

2.1 an "unnatural" rule: $0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$.

$f: E \rightarrow [-\infty, \infty]$

(effective) domain: $\underbrace{\text{dom}(f)}_{\text{def}} = \{x \in E : f(x) < \infty\}$.

e.g. Indicator functions.

$$C \subseteq E \quad \delta_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

$$\Rightarrow \text{dom}(\delta_C) = C$$

Epigraph: (for $f: E \rightarrow [-\infty, \infty]$)

$$\underbrace{\text{epi}(f)}_{\text{def}} = \{(x, y) : f(x) \leq y, x \in E, y \in \mathbb{R}\}. \quad \text{点集.}$$

epigraph is a subset of $E \times \mathbb{R}$.

$$(x, y) \in \text{epi}(f) \Rightarrow x \in \text{dom}(f)$$

$f: E \rightarrow [-\infty, \infty]$ is proper: ① does not attain $-\infty$
 ② $\exists \bar{x} \in E. f(\bar{x}) < \infty$ ($\text{dom}(f) \neq \emptyset$)

Def 2.2. Closed functions.

$f: E \rightarrow [-\infty, \infty]$ is closed if its epigraph is closed.

Prop. 2.3 The indicator function δ_C is closed iff C is

a closed set.

Proof: $\text{epi}(S_c) = \{(x, y) \in E \times \mathbb{R} \mid S_c(x) \leq y\} = C \times \mathbb{R}_f$

If C is closed, then $C \times \mathbb{R}_f^+$ is closed,

S_c is closed.

\Rightarrow "The domain of a closed indicator function is necessarily closed"

But in general, the domain of a closed function might not be closed.

e.g. $f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ \infty & \text{else} \end{cases}$ $(0, +\infty)$

Def 2.5 (lower semicontinuity)

$f: E \rightarrow [-\infty, \infty]$ is called lower semicontinuous at $x \in E$
if $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$ for any seq $\{x_n\}_{n \geq 1} \in E$
for which $x_n \rightarrow x$ as $n \rightarrow \infty$.

If $\forall x \in E$, $f: E \rightarrow [-\infty, \infty]$ is lower-semicontinuous
at $x \in E$, then we say f is lower-semicontinuous

Def: α -level set

for $f: E \rightarrow [-\infty, \infty]$, its α -level set is

$$\text{Lev}(f, \alpha) = \{x \in E : f(x) \leq \alpha\}.$$

Important Proposition:

Closedness \Leftrightarrow lower semicontinuity \Leftrightarrow Any level sets are closed.

Th 2-f Equivalence of closedness, lower semicontinuity, closedness of level set.

Let $f: E \rightarrow [-\infty, \infty]$, the following 3 claims are equivalent:

(1) f is lower semicontinuous

(2) f is closed

(3) $\forall \alpha \in \mathbb{R}$, the level set $\text{Lev}(f, \alpha) = \{x \in E : f(x) \leq \alpha\}$ is closed

Proof:

(1) \Rightarrow (2) Suppose f is lower semicontinuous, we show

$\text{epi}(f)$ is closed. \Rightarrow show all limit points of $\text{epi}(f)$ is inside $\text{epi}(f)$.

Let seq $\{(x_n, y_n)\}_{n \geq 1} \subseteq \text{epi}(f)$

as $x_n \rightarrow x^*; y_n \rightarrow y^*$



so, we have $f(x_n) \leq y_n$.

So, by semi continuity, $f(\bar{x}) \leq \liminf_{n \rightarrow \infty} f(x_n) \leq \liminf_{n \rightarrow \infty} y_n = y^*$

which means that, $(\bar{x}, y^*) \in \text{epi}(f)$
↓ limit points

In that case, all limit points are in $\text{epi}(f)$

$\Rightarrow \text{epi}(f)$ is closed $\Rightarrow f$ is closed.

$\langle 2 \rangle \Rightarrow \langle 3 \rangle$ We show that limit points of $\text{Lev}(f, \alpha)$
are always in $\text{Lev}(f, \alpha)$.

Proof: Let seq $\{(x_n, \alpha)\}$, and when $n \rightarrow \infty$, $x_n \rightarrow \bar{x}$.
 $\forall n, (x_n, \alpha) \in \text{epi}(f)$. By closedness of f , $\text{epi}(f)$
is closed, then $(\bar{x}, \alpha) \in \text{epi}(f)$
that is $f(\bar{x}) \leq \alpha, \bar{x} \in E$. So (\bar{x}, α) is in
 $\text{Lev}(f, \alpha)$

Since $\{x_n\}$ are arbitrarily chosen, limit points
of $\text{Lev}(f, \alpha)$ are always in $\text{Lev}(f, \alpha)$.

$\langle 3 \rangle \Rightarrow \langle 1 \rangle$ Suppose: $\forall \alpha, \text{Lev}(f, \alpha)$ is closed. But f is
not lower semicontinuous.

$\Rightarrow \exists \bar{x}^* \in E$ and corresponding $\{x_n\}$ ($n \rightarrow \infty, x_n \rightarrow \bar{x}^*$)
s.t. $f(\bar{x}^*) > \liminf_{n \rightarrow \infty} f(x_n)$

Take α s.t. $f(\bar{x}^*) > \alpha > \liminf_{n \rightarrow \infty} f(x_n)$

Then there exist a subseq, $\{x_{n_k}\}_{k \geq 1}$

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s.t. $\forall k \geq 1$, $f(x_{n_k}) \leq d$ And, as $k \rightarrow \infty$, $x_{n_k} \rightarrow x^*$
 By Closedness of $\text{Lev}(f, d)$, $(x^*, d) \in \text{Lev}(f, d)$
 $\Rightarrow f(x^*) \leq d$ contradicts!

So f has to be lower semi continuous.

Th 2.7 operations preserving closedness-

(a) Linear Transformation.

Let $A: E \rightarrow V$ be a LT. $b \in V$. $f: V \rightarrow [-\infty, \infty]$
 is an extended real-valued closed function. Then,

$g: E \rightarrow [-\infty, \infty]$ given by

$$g(x) = f(A(x) + b)$$
 is closed.

