda_{ki}\langle y, z_{ki}\rangle$$

$$\leq \sigma_A(y) \underbrace{\sum_{i=1}^n \lambda_{ki}}_{=1} = \sigma_A(y)$$

Take $k\to\infty$, we have $\sigma_{conv(A)}(y) \leq \sigma_A(y)$.

$\Rightarrow \sigma_{conv(A)}(y) = \sigma_A(y)$.

eg 2.36. Support function of the $\boxed{\text{unit simplex}}$.

Suppose underlying space in $\mathbb{R}^n$, consider the unit simplex
$\Delta_n = \{x \in \mathbb{R}^n : e^T x = 1,\ x \geq 0\}$.

$\Delta_n$ can be written as the convex hull of standard basis of $\mathbb{R}^n$
$\Delta_n = \operatorname{conv}\{e_1, \dots e_n\}$.

By Lemma 2.35,

$$\sigma_{\Delta_n}(y) = \sigma_{\{e_1, \dots e_n\}}(y) = \max\{\langle y, e_1\rangle, \dots \langle y, e_n\rangle\}$$
$$= \max\{y_1, \dots y_n\}.$$

Summary of support functions:

The table below summarizes the main support function computations that were considered in this section.

| $C$ | $\sigma_C(\mathbf{y})$ | Assumptions | Reference |
|---|---|---|---|
| $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ | $\max_{i=1,2,\dots,n}\langle \mathbf{b}_i, \mathbf{y}\rangle$ | $\mathbf{b}_i \in \mathbb{E}$ | Example 2.25 |
| $K$ | $\delta_{K^\circ}(\mathbf{y})$ | $K$—cone | Example 2.26 |
| $\mathbb{R}^n_+$ | $\delta_{\mathbb{R}^n_-}(\mathbf{y})$ | $\mathbb{E} = \mathbb{R}^n$ | Example 2.27 |
| $\Delta_n$ | $\max\{y_1, y_2, \dots, y_n\}$ | $\mathbb{E} = \mathbb{R}^n$ | Example 2.36 |
| $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leq \mathbf{0}\}$ | $\delta_{\{\mathbf{A}^T\boldsymbol{\lambda} : \boldsymbol{\lambda} \in \mathbb{R}^m_+\}}(\mathbf{y})$ | $\mathbb{E} = \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$ | Example 2.29 |
| $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Bx} = \mathbf{b}\}$ | $\langle \mathbf{y}, \mathbf{x}_0\rangle + \delta_{\text{Range}(\mathbf{B}^T)}(\mathbf{y})$ | $\mathbb{E} = \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{Bx}_0 = \mathbf{b}$ | Example 2.30 |
| $B_{\lVert\cdot\rVert}[\mathbf{0}, 1]$ | $\lVert\mathbf{y}\rVert_*$ | - | Example 2.31 |