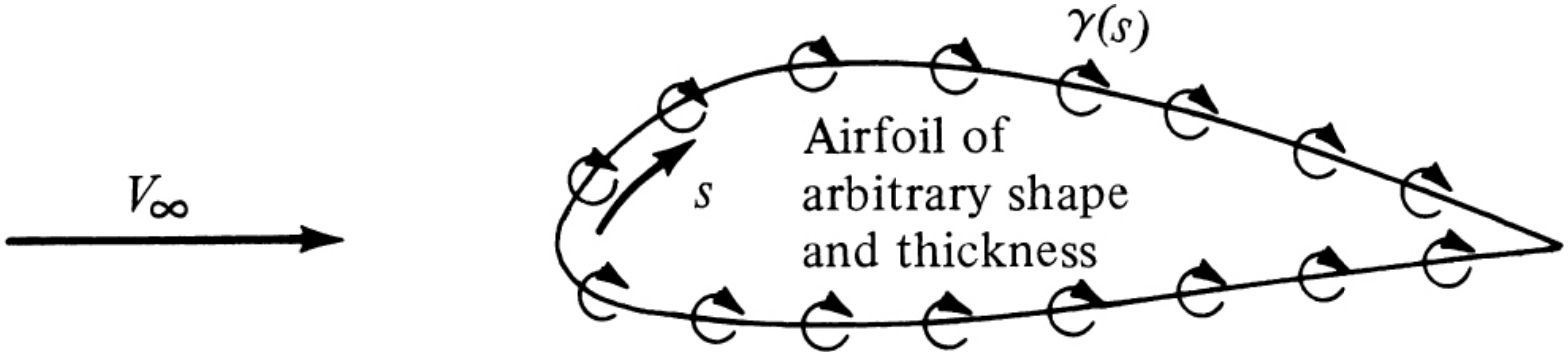
Computing Lifting Flow: The Vortex Panel Method

ASEN 3111

How Do We Induce Lifting Flow Over Aerodynamic Bodies?

Using a *vortex sheet*!



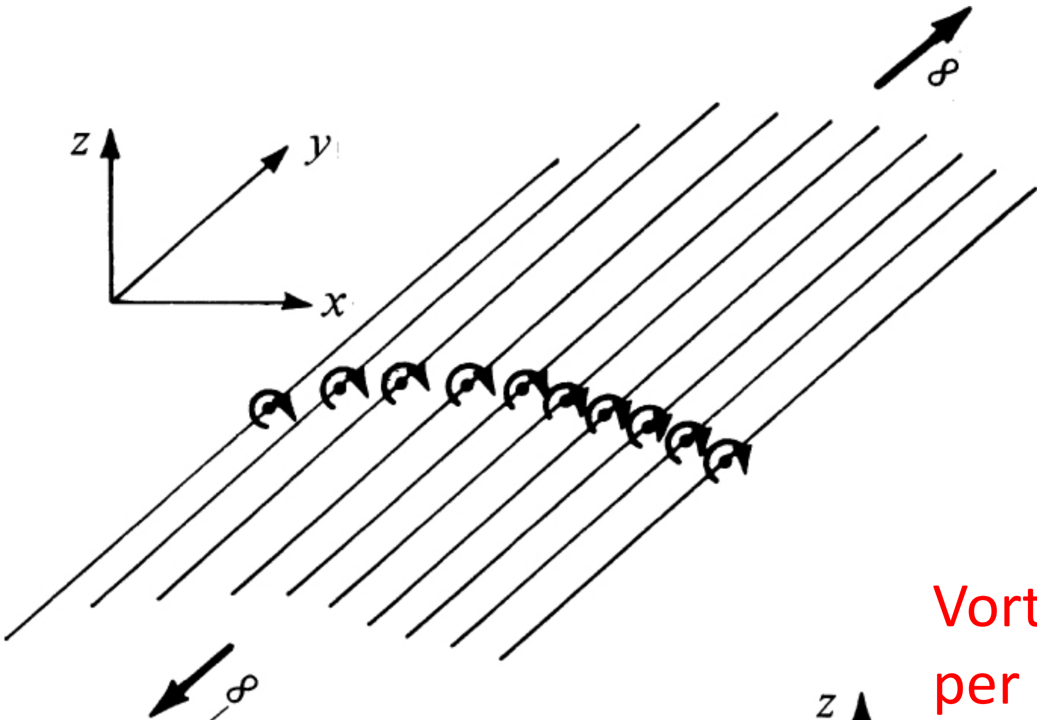
Objective: Add a uniform flow and a vortex sheet on a body of given shape to make shape a streamline subject to the Kutta condition.