Proof: To show g is closed, take a seq $\{(x_n, y_n)\}$
 as $n \rightarrow \infty$, $(x_n, y_n) \rightarrow (x^*, y^*)$. we show $f(x^*) \leq y^*$

$$\{(x_n, y_n)\} \subseteq \text{epi}(g) \Rightarrow g(x_n) = f(A(x_n) + b) \leq y_n$$

By continuity of LT, $A(x_n) + b \rightarrow A(x^*) + b$ as $x_n \rightarrow x^*$
] closedness of f , $(A(x^*) + b, y^*) \in \text{epi}(f)$

$$\Rightarrow f(A(x^*) + b) \leq y^*$$

$$g(x^*)$$

$\Rightarrow \text{epi}(g)$ is closed

$\Rightarrow g$ is closed.

(Positive weighted)

(b) Composition of closed functions

Let $f_1, f_2, \dots, f_m : E \rightarrow [-\infty, \infty]$ be extended real-valued closed functions. Let $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}_+$. Then, the function $f = \sum_{i=1}^m \alpha_i f_i$ is closed.

proof: $\forall i$, we can find $\{(x_n, y_n)\}$ $n \rightarrow \infty$, $x_n \rightarrow x^*$
 $y_n \rightarrow y_i^*$

Since closedness \Leftrightarrow semi continuity,

$$f(x^*) \leq \liminf_{n \rightarrow \infty} f(x_n)$$

$$\Rightarrow (\sum_{i=1}^m \alpha_i f_i)(x^*) \leq \sum_{i=1}^m \liminf_{n \rightarrow \infty} \alpha_i f_i(x_n) \leq \liminf_{n \rightarrow \infty} (\sum_{i=1}^m \alpha_i f_i)(x_n)$$

Fact: $\{a_n\}, \{b_n\}$ 2 seqs.

$$\liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n \leq \liminf_{n \rightarrow \infty} (a_n + b_n)$$

Note: Def. of $\liminf_{n \rightarrow \infty} a_n$ Let $\{a_k\}$ be a subseq of $\{a_n\}$, each $a_k \rightarrow x$.

Let E be the set of such x . We denotes $\liminf_{n \rightarrow \infty} a_n = \inf E$
 $\limsup_{n \rightarrow \infty} a_n = \sup E$

(c) Maximization

Let $f_i: E \rightarrow (-\infty, \infty]$, $i \in I$ be ERV closed function, where I is a given index set.

Then the function $f(x) = \max_{i \in I} f_i(x)$ is closed.

proof: Viz I , $\text{epi}(f_i)$ is closed.

$$\Rightarrow \text{epi}(f) = \bigcap_{i=1}^n \text{epi}(f_i)$$

Intersection of closed sets is closed

$\Rightarrow \text{epi}(f)$ is closed

$\Rightarrow f$ is closed.

Consider a fixed y :

$$f_i(x) \leq y.$$

$\Rightarrow \max_{i \in I} f_i(x)$ has the

smallest epigraph

which is a intersection
of all $\text{epi}(f_i)$.

2.2 Closedness Versus Continuity.

Thm 2.8

continuous
closed \Rightarrow semi-continuous 充分条件.

Let $f: E \rightarrow (-\infty, \infty]$ be an extended real-valued function

that continuous over its closed domain $\text{dom}(f)$, then f is closed.

Note: Continuous: A function g is continuous over its domain $\text{dom}(g)$ if for any sequence $\{x_n\}_{n \geq 1} \subseteq \text{dom}(g)$ satisfying $x_n \rightarrow x^*$ for some $x^* \in \text{dom}(g)$, then $g(x_n) \rightarrow g(x^*)$ as $n \rightarrow \infty$. (连续性强调 domain 与 codomain)

proof: We show that $\text{epi}(f)$ is closed. 超向的一致性

Take a sequence $\{(x_n, y_n)\} \subset \text{epi}(f)$, let $(x_n, y_n) \rightarrow (x^*, y^*)$, we show $(x^*, y^*) \in \text{epi}(f)$, that is $f(x^*) \leq y^*$.

Since $\text{dom}(f)$ is closed, $x^* \in \text{dom}(f)$

$f(x_n) \leq y_n$, $x_n \rightarrow x^* \Rightarrow f(x_n) \rightarrow f(x^*)$. By assumption, $y_n \rightarrow y^*$
(Continuity)

$$\Rightarrow f(x^*) \leq y^*. \Rightarrow (x^*, y^*) \in \text{epi}(f)$$

Any limit point of $\text{epi}(f)$ belongs to $\text{epi}(f)$, implying $\text{epi}(f)$ is closed, then f is closed.

In particular, any real-valued continuous function over E is closed.

Corollary 2.9 Let $f: E \rightarrow \mathbb{R}$ be continuous. Then f is closed.

Note: Closedness \Rightarrow Continuity.



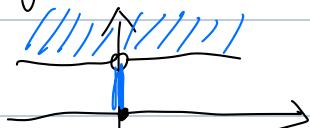
$\alpha \geq 0 \Rightarrow$ closed

eg 2-10 Consider $f_\alpha(x) = \begin{cases} \alpha & x \geq 0 \\ x & 0 \leq x \leq 1 \\ \alpha & \text{else.} \end{cases}$ $\alpha \geq 0 \Rightarrow$ continuous.

eg 2-11 l_0 -norm, $f: \mathbb{R}^n \rightarrow \mathbb{R}$. $f(x) = \|x\|_0 = \#\{i : x_i \neq 0\}$.

Actually $f = \sum_{i=1}^n I(x_i)$, where $I(y) = \begin{cases} 0 & y=0 \\ 1 & y \neq 0 \end{cases}$ (Indicator)

All $I(x_i)$ ($\mathbb{R} \rightarrow \mathbb{R}$) are closed



Since $\text{epi}(f)$ is closed

blue: $\text{epi}(f)$.

By Thm 2.7(b), f is closed.

