1.

for the left, $x=0$.

for the right, $x = \mathbb{1}$. $(\forall i \in [d],\ x_i = 1)$.

Right: $\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{\frac{1}{p}}$ $\quad \|x\|_p^p = \sum_{i=1}^{d} |x_i|^p = \sum_{i=1}^{d} (|x_i|^q)^{\frac{p}{q}}$

$f(t) = t^{\frac{p}{q}}$ concave. $\quad |x_i|^q = 1$.

$\mathbb{E} f(x) \le f(\mathbb{E} X)$

$$\frac{1}{d} \sum_{i=1}^{d} (|x_i|^q)^{\frac{p}{q}} \le f\left(\frac{1}{q}\sum_{i=1}^{d} |x_i|^q\right) = \left(\frac{1}{d}\sum_{i=1}^{d} |x_i|^q\right)^{\frac{p}{q}}$$

$$= \left(\frac{1}{d}\right)^{\frac{p}{q}} \left(\sum_{i=1}^{d} |x_i|^q\right)^{\frac{p}{q}}$$

$$\|x\|_p^p \le d^{1-\frac{p}{q}} \|x\|_q^p$$

$$\|x\|_p \le \left(d^{1-\frac{p}{q}}\right)^{\frac{1}{p}} \|x\|_q = d^{\frac{1}{p}-\frac{1}{q}} \|x\|_q.$$

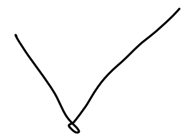
Left: PMI. $\quad$ for $a_1, \cdots, a_d \ge 0$ $\qquad \underline{q > p > 0}$.

$$\left(\frac{1}{d}\sum_{i=1}^{d} a_i^q\right)^{\frac{1}{q}} \le \left(\frac{1}{d}\sum_{i=1}^{d} a_i^p\right)^{\frac{1}{p}}.$$

Set $a_i = |x_i|$ $\qquad \|x\|_q \cdot \left(\frac{1}{d}\right)^{\frac{1}{q}} \le \|x\|_p \cdot \left(\frac{1}{d}\right)^{\frac{1}{p}}$.

$$\|x\|_p \ge \left(\frac{1}{d}\right)^{\frac{1}{q}-\frac{1}{p}} \|x\|_q = d^{\frac{1}{p}-\frac{1}{q}} \|x\|_q \ge \|x\|_q.$$

2. $f^*(z) = \frac{1}{2}\left(\sum_{i=1}^{d} |z_i|^q\right)^{\frac{2}{q}}$. $\quad |z_i|^q \quad |z_i|$



$$\frac{\partial f^*(z)}{\partial z_i} = \frac{1}{2}\cdot\frac{2}{q}\left(\sum_{i=1}^{d}|z_i|^q\right)^{\frac{2}{q}-1} \cdot q\,|z_i|^{q-1} \cdot \mathbb{I}\{z_i \geq 0\}.$$

$$p = \frac{q}{q-1}$$

$$= \left(\sum_{i=1}^{d}|z_i|^q\right)^{\frac{2}{q}-1} |z_i|^{q-1}\,\mathbb{I}\{z_i \geq 0\}.$$

$$\|\nabla f^*(z)\|_p = \left(\sum_{i=1}^{d}\left|\frac{\partial f^*(z)}{\partial z_i}\right|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{d}\left(\left(\sum_{i=1}^{d}|z_i|^q\right)^{\frac{2}{q}-1}\right)^p \left(|z_i|^{q-1}\right)^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{d}|z_i|^q\right)^{\frac{2}{q}-1}\left(\sum_{i=1}^{d}|z_i|^{p(q-1)}\right)^{\frac{1}{p}}$$

$p(q-1) = q.$

$$= \left(\sum_{i=1}^{d}|z_i|^q\right)^{\frac{2}{q}-1}\left(\sum_{i=1}^{d}|z_i|^q\right)^{\frac{1}{p}}$$

$$\frac{1}{p} + \frac{2}{q} - 1 = 1 + \frac{1}{q} - 1 = \frac{1}{q}$$

$$= \left(\sum_{i=1}^{d}|z_i|^q\right)^{\frac{1}{q}} = \|z\|_q.$$

By symmetricity, $\quad \|\nabla f(x)\|_q = \|x\|_p$.

3. Proof by induction. of # vecs.

$k=1$.

Suppose Jensen holds for $n \leq k$. Let's consider case of $k+1$.

$$f\left(\sum_{i=1}^{k+1}\alpha_i x_i\right) = f\left(\sum_{i=1}^{k}\alpha_i x_i + \alpha_{k+1}x_{k+1}\right) = \underline{f\left((1-\alpha_{k+1})\sum_{i=1}^{k}\frac{\alpha_i}{1-\alpha_{k+1}}x_i + \alpha_{k+1}x_{k+1}\right)}$$

$$\begin{cases}\sum_{i=1}^{k}\frac{\alpha_i}{1-\alpha_{k+1}} = 1. \\ (1-\alpha_{k+1}) + \alpha_{k+1} = 1\end{cases}$$

$$\underset{n=2}{\leq} (1-\alpha_{k+1})\, f\left(\sum_{i=1}^{k}\frac{\alpha_i}{1-\alpha_{k+1}}x_i\right) + \alpha_{k+1} f(x_{k+1})$$

Jensen.

$$\underset{n=k}{\leq} (1-\alpha_{k+1})\sum_{i=1}^{k}\frac{\alpha_i}{1-\alpha_{k+1}} f(x_i) + \alpha_{k+1} f(x_{k+1})$$

$= \sum_{i=1}^{k+1} \alpha_i f(x_i)$. $\square$.

