Proofs and solutions for Elementary Differential Geometry 2nd Ed. by Barrett O'Neil

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§0.1 Introduction

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Problem 1. Let $\phi = y\,\mathrm{d}x + \mathrm{d}z\,, \psi = \sin z\,\mathrm{d}x + \cos z\,\mathrm{d}y\,, \xi = \mathrm{d}y + z\,\mathrm{d}z$. Find the standard expressions (in terms of $\mathrm{d}x\,\mathrm{d}y\,, \ldots$) for
(a) $\phi\wedge\psi$, $\psi\wedge\xi$, $\xi\wedge\phi$. (b) $d\phi$, $d\psi$, $d\xi$.

Solution: (a) Applying the definition

$$\phi\wedge\psi = \sin z\,\mathrm{d}z\,\mathrm{d}x + y\cos z\,\mathrm{d}x\,\mathrm{d}y + \cos z\,\mathrm{d}z\,\mathrm{d}y \tag{1.1}$$

$$\psi\wedge\xi = \sin z\,\mathrm{d}x\,\mathrm{d}y + z\sin z\,\mathrm{d}x\,\mathrm{d}z + z\cos z\,\mathrm{d}y\,\mathrm{d}z \tag{1.2}$$

$$\xi\wedge\phi = \mathrm{d}y\,\mathrm{d}x + yz\,\mathrm{d}z\,\mathrm{d}x + \mathrm{d}y\,\mathrm{d}z \tag{1.3}$$

(b) Using definition 6.3,

$$d\phi = \mathrm{d}y\,\mathrm{d}x \tag{1.4}$$
$$d\psi = \cos z\,\mathrm{d}z\,\mathrm{d}x - \sin z\,\mathrm{d}z\,\mathrm{d}y \tag{1.5}$$
$$d\xi = 0 \tag{1.6}$$

□

Problem 2. Let $\psi = \mathrm{d}x\,/y$ and $\psi = z\,\mathrm{d}y$. Check the Leibnizian formula (3) of Theorem 6.4 in this case by computing each term separately.

Solution: We have $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi\psi$. The first term is

$$d(\phi \wedge \psi) = d(z/y\,\mathrm{d}x\,\mathrm{d}y) \tag{1.7}$$
$$= 1/y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{1.8}$$

The second term is

$$d\phi \wedge \psi = 0 \tag{1.9}$$

The third term is

$$\phi\psi = 1/y\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \tag{1.10}$$
$$= -1/y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{1.11}$$

□

Problem 3. For any function f show that $d(\mathrm{d}f) = 0$. Deduce that $d(f\,\mathrm{d}g) = \mathrm{d}f \wedge \mathrm{d}g$.

Solution: Using $\mathrm{d}f = \frac{\partial f}{\partial x_i} dx_i$,

$$d(\mathrm{d}f) = d(\frac{\partial f}{\partial x_i} dx_i) \tag{1.12}$$
$$= d(\frac{\partial f}{\partial x_i}) x_i \tag{1.13}$$
$$= \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j x_i \tag{1.14}$$
$$= 0 \tag{1.15}$$

by antisymmetry (alternation). Point is the partial derivative is symmetric in i and j but the wedge product is antisymmetric in those indices so the whole term is zero. Combining this result with $\mathrm{d}(f\phi) = \mathrm{d}f \wedge \phi + f d\phi$, we see that the second term is zero so that $\mathrm{d}(f\,\mathrm{d}g) = \mathrm{d}f \wedge \mathrm{d}g$. □

Problem 4. Simplify the following forms
(a) $\mathrm{d}(f\,\mathrm{d}g + g\,\mathrm{d}f)$ (b) $\mathrm{d}((f-g)(\mathrm{d}f + \mathrm{d}g))$
(c) $\mathrm{d}(f\,\mathrm{d}g \wedge g\,\mathrm{d}f)$ (d) $\mathrm{d}(gf\,\mathrm{d}f) + \mathrm{d}(f\,\mathrm{d}g)$

Solution: (a)

$$\begin{aligned}
\mathrm{d}(f\,\mathrm{d}g + g\,\mathrm{d}f) &= \mathrm{d}f \wedge \mathrm{d}g + \mathrm{d}g \wedge \mathrm{d}f && (1.16)\\
&= 0 && (1.17)
\end{aligned}$$

by the alternation rule.