Thm 2.12 (Weierstrass theorem for closed functions)

Let $f: E \rightarrow (-\infty, \infty]$ be a proper? closed function and assume that C is a compact set with $C \cap \text{dom}(f) \neq \emptyset$. Then

- (1) f is bounded below over C
- (2) f attains its minimal value over C .

(a) Proof by contradiction.

Suppose by contradiction, f is not bounded below over C .

Then \exists a seq, $\{x_n\}_{n=1}^{\infty} \subseteq C$ s.t. $\lim_{n \rightarrow \infty} f(x_n) = -\infty$.

By BW (Bolzano-Weierstrass) Thm, since C is compact, $\exists \{x_{n_k}\}_{k=1}^{\infty}$, $x_{n_k} \rightarrow \bar{x}$, $\bar{x} \in C$.

By Thm 2.6, f is lower semicontinuous,
($C \cap \text{dom}(f)$ 保证 \bar{x} 在 $\text{dom}(f)$ 中)

$f(\bar{x}) \leq \liminf_{k \rightarrow \infty} f(x_{n_k})$. Contradicts with

$$\lim_{n \rightarrow \infty} f(x_n) = -\infty.$$

Bolzano-Weierstrass:
Every infinite subset
of compact set has
a limit point in
this compact set.

(b). Denote f_{opt} the minimal value of f

over C . $\Rightarrow \exists$ a seq, $\{x_n\}_{n=1}^{\infty}$,

$$f(x_n) \rightarrow f_{\text{opt}} \text{ as } n \rightarrow \infty.$$

If C is compact,

$\{x_n\}_{n=1}^{\infty} \subseteq K$, then

Similar to (a), take a subseq $\{x_{n_k}\}_{k=1}^{\infty}$ s.t.

$x_{n_k} \rightarrow \bar{x} \in C$. as $k \rightarrow \infty$.

$$\text{then } f(\bar{x}) \leq \liminf_{k \rightarrow \infty} f(x_{n_k}) \leq \lim_{k \rightarrow \infty} f(x_{n_k})$$

\parallel
 f_{opt}

$\Rightarrow f(x_n) \text{ 趋限于 } f_{\text{opt}}$.

$\exists \{x_{n_k}\}_{k=1}^{\infty}$, if
 $x_{n_k} \rightarrow \bar{x}$ then $\bar{x} \in C$

紧致空间中的序
- 序有收敛子列

$\Rightarrow \bar{x}$ is a minimizer of f over C .

Note: Without Compactness of C , the existence of minimizer can not be ensured.

But, if the function has a property called "coerciveness", then "compactness" can be replaced by "closedness".

Def 2.13 Coerciveness.

is called coercive if

A proper function $f: E \rightarrow (-\infty, \infty]$

$$\lim_{\|x\| \rightarrow \infty} f(x) = \infty$$

Thm 2.14 (Attainment under coerciveness). Let $f: E \rightarrow (-\infty, \infty]$
be a proper closed and coercive function, $S \subset E$ is a nonempty
closed set s.t. $S \cap \text{dom}(f) \neq \emptyset$. Then f attains its minimal
value over S .

Proof: Let $x_0 \in S \cap \text{dom}(f)$, an arbitrary point.

By coerciveness of f , $\exists M > 0$ s.t.

$$f(x) > f(x_0) \text{ for any } x \text{ s.t. } \|x\| > M.$$

For any minimizer x^* of f over S , $f(x^*) \leq f(x_0)$

The set of all minimizers of f over S

is the same as the set of all minimizers of f over $S \cap B_{\|\cdot\|}[0, M]$,

which is compact ($B_{\|\cdot\|}[0, M]$ is compact, S is closed,
 $S \cap B_{\|\cdot\|}[0, M]$ is a subset of $B_{\|\cdot\|}[0, M]$)

By Thm 2.12, we can attain the minimizer of S over S

2.3 Convex Functions.

Like closedness, the definition of convexity for ERV function
can be written in terms of the epigraph.

Def 2.15 Convex Functions

A extended real-valued function: $f: E \rightarrow [-\infty, \infty]$ is convex if
 $\text{epi}(f)$ is a convex set.

$$p, q \in \text{dom}(f)$$

$$\text{then } \lambda p + (1-\lambda)q \in \text{dom}(f), \forall \lambda \in [0, 1].$$

For an extended real-valued function $f: E \rightarrow [-\infty, \infty]$,

f is convex $\Leftrightarrow f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$

标准定义

$\forall x, y \in E, \lambda \in [0, 1]$

$\Leftrightarrow \text{dom}(f)$ is convex,

$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$

$\forall x, y \in \text{dom}(f), \lambda \in [0, 1].$

Actually a special case of Jensen's inequality

Any $x_1, x_2, \dots, x_k \in E$ and $\lambda \in \Delta_k$, if f is convex,
then: $\sum_{i=1}^k \lambda_i = 1$

$$f\left(\sum_{i=1}^k \lambda_i x_i\right) \leq \sum_{i=1}^k \lambda_i f(x_i)$$

Thm. 2.1b (Operations preserving Convexity)

Note: The proof below mainly combines 2 facts:

(1) Same properties holds for real-valued convex functions defined on a given convex domain.

(2) A proper extended real-valued function is convex iff its domain is convex & its restriction to its domain is a real-valued convex function.

(1) Linear Transformation -

Let $A: E \rightarrow V$ be a linear transformation (2 underlying spaces) and $b \in V$. Let $f: V \rightarrow [-\infty, \infty]$ be an extended

real-valued convex function.

Then the extended real-valued function $g: E \rightarrow [-\infty, \infty]$ given by $g(x) = f(A(x)+b)$ is convex.

(2) Weighted Composition.

Let $f_1, \dots, f_m: E \rightarrow [-\infty, \infty]$ be extended real-valued functions and let $\alpha_1, \dots, \alpha_m \in \mathbb{R}_+$. Then $\sum_{i=1}^m f_i$ is convex.

(3) Maximization.

