Transshipment graph

S is the set of ships that sails between ports in the set P. The port visits takes place between now (day 0) and a given day $\mathbf{d} \in \mathbb{N}$; we let $D := \{0, 1, \dots, \mathbf{d}\}$ denote these day numbers. Each ship $s \in S$ has a given route of ports that it visits in succession. This route is given by a sequence

$$r_s = ((p_1^s, d_1^s), (p_2^s, d_2^s), \dots, (p_{l_s}^s, d_{l_s}^s)) \in (P \times D)^{l_s},$$

where the i'th element in the sequence, $r_s(i) := (p_i^s, d_i^s)$, indicates the i'th port in the route and the day it is visited. We then define the set of *visitation nodes*, as $V := \bigcup_{s \in S} \{ r_s(i) \mid 0 \le i \le l_s \}$.

Given the routes of the ships, we can define the sequence of visitation nodes at a port $p \in P$ by ordering the set $W_p := \bigcup_{s \in S} \bigcup_{1 \le i \le l_s} \{ r_s(i) \mid r_s(i) = (p, d) \text{ for a } d \in D \}$ according to the day component[1]. For $1 \le i \le |W_p|$, we let $v_p(i)$ denote the i'th elements in this sequence, and we let $day_p(i)$ denote the day-component of $v_p(i)$.

Given the visitation nodes V, we now define the *transshipment graph* $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ as follows.

For each $v \in V$ we create the following set of nodes, $\mathcal{N}_v := \{n_{\mathtt{si}}^v, n_{\mathtt{so}}^v, n_{\mathtt{yi}}^v, n_{\mathtt{yo}}^v, n_{\mathtt{on}}^v, n_{\mathtt{off}}^v\}$.We then we create a set of edges, $\mathcal{E}_v = \{e_{\mathtt{unld}}^v, e_{\mathtt{ship}}^v, e_{\mathtt{load}}^v, e_{\mathtt{yard}}^v, e_{\mathtt{on}}^v, e_{\mathtt{off}}^v\}$, where

$$\begin{aligned} e_{\mathtt{unld}}^v &= (n_{\mathtt{si}}^v, n_{\mathtt{yo}}^v), & e_{\mathtt{ship}}^v &= (n_{\mathtt{si}}^v, n_{\mathtt{so}}^v), & & e_{\mathtt{load}}^v = (n_{\mathtt{yi}}^v, n_{\mathtt{so}}^v), \\ e_{\mathtt{yard}}^v &= (n_{\mathtt{yi}}^v, n_{\mathtt{yo}}^v), & e_{\mathtt{on}}^v &= (n_{\mathtt{on}}^v, n_{\mathtt{yi}}^v), & \text{and} & \quad e_{\mathtt{off}}^v = (n_{\mathtt{yo}}^v, n_{\mathtt{off}}^v); \end{aligned}$$

see Figure 1. The idea is, that $n_{\mathtt{si}}^v$ denotes the ship (giving rise to the visitation node $v = (p, d)$) when it arrives at

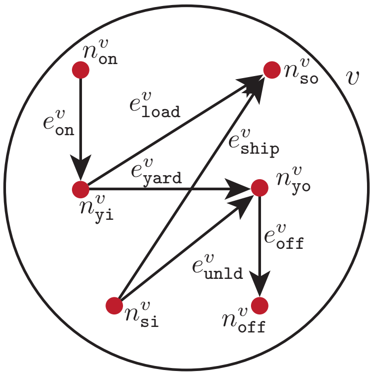

Figure 1: The nodes $\mathcal{N}_v$ and edges $\mathcal{E}_v$ created for the visitation node $v \in V$.

port p at day d ("$\mathtt{si}$" stands for "ship in"), while $n_{\mathtt{so}}^v$ denotes the ship when it leaves ("$\mathtt{so}$" stands for "ship out"). $n_{\mathtt{yi}}^v$ denotes the yard at the port just prior to the ship's arrival, while $n_{\mathtt{yo}}^v$ denotes the yard at the ship's departure. From the yard, containers can be loaded to the ship, corresponding to the edge $e_{\mathtt{load}}^v$, or it can stay on the yard, corresponding to $e_{\mathtt{yard}}^v$. From the ship, containers can be unloaded ($e_{\mathtt{unld}}^v$), or they can stay on the ship ($e_{\mathtt{ship}}^v$). Finally, containers stemming from bookings can be introduced to the yard from "outside" just prior to the ships arrival ($e_{\mathtt{on}}^v$), while others can be removed from the yard when the ships departure ($e_{\mathtt{off}}^v$).