(b)

$$\begin{aligned}
\mathrm{d}((f-g)(\mathrm{d}f + \mathrm{d}g)) &= \mathrm{d}(f-g) \wedge (\mathrm{d}f + \mathrm{d}g) && (1.18)\\
&= \mathrm{d}f \wedge \mathrm{d}g - \mathrm{d}g \wedge \mathrm{d}f && (1.19)\\
&= 2\,\mathrm{d}f \wedge \mathrm{d}g && (1.20)
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{d}(f\,\mathrm{d}g \wedge g\,\mathrm{d}f) &= \mathrm{d}(f\,\mathrm{d}g) \wedge (g\,\mathrm{d}f) - (f\,\mathrm{d}g\,(g\,\mathrm{d}f) && (1.21)\\
&= \mathrm{d}f \wedge \mathrm{d}g \wedge (g\,\mathrm{d}f) - (f\,\mathrm{d}g) \wedge \mathrm{d}g \wedge \mathrm{d}f && (1.22)\\
&= d(fg) \wedge \mathrm{d}f\,\mathrm{d}g && (1.23)
\end{aligned}$$

(d)

$$\begin{aligned}
\mathrm{d}(gf\,\mathrm{d}f) + \mathrm{d}(f\,\mathrm{d}g) &= \mathrm{d}(gf) \wedge \mathrm{d}f + \mathrm{d}f \wedge \mathrm{d}g && (1.24)\\
&= (g\,\mathrm{d}f + f\,\mathrm{d}g) \wedge \mathrm{d}f + \mathrm{d}f \wedge \mathrm{d}g && (1.25)\\
&= (f-1)\,\mathrm{d}g \wedge \mathrm{d}f && (1.26)
\end{aligned}$$

□

Problem 5. For any three 1-forms $\phi_i = \sum_j f_{ij}\,\mathrm{d}x_j$ $(1 \le i \le 3)$, prove

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \begin{pmatrix} f_{11} & f_{22} & f_{33} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,. \tag{1.27}$$

Solution:

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \sum_{i,j,k} f_{1i} f_{2j} f_{3k} \epsilon^{ijk}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{1.28}$$

$$\tag{1.29}$$

because repeats are zero so we need the antisymmetric product of all of the f_ij giving the determinant. □

Problem 6. If r, θ, z are the cylindrical coordinate functions on $\mathbf{R}^2$, then $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Compute the *volume element $dx\ dy\ dz$* of $\mathbf{R}^3$ in cylindrical coordinates. (That is, express $dx\ dy\ dz$ in terms of the functions r, θ, z, and their differentials.)

Solution:

$$\begin{aligned}
\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= (\cos\theta\,\mathrm{d}r - r\sin\theta)(\sin\theta\,\mathrm{d}r + r\cos\theta)\,\mathrm{d}z && (1.30)\\
&= r(-\sin^2\theta\,\mathrm{d}r + \cos^2\theta\,\mathrm{d}r\,\mathrm{d}\theta)\,\mathrm{d}z && (1.31)\\
&= r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z && (1.32)\\
& && (1.33)
\end{aligned}$$

by the alternation rule. □

Problem 7. For a 2-form $\eta = f\,\mathrm{d}x\,\mathrm{d}y + g\,\mathrm{d}x\,\mathrm{d}z + h\,\mathrm{d}y\,dz$, the *exterior derivative $d\eta$* is defined to be the 3-form obtained by replacing f, g, and h by their differentials. Prove that for any 1-form ϕ, $\mathrm{d}\,(\mathrm{d}f) = 0$. Exercises 3 and 7 show that $\mathrm{d}^2 = 0$, that is, for any form ξ, $\mathrm{d}(\mathrm{d}\xi) = 0$. (If ξ is a 2-form, then $d(\mathrm{d}\xi) = 0$, since its degree exceeds 3.)

Solution: We have $df = \sum_i \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$. Then

$$\begin{aligned}
\mathrm{d}(\mathrm{d}f) &= \sum_i \mathrm{d}f_i \wedge \mathrm{d}x_i && (1.34)\\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j && (1.35)
\end{aligned}$$

but the partial derivative is symmetric in i, j and the differentials are antisymmetric so this product is zero. □

Problem 8. Prove that all three operations may be expressed by exterior derivatives as follows:
(a) $df \leftrightarrow \boldsymbol{\nabla} f$.
(b) If $\phi \leftrightarrow V$, then $\mathrm{d}\phi \leftrightarrow \boldsymbol{\nabla} \times V$.
(c) If $\eta \leftrightarrow V$, then $\mathrm{d}\eta = (\boldsymbol{\nabla} \cdot V)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Solution: (a) This one works out by definition. Since $\mathrm{d}x_i$ is dual to U_i.
(b)

$$\begin{aligned}
\mathrm{d}\phi &= \sum_i \mathrm{d}f_i \wedge \mathrm{d}x_i && (1.36)\\
& && (1.37)
\end{aligned}$$

