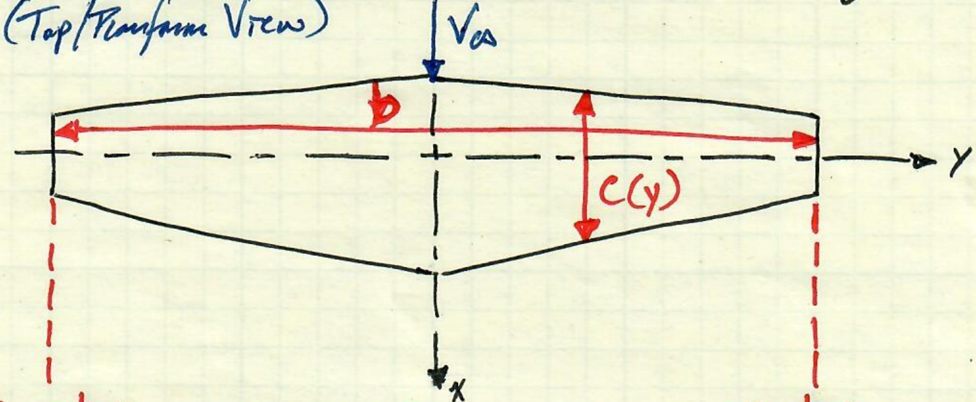
During lecture you derived the general Prandtl Lifting Line Theory which predicts the influence of a General Finite Wing Planform on the total wing Lift Coefficient and Induced Drag Coefficient.

↳ For Computational Assignment 4 we will apply this theory in a discrete computational framework within Matlab.

- Reference: Anderson "Fund. of Aerodynamics" (5th Ed.) Section 5.3.2 - General Lift Distribution

Assume we have some general wing planform,

(Top/Planform View)



$y=-b/2$ | $y=0$ | $y=b/2$

$\theta=\pi$ | $\theta=\pi/2$ | $\theta=0$

↳ Apply following transformation to spanwise position

$$y = +\frac{b}{2}\cos\theta$$ ← Note Anderson defines this with $\theta=0$ @ $y=-b/2$!!

$$y=-\frac{b}{2}\cos\theta$$

so,

$$\theta = \cos^{-1}\left(2\frac{y}{b}\right)$$

Let the wing geometry vary as a function of the wing spanwise position:

1) $c(y) \Rightarrow c(\theta)$ - Chord length varies with wing spanwise position in either y or θ.

Note: $c(y=0) = c(\theta=\pi/2) = c_{root}$

$c(y=\pm b/2) = c(\theta=0,\pi) = c_{tip}$

2) $\alpha(y) \Rightarrow \alpha(\theta)$ - Geometric Angle of Attack

Note: This includes both the physical angle of attack of the wing/plane and any geometric twist applied to the wing!

3) $\alpha_{L=0}(y) \Rightarrow \alpha_{L=0}(\theta)$ - Aerodynamic Zero-Lift Angle of Attack

Note: This accounts for changes in the zero-lift angle of attack caused by changing the airfoil profile/section across the wing!

4) $a_0(y) \Rightarrow a_0(\theta)$ - Aerodynamic Lift-Slope

Note: This accounts for changes in the lift-slope due to changes in the airfoil thickness across the span!

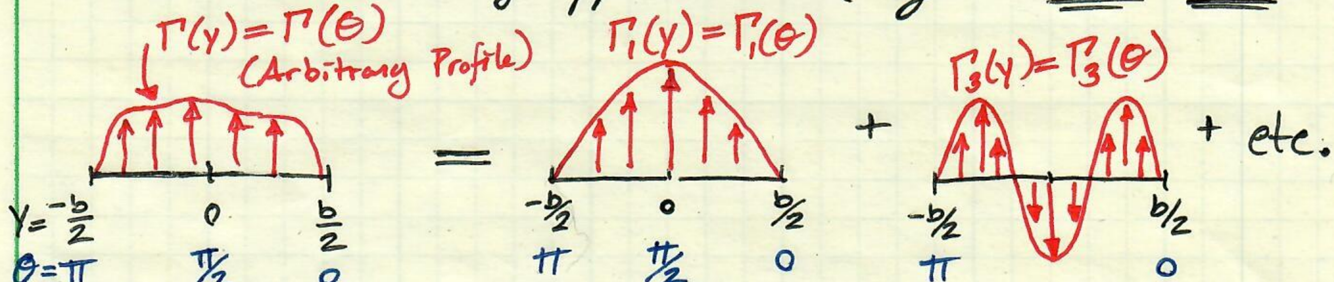
↳ From Thin Airfoil Theory: $a_0 = 2\pi$