Let $f_i: E \rightarrow [-\infty, \infty]$, $i \in I$, be extended real-valued convex functions, where I is a given index set.

Then the function $f(x) = \max_i f_i(x)$ is convex.

Eg. Suppose the underlying space E is Euclidean ($\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$)

let $C \subset E$, $C \neq \emptyset$.

Define the distance function $d_C(x) = \min_{y \in C} \|x - y\|$.

Consider $\varphi_C(x) = \frac{1}{2}(\|x\|^2 - d_C^2(x))$. We show $\varphi(x)$ is convex

proof: $d_C^2(x) = \min_{y \in C} \|x - y\|^2 = \|x\|^2 - \max_{y \in C} [2\langle y, x \rangle - \|y\|^2]$

Hence $\varphi_C(x) = \frac{1}{2} \max_{y \in C} [2\langle y, x \rangle - \|y\|^2]$.

Since $\langle y, x \rangle$, $\|y\|^2$ are both convex, $\varphi_C(x)$ is convex.

Another operation : Partial minimization of jointly convex functions,
also preserve convexity.

Thm 2.18

Let $f: E \times V \rightarrow [-\infty, \infty]$ be a convex function satisfying
the following property :

$\forall x \in E, \exists y \in V$ for which $f(x, y) < \infty$,

let $g: E \rightarrow [-\infty, \infty)$ be defined by $g(x) = \min_{y \in V} f(x, y)$.

Then g is convex.

proof: Let $x_1, x_2 \in E, \lambda \in [0, 1]$. To show the convexity of g , we will prove that:

$$g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda g(x_1) + (1-\lambda)g(x_2) \quad \lambda=0, 1 \Rightarrow \text{Obviously holds}$$

Therefore we assume $\lambda \in (0, 1)$.

Case I. $g(x_1), g(x_2) > -\infty$. Take $\varepsilon > 0 \Rightarrow \exists y_1, y_2 \in V$ s.t.

$$\begin{cases} f(x_1, y_1) \leq g(x_1) + \varepsilon \\ f(x_2, y_2) \leq g(x_2) + \varepsilon \end{cases}$$

By convexity of f ,

$$f(x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \leq \lambda f(x_1, y_1) + (1-\lambda) f(x_2, y_2)$$

$$g(x_1 + (1-\lambda)x_2) \leq$$

$$\leq \lambda(g(x_1) + \varepsilon) + (1-\lambda)(g(x_2) + \varepsilon)$$

$$= \lambda g(x_1) + (1-\lambda)g(x_2) + \varepsilon.$$

Done.

Case 2. At least one of $g(x_1), g(x_2)$ equals to $-\infty$.

Assume $g(x_1) = -\infty$. Then we only need to prove $g(\lambda x_1 + (1-\lambda)x_2) = -\infty$.

$$g(x_1) = -\infty \Rightarrow \forall M \in \mathbb{R}, \exists y_1 \in V \text{ s.t. } f(x_1, y_1) \leq M.$$

By assumption \star , for $x_2, \exists y_2 \in V$ s.t. $f(x_2, y_2) < \infty$.

By convexity of f ,

$$f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \leq \lambda f(x_1, y_1) + (1-\lambda)f(x_2, y_2)$$

$$g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda M + (1-\lambda)f(x_2, y_2)$$

$$\Rightarrow g(\lambda x_1 + (1-\lambda)x_2) = -\infty.$$

2.32. The Infimal Convolution 卷积下确界.

Def. Let $h_1, h_2 : E \rightarrow [-\infty, \infty]$ be 2 proper functions.

The infimal convolution of h_1, h_2 is :

$$(h_1 \square h_2)(x) = \min_{u \in E} \{h_1(u) + h_2(x-u)\}.$$

\rightarrow A direct result of Thm 2.18

Thm 2.18 Convexity of infimal convolution.

Let $h_1 : E \rightarrow (-\infty, \infty]$ be a proper convex function,

$h_2: E \rightarrow \mathbb{R}$, a real-valued convex function.

Then $h_1 \square h_2$ is convex.

Proof: Define $f(x,y) \equiv h_1(x) + h_2(x-y)$

h_1, h_2 is convex $\Rightarrow f$ is convex.

$\forall x \in E, \exists y \in \text{dom}(h_1)$ s.t. $f(x,y) \leq$

$\Rightarrow h_1 \square h_2$ is a partial minimization function of $f(\cdot, \cdot)$
So $h_1 \square h_2$ is convex by Thm 2.18.

Ex 2.20 (Convexity of distance function.)

Let $C \subseteq E$, $C \neq \emptyset$. The distance function can be written as the infimal convolution:

$$d_C(x) = \min_y \{ \|x-y\| : y \in C\} = \min_{y \in E} \{ d_C(y) + \|x-y\| \} = (d_C \square h_1)(x)$$

$$\begin{cases} h_1 = \|\cdot\| & \dots \text{real-valued convex. Thm 2.18} \\ d_C & \dots \text{proper \& convex} \\ \text{indicator function} & \end{cases} \Rightarrow d_C \text{ is convex.}$$

$d_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$

2.3.3 Continuity of Convex Functions

Thm 2.2 Local Lipschitz continuity of convex functions

Let $f: E \rightarrow [-\infty, \infty]$ be convex. Let $x_0 \in \text{int}(\text{dom}(f))$.

Then $\exists \varepsilon > 0, L > 0$ s.t. $B[x_0, \varepsilon] \subset \text{dom}(f)$

$$|f(x) - f(x_0)| \leq L \|x - x_0\|.$$

↓
interior points
in $\text{dom}(f)$

for all $x \in \beta[x_0, \varepsilon]$. L : Lipschitz constant.

Note:

Convex function f is continuous at points in $\text{int}(\text{dom}(f))$

local Lipschitz continuous at nbhds of
points in $\text{int}(\text{dom}(f))$

闭区段

For univariate functions, when the function is closed and convex,
the continuity can be guaranteed.

Thm 2.22 Continuity of closed convex univariate functions.

