$h_y(x) = \langle y, x \rangle$ $\quad h_y : x \to \langle y, x \rangle$ $\quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$.

Dual Norm.

$$\|z\|_* := \sup_{\|x\| \le 1} \langle z, x \rangle.$$

Hölder's Inequality.

$\forall z, x \in \mathbb{R}^d$. $\quad |\langle z, x \rangle| \le \|z\|_* \cdot \|x\|$ $\quad$ fix x

pf: $\|z\|_* = \sup_{\|x\| \le 1} \langle z, x \rangle = \sup_x \langle z, \frac{x}{\|x\|} \rangle \ge \frac{1}{\|x\|} \langle z, x \rangle$. $\quad \langle z, x \rangle \le \|z\|_* \|x\|$.

Set $x \to -x$. $\quad \langle z, x \rangle \ge -\|z\|_* \|x\|$. $\quad \Rightarrow |\langle z, x \rangle| \le \|z\|_* \cdot \|x\|$.

$\|\cdot\|_p$ and $\|\cdot\|_q$ are duals when $\frac{1}{p} + \frac{1}{q} = 1$.

$\|\cdot\|_2$ is its own dual.

$\|\cdot\|_1$ and $\|\cdot\|_\infty$ are dual to each other.

In $\mathbb{R}^d$, all ℓ_p norms are equivalent. In particular,

$$\forall x \in \mathbb{R}^d, p \ge 1, r > p : \quad \|x\|_r \le \|x\|_p \le d^{\frac{1}{p} - \frac{1}{r}} \|x\|_r .$$

However, choice of norm affects how algorithm performance depends on dimension d.

Lower semicontinuous.

$f : \mathbb{R}^d \to \bar{\mathbb{R}}$, $\quad$ l.s.c. at $x \in \mathbb{R}^d$ if $\quad f(x) \le \liminf_{y \to x} f(y)$.

Indicator of a closed set is l.s.c.

$$I_{\mathcal{X}}(x) = \begin{cases} 0 & x \in \mathcal{X} \\ \infty & x \notin \mathcal{X}. \end{cases}$$

$$\min_{x \in \mathcal{X}} f(x) \equiv \min_{x \in \mathbb{R}^d} \{ f(x) + I_{\mathcal{X}}(x) \}.$$

--- Unify constrained and unconstrained optimization.

Cvx Function.

f is cvx $\Longleftrightarrow$ epi(f) is cvx set.

Proper : $f : \mathbb{R}^d \to \overline{\mathbb{R}}$. f is proper means $\exists\, x \in \mathbb{R}^d$ s.t. $f(x) \in \mathbb{R}$.

$f : \mathbb{R}^d \to \mathbb{R}$ is cvx & proper $\Rightarrow$ dom(f) is cvx.