(c) Using $\eta = f\,\mathrm{d}x\,\mathrm{d}y + g\,\mathrm{d}x\,\mathrm{d}z + h\,\mathrm{d}y\,\mathrm{d}z$,

$$\begin{aligned}
\mathrm{d}\eta &= \frac{\partial f}{\partial z}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial g}{\partial y}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z + \frac{\partial h}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && (1.38)\\
& && (1.39)
\end{aligned}$$

and the equality holds using (2). □

Problem 9. Let f and g be real-valued functions on $\mathbf{R}^2 . Provethat \, \mathrm{d}f \wedge \mathrm{d}g = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \mathrm{d}x\,\mathrm{d}y \,. (1.40)$
Solution:

$$\mathrm{d}f \wedge \mathrm{d}g = (\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y)(\frac{\partial g}{\partial x}\,\mathrm{d}x + \frac{\partial g}{\partial y}\,\mathrm{d}y) \tag{1.41}$$

$$= \frac{\partial f}{\partial x}\,\mathrm{d}x\,\frac{\partial g}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial y}\,\mathrm{d}y\,\frac{\partial g}{\partial x}\,\mathrm{d}x \tag{1.42}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \mathrm{d}x\,\mathrm{d}y \tag{1.43}$$

□

§1.7 Mappings

Problem 1. With $F(u, v) = (u^2 - v^2, 2uv)$, find all points $\mathbf{p}$ such that
(a) $F(\mathbf{p}) = (0, 0)$ (b) $F(\mathbf{p}) = (8, 6)$
(c) $F(\mathbf{p}) = \mathbf{p}$.

Solution: (a) $u, v) = (0, 0)$
(b) $u, v) = (\pm 3, \pm 1)$
(c) $(u, v) = (1, 0)$ □

Problem 2. (a)

Problem 3. With $F(u, v) = (u^2 - v^2, 2uv)$, let $\mathbf{v} = (v_1, v_2)$ be a tangent vector to $\mathbf{R}^2 at \mathbf{p} = (p_1, p_2)$. Apply Definition 7.4 directly to express $F * (\mathbf{v})$ in terms of the coordinates of $\mathbf{v}$ and $\mathbf{p}$.

Solution:

$$F * (\mathbf{v}) = (\mathbf{v}[u^2 - v^2], \mathbf{v}[2uv]) \tag{1.44}$$

$$= (2p_1v_1 - 2p_2v_2, 2p_1v_2 + 2p_2v_1) \tag{1.45}$$

□

Problem 4. With $F(u, v) = (u^2 - v^2, 2uv)$, find a formula for the Jacobian matrix of F at all points, and deduce that $F*_p$ is a linear isomorphism at every point of $\mathbf{R}^2 excepttheorigin.$ *Solution:*

$$\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} \tag{1.46}$$

this matrix has rank 2 so given the conditions below Def. 7.9, the map $F*$ is one-to-one. So it is a linear isopmorphism. □

Problem 5. If $F : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, prove that $F * (\mathbf{v}_p) = F(\mathbf{v})_{F(p)}$.

Solution: I don't understand this notation... □

Problem 6. (a) Give an example to demonstrate that a one-to-one and onto mapping need not be a diffeomorphism. (Hint: Take m = n = 1.)
(b) Prove that if a one-to-one and onto mapping F: $\mathbf{R}^n \to \mathbf{R}^n$ is regular, then it is a diffeomorphism.

Solution: (a) Any mapping defined by $f(x) = x^\alpha$ for α is odd doesn't have smooth inverse at the origin but is still one-to-one and onto.
(b) If a mapping is one-to-one at every point and onto, then there is a well-defined inverse mapping making it a diffeomorphism. □

Problem 7. Prove that a mapping $F : \mathbf{R}^n \to \mathbf{R}^m$ preserves directional derivatives in this sense: If $\mathbf{v}_p$ is a tangent vector to $\mathbf{R}^n$ and g is a differentiable function on $\mathbf{R}^m$, then $F * (\mathbf{v}_p)[g] = \mathbf{v}_p[g(F)]$.