Let $f: \mathbb{R} \rightarrow (-\infty, \infty]$ be a proper closed and convex function,
then f is continuous over $\text{dom}(f)$.

Proof: Let $I = \text{dom}(f)$.

1° $\text{int}(I) = \emptyset \Rightarrow I$ is a singleton \Rightarrow Obviously f continuous over I .

2° $\text{int}(I) \neq \emptyset \Rightarrow f$ is continuous over $\text{int}(I)$, by Thm 2.21.

3° We only need to show the continuity of f at
the endpoints of I (if it exists).

Suppose f has a left endpoint a , we prove $\lim_{t \rightarrow a^+} f(t)$ exists.

let $c \in I$, $c < a$. $g(t) = \frac{f(c-t) - f(c)}{t}$, $t \in (0, c-a]$.

Easy to find $g(+)$ ↑ by convexity of f .

$$\Rightarrow g(+ \leq g(c-a), \forall t \in (0, c-a]).$$

$$\Rightarrow \lim_{t \rightarrow (c-a)^-} g(t) \text{ exists. Let } \lim_{t \rightarrow (c-a)^-} g(t) = \ell.$$

$$\Rightarrow f(c-t) = tg(t) + f(c) \rightarrow f(c) + (c-a)\ell, \text{ as } t \rightarrow (c-a)^-$$

$$\Rightarrow \lim_{t \rightarrow c^-} f(t) \rightarrow f(c) + (c-a)\ell. \text{ exist.}$$

$$\begin{aligned} f(c-t) &= tg(t) + f(c) \leq f(c) + (c-a)g(c-a) \\ &= f(c) + f(a) - f(a) = f(a) \end{aligned}$$

$$\Rightarrow \lim_{t \rightarrow (c-a)^-} f(c-t) = \lim_{t \rightarrow c^-} f(t) \leq f(a)$$

On the other hand. f is closed, then f is lower-semicontinuous $\Rightarrow f(a) \leq \liminf_{t \rightarrow a^+} f(t) \leq \lim_{t \rightarrow a^+} f(t)$

$$\Rightarrow \lim_{t \rightarrow a^+} f(t) = f(a). \text{ proving the right continuity of } f \text{ at } a.$$

2.4 Support Functions.

Def. Support Functions.

Let $C \subseteq E$ be a nonempty set, the support function of C is the function $\sigma_C : E^* \rightarrow (-\infty, \infty]$ given by

$$\sigma_C(y) = \max_x \langle y, x \rangle.$$

real: E^* , dual space

$x \in C$

of E .

Lemma 2.23. Closedness & Convexity of support functions.

Let $C \subseteq E$ be a nonempty set, then σ_C is closed and convex.

Proof: The linear function $y \mapsto \langle y, x \rangle$ is obviously closed and convex, by Thm 2.7(c), 2.6(c), support functions

are always closed and convex (regardless of whether C is closed and/or convex)

Note: Minkowski sum:

$$A+B = \{a+b : a \in A, b \in B\}.$$

for a scalar $\alpha \in \mathbb{R}$, a set $A \subseteq E$,

$$\alpha A = \{\alpha a : a \in A\}.$$

Lemma 2.24.

(a) (positive homogeneity) $\forall C \neq \emptyset, C \subseteq E, y \in E^*, \alpha \geq 0$

$$\sigma_C(\alpha y) = \alpha \sigma_C(y)$$

Proof: $\sigma_C(\alpha y) = \max_{x \in C} \langle \alpha y, x \rangle = \alpha \max_{x \in C} \langle y, x \rangle = \alpha \sigma_C(y)$

(b) (subadditivity) $\forall C \neq \emptyset, C \subseteq E, y_1, y_2 \in E^*$,

$$\sigma_C(y_1 + y_2) \leq \sigma_C(y_1) + \sigma_C(y_2)$$

Proof: $\sigma_C(y_1 + y_2) = \max_{x \in C} \langle y_1 + y_2, x \rangle = \max_{x \in C} [\langle y_1, x \rangle + \langle y_2, x \rangle]$

$$\leq \max_{x \in C} \langle y_1, x \rangle + \max_{x \in C} \langle y_2, x \rangle = \sigma_C(y_1) + \sigma_C(y_2)$$

\nearrow

equality holds when $y_1 = y_2$

(c) $\forall C \neq \emptyset, C \subseteq E, y \in E^*, d \geq 0$

$$\sigma_{\alpha C}(y) = d \sigma_C(y).$$

Proof: $\sigma_{\alpha C}(y) = \max_{x \in \alpha C} \langle y, x \rangle = \max_{x \in C} \langle y, dx \rangle$

$$= d \max_{x \in C} \langle y, x \rangle = d \sigma_C(y).$$

(d) $\forall A, B \neq \emptyset, A, B \subseteq E, y \in E^*$

Proof: $\sigma_{A+B}(y) = \max_{x \in A+B} \langle y, x \rangle = \max_{x_1 \in A, x_2 \in B} \langle y, x_1 + x_2 \rangle$

$$= \max_{x_1 \in A} \langle y, x_1 \rangle + \max_{x_2 \in B} \langle y, x_2 \rangle = \sigma_A(y) + \sigma_B(y).$$

Examples of support functions:

e.g. 2.25. Support function on finite set:

$$C = \{b_1, b_2, \dots, b_m\}, b_i \sim b_m \in E.$$

$$\Rightarrow \sigma_C(y) = \max \{ \langle b_1, y \rangle, \langle b_2, y \rangle, \dots, \langle b_m, y \rangle \}.$$

e.g. 2.26. Support functions of cones.

