Larson's Solutions

Shuai Jiang

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Contents

Preface

- This is an attempt at putting together a thorough solution guide to Larson's book.

- Although I dabbled in mathematics all my life, I believe my proof-writing skills are still weak. Please contact me about suggestions and errors!

Chapter 1

Heuristics

1.1   Search for a Pattern

Exercise 1.1.6

We just have to show that there exists an infinite number of the digit 6 in the sequence, and not to find a closed-form expression of it. If we extend the sequence, it should be of the following form $2, 7, 1, 4, 7, 4, 2, 8, 2, 8, 8, 1, 6, 1, 6, 1, 6, 6, 4, 8, 6, 6, 6, 6, \ldots$

The 4 consecutive 6's seems intriguing, so lets study what happens when we expand it out according to our rule: $3, 6, 3, 6, 3, 6, 3, 6$. This is good! Four consecutive 6's produces four more 6's, but the next iteration doesn't seem so promising: $1, 8, 1, 8, 1, 8, 1, 8, 1, 8, 1, 8, 1, 8$. There are no 6's this time.

Let's not give up so early, here are the next few iterations (with truncation):

$$8, 8, \ldots, 8 \to 6, 4, 6, 4, \ldots, 6, 4 \to 2, 4, 2, 4, \ldots, 2, 4 \to 8, 8, \ldots, 8. \qquad (1.1)$$

This is great news! We actually found a cycle which gaurentees us more 6's if we follow it, and hence the exercise is proven (informally). □

Exercise 1.1.7

All prime numbers satisfy the property that the first $n - 1$ digit of S_n are n. This pattern is fairly obvious if one starts to extend the S_n sequences out a bit more. The first conjecture might be that all odd numbers satisfy the property, but the case of S_9 is a counter-example.

The proof is as follows: we first note that if n is in S_n, then it will be incremented to $n + 1$ in S_{n+1}. Next, note that if an element of the sequence is a prime number, it can only increment by one if n (of the index S_n) is the same prime number.

If we put these two facts together, we see that given a prime p, the first $p-1$ numbers will increment along with S_n by the first statement to reach p as $n \to p$. The speed of incrementation might be different, but we really don't care about that by the second statement. If an element of the sequence reached p already, then it'll stay at p since no other numbers will divide it.

Note that the $p, p+1, p+2, \ldots$th numbers will be greater than p by the S_2. $\qquad\square$

Exercise 1.1.8

The intuition for this problem is quite easy to grasp as the patterns are laid out before our eyes. I believe the hard part is putting that solution to words. There's one thing that I used in proving (albeit less rigorous than I like) this problem, which is that the alternate sum of binomial coefficients is 0. This is a commonly used fact, and is quite useful at times.

Start with $n = 3$ for a concrete example. The following sequence is an acceptable solution:

$$01212123 \tag{1.2}$$

where each digit signifies the index in Pascal's triangle. We start with the null set, then pick any set with one element and alternate with subsets of two elements and so on. The algorithm is a greedy one, and we wish to show it satisfies all the conditions.

Condition one is easy to show, but why are we guaranteed that conditions (ii) and (iii) of the problem is satisfied? For condition (iii), note that if we simply alternate in the order of the elements, we will always be able to alternate between a subset of size m and $m + 1$. So in our small example, we would add the subset of just element 1, then add element 2, take away element 1, add element 3 and so on.

To see that we do indeed use every subset, we need to put our algorithm into numbers. Using our small example again, we start with the standard Pascal triangle row:

$$1\|3\|3\|1. \tag{1.3}$$

This means our sequence needs to have one 0, three 1's, three 2's and one 3 (like in equation 1.2). If we alternate between the $\binom{3}{0}$ and $\binom{3}{1}$ we will end up with this:

$$0\|\dot{2}\|3\|1 \tag{1.4}$$

where the dot indicates the size of the subset of the end of our sequence. Alternating between $\binom{3}{1}$ and $\binom{3}{2}$ will result in

$$0\|0\|\dot{0}\|1. \tag{1.5}$$

This configuration means that our sequence's last element is of size 2, and we have one last subset of size 3 to take care of. At the end, all our numbers will be 0; this is what we want to capture in our proof.