- If we assume the wing is composed of only thin sections then:

$$a_0(y) = \text{Constant} = 2\pi$$

5) $\Gamma(y) \Rightarrow \Gamma(\theta)$ - Bound Circulation

We assume the Bound Circulation can vary across the wing span and that the Arbitrary Profile/Distribution can be Mathematically approximated by a Fourier Series!



$$\Gamma(\theta) = \Gamma_1(\theta) + \Gamma_3(\theta) = 2bV_\infty A_1 \sin\theta + 2bV_\infty A_3 \sin 3\theta + \text{etc.}$$

if $N = 4$ $\quad j = \{1, 2, 3, 4\}$

$$\Gamma(\theta) = 2bV_\infty \sum_{j=1}^{N} A_{(2j-1)} \sin((2j-1)\theta)$$

1 2 3 4
1, 3, 5, 7

{1, 3, 5, 7

Now to solve for these "weighting" coefficients (i.e. $A_1, A_3, \ldots, A_{2N-1}$) we need to abide by the following equality:

$$\alpha_{geometric} = \alpha_{effective} + \alpha_{induced}$$

$$\alpha(\theta) = \frac{4b}{a_0(\theta)\, c(\theta)} \sum_{n=1}^{\infty} A_n \sin(n\theta) + \alpha_{L=0}(\theta) + \sum_{n=1}^{\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)}$$

Change from Anderson

Ideally we want to solve this for every (∞) spanwise positions along the wing, but to make the problem tractable we will instead solve this at N control point locations:

$$\theta_i = \frac{i\pi}{2N} \quad \text{where} \quad i = 1, 2, \ldots, N$$

We also limit the above "summations" to N unknown weightings in the fourier series:

$$A_{2j-1} = \underbrace{A_1, A_3, \ldots A_{2N-1}}_{N \text{ odd modes!!}} \quad \text{where} \quad j = 1, 2, \ldots, N$$

Or for an $N=2$ system we have:

$$\underbrace{\begin{bmatrix} \alpha(\theta_1) - \alpha_{L=0}(\theta_1) \\ \alpha(\theta_2) - \alpha_{L=0}(\theta_2) \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} \frac{4b}{a_0(\theta_1)c(\theta_1)}\sin(\theta_1) + 1 & \frac{4b}{a_0(\theta_1)c(\theta_1)}\sin(3\theta_1) + \frac{3\sin(3\theta_1)}{\sin(\theta_1)} \\ \frac{4b}{a_0(\theta_2)c(\theta_2)}\sin(\theta_2) + 1 & \frac{4b}{a_0(\theta_2)c(\theta_2)}\sin(3\theta_2) + \frac{3\sin(3\theta_2)}{\sin(\theta_2)} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} A_1 \\ A_3 \end{bmatrix}}_{x}$$

Solve the linear system: $\mathbf{A}x = b$ for x !!

Now that you have solved for x (or $A_1, A_3, \ldots, A_{2N-1}$), ~~then~~ you can compute:

$$C_L = A_1 \pi AR \longleftarrow AR = \frac{b^2}{S}$$

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1+\delta) = \frac{C_L^2}{\pi e AR}$$

Span Efficiency Factor!!

where

$$\delta = \sum_{j=2}^{N} (2j-1)\left(\frac{A_{(2j-1)}}{A_1}\right)^2$$

$$e = \frac{1}{1+\delta}$$

Note we choose odd A coefficients because they produce symmetric contributions to $\Gamma(\theta)$ that are consistent with level cruise.

When might we want to consider even A coefficients?