Solution:

$$\mathbf{v}_p[g(F)] = \mathrm{d}(g(F))\,[\mathbf{p}] \tag{1.47}$$
$$= \mathrm{d}g\,(\mathrm{d}F\,[\mathbf{p}]) \tag{1.48}$$
$$\equiv F * (\mathbf{v}_p)[g] \tag{1.49}$$

□

Problem 8. In the definition of tangent map (Def. 7.4), the straight line $t \to \mathbf{p} + t\mathbf{v}$ can be replaced by any curve a with initial velocity $\mathbf{v}_p$.

Solution: The function $F*$ gives the initial velocity of a curve $\alpha'(0)$. Any curve $\beta(t)$ with $\beta'(0)$ will have the same tangent vector and so won't affect the value of $F*$. □

Problem 9. Let $F : \mathbf{R}^n \to \mathbf{R}^m$ and $G : \mathbf{R}^m \to \mathbf{R}^p$ be mappings. Prove:
(a) Their composition $GF : \mathbf{R}^n \to \mathbf{R}^p$ is a (differentiable) mapping. (Take m = p = 2 for simplicity.)
(b) $(GF)* = G * F*$. (Hint: Use the preceding exercise.) This concise formula is the general chain rule. Unless dimensions are small, it becomes formidable when expressed in terms of Jacobian matrices.
(c) If F is a diffeomorphism, then so is its inverse mapping F^{-1}.

Solution: (a) A mapping is a function with differentiable coordinate functions. Since differentiation is well-defined for composite functions using chain rule, the composite function GF is a mapping.
(b) By definition,$F * (v) = \beta'(0)$. So,

$$(GF) * (\beta'(0)) = (GF(\beta))'(0) \tag{1.50}$$
$$= G * (F(\beta)'(0)) \tag{1.51}$$
$$= G * F * (\beta'(0)) \tag{1.52}$$

(c) If F is a diffeomorphism, then it has an inverse F^{-1}. The function F^{-1} has an inverse. Its tangent map also has an inverse by the previous exercise that maps only the zero vector to 0 (i.e. $F^{-1}F = I \to (F^{-1}F)* = I* \to F^{-1} * F* = I$). $\square$

Problem 10. Show (in two ways) that the map $F : \mathbf{R}^2 \to \mathbf{R}^2$ such that $F(u, v) = (v\,\mathrm{e}^u, 2u)$ is a diffeomorphism:
(a) Prove that it is one-to-one, onto, and regular; (b) Find a formula for its inverse $F^{-1} : \mathbf{R}^2 \to \mathbf{R}^2$ and observe that F^{-1} is differentiable. Verify the formula by checking that both FF^{-1} and $F^{-1}F$ are identity maps.

Solution: (a) The map is one-to-one and onto since the coordinate functions are one-to-one and onto. The Jacobian matrix is rank two so the tangent map is one-to-one so the map is regular.
(b) Let $F^{-1} = (u\,\mathrm{e}^{-u}/v, v/(2u))$. Then $FF^{-1} = F^{-1}F = I$. $\square$

Chapter 2

Frame Fields

§2.1 Euclidean Space

§2.2 Tangent Vectors

§2.3 Directional Derivatives

§2.4 Curves in $\mathbf{R}^3$

§2.5 1-Forms

§2.6 Differential Forms

Problem 1. Let $\phi = y\,\mathrm{d}x + \mathrm{d}z\,, \psi = \sin z\,\mathrm{d}x + \cos z\,\mathrm{d}y\,, \xi = \mathrm{d}y + z\,\mathrm{d}z$. Find the standard expressions (in terms of $\mathrm{d}x\,\mathrm{d}y\,, ...$) for
(a) $\phi \wedge \psi$, $\psi \wedge \xi$, $\xi \wedge \phi$. (b) $d\phi$, $d\psi$, $d\xi$.

Solution: (a) Applying the definition

$$\phi \wedge \psi = \sin z\,\mathrm{d}z\,\mathrm{d}x + y\cos z\,\mathrm{d}x\,\mathrm{d}y + \cos z\,\mathrm{d}z\,\mathrm{d}y \tag{2.1}$$
$$\psi \wedge \xi = \sin z\,\mathrm{d}x\,\mathrm{d}y + z\sin z\,\mathrm{d}x\,\mathrm{d}z + z\cos z\,\mathrm{d}y\,\mathrm{d}z \tag{2.2}$$
$$\xi \wedge \phi = \mathrm{d}y\,\mathrm{d}x + yz\,\mathrm{d}z\,\mathrm{d}x + \mathrm{d}y\,\mathrm{d}z \tag{2.3}$$

(b) Using definition 6.3,

$$d\phi = \mathrm{d}y\,\mathrm{d}x \tag{2.4}$$
$$d\psi = \cos z\,\mathrm{d}z\,\mathrm{d}x - \sin z\,\mathrm{d}z\,\mathrm{d}y \tag{2.5}$$
$$d\xi = 0 \tag{2.6}$$

□

Problem 2. Let $\psi = \mathrm{d}x\,/y$ and $\psi = z\,\mathrm{d}y$. Check the Leibnizian formula (3) of Theorem 6.4 in this case by computing each term separately.