If one continues this pattern in general for all n, we will have the following equation

$$\binom{n}{n} - \left(\binom{n}{n-1} - \left(\binom{n}{n-2} - \cdots \left(\binom{n}{1} - \binom{n}{0}\right) + 1\right) + \cdots + 1\right) \tag{1.6}$$

I realize that's a big jump, but try to convince yourself that if you work out the pattern, it's what you see. There's really no work left, as that equation simplifies to the alternating sum of binomial coefficients which is 0! This proves[1] that our algorithm will satisfy all three conditions. $\qquad\square$

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[1]Not that rigorous I suppose

Exercise 1.1.9

Two immediate ways to solve this:

- Look at the Pascal triangle in terms of even-odd. You'll quickly notice a pattern that the odd binomial coefficients resemble a Sierpinski triangle. It'll be extremely tedious, but I think given enough patience and organization one can count them in terms of the pattern.

- Note the following pattern starting from row 1: $1, 2, 2, 4, 2, 4, 4, 8, 2$. Instead of looking at the powers of 2, maybe the exponents will give us more insight: $0, 1, 1, 2, 1, 2, 2, 3, 1$. An *extremely* astute observer will note that it's the number of ones in the binary expansion of the row number! For example, row 3 (1,3,3,1) is 11 in binary so the number of odd coefficients is 2^2 for the number of 1s in the binary representation.

  I remembered this trick back in high school, so don't fret if you can't figure this out. The theorem which this is based on is Lucas' theorem.[2] □

Exercise 1.1.10

I'm not sure how to cleverly solve this one. My approach was to look at the equations in various mods (i.e. 2, 4, and 8) and see what I can clobber together. I couldn't prove that all such primes are able to be expressed in the forms though.

Exercise 1.1.11

The conjecture should be fairly simple to make; try writing a few terms of the sequence starting with $1/2, 1, 2, -1$. The limit seems to be 1, which is easily proved with a technique that's covered in 7.6.4 as mentioned in the problem.

In order to prove this though, one can approach the problem like 1.1.3. We note that $a_2 = \frac{1}{2-a_1}$. Continuing the series:

$$a_3 = \frac{1}{2 - a_2} = \frac{1}{2 - \frac{1}{2-a_1}} = \frac{2 - a_1}{3 - 2a_1} \tag{1.7}$$

$$a_4 = \frac{3 - 2a_1}{4 - 3a_1} \tag{1.8}$$

$$a_5 = \frac{4 - 3a_1}{5 - 4a_1} \tag{1.9}$$

The patterns is quite obvious now:

$$a_n = \frac{(n-1) - (n-2)a_1}{n - (n-1)a_1} \tag{1.10}$$

This can be proven by induction quite easily (but a pain to type out). The limit of that as n approaches infinity is 1. □

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[2] `http://en.wikipedia.org/wiki/Lucas'_theorem`

Exercise 1.1.12

This took a considerable amount of time to proof, from just getting a feel of the problem to finally seeing a solution. The key was the expression $(a \cdot b) \cdot ((a \cdot b) \cdot b)$. Note that we can express this in two forms! It can equal b (by considering $a \cdot b$ together) or $(a \cdot b) \cdot a$ (by considering the right most parenthesis first).

Now that we have $b = (a \cdot b) \cdot a$, we simply apply a to both sides to get $b \cdot a = ((a \cdot b) \cdot a) \cdot a = a \cdot b$. $\qquad\square$

1.2   Draw a Figure

1.3   Formulate an Equivalent Problem

Exercise 1.3.8

The hint is a pretty big one; if we use it, we arrive at a polynomial of the form $(y + 1)^7 - 2(y + 1)^5 + 10(y + 1)^2 - 1$. If we naively expand everything, we will see that the corresponding equation is of the form

$$y^7 + 7y^6 + 19y^5 + 25y^4 + 15y^3 + 11y^2 + 17y + 18 \qquad (1.11)$$

This polynomial has coefficients bigger than one, hence for any positive y, our polynomial will be greater than 0. $\qquad\square$

Exercise 1.3.9

The simplest way to do this is to recast this problem as a combinatorics problem. Consider viewing the number n as n individual stones and consider the gaps in between. There is a one-to-one correspondence between choosing whether we want to select the gap (thus partitioning up the stones) and the quantity we are looking for.

For example, in the case of $n = 3$, the case of $1+2$ would be to select the first gap and not select the second. The result is immediate after this recasting. $\qquad\square$

Exercise 1.3.10

Once again, view this an a combinatorics problem, but more sticks-and-balls type.[3] Imagine expressing the number 10 as ten stones and having four sticks which can partition the stones into five. Since we care about order, then it's simply the number of ways we can order the the balls and four sticks $\binom{10+4}{4}$.

