# CS 726: Homework #1

Posted: Jan 21, 2025. Due: Feb 3, 2025 on Canvas

**Please typeset your solutions.**

You should provide sufficient justification for the steps of your solution. The level of detail should be such that your fellow students can understand your solution without asking you for further explanation.

| Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7.1 | Q7.2 | Q8 | Q9.1 | Q9.2 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 10 | 10 | 5 | 10 | 15 | 5 | 10 | 10 | 10 | 10 | 100 pts |

## Q 1 $\ell_p$ Norms

All $\ell_p$ norms are related via the following inequalities:

$$(\forall q > p \geq 1)(\forall \mathbf{x} \in \mathbb{R}^d): \ \|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p \leq d^{\frac{1}{p}-\frac{1}{q}}\|\mathbf{x}\|_q. \tag{1}$$

Provide examples of non-zero vectors (vectors whose elements are not all zeros) for which these inequalities are tight (satisfied with equality). **Your examples must work for any dimension $d \geq 1$.**

**Note:** Obviously, the left and the right inequality cannot be both satisfied at the same time, so you need to come up with two separate vectors for which the left and the right inequalities are tight.

## Q 2 Squared $\ell_p$ and $\ell_q$ Norms

Let $p > 1$ be a parameter and let $q = \frac{p}{p-1}$ (so that $\frac{1}{p} + \frac{1}{q} = 1$). Consider the functions $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_p^2$ and $f^*(\mathbf{z}) = \frac{1}{2}\|\mathbf{z}\|_q^2$ defined on $\mathbb{R}^d$. Both functions are continuously differentiable over $\mathbb{R}^d$ (you do not need to prove this).

Prove that for all $\mathbf{x}, \mathbf{z} \in \mathbb{R}^d$, we have

$$\|\nabla f^*(\mathbf{z})\|_p = \|\mathbf{z}\|_q \tag{2}$$

$$\|\nabla f(\mathbf{x})\|_q = \|\mathbf{x}\|_p. \tag{3}$$

## Q 3 Jensen's Inequality

Let $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ be a convex function. Prove that for any sequence of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$ and any sequence of non-negative scalars $\alpha_1, \ldots, \alpha_k \geq 0$ such that $\sum_{i=1}^k \alpha_i = 1$ we have:

$$f\Big(\sum_{i=1}^k \alpha_i \mathbf{x}_i\Big) \leq \sum_{i=1}^k \alpha_i f(\mathbf{x}_i) \tag{4}$$

(Note: you are asked to prove the Jensen's inequality, so your proof cannot be "this follows from Jensen's inequality".)

## Q 4 Continuous Functions

Suppose the function $f : \mathbb{R}^d \to \mathbb{R}$ is continuous at $x_0$ and $f(x_0) > 0$.

### Q 4.1

Write down the $\epsilon$-$\delta$ definition of $f$ being continuous at a point $x_0$.

### Q 4.2

Use the definition above to prove the following: for any number $\alpha \in (0, 1)$, there exists a radius $r > 0$ such that $f(x) \geq \alpha f(x_0)$ for all $x$ with $\|x - x_0\| \leq r$.

## Q 5 Convex Functions

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a *convex* function. If, $\forall \mathbf{x} \in \mathbb{R}^d$, $|f(\mathbf{x})| \leq M$, for some constant $M < \infty$, then $f$ must be a constant function (i.e., taking the same value for all $\mathbf{x} \in \mathbb{R}^d$).

## Q 6 Twice Differentiable Convex Functions

Suppose $\mathrm{dom} f$ is an open set and suppose $f$ is twice continuously differentiable on $\mathrm{dom} f$. Prove that $f$ is convex if and only if its domain is convex and $\nabla^2 f(\mathbf{x})$ is positive semi-definite (p.s.d.) for all $\mathbf{x} \in \mathrm{dom} f$.

**Hint:** You can use the first-order condition for convexity: $f$ is convex over a convex set $C$ if and only if

$$f(\mathbf{z}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{z} - \mathbf{x}) \qquad \forall \mathbf{x}, \mathbf{z} \in C \tag{5}$$

## Q 7 Convexity and Sublevel Sets

A sublevel set of $f$ is defined as

$$L_c^-(f) := \{\mathbf{x} | f(\mathbf{x}) \leq c\} \tag{6}$$

for each $c \in \mathbb{R}$.

### Q 7.1

Prove that the sublevel sets of a convex function are convex.

### Q 7.2

True or False: If all the sublevel sets of a function are convex, then this function must be convex.
If true, prove it. If false, provide a counter example.

## Q 8 Matrix Square Root

Let $\mathbf{A}$ be a symmetric p.s.d. matrix. Show that its matrix square root exists, i.e., there exists a symmetric p.s.d. matrix $\mathbf{Z}$ such that $\mathbf{Z}^2 = \mathbf{A}$.

## Q 9 Eigenvalues

### Q 9.1

Let $\mathbf{A}$ be a real symmetric $d \times d$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$. Prove that, $\forall \mathbf{x} \in \mathbb{R}^d$ :

(i) $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq \lambda_1 \|\mathbf{x}\|_2^2$;

(ii) $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq \lambda_d \|\mathbf{x}\|_2^2$.

## Q 9.2

Let $\mathbf{A}$ and $\mathbf{B}$ be $d \times d$ symmetric matrices. Prove that

$$\lambda_1(\mathbf{A} - \mathbf{B}) \leq \lambda_d(\mathbf{A}) - \lambda_d(\mathbf{B}) \leq \lambda_d(\mathbf{A} - \mathbf{B}), \tag{7}$$

where $\lambda_i(\mathbf{M})$ denotes the $i$-th smallest eigenvalue of a matrix $\mathbf{M}$.