Solution: We have $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi\psi$. The first term is

$$d(\phi \wedge \psi) = d(z/y\,\mathrm{d}x\,\mathrm{d}y) \tag{2.7}$$
$$= 1/y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{2.8}$$

The second term is

$$d\phi \wedge \psi = 0 \tag{2.9}$$

The third term is

$$\phi\psi = 1/y\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \tag{2.10}$$
$$= -1/y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{2.11}$$

□

Problem 3. For any function f show that $d(\mathrm{d}f) = 0$. Deduce that $d(f\,\mathrm{d}g) = \mathrm{d}f \wedge \mathrm{d}g$.

Solution: Using $\mathrm{d}f = \frac{\partial f}{\partial x_i} dx_i$,

$$d(\mathrm{d}f) = d(\frac{\partial f}{\partial x_i} dx_i) \tag{2.12}$$
$$= d(\frac{\partial f}{\partial x_i})x_i \tag{2.13}$$
$$= \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j x_i \tag{2.14}$$
$$= 0 \tag{2.15}$$

by antisymmetry (alternation). Point is the partial derivative is symmetric in i and j but the wedge product is antisymmetric in those indices so the whole term is zero. Combining this result with $\mathrm{d}(f\phi) = \mathrm{d}f \wedge \phi + f d\phi$, we see that the second term is zero so that $\mathrm{d}(f\,\mathrm{d}g) = \mathrm{d}f \wedge \mathrm{d}g$. □

Problem 4. Simplify the following forms
(a) $\mathrm{d}(f\,\mathrm{d}g + g\,\mathrm{d}f)$ (b) $\mathrm{d}((f-g)(\mathrm{d}f + \mathrm{d}g))$
(c) $\mathrm{d}(f\,\mathrm{d}g \wedge g\,\mathrm{d}f)$ (d) $\mathrm{d}(gf\,\mathrm{d}f) + \mathrm{d}(f\,\mathrm{d}g)$

Solution: (a)

$$\mathrm{d}(f\,\mathrm{d}g + g\,\mathrm{d}f) = \mathrm{d}f \wedge \mathrm{d}g + \mathrm{d}g \wedge \mathrm{d}f \tag{2.16}$$
$$= 0 \tag{2.17}$$

by the alternation rule.

(b)

$$\begin{aligned}
\mathrm{d}((f-g)(\mathrm{d}f+\mathrm{d}g)) &= \mathrm{d}(f-g)\wedge(\mathrm{d}f+\mathrm{d}g) &(2.18)\\
&= \mathrm{d}f\wedge\mathrm{d}g-\mathrm{d}g\wedge\mathrm{d}f &(2.19)\\
&= 2\,\mathrm{d}f\wedge\mathrm{d}g &(2.20)
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{d}(f\,\mathrm{d}g\wedge g\,\mathrm{d}f) &= \mathrm{d}(f\,\mathrm{d}g)\wedge(g\,\mathrm{d}f)-(f\,\mathrm{d}g\,(g\,\mathrm{d}f) &(2.21)\\
&= \mathrm{d}f\wedge\mathrm{d}g\wedge(g\,\mathrm{d}f)-(f\,\mathrm{d}g)\wedge\mathrm{d}g\wedge\mathrm{d}f &(2.22)\\
&= d(fg)\wedge\mathrm{d}f\,\mathrm{d}g &(2.23)
\end{aligned}$$

(d)

$$\begin{aligned}
\mathrm{d}(gf\,\mathrm{d}f)+\mathrm{d}(f\,\mathrm{d}g) &= \mathrm{d}(gf)\wedge\mathrm{d}f+\mathrm{d}f\wedge\mathrm{d}g &(2.24)\\
&= (g\,\mathrm{d}f+f\,\mathrm{d}g)\wedge\mathrm{d}f+\mathrm{d}f\wedge\mathrm{d}g &(2.25)\\
&= (f-1)\,\mathrm{d}g\wedge\mathrm{d}f &(2.26)
\end{aligned}$$