See the footnotes for a bit more detailed solution on how the whole sticks-and-ball problems work. $\qquad\square$

Exercise 1.3.11

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[3]http://www.math.illinois.edu/~hildebr/putnam/training/combinatorial13-2sol.pdf

Using the hint, we can view this problem as a geometry exercise. The first expression $x^2 = y^2$ is simply two intersecting lines at the origin, while the second expression is that of a circle centered at $(a, 0)$ with radius 1.

From this, it is easy to see that there are four solutions when $\{-\sqrt{2} < a < \sqrt{2}\} - -1, 1$, three solutions at $1, -1$, two solutions $a = \pm\sqrt{2}$, and zero solutions elsewhere. Nowhere is there only one solution due to symmetry of the problem. □

Exercise 1.3.12

TODO

Exercise 1.3.13

TODO

Exercise 1.3.14

First, some notation. Let a, b, c be the sides which correspond to p_1, p_2, p_3 respectively. We use the hint:

$$2A = ap_1 = bp_2 = cp_3 \tag{1.12}$$
$$2A = r(a + b + c) \tag{1.13}$$

The first equation comes from basic geometry, while the second comes form noticing that the incircle is tangent to each side; hence a right angle with each side.

Now, we simply substitue the $a = \frac{2A}{p_1}$ into the second equation, along with b, c. Cancelling the $2A$, from both sides of the equation and moving r to the other side gives us the answer. □

1.4   Modify the Problem

Since there are no exercises in this section, I'll go through two of the problems.

Exercise 1.4.1

The last inequality is proven in the book; a big hint is to use Cauchy-Schwarz.

Exercise 1.4.2

One key fact is that the hypotenuse of a circumscribed right triangle is the diameter of the circle.

Exercise 1.4.3

Consider the modulo of the problem with respect to a number.

1.5   Choose Effective Notation

Exercise 1.5.5

This problem is quite simple.

1. $4c = 5s$ where c is the number of people of ordered cheesecake, and s the number who ordered strudel.

2. $6s = p$ where p, s is the number of professors and students respectively.

Exercise 1.5.6

Let the height of the crossing be denoted h, and let the left pole be of height x and the right pole of height y. We seek to establish a relationship between the three variables, which we will do using similar triangles.

Let the distance between the bases of the two pole be denoted $c = c_1 + c_2$ where c_1 is the segment from the left pole to the crossing, and c_2 the segment from the right pole to the crossing. We can then write the following expression from viewing the similar triangles:

$$\frac{c_2 x}{c_1 + c_2} = \frac{h}{c_2} \tag{1.14}$$

Similarly for the other variable:

$$\frac{c_1 y}{c_1 + c_2} = \frac{h}{c_1}. \tag{1.15}$$

Combining the two and solving for $\frac{x}{y}$, we have $\frac{x}{y} = \frac{c_1}{c_2}$. Now:

$$h = \frac{c_2 x}{c_1 + c_2} \tag{1.16}$$

$$= \frac{x}{\frac{c_1}{c_2} + 1} \tag{1.17}$$

$$= \frac{x}{\frac{x}{y} + 1} \tag{1.18}$$

$$= \frac{xy}{x + y} \tag{1.19}$$

$\square$

Exercise 1.5.7

(A bit lazy with the figures for this one, so I'll try to explain this with words). Note that there are probably many solutions to this. Let k be the length of the right triangle that is opposite angle θ. We can express k as a $\tan(\theta) = \frac{k}{L}$.

Since we know the width of the paper, the remaining length is $8 - k$. The hypotenuse of the right triangle at the bottom left corner of figure 1.17 is of length k also, as it's simply folded up. Only one thing remains, which is that the angle for the right triangle at the bottom is of 2θ, by simply tallying up the angles.

