Let V be a normed space.

Def Let $\{V_n\}_n \subset V$. The series $\sum_n V_n$ is summable if $\left\{\sum_{n=1}^{m} V_n\right\}_{m=1}^{\infty}$ converges. $\sum_n V_n$ is absolutely summable if $\sum_n \|V_n\|$ converges.

Thm:

$\sum_n V_n$ is abs. summable $\Rightarrow$ $\left\{\sum_{n=1}^{m} V_n\right\}_{m=1}^{\infty}$ is Cauchy in V.

pf: Ex, same as $V=\mathbb{R}$.

Assume $\underbrace{\sum_{k=1}^{m} \|V_k\|}_{S_m} \to S$ as $m \to \infty$. $\forall \varepsilon > 0, \exists M$ s.t. $\forall m > M$, $\left|\sum_{k=1}^{m} \|V_k\| - S\right| < \frac{\varepsilon}{2}$.

Pick $n > M$ $\left|\sum_{k=1}^{n} \|V_k\| - S\right| < \frac{\varepsilon}{2}$.

$\therefore$ $|S_m - S_n| = |S_m - S - (S_n - S)| \le |S_m - S| + |S_n - S| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Hence $\{S_m\}$ is Cauchy.

Thm. V is a Bsp. $\Leftrightarrow$ Every abs summable series is summable

pf: $(\Rightarrow)$ V is a Bsp.

$\sum_n V_n$ is abs. summable $\overset{\text{Prev}}{\underset{\text{Thm}}{\Longrightarrow}}$ $\{\sum V_m\}$ is Cauchy. hence converges. in V

$\Rightarrow$ summable

$(\Leftarrow)$ Suppose Every $\underbrace{\text{abs summable series}}_{}$ is summable.

Construct this series

Let $\{V_n\}$ be a Cauchy seq in V. We are going to show that this seq has a convergent subseq. Then $\{V_n\}$ converges.

$\{v_n\}$ is Cauchy $\Rightarrow$ $\forall k \in \mathbb{N}$ $\exists N_k \in \mathbb{N}$ s.t. $\forall n, m \geq N_k$ $\|v_n - v_m\| < \underbrace{2^{-k}}_{\uparrow}$

summable, well-chosen.

Define $n_k = \sum_{m=1}^{k} N_m$. $n_1 < n_2 < \cdots$ and $\forall k$, $n_k \geq N_k$.

Thus $\forall k \in \mathbb{N}$, $\|v_{n_{k+1}} - v_{n_k}\| < 2^{-k}$

Thus $\sum_k (v_{n_{k+1}} - v_{n_k})$ is abs summable.

By our assumption, $\sum_k (v_{n_{k+1}} - v_{n_k})$ is summable

i.e. $\left\{ \sum_{k=1}^{m} (v_{n_{k+1}} - v_{n_k}) \right\}_{m=1}^{\infty}$ converges in V. as $m \to \infty$.

$$\begin{gathered} \| \\ v_{n_{m+1}} - \underbrace{v_{n_1}}_{\substack{\uparrow \\ \text{fixed.}}} \end{gathered}$$

$\Rightarrow \{v_m\}_{m=1}^{\infty}$ converges as $m \to \infty$

$\Rightarrow V$ is a Bsp.

Operators & Functionals.

$\downarrow$ Matrices?

$\downarrow$ Inner product when fixed 1 element?

$\boxed{\text{Ex. in mind}}$:

Let $K : [0,1] \times [0,1] \to \mathbb{C}$. be $\underline{\text{conts}}$. (continuous)

For $f \in C([0,1])$ $\quad Tf(x) = \int_0^1 K(x,y) f(y) dy$. $\quad$ T: a linear operator.

Then $Tf \in C([0,1])$ and linear w.r.t. f.

$$T\left(\sum_{i=1}^{n} \lambda_i f_i\right) = \sum_{i=1}^{n} \lambda_i T f_i$$

__Defn.__ Let V, W be v.s.p. We say $T : V \to W$ is __linear__ if

$\forall \lambda_1, \lambda_2 \in \mathbb{K}$ (field of scalars), $\forall v_1, v_2 \in V$, $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T v_1 + \lambda_2 T v_2$

We are interested in some "class" of linear ops.

Recall that $T : V \to W$ (a map)

is conts on V if

$\forall v \in V$, $\forall \{v_n\}$ with $v_n \to v \Rightarrow Tv_n \to Tv$.

or equivalently,

$\forall$ open $U \in W$, the set $T^{-1}(U) = \{v \in V \mid Tv \in U\}$ is open in V.

Thm. A linear op. $T: V \to W$ is conts iff $\exists C > 0$, s.t. $\forall v \in V$,

$\|Tv\|_W \le C\|v\|_V$ $(*)$

Lin op: Bounded $\Leftrightarrow$ Conts.