□

Problem 5. For any three 1-forms $\phi_i = \sum_j f_{ij}\,\mathrm{d}x_j$ $(1 \le i \le 3)$, prove

$$\phi_1\wedge\phi_2\wedge\phi_3 = \begin{pmatrix} f_{11} & f_{22} & f_{33} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,. \tag{2.27}$$

Solution:

$$\phi_1\wedge\phi_2\wedge\phi_3 = \sum_{i,j,k} f_{1i}f_{2j}f_{3k}\epsilon^{ijk}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{2.28}$$

$$\tag{2.29}$$

because repeats are zero so we need the antisymmetric product of all of the f_ij giving the determinant. □

Problem 6. If r, θ, z are the cylindrical coordinate functions on $\mathbf{R}^2$, then $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Compute the *volume element* $dx\ dy\ dz$ of $\mathbf{R}^3$ in cylindrical coordinates. (That is, express $dx\ dy\ dz$ in terms of the functions r, θ, z, and their differentials.)

Solution:

$$\begin{aligned}
\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= (\cos\theta\,\mathrm{d}r - r\sin\theta)(\sin\theta\,\mathrm{d}r + r\cos\theta)\,\mathrm{d}z &(2.30)\\
&= r(-\sin^2\theta\,\mathrm{d}r + \cos^2\theta\,\mathrm{d}r\,\mathrm{d}\theta)\,\mathrm{d}z &(2.31)\\
&= r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z &(2.32)\\
& &(2.33)
\end{aligned}$$

by the alternation rule. □

Problem 7. For a 2-form $\eta = f\,\mathrm{d}x\,\mathrm{d}y + g\,\mathrm{d}x\,\mathrm{d}z + h\,\mathrm{d}y\,dz$, the *exterior derivative $d\eta$* is defined to be the 3-form obtained by replacing f, g, and h by their differentials. Prove that for any 1-form ϕ, $\mathrm{d}\,(\mathrm{d}f) = 0$. Exercises 3 and 7 show that $\mathrm{d}^2 = 0$, that is, for any form ξ, $\mathrm{d}(\mathrm{d}\xi) = 0$. (If ξ is a 2-form, then $d(\mathrm{d}\xi) = 0$, since its degree exceeds 3.)

Solution: We have $df = \sum_i \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$. Then

$$\mathrm{d}(\mathrm{d}f) = \sum_i \mathrm{d}f_i \wedge \mathrm{d}x_i \tag{2.34}$$

$$= \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j \tag{2.35}$$

but the partial derivative is symmetric in i, j and the differentials are antisymmetric so this product is zero. □

Problem 8. Prove that all three operations may be expressed by exterior derivatives as follows:
(a) $df \leftrightarrow \boldsymbol{\nabla} f$.
(b) If $\phi \leftrightarrow V$, then $\mathrm{d}\phi \leftrightarrow \boldsymbol{\nabla} \times V$.
(c) If $\eta \leftrightarrow V$, then $\mathrm{d}\eta = (\boldsymbol{\nabla} \cdot V)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Solution: (a) This one works out by definition. Since $\mathrm{d}x_i$ is dual to U_i.
(b)

$$\mathrm{d}\phi = \sum_i \mathrm{d}f_i \wedge \mathrm{d}x_i \tag{2.36}$$

$$\tag{2.37}$$

(c) Using $\eta = f\,\mathrm{d}x\,\mathrm{d}y + g\,\mathrm{d}x\,\mathrm{d}z + h\,\mathrm{d}y\,\mathrm{d}z$,

$$\mathrm{d}\eta = \frac{\partial f}{\partial z}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial g}{\partial y}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z + \frac{\partial h}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2.38}$$

$$\tag{2.39}$$

and the equality holds using (2). □

Problem 9. Let f and g be real-valued functions on $\mathbf{R}^2.Provethat\,\mathrm{d}f \wedge \mathrm{d}g = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \mathrm{d}x\,\mathrm{d}y\,.(2.40)$
Solution:

$$\mathrm{d}f \wedge \mathrm{d}g = (\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y)(\frac{\partial g}{\partial x}\,\mathrm{d}x + \frac{\partial g}{\partial y}\,\mathrm{d}y) \tag{2.41}$$

$$= \frac{\partial f}{\partial x}\,\mathrm{d}x\,\frac{\partial g}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial y}\,\mathrm{d}y\,\frac{\partial g}{\partial x}\,\mathrm{d}x \tag{2.42}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \mathrm{d}x\,\mathrm{d}y \tag{2.43}$$

□

§2.7 Mappings

Problem 1. With $F(u,v) = (u^2 - v^2, 2uv)$, find all points $\mathbf{p}$ such that
(a) $F(\mathbf{p}) = (0,0)$ (b) $F(\mathbf{p}) = (8,6)$
(c) $F(\mathbf{p}) = \mathbf{p}$.