Finally, we have the expression:

$$\cos(2\theta) = \frac{k}{8 - k} \tag{1.20}$$

$$\frac{8\cos(2\theta)}{1 + \cos(2\theta)} = k \tag{1.21}$$

$$\frac{8\cos(2\theta)}{1 + \cos(2\theta)} = k \tag{1.22}$$

$$\frac{8\cos(2\theta)}{1 + \cos(2\theta)} = L\tan(\theta) \tag{1.23}$$

$$L = \frac{1}{\tan(\theta)} \frac{8\cos(2\theta)}{1 + \cos(2\theta)} \tag{1.24}$$

One can simplify this more, but the main goal is accomplished. $\qquad\square$
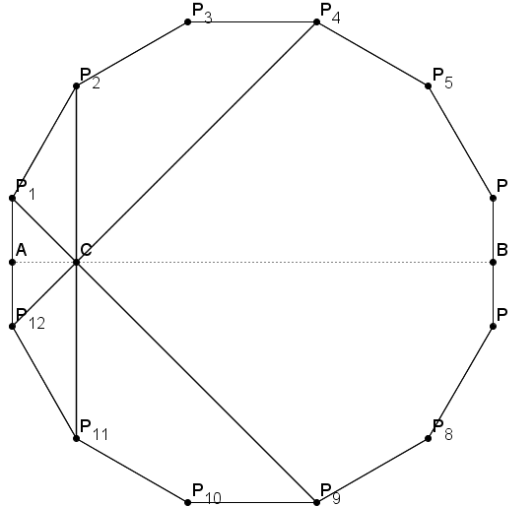

Figure 1.1: Exercise 1.5.8

Exercise 1.5.8

We see from figure 1.1 that indeed they are concurrent, but we need to prove this rigorously.

The angle of $CP_1P_{12} = P_{12}P_1C = 45°$ by looking at polygons created by the diagonals itself and the dodecagon. If we let the side length of the dodecagon be x, then P_1P_9 and P_4P_{12} intersects at exactly $x/2$ distance away from the segment P_1P_{12} by the properties of right triangle ($P_1P_{12}C$ is the special right triangle).

It is obvious that P_2P_{11} creates a trapezoid with angles 150 and 30. Since we have the angle, we can now find the height of the trapezoid, which is exactly $x/2$ as predicted (using $\sin 30°$).

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One has to tie this proof a bit tighter for the Putnam exam though by proving the diagonals first intersect line AB, [4] but this is a good enough sketch I think. □

Exercise 1.5.9

1. Average rate by definition means the total distance over total time. The key point to note is that the time spent driving from point A to B will be *more* then the other way around, since the speed is slower. Henyce, the average rate is less than 50 miles per hour.

   Algebraically, let x be the distance between A and B. We have the following:

$$s = \frac{d}{t} \tag{1.25}$$

$$= \frac{2x}{\frac{x}{40} + \frac{x}{60}} \tag{1.26}$$

$$= \frac{2(2400)}{40 + 60} \tag{1.27}$$

   and we see that it's the harmonic average of the two speeds. The cool thing is that the distance doesn't matter at all! □

2. Of course, we see physically that the two must equal, but we should do this algebraically. We'll do this the hard way, by keeping both the amounts of liquid and the volume of the spoon as variables (instead of scaling the spoon volume to 1).

   Let V be the volume of the coffee (and the cup of cream), then let x be the volume of the spoon. The initial scoop will result in V coffee and x cream in the coffee cup and $V - x$ cream in the cream cup. Now the scoop from the coffee cup will contain $x \cdot \frac{x}{V+x}$ amount of cream, and $x \cdot \frac{V}{V+x}$ amount of coffee.

   This means that the final amount of cream left in the coffee cup is $x - x \cdot \frac{x}{V+x}$, while the total amount of coffee in the coffee cup is $0 + x \cdot \frac{V}{V+x}$. One can verify easily that they are equal. □

3. Let the radius of the Earth in feet be R, then the original length of the rope is $2\pi R$. Extending that string by 6 feet will result in the new length to be $2\pi R + 6$.

   We wish to find the new radius which have that value as its circumference; this is done by dividing by 2π. Thus, the string is lifted by $\frac{6}{2\pi}$ total feet, which is around 1 foot and greater than an inch. □

Exercise 1.5.10

[4]a symmetry argument should do here

1. Note the residuals of the odd numbers squared in mod 8 is always 1. We also know that there exists m, n such that $m^2 + n^2 = p$ for p prime and one more than a multiple of 8. If we simply examine the residuals, we see that both one of m, n must be odd and the other must be 0 or 4 mod 8. In both cases, we can rewrite $n := 4y$ (i.e. it's a multiple of 4), and thus we have our form. $\square$

2. Similar to above, we see that $4y^2$ will result in residuals of 0 or 4, meaning that it *has* to be of 4 mod 8. This is because there is not a residual of 5 in the mod 8!