[We say T is a bounded lin op]

A bounded lin op takes a bd subset of V to a bd subset of W.

pf: $(\Leftarrow)$ Assume $(*)$. Let $v_n \to v \in V$. Then by $(*)$

$0 \le \|Tv_n - Tv\|_W = \|T(v_n - v)\|_W \le C\|v_n - v\|_V \to 0$.

Set $\delta = \frac{\varepsilon}{C}$. Then $Tv_n \to Tv$ is obvious. (by sandwich thm).

$(\Rightarrow)$ Here we adopt $\boxed{\text{open set def}}$ of conts.

Assume T is conts.

Then $T^{-1}(B_W(0,1)) = \{v \in V : T(v) \in B_W(0,1)\}$ is open in V.

Since $T0 = 0$. $0 \in T^{-1}(B_W(0,1)) \Rightarrow \exists r > 0$ s.t.

$B_V(0,r) \subset T^{-1}(B_W(0,1))$.

Let $v \in V \setminus \{0\}$. Then $\left\|\frac{r}{2\|v\|_V} \cdot v\right\|_V = \frac{r}{2} < r$. $\Rightarrow \frac{r}{2\|v\|_V} \cdot v \in B_V(0,r)$

$\Rightarrow T\left(\frac{r}{2\|v\|_V} \cdot v\right) \in B_W(0,1)$

$\Rightarrow \left\|T\left(\frac{r}{2\|v\|_V} \cdot v\right)\right\|_W \le 1$ By homo of norm, $\|T(v)\|_W \le \frac{2}{r}\|v\|_V$.

Denote $C = \frac{2}{\delta}$. we are done.

Eg. $T: C([0,1]) \to C([0,1])$ given by $Tf(x) = \int_0^1 k(x,y) f(y)\,dy$

where $k: [0,1] \times [0,1] \to \mathbb{K}$. Let's prove T is a $\boxed{\text{bd lin op}}$

pf: Ez: T is lin. ✓.

for $f \in C([0,1])$. $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$.

Then $|Tf(x)| = \left| \int_0^1 k(x,y) f(y)\,dy \right| \le \int_0^1 |k(x,y)|\,|f(y)|\,dy \le \|f\|_\infty \int_0^1 |k(x,y)|\,dy$

$\le \|f\|_\infty \|k\|_\infty \int_0^1 dy = \|k\|_\infty \|f\|_\infty$ holds for $\forall x$.

$\Rightarrow \|Tf\|_\infty \le \|k\|_\infty \|f\|_\infty$.

$\Rightarrow \|T\| \le \|k\|_\infty$

k: $\boxed{\text{kernel}}$ of lin op T.

Dfn. $B(V,W) = \{T: V \to W : T \text{ is a bd lin op}\}$.

Clearly, $B(V,W)$ is a v.s.p. (sum, sca mult closedness).

the operator norm via

$$\|T\| := \sup_{\|v\|=1} \|Tv\|.$$

Rmk: for $v \in V$.
Since $\|T\| \ge \|T\frac{v}{\|v\|}\|$ by def,
$\|Tv\| \le \|T\|\,\|v\|$.

Thm. The operator norm is a norm. So $B(V,W)$ is a normed space.

pf: 1) Prove $\|T\| = 0 \Leftrightarrow T = 0$.

($\Rightarrow$) $\|T\| = 0 \Rightarrow \forall v$ s.t. $\|v\| = 1$, $\|Tv\| = 0$. Since W is a normed v.sp,

$Tv = 0$, $\forall v$ s.t. $\|v\| = 1$. Then $\forall v \in V$, $Tv = \|v\| \cdot T(\frac{v}{\|v\|}) = \|v\| \cdot 0 = 0$.

$\Rightarrow T = 0$.

($\Leftarrow$) $T = 0 \Rightarrow \forall v \in V$, $\|Tv\| = 0 \Rightarrow \|T\| = \sup_{\|v\|=1} \|Tv\| = 0$.

2) Homo. Easy to follow the homo of $\|\cdot\|_W$.

3) Triange.

If $S, T \in B(V, W)$. $v \in V$, $\|v\| = 1$,

$$\|(S+T)v\| = \|Sv + Tv\| \leq \|Sv\| + \|Tv\| \leq \|S\| + \|T\|, \quad \forall v \in V \text{ s.t. } \|v\| = 1.$$

$$\Rightarrow \|S+T\| \leq \|S\| + \|T\|.$$

Thm: W is a Bsp $\Rightarrow$ $B(V, W)$ is a Bsp. (no matter V)

pf:

Suppose $\{T_n\}_n \subset B(V, W)$. s.t. $C = \sum_n \|T_n\| < \infty$. (i.e. $\sum T_n$ is abs. summable)

We want to show $\sum T_n$ is summable.