Let $K \subseteq E$ be a cone.

$$S \cap T : \{u \in S \mid u \in T\} \geq 0.$$

Define polar cone of K as: $S \subseteq E$, if $\forall x \in S, \forall y \in K^{\circ}$
 $\lambda x \in S$, then S is "cone"

$$K^{\circ} = \{y \in E^*: \langle y, x \rangle \leq 0 \text{ for all } x \in K\}.$$

We show that: $\sigma_K(y) = \delta_{K^{\circ}}(y).$

prof: 1° $y \in K^{\circ} \Rightarrow \langle y, x \rangle \leq 0, \forall x \in K$

$$\downarrow \\ \delta_{K^{\circ}}(y) = 0$$

$$\downarrow \\ \sigma_K(y) = \max_{x \in K} \langle y, x \rangle = 0$$

$$\Rightarrow \sigma_K(y) = \delta_{K^{\circ}}(y).$$

2° $y \notin K^{\circ} \Rightarrow \exists \tilde{x} \in K, \langle y, \tilde{x} \rangle > 0 \quad \lambda \tilde{x} \in K, \lambda \geq 0$
 since K is cone.

$$\downarrow \\ \delta_{K^{\circ}}(y) = \infty$$

$$\Rightarrow \sigma_K(y) \geq \langle y, \lambda \tilde{x} \rangle = \lambda \langle y, \tilde{x} \rangle, \forall \lambda \geq 0$$

Take $\lambda \rightarrow \infty \Rightarrow \sigma_K(y) = \infty$ for $y \notin K^{\circ}$

$$\Rightarrow \sigma_K(y) = \delta_{K^{\circ}}(y).$$

e.g. 2.27. support function of the nonnegative orthant

Consider $E = \mathbb{R}^n$. As a special case of e.g 2.26,
 since $(\mathbb{R}_+^n)^{\circ} = \mathbb{R}_-^n$ (obviously)

$$\text{Then } \sigma_{\mathbb{R}_+^n}(y) = \delta_{\mathbb{R}_-^n}(y)$$

Before ex 2.28, we introduce a lemma:

Lemma 2.28 Farka's lemma - second formulation.

Let $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$. Then, the following (1), (2) is equivalent

(1) If $Ax \leq 0$ then $c^T x \leq 0$

(2) $\exists y \in \mathbb{R}_+^m$ s.t. $A^T y = c$.

(A kind of cone)

ex 2.29. support functions of convex polyhedral cones.

Let $E = \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$.

Define $S = \{x \in \mathbb{R}^n : Ax \leq 0\}$. Obviously, S is a cone.

Therefore $\sigma_S(y) = \delta_{S^\circ}(y)$

$y \in S^\circ$ iff $\forall x \text{ s.t. } Ax \leq 0 \text{ we have } \langle y, x \rangle \leq 0$.
 $(\forall x \in S)$

By Farka's Lemma,

The claim ~~is~~ $\Leftrightarrow \exists \lambda \in \mathbb{R}_+^m$ s.t. $A^T \lambda = y$

$$\Rightarrow S^\circ = \{A^T \lambda : \lambda \in \mathbb{R}_+^m\}.$$

To conclude: $\sigma_S(y) = \delta_{\{A^T \lambda : \lambda \in \mathbb{R}_+^m\}}(y)$.

example of polyhedral cone:

$$C = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \begin{array}{l} 2x_1 - x_2 = 0, \\ x_1 + 3x_2 \leq 0 \end{array} \right\}.$$

$$\begin{cases} 2x_1 - x_2 \leq 0 \\ -2x_1 + x_2 \leq 0 \end{cases} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \quad AX \leq 0$$

e.g. 2-30 support functions of affine sets.

Let underlying space be $E = \mathbb{R}^n$ and let $B = \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Define the affine set:

$$C = \{x \in \mathbb{R}^n : Bx = b\}.$$

$$\sigma_C(y) = \max_{x \in C} \langle y, x \rangle$$

We assume $C \neq \emptyset \Rightarrow \exists x_0 \in \mathbb{R}^n$ s.t. $Bx_0 = b$.

$$\text{Then } \sigma_C(y) = \max_x \{ \langle y, x \rangle : Bx = b \},$$

换元, 转换为求

make the change of variable: $X = Z + X_0 \Rightarrow BZ = BX - BX_0 = 0$

$$\begin{aligned} \sigma_C(y) &= \max_Z \{ \langle y, Z \rangle + \langle y, X_0 \rangle : BZ = 0 \} \\ &= \langle y, X_0 \rangle + \max_Z \{ \langle y, Z \rangle : BZ = 0 \} \\ &= \langle y, X_0 \rangle + \sigma_{\tilde{C}}(y) \end{aligned}$$

$$\text{where } \tilde{C} = \{x \in \mathbb{R}^n : Bx = 0\}.$$

$$= \{x \in \mathbb{R}^n : Ax \leq 0\} \quad A = \begin{pmatrix} -B \\ B \end{pmatrix}$$

a convex

it follows that $\sigma_{\tilde{C}} = \delta_{\tilde{C}^\circ}$, polyhedral cone.

where \tilde{C}° is a polar cone of C°

$$\tilde{C}^\circ = \{B^\top \lambda_1 - B^\top \lambda_2 : \lambda_1, \lambda_2 \in \mathbb{R}_+^m\}.$$

We will show that $C^\circ = \text{Range}(B^T)$.

$\Rightarrow C^\circ \subset \text{Range}(B^T)$.

If $x \in C^\circ$, then $\exists \lambda_1, \lambda_2 \in \mathbb{R}^m$ s.t. $x = B^T(\lambda_1 - \lambda_2) \in \text{Range}(B^T)$

$\Leftarrow \text{Range}(B^T) \subset C^\circ$.

If $x \in \text{Range}(B^T)$, then $\exists \lambda \in \mathbb{R}^m$ s.t. $x = B^T\lambda$

Refine $\lambda_1 = [\lambda]_+$, $\lambda_2 = [\lambda]_-$, we obtain that $\lambda = \lambda_1 - \lambda_2$
with $\lambda_1, \lambda_2 \in \mathbb{R}_+^m$. Here $x = B^T(\lambda_1 - \lambda_2) \in C$

$$\Rightarrow \mathcal{T}_C(y) = \langle y, x_0 \rangle + \mathcal{S}_{\text{Range}(B^T)}(y)$$

$$([\lambda]_*)_i = \begin{cases} \lambda_i & \text{if } \lambda_i > 0 \\ 0 & \text{if } \lambda_i \leq 0 \end{cases}$$

e.g 2.32. support functions of unit balls.