Solution: (a) $u, v) = (0,0)$
(b) $u, v) = (\pm 3, \pm 1)$
(c) $(u,v) = (1,0)$ □

Problem 2. (a)

Problem 3. With $F(u,v) = (u^2 - v^2, 2uv)$, let $\mathbf{v} = (v_1, v_2)$ be a tangent vector to $\mathbf{R}^2 at \mathbf{p} = (p_1, p_2)$. Apply Definition 7.4 directly to express $F * (\mathbf{v})$ in terms of the coordinates of $\mathbf{v}$ and $\mathbf{p}$.

Solution:

$$F * (\mathbf{v}) = (\mathbf{v}[u^2 - v^2], \mathbf{v}[2uv]) \tag{2.44}$$
$$= (2p_1v_1 - 2p_2v_2, 2p_1v_2 + 2p_2v_1) \tag{2.45}$$

□

Problem 4. With $F(u,v) = (u^2 - v^2, 2uv)$, find a formula for the Jacobian matrix of F at all points, and deduce that $F*_p$ is a linear isomorphism at every point of $\mathbf{R}^2 excepttheorigin$. *Solution:*

$$\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} \tag{2.46}$$

this matrix has rank 2 so given the conditions below Def. 7.9, the map $F*$ is one-to-one. So it is a linear isopmorphism. □

Problem 5. If $F : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, prove that $F * (\mathbf{v}_p) = F(\mathbf{v})_{F(p)}$.

Solution: I don't understand this notation... □

Problem 6. (a) Give an example to demonstrate that a one-to-one and onto mapping need not be a diffeomorphism. (Hint: Take m = n = 1.)
(b) Prove that if a one-to-one and onto mapping F: $\mathbf{R}^n \to \mathbf{R}^n$ is regular, then it is a diffeomorphism.

Solution: (a) Any mapping defined by $f(x) = x^\alpha$ for α is odd doesn't have smooth inverse at the origin but is still one-to-one and onto.
(b) If a mapping is one-to-one at every point and onto, then there is a well-defined inverse mapping making it a diffeomorphism. □

Problem 7. Prove that a mapping $F : \mathbf{R}^n \to \mathbf{R}^m$ preserves directional derivatives in this sense: If $\mathbf{v}_p$ is a tangent vector to $\mathbf{R}^n$ and g is a differentiable function on $\mathbf{R}^m$, then $F * (\mathbf{v}_p)[g] = \mathbf{v}_p[g(F)]$.

Solution:

$$
\begin{aligned}
\mathbf{v}_p[g(F)] &= \mathrm{d}(g(F))\,[\mathbf{p}] && (2.47)\\
&= \mathrm{d}g\,(\mathrm{d}F\,[\mathbf{p}]) && (2.48)\\
&\equiv F * (\mathbf{v}_p)[g] && (2.49)
\end{aligned}
$$

□

Problem 8. In the definition of tangent map (Def. 7.4), the straight line $t \to \mathbf{p} + t\mathbf{v}$ can be replaced by any curve a with initial velocity $\mathbf{v}_p$.

Solution: The function $F*$ gives the initial velocity of a curve $\alpha'(0)$. Any curve $\beta(t)$ with $\beta'(0)$ will have the same tangent vector and so won't affect the value of $F*$. □

Problem 9. Let $F : \mathbf{R}^n \to \mathbf{R}^m$ and $G : \mathbf{R}^m \to \mathbf{R}^p$ be mappings. Prove:
(a) Their composition $GF : \mathbf{R}^n \to \mathbf{R}^p$ is a (differentiable) mapping. (Take m = p = 2 for simplicity.)
(b) $(GF)* = G * F*$. (Hint: Use the preceding exercise.) This concise formula is the general chain rule. Unless dimensions are small, it becomes formidable when expressed in terms of Jacobian matrices.
(c) If F is a diffeomorphism, then so is its inverse mapping F^{-1}.