This requires finding the ***right*** *value for the sheet strength.*

The Vortex Sheet

Total Potential:

$$\phi(x,y) = -\int_a^b \frac{\gamma(s)ds}{2\pi}\theta$$



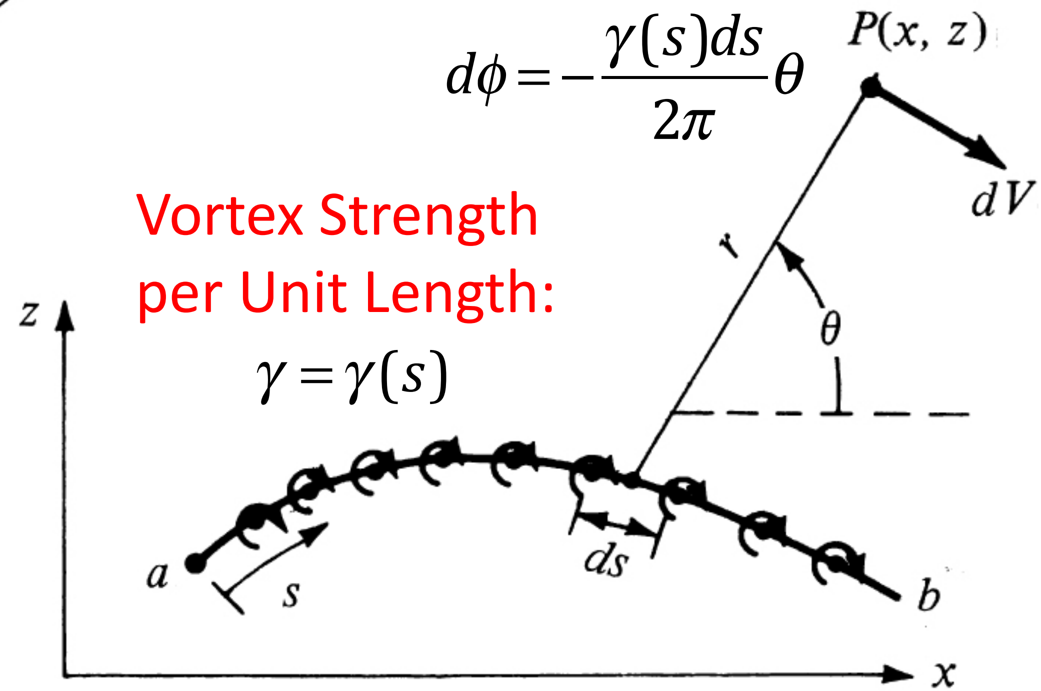
Vortex sheet in perspective

Induced Potential:

$$d\phi = -\frac{\gamma(s)ds}{2\pi}\theta$$

Vortex Strength per Unit Length:

$$\gamma = \gamma(s)$$



Edge view of sheet

The Vortex Sheet "Problem"

Problem: Find the sheet strength $\gamma(s)$ such that:

$$\mathbf{V}\cdot\mathbf{n}=\mathbf{V}_{\infty}\cdot\mathbf{n}+\frac{\partial\phi}{\partial n}=0$$

along the airfoil surface and the Kutta condition:

$$\gamma(\mathrm{TE})=0$$

is satisfied.

Total Potential:

$$\phi(x,y)=-\oint_{C}\frac{\gamma(s)ds}{2\pi}\theta$$

The Vortex Sheet “Problem”