Suppose E is the underlying space endowed with a norm $\|\cdot\|$.
Consider the unit ball given by $B_{\|\cdot\|}[0, 1] = \{x \in E : \|x\| \leq 1\}$.

$$\Rightarrow \mathcal{T}_{B_{\|\cdot\|}[0,1]}(y) = \max_{\|x\| \leq 1} \langle y, x \rangle = \underbrace{\|y\|_*}_{\downarrow \text{dual norm.}}$$

For example, for space \mathbb{R}^n we have

$$\mathcal{T}_{B_{\|\cdot\|}[0,1]}(y) = \|y\|_q \quad (1 \leq p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1)$$

$$\sigma_{\| \cdot \|_Q, \mathbb{D}^n}(\mathbf{y}) = \|\mathbf{y}\|_{Q^{-1}} \quad (\mathbf{Q} \in \mathbb{S}_{++}^n).$$

e.g. 2.32.

Consider $C \subset \mathbb{R}^2$, $C = \left\{ (\mathbf{x}_1, \mathbf{x}_2)^T : \mathbf{x}_1 + \frac{\mathbf{x}_2^2}{2} \leq 0 \right\}$.

$$f_C(\mathbf{y}) = \max_{\mathbf{x}_1, \mathbf{x}_2} \left\{ \mathbf{x}_1 y_1 + \mathbf{x}_2 y_2 : \mathbf{x}_1 + \frac{\mathbf{x}_2^2}{2} \leq 0 \right\}.$$

$$f_C(\mathbf{0}) = 0.$$

We only consider $\boxed{\mathbf{y} \neq \mathbf{0}}$. The maximum is attained at the boundary of C .

Suppose we attain the maximum at an interior point of C . $\frac{\partial}{\partial \mathbf{x}} \langle \mathbf{x}, \mathbf{y} \rangle = \frac{\partial}{\partial \mathbf{x}} \mathbf{y}^T \mathbf{x} = \mathbf{y} = \mathbf{0}$

contradicts with $\mathbf{y} \neq \mathbf{0}$.

$$\begin{aligned} \Rightarrow f_C(\mathbf{y}) &= \max_{\mathbf{x}_1, \mathbf{x}_2} \left\{ \mathbf{x}_1 y_1 + \mathbf{x}_2 y_2 : \mathbf{x}_1 + \frac{\mathbf{x}_2^2}{2} \leq 0 \right\} \\ &= \max_{\mathbf{x}_2} \left\{ -\frac{y_1}{2} \mathbf{x}_2^2 + y_2 \mathbf{x}_2 \right\} \end{aligned}$$

$$1^\circ y_1 < 0 \Rightarrow f_C(\mathbf{y}) = 0.$$

$$2^\circ y_1 = 0, y_2 \neq 0 \Rightarrow f_C(\mathbf{y}) = 0$$

$$3^\circ y_1 > 0 \quad \text{when} \quad \mathbf{x}_2 = -\frac{y_2}{y_1}, \quad f_C(\mathbf{y}) = \frac{y_2^2}{2y_1}$$

$$\Rightarrow f_C(\mathbf{y}) = \begin{cases} \frac{y_2^2}{2y_1} & y_1 > 0 \\ 0 & y_1 = y_2 = 0 \end{cases}$$

on else.

By Thm 2.23, σ_C is convex and closed.
(but not continuous on $(0,0)$)

(remember for 1-dim function,
convexity and closedness imply continuity)

Take $a > 0$, $\begin{cases} y_1(t) = \frac{t^2}{2a} \\ y_2(t) = t \end{cases} \quad (t \geq 0) \Rightarrow \sigma_C(y_1(t), y_2(t)) = a$.

$\lim_{t \rightarrow 0^+} \sigma_C(y_1(t), y_2(t)) = a \neq 0 \Rightarrow$ discontinuity
on $(0,0)$

Important property of support functions:

\Rightarrow Support functions are completely determined by
their underlying sets as long as their sets are closed and convex
(We prove it by using Thm 2.33)

Thm 2.33 Strictly separation theorem. Let $C \subseteq E$ be
a nonempty closed and convex set, let $y \notin C$.
Then $\exists p \in E^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ s.t.

$$\begin{cases} \langle p, y \rangle > \alpha \\ \langle p, x \rangle \leq \alpha, \forall x \in C. \end{cases}$$

$y \notin$ 凸集的
等价表达

Proof: By second projection thm [[0, Thm 9.8]]

证明 R 难 part.

the vector $\bar{x} = p_C(y) \in C$ satisfies

$$(y - \bar{x})^\top (x - \bar{x}) \leq 0, \forall x \in C.$$

$$\Leftrightarrow (y - \bar{x})^\top x \leq (y - \bar{x})^\top \bar{x}, \forall x \in C.$$

$$\begin{cases} p = y - \bar{x} \neq 0 \quad (\text{since } y \notin C) \\ \alpha = (\bar{y} - \bar{x})^\top \bar{x} \end{cases}$$

so we have $\langle p, x \rangle \leq \alpha, \forall x \in C$.

$$\text{For } y \notin C, \langle p, y \rangle = (y - \bar{x})^\top y$$

$$= (y - \bar{x})^\top (y - \bar{x}) + (y - \bar{x})^\top \bar{x}$$

$$= \|y - \bar{x}\|^2 + (y - \bar{x})^\top \bar{x} > (y - \bar{x})^\top \bar{x} = \alpha.$$

(since $y \notin C, \|y - \bar{x}\| > 0$)

Then we are done.



Lemma 2.34 Let $A, B \subseteq E$ be nonempty closed and convex sets. Then $A = B$ iff $\overline{\cup A} = \overline{\cup B}$.

Proof: (\Rightarrow) $A = B \Rightarrow \overline{\cup A} = \overline{\cup B}$ is obvious.

(\Leftarrow) Suppose $\sigma_A = \sigma_B$, we show that $A = B$.