To model the routes of the ships, we firstly create a distinguished "start node", $n_{\mathtt{start}}^s$ and a distinguished "end node", $n_{\mathtt{end}}^s$ for each $s \in S$, and we create edges $e_0^s = (n_{\mathtt{start}}^s, n_{\mathtt{si}}^{r_s(1)})$ and $e_{l_s}^s = (n_{\mathtt{so}}^{r_s(l_s)}, n_{\mathtt{end}}^s)$. We then "connect" the route by creating edges $e_i^s = (n_{\mathtt{so}}^{r_s(i)}, n_{\mathtt{si}}^{r_s(i+1)})$ for each $1 \le i < l_s$; see Figure 2. We let $\mathcal{E}_{\mathtt{route}} = \bigcup_{s \in S} \{ e_i^s \mid 0 \le i \le l_s \}$.

To model the possibility of transshipments, we similarly create a node $n_{\mathtt{start}}^p$ and a node $n_{\mathtt{end}}^p$ for all $p \in P$. We define $e_0^p = (n_{\mathtt{start}}^p, n_{\mathtt{yi}}^{v_p(1)})$ and $e_{|W_p|}^p = (n_{\mathtt{yo}}^{v_p(|W_p|)}, n_{\mathtt{end}}^p)$, and we further define $e_i^p := (n_{\mathtt{yo}}^{v_p(i)}, n_{\mathtt{yi}}^{v_p(i+1)})$ for each $1 \le i < |W_p|$.

[1] For simplicity, we assume that $r_s(i) \ne r_{s'}(j)$ for any $s \ne s'$ and any i, j. Otherwise, each elements in V are augmented with the ship s that makes the port call, and an ordering on S is assumed (and used for ordering W_p), too.

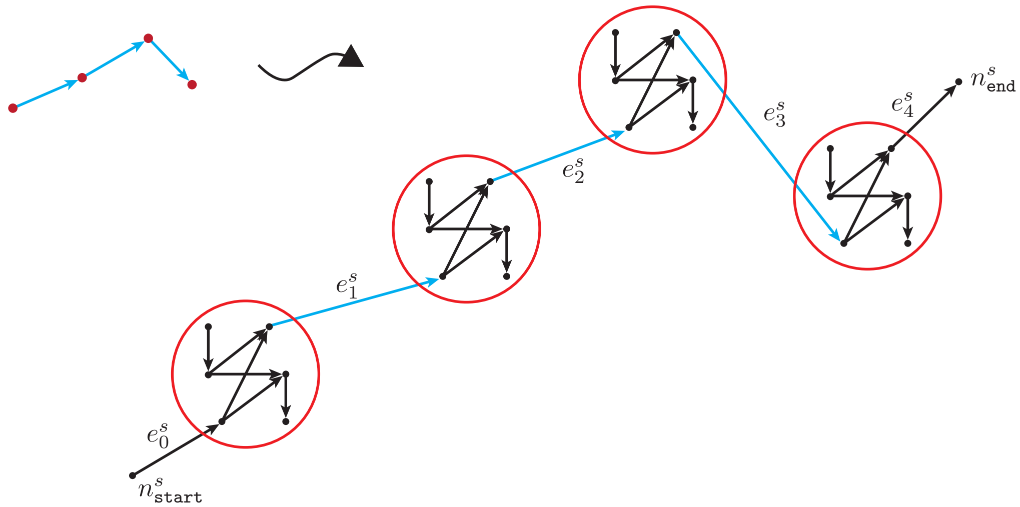

Figure 2: Assuming that s's entire route consists of four visitation nodes as depicted in the top left corner, then the corresponding nodes and edges in $\mathcal{G}$ made from this route are depicted in the rest of the figure.

These constitute all the nodes and edges in $\mathcal{G}$, i.e.

$$\mathcal{N} = \bigcup_{v \in V} \mathcal{N}_v \,\dot\cup\, \bigcup_{s \in S} \{n_{\texttt{start}}^s, n_{\texttt{end}}^s\} \,\dot\cup\, \bigcup_{p \in P} \{n_{\texttt{start}}^p, n_{\texttt{end}}^p\}, \text{ and}$$

$$\mathcal{E} = \bigcup_{v \in V} \mathcal{E}_v \,\dot\cup\, \mathcal{E}_{\texttt{route}} \,\dot\cup\, \bigcup_{p \in P} \{\, e_i^p \mid 0 \le i \le |W_p| \,\}.$$

Model

Parameters

Γ	Set of types of containers
$\Gamma_{20}, \Gamma_{40}, \Gamma_{\text{R}} \subseteq \Gamma$	Set of container types with length $20'$, length $40'$, and that are reefers, respectively.
$C_{\text{M}}^{\tau,p} \in \mathbb{R}$	The cost of moving (loading or unloading) a container of type $\tau \in \Gamma$ in port $p \in P$.
$C_{\text{L}}^{\tau,p} \in \mathbb{R}$	The cost per day of having a container of type $\tau \in \Gamma$ lying at port $p \in P$.
$R^{\tau,p,p'} \in \mathbb{R}$	The revenue for transporting a container of type $\tau \in \Gamma$ from port $p \in P$ to port $p' \in P$.
CN	Set of constraints on the relationship between the number of each type of containers in a feasible stowing. Each $c \in C$ is a function $c : \Gamma \cup \{\texttt{rhs}\} \to \mathbb{R}$, where $c(\tau)$ is the coefficient for containers of the type τ, and $c(\texttt{rhs})$ is the right-hand-side of the less-or-equal-to constraint (see later).