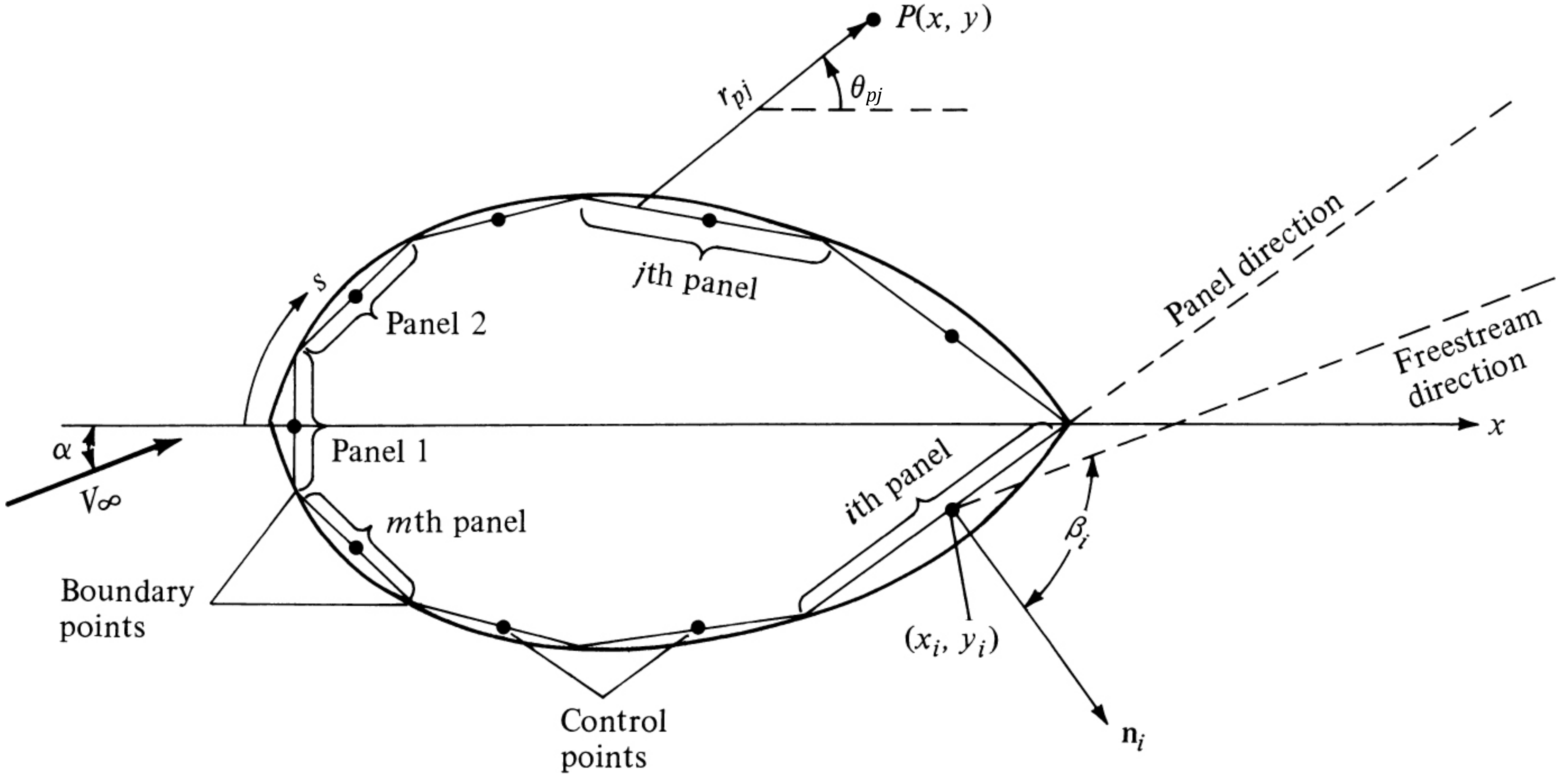
To solve this problem numerically, we must:

Step 1. Discretize the geometry.
Step 2. Discretize the solution field.
Step 3. Discretize the governing equation

This forms the basis of the *vortex panel method*.

Step 1: Discretize the Geometry

Our first step involves discretizing the geometry into *panels*:



Step 2: Discretize the Solution Field

Our second step involves discretizing the *solution field*.

A first option would be to approximate the vortex strength per unit length as *constant* over each panel. This, coupled with our prior geometry discretization, yields the approximation:

$$\phi\left(x,y\right)=-\oint_{C}\frac{\gamma(s)ds}{2\pi}\theta\approx-\sum_{j=1}^{m}\frac{\gamma_{j}}{2\pi}\int_{j}\theta_{pj}ds_{j}$$