Solution: (a) A mapping is a function with differentiable coordinate functions. Since differentiation is well-defined for composite functions using chain rule, the composite function GF is a mapping.
(b) By definition,$F * (v) = \beta'(0)$. So,

$$
\begin{aligned}
(GF) * (\beta'(0)) &= (GF(\beta))'(0) && (2.50)\\
&= G * (F(\beta)'(0)) && (2.51)\\
&= G * F * (\beta'(0)) && (2.52)
\end{aligned}
$$

(c) If F is a diffeomorphism, then it has an inverse F^{-1}. The function F^{-1} has an inverse. Its tangent map also has an inverse by the previous exercise that maps only the zero vector to 0 (i.e. $F^{-1}F = I \to (F^{-1}F)* = I* \to F^{-1} * F* = I$). □

Problem 10. Show (in two ways) that the map $F : \mathbf{R}^2 \to \mathbf{R}^2$ such that $F(u, v) = (v\,\mathrm{e}^u, 2u)$ is a diffeomorphism:
(a) Prove that it is one-to-one, onto, and regular; (b) Find a formula for its inverse $F^{-1} : \mathbf{R}^2 \to \mathbf{R}^2$ and observe that F^{-1} is differentiable. Verify the formula by checking that both FF^{-1} and $F^{-1}F$ are identity maps.

Solution: (a) The map is one-to-one and onto since the coordinate functions are one-to-one and onto. The Jacobian matrix is rank two so the tangent map is one-to-one so the map is regular.
(b) Let $F^{-1} = (u\,e^{-u}/v, v/(2u))$. Then $FF^{-1} = F^{-1}F = I$. □

§2.8 Dot Product

Problem 1. Let $\mathbf{v} = (1, 2, -1)$ and $\mathbf{w} = (-1, 0, 3)$ be tangent vectors at a point of $\mathbf{R}^3$.
(a) $\mathbf{v} \cdot \mathbf{w}$ (b) $\mathbf{v} \times \mathbf{w}$
(c) $\mathbf{v}/||\mathbf{v}||$, $\mathbf{w}/||\mathbf{w}||$ (d)$||\mathbf{v} \times \mathbf{w}$
(e) the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

Solution: (a) $-1 + 0 - 3 = -4$.
(b) $6\mathbf{e_1} - 2\mathbf{e_2} + 2\mathbf{e_3}$.
(c) $\mathbf{v}/||\mathbf{v}|| = \mathbf{v}/\sqrt{6}$, $\mathbf{w}/||\mathbf{w}|| = \mathbf{w}/\sqrt{10}$
(d) $2\sqrt{10}$.
(e)$-4/\sqrt{60}$.

□

Problem 2. Prove that Euclidean distance has the properties
(a) $d(\mathbf{p}, \mathbf{q}) \geq 0; d(\mathbf{p}, \mathbf{q}) = 0$ if and only if $\mathbf{p} = \mathbf{q}$,
(b) $d(\mathbf{p}, \mathbf{q}) = d(\mathbf{q}, \mathbf{p})$,
(c) $d(\mathbf{p}, \mathbf{q}) + d(\mathbf{q}, \mathbf{r}) \geq d(\mathbf{p}, \mathbf{r})$, for any points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ in$\mathbf{R}^3$.

Solution: (a) The norm is the square root of the sum of squares. Squares in $\mathbf{R}^3$ are positive and taking the positive branch of the square root function, the result is proven. If $d(\mathbf{p}, \mathbf{q}) = 0$, then $(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2 = 0$ but since all squares are positive, each term much be zero, thus $\mathbf{p} = \mathbf{q}$. If $\mathbf{p} = \mathbf{q}$, from the definition of the norm $d(\mathbf{p}, \mathbf{q}) = 0$.
(b) The norm is symmetric under interchange of its arguments since the square of a difference is symmetric under interchange of its arguments.
(c) Squaring both sides gives, on the right $\mathbf{p} \cdot \mathbf{p} - 2\mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{q} - 2\mathbf{q} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}$, and on the left $\mathbf{p} \cdot \mathbf{p} - 2\mathbf{p} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}$. The difference is $2\mathbf{q} \cdot \mathbf{q} - 2(\mathbf{p} + \mathbf{r}) \cdot \mathbf{q} + 2\mathbf{p} \cdot \mathbf{r} = 2(\mathbf{p} \cdot (\mathbf{r} - \mathbf{q}) + \mathbf{q} \cdot (\mathbf{q} - \mathbf{r})) = 2(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{r} - \mathbf{q})$. In the minimal case, the points are on a line. Then using $\mathbf{r} = \mathbf{q} + \mathbf{p}$ gives zero. □