Variables The decision variables are

$$x_e^{\tau,v} \in \mathbb{R}^+ \qquad \text{for all } \tau \in \Gamma, v \in V, e \in \mathcal{E}.$$

This denotes the number of containers of type τ that should be transported to port p at time d that "flows" on the edge e.

Likewise we define the auxiliary variables

$$w_e^\tau = \sum_{v \in V} x_e^{\tau,v} \in \mathbb{R}^+ \qquad \text{for all } \tau \in \Gamma, e \in \mathcal{E}_{\texttt{route}}.$$

For a $c \in CN$, $c(\tau)$ corresponds to the coefficient of the variable w_e^τ for any edge e in the transshipment graph where cargo is transported on a ship, i.e. $\mathcal{E}_{\texttt{route}}$.

Objective function Income is made on all booking that are taken (all containers stowed on any of the ships), while there is a cost associated with loading and unloading containers (including transshipments), and for having the containers standing at the yard. Thus, the following is maximized [NB: where $D' < \frac{\mathbf{d}}{2}$ is a given deadline not previously described]:

$$
\begin{aligned}
&\sum_{\tau\in\Gamma}\sum_{(p',d')\in V}\Big(\sum_{(p,d)\in V.p<D'} x^{\tau,(p',d')}_{e^{(p,d)}_{\text{on}}}\cdot R^{\tau,p,p'} + \sum_{s\in S} x^{\tau,(p',d')}_{e^{s}_{0}}\cdot R^{\tau,p^s_1,p'} + \sum_{p\in P} x^{\tau,(p',d')}_{e^{p}_{0}}\cdot R^{\tau,p,p'}\Big)\\
&-\sum_{(p,d)\in V}\sum_{\tau\in\Gamma}\sum_{v'\in V}\Big(x^{\tau,v'}_{e^{(p,d)}_{\texttt{load}}} + x^{\tau,v'}_{e^{(p,d)}_{\texttt{unld}}}\Big)\cdot C^{\tau,p}_{\texttt{M}}\\
&-\sum_{p\in P}\sum_{i=1}^{|W_p|-1}\sum_{\tau\in\Gamma}\sum_{v'\in V} x^{\tau,v'}_{e^{v_p(i)}_{\texttt{nxP}}}\cdot\big(day_p(i+1)-day_p(i)\big)\cdot C^{\tau,p}_{\texttt{L}}
\end{aligned}
$$

Constraints

Flow conservation At all internal nodes in the transshipment graph, the flow of containers of each type and destination should be preserved. I.e. for nodes with both ingoing and outgoing edges, the incoming flow of containers (of a specific type and a specific destination-node) should equal the outgoing flow of containers (of that type and destination-node). That is,

$$\forall n \in \bigcup_{v\in V}\{n^v_{\texttt{si}}, n^v_{\texttt{so}}, n^v_{\texttt{yi}}, n^v_{\texttt{yo}}\}\ \forall\tau\in\Gamma\ \forall v\in V: \quad \sum_{e\in In(n)} x^{\tau,v}_e = \sum_{e\in Out(n)} x^{\tau,v}_e,$$

where $In(n) = \{\ (n',n'') \in \mathcal{E} \mid n'' = n\ \}$, and $Out(n) = \{\ (n'',n') \in \mathcal{E} \mid n'' = n\ \}$ for all $n \in \mathcal{N}$.

Outflow Every container taken on board a ship must be delivered (at the right destination):

$$\forall v \in V\ \forall \tau\in\Gamma: \quad x^{\tau,v}_{e^v_{\texttt{off}}} = \sum_{e\in InE} x^{\tau,v}_e,$$

where $InE = \bigcup_{v\in V}\{e^v_{\texttt{on}}\} \cup \bigcup_{s\in S}\{e^s_0\} \cup \bigcup_{p\in P}\{e^p_0\}$.

Capacity constraints

$$\forall c \in CN\ \forall e\in\mathcal{E}_{route}: \quad \sum_{\tau\in\Gamma} c(\tau)\cdot w^\tau_e \le c(\texttt{rhs})$$

Restriction on variables To help with preprocessing, we add the following (redundant) constraints saying that a container is not transported later then it shold be delivered:

$$\forall (p,d)\in V\ \forall e\in\mathcal{E}_{(p,d)}\ \forall\tau\in\Gamma\ \forall(p',d')\in V\ .\ d' < d: \quad x^{\tau,(p',d')}_e = 0.$$