4.

1. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\|x - x_0\| \leq \delta$ implies $|f(x) - f(x_0)| \leq \varepsilon$.

2. $f(x_0) - f(x) \leq \varepsilon$. $\forall \varepsilon$.

$f(x) \geq f(x_0) - \varepsilon = \alpha f(x_0)$ $\alpha \in (0,1)$

set $0 < \varepsilon < f(x_0)$

set $\delta = r$.

5. If not const. $\Rightarrow \exists x_0$, $\nabla f(x_0) \neq 0$.

$f(x) \geq f(x_0) + \nabla f(x_0)^T (x - x_0)$

Set $x = x_0 + \nu \nabla f(x_0)$. $\in \mathbb{R}^d$. $(\nu \in \mathbb{R})$

$\Rightarrow f(x) \geq f(x_0) + \nu \|\nabla f(x_0)\|^2 \geq \nu \|\nabla f(x_0)\|^2 - M$.

when $\nu \geq \frac{2M}{\|\nabla f(x_0)\|^2}$, $f(x) \geq M$. Contradicts with our assumption.

Hence must be const.

6. ($\Rightarrow$) Taylor's thm, $f(z) = f(x) + \langle \nabla f(x), z - x \rangle + \langle \nabla^2 f(x + \nu(z-x))(z-x), z - x \rangle$
$\exists \nu \in (0,1)$

By convexity, $f(z) - f(x) - \langle \nabla f(x), z - x \rangle = \langle \nabla^2 f(x + \nu(z-x))(z-x), z - x \rangle \geq 0$.
1-order

set $x \to 0$. $\underline{u = z - x}$. (arbitrary, since z, x arbitrary) $u^T \nabla^2 f(x) u \geq 0 \quad \forall u$.

$\Downarrow$

$\nabla^2 f(x)$ p.s.d.

($\Leftarrow$). Apply directly to Taylor's Thm.

7.

1. Let $x, y \in L_c^-(f)$. $\alpha+\beta=1$, $\alpha, \beta \in \mathbb{R}$. Let's prove $\alpha x + \beta y \in L_c^-(f)$.

$f(x), f(y) \leq c \quad \Rightarrow \quad f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \leq (\alpha + \beta) c = c.$

$\Rightarrow \alpha x + \beta y \in L_c^-(f)$.

2. False.

$f(x) = x^3$. $\quad f(x) \leq c \Rightarrow x \in (-\infty, c^{\frac{1}{3}}]$

$\Rightarrow \forall c$, $L_c(x^3)$ is cvx.

But $f(x) = x^3$ is not cvx

because $\nabla^2 f(x) = 6x$. $\quad \nabla^2 f(x) \succeq 0$ not always holds.

8. $\underset{d\times d}{A} = \underset{d\times n}{U^T}\underset{n\times n}{\Lambda}\underset{n\times d}{U}$. $\quad U^T U = I$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, $n \le d$.

$\Lambda^{\frac{1}{2}}$ exists. $\quad$ Let $Z = \underset{d\times n}{U^T}\Lambda^{\frac{1}{2}}\underset{n\times d}{U}$. $\quad \cdots$ symmetric.

$$Z^2 = Z^T Z = U^T \Lambda^{\frac{1}{2}} U U^T \Lambda^{\frac{1}{2}} U = U^T \Lambda U = A. \qquad \square.$$

9.

1. $x^T A x = \sum_i \sum_j a_{ij} x_i x_j$

$$\frac{x^T A x}{\|x\|_2^2} = \frac{x^T A x}{\|x\|_2^2} = \frac{x^T U^T \Lambda U x}{\|x\|_2^2} = \frac{y^T \Lambda y}{\|x\|_2^2} = \frac{\sum_{i=1}^d \lambda_i y_i^2}{\sum_{i=1}^d y_i^2} \in \left[\frac{\lambda_1 \sum y_i^2}{\sum y_i^2}, \frac{\lambda_d \sum y_i^2}{\sum y_i^2}\right]$$

$\|y\|_2 = \|x\|_2 \qquad = [\lambda_1, \lambda_d]$.

$y = Ux \quad \|y\|^2 = \|Ux\|^2 = x^T \underbrace{U^T U}_{I} x = x^T x = \|x\|_2^2$

2. $\lambda_1(M) = \min_u \{u^T M u : \|u\|_2 = 1\}$.

$\lambda_d(M) = \max_u \{u^T M u : \|u\|_2 = 1\}$.

Set u s.t. $\|u\|_2 = 1$, $u^T B u = \lambda_d(B)$. $\quad$ (Fixed u)

$\lambda_1(A-B) \le u^T(A-B)u = u^T A u - u^T B u = u^T A u - \lambda_d(B) \le \lambda_d(A) - \lambda_d(B)$. $\quad \checkmark$.

Set v s.t. $\|v\|_2 = 1$. $v^T A v = \lambda_d(A)$.

$\lambda_d(A-B) \ge v^T(A-B)v = \lambda_d(A) - v^T B v \ge \lambda_d(A) - \lambda_d(B)$. $\quad \square$.