   Since the residual of 4 only comes in when n is divisible by 2, we can rewrite $n := 2y$.

   The first term is kind of a mis-nomer... as it can be simply $x^2 + 4y^2$... $\square$

1.6   Exploit Symmetry

Exercise 1.6.6

1. By symmetry, we can simply expand the first term and know what the other terms will be like. Note that by expanding the x^2y element in the first parenthesis, we have $x^3y^3, x^2y^2z^2$ and x^4yz. Thus, we will have $3x^3y^2z^2 + x^3y^3 + x^3z^3 + y^3z^3 + x^4yz + xy^4z + xyz^4$.

2. There's a typo in my version of the book. It should be $x^5 + y^5 + z^5$ instead of $x^2 + y^5 + z^5$.

   If we actually use the proposed substitution, this turns into a tedious problem in algebra. Fortunately, symmetry gives us a few shortcuts. First, notice that the leftmost group gives $x^2 + xy + y^2$ and the middle group is the binomial expansion of $(-x - y)^5$ without the initial x^5 or y^5 term all over 5. THe RHS is of the same format: binomial expansion of $(-x - y)^7$ without x^7, y^7 term divided by 7.

   Now by symmetry, we can easily see that the two equations are equal (if you don't trust me, just expand it out).

Exercise 1.6.7

This is a hard figure to draw, so I'll introduce some notation for me to not draw it. We'll use ordered pair notation, where the first coordinate is which row from the bottom, and the second coordinate is which column from the left. For example, $(0, 0)$ would be the bottom-right most penny, $(0, 1)$ would be to the right of it. Now $(1, 0)$ would be the first penny in the second row meaning that $(5, 0)$ is the only element with 5 in the first coordinate (since there's only one penny on the fifth row). Be aware this is zero-indexing.

We will prove by contradiction and let's examine the inner-most triangle $(1, 1), (1, 2), (2, 1)$. WLOG let's say that $(2, 1)$ is of the color white, and the other two points are black by
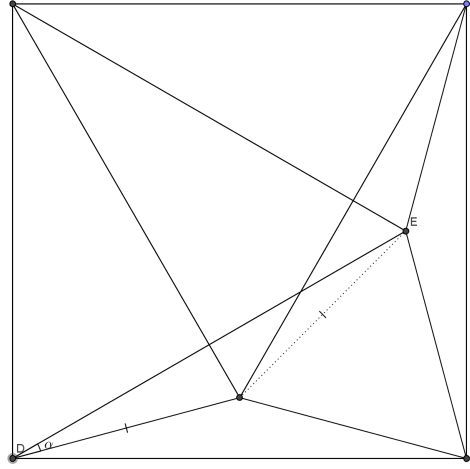
Figure 1.2: Exercise 1.6.10

pigeon-hole principal. This then forces $(0, 2)$ to be also white as if it's black, then $(0, 2), (1, 1), (1, 2)$ will form a triangle.

Now note that $(1, 0), (1, 3)$ MUST be black also as there's a equilateral triangle with the color white with $(1, 0), (2, 1), (0, 2)$ (and by symmetry, the other one too). Finally, if we look at $(0, 1)$ or $(0, 3)$, they can't be black or white! Thus, contradiction and we are done. $\qquad\square$

Exercise 1.6.8

Exercise 1.6.9

Exercise 1.6.10

We exploit the symmetry and draw another set of triangles on the bottom side of the square. See figure 1.2 for reference.

The first thing we note is that the dotted line connecting the vertex of the two 15 degree triangles will result in an equilateral triangle. Now by symmetry, if angle α is 15 degrees then we will have proven what we wanted to prove.

Triangle ADE contains α twice, with all the other parts known, as they are either 15 degrees or equilateral triangles. The expression we have is $2\alpha + 60 + 60 + 15 + 15 = 180 \rightarrow \alpha = 15$. See Geometry Revisited for another view on this problem. $\qquad\square$

Exercise 1.6.11

Let a be the second term, and d be the difference. We wish to show that $(a - d)a(a +$

$d)(a + 2d) + d^4$ is a perfect square.

$$\begin{aligned}
(a - d)a(a + d)(a + 2d) + d^4 &= (a^2 - d^2)(a^2 + 2da) + d^4 \\
&= a^4 + 2da^3 - d^2a^2 - 2d^3a + d^4 \\
&= (a^2 + ad - d^2)^2
\end{aligned}$$

$\square$