Assume for contradiction $A \neq B$.

$\Rightarrow \exists y \in A$ s.t. $y \notin B$.

B is nonempty closed and convex, by Thm 2.33,

$\exists p \in E^* \setminus \{0\}$, $\alpha > 0$

s.t. $\langle p, x \rangle \leq \alpha < \langle p, y \rangle$, $\forall x \in B$.

Taking the maximum over $x \in B$.

$$\sigma_B(p) \leq \alpha < \langle p, y \rangle \leq \sigma_A(p).$$

then $\sigma_A \neq \sigma_B$, contradicts! $\Rightarrow A = B$,

Lemma 2.35. Let $A \subseteq E$ be nonempty. Then.

(a) $\sigma_A = \sigma_{\text{Cl}(A)}$

The support function stays the same
under the operations of closure $\text{Cl}(E)$
convex hull.

(b) $\sigma_A = \sigma_{\text{conv}(A)}$

proof:

(a) Since $A \subseteq \text{Cl}(A)$, $\sigma_A(y) \leq \sigma_{\text{Cl}(A)}(y)$, $\forall y \in E^*$.
 $\Rightarrow \sigma_A \leq \sigma_{\text{Cl}(A)}$. Note: def of convex hull of set E

We will show the reverse inequality. ... The smallest convex polygon that contains all the points in the set.

Let $y \in E^*$, by def of

support functions, $\exists a$ s.t.

$$\{x_k\}_{k \geq 1} \subseteq \text{Cl}(A) \text{ s.t. } \langle y, x_k \rangle \rightarrow \sigma_{\text{Cl}(A)}(y) \text{ as } k \rightarrow \infty.$$

类似闭包，指包含所有 E 的点的最小凸集。

By def of closure, $\exists \{z_k\}_{k \geq 1} \subseteq A$ s.t. $z_k \rightarrow x_k$ as $k \rightarrow \infty$.

Since $z_k \in A$, $\sigma_A(y) \geq \langle y, z_k \rangle = \langle y, x_k \rangle + \langle y, z_k - x_k \rangle$

take $k \rightarrow \infty \Rightarrow \sigma_A(y) \geq \sigma_{\text{cl}(A)}(y) + 0 = \sigma_{\text{cl}(A)}(y)$.

$\Rightarrow \sigma_A = \sigma_{\text{cl}(A)}$.

(b). Since $A \subseteq \text{conv}(A)$, $\sigma_A(y) \leq \sigma_{\text{conv}(A)}(y)$, $\forall y \in E^*$.

We will show the reverse inequality.

Similarly to A, $\exists \{x_k\}_{k \geq 1} \subseteq \text{conv}(A)$ s.t. $\langle y, x_k \rangle \rightarrow \sigma_{\text{conv}(A)}(y)$ as $k \rightarrow \infty$.

$x_k \in \text{conv}(A)$, by def of convex hull, for $\forall k$,
 $\exists x_{k1}, x_{k2}, \dots, x_{kn} \in A$ and $\lambda_k \in \Delta_{nk}$ s.t.

$$x_k = \sum_{i=1}^n \lambda_{ki} z_{ki}$$

$$\begin{aligned} \Rightarrow \langle y, x_k \rangle &= \langle y, \sum_{i=1}^n \lambda_{ki} z_{ki} \rangle = \sum_{i=1}^n \lambda_{ki} \langle y, z_{ki} \rangle \\ &\leq \sigma_A(y) \underbrace{\sum_{i=1}^n \lambda_{ki}}_{\geq 1} = \sigma_A(y) \end{aligned}$$

Take $k \rightarrow \infty$, we have $\sigma_{\text{conv}(A)}(y) \leq \sigma_A(y)$.

$\Rightarrow \sigma_{\text{conv}(A)}(y) = \sigma_A(y)$.

e.g 2-3b. Support function of the Unit simplex.

Suppose underlying space in \mathbb{R}^n , consider the unit simplex
 $\Delta_n = \{x \in \mathbb{R}^n : e^T x = 1, x \geq 0\}$.

Δ_n can be written as the convex hull of standard basis of \mathbb{R}^n

$$\Delta_n = \text{conv}\{e_1, \dots, e_n\}.$$

By Lemma 2.35,

$$\begin{aligned} \underline{\sigma}_{\Delta_n}(y) &= \sigma_{\{e_1, \dots, e_n\}}(y) = \max \{ \langle y, e_1 \rangle, \dots, \langle y, e_n \rangle \} \\ &= \max \{ y_1, \dots, y_n \}. \end{aligned}$$

Summary of support functions:

The table below summarizes the main support function computations that were considered in this section.

C	$\sigma_C(\mathbf{y})$	Assumptions	Reference
$\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$	$\max_{i=1,2,\dots,n} \langle \mathbf{b}_i, \mathbf{y} \rangle$	$\mathbf{b}_i \in \mathbb{E}$	Example 2.25
K	$\delta_K(\mathbf{y})$	K -cone	Example 2.26
\mathbb{R}_+^n	$\delta_{\mathbb{R}_+^n}(\mathbf{y})$	$\mathbb{E} = \mathbb{R}^n$	Example 2.27
Δ_n	$\max\{y_1, y_2, \dots, y_n\}$	$\mathbb{E} = \mathbb{R}^n$	Example 2.36
$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{0}\}$	$\delta_{\{\mathbf{A}^T \lambda : \lambda \in \mathbb{R}_+^m\}}(\mathbf{y})$	$\mathbb{E} = \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}$	Example 2.29
$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{B}\mathbf{x} = \mathbf{b}\}$	$\langle \mathbf{y}, \mathbf{x}_0 \rangle + \delta_{\text{Range}(\mathbf{B}^T)}(\mathbf{y})$	$\mathbb{E} = \mathbb{R}^n, \mathbf{B} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{B}\mathbf{x}_0 = \mathbf{b}$	Example 2.30
$B_{\ \cdot\ } [0, 1]$	$\ \mathbf{y}\ _*$	-	Example 2.31