• Normed Space.

V, a vec space over $\underline{\mathbb{R} \text{ or } \mathbb{C}}$ $(\mathbb{F})$. V comes with 2 ops.

1. $+ : V \times V \to V \quad (V_1, V_2) \mapsto V_1 + V_2$.

2. $\cdot : \mathbb{F} \times V \to V \quad (\alpha, V) \mapsto \alpha \cdot V$.

• <u>Defn.: Finite dim</u>

V is finite dim, if every linearly independent set is a finite set.

i.e. $\forall E \subset V$ s.t. $\forall V_1, \cdots V_N \in E$, if $\sum_{i=1}^{N} \alpha_i V_i = 0 \Rightarrow \alpha_1 = \cdots = \alpha_N = 0$,

then E is finite.

V is infinite-dim, if V is not finite dim.

Examples of infi dim:

$C([0,1])$.

explain: $E = \{f_n(x) = x^n \quad n \in \mathbb{N} \cup \{0\}\}$. is linearly independent & infinite.

and $E \subset C([0,1])$

<u>Defn.: norm.</u>

$\|\cdot\| : V \to [0, +\infty)$ with 3 properties:

1. $\|V\| = 0 \Leftrightarrow V = 0$. (Definiteness)
2. $\|\lambda V\| = |\lambda| \|V\|$. $\forall \lambda \in \mathbb{F}, \forall v \in V$. (Homogenity)
3. $\|V_1 + V_2\| \le \|V_1\| + \|V_2\|$ $\forall V_1, V_2 \in V$. (Triangle inequality).

A vec space V endowed with norm is a normed vector space.

<u>semi-norm.</u> Satisfy 2, 3. but not necesserily 1.

Defn. <u>metric</u> A metric on a set X is a map: $d : X \times X \to [0, +\infty)$

s.t. a. $d(x,y) = 0 \iff x = y$.

b. $d(x,y) = d(y,x)$

c. $d(x,y) \le d(x,z) + d(z,y)$

<u>**Thm.**</u> If $\|\cdot\|$ is a norm on V. then $d(v,w) = \|v-w\|$ defines a metric on V. (metric induced by norm)

Pf: 1) $\Rightarrow$ a. 3) $\Rightarrow$ c. immediately.

From 2), $\underset{d(v,w)}{\underbrace{\|v-w\|}} = |-1|\,\|v-w\| = \|(-1)(v-w)\| = \|w-v\| = d(w,v)$. $\Rightarrow$ b. $\square$.

Eg. $\mathbb{R}^n$ or $\mathbb{C}^n$ with Euclidean norm.

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}.$$

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

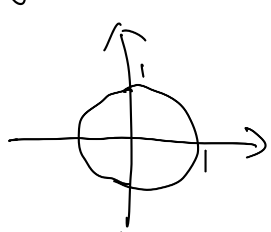
More generally, $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$. $1 \le p < \infty$.

$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$. not hard to show.

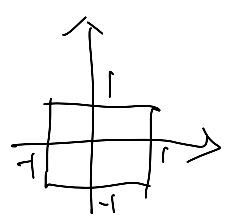
$$B(x,r) = \{ y \in X : d(y,x) \le r \}.$$

For $X = \mathbb{R}^2$

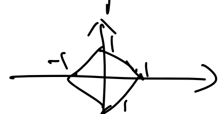
$B_2(0,1)$:



$B_\infty(0,1)$:



$B_1(0,1)$:



Eg. If X is a metric space,

$C_\infty(X) := \{ f: X \to \mathbb{C} \mid f \text{ is continuous and bounded} \}$.

eg. $C_\infty([0,1]) = C([0,1])$ （闭区间连续函数一定有界）

<u>Thm.</u> $C_\infty(X)$ is a <u>vector space</u>. (because $\begin{cases} + \\ \cdot \end{cases}$ 2 ops can be satisfied)

closeness

$\|u\|_\infty = \sup\limits_{x\in X} |u(x)|$ is a norm on $C_\infty(X)$.

Pf: 1) $\|u\|_\infty = 0 \Rightarrow \sup\limits_{x\in X} |u(x)| = 0$. Suppose for contra,

$\exists x_0 \in X$ s.t. $u(x_0) \neq 0$. $\Rightarrow \sup\limits_{x\in X} |u(x)| \geq |u(x_0)| \neq 0$. Contradicts. $\Rightarrow u(x) = 0$, $\forall x$.

2) $\|\lambda u\|_\infty = \sup\limits_{x\in X} |\lambda u(x)| = |\lambda| \sup\limits_{x\in X} |u(x)| = |\lambda| \|u\|_\infty$.

3) $\|u+v\|_\infty = \sup\limits_{x\in X} |u(x)+v(x)| \leq \sup\limits_{x\in X} |u(x)|+|v(x)| \leq \sup\limits_{x\in X} |u(x)| + \sup\limits_{x\in X} |v(x)|$.

Triangle

$\sup(a+b) \leq \sup a + \sup b$

(Discuss)

Note: <u>seq convergence in $C_\infty(X)$.</u> i.e. $u_n \to u$ as $n \to \infty$.

$\|u_n - u\|_\infty \to 0$ as $n \to \infty$. $\Leftrightarrow \forall \varepsilon, \exists N \in \mathbb{N}$ s.t. $\forall n > N$,
$\|u_n - u\|_\infty < \varepsilon$.