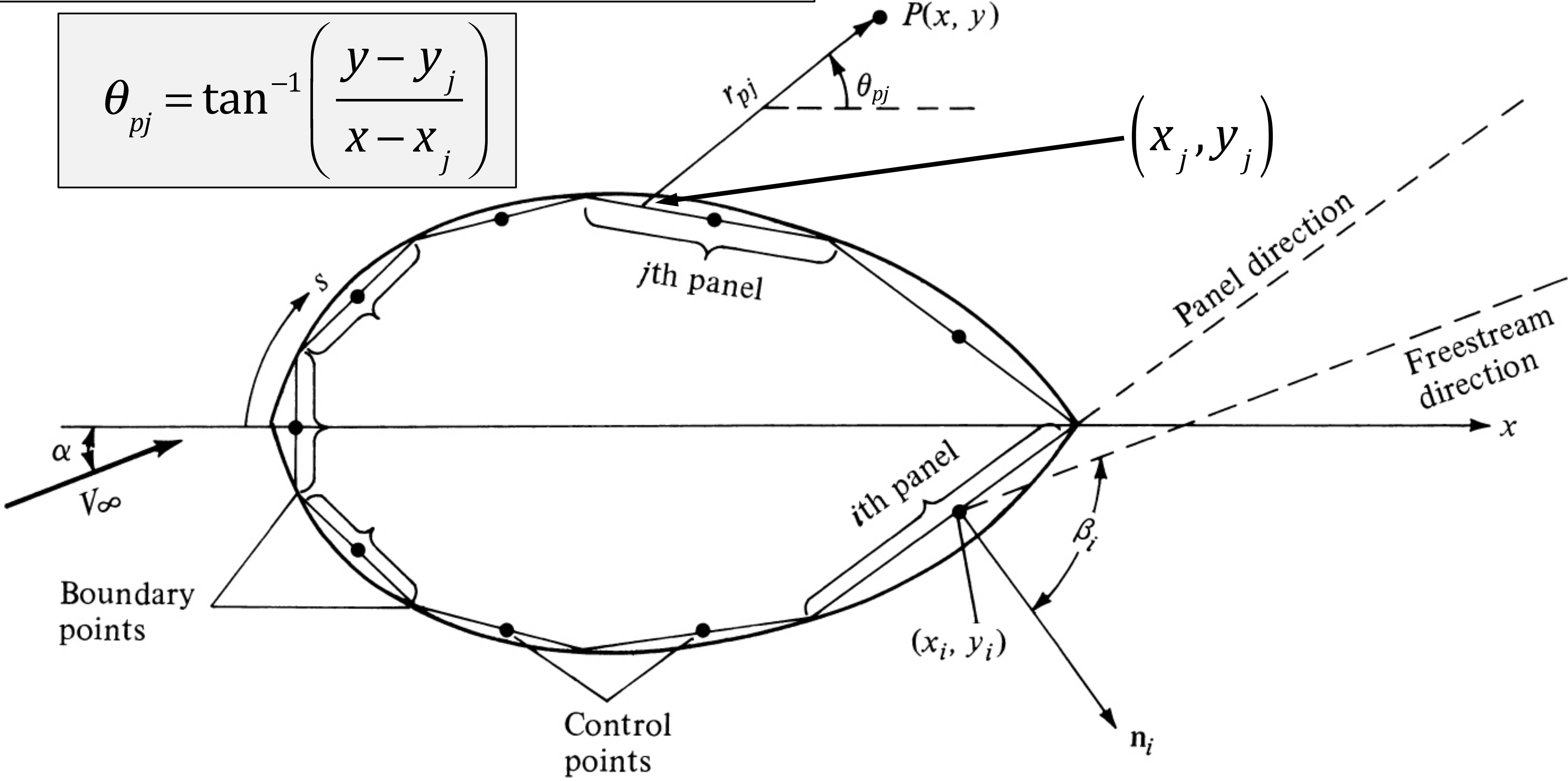
where:

$$\theta_{pj}=\tan^{-1}\left(\frac{y-y_{j}}{x-x_{j}}\right)$$

Step 2: Discretize the Solution Field

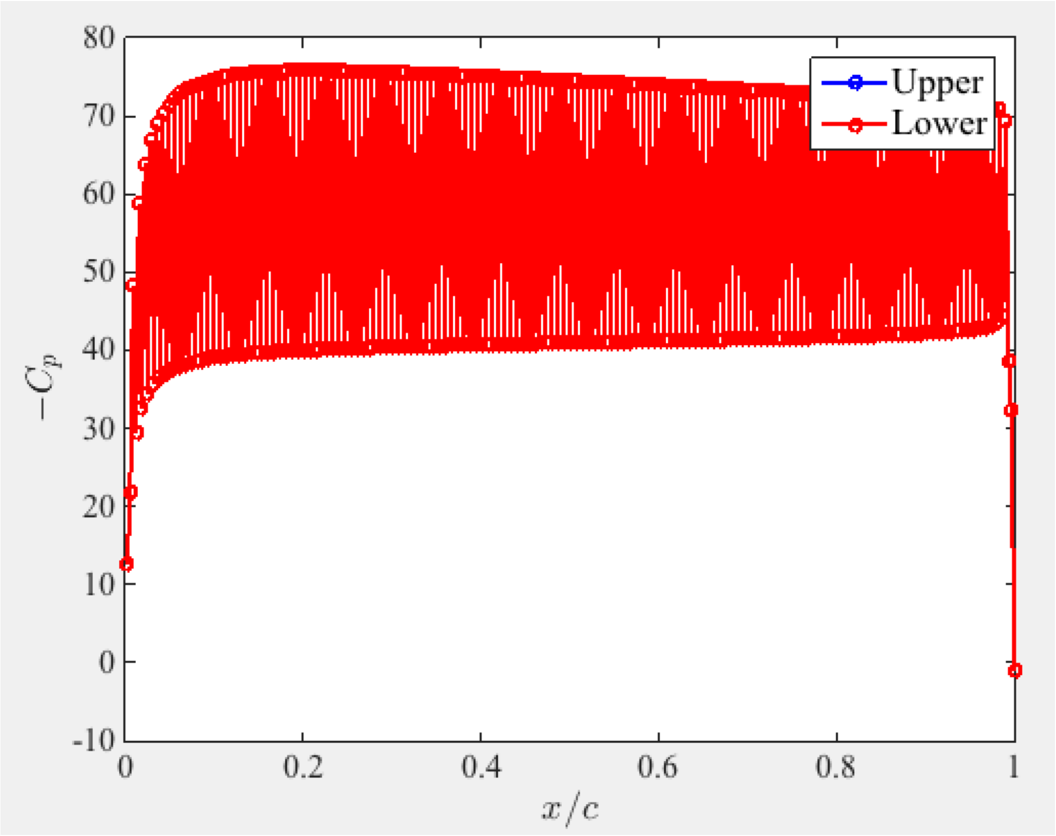
$$\phi(x,y) = -\oint_C \frac{\gamma(s)ds}{2\pi}\theta \approx -\sum_{j=1}^{m} \frac{\gamma_j}{2\pi}\int_j \theta_{pj} ds_j$$

$$\theta_{pj} = \tan^{-1}\left(\frac{y - y_j}{x - x_j}\right)$$



Step 2: Discretize the Solution Field

Unfortunately, this approach yields *highly unstable* numerical results:



Coefficient of Pressure for a NACA 0012 Airfoil at 0° AOA

Step 2: Discretize the Solution Field

Unfortunately, this approach yields *highly unstable* numerical results:

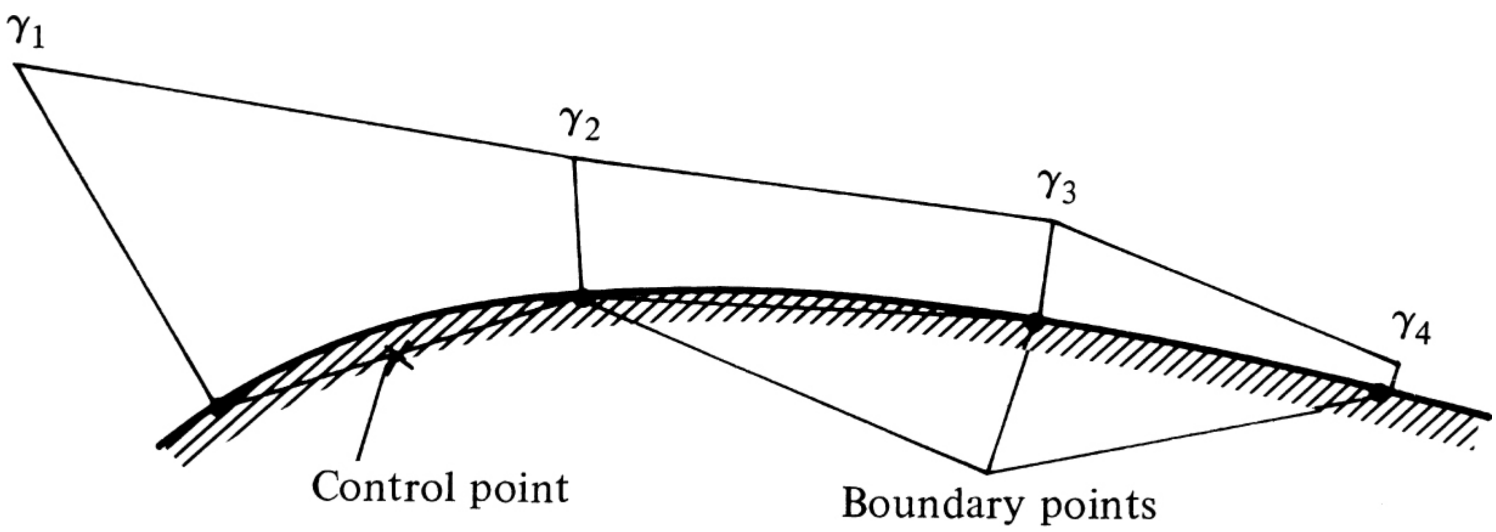
Per Anderson…

> Moreover, the resulting numerical distributions for γ are not always smooth, but rather, they have oscillations from one panel to the next as a result of numerical inaccuracies.

No kidding!

Step 2: Discretize the Solution Field

An alternative option for discretizing the *solution field* is to approximate the vortex strength per unit length as *linearly varying* over each panel:



This yields a more accurate (and stable!) *second-order* approximation.

Step 2: Discretize the Solution Field

As the trailing edge is not smooth, we employ two different values of strength there. Also, it will be convenient to start our “numbering” scheme there.