Let $v \in V$. Let $m \in \mathbb{N}$

$$\sum_{n=1}^{m} \|T_n v\| \leq \sum_{n=1}^{m} \|T_n\| \|v\| = \|v\| \sum_{n=1}^{m} \|T_n\| = C\|v\|.$$

$\Rightarrow \left\{\sum_{n=1}^{m} \|T_n v\|\right\} \in \mathbb{R}$ is bounded $\Rightarrow \sum_n \|T_n v\|$ converges. in $\mathbb{R}$.

Thus $\sum_n T_n v$ is abs. summable in W.

$\because$ W is Bsp $\therefore$ $\sum_n T_n v$ is summable lim exists.

Define $T : V \to W$ via $Tv := \lim_{m \to \infty} \sum_{n=1}^{m} T_n v$ ... Our candidate.

First, T is lin. $\forall \lambda_1, \lambda_2 \in \mathbb{K}$, $v_1, v_2 \in V$,

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \lim_{m \to \infty} \sum_{n=1}^{m} T_n(\lambda_1 v_1 + \lambda_2 v_2) = \lim_{m \to \infty} \lambda_1 \sum_{n=1}^{m} T_n(v_1) + \lambda_2 \sum_{n=1}^{m} T_n(v_2)$$

$$= \lambda_1 T v_1 + \lambda_2 T v_2.$$

Second, T is bdd.

Let $v \in V$, $\|v\| = 1$ $\quad \|Tv\| = \left\|\lim_{m \to \infty} \sum_{n=1}^{m} T_n v\right\| = \lim_{m \to \infty} \left\|\sum_{n=1}^{m} T_n v\right\| \leq \lim_{m \to \infty} \sum_{n=1}^{m} \|T_n v\|$

$$\leq \underbrace{\left(\lim_{m \to \infty} \sum_{n=1}^{m} \|T_n\|\right)}\ \underbrace{\|v\|}_{1} = \lim_{m \to \infty} \sum_{n=1}^{m} \|T_n\| = C.$$

$\therefore$ $\|Tv\| \leq C$, $\forall v$ s.t. $\|v\| = 1$ $\Rightarrow$ $\|Tv\| \leq C\|v\|$, $\forall v \in V$.

Hence, $T \in B(V, W)$. Now we claim, $T = \lim_{m\to\infty} \sum_{n=1}^{m} T_n$ in $B(V, W)$.

Let $v \in V$ s.t. $\|v\| = 1$.

Then $\|Tv - \sum_{n=1}^{m} T_n v\| = \|(T - \sum_{n=1}^{m} T_n) v\|$

$$= \left\| \lim_{m'\to\infty} \sum_{n=1}^{m'} T_n v - \sum_{n=1}^{m} T_n v \right\|$$

$$= \left\| \lim_{m'\to\infty} \sum_{n=m+1}^{m'} T_n v \right\| \leq \lim_{m'\to\infty} \sum_{n=m+1}^{m'} \|T_n v\| \leq \sum_{n=m+1}^{\infty} \|T_n\| \quad \forall v \in V \text{ s.t. } \|v\| = 1$$

$$\therefore\ 0 \leq \left\| T - \sum_{n=1}^{m} T_n \right\| = \sup_{\|v\|=1} \left\| \left(T - \sum_{n=1}^{m} T_n\right) v \right\| \leq \underbrace{\sum_{n=m+1}^{\infty} \|T_n\|}_{\downarrow\ 0}$$

as $\sum_{n=1}^{\infty} \|T_n\| = c$ ($\sum_n T_n$ abs summable as we assumed)

Sand-wich $\Rightarrow \|T - \sum_{n=1}^{m} T_n\| \to 0$ as $m \to \infty$.

i.e. $T = \lim_{m\to\infty} \sum_{n=1}^{m} T_n$,

implying $\sum_n T_n$ is summable. Hence $B(V, W)$ is a Bsp.

Basic steps:

1. Find a <u>candidate</u>.
2. Show it in the sp.
3. Show convergence in the space.

<u>Defn.</u> If V is a normed space.

$$V' = B(V, \mathbb{K})$$

is the <u>dual space</u> of V.

Normally speaking, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, which are complete. (Bsp).

Then by above thm, V' is always Bsp.

Eg. $\forall 1 \le p < \infty$

$$(\ell_p') = \ell^{p'} \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

$(l^1)' = l^\infty$, $(l^2)' = \ell^2$. But $(\ell^\infty)' \neq \ell^1$.

$\Downarrow$

Only ℓ^p that has this property.