$\Leftrightarrow \forall \varepsilon, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $\forall x \in X$, $|u_n(x) - u(x)| < \varepsilon$.

$\Leftrightarrow$ uniform convergence in X.

在 $C_\infty(X)$ 上序列收敛 $\Leftrightarrow$ 在 X 上一致收敛.

(注. 必须用无穷范数作为 norm 才ok).

sup ...

More examples on normed spaces.

(ℓ^p 实际上是所有实数序列的1个子列).

$$\ell^p = \{ \underbrace{\{a_j\}_{j=1}^{\infty}}_{a} \mid \|a\|_p < \infty \}.$$

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

$$\|a\|_\infty = \sup_{1 \le j \le \infty} |a_j|.$$

Eg. $\{\frac{1}{j}\}_{j=1}^{\infty} \subset \ell^p$. $\forall p > 1$ (but not for $p = \infty$) (Infinite series).

Banach spaces.

Defn. A normed space is a Banach space if it is $\boxed{\text{complete}}$ w.r.t the metric induced by norm.

(Cauchy seq w.r.t the metric induced by norm always converge)

Eg. $\mathbb{R}^n$, $\mathbb{C}^n$ are complete w.r.t. any of $\|\cdot\|_p$ norm.

Thm. If X is a metric space, then $C_\infty(X)$ is a Bsp. (Banach space).

X上有界连续函数

Pf: We gonna prove: Every Cauchy seq u_n in $C_\infty(X)$ converges to a pt in $C_\infty(X)$.

Take cauchy seq. $\{u_n\}$ in $C_\infty(X)$.

We show $\{u_n\}$ is bounded in $C_\infty(X)$

$\exists N_0 \in \mathbb{N}$ s.t. $\forall n, m \geq N_0$ $\|u_m - u_n\|_\infty \leq 1$. (Cauchy seq's def, set $\varepsilon = 1$).

Then, $\forall n \geq N_0$, $\|u_n\| \leq \|u_n - u_{N_0}\| + \|u_{N_0}\| \leq \|u_{N_0}\|_\infty + 1$.

$\Rightarrow \forall n \in \mathbb{N}$, $\|u_n\|_\infty \leq \underbrace{\|u_1\|_\infty + \|u_2\|_\infty + \cdots + \|u_{N_0}\| + 1}_{= B}$. Put all upper together.

$\forall x \in X$, $|u_n(x) - u_m(x)| \leq \|u_n - u_m\|_\infty$.

Therefore, for fixed x, $\{u_n(x)\}_{n=1}^\infty$ is a cauchy seq. in $\mathbb{C}$ (Since $\{u_n\}_{n=1}^\infty$ is cauchy).

Note that $\{u_n(x)\}_{n=1}^\infty \subset \mathbb{C}$, and $\mathbb{C}$ is complete.

$\Rightarrow \forall x \in X$, $\{u_n(x)\}_{n=1}^\infty$ has a limit in $\mathbb{C}$.

Define $u: X \to \mathbb{C}$ as $u(x) = \lim_{n\to\infty} u_n(x)$ (pt-wise limit).

Now we show that. 1. $u \in C^\infty(X)$ 2. $\{u_n\}_{n=1}^\infty$ converges to u.

$\Updownarrow$

u is $\begin{cases} \text{bounded} ✓ \\ \text{continuous} ✓ \end{cases}$

$\forall x \in X$, $|u(x)| = \left|\lim_{n\to\infty} u_n(x)\right| = \lim_{n\to\infty} |u_n(x)| \leq B$.

$\Rightarrow \|u\|_\infty = \sup_{x\in X} |u(x)| \leq B$. $\Rightarrow$ u is bounded. in $\mathbb{C}$.

Now show $\|u - u_n\|_\infty \to 0$. as $n \to \infty$.

Let $\varepsilon > 0$, Since $\{u_n\}$ is Cauchy in $C^\infty(X)$. $\forall N$ s.t. $\forall n, m \geq N$

$\|u_n - u_m\|_\infty \leq \frac{\varepsilon}{2}$.

Let $x \in X$. $\forall x$, $|u_n(x) - u_m(x)| \leq \frac{\varepsilon}{2}$.

Let $m \to \infty$. $\Rightarrow \forall n \geq N$, $\forall x$, $|u_n(x) - u(x)| \leq \frac{\varepsilon}{2}$.

$\Rightarrow \|u_n - u\|_\infty = \sup_{x\in X} |u_n(x) - u(x)| \leq \frac{\varepsilon}{2}$. $\Rightarrow \|u_n - u\|_\infty \to 0$.

u_n converges to u.

Last, we show u is continuous.

$\|u_n - u\|_\infty \to 0 \Rightarrow u_n \to u$ uniformly at x.

Since all u_n are continuous, u is continuous.

Thus. $u \in C^{\infty}(X)$. and Cauchy seq $u_n \to u$.

$\Rightarrow C^{\infty}(X)$ is a Banach space. $\square$

Eg.

1. ℓ_p is a Bsp. for all $1 \le p < \infty$.
2. $c_0 = \{ a \in \ell^{\infty} : \lim_{j \to \infty} a_j = 0 \}$ is a Bsp. with norm $\|a\|_{\infty} = \sup_{1 \le j \le \infty} |a_j|$.

1.

pf: $\ell_p = \{ \{a_j\}_{j=1}^{\infty} : \|a\|_p < \infty \}$.

For Cauchy seq $\{u_n\}_{n=1}^{\infty} \subset \ell_p$.