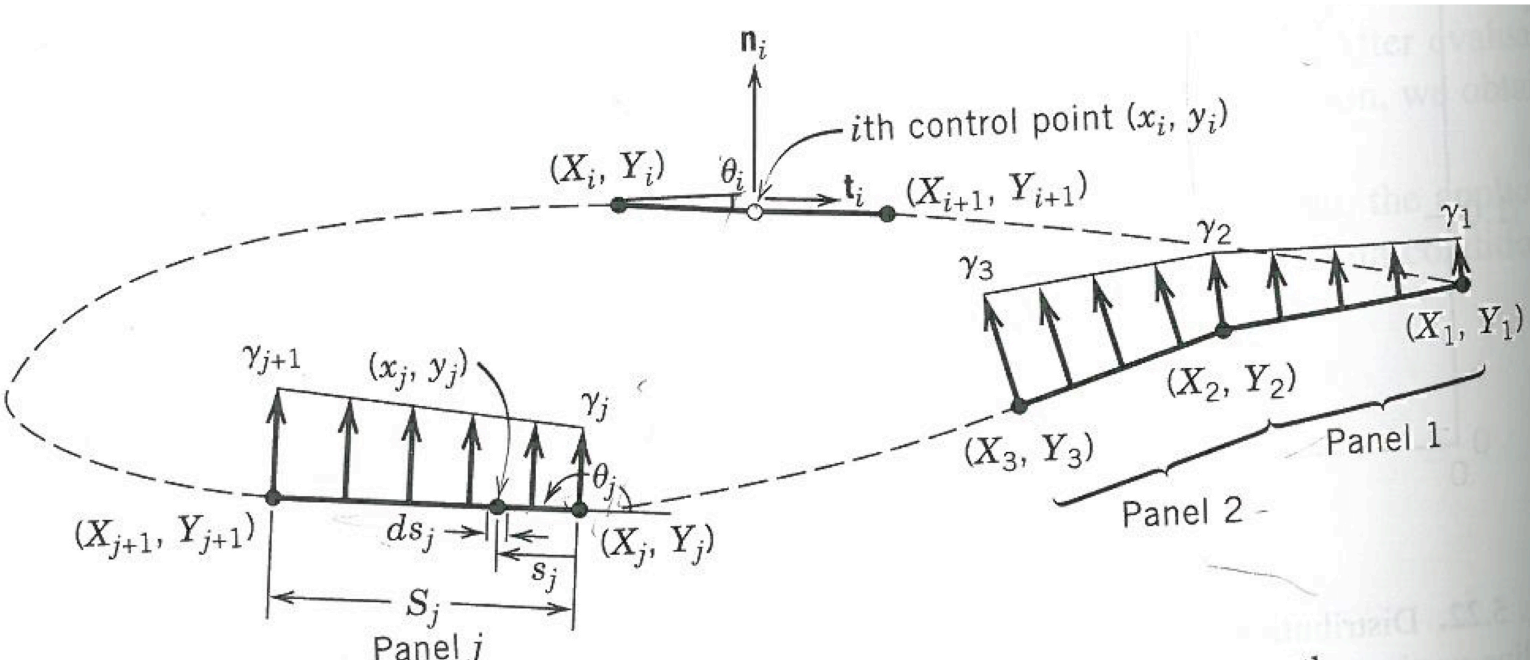


Fig. 5.23. Replacement of an airfoil by vortex panels of linearly varying strength.

Step 2: Discretize the Solution Field

As the trailing edge is not smooth, we employ two different values of strength there. Also, it will be convenient to start our "numbering" scheme there.

Moreover, as a consequence of having two values of strength at the trailing edge, there are a total of $m + 1$ unknowns where m is the number of panels:

$$\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_j, \ldots, \gamma_m, \gamma_{m+1}$$

Strength Associated with jth Boundary Point

Step 2: Discretize the Solution Field

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$$\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_j, \ldots, \gamma_m, \gamma_{m+1}$$

Strengths Associated with Trailing Edge

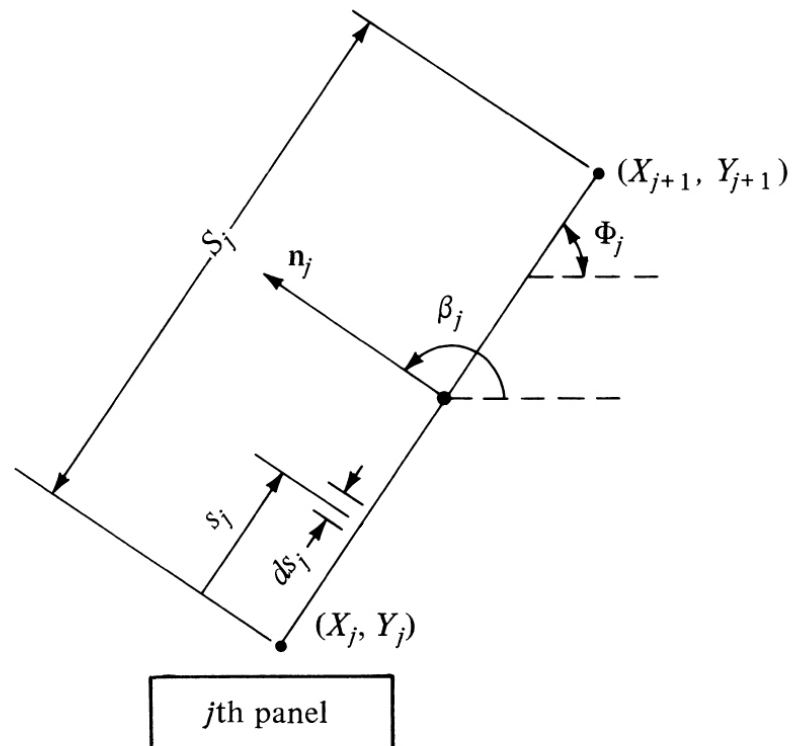
Step 2: Discretize the Solution Field

Mathematically, we have:

$$\phi\left(x,y\right)=-\oint_C \frac{\gamma(s)ds}{2\pi}\theta \approx -\sum_{j=1}^{m}\int_j \frac{\gamma(s_j)}{2\pi}\theta_{pj} ds_j$$

where:

$$\gamma(s_j)=\gamma_j+\left(\gamma_{j+1}-\gamma_j\right)\frac{s_j}{S_j}$$



Step 3: Discretize the Governing Equation

Our last step involves discretizing the *governing equations*. We choose to enforce the integral equation at *control points* lying in the middle of each panel. This yields, after some algebra and simplifications:

$$V_\infty \cos\beta_i - \sum_{j=1}^{m} \int_j \frac{\gamma(s_j)}{2\pi} \frac{\partial}{\partial n_i}\left(\theta_{ij}\right) ds_j = 0$$

$$\text{for } i = 1,\ldots,m$$

Angle Between Freestream and Normal

Control Point Location

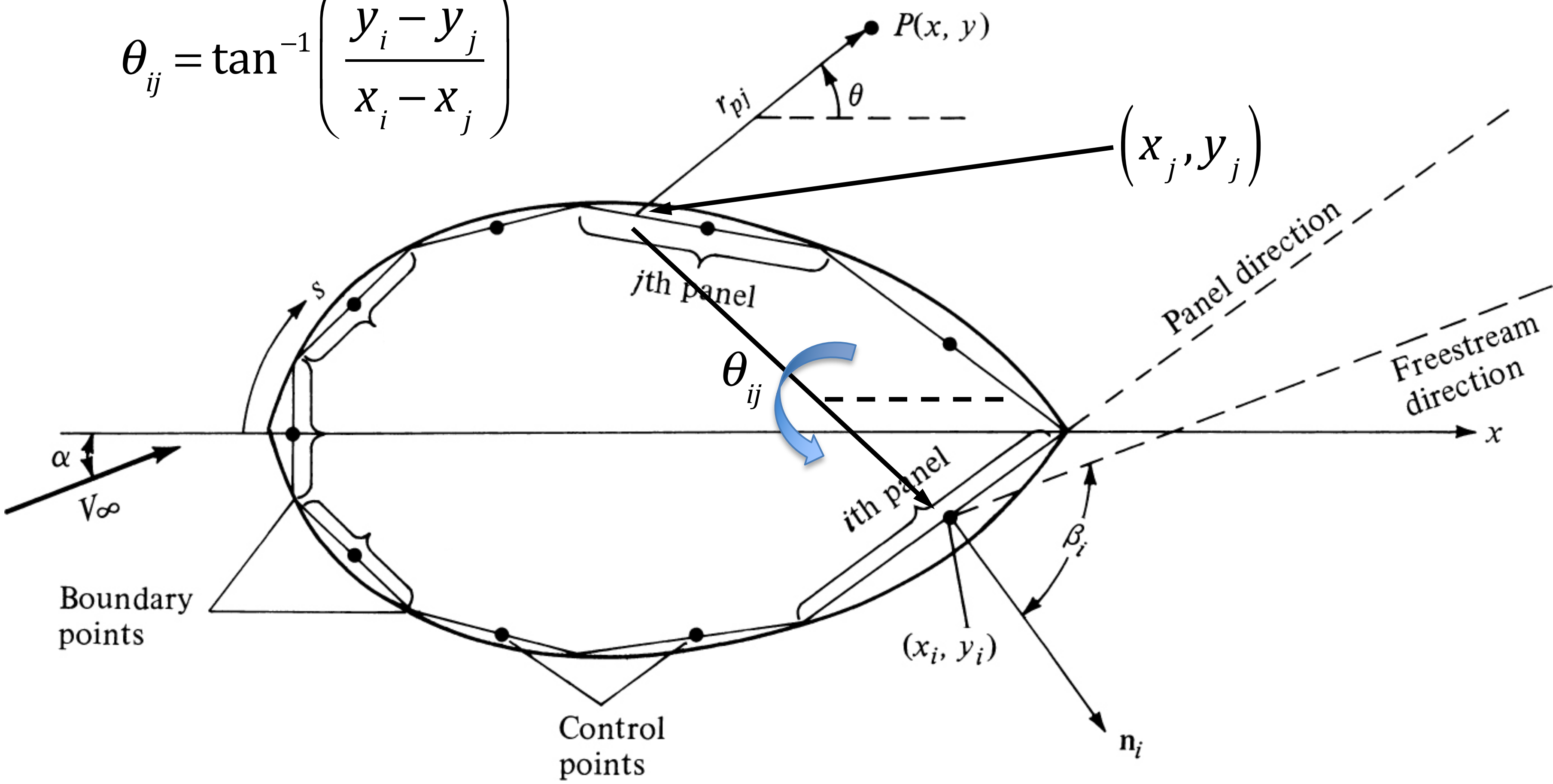
where:

$$\theta_{ij} = \tan^{-1}\left(\frac{y_i - y_j}{x_i - x_j}\right)$$

Step 3: Discretize the Governing Equation

$$V_\infty \cos\beta_i - \sum_{j=1}^{m} \int_j \frac{\gamma(s_j)}{2\pi} \frac{\partial}{\partial n_i}\left(\theta_{ij}\right) ds_j = 0 \quad \text{for } i = 1,\ldots,m$$

$$\theta_{ij} = \tan^{-1}\left(\frac{y_i - y_j}{x_i - x_j}\right)$$



Step 3: Discretize the Governing Equation

We also need to enforce the *Kutta condition*. To ensure a smooth flow at the trailing edge, we need to enforce that the two values of the vortex strength at the trailing edge are equal in magnitude and opposite in sign:

$$\gamma_1 + \gamma_{m+1} = 0$$

Solving the Linear System of Equations

The system:

$$V_\infty \cos\beta_i - \sum_{j=1}^{m} \int_j \frac{\gamma(s_j)}{2\pi} \frac{\partial}{\partial n_i}\left(\theta_{ij}\right) ds_j = 0$$

$$\text{for } i = 1,\ldots,m$$

$$\gamma_1 + \gamma_{m+1} = 0$$

constitutes m + 1 equations for our m + 1 unknowns. We can thus repose the above as a matrix system:

$$\mathbf{Ax} = \mathbf{b}$$

Solving the Linear System of Equations

The matrix components are equal to:

$$\mathbf{A}_{ij} = \begin{cases} \left(I_{i,j-1} + J_{i,j}\right)/\left(2\pi\right) & \text{if} \quad i < m+1 \\ 1 & \text{if} \quad i = m+1 \text{ and } j = 1, m+1 \\ 0 & \text{if} \quad i = m+1 \text{ and } j \neq 1, m+1 \end{cases}$$

where:

$$I_{i,j} = \begin{cases} \int_j \frac{s_j}{S_j} \frac{\partial}{\partial n_i}\left(\theta_{ij}\right) ds_j & \text{if } 1 \le j \le m \\ 0 & \text{otherwise} \end{cases} \qquad J_{i,j} = \begin{cases} \int_j \left(1 - \frac{s_j}{S_j}\right) \frac{\partial}{\partial n_i}\left(\theta_{ij}\right) ds_j & \text{if } 1 \le j \le m \\ 0 & \text{otherwise} \end{cases}$$

Solving the Linear System of Equations

while the vector components are equal to:

$$\mathbf{x}_j = \gamma_j$$

$$\mathbf{b}_i = \begin{cases} V_\infty \cos\beta_i & \text{if } i < m+1 \\ 0 & \text{otherwise} \end{cases}$$

Solving the Linear System of Equations

Evaluation of the integrals listed above is rather involved and contained in the supplemental document:

"VortexPanelFromKuetheAndChow.pdf"

As the name indicates, this document is a scan from the classic Aerodynamics text by *Kuethe and Chow*.

This scan also contains a *Fortran implementation* of the second-order vortex panel method which may serve as a guide for your own implementation.

Postprocessing: Velocity and Pressure

Finally, after solving for the sheet strength, one can obtain the tangential velocity at each control point by:

$$V_i = V_\infty \sin\beta_i - \sum_{j=1}^{m} \int_j \frac{\gamma(s_j)}{2\pi} \frac{\partial}{\partial s}\left(\theta_{ij}\right) ds_j$$

and the corresponding pressure coefficient by:

$$C_{p,i} = 1 - \left(\frac{V_i}{V_\infty}\right)^2$$

The integrals above, as before, can be evaluated explicitly, as detailed in the scan from Kuethe and Chow.

Postprocessing: Coefficient of Lift

After obtaining the tangential velocity, the circulation may be obtained via the equation:

$$\Gamma = -\oint_C \mathbf{V} \cdot \mathbf{ds} \approx \sum_{j=1}^{m} V_j S_j$$

and the sectional coefficient of lift via:

$$c_l = \frac{L'}{q_\infty c} = \frac{\rho_\infty V_\infty \Gamma}{0.5 \rho_\infty V_\infty^2 c} = \frac{2\Gamma}{V_\infty c}$$

Note: The circulation may also be directly obtained from the sheet strength!

Last Words of Caution

Many of the expressions in Kuethe and Chow are given in *dimensionless* form. This is quite powerful, but you need to be careful to convert back and forth when working with both ***dimensional*** (as presented here) and ***dimensionless*** (as in Kuethe and Chow) variables.

Besides that, the TA, TFs, and I will be available to help the next three weeks for this lab!